On complete classes of valuated matroids

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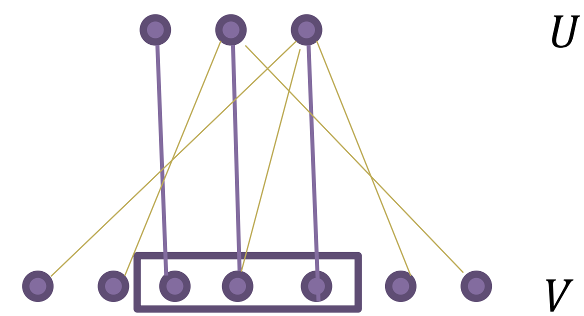
Warm up:
Complete classes of matroids

Transversal matroids and gammoids

Transversal matroid

Bipartite graph $G = (V, U; E), |U| = d$

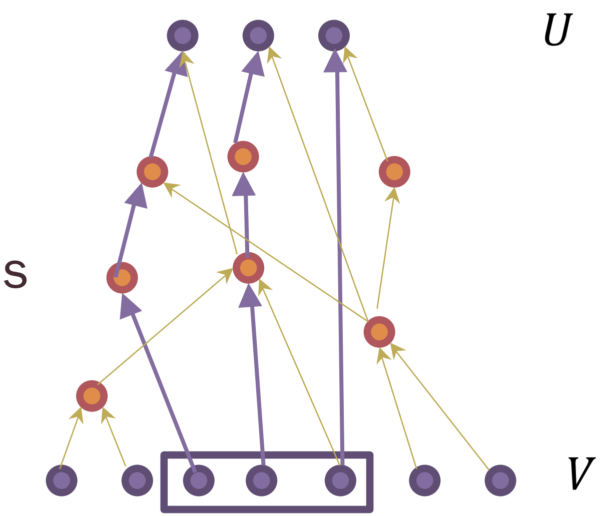
$X \subseteq V$ is a basis if there is a perfect matching between X and U



Gammoid

Directed graph $G = (N, A), U, V \subseteq N, |U| = d$

$X \subseteq V$ is a basis if there exist d node−disjoint paths between X and U



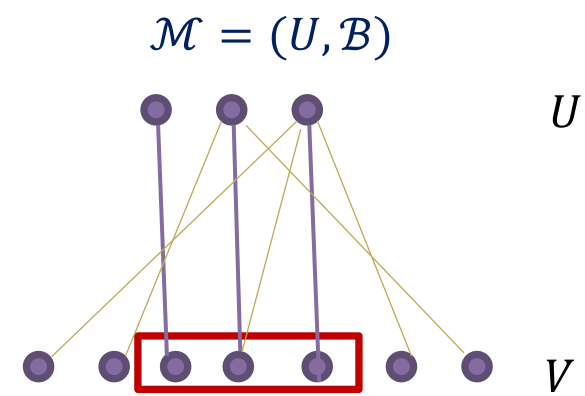
Operations on matroids

Deletion, Contraction, Duality, Direct sum, Truncation

Induction by bipartite graph

Bipartite graph $G = (V, U; E), \mathcal{M} = (U, \mathcal{B})$

$X \subseteq V$ is a basis if there is a perfect matching between X and a basis in $\mathcal{M}$

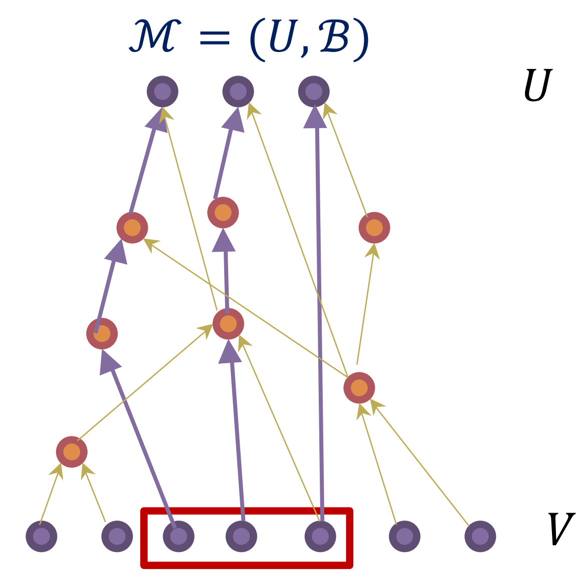


Transversal matroid = induction by bipartite graph from a free matroid

Induction by network

Directed graph $G = (N, A), U, V \subseteq N, \mathcal{M} = (U, \mathcal{B})$

$X \subseteq V$ is a basis if there exist d node−disjoint paths between X and a basis in $\mathcal{M}$



Gammoid = induction by network from a free matroid

Complete classes of matroids

- Ingleton 1977:

> Let us call a class of matroids complete if it is closed under the operations of restriction, contraction, dualization, direct sum, truncation and induction. (The same concept could be defined by various different lists of permissible operations, depending on personal taste. This list certainly includes some logically redundant items.)

- It suffices: induction by bipartite graph, direct sum and contraction.

- Gammoids are the smallest complete class containing the free matroid on 1 element. Not all matroids are gammoids.

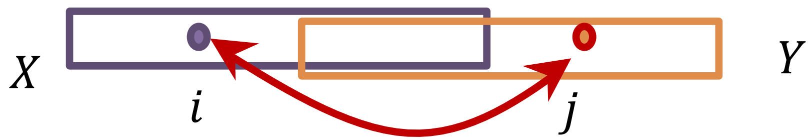
Valuated matroids

Valuated matroids

$\mathcal{M} = (V, \mathcal{B})$, ground set V, bases $\emptyset \neq \mathcal{B} \subseteq 2^V$ such that

$$\forall X, Y \in \mathcal{B}, \forall i \in X \setminus Y \;\; \exists j \in Y \setminus X$$
$$X - i + j, Y + i - j \in \mathcal{B}$$

symmetric basis exchange axiom



All bases have the same cardinality $d = \mathrm{rk}(\mathcal{M})$, the rank of $\mathcal{M}$

Dress & Wenzel 1990. $f : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ is a valuated matroid if

$$\forall X, Y \in \binom{V}{d}, \forall i \in X \setminus Y \;\; \exists j \in Y \setminus X$$
$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$

- The support $\left\{X \in \binom{V}{d} : f(X) > -\infty\right\}$ forms the basis of a matroid.

Examples

- $f : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$

- Trivially valuated matroid: $\mathcal{M} = (V, \mathcal{B})$, $\mathrm{rk}(\mathcal{M}) = d$

$$f(X) = \begin{cases} 0 \text{ if } X \in \mathcal{B} \\ -\infty \text{ if } X \notin \mathcal{B} \end{cases}$$

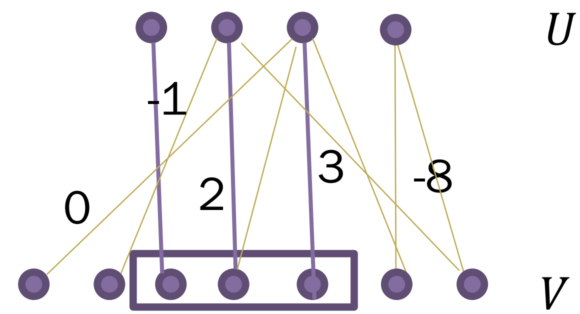
- Weighted matroids: $\mathcal{M} = (V, \mathcal{B})$, $\mathrm{rk}(\mathcal{M}) = d, w \in \mathbb{R}^V$

$$f(X) = \begin{cases} \sum_{i \in B} w_i \text{ if } X \in \mathcal{B} \\ -\infty \text{ if } X \notin \mathcal{B} \end{cases}$$

$$f\colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$$

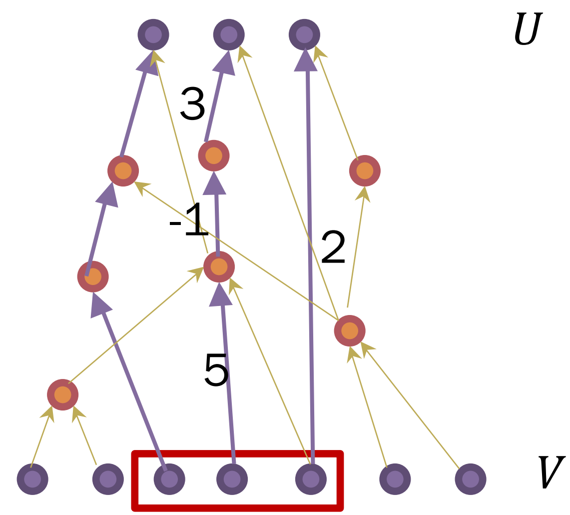
- **Valuated transversal matroid:** Bipartite graph $G = (V, U; E), |U| = d, w \in \mathbb{R}^E$

  $f(X) =$max cost of a perfect matching between X and U



- **Valuated gemmoid:** Directed graph $G = (N, A)$, $U, V \subseteq N, |U| = d$

  $f(X) =$max cost of a d node−disjoint paths between X and U



- ## Matrices with polynomial entries

A: $d \times n$ matrix over $\mathbb{R}[t]$

$$f(X) = \text{degree of the determinant of } A_X \text{ for } X \in \binom{V}{d}$$

$$\begin{array}{cccc} a & b & c & d \\ \end{array}$$

$$\begin{pmatrix} t^2 & 1 & 1 & 1 \\ 1+t & 0 & 0 & 1 \end{pmatrix}$$

	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$
det	$-t-1$	$-t-1$	t^2-t-1	0	1	1
deg(t)	1	1	2	$-\infty$	0	0

Valuated matroids and matroids

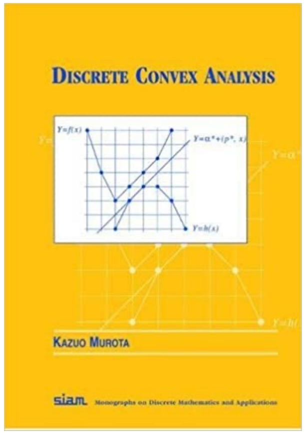
$f: \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ is a valuated matroid if

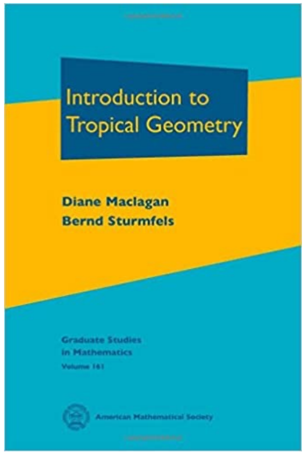
$$\forall X, Y \in \binom{V}{d}, \forall i \in X \setminus Y \ \ \exists j \in Y \setminus X$$
$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$

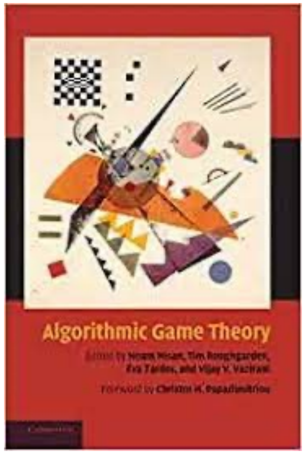
- The support $\left\{X \in \binom{V}{d} : f(X) > -\infty\right\}$ forms the basis of a matroid
- For any cost function $p \in \mathbb{R}^V$, the set $\arg\max f(X) - p(X)$ forms the basis of a matroid as
  - $f(X) - p(X)$ is also valuated matroid, and
  - $\arg\max f(X)$ forms the basis of a matroid
- Dress & Wenzel 1990: the greedy algorithm naturally extends to valuated matroids, i.e., greedy solves $\max f(X)$

Valuated matroids everywhere

- Dress & Wenzel 1990: greedy algorithm naturally extends to valuated matroids
- Discrete convex analysis, Murota 1996-: Valuated matroids = M-concave functions on $\{0,1\}^n$
- Tropical geometry: Valuated matroids ~ Tropical linear spaces
- Game theory and mechanism design Generalized valuated matroids~ **Gross substitute valuations**







Operations on valuated matroids

$$f\colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$$

- Deletion $Z \subseteq V$:

  $$V' = B \setminus Z, d' = d, f'(X) = f(X)$$

- Contraction $Z \subseteq V$:

  $$V' = B \setminus Z, d' = d - |Z|, f'(X) = f(X \cup Z)$$

- Duality

  $$V' = V,\ d' = |V| - d, f'(X) = f(V - X)$$

- Direct sum $f_1\colon \binom{V_1}{d_1} \to \mathbb{R} \cup \{-\infty\}$, $f_2\colon \binom{V_2}{d_2} \to \mathbb{R} \cup \{-\infty\}$

  $$V' = V_1 \cup V_2, \qquad d' = d_1 + d_2,$$
  $$f'(X_1 \cup X_2) = f_1(X_1) + f_2(X_2)$$

- Truncation:

  $$V' = V,\ d' = d - 1, f'(X) = \max_{v} f(X + v)$$

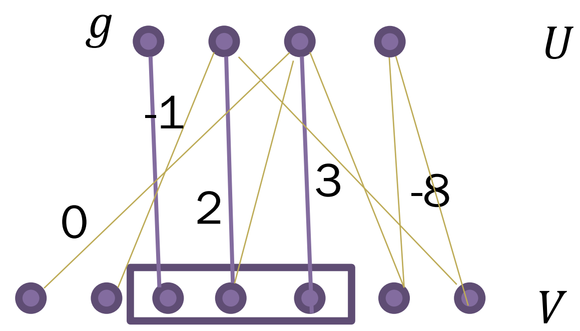
Operations on valuated matroids

Induction by bipartite graph

Bipartite graph $G = (V, U; E)$,

g valuated matroid on $U, w \in \mathbb{R}^E$

$f(X) = \max\{w(M) + g(Y)$:



M matching between X and $Y \subseteq U\}$ and $Y \subseteq U\}$

Valuated transversal matroid = induction by bipartite graph from a trivially valuated free matroid

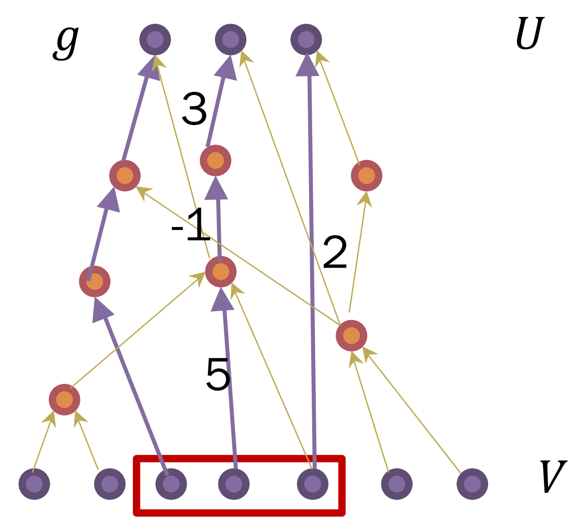
Induction by network

Directed graph $G = (N, A), U, V \subseteq N$,

g valuated matroid on $U, w \in \mathbb{R}^A$

$f(X) = \max\{w(F) + g(Y)$:

F node disjoint paths between X and $Y \subseteq U\}$

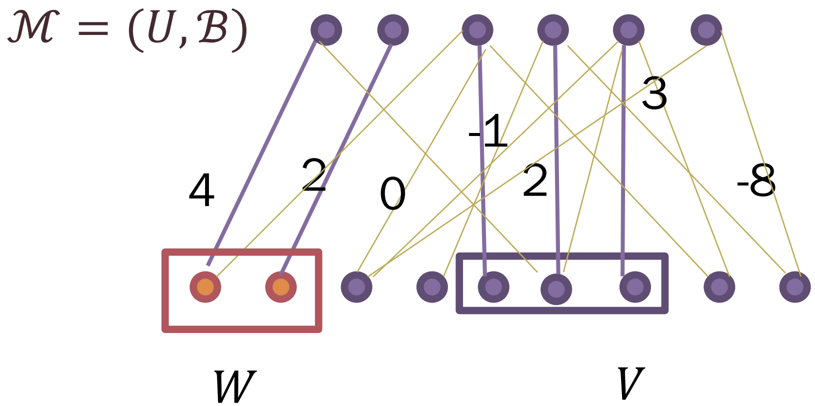


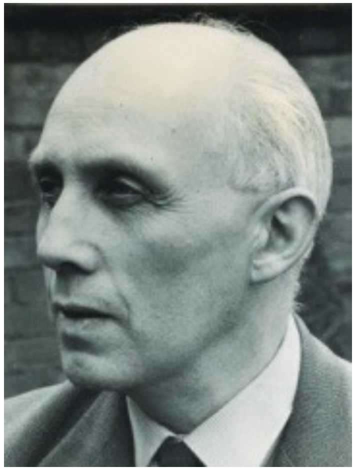
Complete classes of valuated matroids

A class of valuated matroids is complete, if it is closed under contraction, deletion, duality, truncation, and induction by network.

- Induction by bipartite graph, direct sum and contraction suffice.
- Smallest complete class: valuated gammoids
- *What is the smallest complete class containing all trivially valuated matroids?*

R-minor valuated matroids





Richard Rado
(1906-1989)

- Bipartite graph $G = (W \cup V, U; E), \mathcal{M} = (U, \mathcal{B}), \mathrm{rk}(M) = d + |W|$
- $f: \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$

$$f(X) = \max\{w(M): M \text{ is a perfect matching between } X \cup W \text{ and some } Y \in \mathcal{B}\}$$

THEOREM (HLSV'21): R-minor valuated matroids are the smallest complete class containing all trivially valuated matroids

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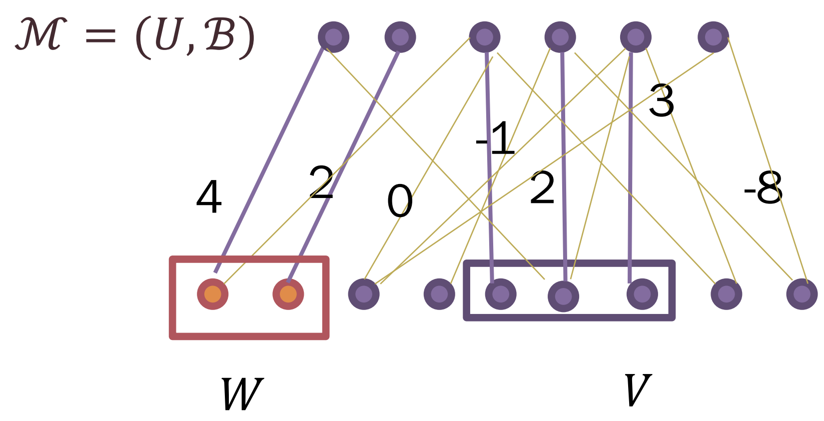
Is every valuated matroid R-minor?

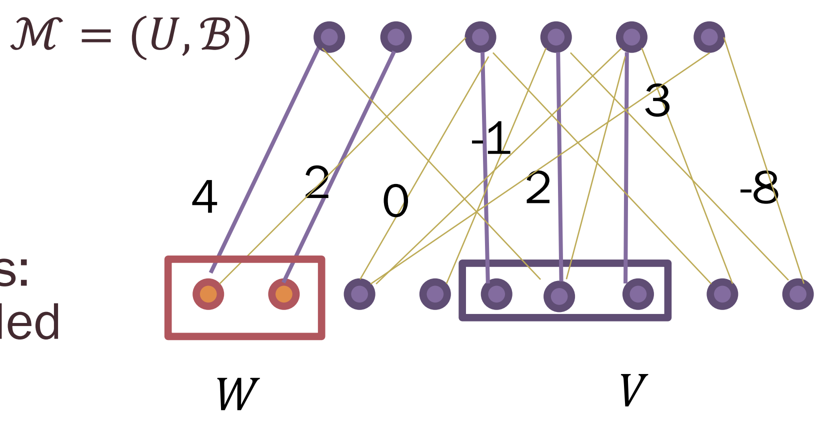
- Variant of this question asked by Frank in 2003, popularized later by Murota
- Closely related questions on gross substitutes valuations: Hatfield & Milgrom 2005, Ostrovsky & Paes Leme 2015
- If the answer is yes…
  Valuated matroids = matroid + graphs + weights
- And the answer is…



NO

THEOREM (HLSV'21): Not every valuated matroid is R-minor.

There are valuated matroids that do not arise from unvaluated matroids using induction by network and other simple operations.



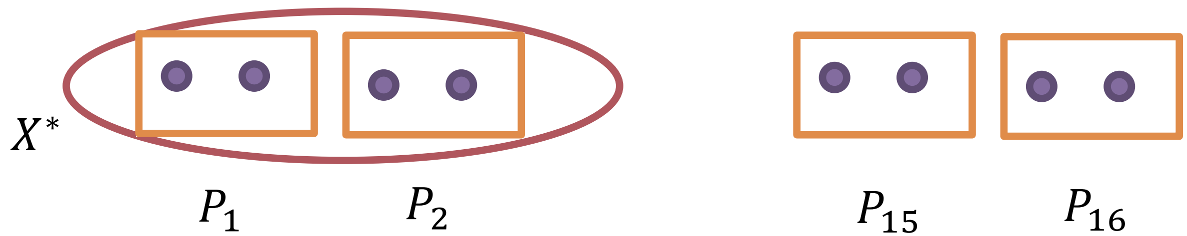
Difficulties

- Cannot argue with matroid invariants: every matroid rank function is included
- No (obvious) information theoretic arguments: the contracted set W can be arbitrarily large

THEOREM (HLSV'21): Not every valuated matroid is R-minor.

- Ground set $V = \{1,2,\ldots,32\} = P_1 \cup P_2 \cup \cdots \cup P_{16}$, $d = 4$

$$\mathcal{H} = \{P_i \cup P_j : \text{either } i \text{ or } j \text{ } is \text{ even}\}$$
$$X^* = P_1 \cup P_2 = \{1,2,3,4\}$$



$$f(X) = \begin{cases} 0 & \text{if } X \notin \mathcal{H} \\ -1 & \text{if } X = X^* \\ < -1 & \text{otherwise} \end{cases}$$

THEOREM (HLSV'21): Any such f is a valuated matroid that is not R-minor.

- Ground set $V = \{1,2,\ldots,32\} = P_1 \cup P_2 \cup \cdots \cup P_{16}$, $d = 4$

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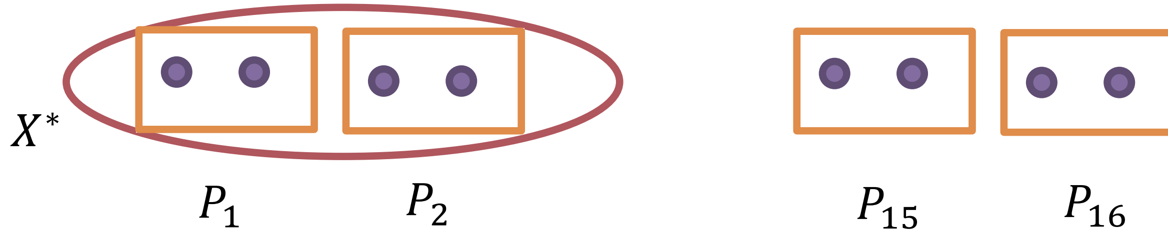

- $\mathcal{M} = (V, \mathcal{H})$ is a sparse paving matroid
- Careful smallest counter-example
- LP relaxation of R-minor representation, dual Lovász extension
- The dual matroid of $\mathcal{H}$ (maximizers of f) is not fully reducible: not a sum of smaller matroids.
- Uncrossing, uncrossing & uncrossing:
  - Case $W = \emptyset$: independent matching for X^* and X can be recombined to give matching of weight ≥ -1 for some Y with $f(Y) < -1$
  - Case $W \neq \emptyset$: showing that nodes in W can be removed

Connections to mathematical economics

Gross substitutes valuations

- $v: 2^V \to \mathbb{R}_+$ valuation function over a set of n indivisible items
- Given prices $p \in \mathbb{R}^V$, the agent wishes to buy a set of goods from

$$D(v,p) = \arg\max_S v(S) - p(S)$$

- **Gross substitutes property**: if the price of some items goes up, then the demand for all other goods may not decrease.

Gross substitutes valuations

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- **Gross substitutes property:** if the price of some items goes up, then the demand for all other goods may not decrease.
- Introduced by Kelso & Crawford in 1982 in the context of matching markets & auction algorithms
- Gül & Stacchetti 1999: Set of valuations for which **Walrasian equilibrium** exists and can be found in poly-time.
- For GS valuations, $D(v,p)$ can be found by greedy algorithm.
- GS are valuations under which $\sum_{\{i\}} v_i(S_i)$ where $\{S_i\}_i$ is an allocation of some items is efficiently computable.

Gross substitutes valuations

$f: 2^V \rightarrow \mathbb{R} \cup \{-\infty\}$ is a valuated generalized matroid if

- $\forall X, Y \subseteq V, \quad |X| < |Y| \; \exists j \in Y \setminus X$
$$f(X) + f(Y) \leq f(X + j) + f(Y - j)$$
- $\forall X, Y \subseteq V, \; |X| = |Y|, \quad \forall i \in X \setminus Y \;\; \exists j \in Y \setminus X$
$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$

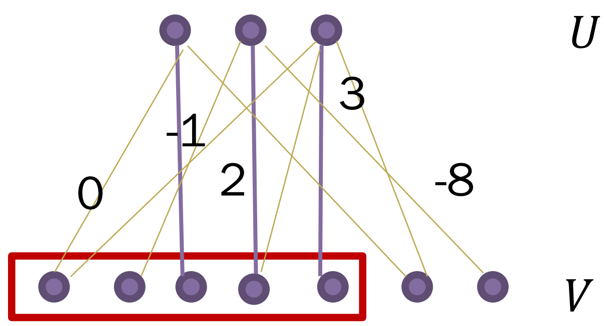
- Valuated matroids = M-concave set functions ~ bases
- Valuated generalized matroids = M♮-concave set functions ~ independent sets

- Valuated generalized matroids are submodular functions

THEOREM (Fujishige & Yang 2003): Valuated generalized matroids = Gross substitutes valuation functions

Assignment valuations

~ transversal valuated matroids



Bipartite graph $G = (V, U; E), w \in \mathbb{R}^E$

$v(X) =$max cost of a matching with endpoints in X

Construction of substitutes

- Hatfield & Milgrom 2005: does every GS valuation arise from assignment valuations using endowment (~contraction)?

- Ostrovsky & Paes Leme 2015: no, because not all matroid rank functions arise in this form

Matroid based valuation conjecture: every GS valuation arises from weighted matroid rank functions using endowment (~contraction) and merge (~convolution)

- Tran 2020: merge only does not suffice
- Why care? GS valuations are central in auction design. Constructive description of GS valuations specifies a succinct language in which agents express their valuations.

Matroid based valuation conjecture: every GS valuation arises from weighted matroid rank functions using endowment (~contraction) and merge (~convolution)

- Refined version: every GS valuation arises from matroid rank functions using endowment and induction by bipartite graph
- A strengthening of the conjecture valuated matroids = R-minor
- Take a counterexample f from our construction scaled to value range $(-1,0]$

$$h(X) = \begin{cases} |X| \text{ if } |X| \leq 3 \\ 4 + f(X) \text{ if } |X| = 4 \\ 4 \text{ if } |\mathrm{X}| \geq 5 \end{cases}$$

How did we get interested in this problem?

- Nash Social Welfare problem: given m indivisible items and n agents with valuation functions $v_i: 2^m \to \mathbb{R}_+$, find an allocation $[m] = S_1 \cup S_2 \cup \cdots \cup S_n$ that maximizes

$$\prod_{i=1}^{n} v_i(S_i)^{\frac{1}{n}}$$

- Cole & Gkatzelis 2015, AGSS 2017, BKV 2018: constant factor approximations for additive utilities
- Li & Vondrák 2021: estimation algorithm for conic combinations of Rado valuations
- Garg, Husić, V. 2021: constant factor approximation for Rado valuations
- Li & Vondrák 2021: constant factor approximation for general submodular valuations

Summary

- Valuated matroids $\neq$ R-minor: valuated matroids are (much?) more complex objects than matroids.

- The matroid based valuation conjecture is false.

- Implication on Lorentzian polynomials.

Open questions

- Can we formulate a succinct property satisfied by R-minor but not all valuated matroids?
- Can we bound the size of the contracted set W in R-minor representations by poly($|V|$)?
  Kratsch & Wahlström 2020: bound for gammoids
- Can we 'well-approximate' valuated matroids using R-minor functions?
- How well can Rado/gross substitutes approximate nonnegative submodular functions?
  Dobzinski, Feige, Feldman 2020: lower bound $\Omega\left(\frac{\log n}{\log \log n}\right)$

Thank you!