On the Identifiability of Sparse Vectors from Modulo Compressed Sensing Measurements

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***Abstract*—Compressed sensing deals with recovery of sparse signals from low dimensional projections, but under the assumption that the measurement setup has infinite dynamic range. In this paper, we consider a system with finite dynamic range, and to counter the clipping effect, the measurements crossing the range are folded back into the dynamic range of the system through modulo arithmetic. For this setup, we derive theoretical results on the minimum number of measurements required for unique recovery of sparse vectors. We also show that recovery using the minimum number of measurements is achievable by using a measurement matrix whose entries are independently drawn from a continuous distribution. Finally, we present an algorithm based on convex relaxation and develop a mixed integer linear program (MILP) for recovering sparse signals from the modulo measurements. Our empirical results demonstrate that the minimum number of measurements required for recovery using the MILP algorithm is close to the theoretical result for signals with low variance.**

***Index Terms*—Modulo compressed sensing, ℓ_1 recovery.**

I. Introduction

The effect of dynamic range in data acquisition systems has been an important research area in signal processing [1]–[4]. Systems with low dynamic range lead to signal loss due to clipping, and high dynamic range systems with finite resolution sampling are affected by high quantization noise. A direction of research in recent years to counter this problem has been the so-called self-reset analog to digital converters (SR-ADCs) [5], [6], which fold the amplitudes back into the dynamic range of the ADCs using the modulo arithmetic, thus mitigating the clipping effect. However, these systems encounter information loss due to the modulo operation. The transfer function of the SR-ADC with parameter λ is

$$\mathcal{M}_\lambda(t) = 2\lambda\left(\left[\!\left[\frac{t}{2\lambda}+\frac{1}{2}\right]\!\right]-\frac{1}{2}\right), \quad (1)$$

where $[\![t]\!] \triangleq t - \lfloor t \rfloor$ is the fractional part of t [7].

In the context of SR-ADCs, an alternative sampling theory called the *unlimited sampling framework* was developed in [7], [8], which provides sufficient conditions on the sampling rate for guaranteeing the recovery of band-limited signals from its folded samples. Extending these results, the work in [9] considered the inverse problem of recovering K low pass filtered spikes in a continuous-time sparse signal, and developed a new sampling theorem and a signal recovery algorithm. In [10], the authors studied the quantization of oversampled signals in the SR-ADC architecture with the goal of reducing the overload distortion error.

A novel HDR imaging system that employs SR-ADCs to overcome limitations due to limited dynamic range was studied in [2], [11]. Mathematically, this involves applying an SR-ADC individually to multiple linear measurements of the images, and is termed as modulo compressed sensing (modulo-CS) [11]. Modulo-CS can also help overcome issues introduced by signal clipping in other signal acquisition systems where compressed measurements of sparse signals are available, such as communication systems [4], [12]. By exploiting the sparsity of the signal, the modulo-CS setup can be used to overcome the losses caused by limited dynamic range. In the somewhat restrictive setting where the modulo-CS measurements are assumed to span at most two periods, [11] proposes an algorithm and analyzes the sample complexity under Gaussian measurement matrices. In [12], a generalized approximate message passing algorithm tailored to modulo-CS was proposed by assuming a Bernoulli-Gaussian distribution on the sparse signal. The results in these papers suggest that sparsity is very useful in the recovery of signals from modulo-CS measurements.

In the context of the above, our contributions in this paper are twofold: (a) we derive necessary and sufficient conditions on the measurement matrix under which sparse signals are identifiable under modulo-CS measurements, and (b) we present a novel algorithm for modulo-CS recovery and derive its theoretical guarantees. To elaborate:

1) We derive necessary and sufficient conditions for unique recovery of sparse signals in the modulo-CS setup.
2) We show that the minimum number of measurements m to uniquely reconstruct every s-sparse signal from modulo measurements is $2s+1$.
3) We also show that $m = 2s+1$ is sufficient, and that a measurement matrix with $2s+1$ rows and entries drawn independently from any continuous distribution satisfies the identifiability conditions with high probability.
4) We present a mixed integer linear program for modulo-CS recovery via convex relaxation. We also identify an integer range space property, which guarantees exact sparse signal recovery via the relaxed problem.

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Notation: Bold lowercase and uppercase letters denote vectors and matrices, respectively, and script styled letters denote sets. A vector supported on index set $\mathcal{S}$ is denoted as $\boldsymbol{x}_{\mathcal{S}}$ and $\boldsymbol{A}_{\mathcal{S}}$ denotes the sub-matrix with columns of $\boldsymbol{A}$ corresponding to set $\mathcal{S}$. The ℓ_0-norm of a vector, $\|\boldsymbol{x}\|_0$, is the number of nonzero entries in $\boldsymbol{x}$. The inner product between $\boldsymbol{a}$ and $\boldsymbol{x}$ is denoted by $\langle \boldsymbol{a}, \boldsymbol{x} \rangle$. For a set $\mathcal{S}$, $|\mathcal{S}|$ and $\mathcal{S}^{\mathrm{c}}$ denotes the cardinality and the complement of the set, respectively.

II. Modulo Compressed Sensing

Let $\boldsymbol{x} \in \mathbb{R}^N$ denote an s-sparse vector, i.e., $\|\boldsymbol{x}\|_0 \leq s$, with $s < \frac{N}{2}$. For ease of exposition, instead of SR-ADC transfer function given in (1), we consider an equivalent modular arithmetic which returns the fractional part of a real number, i.e., it returns $[\![t]\!] \triangleq t - \lfloor t \rfloor$. We obtain m projections of $\boldsymbol{x}$ as follows:

$$z_i = [\![\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle]\!], \; i = 1, 2, \ldots, m. \tag{2}$$

Usually, $m \leq N$ in the compressed sensing paradigm, but we will also present extensions to dense vectors ($s \geq \frac{N}{2}$) in the overdetermined system setup ($m > N$).

Stacking the projections $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle$ as a vector $\boldsymbol{y}$, we can rewrite (2) in a form similar to the CS framework as

$$\boldsymbol{z} = [\![\boldsymbol{y}]\!] = [\![\boldsymbol{A}\boldsymbol{x}]\!], \tag{3}$$

where $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_m \end{bmatrix}^T \in \mathbb{R}^{m \times N}$ is the measurement matrix and $[\![\cdot]\!]$ represents the element wise modulo-1 operation on a vector, as before.

The non-linearity introduced by the modulo operation along with the underdetermined compressive measurements could lead to an indeterminate system, i.e., it may not have a unique solution. In this paper, we explore the role of sparsity in uniquely recovering an s-sparse input signal $\boldsymbol{x}$ from the modulo-CS measurements $\boldsymbol{z}$ obtained using (3).

P_0 formulation: Any real valued vector $\boldsymbol{y} \in \mathbb{R}^m$ can be uniquely decomposed as $\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{v}$, where $\boldsymbol{z} \in [0, 1)^m$ and $\boldsymbol{v} \in \mathbb{Z}^m$ denote the fractional part and integer part (the floor function) of $\boldsymbol{y}$, respectively. Using this decomposition, the non-linearity in (3) can be represented using a linear equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{v}$. Now, consider the optimization problem:

$$\arg\min_{\boldsymbol{w}, \boldsymbol{v}} \|\boldsymbol{w}\|_0 \text{ subject to } \boldsymbol{A}\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{v}; \boldsymbol{v} \in \mathbb{Z}^m. \tag{P$_0$}$$

Any s'-sparse solution $\boldsymbol{x}^*$ to (P$_0$) satisfies $s' \leq s$ (i.e., $\boldsymbol{x}^*$ is s-sparse), since $\boldsymbol{x}$ is s-sparse and satisfies the constraints of (P$_0$). Thus, unique identifiability of an s-sparse $\boldsymbol{x}$ from modulo-CS measurements is equivalent to the existence of a unique solution to (P$_0$), which we discuss next.

III. Identifiability

In this section, we derive conditions under which (P$_0$) admits a unique s-sparse solution.

Lemma 1 (Necessary and sufficient conditions)**.** *Any vector $\boldsymbol{x}$ satisfying $\|\boldsymbol{x}\|_0 \leq s < \frac{N}{2}$ is an unique solution to the optimization problem* (P$_0$) *if and only if any $2s$ columns of matrix $\boldsymbol{A}$ are linearly independent of all $\boldsymbol{v} \in \mathbb{Z}^m$.*

Proof. We first prove sufficiency by contradiction. Let $\boldsymbol{z} = [\![\boldsymbol{A}\boldsymbol{x}]\!]$, $\boldsymbol{x}$ is an s-sparse vector, and $\boldsymbol{A} \in \mathbb{R}^{m \times N}$. Suppose the optimization problem (P$_0$) returned another s-sparse vector $\boldsymbol{x}^{\#}$ (so that $\|\boldsymbol{x}^{\#}\|_0 \leq s$), then

$$\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}^{\#}) = \boldsymbol{v} \Rightarrow \boldsymbol{A}_{\mathcal{S}}(\boldsymbol{x}_{\mathcal{S}} - \boldsymbol{x}^{\#}_{\mathcal{S}}) = \boldsymbol{v} \in \mathbb{Z}^m,$$

where the set $\mathcal{S}$ is the union of the supports of $\boldsymbol{x}$ and $\boldsymbol{x}^{\#}$. Since $|\mathcal{S}| \leq 2s$, a set of $2s$ columns of $\boldsymbol{A}$ span an integer vector $\boldsymbol{v}$, which violates the condition in the Lemma.

To prove the necessary part, suppose $\exists\, \mathcal{S}$ such that $|\mathcal{S}| = 2s$ and $\boldsymbol{0} \neq \boldsymbol{u} \in \mathbb{R}^{2s}$ such that $\boldsymbol{A}_{\mathcal{S}}\boldsymbol{u} = \boldsymbol{v}$, where $\boldsymbol{v} \in \mathbb{Z}^m$. We construct two s-sparse vectors $\boldsymbol{x}^0, \boldsymbol{x}^{\#} \in \mathbb{R}^N$ from $\boldsymbol{u}$, where the first s indices of $\mathcal{S}$ constitute the support of $\boldsymbol{x}^0$ with the values equal to first s entries of $\boldsymbol{u}$, and the remaining s entries of $\mathcal{S}$ constitute the support of $\boldsymbol{x}^{\#}$ with the values equal to the corresponding s entries of $\boldsymbol{u}$. Also, define $\boldsymbol{z} = [\![\boldsymbol{A}\boldsymbol{x}^0]\!]$ and $\boldsymbol{y}^0 = \boldsymbol{A}\boldsymbol{x}^0$, so that $\boldsymbol{y}^0 = \boldsymbol{z} + \boldsymbol{v}^0$ for some $\boldsymbol{v}^0 \in \mathbb{Z}^m$. Then, using $\boldsymbol{A}_{\mathcal{S}}\boldsymbol{u} = \boldsymbol{v}$, we have $\boldsymbol{A}\boldsymbol{x}^{\#} + \boldsymbol{A}\boldsymbol{x}^0 = \boldsymbol{v}$ which implies $\boldsymbol{y}^0 - \boldsymbol{v} = -\boldsymbol{A}\boldsymbol{x}^{\#}$. Thus, $-\boldsymbol{x}^{\#}$ is also a solution to the optimization problem since $\|\boldsymbol{x}^{\#}\|_0 \leq s$ and $[\![-\boldsymbol{A}\boldsymbol{x}^{\#}]\!] = [\![\boldsymbol{y}^0 - \boldsymbol{v}]\!] = \boldsymbol{z}$, which is a contradiction. $\square$

The following corollary presents a similar result for the recovery of dense vectors, which requires $m > N$.

Corollary 1. *Any vector $\boldsymbol{x}$ satisfying $\|\boldsymbol{x}\|_0 \geq \frac{N}{2}$ is a unique solution to $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{y}$ with $\boldsymbol{y} = [\![\boldsymbol{A}\boldsymbol{x}]\!] + \boldsymbol{v}$ and $\boldsymbol{v} \in \mathbb{Z}^m$ if only if the columns of matrix $\boldsymbol{A}$ are linearly independent of all $\boldsymbol{v} \in \mathbb{Z}^m$. Consequently, the minimum number of measurements required for unique recovery is $m = N + 1$.*

Proof. The proof is similar to Lemma 1, with the observation that when $\|\boldsymbol{x}\|_0 \geq \frac{N}{2}$ and $\|\boldsymbol{x}^{\#}\|_0 \geq \frac{N}{2}$, $\boldsymbol{x} - \boldsymbol{x}^{\#}$ can be any N length real vector. $\square$

To compare the modulo-CS problem to the standard CS problem, we state two necessary conditions for modulo-CS recovery below. The proof is immediate from Lemma 1.

Corollary 2. *The following two conditions are necessary for recovering any vector $\boldsymbol{x}$ satisfying $\|\boldsymbol{x}\|_0 \leq s$ as a unique solution of the optimization problem* (P$_0$)*:*

1) *$m \geq 2s + 1$, and*
2) *Any $2s$ columns of $\boldsymbol{A}$ are linearly independent.*

We recall that $m = 2s$ is necessary and sufficient for unique sparse signal recovery in the standard CS setup [13, Theorem 2.14]. The above corollary shows that the minimum number of measurements needed to reconstruct all s-sparse vector from modulo measurements is $2s+1$. We now show that $m = 2s+1$ measurements are also sufficient. Thus, we see that the penalty for unique sparse signal recovery due to the modulo operation is just one additional measurement.

Theorem 1 (Sufficiency)**.** *For any $N \geq 2s+1$, there exists a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times N}$ with $m = 2s+1$ rows such that every s-sparse $\boldsymbol{x} \in \mathbb{R}^N$ can be uniquely recovered from its modulo measurements $\boldsymbol{z} = [\![\boldsymbol{A}\boldsymbol{x}]\!]$ as a solution to* (P$_0$)*.*

Proof. Let $\boldsymbol{A} \in \mathbb{R}^{(2s+1) \times N}$ be a matrix for which at least one s-sparse vector $\boldsymbol{x}$ cannot be recovered from $\boldsymbol{z} = [\![\boldsymbol{A}\boldsymbol{x}]\!]$ via (P$_0$). Hence, $\boldsymbol{A}$ does not satisfy the condition in Lemma 1. We will show that the set of all such matrices is of Lebesgue measure 0. To this end, we define two sets:

1) Let $\mathcal{V} = \{\boldsymbol{v} | \boldsymbol{v} \in \mathbb{Z}^m\}$ denote the countably infinite set of all integer vectors.
2) Let $\mathfrak{S} = \{T | T \subset [N], |T| = 2s\}$ denote the set of all index sets on $[N]$ whose cardinality is $2s$.

For a given $\boldsymbol{u} \in \mathcal{V}$ and $\mathcal{S} \in \mathfrak{S}$, construct $\boldsymbol{B}(\boldsymbol{u}, \mathcal{S}) = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{A}_{\mathcal{S}} \end{bmatrix}$. Hence, the condition in Lemma 1 fails if $\det(\boldsymbol{B}(\boldsymbol{u}, \mathcal{S})) = 0$. This is a nonzero polynomial function of the entries of $\boldsymbol{A}_{\mathcal{S}}$, and therefore the set of matrices which satisfy this condition have Lebesgue measure 0. Now, consider $\cup_{\mathcal{S} \in \mathfrak{S}} \cup_{\boldsymbol{u} \in \mathcal{V}} \{\boldsymbol{A} | \det(\boldsymbol{B}(\boldsymbol{u}, \mathcal{S})) = 0\}$. This is a finite union of countable unions of Lebesgue measure 0 sets and hence is also of Lebesgue measure 0. Hence, a matrix $\boldsymbol{A}$ chosen outside of this set will ensure that any s-sparse vector $\boldsymbol{x}$ can be recovered from its modulo measurements $\boldsymbol{y} = [\![\boldsymbol{A}\boldsymbol{x}]\!]$. $\square$

Remark 1: If the entries of $\boldsymbol{A}$ are drawn independently from any continuous distribution, $\boldsymbol{A}$ lies outside the set of Lebesgue measure 0 described in Theorem 1 and hence is a valid candidate for modulo-CS recovery.

Remark 2: From [14, Proposition 1], for any integer vector $\boldsymbol{a} \in \mathbb{Z}^K$ and $\boldsymbol{x} \in \mathbb{R}^K$ it holds that $[\![\boldsymbol{a}^T [\![\boldsymbol{x}]\!]]\!] = [\![\boldsymbol{a}^T \boldsymbol{x}]\!]$. As consequence, if the entries of $\boldsymbol{A}$ are integers, then $\boldsymbol{x}$ and $[\![\boldsymbol{x}]\!]$ result in the same modulo measurements, and unique recovery is not possible. Hence, integer matrices cannot be used as candidate measurement matrices for modulo-CS.

Remark 3: Extending Theorem 1 to dense vectors similar to Corollary 1, it can be shown that $m = N+1$ suffices for unique recovery of all $\boldsymbol{x} \in \mathbb{R}^N$.

We next study the recoverability of sparse vectors when the ℓ_0-norm in (P$_0$) is replaced with the ℓ_1-norm, thus making the objective function convex.

IV. Convex Relaxation

In the previous section, we derived conditions for unique sparse vector recovery from modulo-CS measurements via (P$_0$). However, both the objective function and the constraint set of (P$_0$) are non-convex, and solving it requires an exhaustive search over all possible index sets and integer vectors of length m. Replacing the ℓ_0-norm in (P$_0$) with the ℓ_1-norm, we obtain the combinatorial optimization problem:

$$\arg\min_{\boldsymbol{x}, \boldsymbol{v}} \|\boldsymbol{x}\|_1 \text{ subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{v}; \ \boldsymbol{v} \in \mathbb{Z}^m. \qquad \text{(P}_1\text{)}$$

A. Integer Range Space Property

In order to develop conditions on $\boldsymbol{A}$ for unique recoverability of the original sparse vector via (P$_1$), we introduce the following property.

Definition 1 (Integer range space property (IRSP))**.** *A matrix $\boldsymbol{A}$ is said to satisfy the IRSP of order s if, for all sets $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \leq s$,*

$$\|\boldsymbol{u}_{\mathcal{S}}\|_1 < \|\boldsymbol{u}_{\mathcal{S}^c}\|_1,$$

holds for every $\boldsymbol{u} \in \mathbb{R}^N$ with $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{v} \in \mathbb{Z}^m$.

Remark 4: In the above, if the integer vector $\boldsymbol{v}$ is replaced with the all zero vector, the IRSP boils down to the null space property, which is necessary and sufficient for the ℓ_1 norm based relaxation of the standard CS problem.

Theorem 2 (ℓ_1 recovery from modulo-CS)**.** *Every s-sparse $\boldsymbol{x}$ is the unique solution of* (P$_1$) *if and only if the matrix $\boldsymbol{A}$ satisfies the IRSP of order s.*

Proof. Consider a fixed index set $\mathcal{S}$ with $|\mathcal{S}| \leq s$, and suppose that every $\boldsymbol{x}$ supported on $\mathcal{S}$ is a unique minimizer of (P$_1$). Then, for any $\boldsymbol{u}$ such that $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{v} \in \mathbb{Z}^m$, the vector $\boldsymbol{u}_{\mathcal{S}}$ is the unique minimizer of (P$_1$). But, $\boldsymbol{A}(\boldsymbol{u}_{\mathcal{S}} + \boldsymbol{u}_{\mathcal{S}^c}) = \boldsymbol{v}$. Thus, $\|\boldsymbol{u}_{\mathcal{S}}\|_1 < \|\boldsymbol{u}_{\mathcal{S}^c}\|_1$, which proves the necessary condition. Conversely, suppose that the IRSP holds with respect to the set $\mathcal{S}$. Consider $\boldsymbol{x}$ supported on $\mathcal{S}$ and another vector $\boldsymbol{x}^\#$ that result in the same modulo measurements, i.e., $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}_1 = \boldsymbol{A}\boldsymbol{x}^\# + \boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are integer vectors. Letting $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}^\#$, the vector $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{v}_2 - \boldsymbol{v}_1 = \boldsymbol{v} \in \mathbb{Z}^m$. Hence, by virtue of the IRSP, $\|\boldsymbol{u}_{\mathcal{S}}\|_1 < \|\boldsymbol{u}_{\mathcal{S}^c}\|_1$. Then,

$$\begin{aligned}\|\boldsymbol{x}\|_1 &\leq \|\boldsymbol{x} - \boldsymbol{x}^\#_{\mathcal{S}}\|_1 + \|\boldsymbol{x}^\#_{\mathcal{S}}\|_1 \\ &= \|\boldsymbol{u}_{\mathcal{S}}\|_1 + \|\boldsymbol{x}^\#_{\mathcal{S}}\|_1 \\ &< \|\boldsymbol{u}_{\mathcal{S}^c}\|_1 + \|\boldsymbol{x}^\#_{\mathcal{S}}\|_1 = \|-\boldsymbol{x}^\#_{\mathcal{S}^c}\|_1 + \|\boldsymbol{x}^\#_{\mathcal{S}}\|_1 = \|\boldsymbol{x}^\#\|_1.\end{aligned}$$

Thus, $\boldsymbol{x}$ is the unique minimizer of (P$_1$), and IRSP relative to $\mathcal{S}$ is sufficient. Finally, letting $\mathcal{S}$ vary, we see that $\boldsymbol{A}$ satisfying IRSP of order s is necessary and sufficient. $\square$

The above theorem shows that the IRSP is a key property for guaranteeing sparse vector recovery from modulo-CS measurements via (P$_1$). Next, we present a practical algorithm for solving (P$_1$) based on mixed-integer linear programming and empirically evaluate its performance.

B. Mixed Integer Linear Program (MILP)

The ℓ_1 norm in the (P$_1$) problem can be rewritten as a linear function using two positive vectors $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$ as

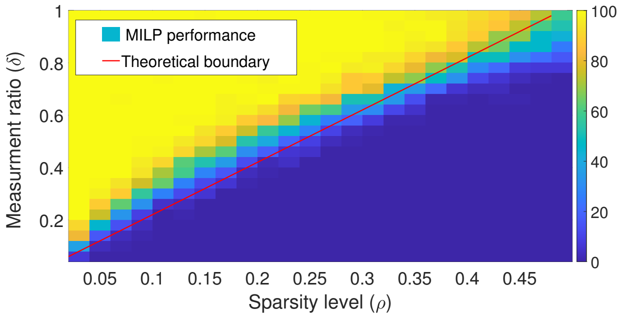

Fig. 1. Percentage of success recovery for MILP.

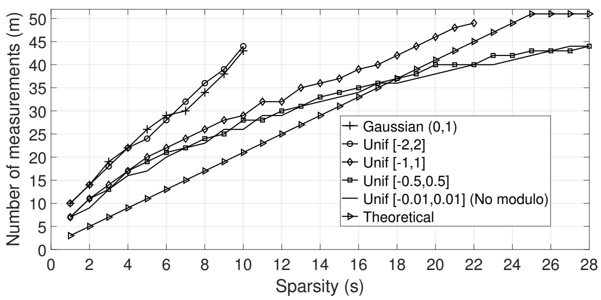

Fig. 2. Phase transition curves for 80% recovery accuracy.

$\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^-$. This leads to the MILP formulation:

$$\begin{aligned} &\min_{\boldsymbol{x}^+, \boldsymbol{x}^-, \boldsymbol{v}} \quad \mathbf{1}^T \left(\boldsymbol{x}^+ + \boldsymbol{x}^-\right) \\ &\text{subject to } \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A} & -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^+ \\ \boldsymbol{x}^- \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{z}; \\ &\boldsymbol{v} \in \mathbb{Z}^m; \quad \boldsymbol{x}^+, \boldsymbol{x}^- \geq 0. \end{aligned} \tag{4}$$

The MILP can be solved efficiently using the branch-and-bound algorithm [15]. Once $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$ are obtained, we can solve for $\boldsymbol{x}$ as $\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^-$.

V. Simulation Results

We now empirically evaluate performance of the MILP for the modulo-CS problem. We solve the MILP using the *intlinprog* function in the MATLAB optimization toolbox. We set $N = 50$, and for a given sparsity level $\rho = \frac{s}{N}$, we randomly select ρN indices of the input signal to be nonzero, and set the others to zero. The nonzero entries are drawn from either the uniform or Gaussian distributions with zero-mean and different variances as specified in the figures. For a given measurement ratio $\delta = \frac{m}{N}$, the entries of the measurement matrix with δN rows are drawn from an i.i.d. Gaussian distribution with mean zero and variance $\frac{1}{m}$.

We first present the phase transition curve of the MILP problem by plotting the success rate over 1000 Monte Carlo simulations when the nonzero entries of the sparse signal are obtained from a uniform distribution on $[-1, 1]$, denoted by Unif $[-1, 1]$. From Fig. 1, we see that the transition region between success and failure roughly follows the theoretical result (solid line in red) in Theorem 1. In particular, the MILP formulation of the algorithm based on convex relaxation performs close to the theoretical bound for the (P_0) problem, i.e., it is near-optimal in the settings considered here.

In the next experiment, we compare the performance of the MILP algorithm when the measurements span different number of modulo periods. To this end, we evaluate the performance of the MILP for different distributions on the nonzero entries of the sparse signal. We plot the minimum value of m required for 80% recovery accuracy (i.e., exact recovery of the sparse signals in 80% of the random experiments) for sparsity levels varying from 1 to $\frac{N}{2}$ in Fig. 2. For the Unif $[-0.01, 0.01]$ curve, the measurements are always in the range $[-0.5, 0.5]$, and by shifting by 0.5, we obtain all measurements within a single modulo period, hence the modulo operation does not introduce any nonlinearity. As seen in the figure, the curve for Unif $[-0.5, 0.5]$ which spans at least 2 modulo periods is close to the Unif $[-0.01, 0.01]$ case without the modulo operation. When the variance of the signal is low, MILP performs close to the theoretical limit for the minimal number of measurements required. However, with increase in variance of the input signal, the measurements span a larger number of modulo periods, and the performance starts to deteriorate. We also notice that the simulated curves for the Unif $[-0.01, 0.01]$ and Unif $[-0.5, 0.5]$ cases cross the theoretical bound for sparsity levels beyond $s = 18$. There are two reasons for this. First, the simulated curves correspond to 80% recovery success rate, while the theoretical results were for perfect recovery of all sparse signals. Second, the simulated curves are for specific source distributions, while the theoretical result is for arbitrary (even adversarially chosen) sparse vectors. Nonetheless, the theoretical curve forms a useful benchmark for the performance of modulo-CS recovery algorithms.

VI. Conclusions

In this work, we considered the problem of recovering sparse signals from modulo compressed sensing measurements. We presented an equivalent optimization problem for the modulo-CS setup using the ℓ_0 norm. For this optimization problem, we showed that the $2s+1$ measurements are necessary and sufficient for recovering s-sparse signals. Finally, we considered a convex relaxation for the ℓ_0-norm and presented an algorithm based on mixed-integer linear programming. For this algorithm, we obtained theoretical guarantees as a property of the measurement matrix.

In this work, we have considered the case of noiseless measurements. These results and the MILP algorithm need to be extended to the case of noisy measurements. Further, deriving the sample complexity of the MILP algorithm is an interesting direction for future work.

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