01.110: Computational Fabrication

Week 5: Planar Kinematics

Hassan Hariri

+65 8401 0454
Hassan_hariri@sutd.edu.sg

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Established in collaboration with MIT

Overview

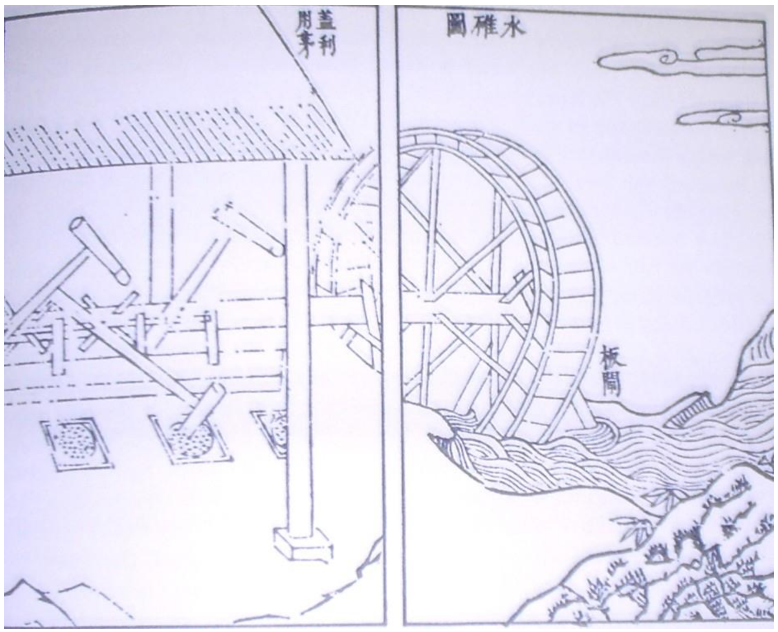
- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

Mechanisms

- **Mechanisms** are mechanical systems comprised of interconnected machine elements, such as rods, beams, springs, and pivots, used to transmit power or motion.
- The purpose is to transform input power into a useful application of forces/torque combined with a desired movement.
- Able to control force and motion in relatively complex ways using parts and interconnections that were simple to fabricate.

Historical Motivation

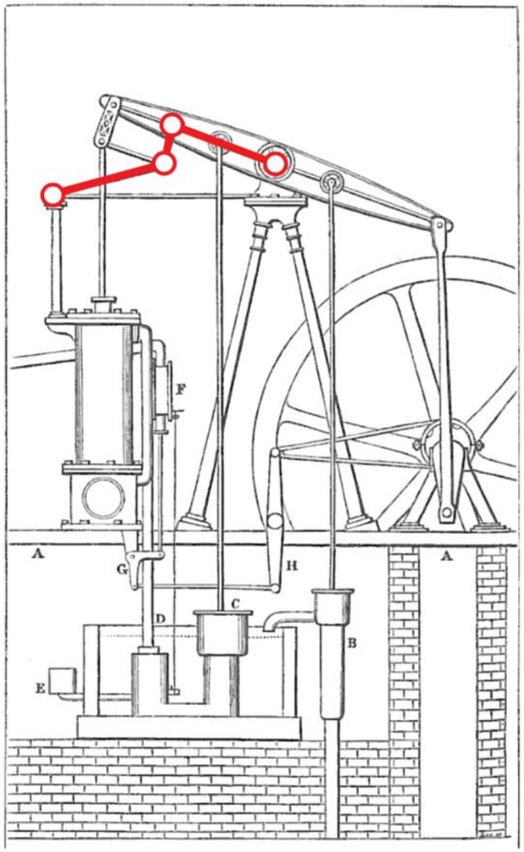
- Relieve people from physically and mentally tedious labor task.

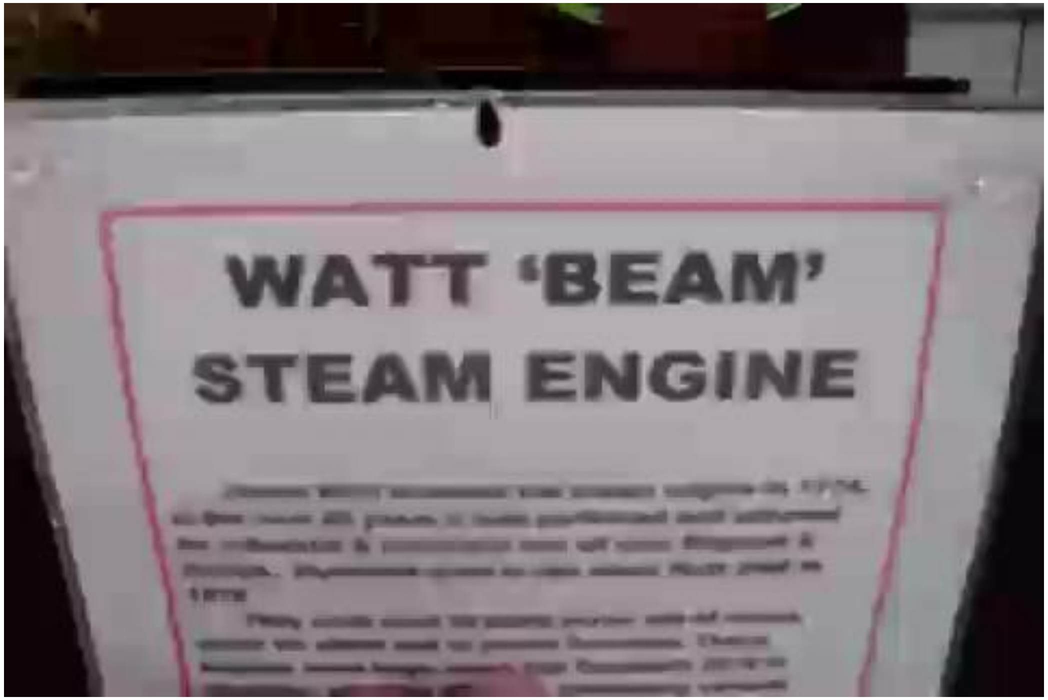


Converting continuous rotary motion into oscillating motions

Historical Motivation

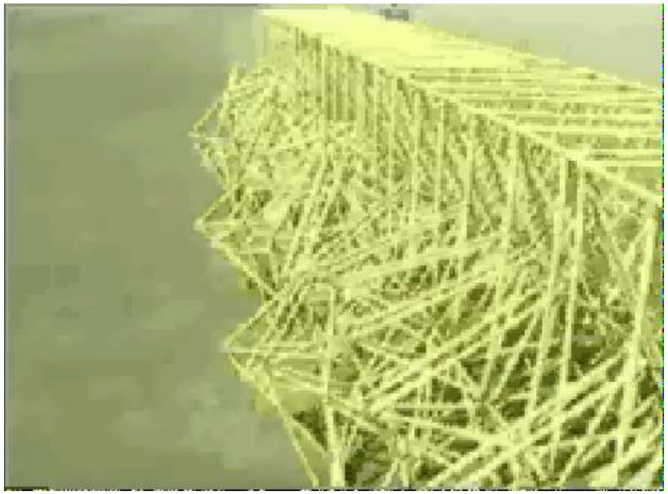
- To provide mechanize solutions (machines) for Industrialization or Transportation.





Steam Engines

Modern Use of Mechanisms



Theo Jansen Kinetic Sculpture

Robotic Manipulators

Convertible hardtops

Exoskeletons

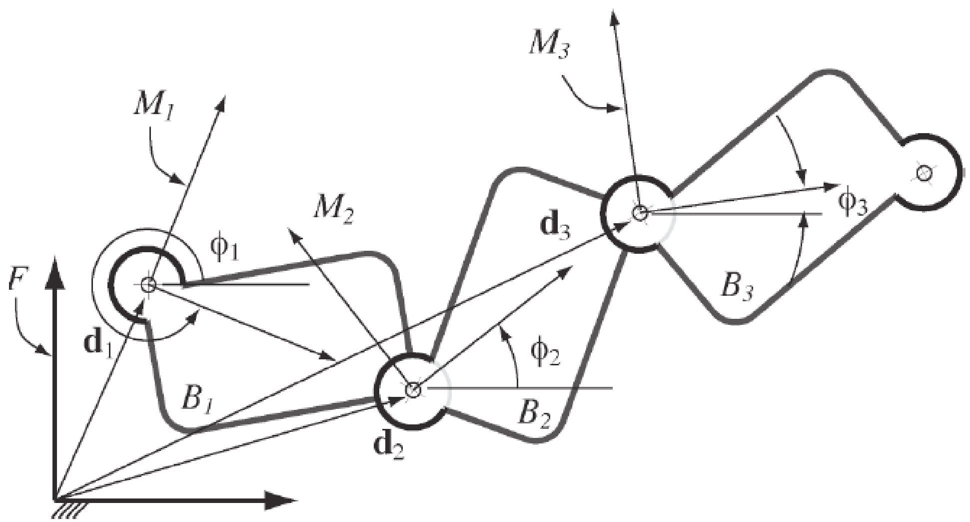
Overview

- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

Linkages

- A **linkage** is a kinematic chain with one of its links fixed to the ground.

- A **kinematic chain** is an assembly of links connected by joints into a system. It is a collection of n rigid bodies, B_i, i = 1, …, n that are connected by j hinges or sliding joints.



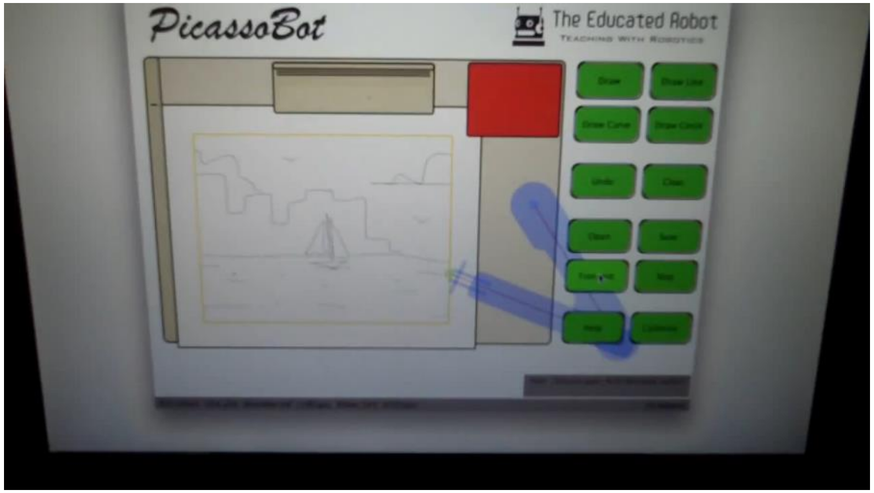
Kinematics of Linkages

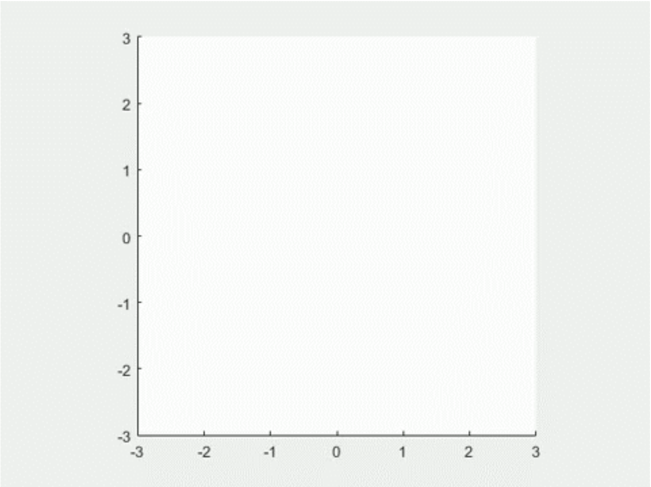
- **Kinematics**: The study of motion without regard for the causes of that motion (no forces, no moments, no masses).
- We are going to focus on understanding the **Planar kinematics of linkages**

Serial (open loop) Link Mechanisms

Three-dimensional Kinematics

Planar Kinematics



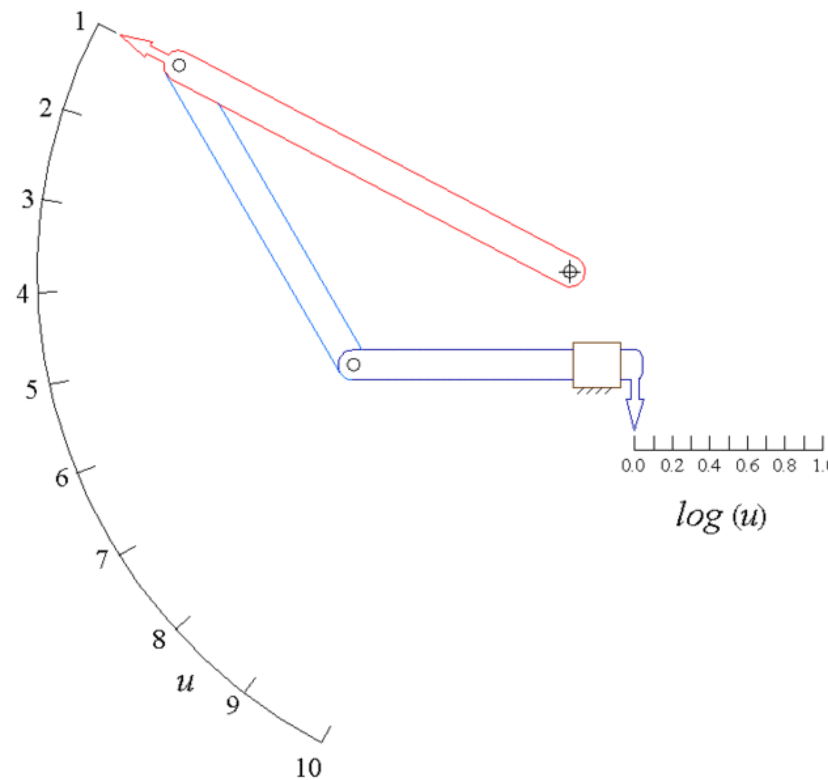


Parallel (closed loop) Link Mechanisms

Three-dimensional Kinematics

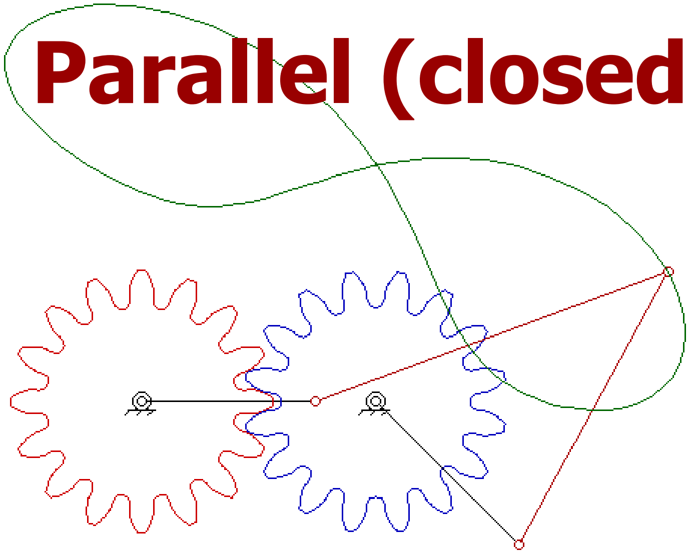
Planar Kinematics

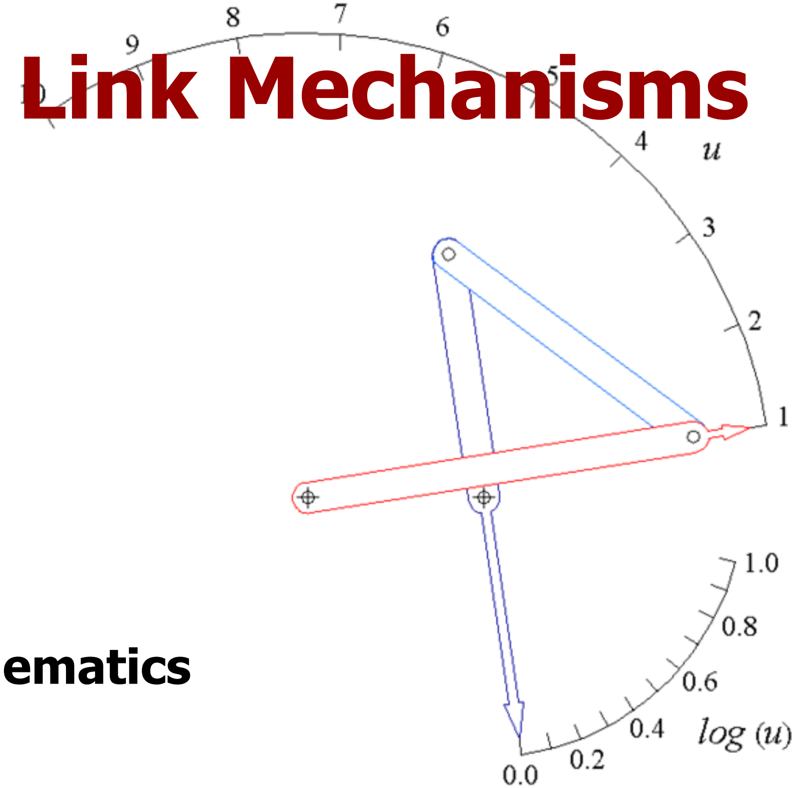
Parallel (closed loop) Link Mechanisms



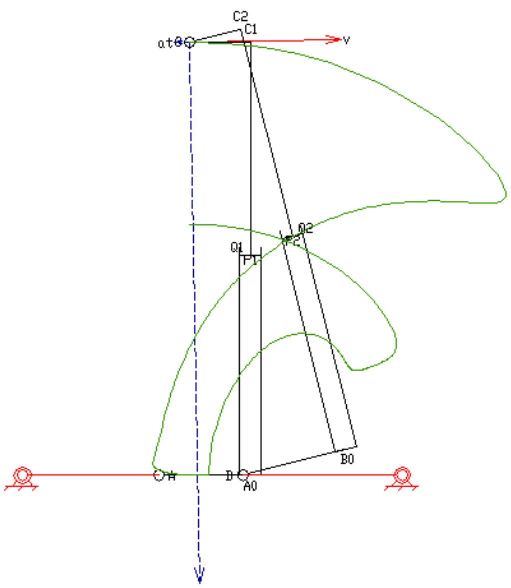
Planar Kinematics

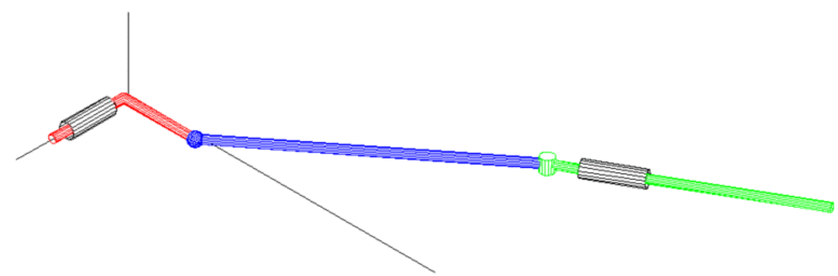
Parallel (closed loop) Link Mechanisms





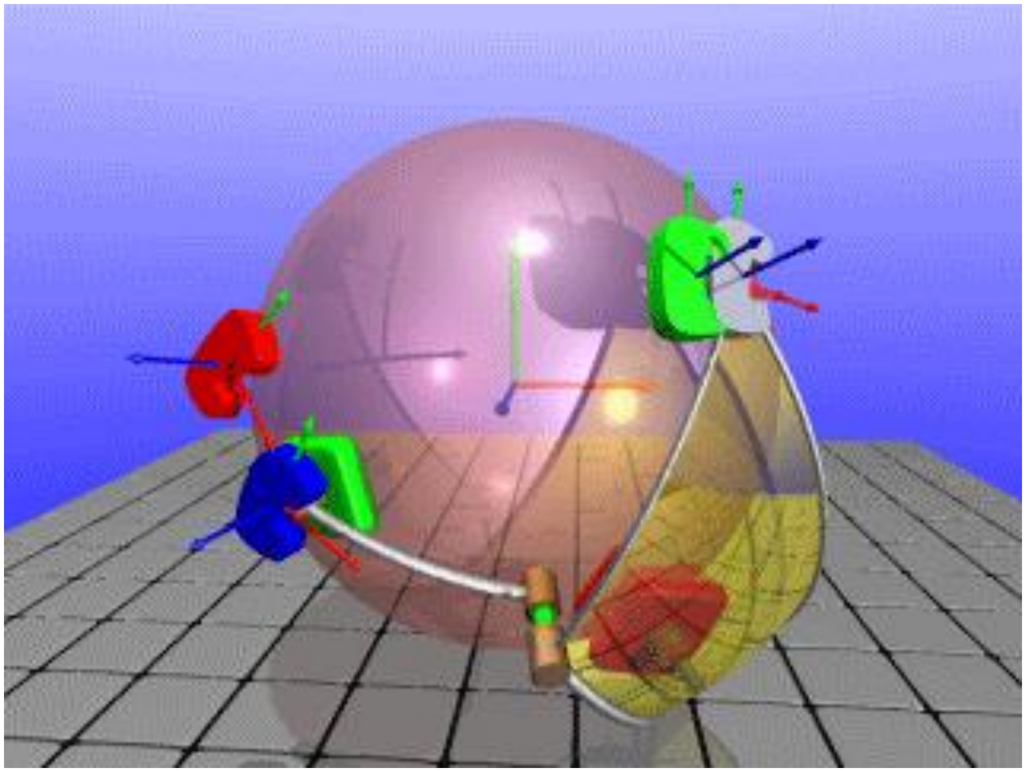
Planar Kinematics



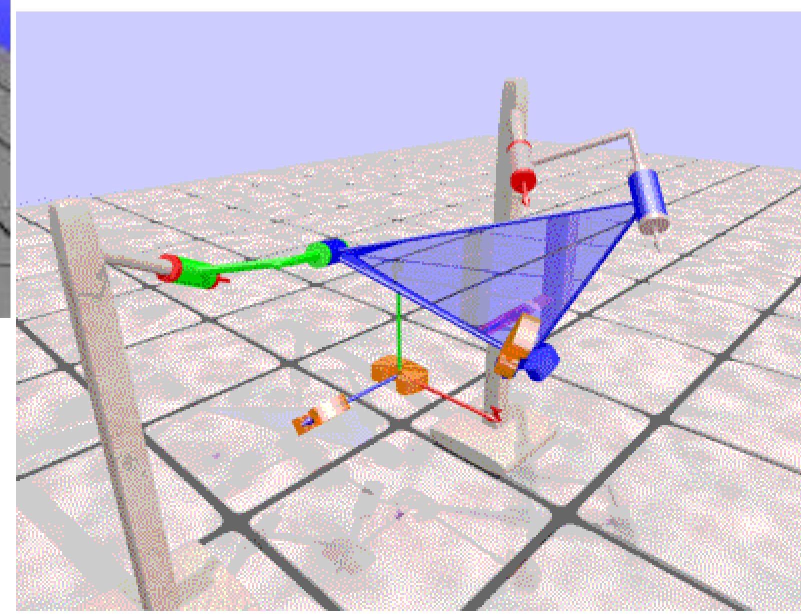


(c) PA Simionescu 2006

Parallel (closed loop) Link Mechanisms

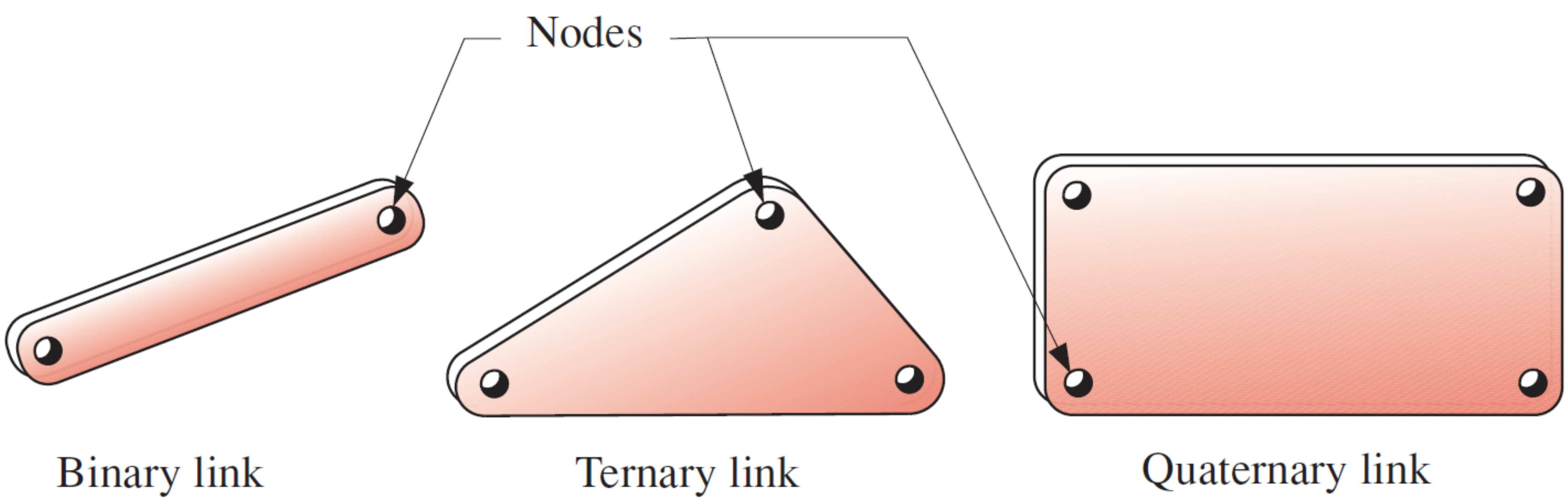


Spatial (Three-dimensional) Kinematics



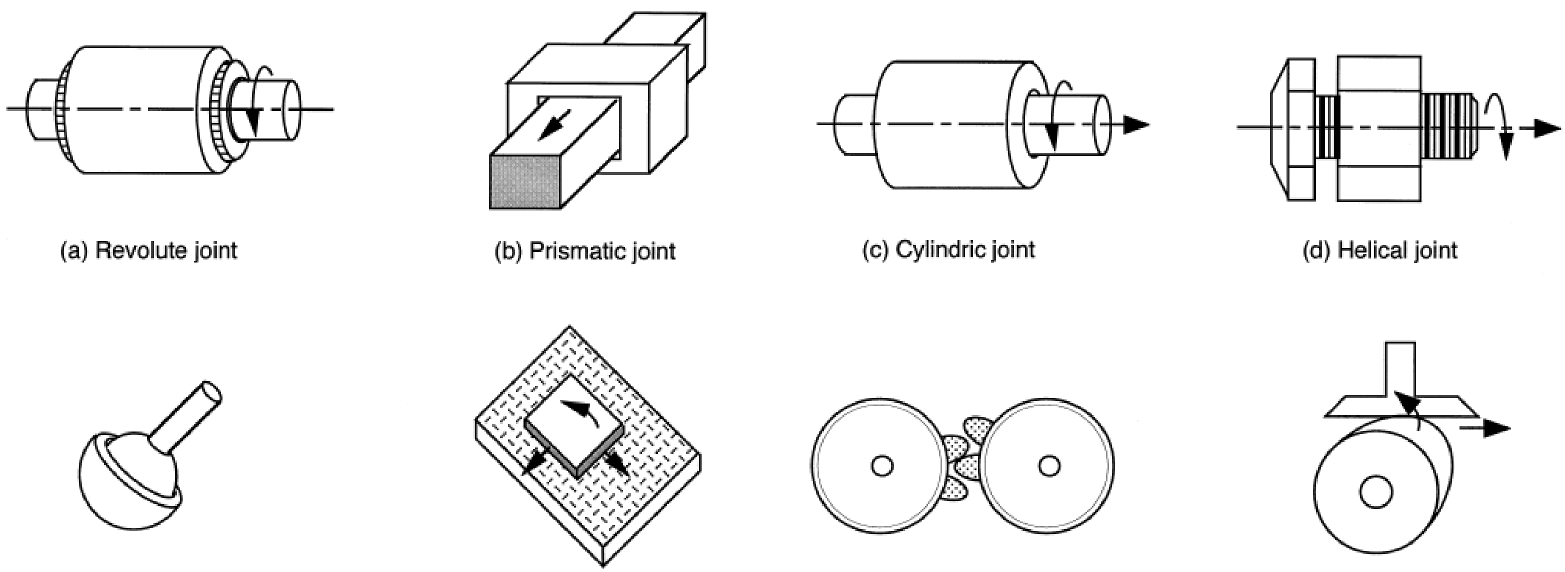
Links

- A **link** is a rigid body that posses two or more nodes that serves as point of attachment to other links.



Joints

- The connection between two links is called a **joint**.
- A joint provides some physical constraints on the relative motion between two members.



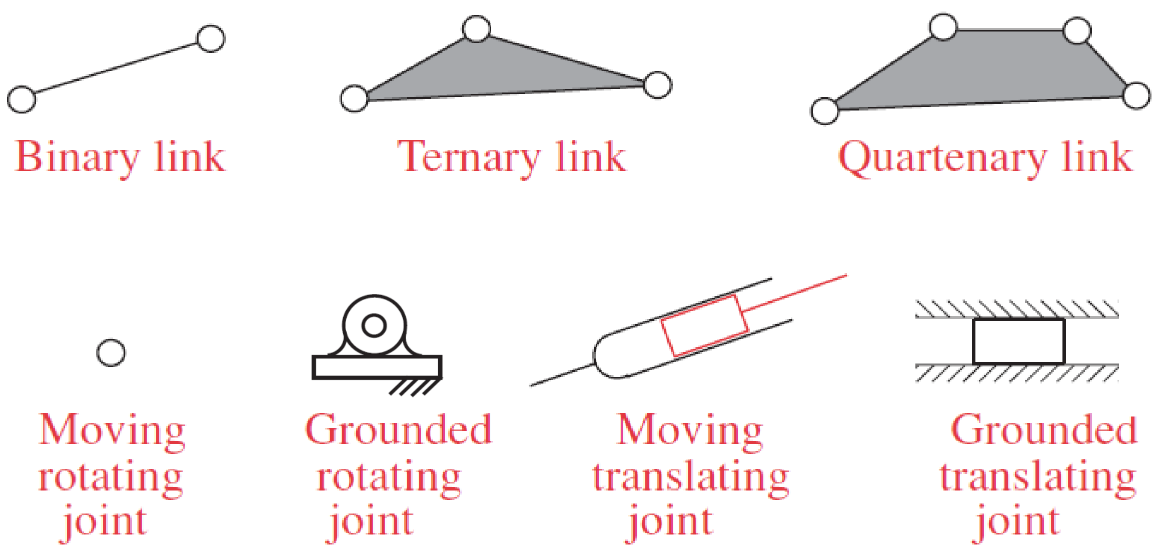
(a) Revolute joint (b) Prismatic joint (c) Cylindric joint (d) Helical joint

(e) Spherical joint (f) Plane pair (g) Gear pair (h) Cam pair

Kinematic Diagram

- The **kinematic diagram** of a linkage is a stick diagram that displays only the essential skeleton of the mechanism.

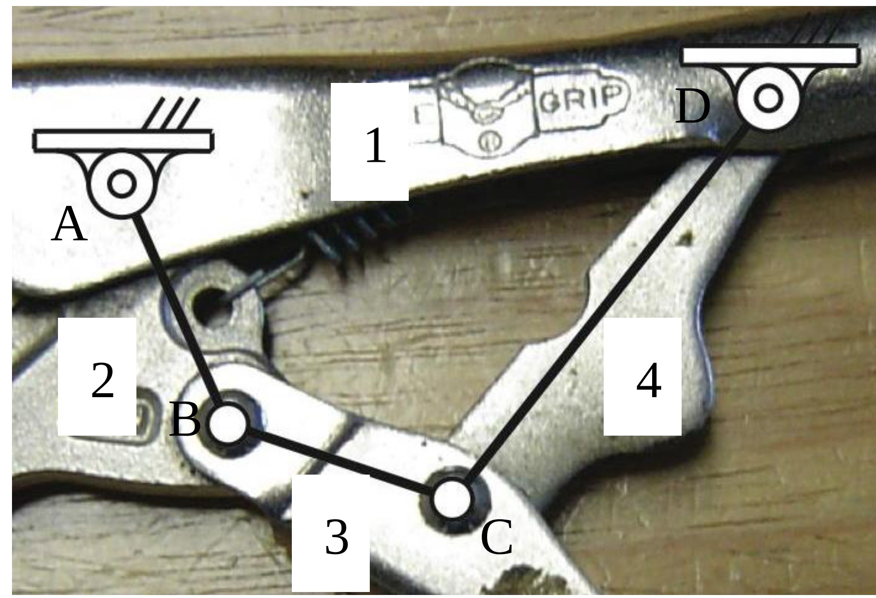
- It consists of the key dimensions that affect the linkage motion.



Kinematic Diagram

- All links are numbered and joints lettered in a kinematic diagram.

The vise grip



Associated Kinematic diagram

Overview

- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

Linkage Mobility

- The **degrees of freedom (dof) or mobility** of a system is equal to the number of coordinates needed to define its position in space.

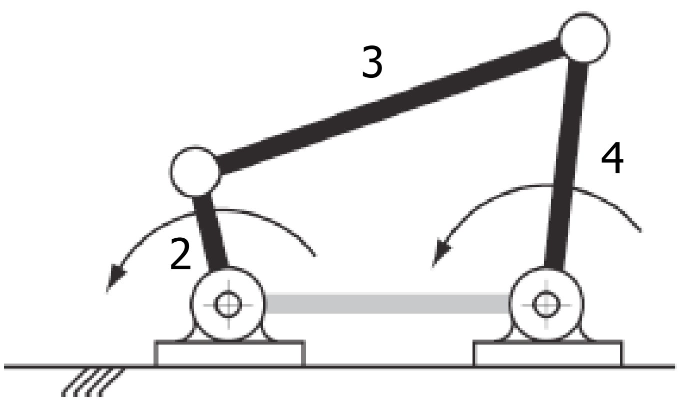
- For planar linkage, it depends on the number of links and joints and the types of joints used to construct the linkage.

$$\boxed{F = 3n - 2j - 3}$$

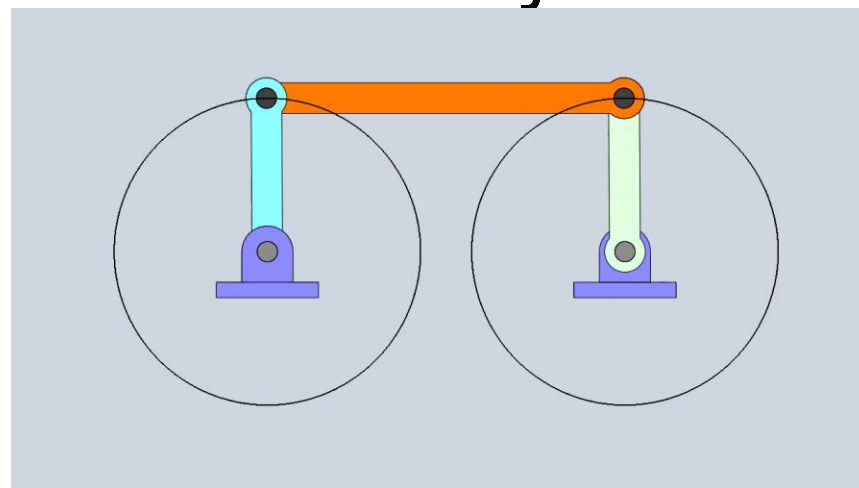
F = Degrees of frredom
n = number of links
j = number of revolute/prismatic joints

Linkage Mobility Example 1

- The *four-bar linkage* is a closed-loop kinematic chain that consist of four links and four joints.



Ground (link 1)



$$n = 4, j = 4$$

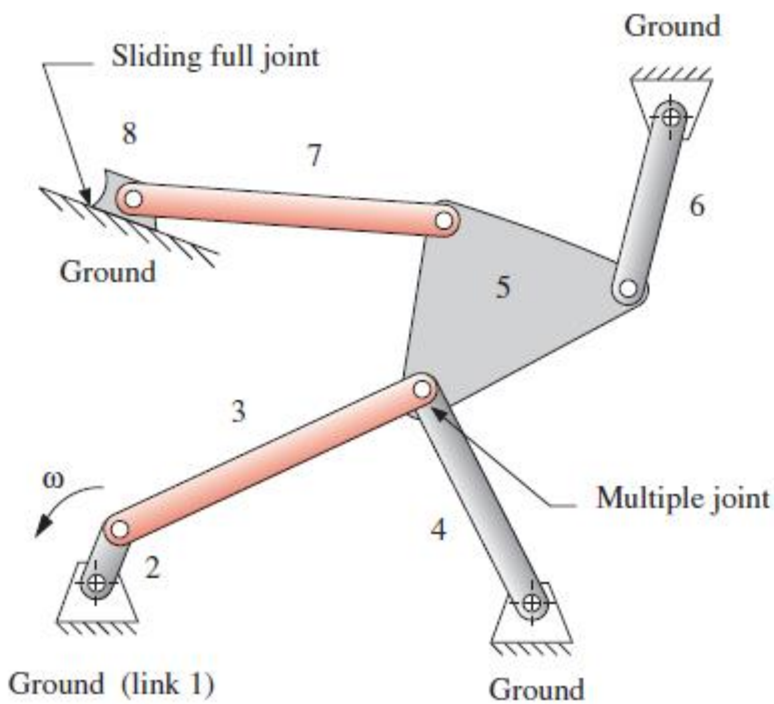
$$F = 3n - 2j - 3$$

$$= 3(4) - 2(4) - 3$$

$$= 1$$

Linkage Mobility Example 2

- When *n* links are connected together at a particular joint, the number of joints at that connection is *n-1.*

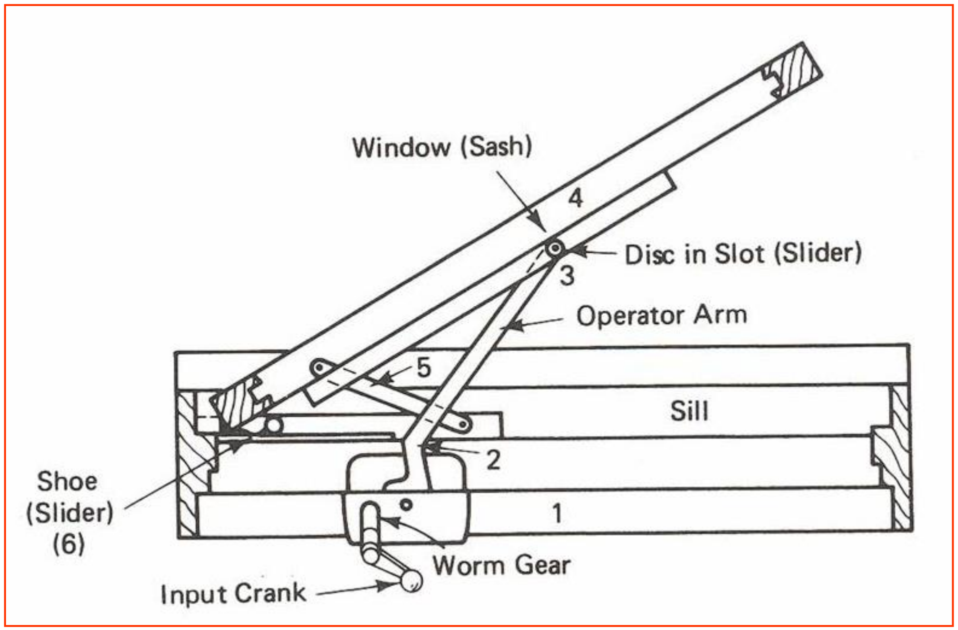


$$n = 8, j = 10$$

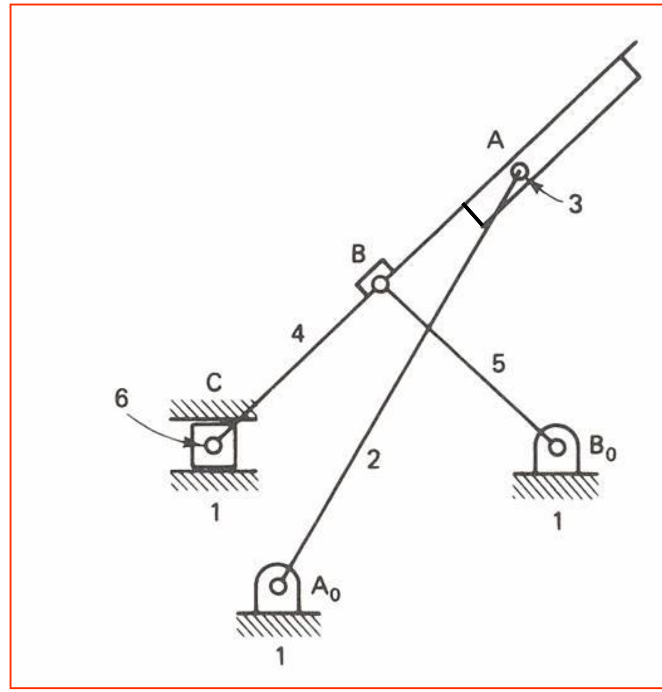
$$F = 3n - 2j - 3$$
$$= 3(8) - 2(10) - 3$$
$$= 1$$

Exercise 1

- Determine the degrees of freedom for the window mechanism shown below



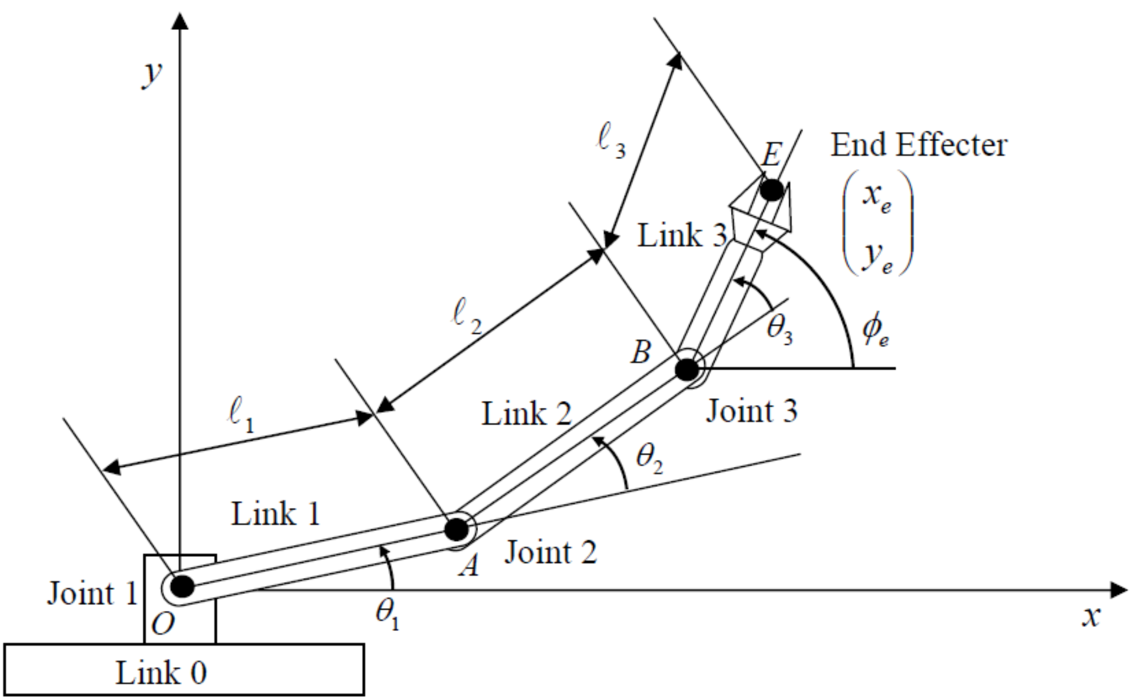
Mechanism to open and close a window

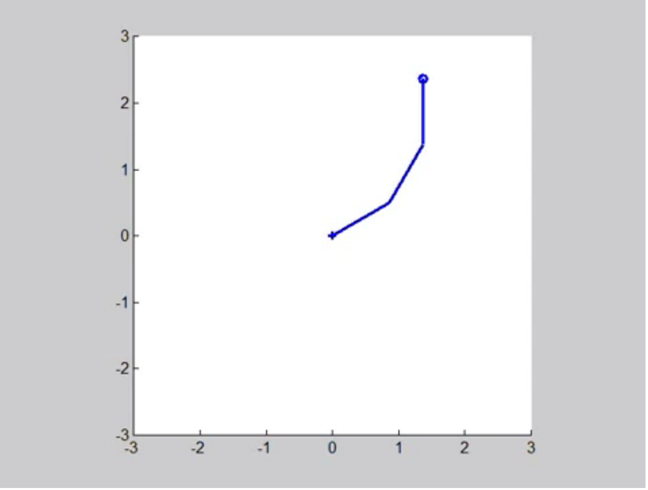


Associated Kinematic diagram

Exercise 2

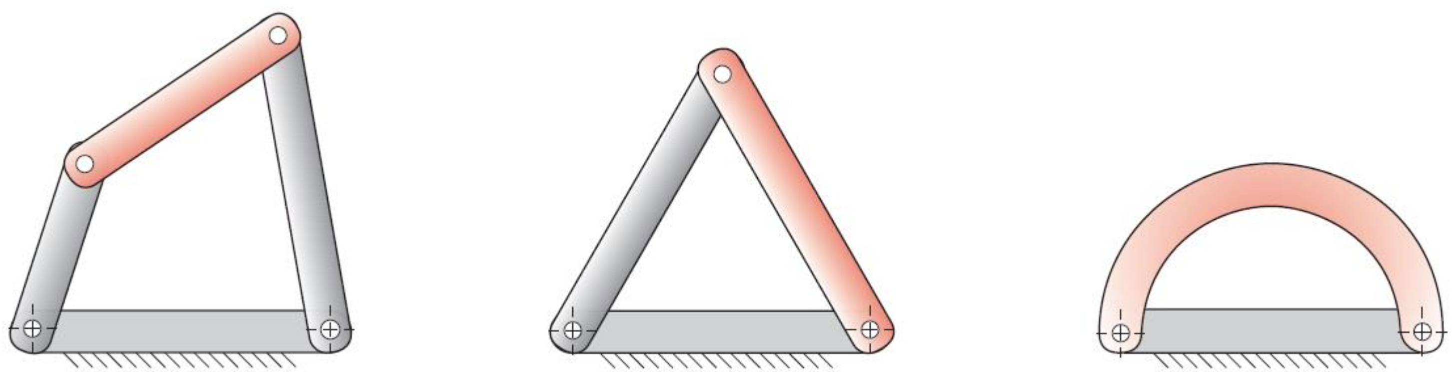
- The planar robot arm is an open loop kinematic chain that consists of four links and three joints. Determine its degree-of-freedom





Linkages and Structures

- The mobility of an assembly of links completely predicts the mechanism behaviour.



(a) Mechanism—*DOF* = +1 (b) Structure—*DOF* = 0 (c) Preloaded structure—*DOF* = −1

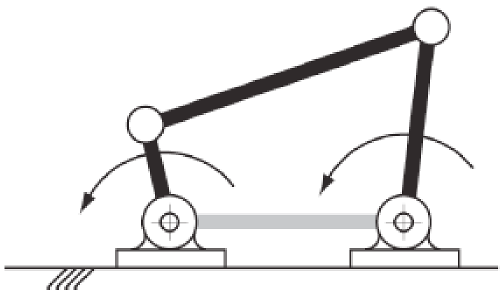
$F > 0$ $F = 0$ $F < 0$

Overview

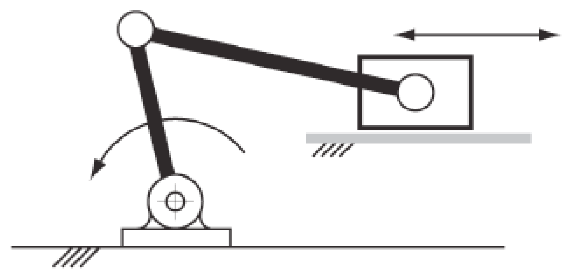
- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

The Four-Bar Mechanism

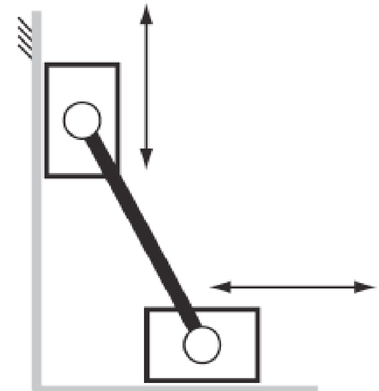
- The simplest & most versatile 1-dof mechanism is a four bar linkage.



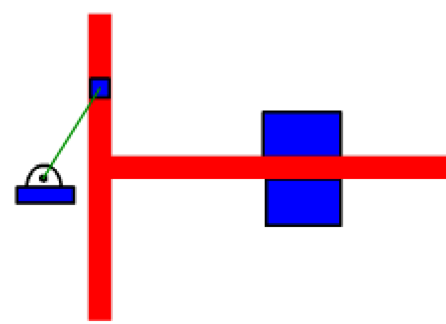
Four-bar linkage



Slider Crank linkage

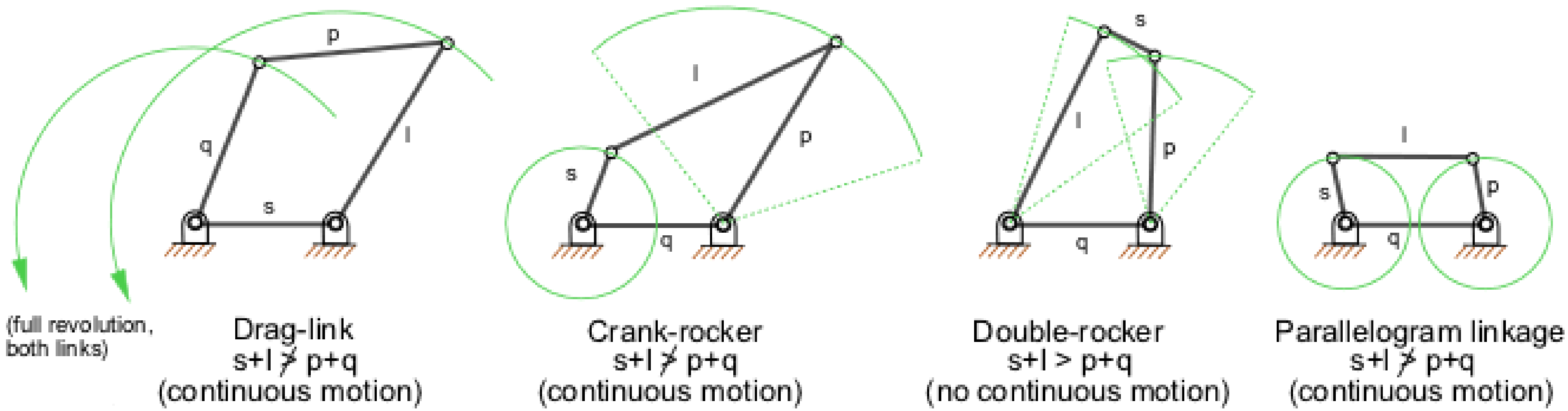


Double slider linkage



Grashof Condition

- The behavior of a four-bar linkage is dependent upon the relative lengths of its four links.



Case 1 **Case 2** **Case 3** **Case 4**

Grashof Condition

- The Grashof's condition for a four-bar linkage states that "If the sum of the shortest and longest link of a planar four bar linkage is less than or equal to the sum of the remaining two links, then the shortest link can rotate fully with respect to a neighbouring link.
- Condition

$$S+L \leq P+Q$$

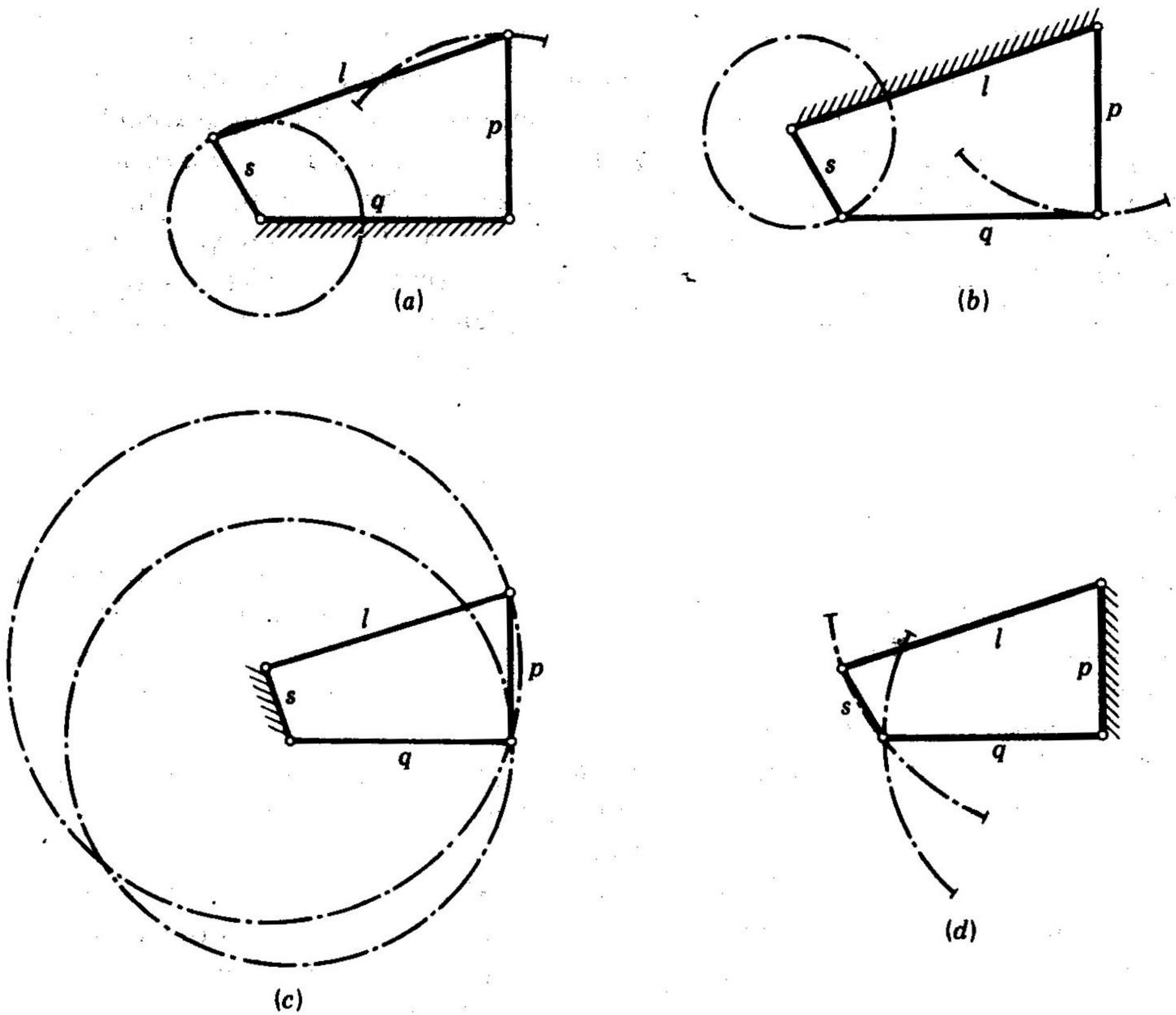
where S is the shortest link, L is the longest, and P and Q are the other links.

Grashof Condition

Inversion of Mechanism

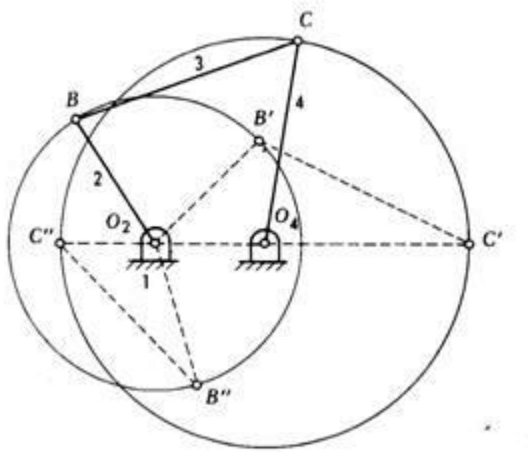
- A mechanism is one in which one of the links of a kinematic chain is fixed.
- Different mechanisms can be obtained by fixing different links of the same kinematic chain.
- These are called as inversions of the mechanism.
- By changing the fixed link, the number of mechanisms which can be obtained is equal to the number of links.
- Except the original mechanism, all other mechanisms will be known as inversions of original mechanism.
- The inversion of a mechanism does not change the motion of its links relative to each other.

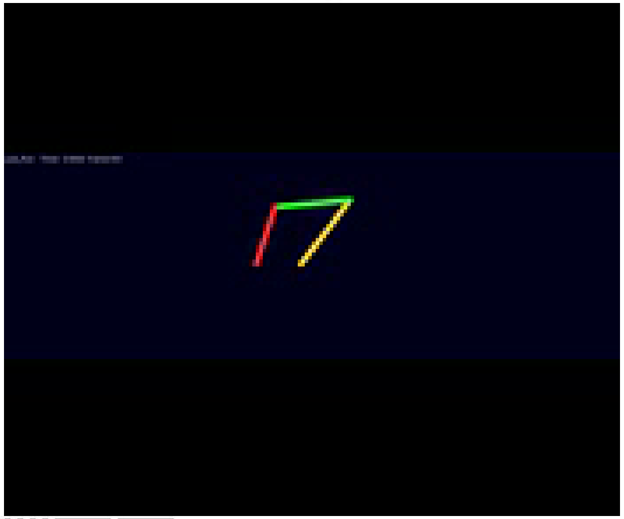
Inversion of 4 bar linkages

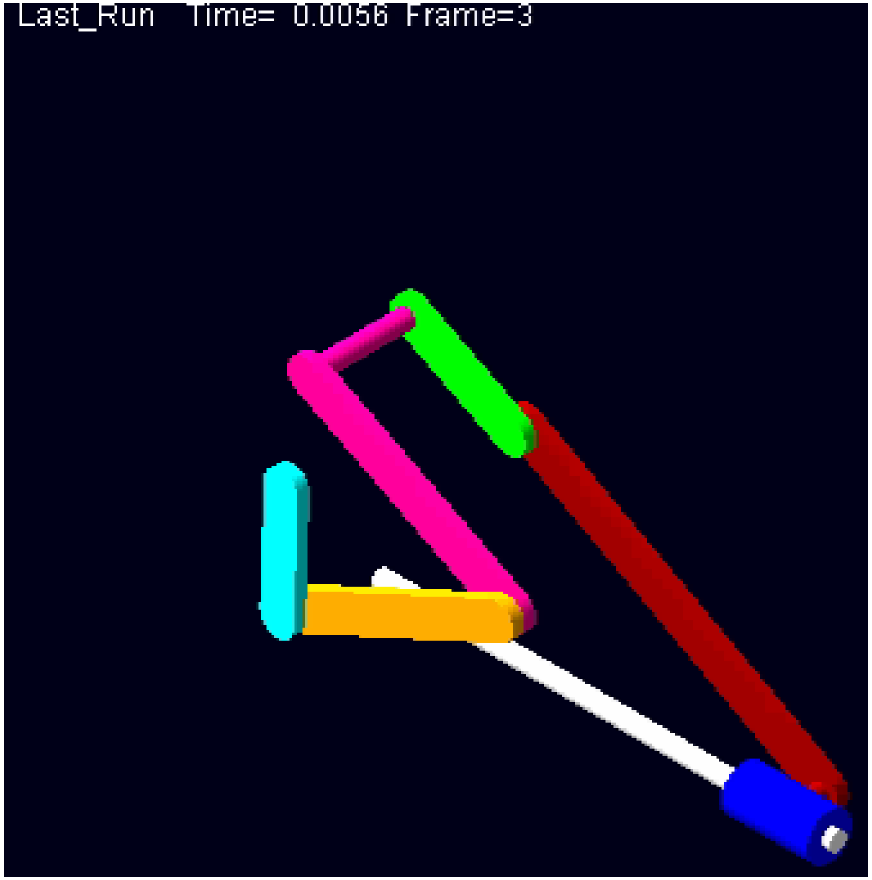


Grashof Condition-Case 1

- Drag link mechanism:

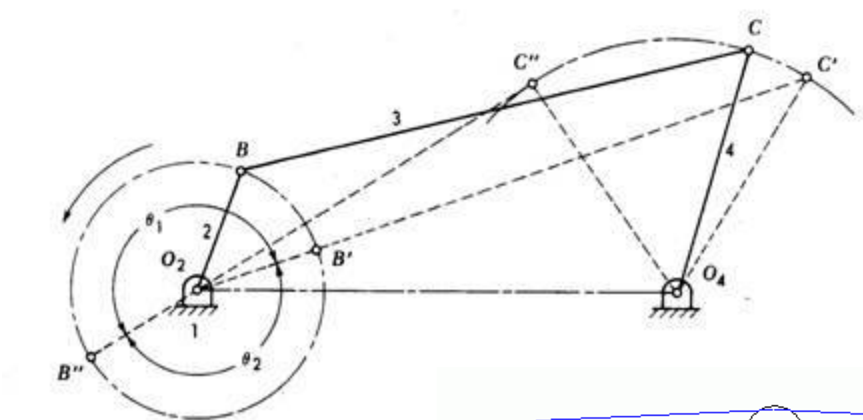


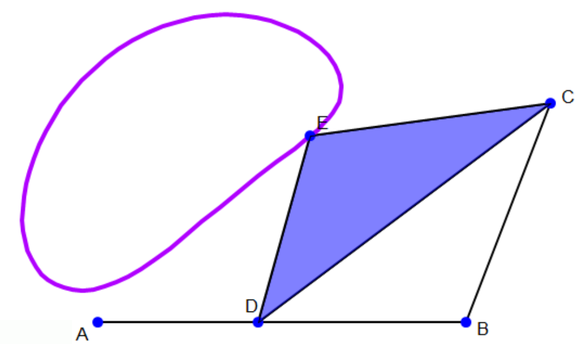


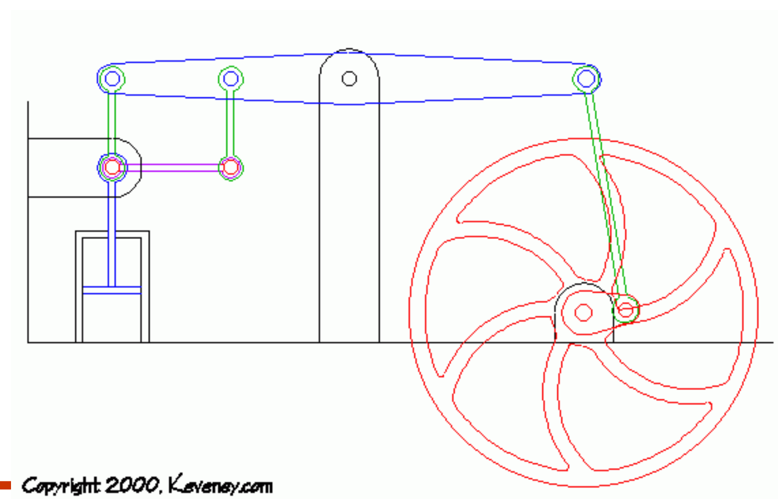


Grashof Condition-Case 2

- Crank-Rocker mechanism:in this mechanism, either link 1 or link 3 is fixed. Link 2 (crank) rotates completely and link 4 (rocker) oscillates.

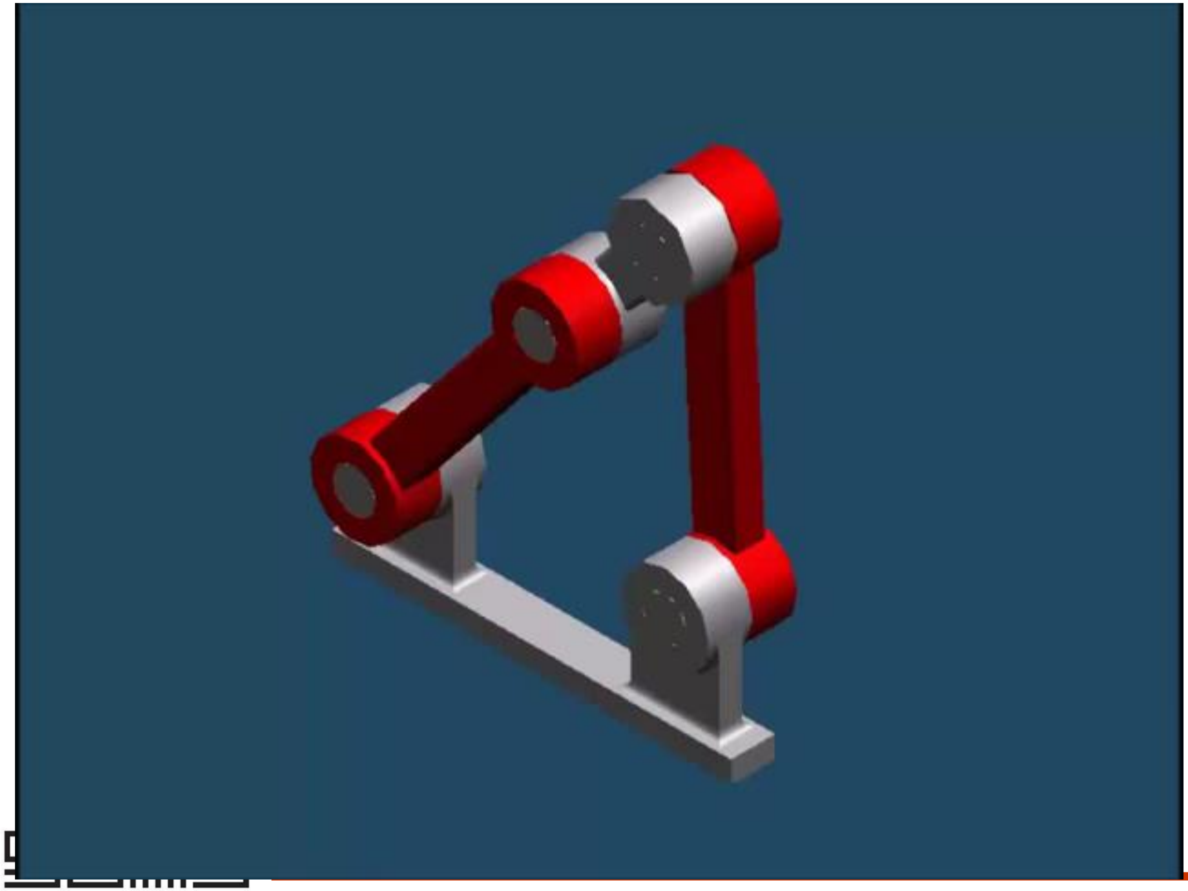


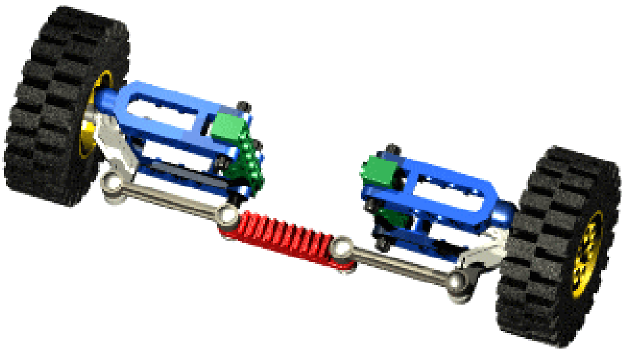




Grashof Condition-Case 3

- Double-rocker mechanism: in this mechanism, link 4 is fixed. Link 2 makes complete rotation, whereas links 3 & 4 oscillate.



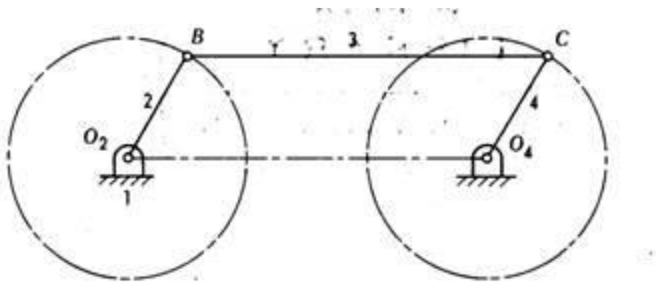


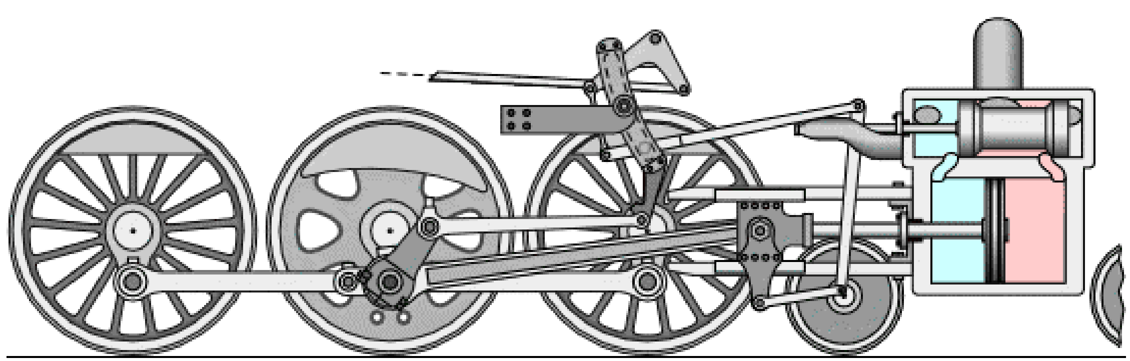
Grashof Condition- Case 3

❖ Double-rocker mechanism

Grashof Condition-Case 4

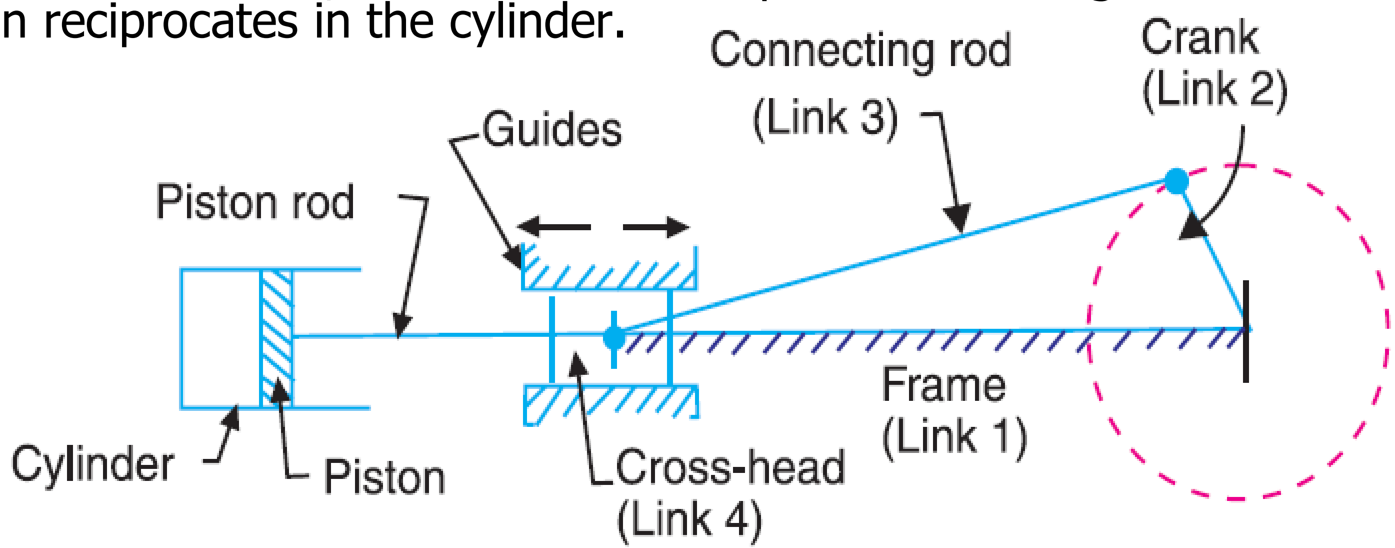
- Double-Crank mechanism: this is one type of drag link mechanism, where, links 1& 3 are equal and parallel and links 2 & 4 are equal and parallel.





Single slider crank chain

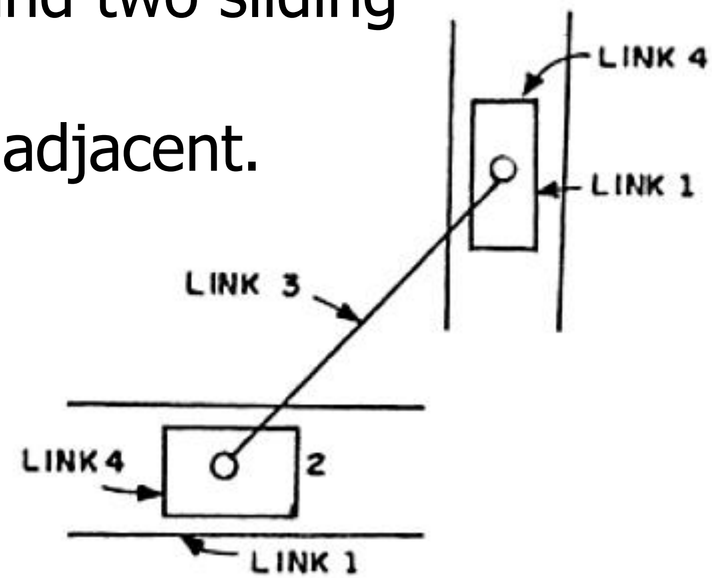
- A single slider crank chain is a modification of the basic four bar chain.
- It consist of one sliding pair and three turning pairs. It is, usually, found in reciprocating steam engine mechanism.
- This type of mechanism converts rotary motion into reciprocating motion and vice versa.
- In a single slider crank chain, as shown in Fig., the links 1 and 2, links 2 and 3, and links 3 and 4 form three turning pairs while the links 4 and 1 form a sliding pair.
- The link 1 corresponds to the frame of the engine, which is fixed. The link 2 corresponds to the crank ; link 3 corresponds to the connecting rod and link 4 corresponds to cross-head.
- As the crank rotates, the cross-head reciprocates in the guides and thus the piston reciprocates in the cylinder.



Single slider crank chain

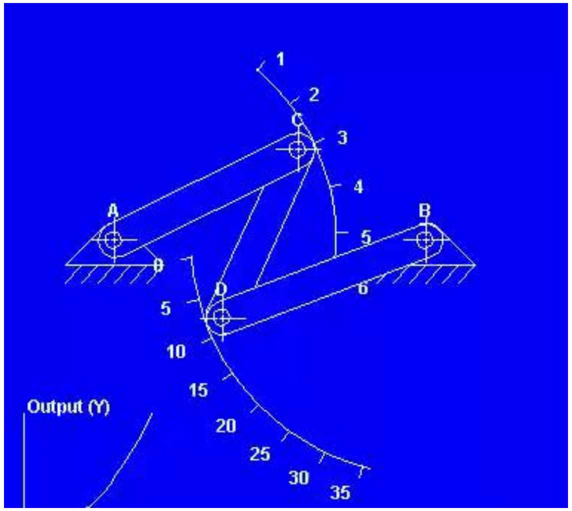
Double-Slider crank Chain

- It consists of four pairs out of which two are turning pairs and two others are sliding pairs.
- Also the two pairs of the same kind are adjacent.
- Double-slider crank chain is shown in Fig. 1.28 (a).
- Link 1 and link 2 is sliding pair, link 2 and link 3 is turning pair, link 3 and link 4 is second turning pair, link 4 and link 1 is second sliding pair.
- Hence there are two turning pairs and two sliding pairs.
- Also the pairs of the same kind are adjacent.

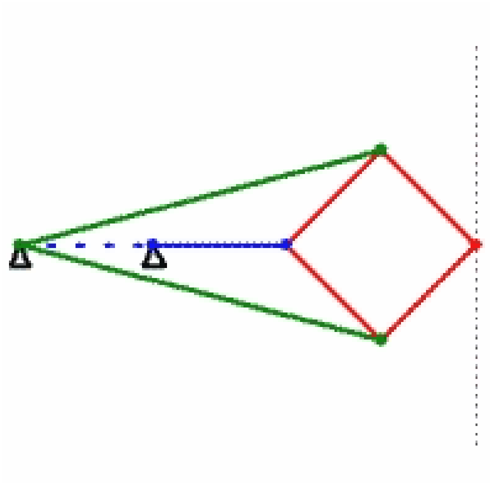


Linkage Design

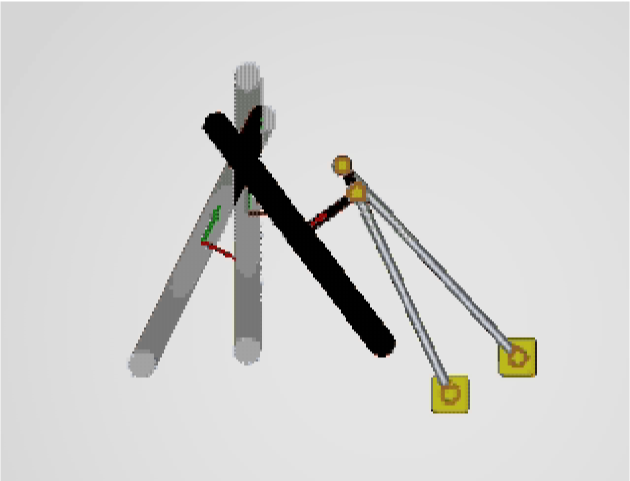
❖ In linkage design, we determine the dimensions of the linkage mechanism to perform a given task based on the functional requirements.



Function Generation



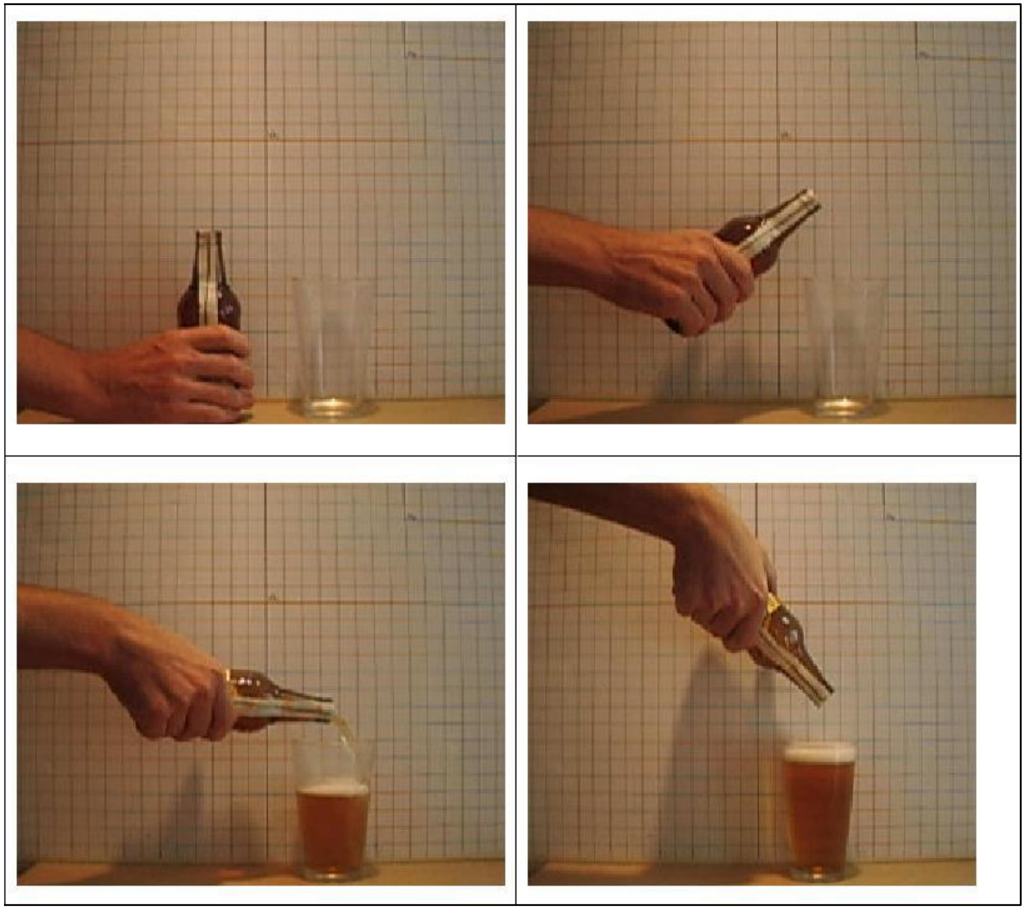
Path Generation



Rigid Body Guidance

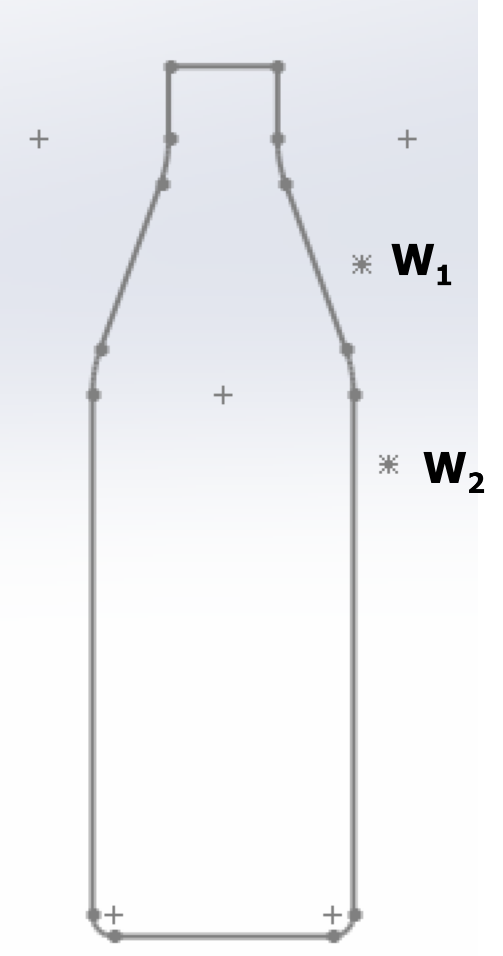
Geometric Design of Linkages

- Say we want to design a mechanism to guide a body (bottle) in a prescribed way.



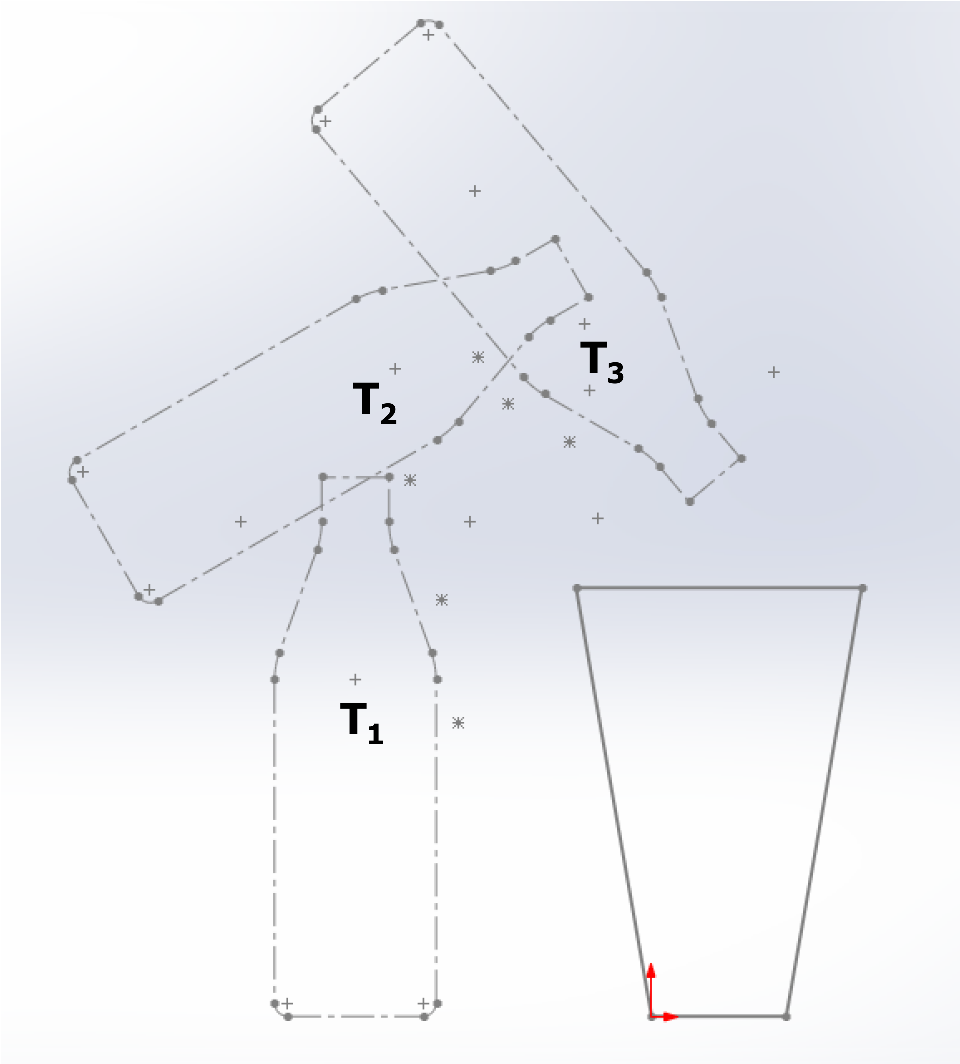
3 Position Synthesis

- Add two attachment points W_1 and W_2 to your body.
- These will be the moving pivots of your mechanism.
- You are free to choose this.



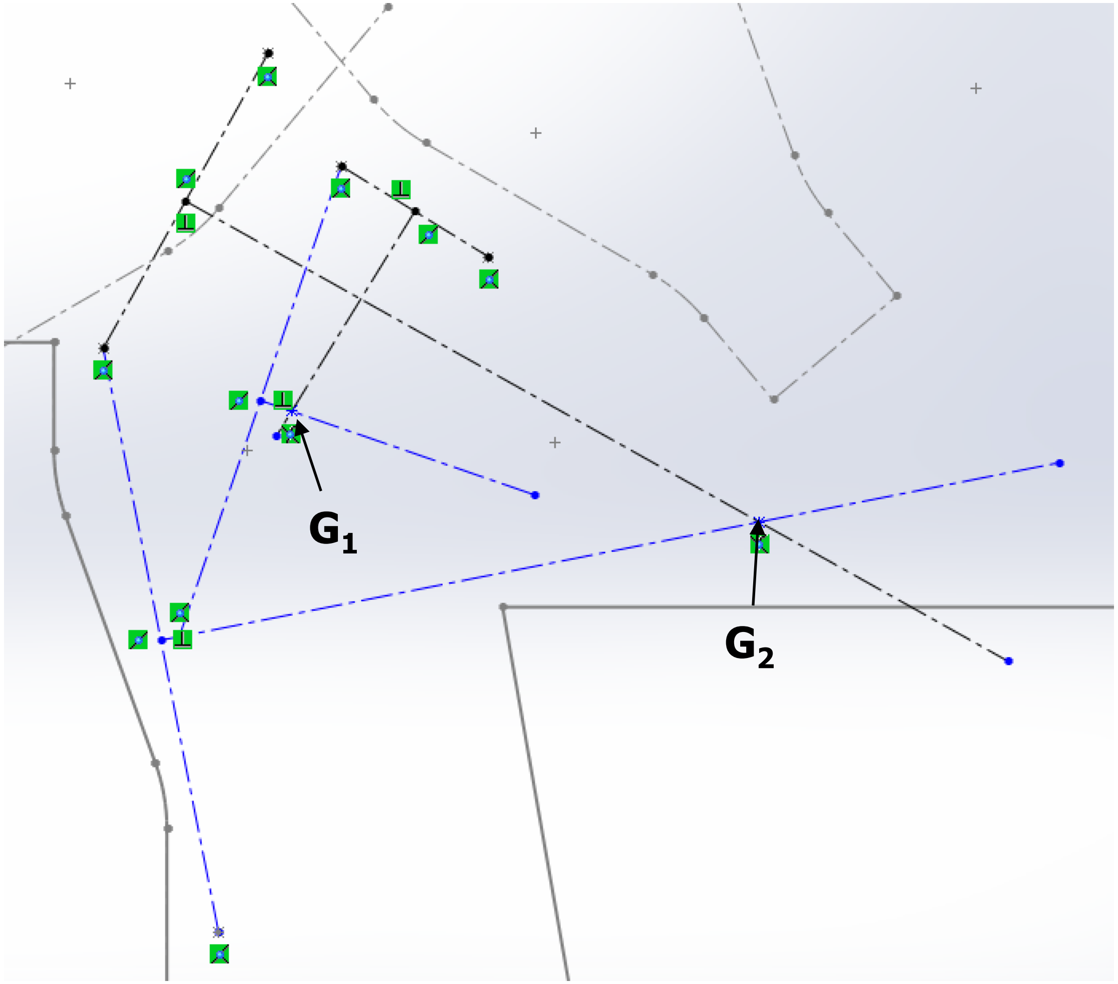
3 Position Synthesis

- Pick 3 task positions



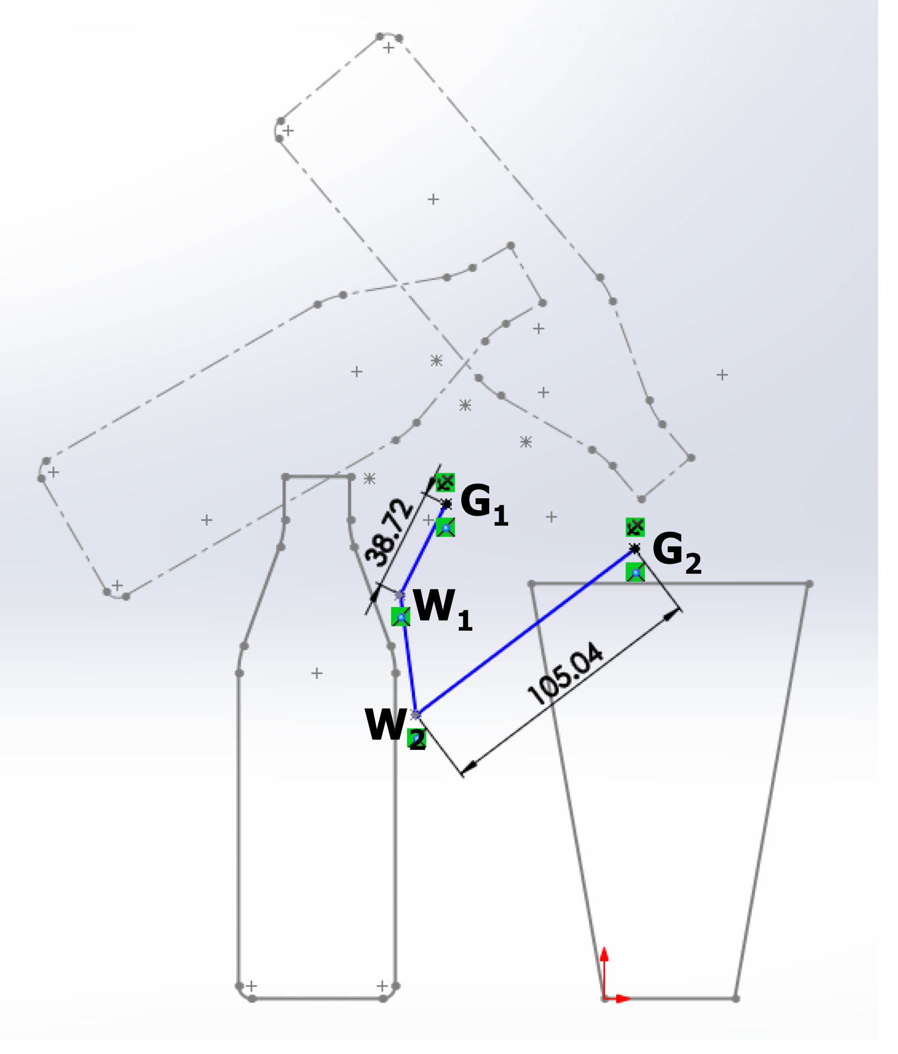
3 Position Synthesis

- For each of these attachment points, draw a perpendicular bisector to locate the ground pivots G_1 and G_2.



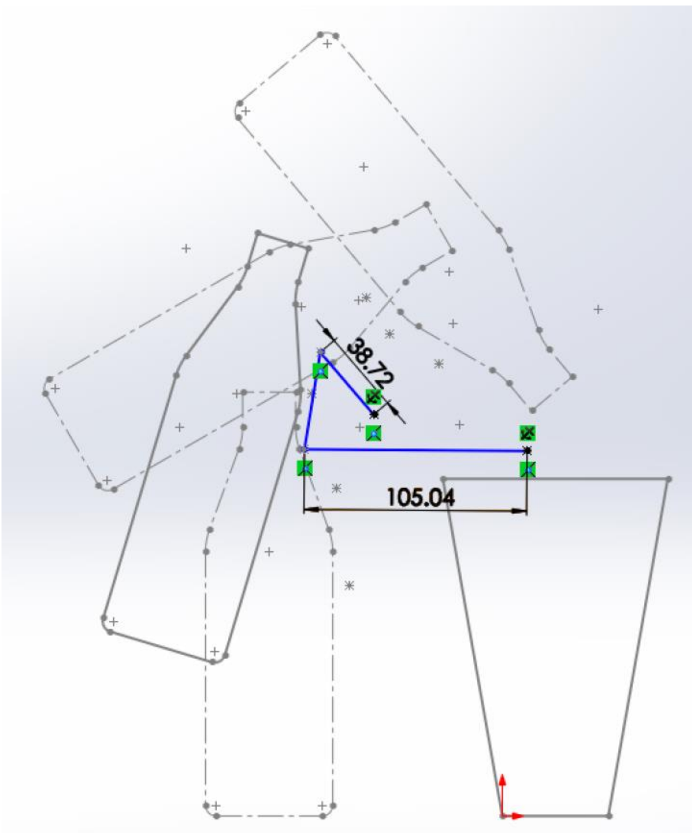
3 Position Synthesis

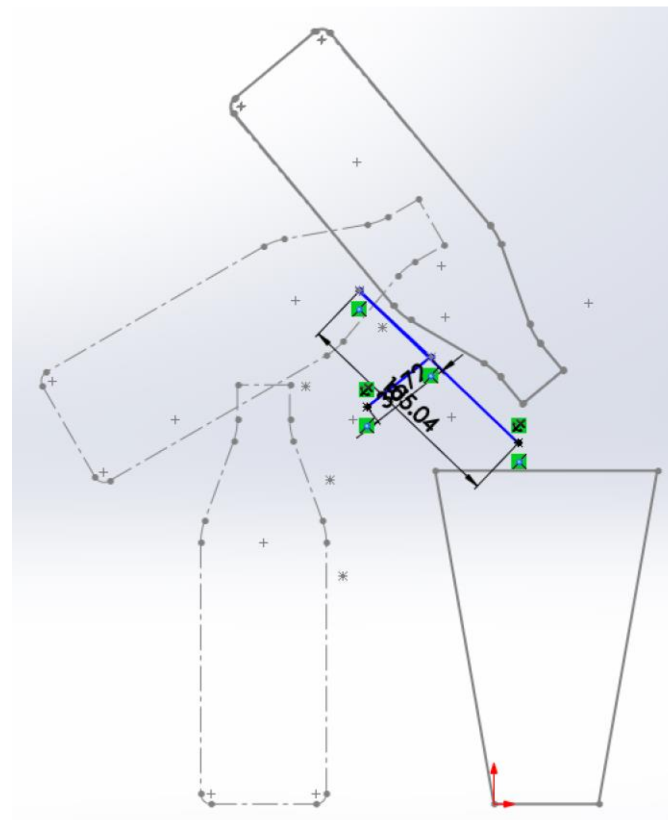
- Delete your construction lines and fixed the two newly found ground pivots G_i.
- Draw lines to connect your pivots $G_1W_1W_2G_2$ as shown.
- Dimension G_1W_1 and G_2W_2 as shown to obtain your four bar linkage.



3 Position Synthesis

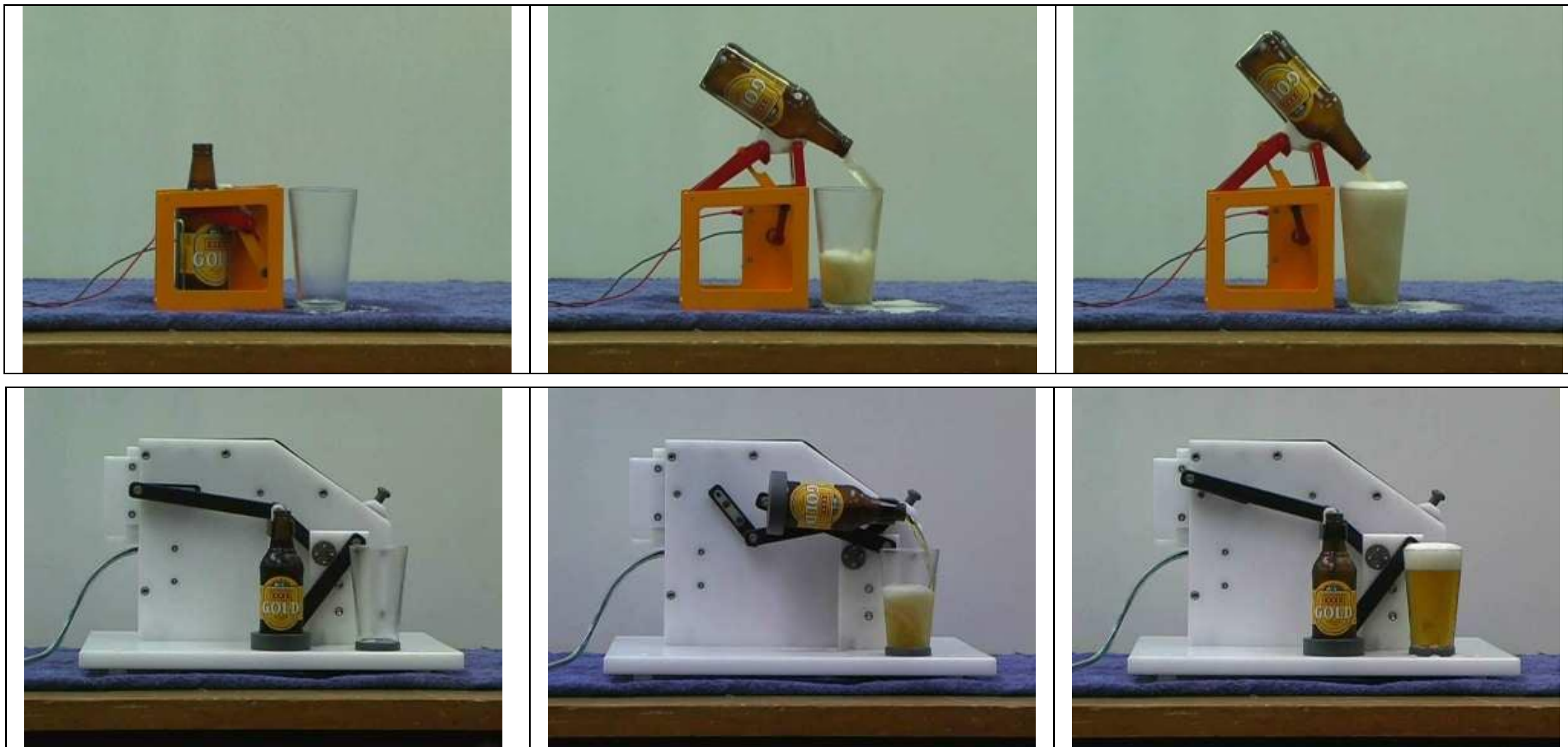
- Now move your block to see how your four-bar linkage behave.





Example

- Try to adjust your attachment pivots to see if you could get a feasible solution.



Overview

- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

Linkage Analysis

- Once you have designed your linkage, you need to be able to do two things before performing stress analysis.
  - Know the angle, angular velocity and angular acceleration of every link.
  - Determine the position, velocity and acceleration of any point on the linkage.

- Once the above is determined, you can solve it using Newton's Law treating the 4 bar linkage as a multi-component structure with

$$\boxed{\sum F = m\vec{a}}$$

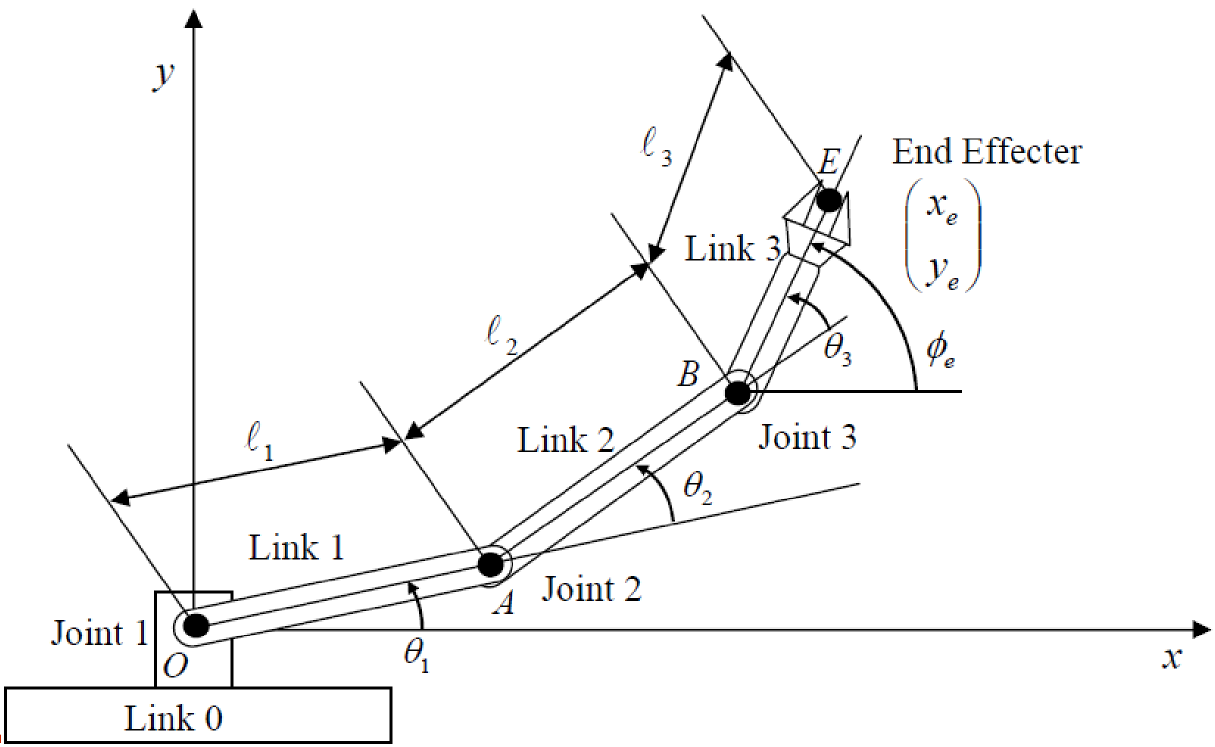
$$\boxed{\text{å}\, T = I a}$$

Overview

- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

Planar kinematics of Serial Link Mechanisms

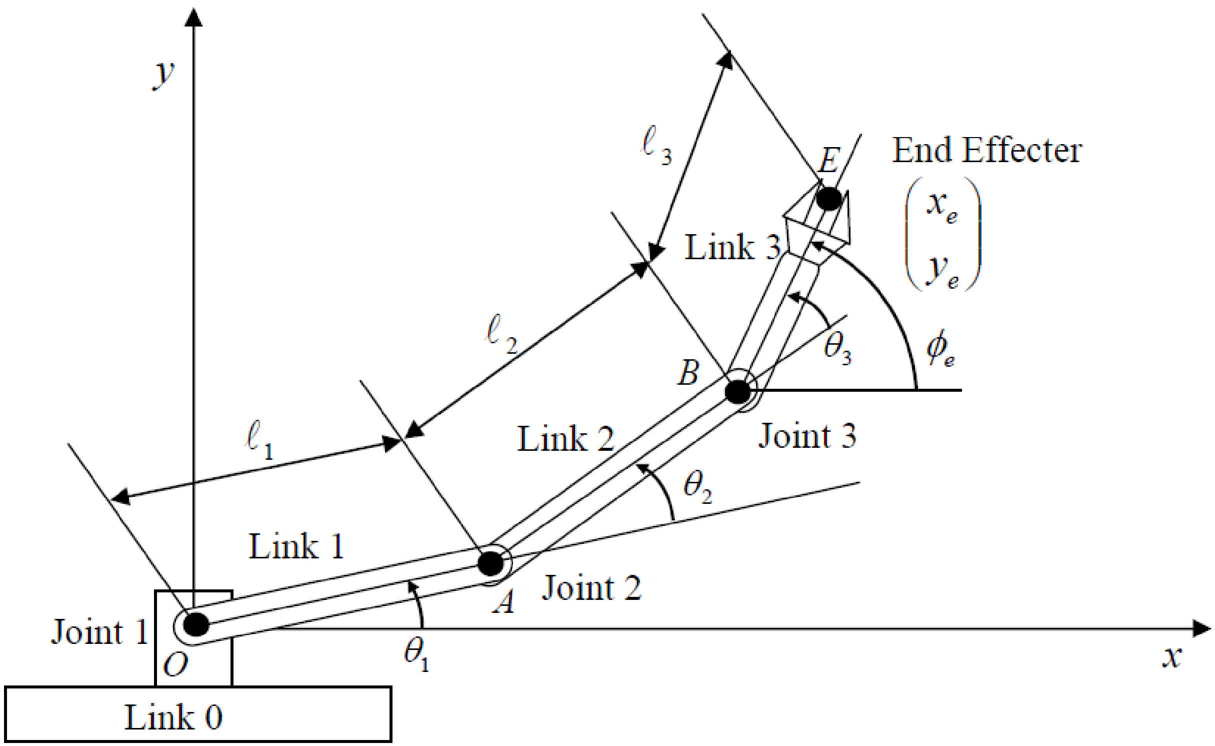
- Consider a three degree-of-freedom, planar robot arm shown in Figure aabove. The arm consists of one fixed link and three movable links that move within the plane. All the links are connected by revolute joints whose joint axes are all perpendicular to the plane of the links. There is no closed-loop kinematic chain; hence, it is a serial link mechanism.



Planar kinematics of Serial Link Mechanisms

- Since this robot arm performs task by moving its end-effecter at point E, we are concerned with the location of the end-effecter. To describe its location, we use a coordinate system, O-xy, fixed to the base link with the origin at the first joint, and describe the end-effecter position with coordinates xe and ye . We can relate the end-effecter coordinates to the joint angles determined by the three actuators by using the link lengths and joint angles

Planar kinematics of Serial Link Mechanisms



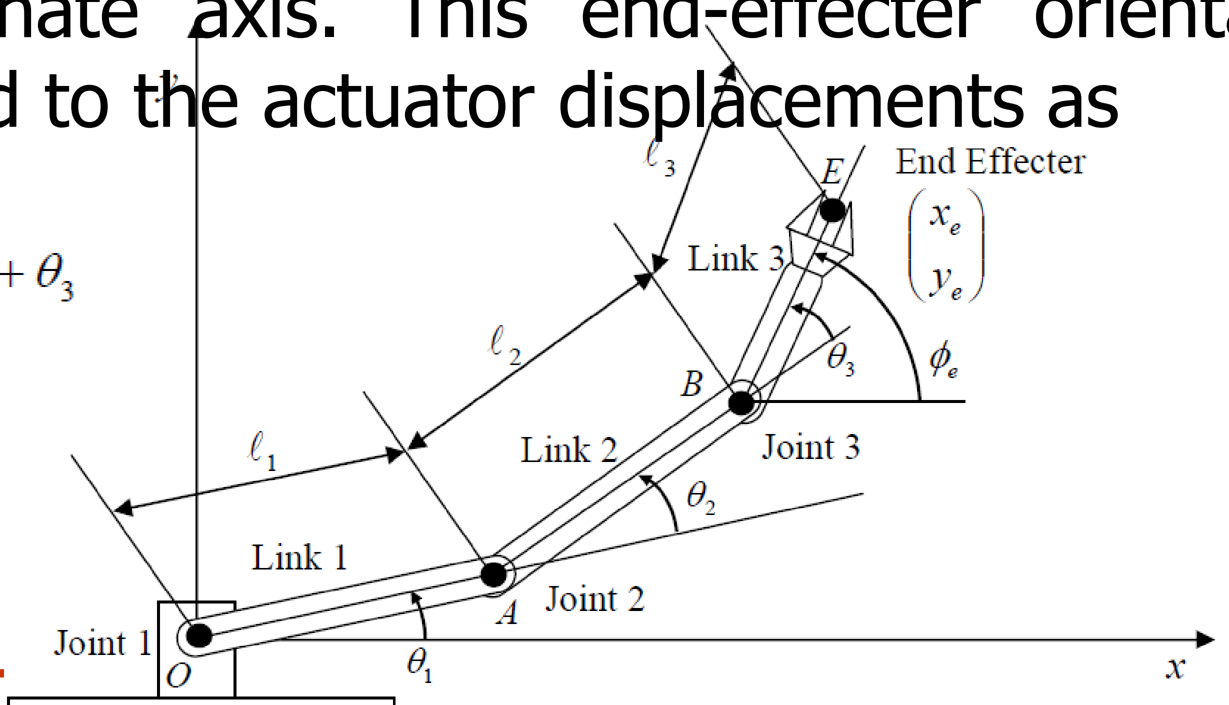
$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

Planar kinematics of Serial Link Mechanisms

- This three dof robot arm can locate its end-effecter at a desired orientation as well as at a desired position. The orientation of the end-effecter can be described with the angle of the centerline of the end-effecter measured from the positive x coordinate axis. This end-effecter orientation is related to the actuator displacements as

$$\phi_e = \theta_1 + \theta_2 + \theta_3$$



Planar kinematics of Serial Link Mechanisms

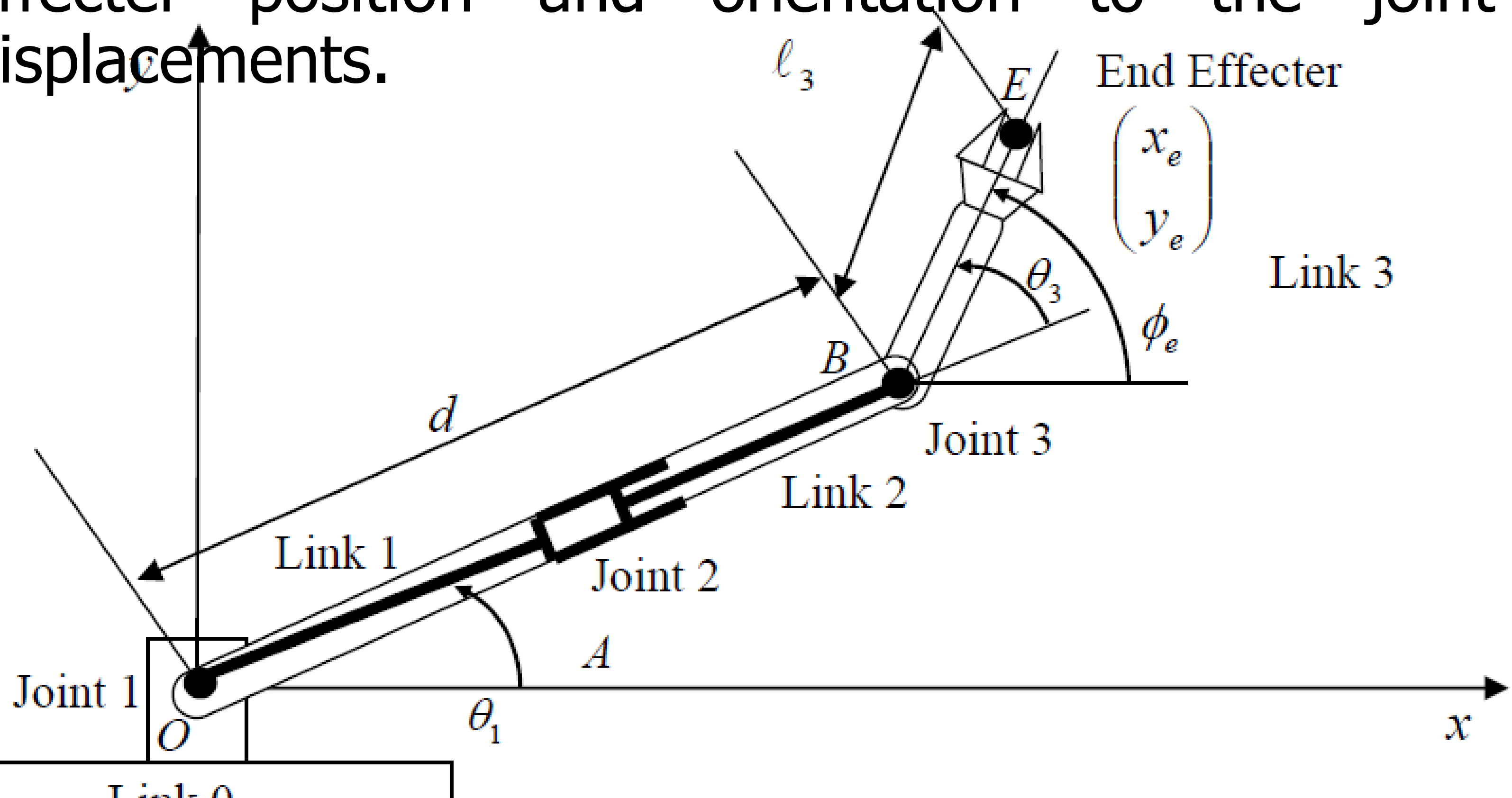
$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi_e = \theta_1 + \theta_2 + \theta_3$$

- The above three equations describe the position and orientation of the robot end-effecter viewed from the fixed coordinate system in relation to the actuator displacements. In general, a set of algebraic equations relating the position and orientation of a robot end-effecter, or any significant part of the robot, to actuator displacements, or displacements of active joints, is called **Kinematic Equations**, or more specifically, **Forward Kinematic Equations**

Exercise 3

- Shown below in Figure is a planar robot arm with two revolute joints and one prismatic joint. Using the geometric parameters and joint displacements, obtain the kinematic equations relating the end-effecter position and orientation to the joint displacements.



Formal Expression of Kinematic Equations

- Two types of joints, prismatic and revolute joints, constitute robot mechanisms in most cases. The displacement of the *i-th* joint is described by distance d_i if it is a prismatic joint, and by angle θ_i for a revolute joint. For **formal expression**, let us use a generic notation: q_i. Namely, joint displacement q_i represents either distance d_i or angle θ_i depending on the type of joint.

$$q_i = \begin{cases} d_i & \text{Prismatic joint} \\ \theta_i & \text{Revolute joint} \end{cases}$$

Formal Expression of Kinematic Equations

- We collectively represent all the joint displacements involved in a robot mechanism with a column vector: $q = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$, where n is the number of joints. Kinematic equations relate these joint displacements to the position and orientation of the end-effecter. Let us collectively denote the end-effecter position and orientation by vector p. For planar mechanisms, the end-effecter location is described by three variables:

$$p = \begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix}$$

Formal Expression of Kinematic Equations

- Using these notations, we represent kinematic equations as a vector function relating p to q:

$$p = f(q), \quad p \in \Re^{3x1}, q \in \Re^{nx1}$$

Inverse Kinematics of Planar Serial Link Mechanisms

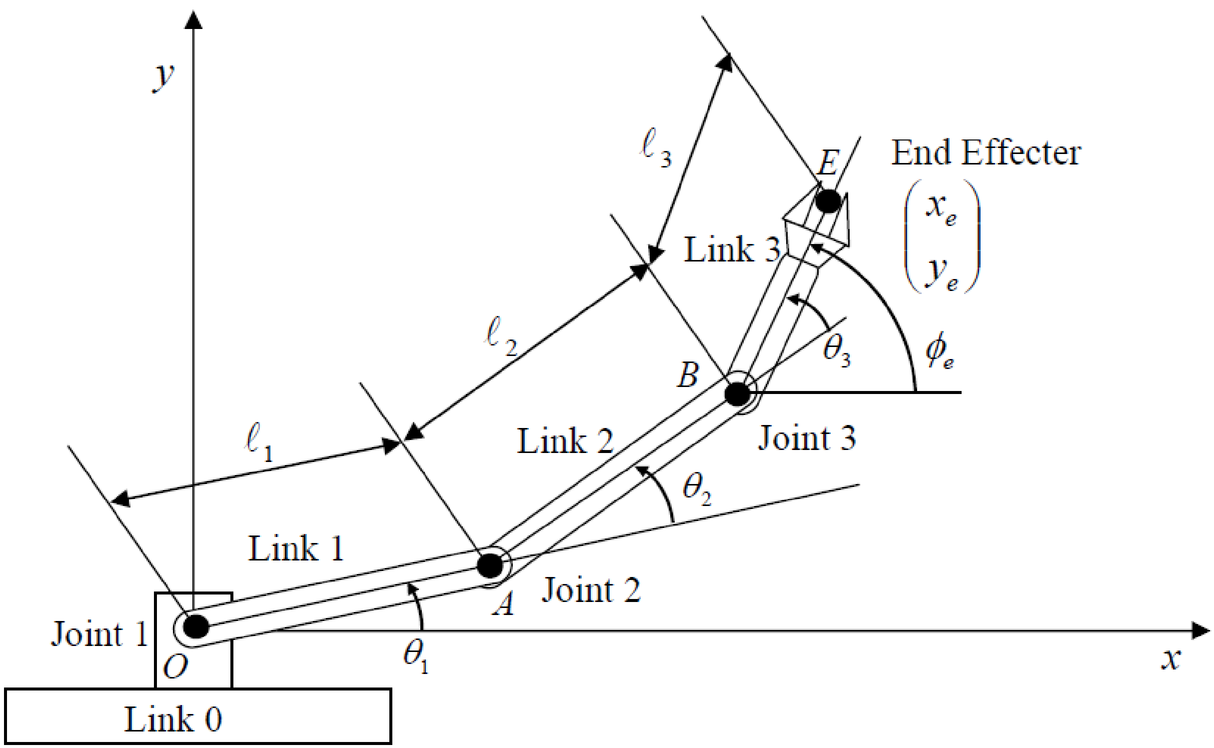
- The problem of finding the end-effecter position and orientation for a given set of joint displacements is referred to as the **direct/ or forward** kinematics **problem**.
- we discuss now the problem of moving the end-effecter of a manipulator arm to a specified position and orientation. We need to find the joint displacements that lead the end-effecter to the specified position and orientation. This is the inverse of the previous problem, and is thus referred to as the **inverse kinematics problem**.

Inverse Kinematics of Planar Serial Link Mechanisms

- The kinematic equation must be solved for joint displacements, given the end-effecter position and orientation. Once the kinematic equation is solved, the desired end-effecter motion can be achieved by moving each joint to the determined value.
- In the direct kinematics problem, the end-effecter location is determined uniquely for any given set of joint displacements. On the other hand, the inverse kinematics is more complex in the sense that ***multiple solutions*** may exist for the same end-effecter location. Also, solutions may not always exist for a particular range of end-effecter locations and arm structures. Further, since the kinematic equation is comprised of ***nonlinear*** simultaneous equations with many trigonometric functions, it is not always possible to derive a ***closed-form solution***, which is the explicit inverse function of the kinematic equation. When the kinematic equation cannot be solved analytically, ***numerical methods*** are used in order to derive the desired joint displacements

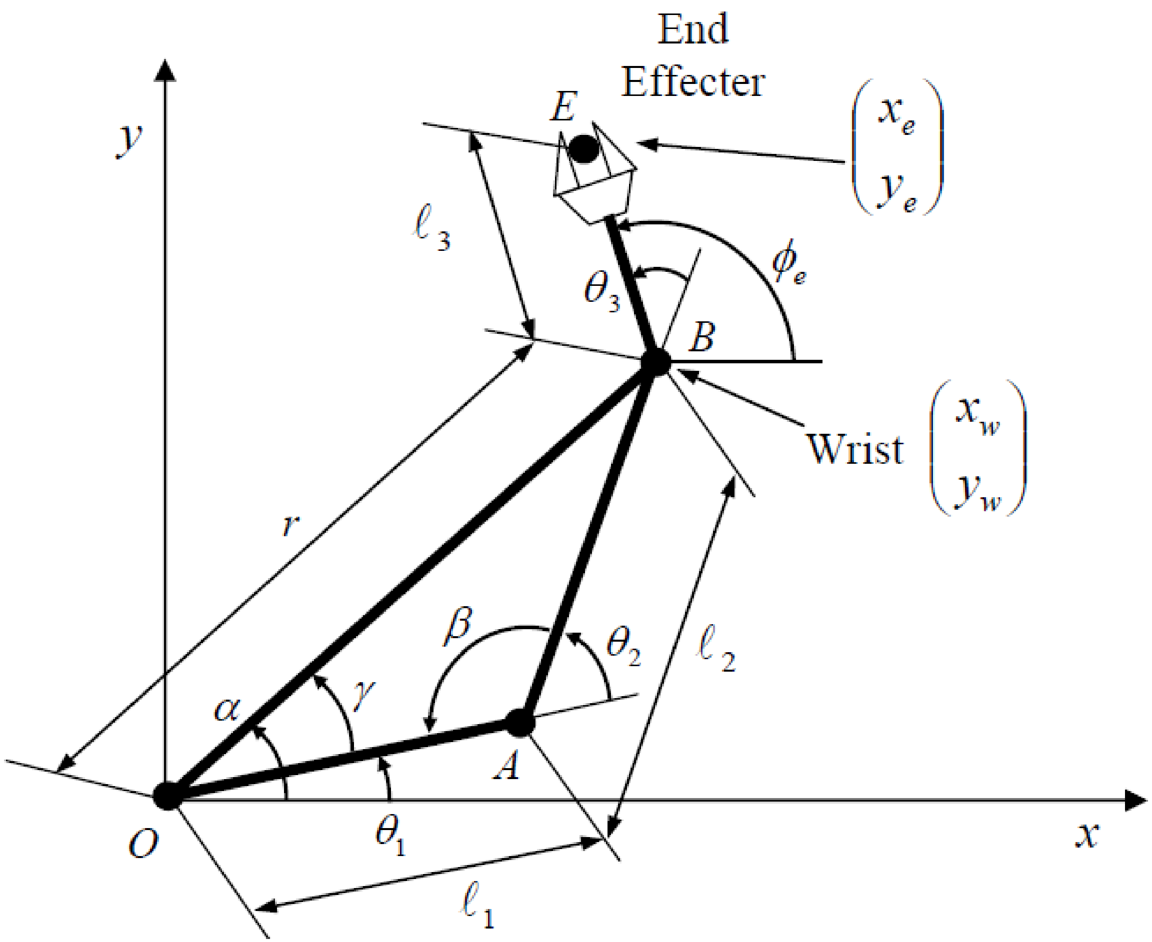
Example 3

- Consider the three dof planar arm shown in Figure below.



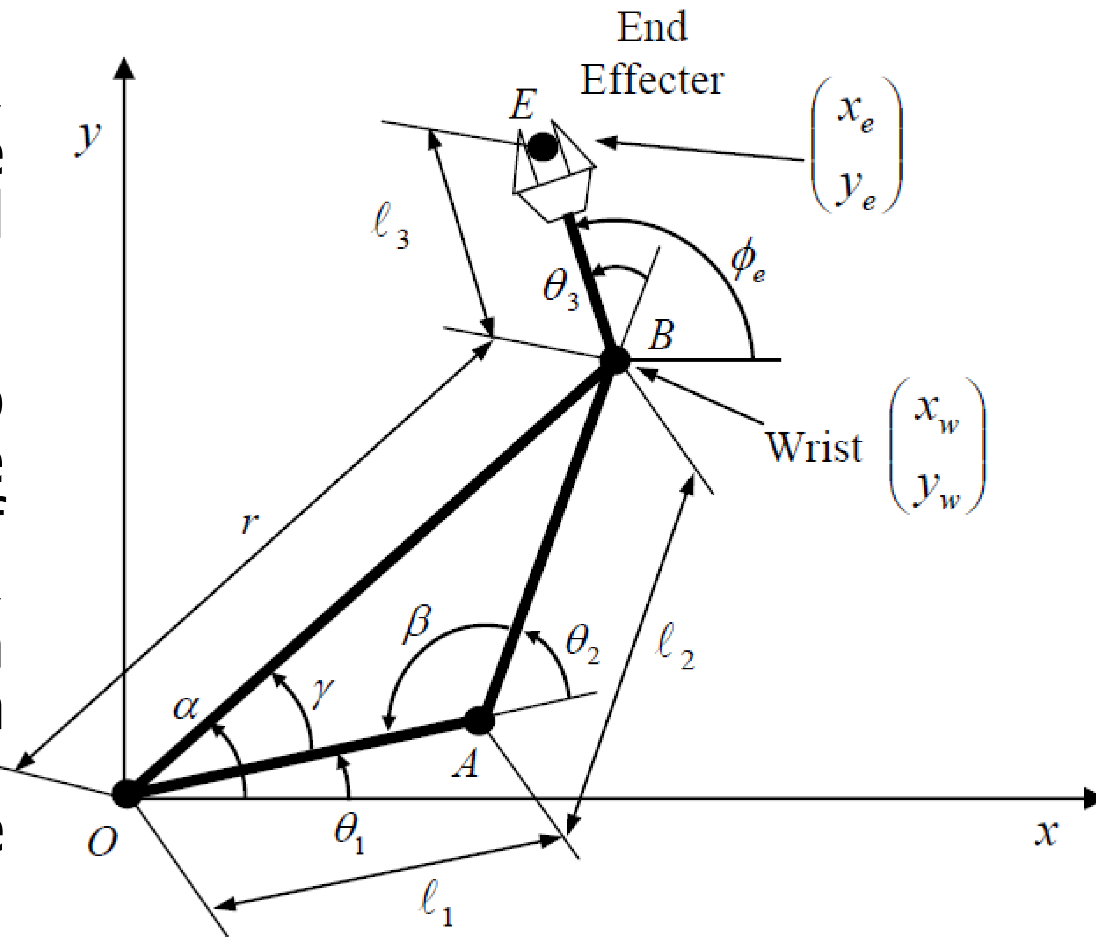
Example 3

- To solve its inverse kinematics problem the kinematic structure is redrawn in Figure below



Example 3

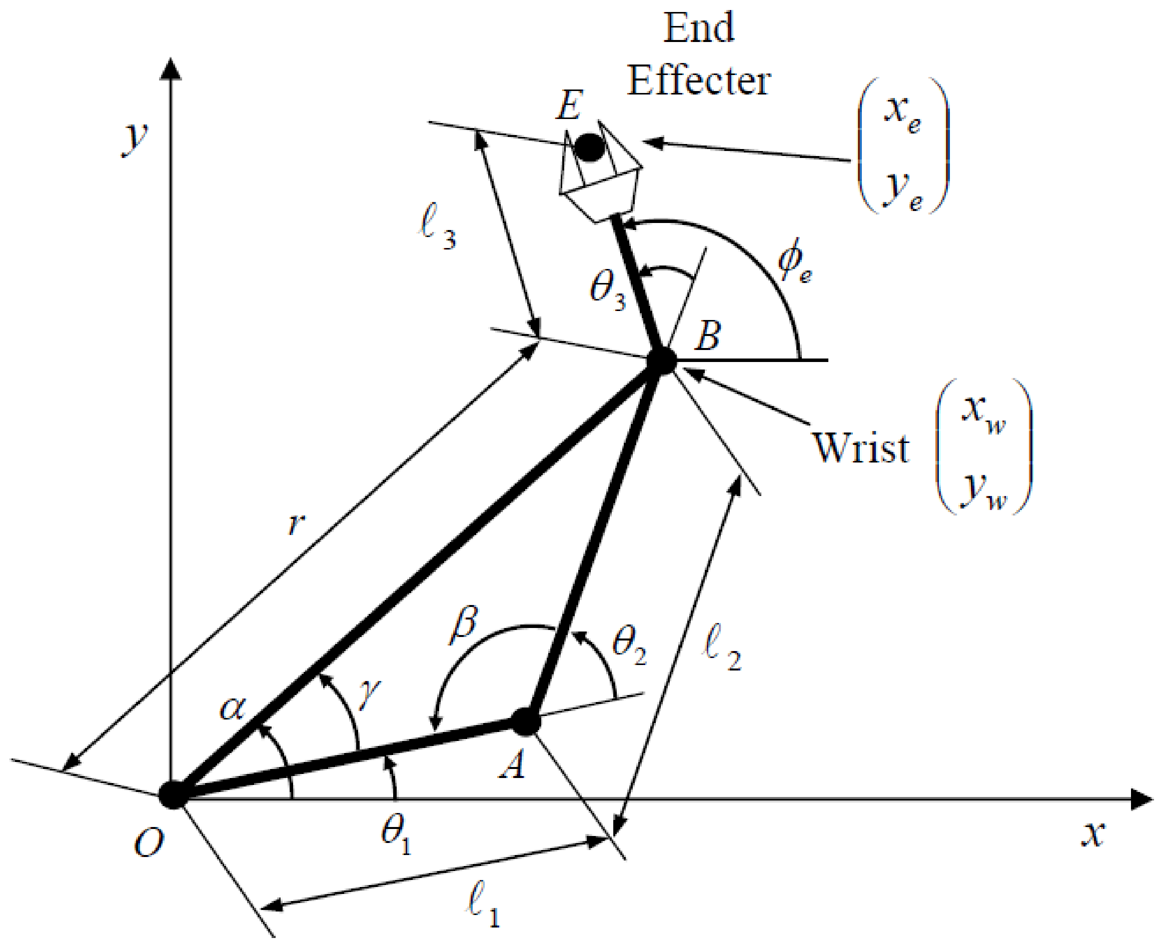
- The problem is to find three joint angles θ_1, θ_2, θ_3 that lead the end effecter to desired position and orientation, x_e, y_e, θ_e. We take a two-step approach. First, we find the position of the wrist, point B, from , x_e, y_e, θ_e. Then we find θ_1, θ_2 from the wrist position. Angle θ_3 can be determined immediately from the wrist position.



Example 3

$$x_w = x_e - \ell_3 \cos\phi_e$$
$$y_w = y_e - \ell_3 \sin\phi_e$$

$$\alpha = \tan^{-1}\frac{y_w}{x_w}$$



- Let us consider the triangle OAB and applying the law of cosines to angle β yields

$$\ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos\beta = r^2$$

$$r^2 = x_w^2 + y_w^2$$

$$\boxed{\theta_2 = \pi - \beta = \pi - \cos^{-1}\frac{\ell_1^2 + \ell_2^2 - x_w^2 - y_w^2}{2\ell_1\ell_2}}$$

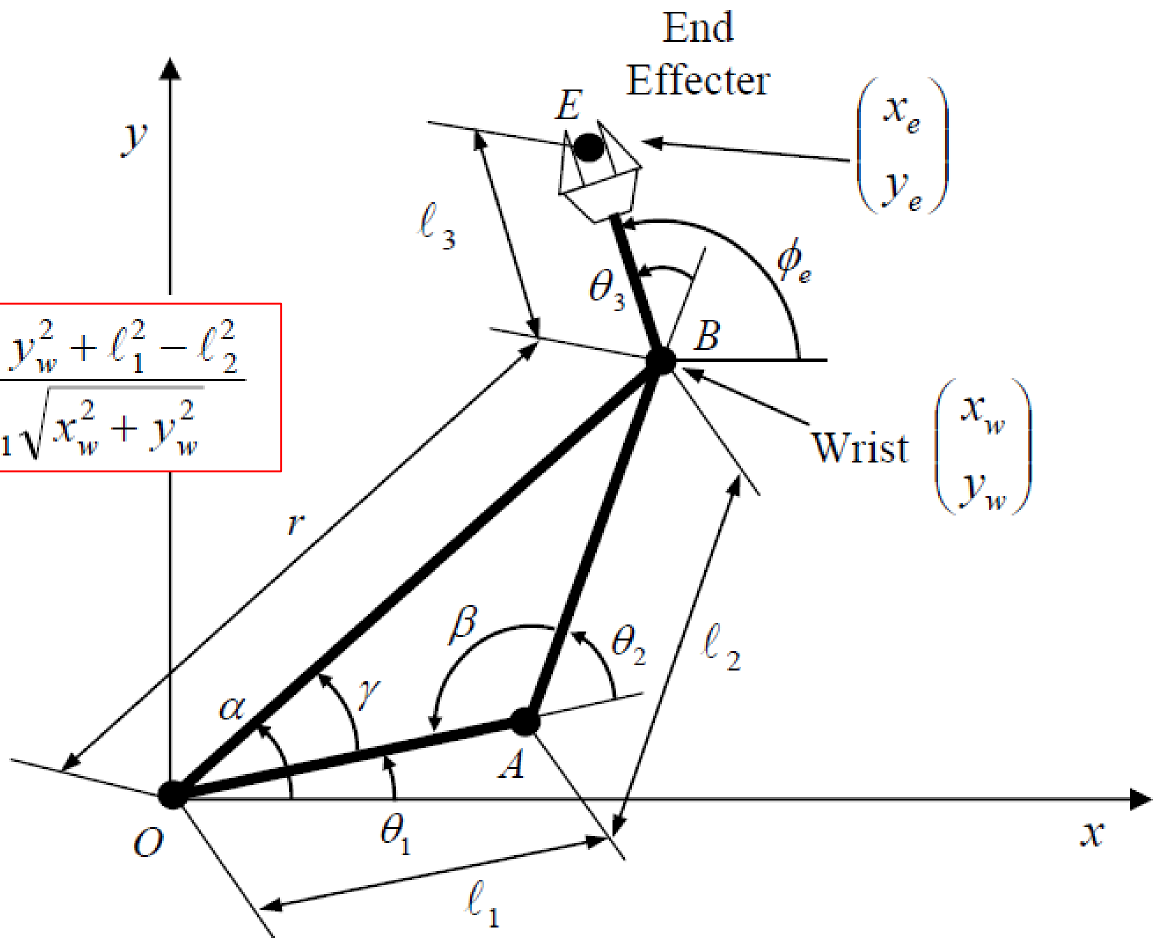
Example 3

- Similarly,

$$r^2 + \ell_1^2 - 2r\ell_1 \cos\gamma = \ell_2^2$$

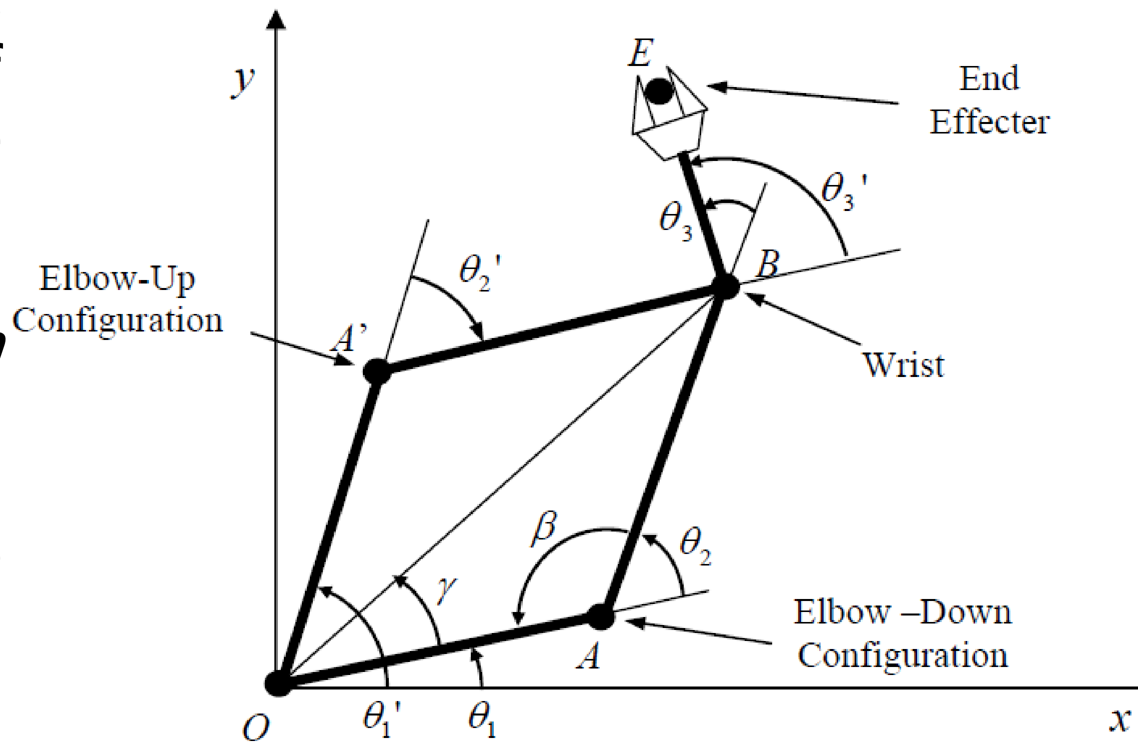
$$\boxed{\theta_1 = \alpha - \gamma = \tan^{-1}\frac{y_w}{x_w} - \cos^{-1}\frac{x_w^2 + y_w^2 + \ell_1^2 - \ell_2^2}{2\ell_1\sqrt{x_w^2 + y_w^2}}}$$

$$\boxed{\theta_3 = \phi_e - \theta_1 - \theta_2}$$



Example 3

- It is interesting to note that there is another way of reaching the same end-effecter position and orientation, i.e. ***another solution*** to the inverse kinematics problem. Figure shows two configurations of the arm leading to the same end-effecter location.



$$\theta_1' = \theta_1 + 2\gamma$$

$$\theta_2' = -\theta_2$$

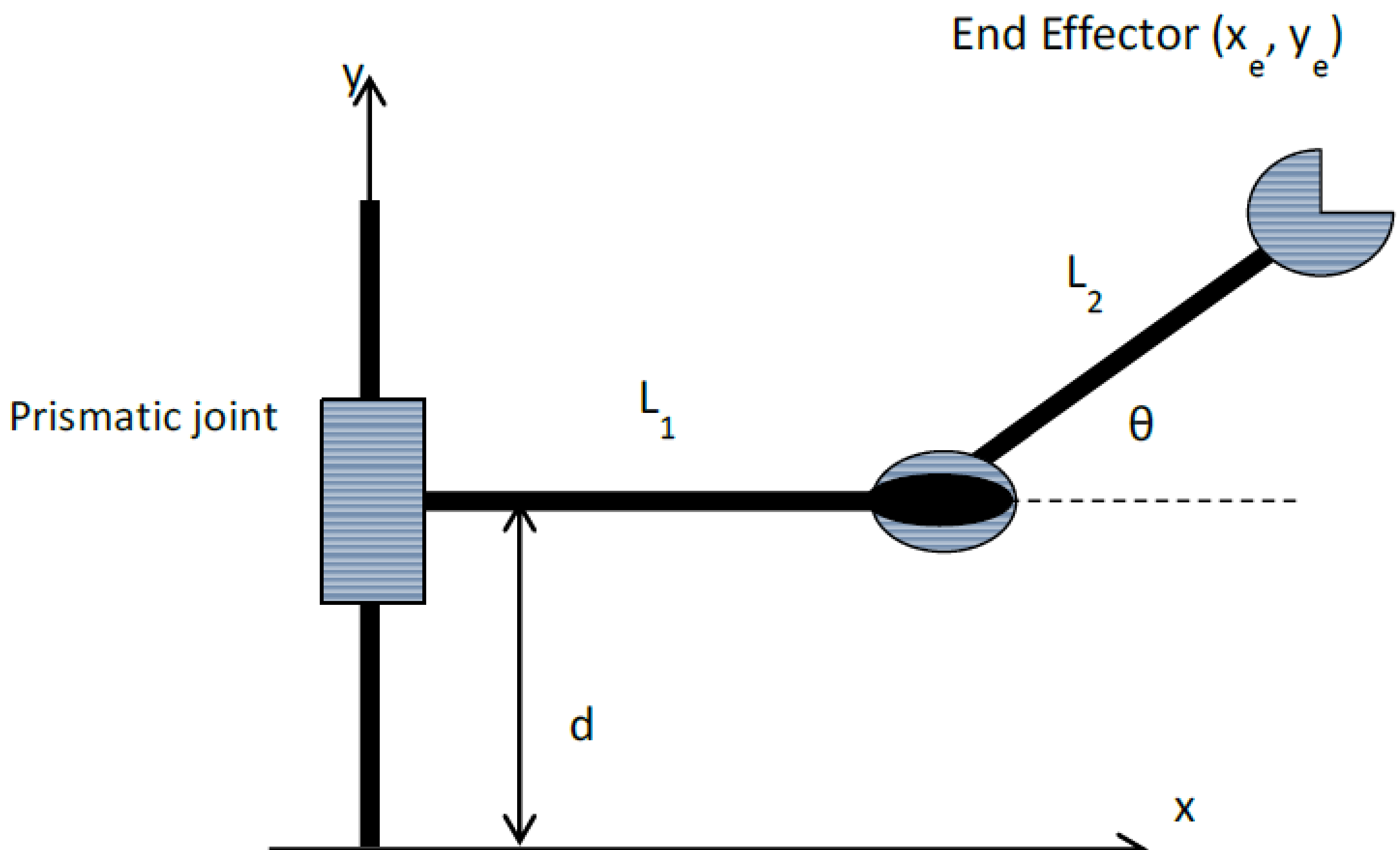
$$\theta_3' = \phi_e - \theta_1' - \theta_2' = \theta_3 + 2\theta_2 - 2\gamma$$

Multiple solutions to the inverse kinematics problem

- Inverse kinematics problems often possess multiple solutions, like the above example, since they are nonlinear. Specifying end-effecter position and orientation does not uniquely determine the whole configuration of the system.
- The existence of multiple solutions, however, provides the robot with an extra degree of flexibility. Consider a robot working in a crowded environment. If multiple configurations exist for the same end-effecter location, the robot can take a configuration having no interference with the environment. Due to physical limitations, however, the solutions to the inverse kinematics problem do not necessarily provide feasible configurations. We must check whether each solution satisfies the constraint of movable range, i.e. stroke limit of each joint.

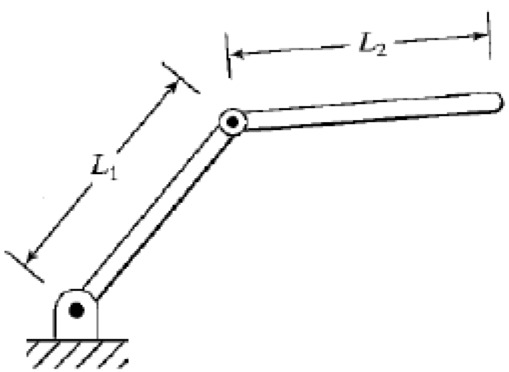
Exercise 4

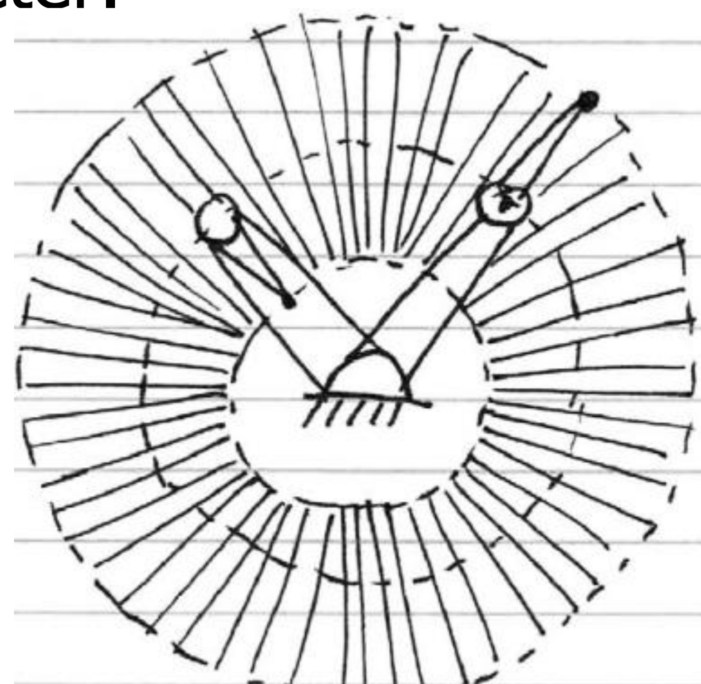
- Given the desired end effector position, calculate the parameters d and θ for the robot arm shown



Workspace

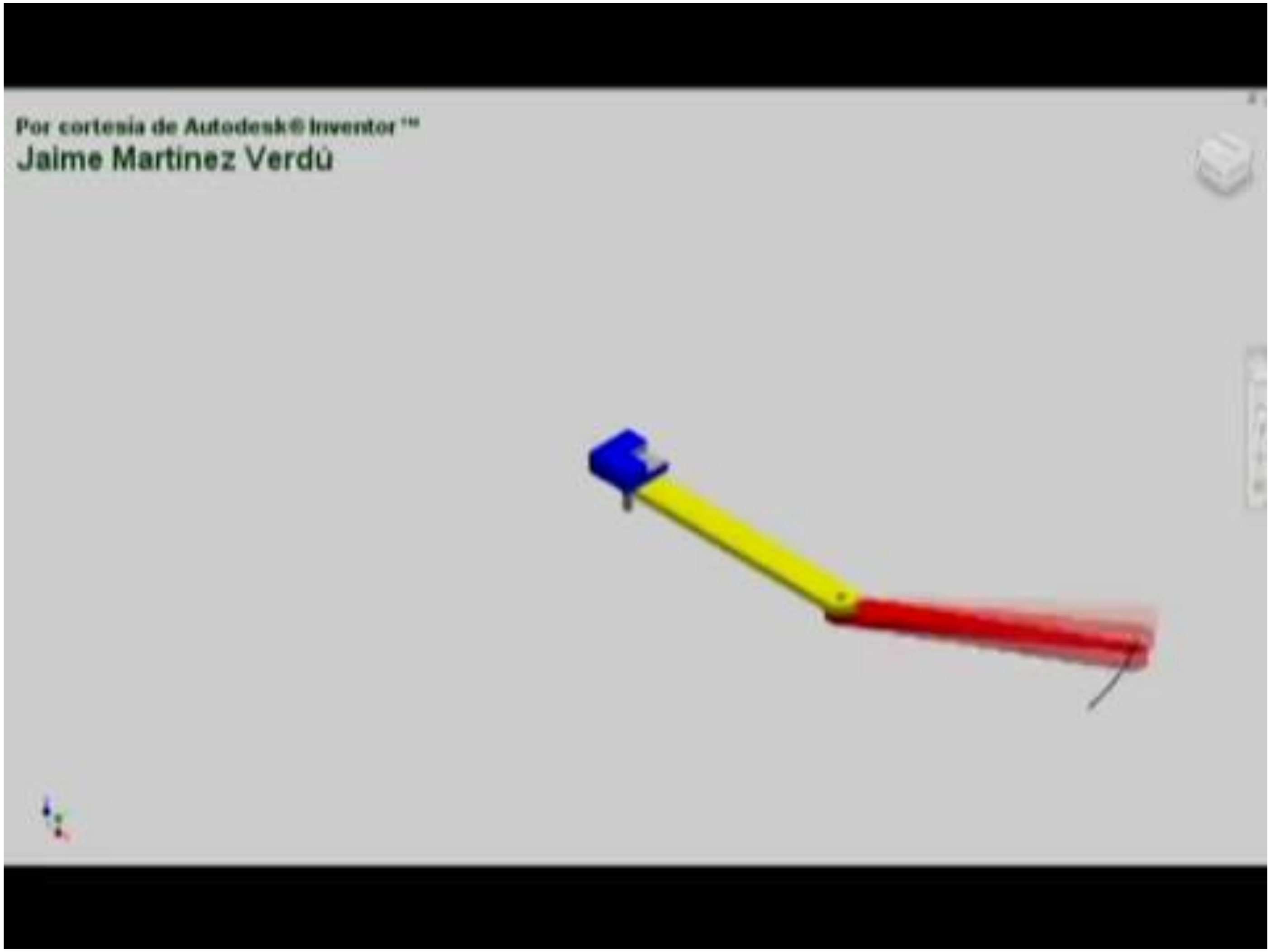
- **Workspace**: Range of reachable manipulator configurations
- **Dexterous workspace**: Range of reachable configurations while retaining freedom in the orientation of the end-effecter.





- Outer radius of $l_1 + l_2$
- If $l_1 \neq l_2$, inner radius of $|l_1 - l_2|$

Dexterous workspace

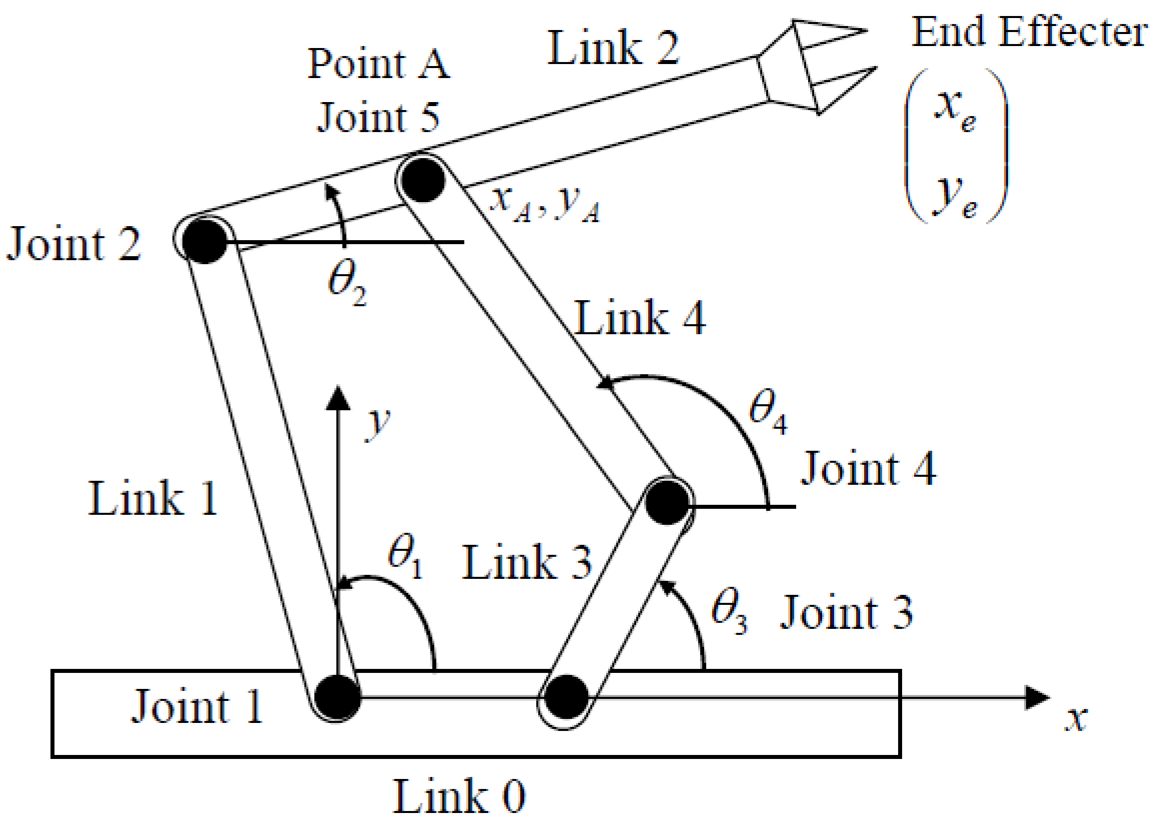


Overview

- Mechanisms
- Linkages
- Linkage Mobility
- Geometric Design of Linkages
- Linkage analysis
- Planar kinematics of Serial Link Mechanisms
- Planar Kinematics of Parallel Link Mechanisms

Planar Kinematics of Parallel Link Mechanisms

- Consider the five-bar-link planar robot arm shown in Figure below



Kinematics of Parallel Link Mechanisms

Forward kinematic equations

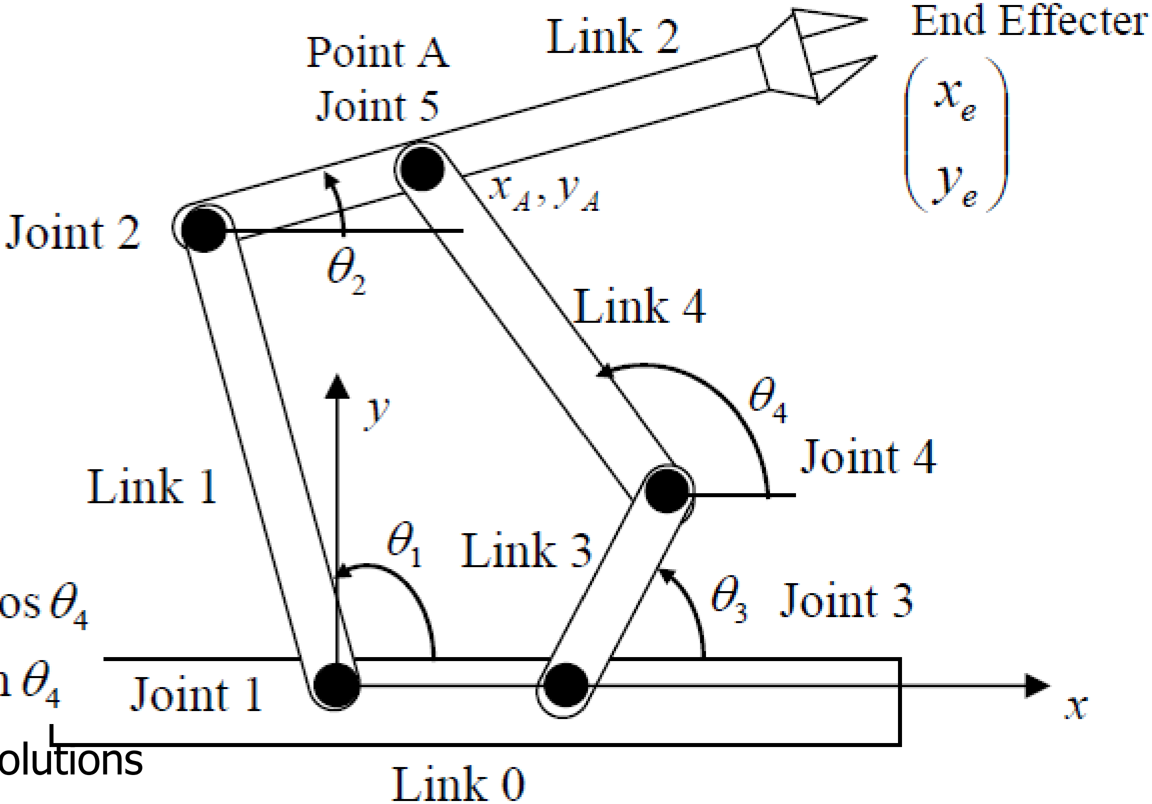
- Determine the location of the end-effecter if θ_1, θ_3 are known

$$\boxed{\begin{aligned} x_e &= \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2 \\ y_e &= \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \end{aligned}}$$

$$x_A = \ell_1 \cos\theta_1 + \ell_5 \cos\theta_2$$
$$y_A = \ell_1 \sin\theta_1 + \ell_5 \sin\theta_2$$

$$x_A = \ell_3 \cos\theta_3 + \ell_4 \cos\theta_4$$
$$y_A = \ell_3 \sin\theta_3 + \ell_4 \sin\theta_4$$

$$\ell_1 \cos\theta_1 + \ell_5 \cos\theta_2 = \ell_3 \cos\theta_3 + \ell_4 \cos\theta_4$$
$$\ell_1 \sin\theta_1 + \ell_5 \sin\theta_2 = \ell_3 \sin\theta_3 + \ell_4 \sin\theta_4$$

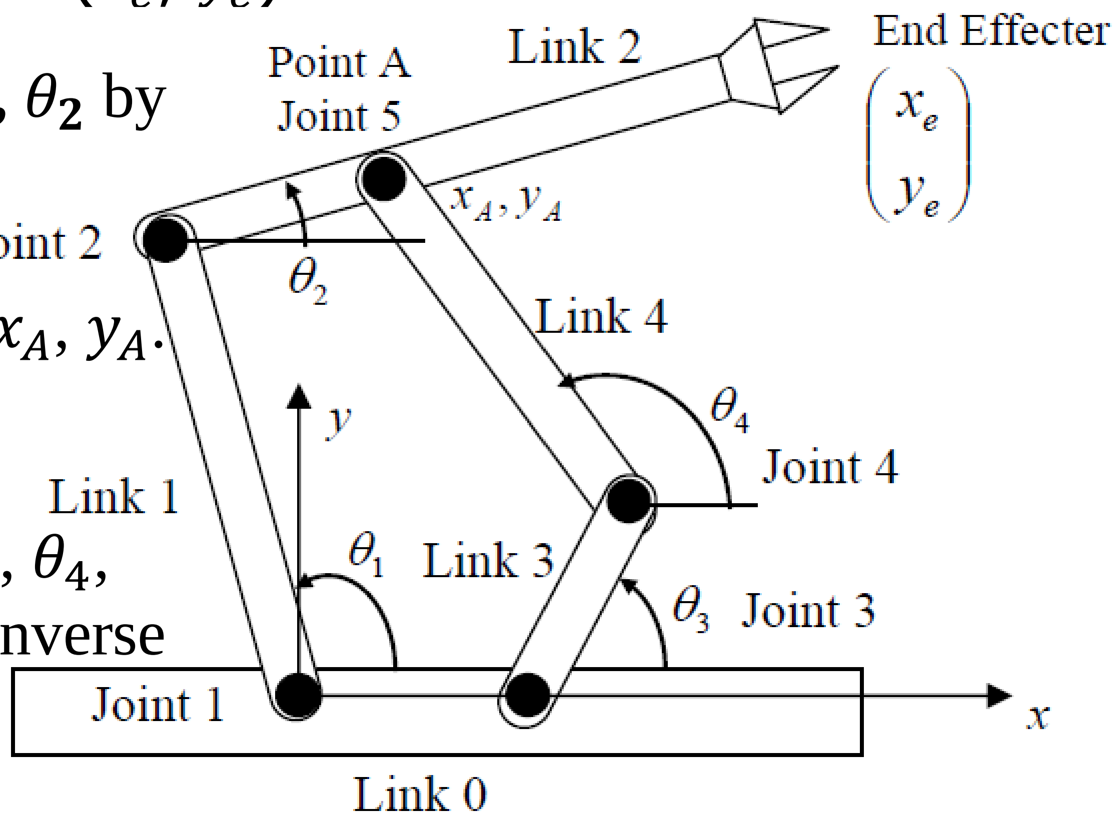


It should also be noted that multiple solutions exist for these constraint equations

Planar Kinematics of Parallel Link Mechanisms

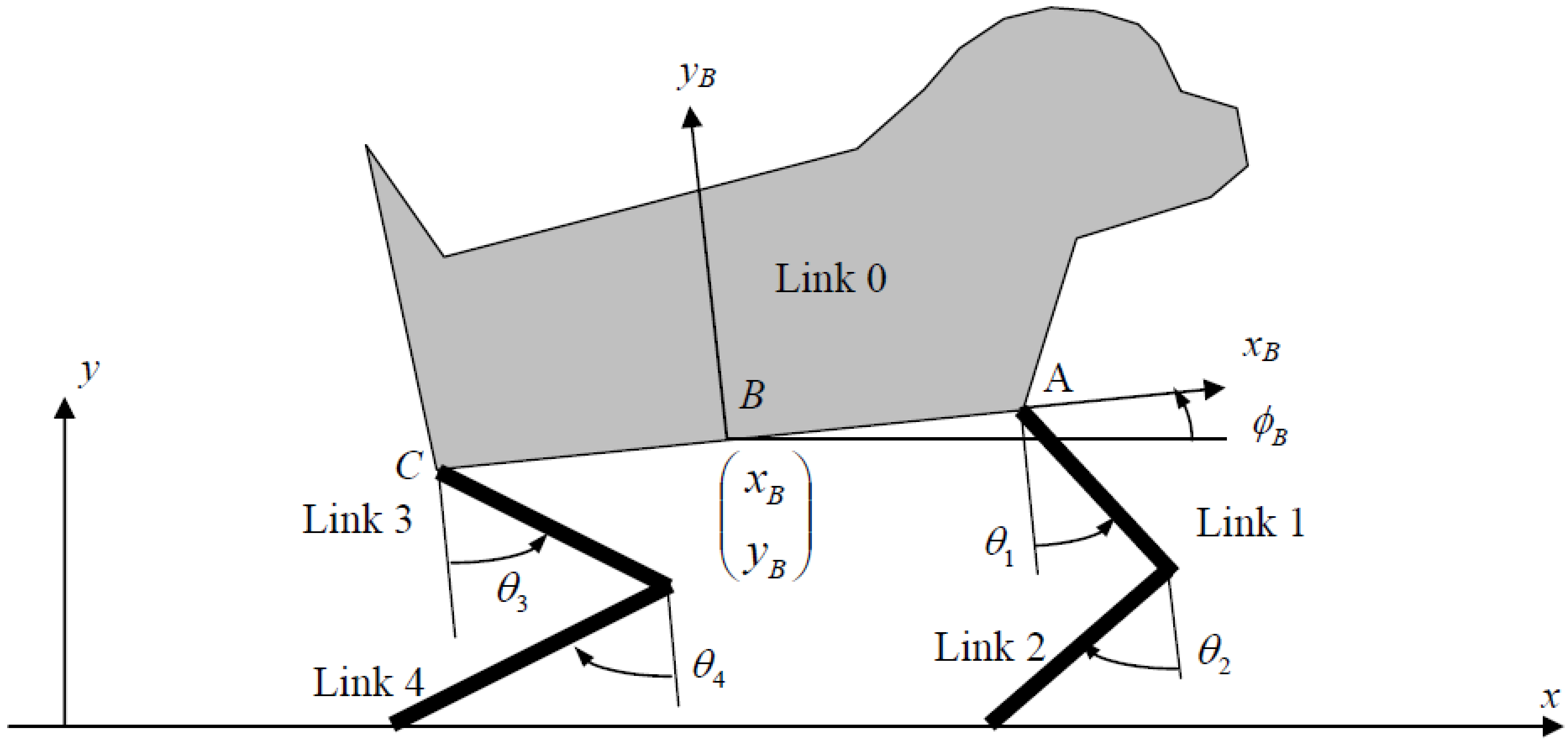
- **Inverse kinematic equations**
  - The problem is to find θ_1, θ_3 that lead the end point to a desired position (x_e, y_e)

Step 1: Given x_e, y_e, find θ_1, θ_2 by solving the two-link inverse kinematics problem.
Step 2: Given θ_1, θ_2, obtain x_A, y_A. This is a forward kinematics problem.
Step 3: Given x_A, y_A, find θ_3, θ_4, by solving another two-link inverse kinematics problem.



Exercise 5

- Obtain the joint angles of the dog's legs, given the body position and orientation (x_B, y_B, θ_B, x_{Link4}, x_{Link2})



Serial versus Parallel Link Mechanisms

Open Chain	vs.	Closed Chain
→ Larger workspace (+ dexterity)		→ More constrained in reachability
→ Simpler Kinematics, Dynamics and Control		→ Complex Geometry (Difficult to find closed form solutions)
→ Lower cost		
→ Low stiffness		→ High stiffness
		→ Higher FORCE or VELOCITY (low inertia)

QUESTIONS?