Nonlinear Finite Elements and Material Models for Continuum Mechanics

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Computational Fabrication (ISTD 01.110)

Monday, June 19, 2017

Nonlinear continuum mechanics

- So far, we have only considered small (infinitesimal) deformations and strains ($\varepsilon \ll 1$, typically $\varepsilon < 0.05$) and a linear elastic constitutive law
- Valid for many engineering applications, e.g. steel bridges and structures, dynamics of cars, …
- What happens, if we have large (finite) deformations and strains and other types of material behaviour (hyperelastic, plastic, …), e.g. deformation of soft, rubbery objects, car crash simulation, …?

$\rightarrow$ Nonlinear continuum mechanics and finite elements

Strain measures

- Linear strain:

$$\boldsymbol{\varepsilon}(\boldsymbol{X}) = \frac{1}{2}\left(\nabla\boldsymbol{u}(\boldsymbol{X}) + \nabla\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{X})\right)$$

- Deformation gradient:

$$\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{I} + \nabla\boldsymbol{u}(\boldsymbol{X}) = \nabla\boldsymbol{x}$$
$$J = \det\boldsymbol{F}$$

- Right Cauchy-Green tensor:

$$\boldsymbol{C}(\boldsymbol{X}) = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$$

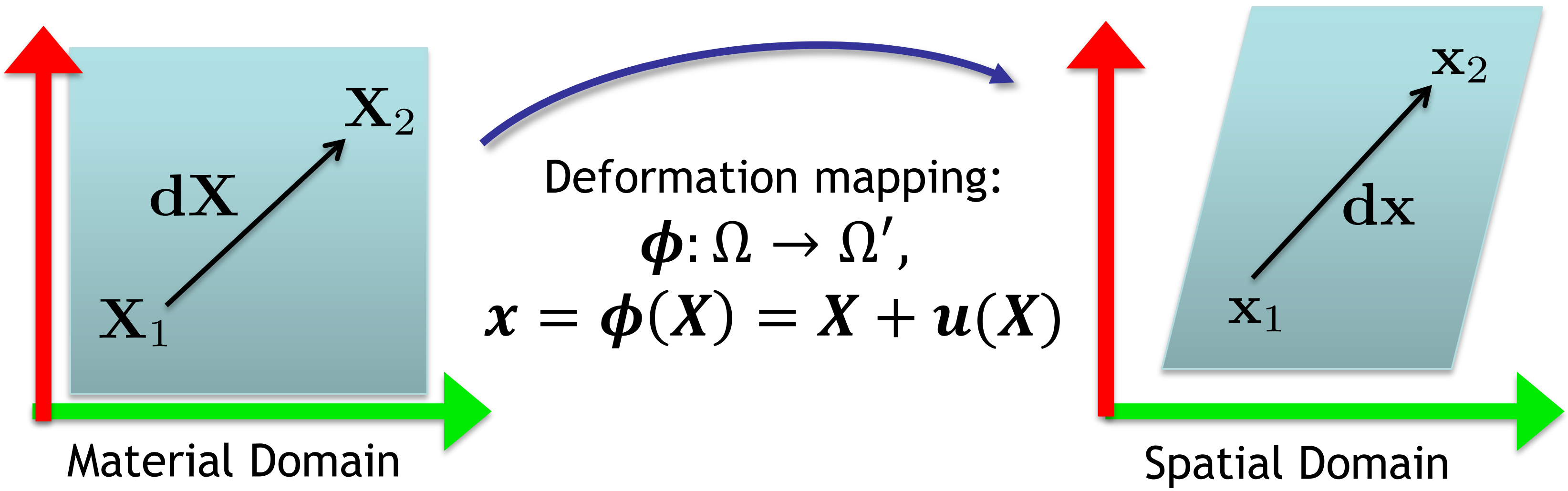
- Green-Lagrange strain tensor:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{X}) &= \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right)\\ &= \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}} + \nabla\boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{u}\right)\end{aligned}$$

nonlinear (quadratic) contribution

- <u>Geometric nonlinearity</u> in continuum mechanics

Derivation of deformation gradient



- Mapping of a vector: $\boldsymbol{x_2} = \boldsymbol{x_1} + \boldsymbol{dx} = \boldsymbol{\phi}(\boldsymbol{X_1} + \boldsymbol{dX})$
- Apply Taylor expansion: $\boldsymbol{x_1} + \boldsymbol{dx} \approx \boldsymbol{\phi}(\boldsymbol{X_1}) + \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}}(\boldsymbol{X_1})\, \boldsymbol{dX}$
- Use $\boldsymbol{x_1} = \boldsymbol{\phi}(\boldsymbol{X_1})$: $\boldsymbol{dx} \approx \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}}(\boldsymbol{X_1})\, \boldsymbol{dX} = \boldsymbol{F}(\boldsymbol{X})\, \boldsymbol{dX}$
- Deformation gradient: $\boldsymbol{F}(\boldsymbol{X}) = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}}(\boldsymbol{X}) = \boldsymbol{\nabla x}(\boldsymbol{X}) = \boldsymbol{I} + \boldsymbol{\nabla u}$

Strain measures

- Linear strain:

$$\boldsymbol{\varepsilon}(\boldsymbol{X}) = \frac{1}{2}\left(\nabla\boldsymbol{u}(\boldsymbol{X}) + \nabla\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{X})\right)$$

- Deformation gradient:

$$\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{I} + \nabla\boldsymbol{u}(\boldsymbol{X}) = \nabla\boldsymbol{x}$$

Measure of change of length

$$J = \det\boldsymbol{F}$$

Measure of volume change

- Right Cauchy-Green tensor:

$$\boldsymbol{C}(\boldsymbol{X}) = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$$

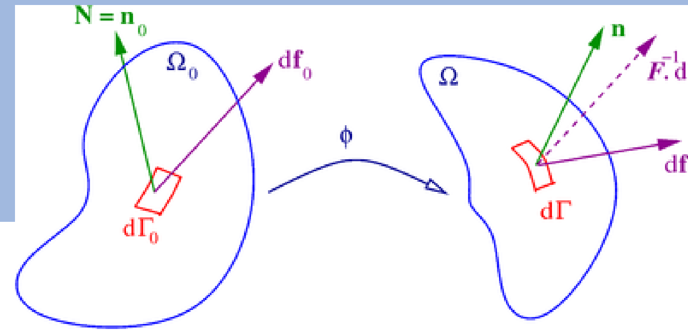
- Green-Lagrange strain tensor:

$$\boldsymbol{E}(\boldsymbol{X}) = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right)$$
$$= \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}} + \nabla\boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{u}\right)$$

nonlinear (quadratic) contribution

- <u>Geometric nonlinearity</u> in continuum mechanics

Stress measures



- Cauchy stress:

$$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$$

- Cauchy stress and linear constitutive law:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}, \qquad \boldsymbol{C}\text{:3x3x3x3-tensor}$$

- First Piola-Kirchoff stress:

$$\boldsymbol{t} = \boldsymbol{P} \cdot \boldsymbol{n}_0 \rightarrow \quad \boldsymbol{P} = J\, \boldsymbol{\sigma}\, \boldsymbol{F}^{-\mathrm{T}}$$

- Second Piola-Kirchoff stress:

$$\boldsymbol{F}^{-1} \cdot \boldsymbol{t}_0 = \boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{n}_0 \;\rightarrow\; \boldsymbol{S} = J\, \boldsymbol{F}^{-1} \boldsymbol{\sigma}\, \boldsymbol{F}^{-\mathrm{T}} = \boldsymbol{F}^{-1} \boldsymbol{P}$$

- Nonlinear, hyperelastic material models:
  $\boldsymbol{S} = \psi(\boldsymbol{F})$, or $\boldsymbol{S} = \psi(\boldsymbol{E})$, or $\boldsymbol{P} = \psi(\boldsymbol{F})$, or $\boldsymbol{P} = \psi(\boldsymbol{E})$

- <u>Material nonlinearity</u> (more details later)

Balance equations

- Lagrangian form of balance of linear momentum:

$$\mathrm{div}\, \boldsymbol{F}\, \boldsymbol{S} + \rho_0 \boldsymbol{b} = \boldsymbol{0}$$

- Weak form:

$$\underbrace{\int_{\Omega} \partial \boldsymbol{E}(\boldsymbol{w}) \cdot \boldsymbol{S}(\boldsymbol{u}) \boldsymbol{dX}}_{\Rightarrow \boldsymbol{f}(\boldsymbol{u})} - \underbrace{\int_{\Omega} \rho_0 \boldsymbol{w}^T \boldsymbol{b}\, \boldsymbol{dX} - \int_{\Gamma_n} \boldsymbol{w}^T \boldsymbol{t}\, \boldsymbol{dX}}_{\Rightarrow \boldsymbol{b}} = \boldsymbol{0} \;\; \forall \boldsymbol{w}$$

- Discretized form using $\boldsymbol{u}^h = \boldsymbol{N}(\boldsymbol{\xi})\, \boldsymbol{u}^e$:

$$\boldsymbol{r}(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{b} \triangleq \boldsymbol{0}$$

Nonlinear FEM

- Similar steps as before:
  1. Meshing: discretization of domain into small elements
  2. Discretization of displacement in each element
  3. Element-wise valuation of weak form of equilibrium equations leads to nonlinear force vectors and tangent stiffness matrices (evaluation of strains, stresses, etc.)
  4. Assembly of global nonlinear force and residual vectors and tangent stiffness matrices:
  $$\mathbf{r}(\mathbf{u}) = \mathbf{0}, \qquad \frac{\mathbf{dr}}{\mathbf{du}} = \frac{\mathbf{df}}{\mathbf{du}} = \mathbf{K}(\mathbf{u})$$
  5. Iterative solution of nonlinear system for nodal displacement vector (e.g. using Newton's method)
  6. Post-processing: evaluation of displacements, stresses etc.

FEM pseudo-code

- Initialize $\mathbf{r}, \mathbf{u}, \Delta\mathbf{u} = \mathbf{0}$
- While $\|\mathbf{r}\| > \epsilon \wedge \|\Delta\mathbf{u}\| > \epsilon$
  - Set $\mathbf{K}, \mathbf{r} = \mathbf{0}$
  - For every element $e = 1, \dots, \ell$
    - Evaluate $\boldsymbol{u}^h(\boldsymbol{\xi})|_{\Omega^{\mathrm{e}}}$ using $\mathbf{u}$
    - Evaluate $\mathbf{F}, \mathbf{E}, \mathbf{S}, \dots$
    - Local assembly of $\mathbf{K}^{\mathrm{e}}, \mathbf{f}^{\mathrm{e}}, \mathbf{b}^{\mathrm{e}}$using weak form
    - Global assembly $\mathbf{K} \longleftarrow \mathbf{K}^{\mathrm{e}}, \mathbf{r} \longleftarrow \mathbf{f}^{\mathrm{e}} - \mathbf{b}^{\mathrm{e}}$ (apply essential BC)
  - Solve linear system $\mathbf{K}\,\Delta\mathbf{u} = -\mathbf{r}$
  - Update $\mathbf{u} \longleftarrow \mathbf{u} + \Delta\mathbf{u}$
- Use $\mathbf{u}$ to evaluate $\boldsymbol{u}^h, \boldsymbol{F}, \boldsymbol{E}, \boldsymbol{S}, \boldsymbol{\sigma}, \dots$

Nonlinear Continuum Mechanics

Three important properties to compute:

1. Deformation: $\mathbf{F}$
2. Strain: $\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right)$
3. Stress: $\mathbf{S} = \boxed{\boldsymbol{\psi}}(\mathbf{E})$

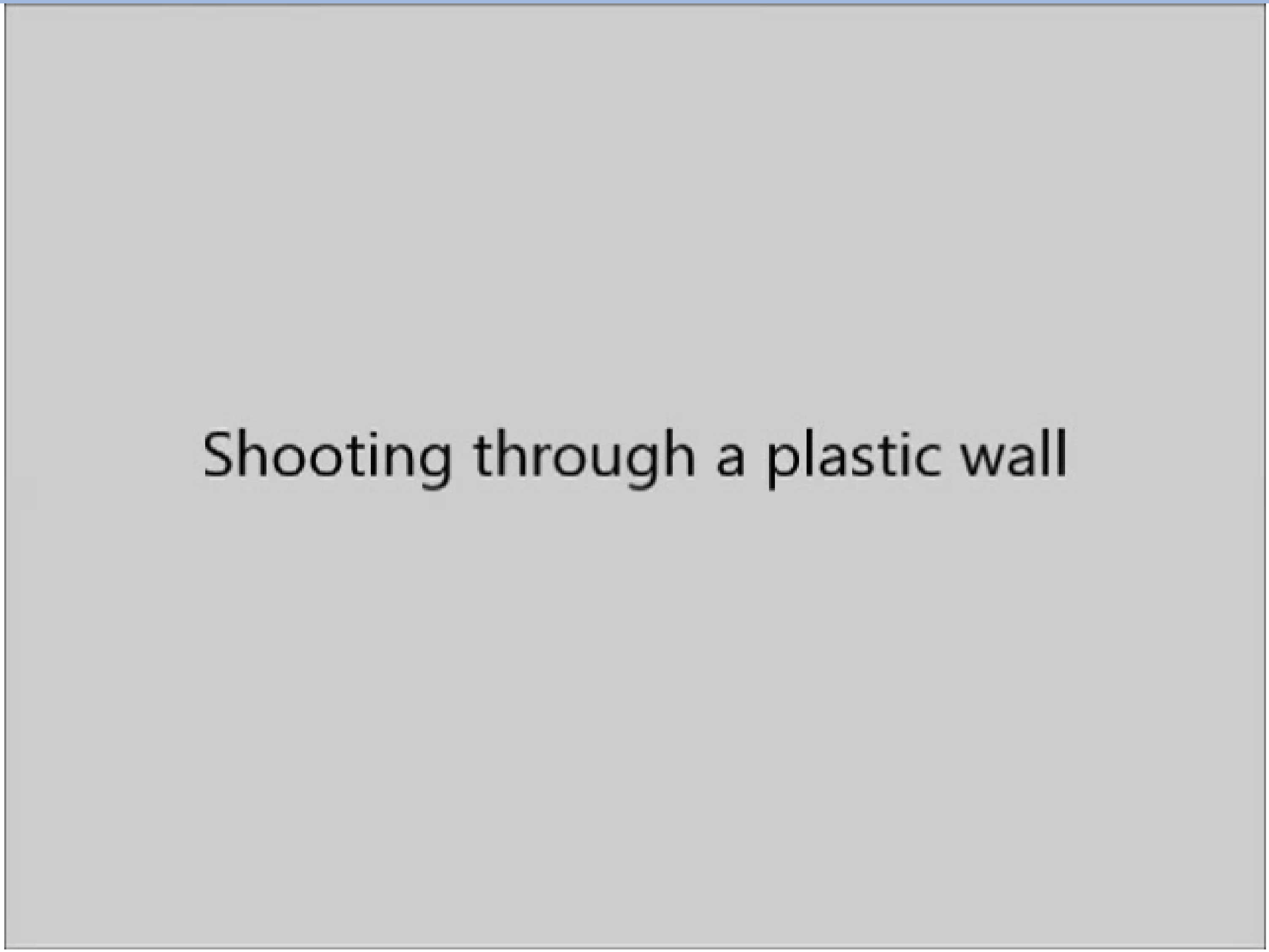
Material
Model

The Importance of Material Models

- The material model really determines the behavior of the finite element solver
- It lets us simulate a wide range of materials

The Importance of Material Models

The Importance of Material Models



Types of Materials

- In this lecture we’ll discuss the broad properties of a variety of materials
- The mathematical models people use to describe them
- How we measure these properties from real world materials

Types of Materials

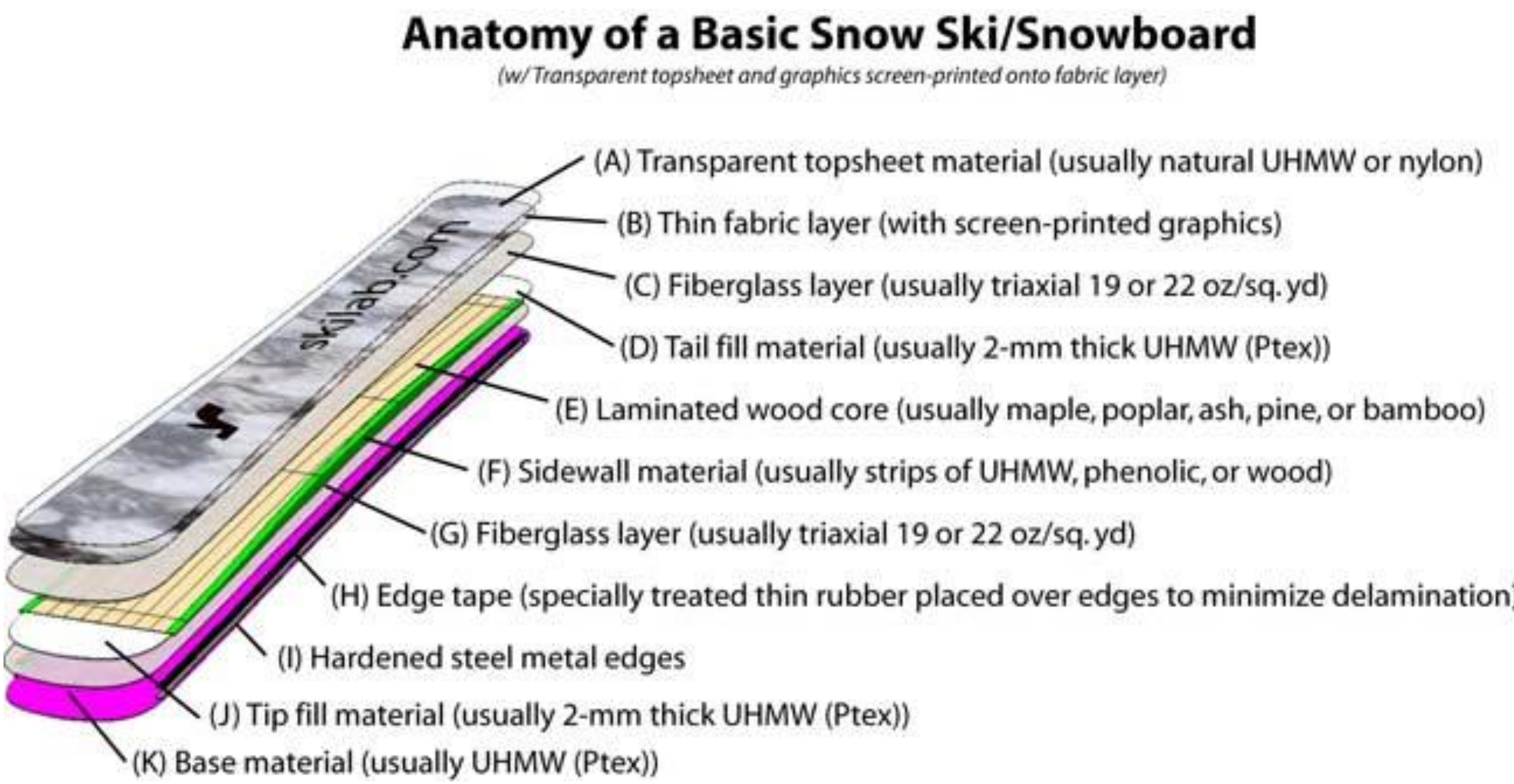
“Soft”

“Hard”

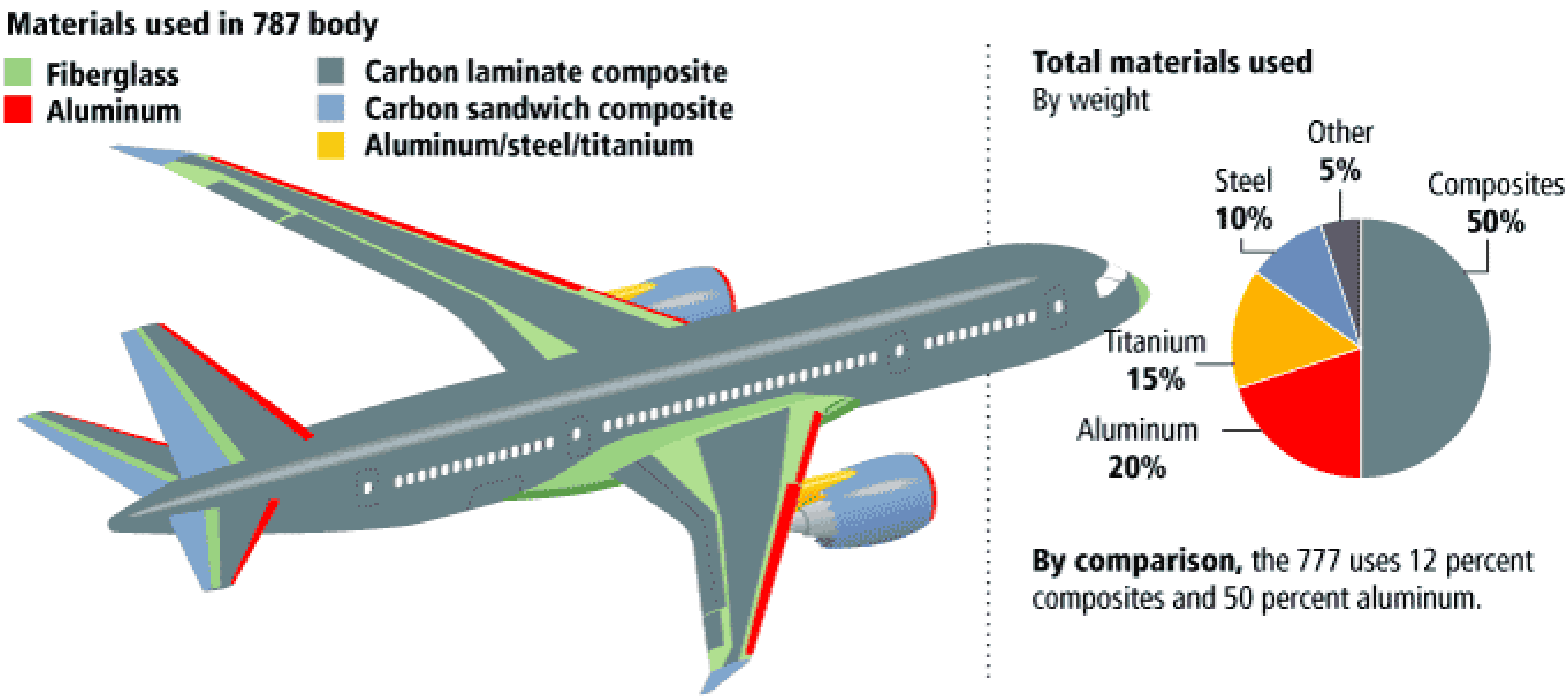
Types of Materials

- There are many types of materials
  - Elastic
  - Plastic
  - Viscous
  - Composites
  - Cellular Materials
  - Lattice Structures
- Each one has different mechanical properties
- When we fabricate things we exploit these properties to achieve optimal results

Example: A Ski



Example: Boeing 787 Dreamliner



Let’s build an Airplane

It’s important to get your material models right!

Boeing 787 Dreamliner planes in production found with wing cracks

CBC News Posted: Mar 07, 2014 7:55 PM ET | Last Updated: Mar 07, 2014 7:55 PM ET

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- New Boeing 787 Dreamliner lands at Vancouver airport
- Timeline: The Boeing 787 Dreamliner

Boeing said that roughly 40 airplanes might be affected and that it will take one to two weeks to inspect each plane and fix any cracks found on shear ties on a wing rib (Elaine Thompson/Associated Press)

Boeing's much-delayed 787 Dreamliner has hit another production snafu.

Hairline cracks have been discovered in the wings of some 787s that are being built. The Chicago-based manufacturer said none of the 122 jets already flown by airlines around the world are affected.

"We are confident that the condition does not exist in the in-service fleet," Boeing spokesman Doug Alder said in an email. "We understand the issue, what must be done to correct it and are completing inspections of potentially affected airplanes."

New Boeing 787 Dreamliner lands at Vancouver airport

Timeline: The Boeing 787 Dreamliner

The production problem was first reported by The Wall Street Journal.

Boeing said that roughly 40 airplanes might be affected and that it will take one to two weeks to inspect each plane and fix any cracks found on shear ties on a wing rib. A shear tie is an attachment fitting. It is part of the rib — and connects the rib to the wing skin. The company would not give

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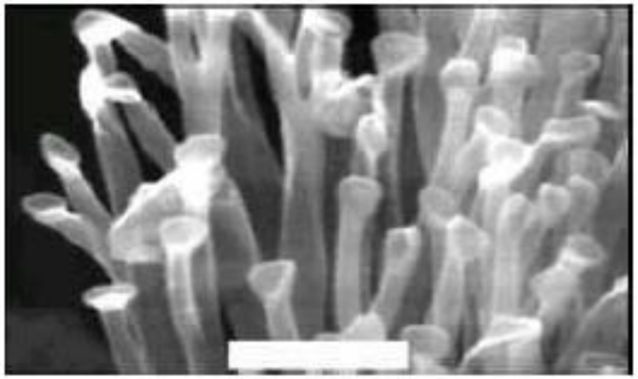
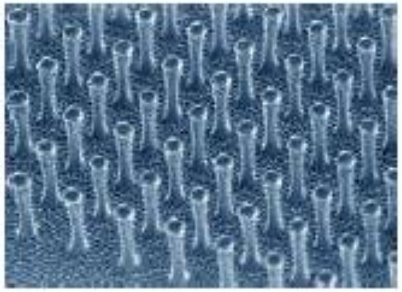
Bioinspired Materials

The example of Gecko

setae

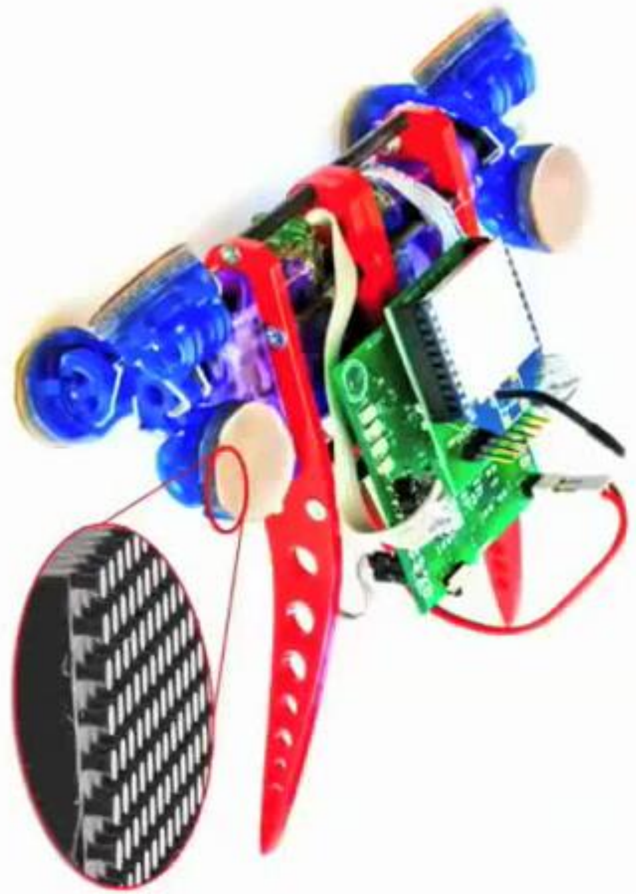


spatulae



Scanning electron microscope image of a 1cm^2 section of the Gecko-sticky tape made of polyimide fibers

Bioinspired Materials



Types of Materials

- There are many types of materials
  - Elastic
  - Plastic
  - Viscous

    TODAY

  - Composites
  - Cellular Materials
  - Lattice Structures

    NEXT LECTURE

- Each one has different mechanical properties
- When we fabricate things we exploit these properties to achieve optimal results

Types of Materials

- There are many types of materials
  - Elastic
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  - Composites
  - Cellular Materials
  - Lattice Structures
- Each one has different mechanical properties
- When we fabricate things we exploit these properties to achieve optimal results

Elastic Materials

- Defining Properties:
  - Stress is only dependent on deformation (strain)
  - Object always returns to its original shape

Material Models for Elastic Materials

- We've already seen one
- It's similar to a linear spring
- Linear Elasticity (Hooke's Law)

$$\sigma = \boxed{[\mathbf{C}]}\epsilon$$

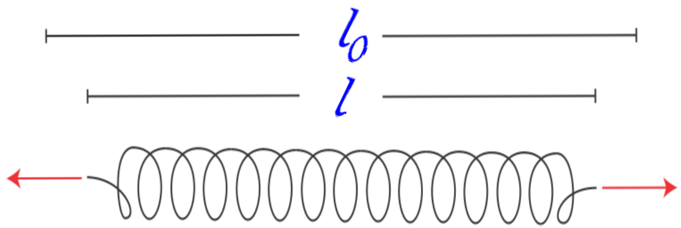
Material
Tensor

Constitutive Model

- Continuum Model:

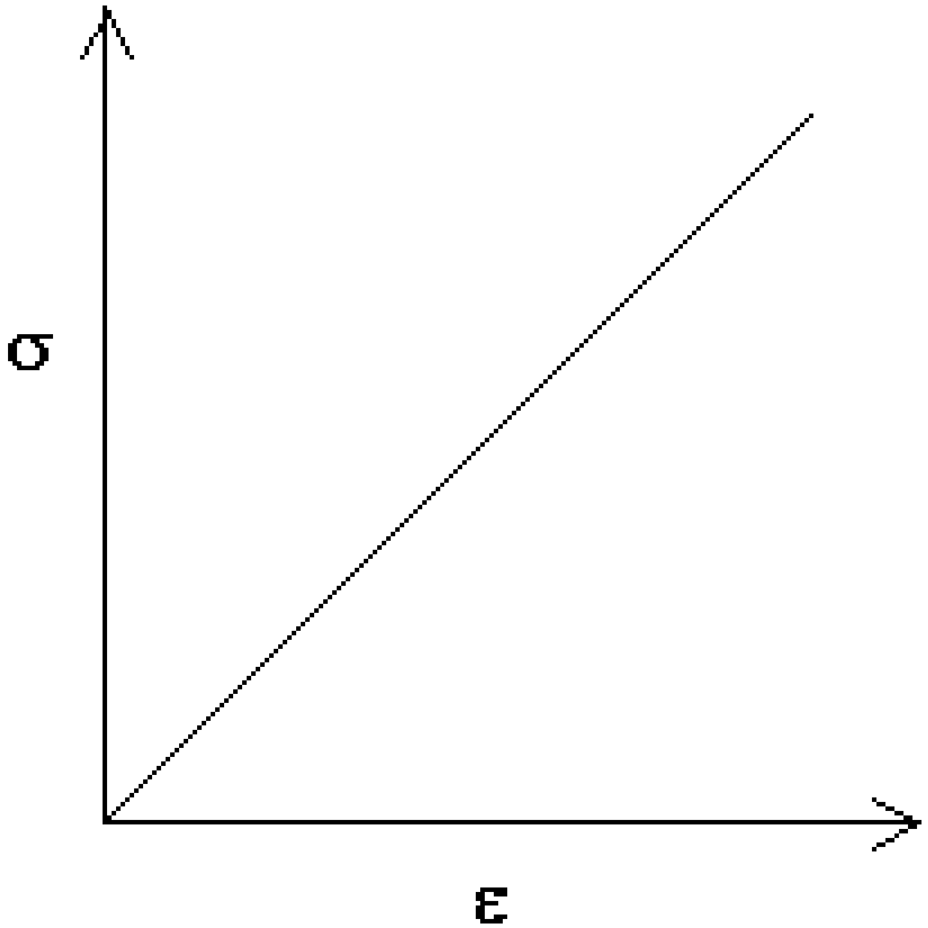
$$\sigma = \boxed{[\mathbf{C}]}\epsilon$$

- Mass Spring:

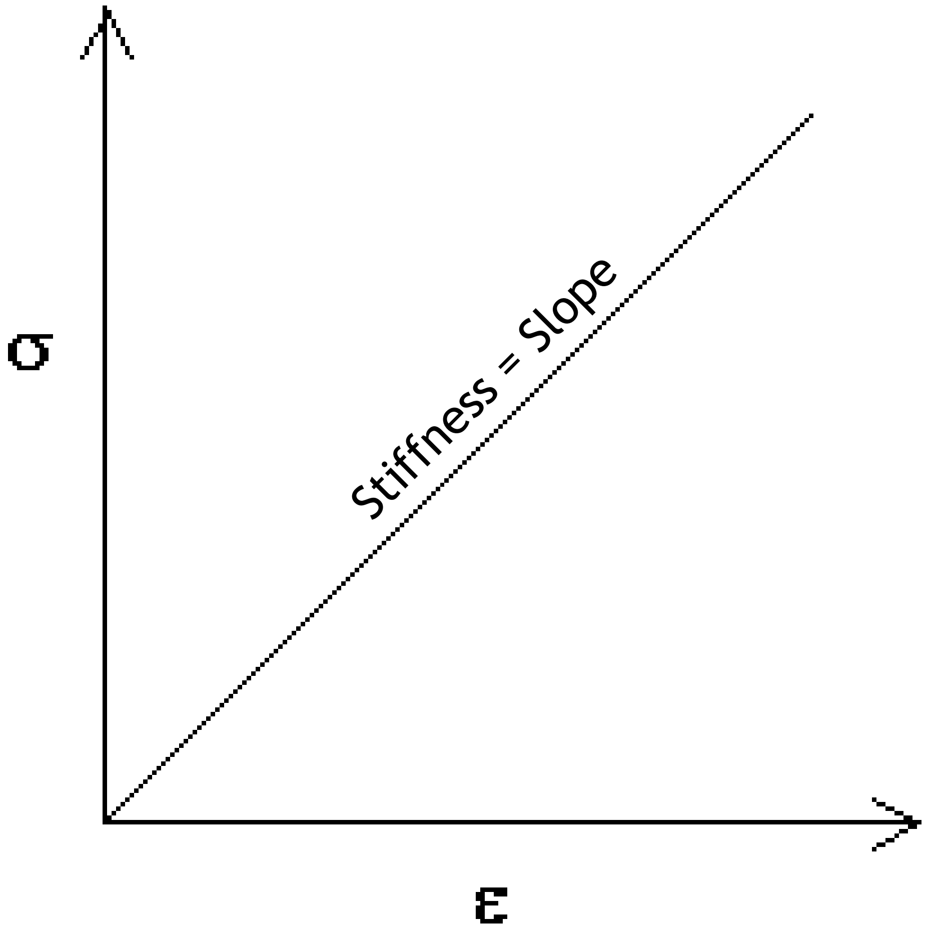


$$T = \boxed{k}\left(\frac{l}{l_0} - 1\right)$$

Linear Elasticity



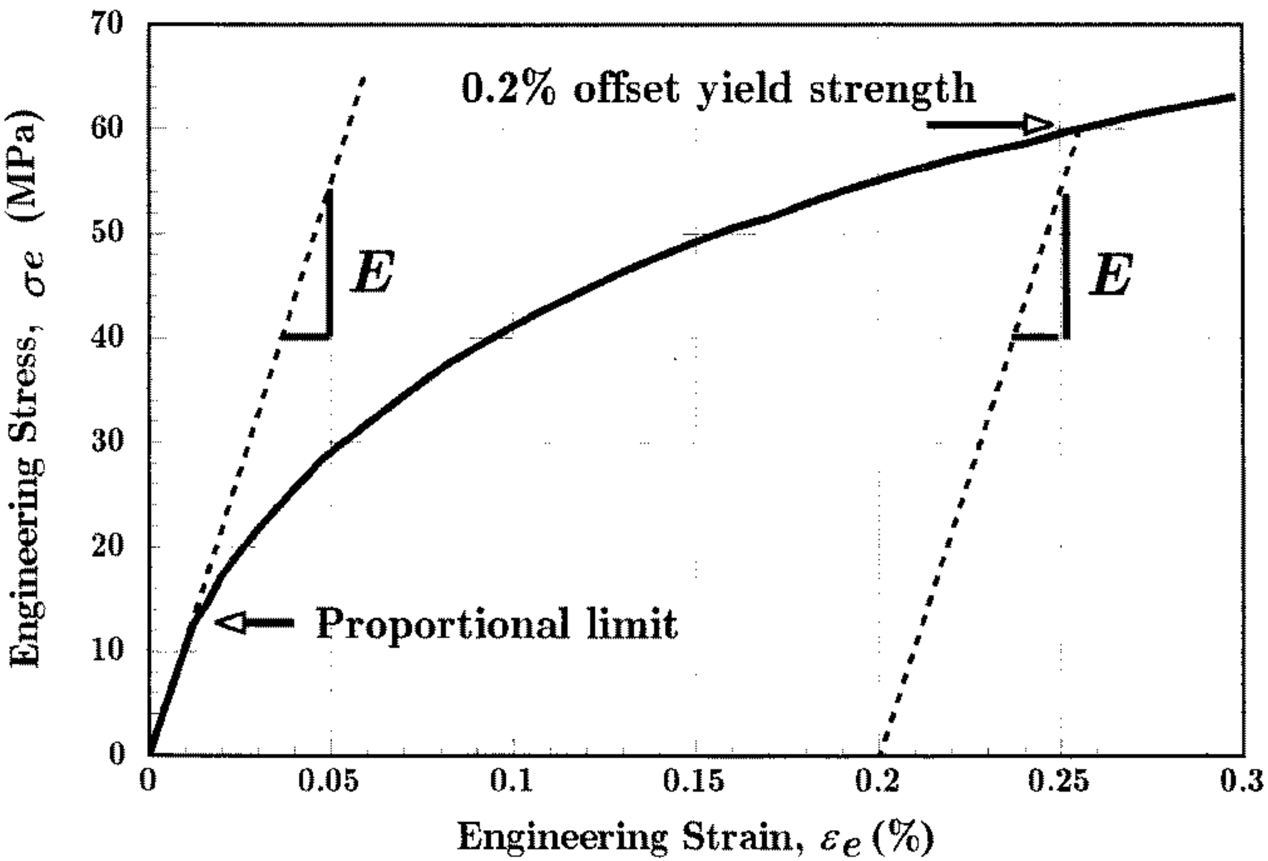
Linear Elasticity



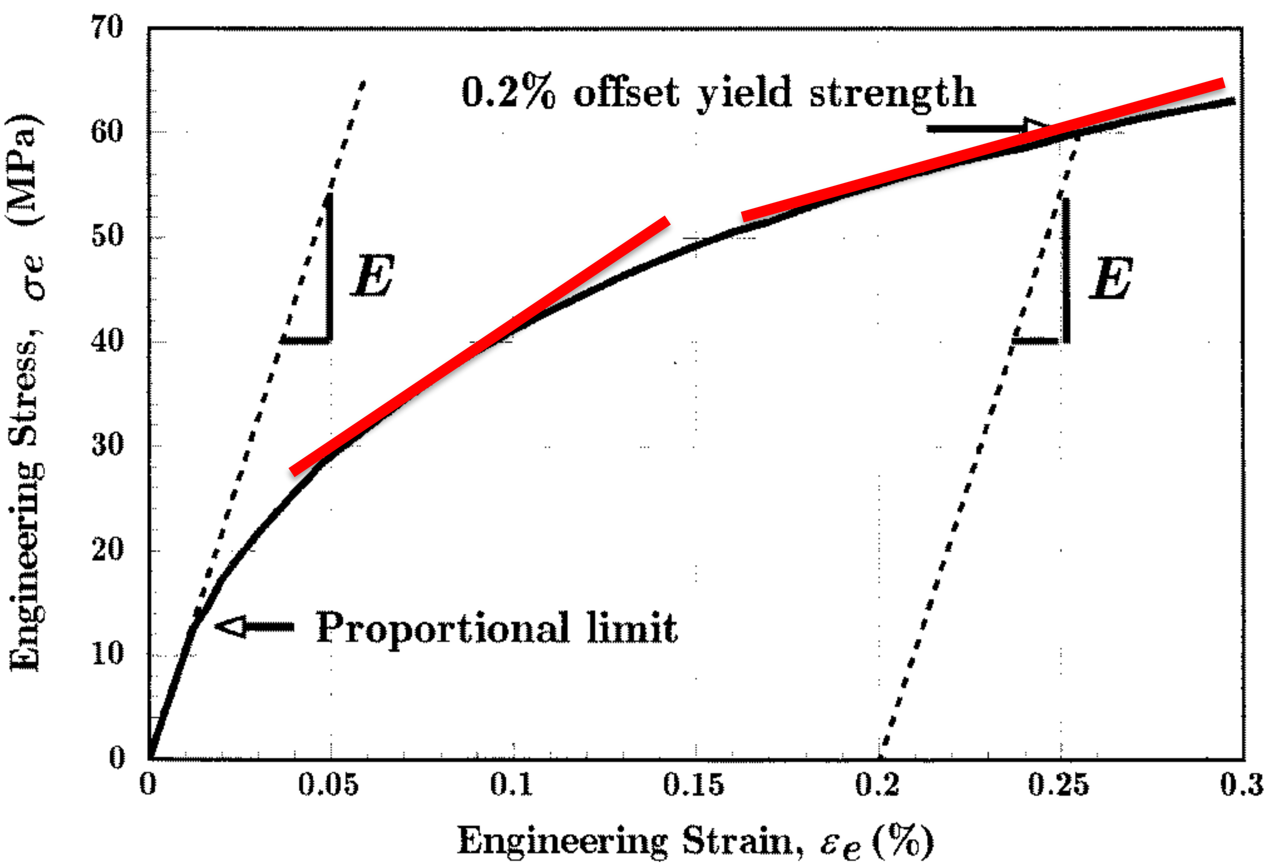
Nonlinear Elasticity

- Linear Elasticity is only a good approximation over a small range of deformations
- For instance, if we measure real-world materials we find that the stress-strain curves are highly non-linear
- <u>Material nonlinearity</u> in continuum mechanics

Nonlinear Elasticity: Stiffness



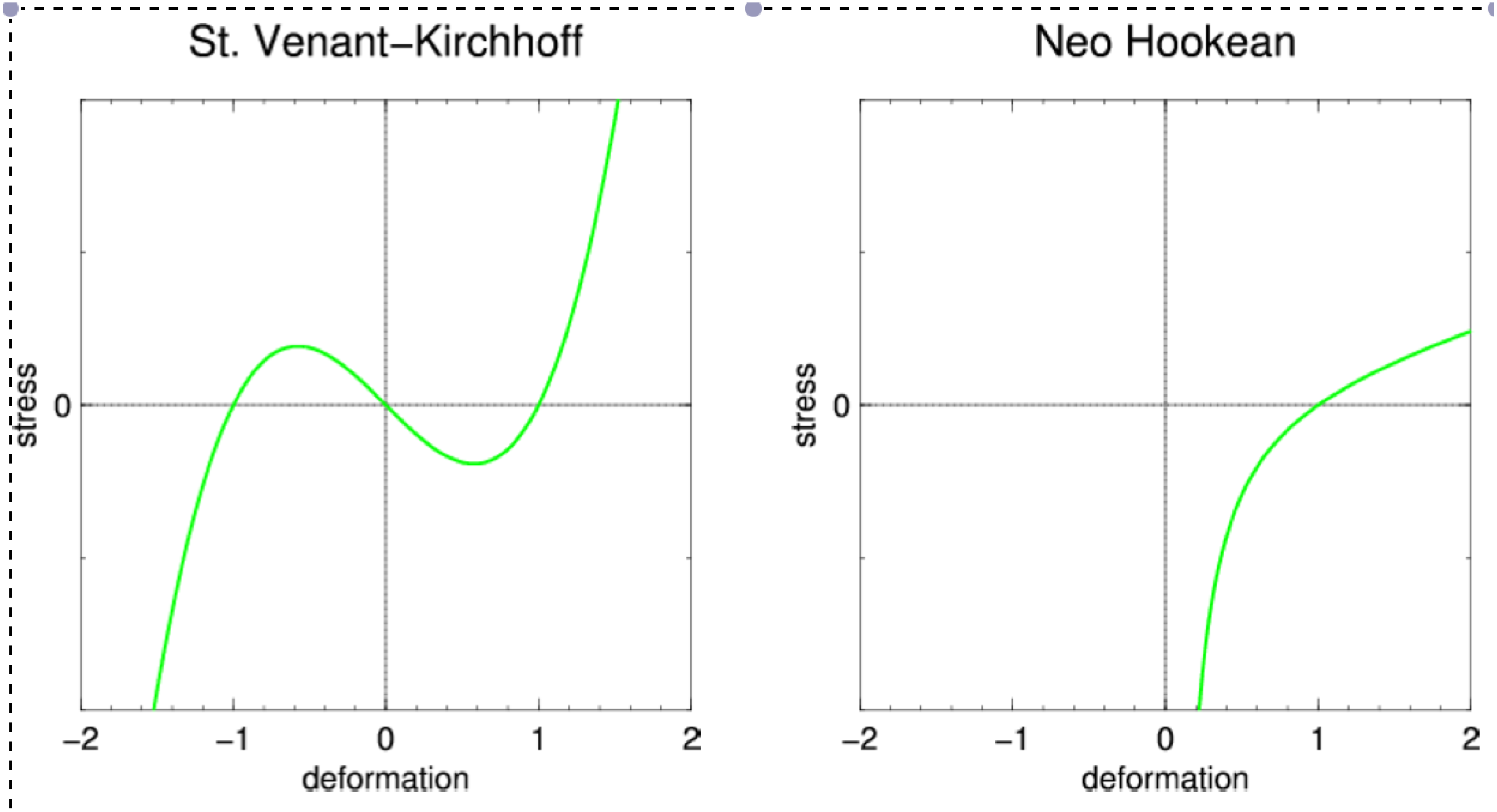
Nonlinear Elasticity



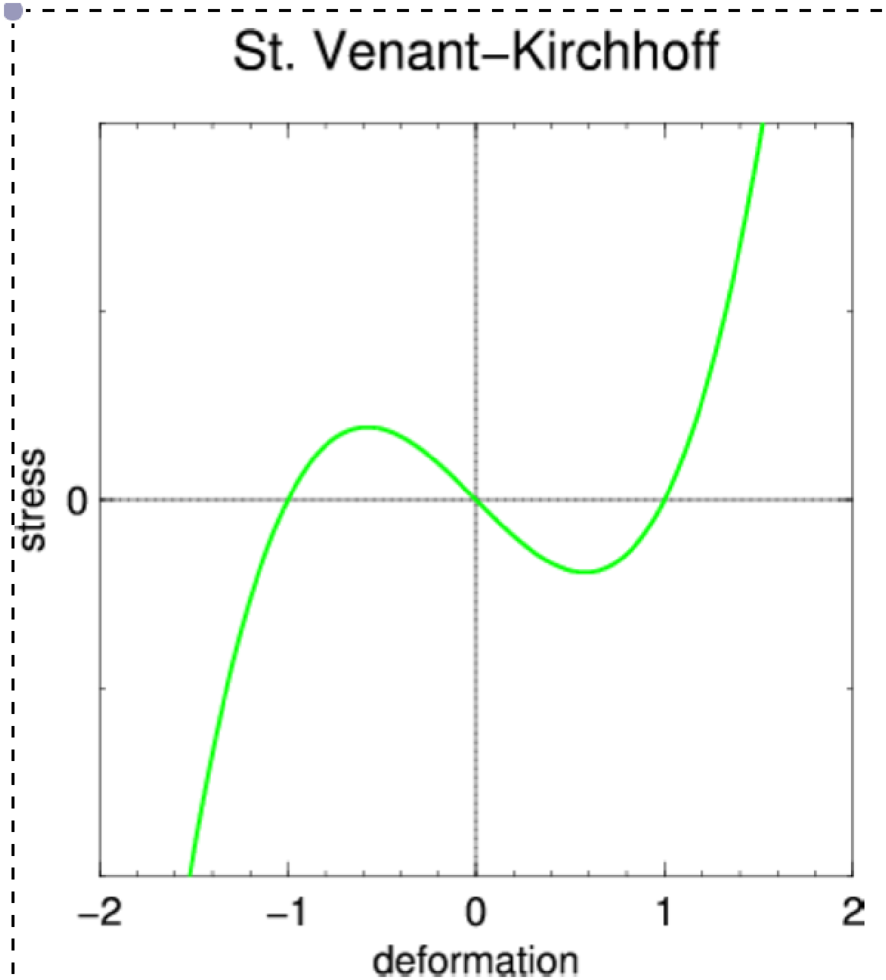
Hyperelastic Materials

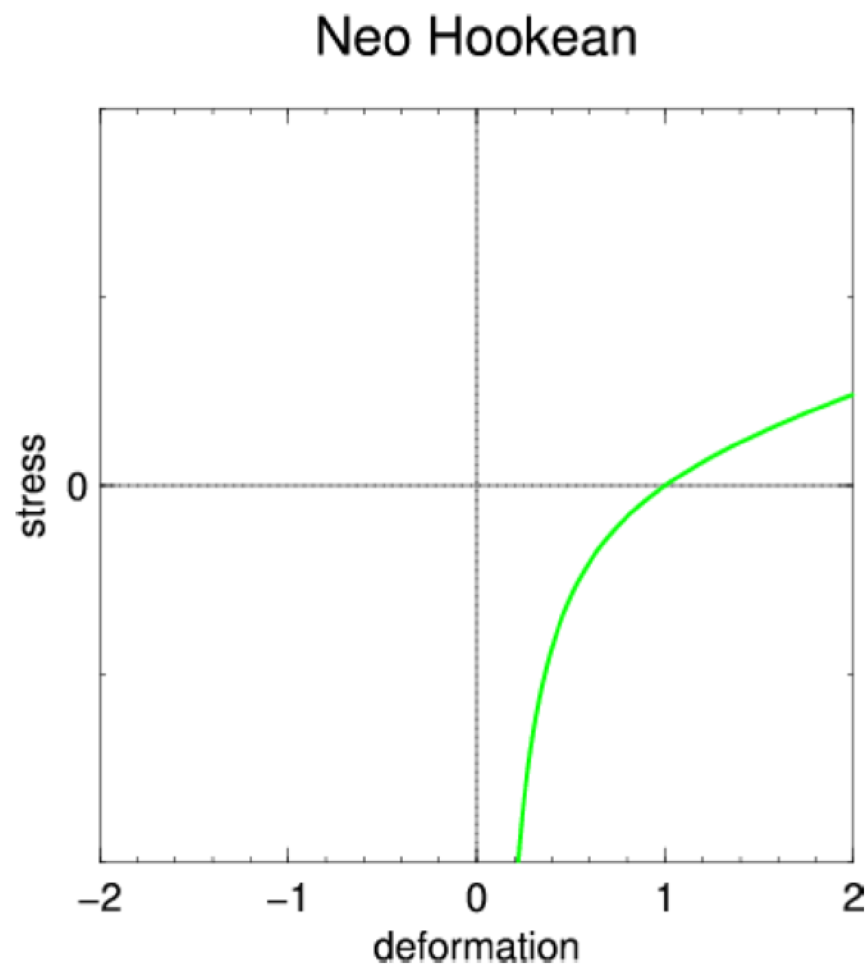
- Specific Type of Nonlinear Material
- Elastic
- Derived from a Strain Energy which is a single scalar valued function

Common Hyperelastic Models



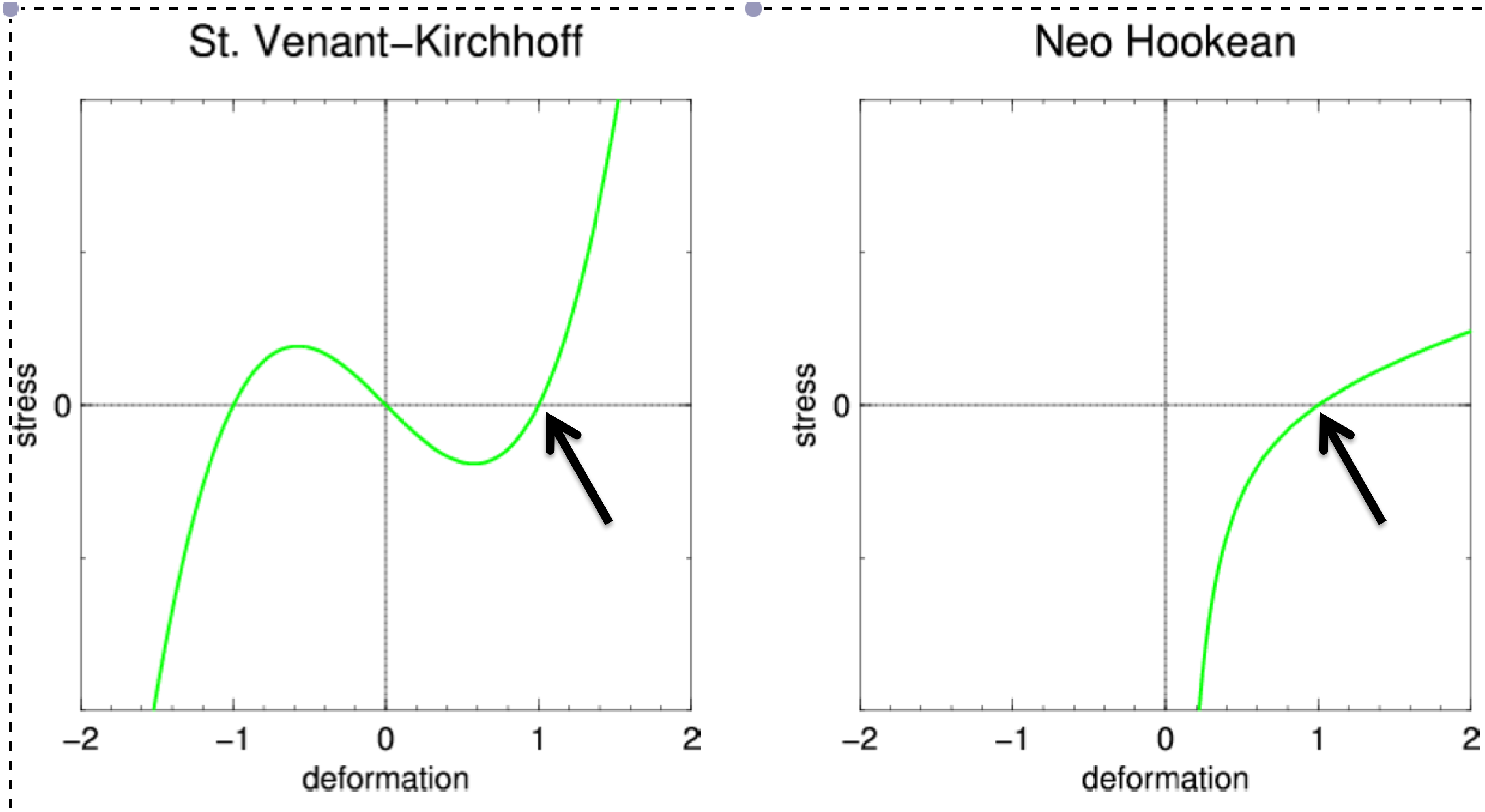
Common Hyperelastic Models





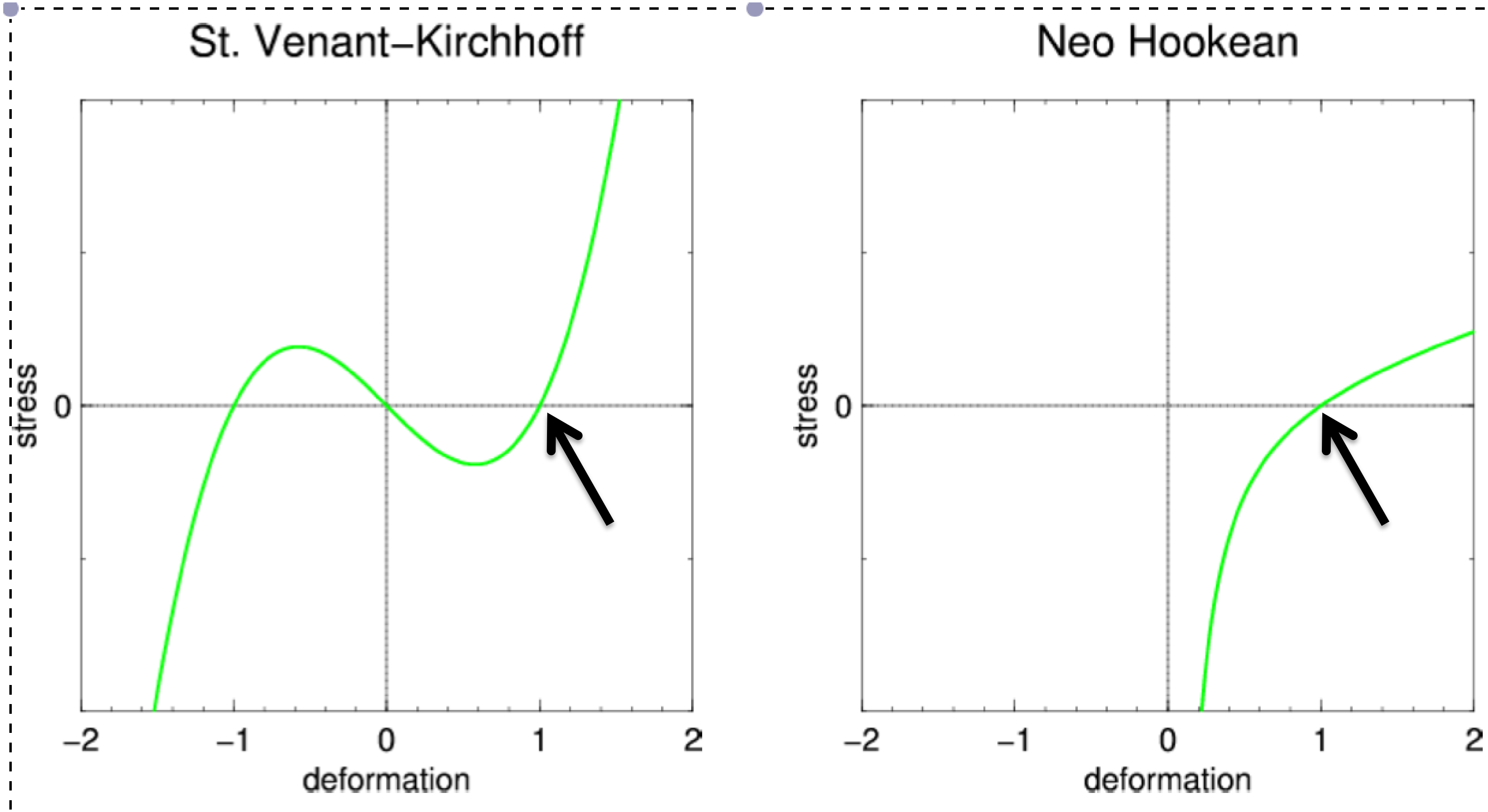
No Force when Object is Undefined

Common Hyperelastic Models



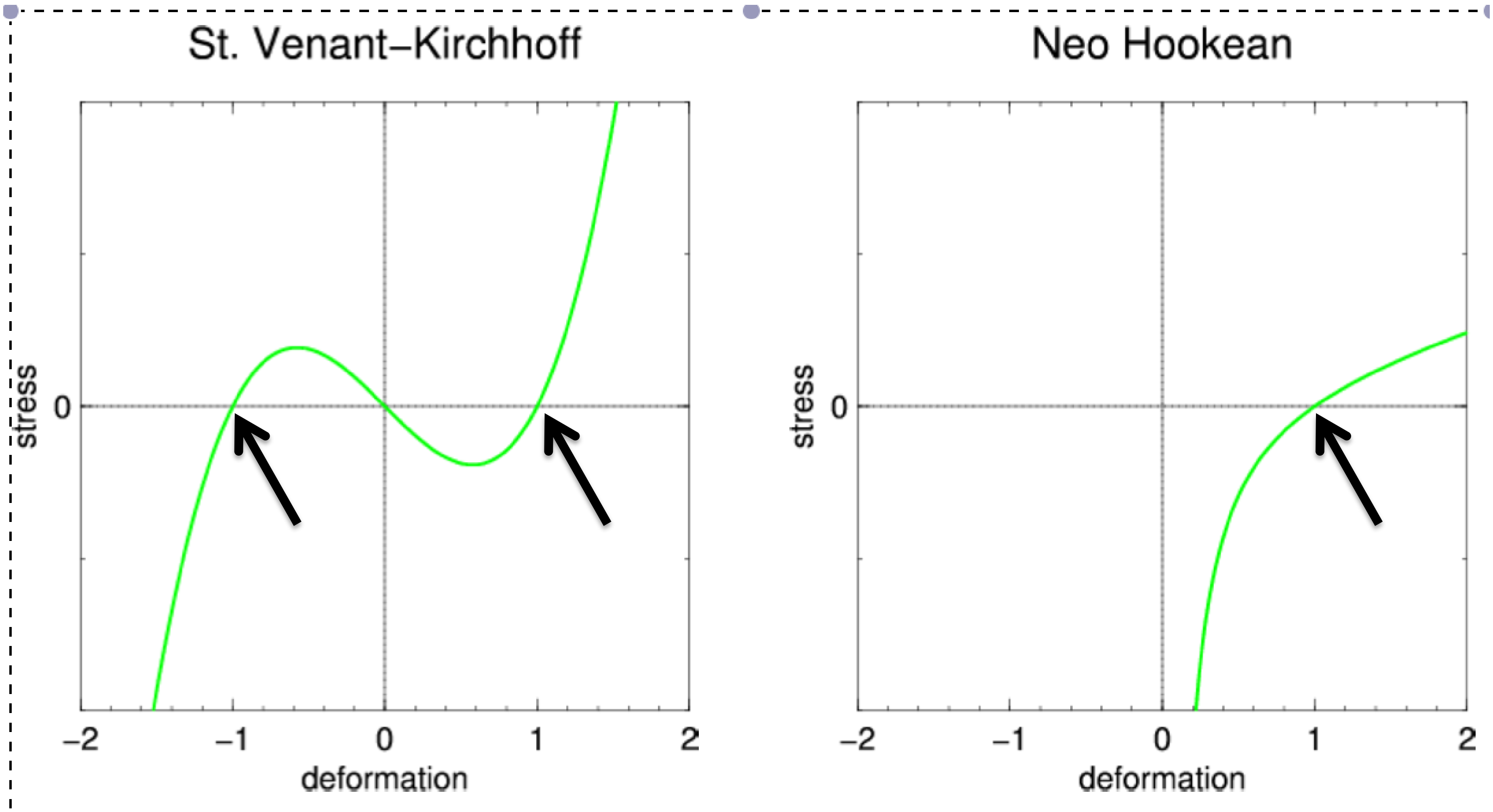
No Force when Object is Undefined

Common Hyperelastic Models



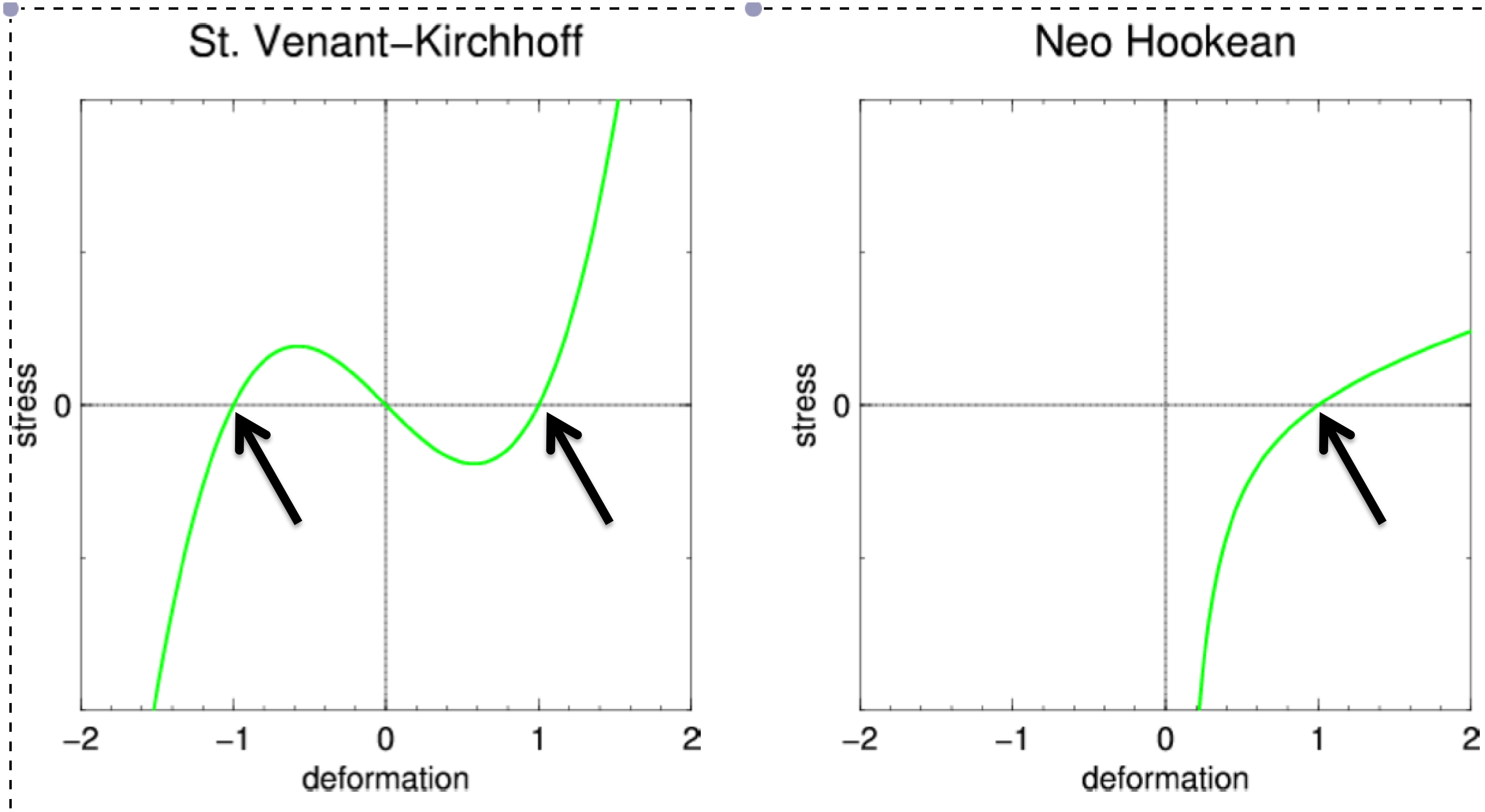
Only one zero point

Common Hyperelastic Models



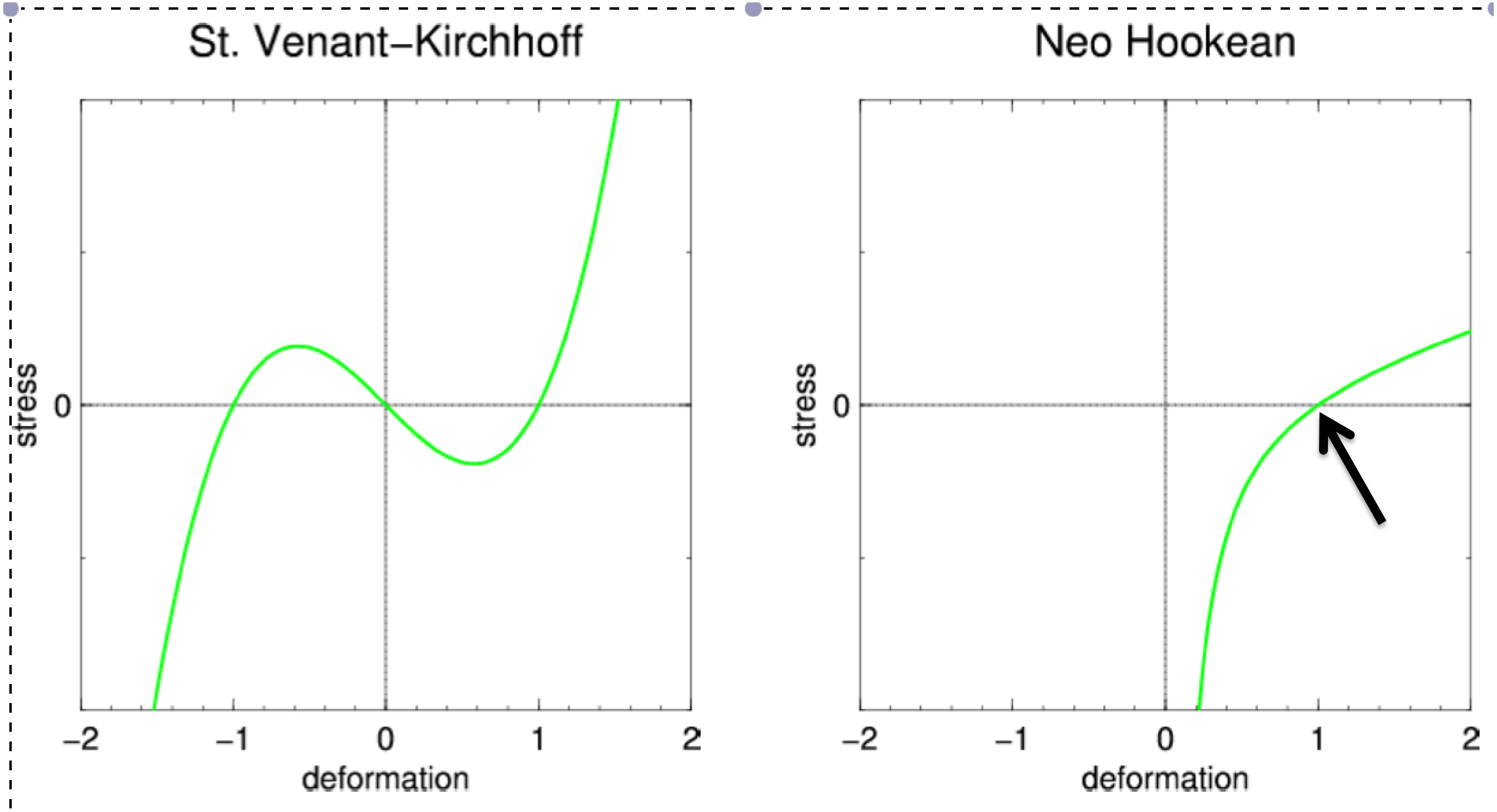
Only one zero point

Hyperelastic Models: Other Problems



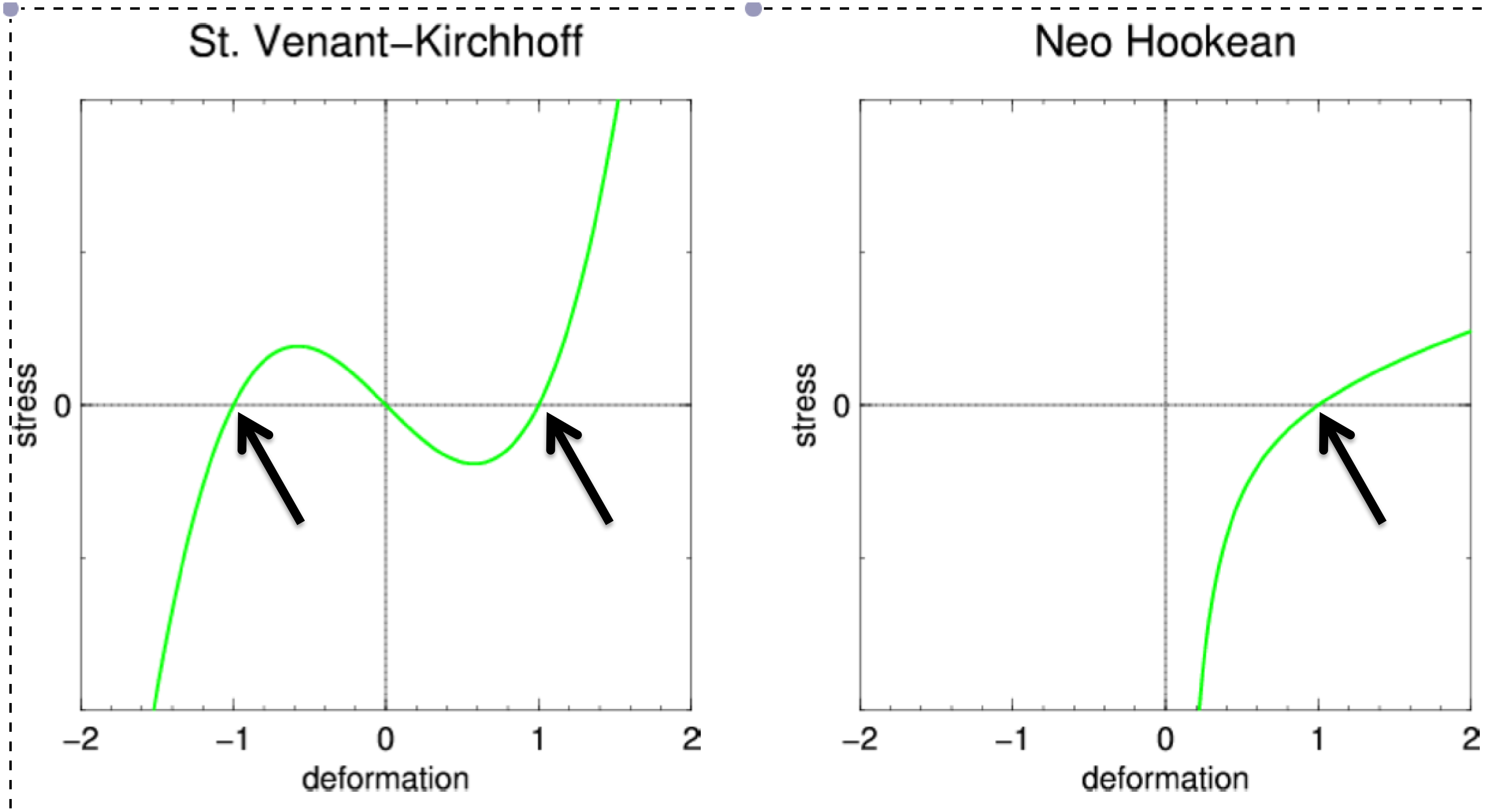
What happens to the St. Venant Kirchhoff model near 0 deformation?

Hyperelastic Models: Other Problems



Stiffness is negative = unstable !!!!!!!

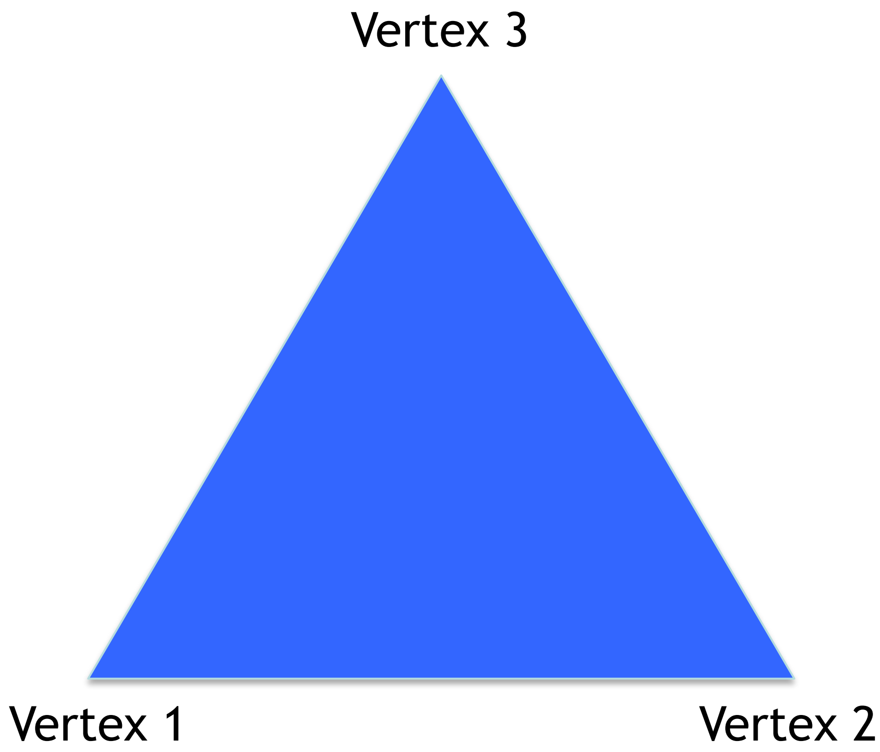
Hyperelastic Models: Other Problems



Leads to Element Inversion

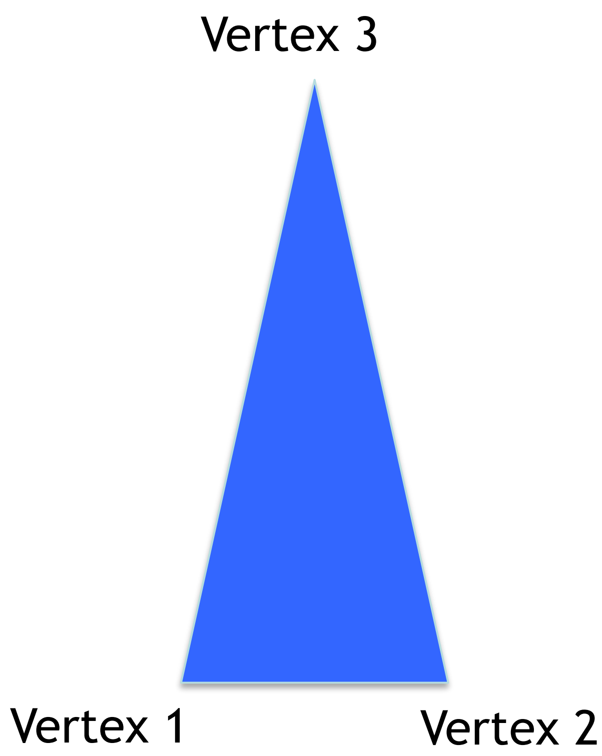
Hyperelastic Models: Element Inversion

- Start with a triangle



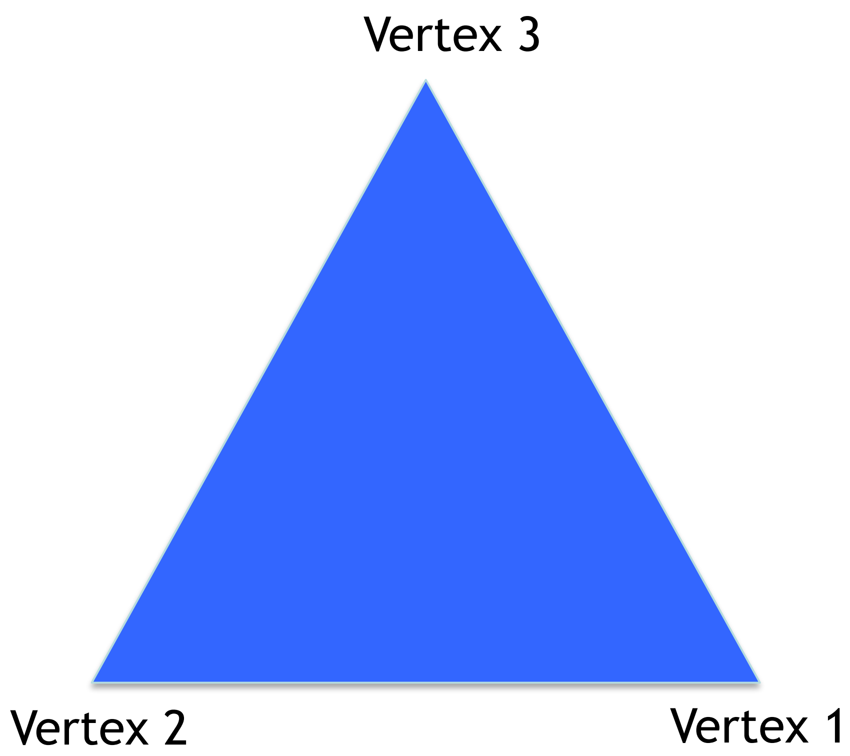
Hyperelastic Models: Element Inversion

- Squish it …



Hyperelastic Models: Element Inversion

- It gets sucked inside out



2D/3D Elasticity - Material models

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu \|\mathbf{E}\|_F^2 + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P} = \mathbf{F}\,[2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}]$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det\mathbf{F}$$

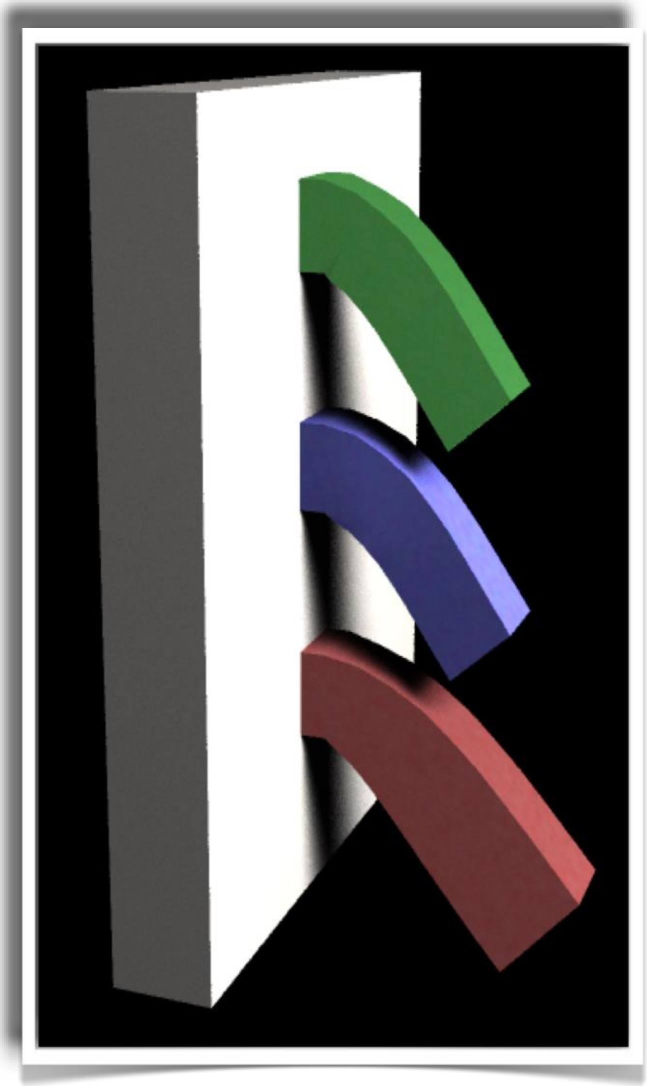
$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu\log(J) + \frac{\lambda}{2}\log^2(J)$$

$$\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda\log(J)\mathbf{F}^{-T}$$

Formulae for 1st Piola-Kirchoff Stress

$(\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P})$

Different Models

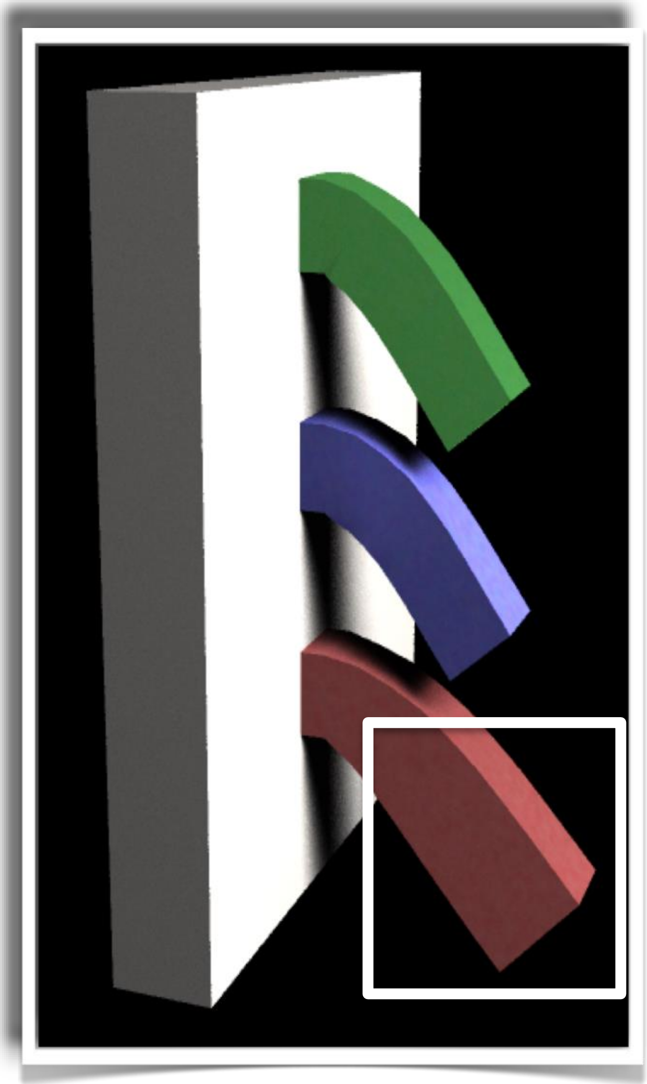


St. Venant-Kirchoff

Neo-Hookean

Linear

Different Models



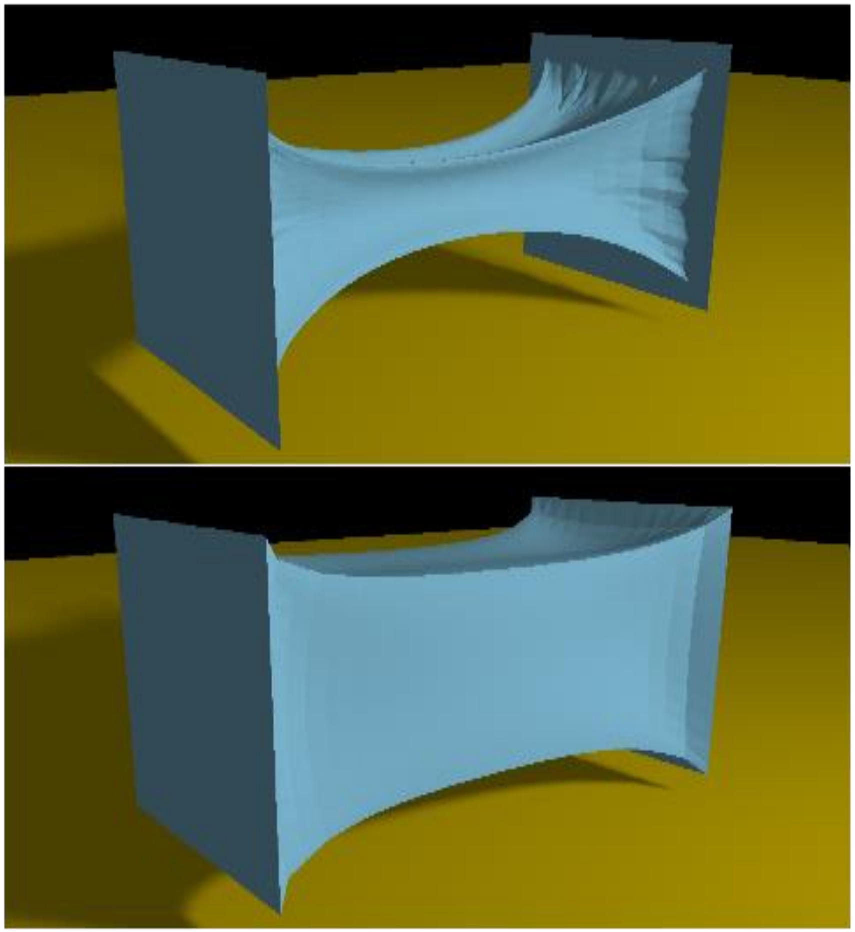
St. Venant-Kirchoff

Neo-Hookean

Linear

Choosing the wrong material model leads to artifacts!!!!

Hyperelastic Models: Differences



St. Venant-Kirchoff

NeoHookean

2D/3D Elasticity - Material models

St. Venant-Kirchhoff material

$$\mathbf{P} = \mathbf{F}\left[2\boxed{\mu}\mathbf{E} + \boxed{\lambda}\mathrm{tr}(\mathbf{E})\mathbf{I}\right]$$

Neohookean elasticity

$$\mathbf{P} = \boxed{\mu}(\mathbf{F} - \mathbf{F}^{-T}) + \boxed{\lambda}\log(J)\mathbf{F}^{-T}$$

Each model as 2 parameters:

2D/3D Elasticity - Material models

St. Venant-Kirchhoff material

$$\mathbf{P} = \mathbf{F}\,[2\boxed{\mu}\,\mathbf{E} + \boxed{\lambda}\,\mathrm{tr}(\mathbf{E})\mathbf{I}]$$

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$$\mathbf{P} = \boxed{\mu}\,(\mathbf{F} - \mathbf{F}^{-\mathsf{T}}) + \boxed{\lambda}\,\log(J)\mathbf{F}^{-\mathsf{T}}$$

Each model as 2 parameters:
μ are λ Lamé parameters

2D/3D Elasticity - Material models

St. Venant-Kirchhoff material

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Neohookean elasticity

$$\mathbf{P} = \boxed{\mu}(\mathbf{F} - \mathbf{F}^{-\mathsf{T}}) + \boxed{\lambda}\log(J)\mathbf{F}^{-\mathsf{T}}$$

Each model as 2 parameters:
μ are λ Lamé parameters

They are related to the fundamental physical parameters:

ν – the Poisson's Ratio
E – the Young's Modulus (Stiffness)

Online conversion tool: http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/calc_elastic_constants.cfm

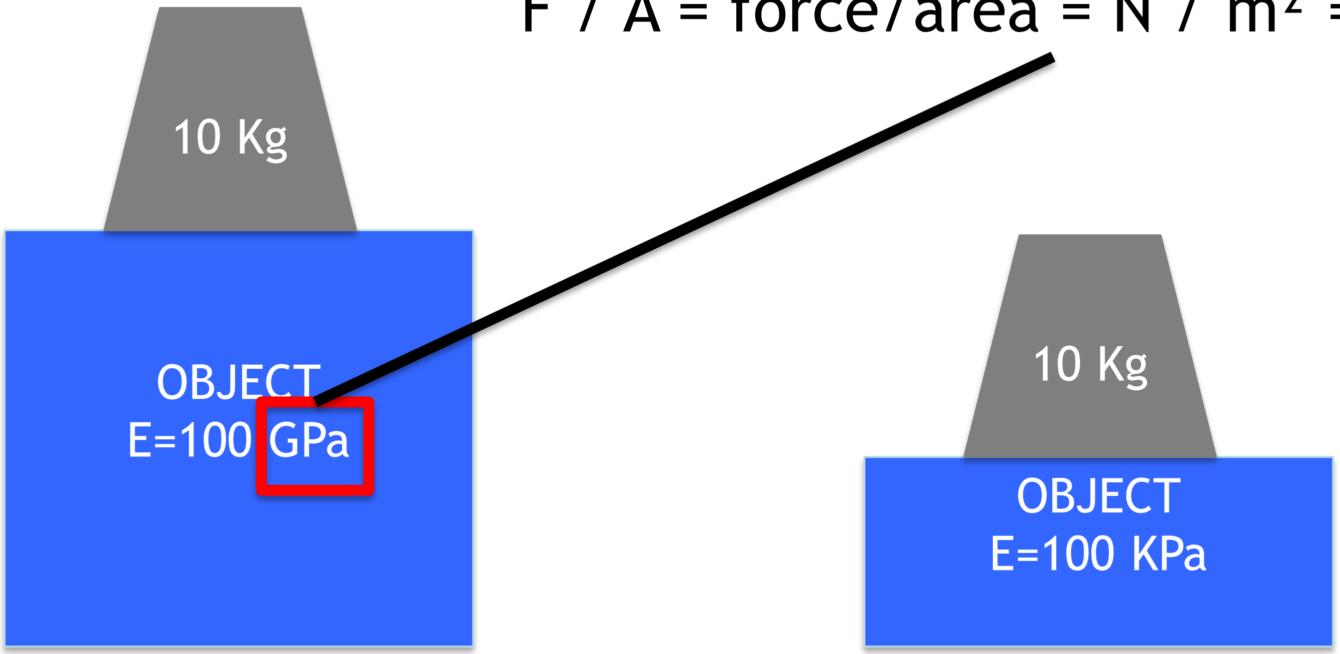
Hyperelastic Models: Parameter Measurement

- Let’s see how to measure both the Young’s Modulus and the Poisson’s Ratio

What Do These Parameters Mean

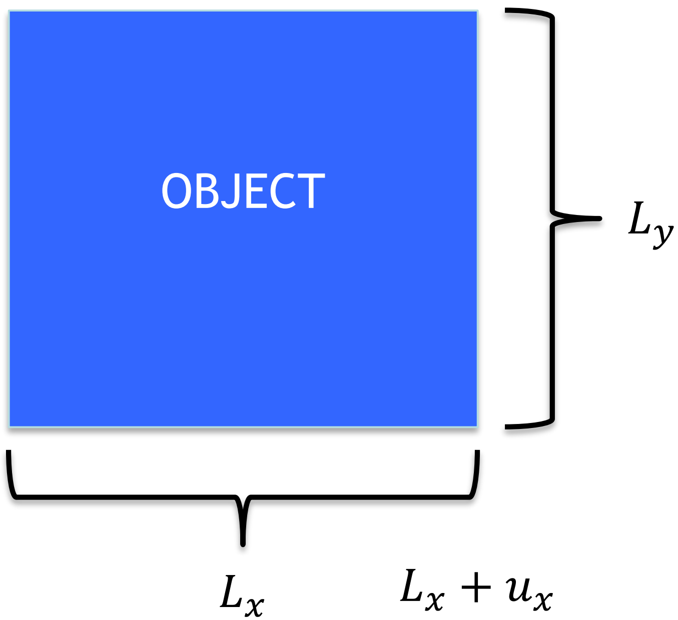
- Stiffness is pretty intuitive

$F \ / \ A$ = force/area = $N \ / \ m^2$ = Pascals

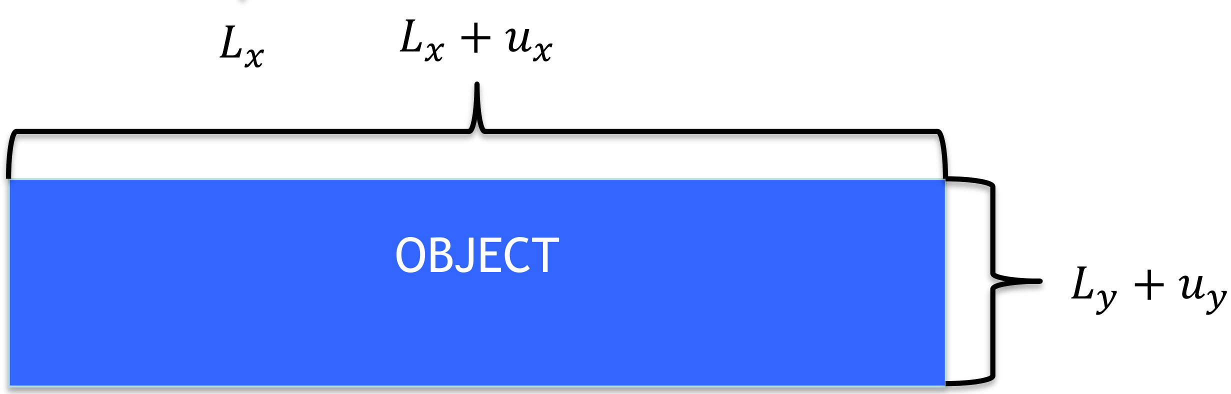


What Do These Parameters Mean

- Poisson's Ratio controls volume preservation



$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} \approx \frac{u_y / L_y}{u_x / L_x}$$



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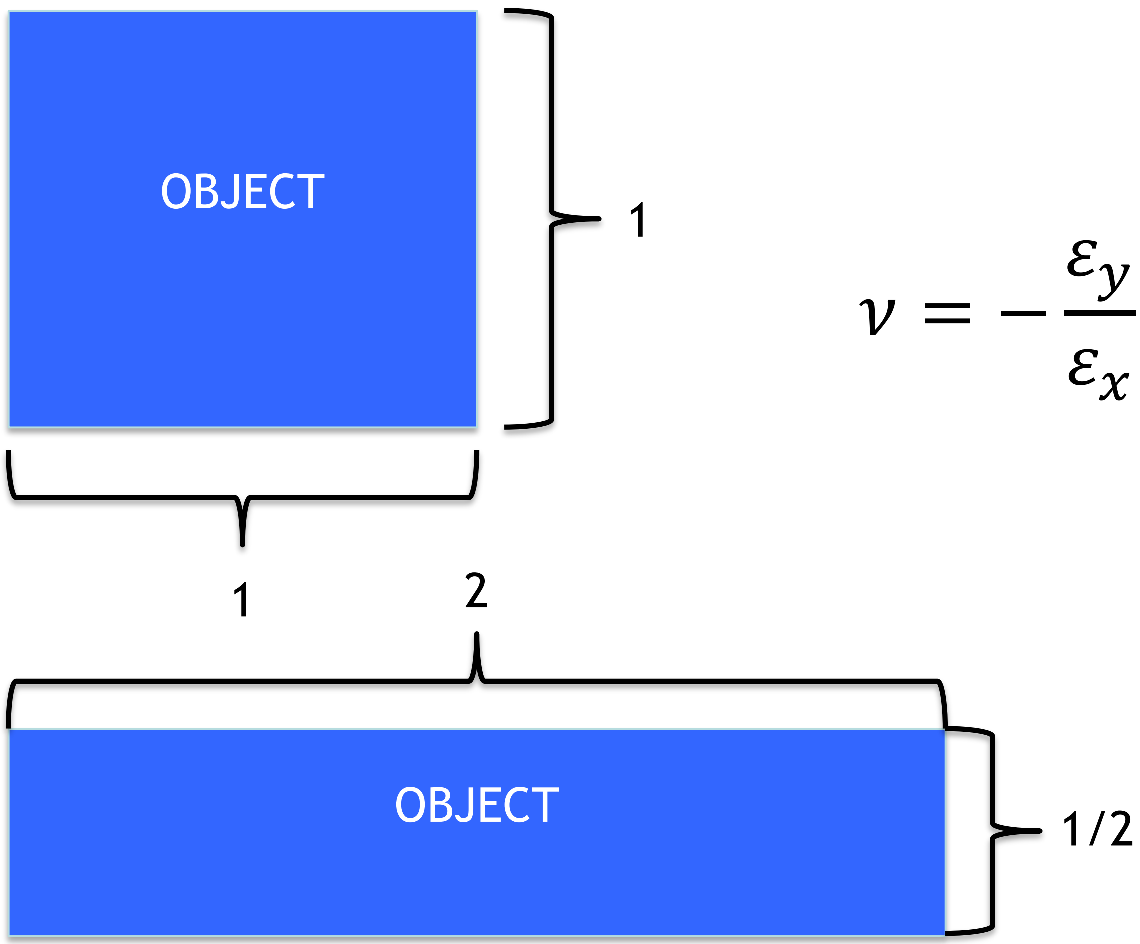

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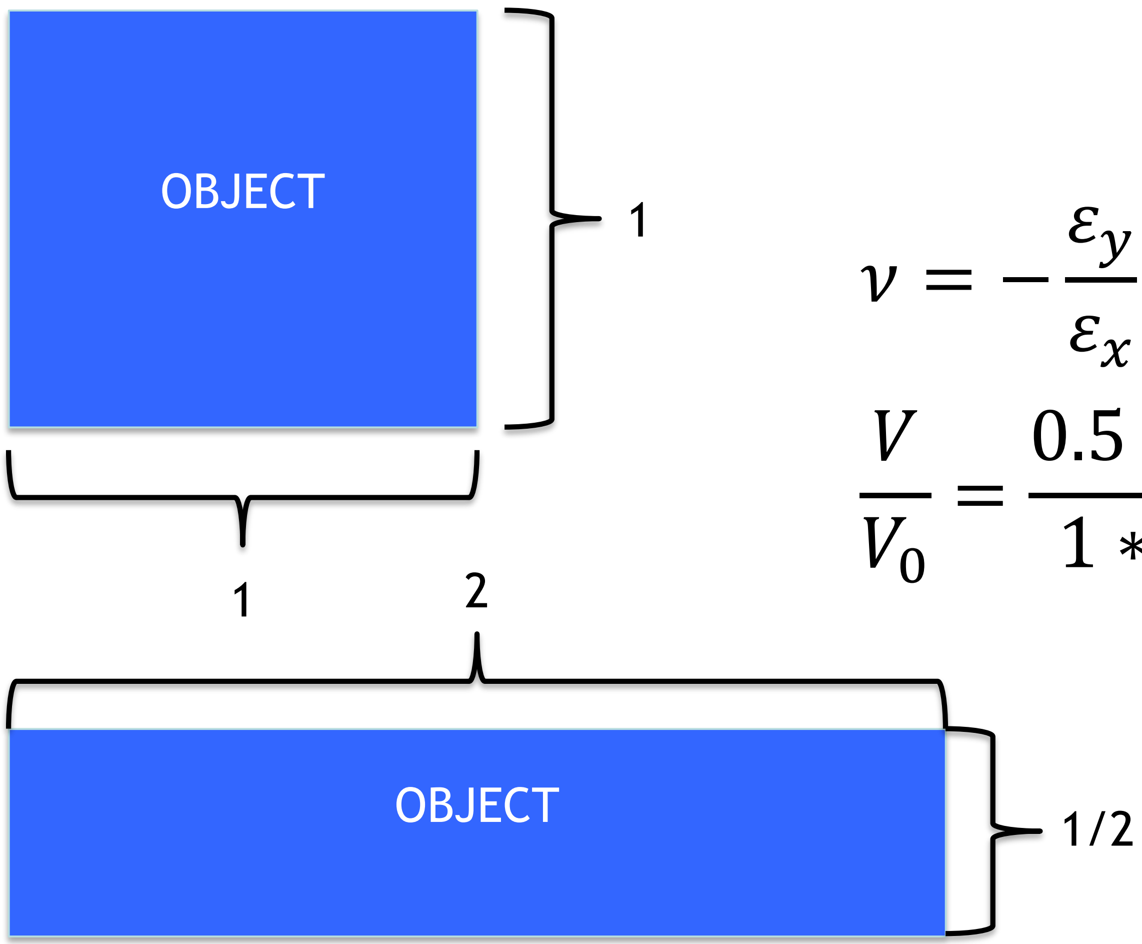
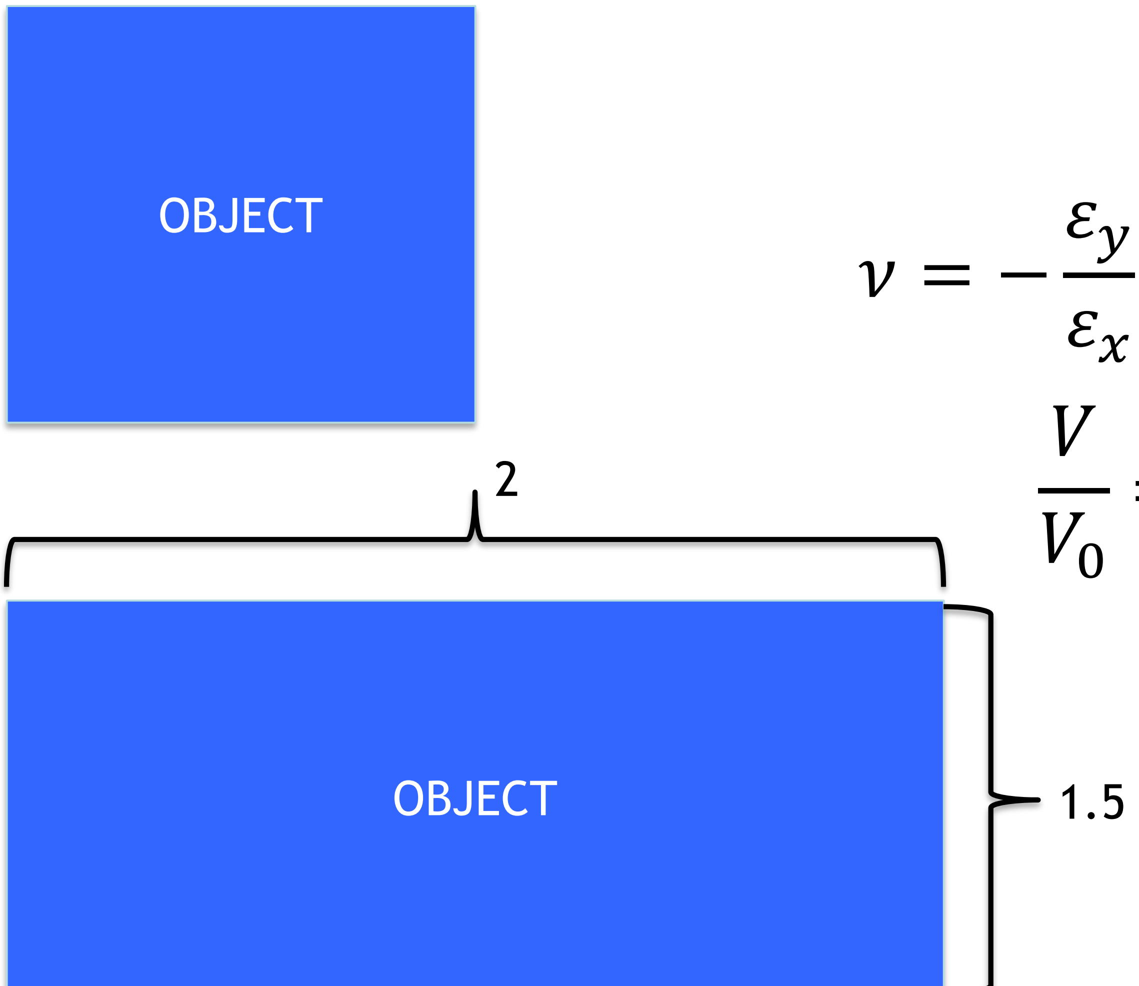


$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} \approx -\frac{-0.5/_{1}}{1/_{1}} = 0.5$$

$$\frac{V}{V_0} = \frac{0.5 * 2}{1 * 1} = 1$$

What Do These Parameters Mean

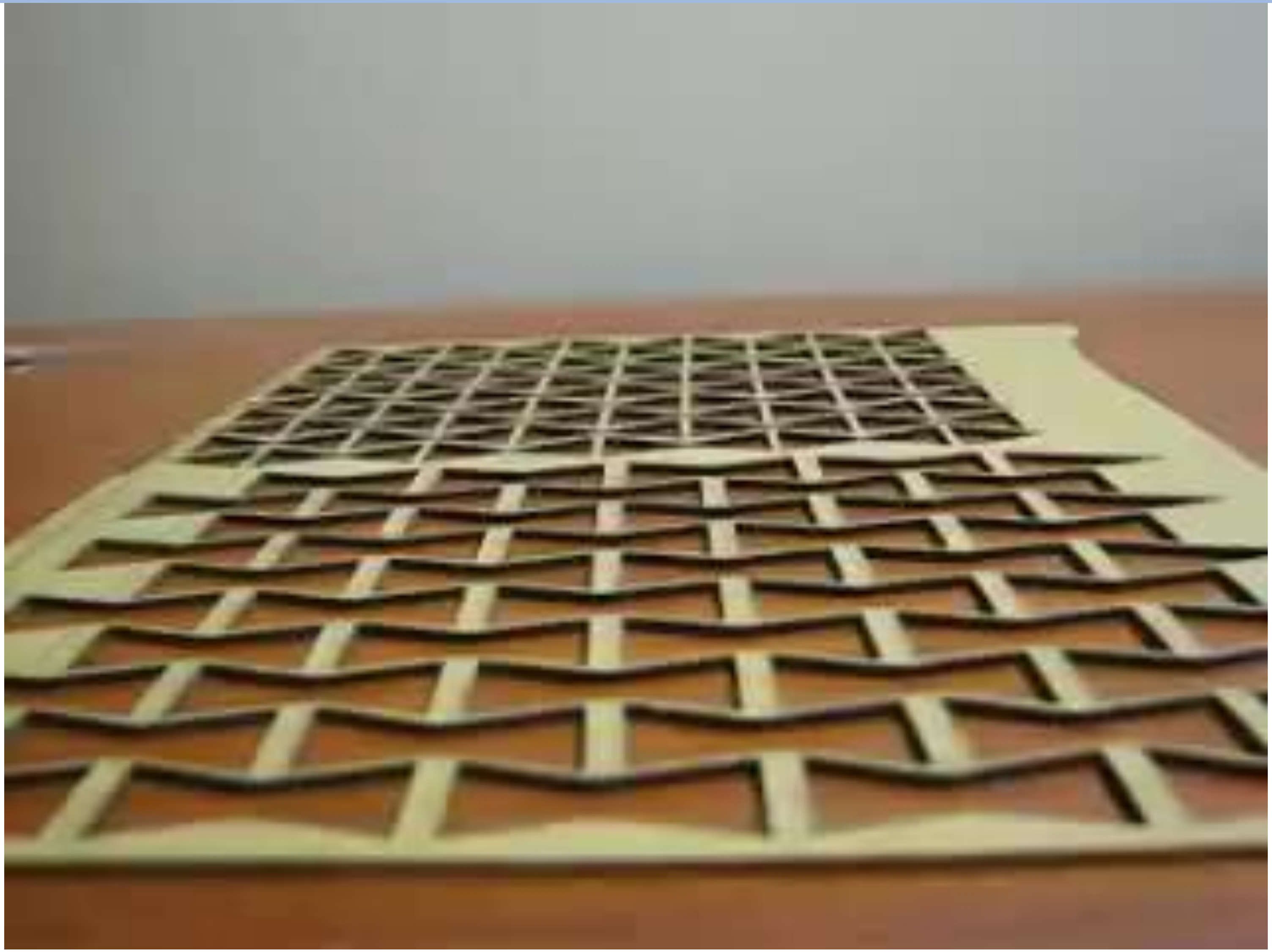
- You can also have objects with negative Poisson's ratio



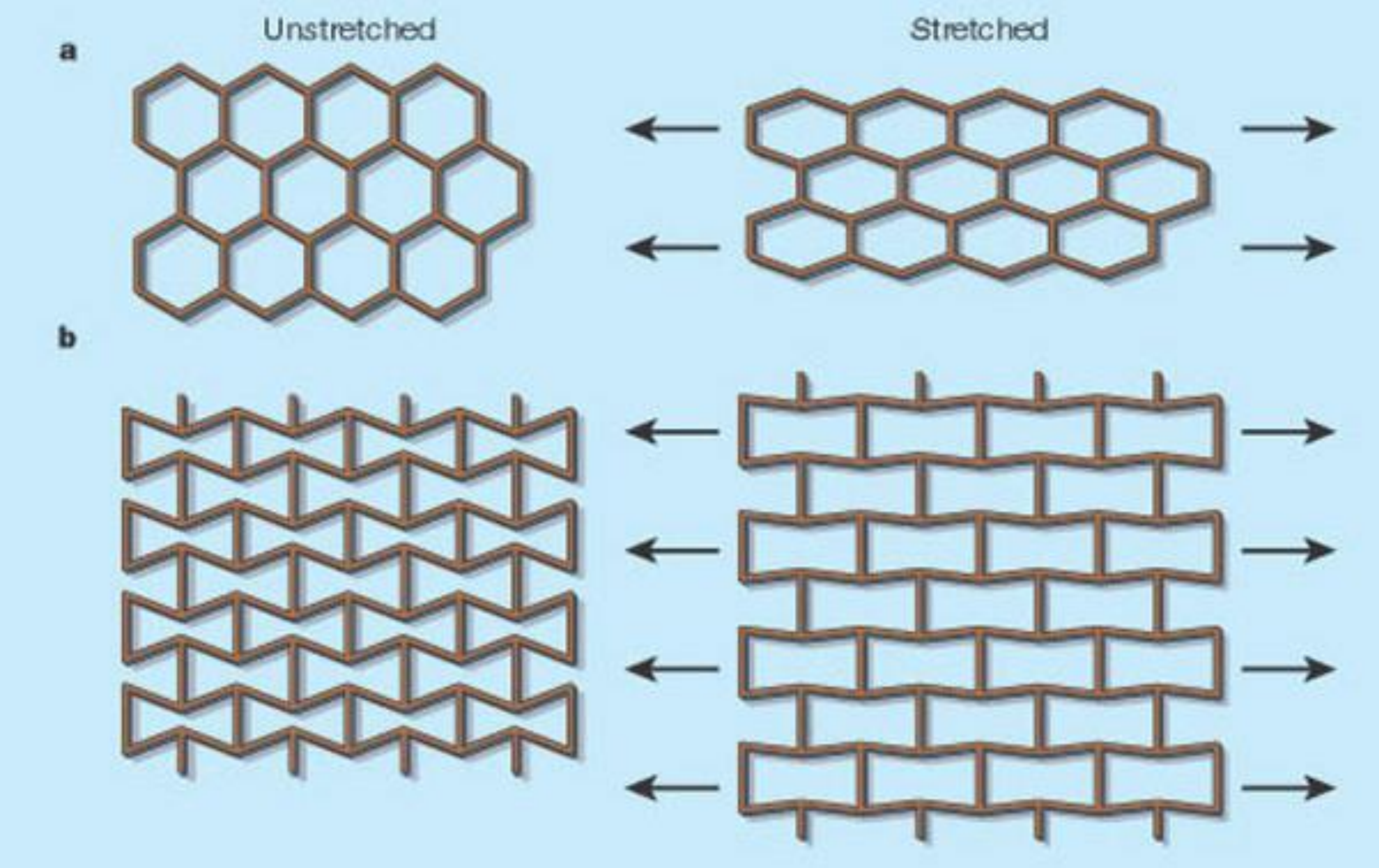
$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} \approx -\frac{0.5/_1}{1/_1} = -0.5$$

$$\frac{V}{V_0} = \frac{1.5 * 2}{1 * 1} = 3$$

Negative Poisson’s Ratio



Negative Poisson's Ratio



Negative Poisson’s Ratio

Measurement

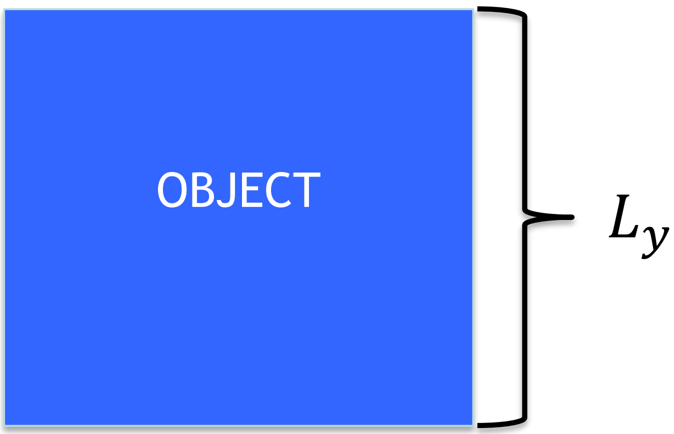
- How do we measure parameters of models ?

Measurement

- How do we measure parameters of models ?
- The units of stress are $Pa = \frac{N}{m^2}$
- Strain is unitless and it equals $\frac{u}{L}$

Simple Measurement: Stiffness

Uniaxial compression test
(or uniaxial tension test)



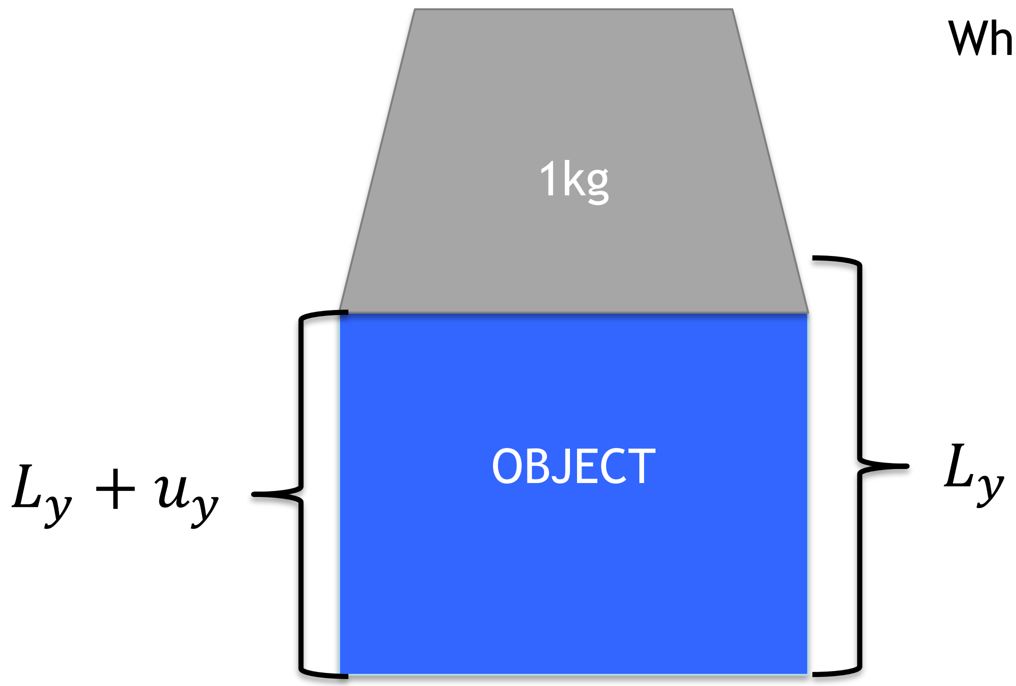
Simple Measurement

Uniaxial compression test
(or uniaxial tension test)

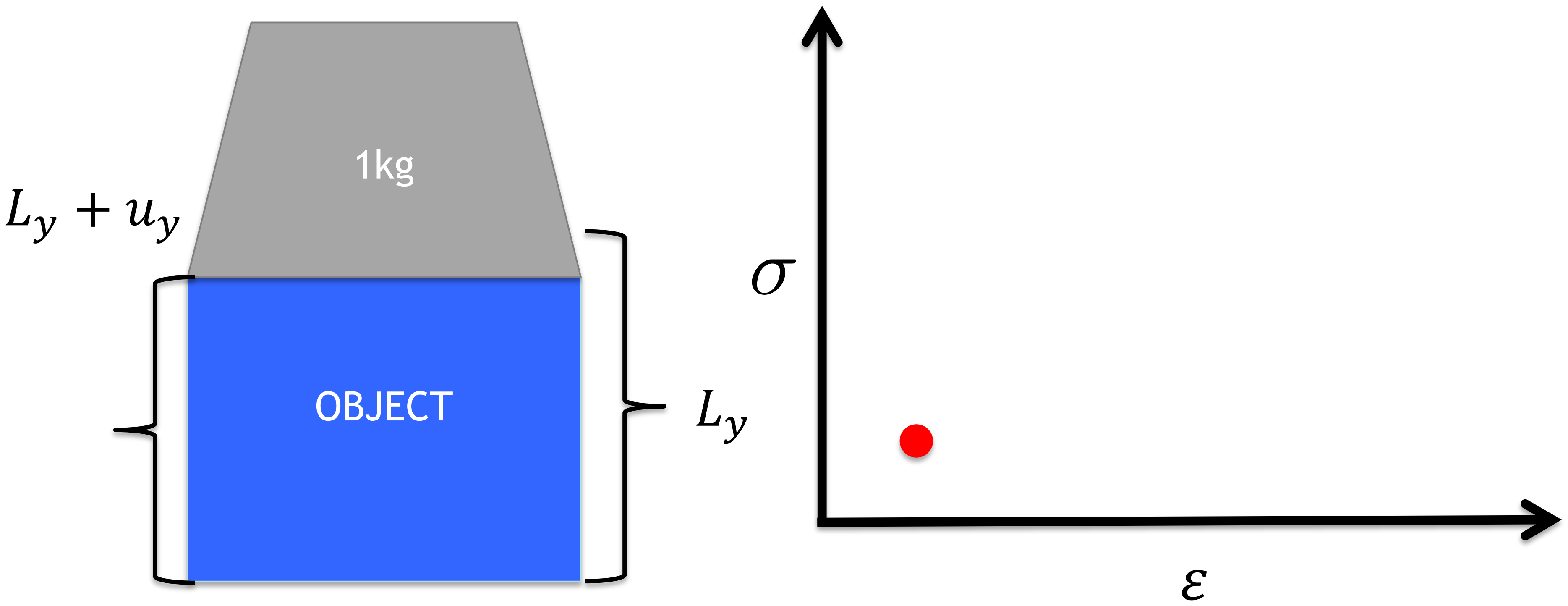
What's the Force ?

What's the Area?

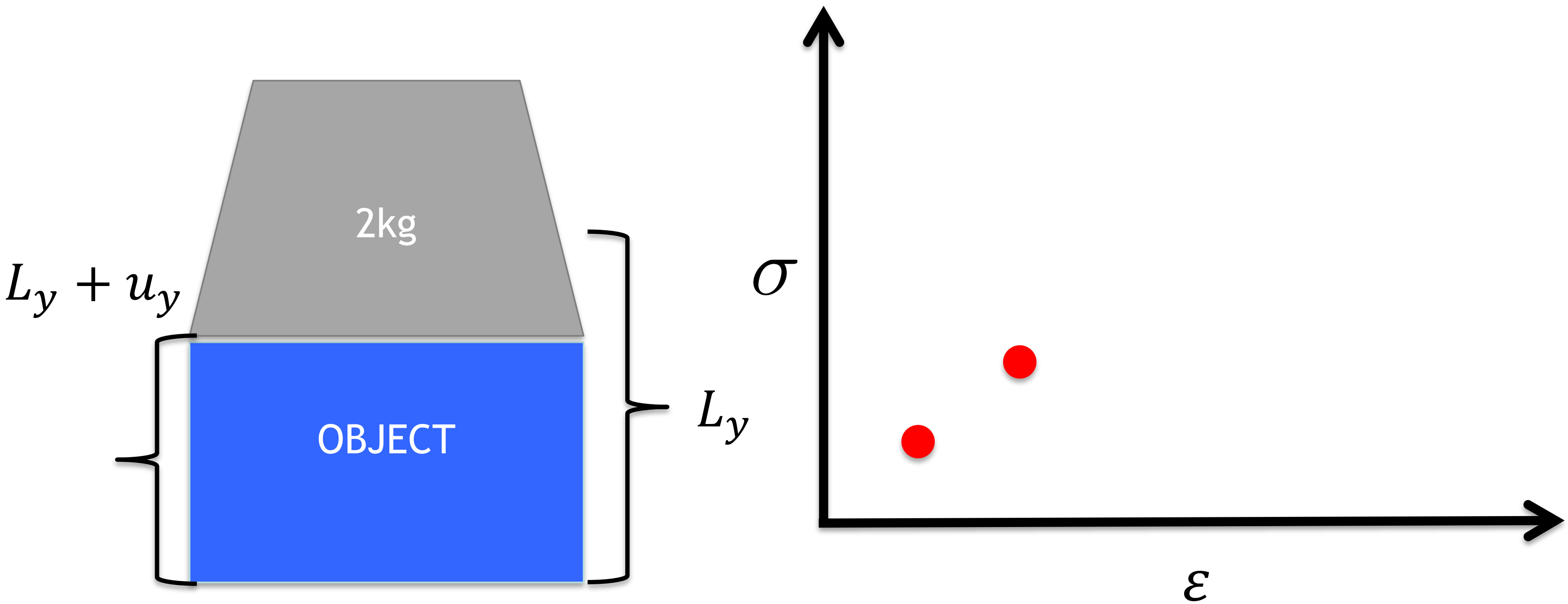
What's the Stress ?



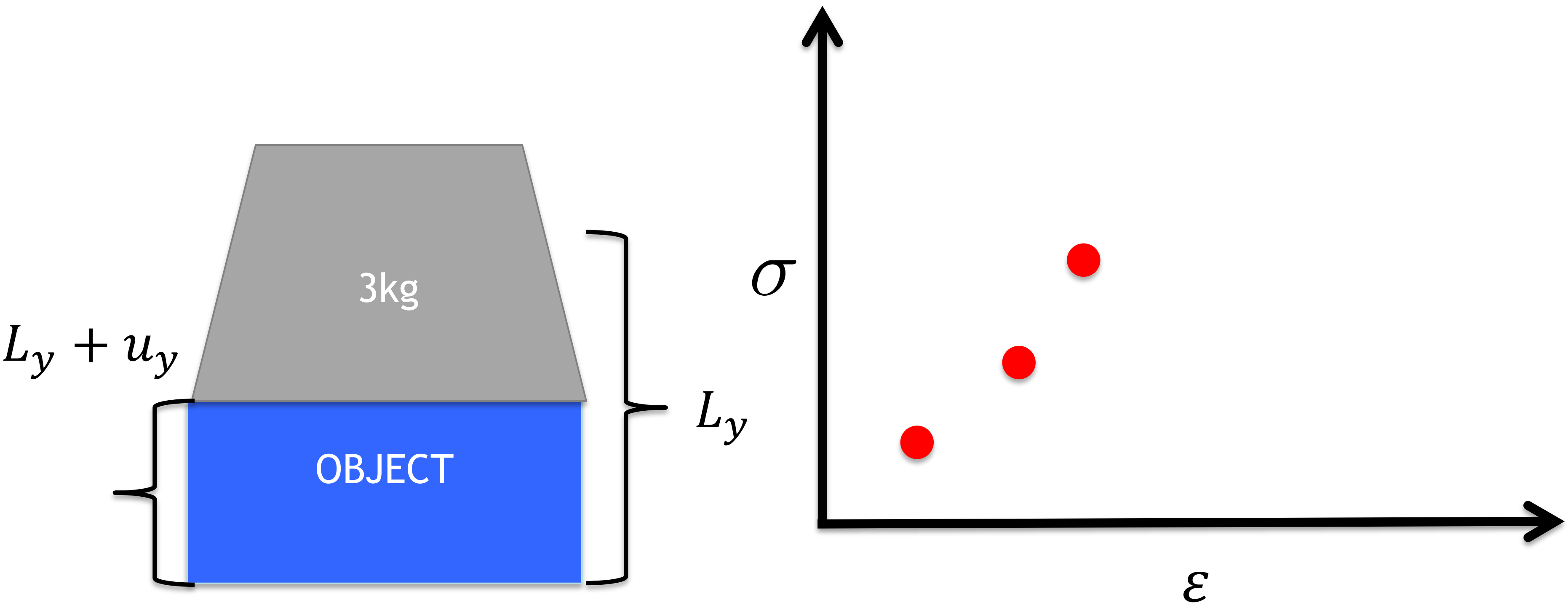
Simple Measurement



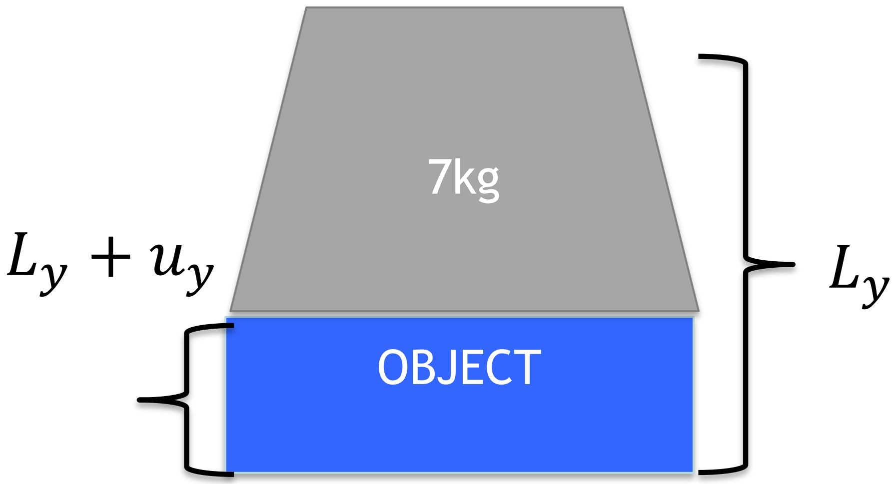
Simple Measurement

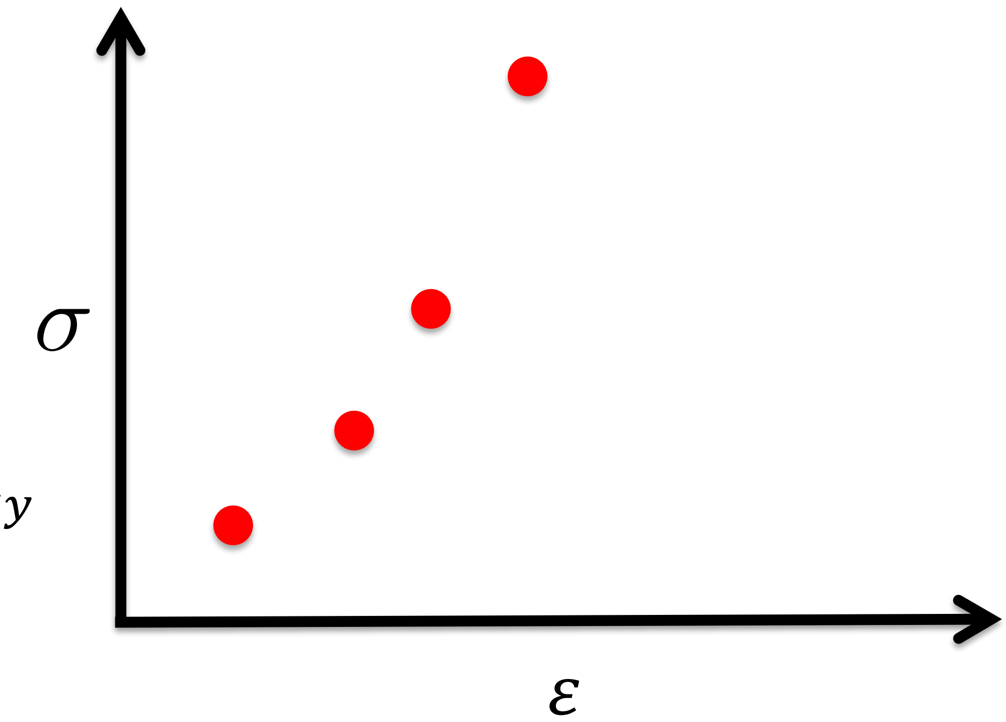


Simple Measurement



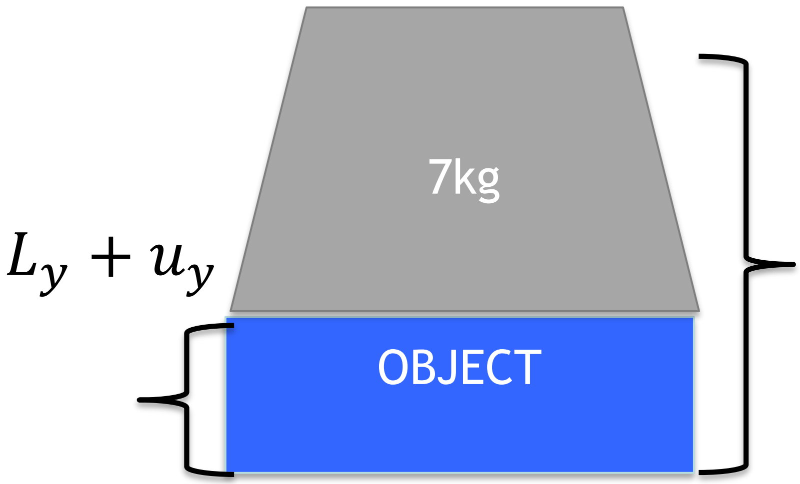
Simple Measurement

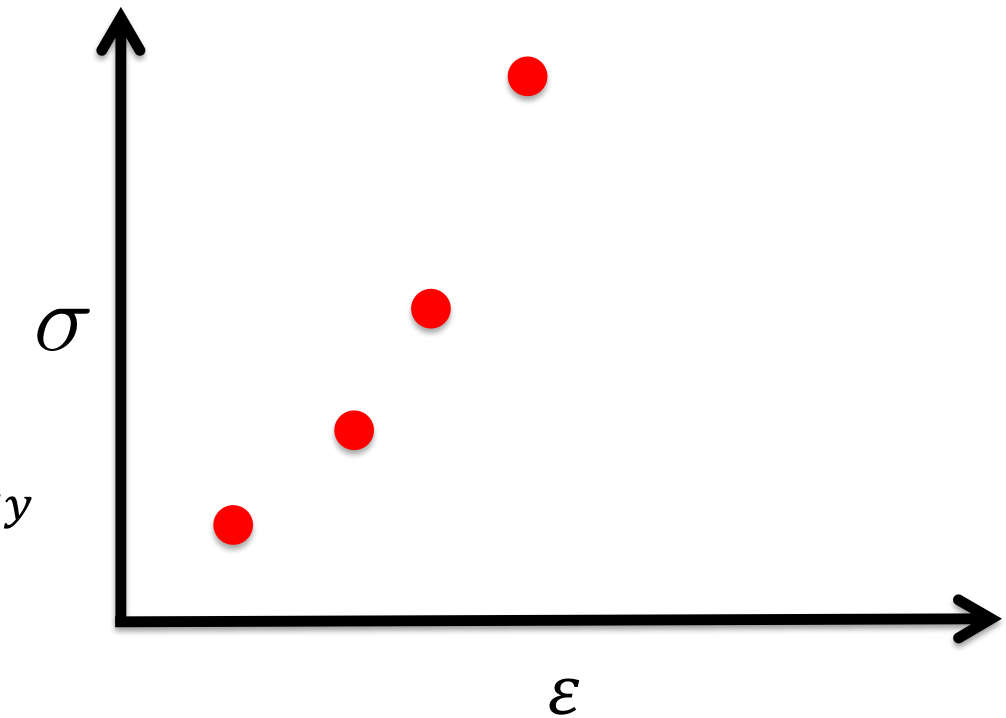




Simple Measurement

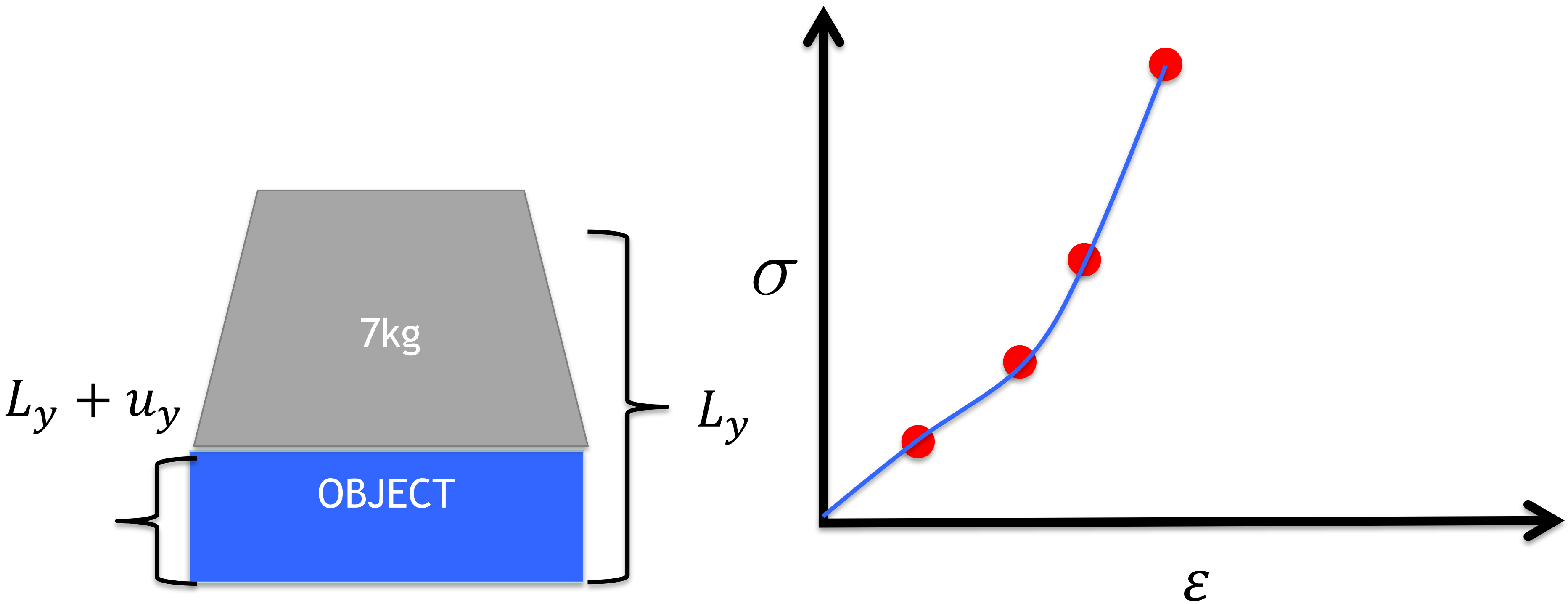
How do we get the stiffness ?





Simple Measurement

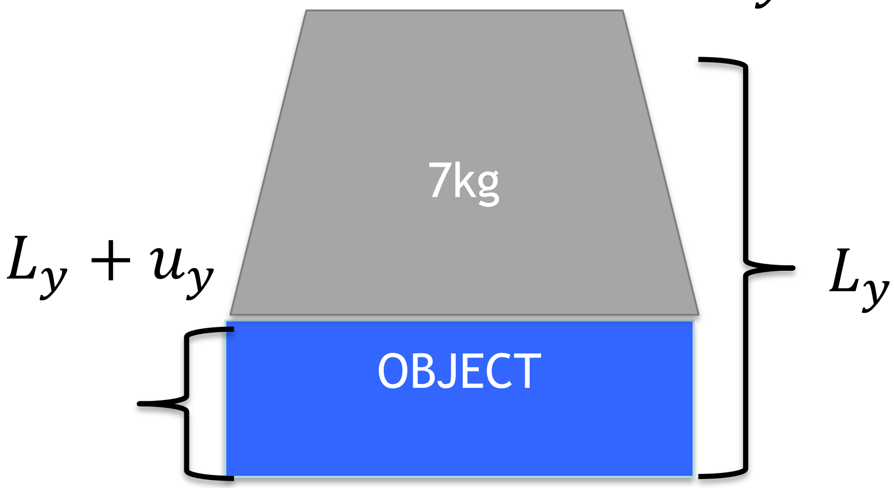
How do we get the stiffness ?

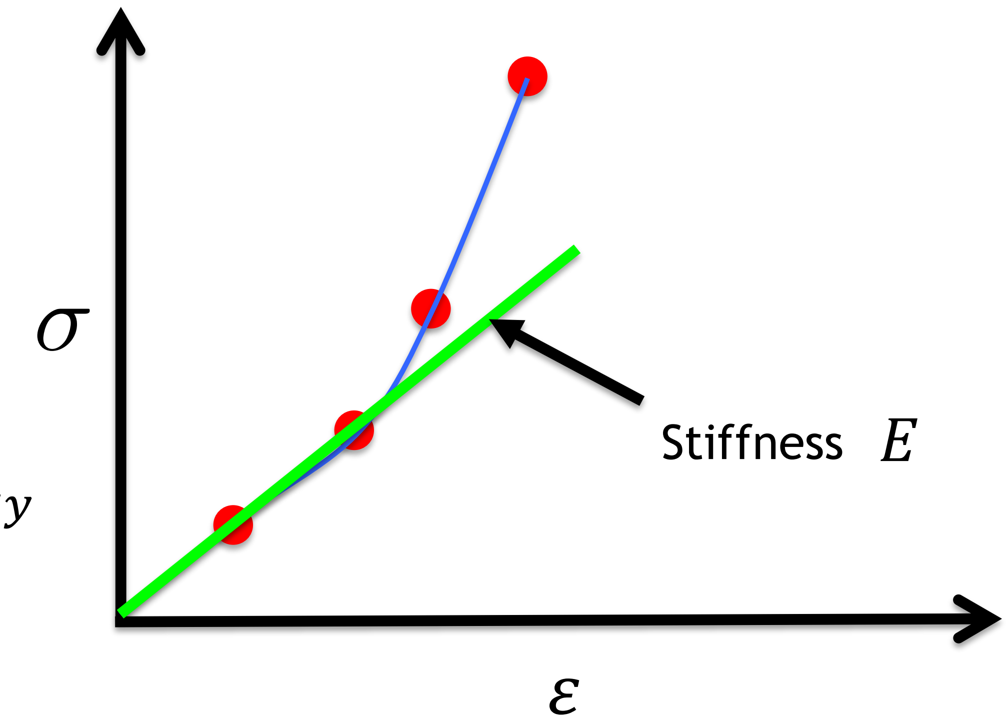


Simple Measurement

How do we get the Young's modulus?

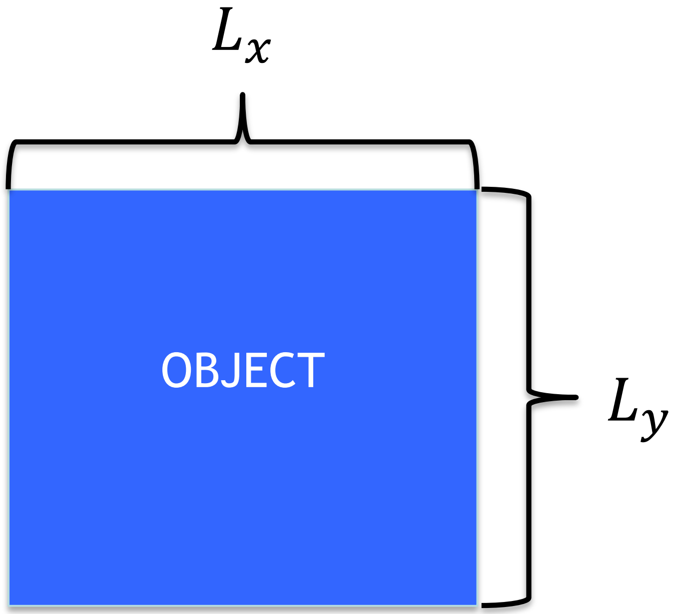
$$E = \frac{\sigma}{\varepsilon} = \frac{F}{A} \Big/ \frac{u_y}{L_y}$$





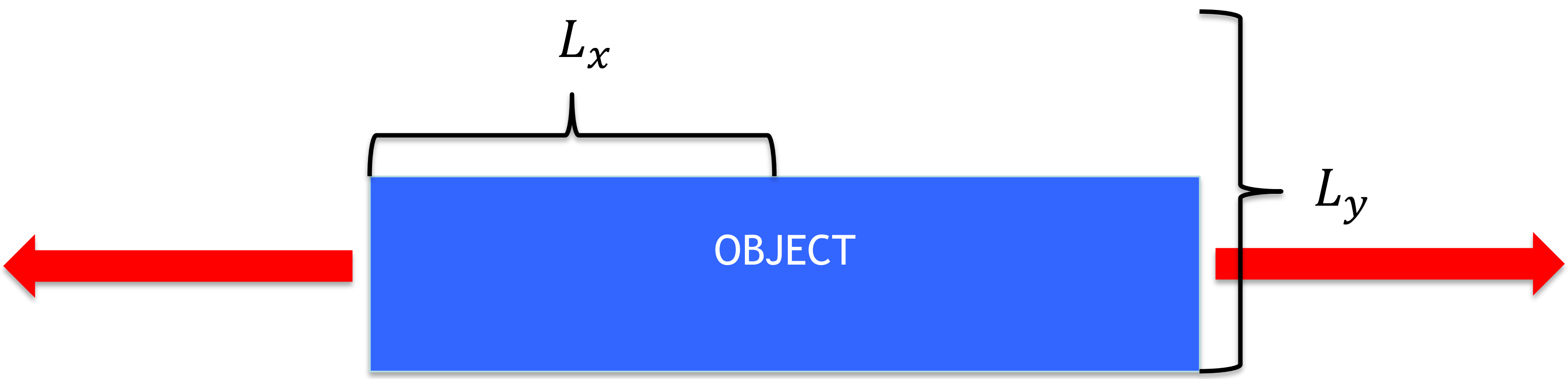
Simple Measurement: Poisson's Ratio

Again:
Uniaxial compression test
(or uniaxial tension test)



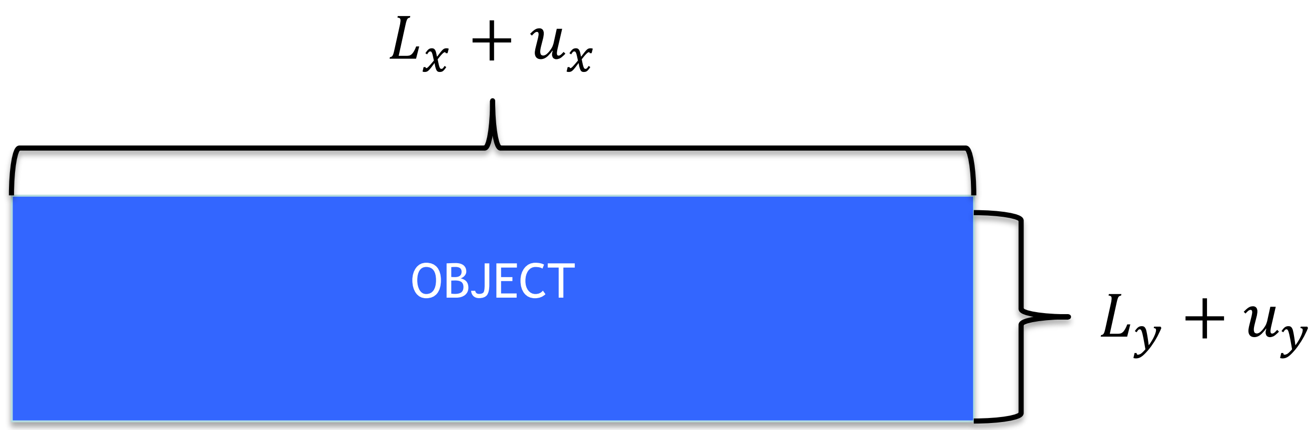
Simple Measurement: Poisson's Ratio

Again:
Uniaxial compression test
(or uniaxial tension test)



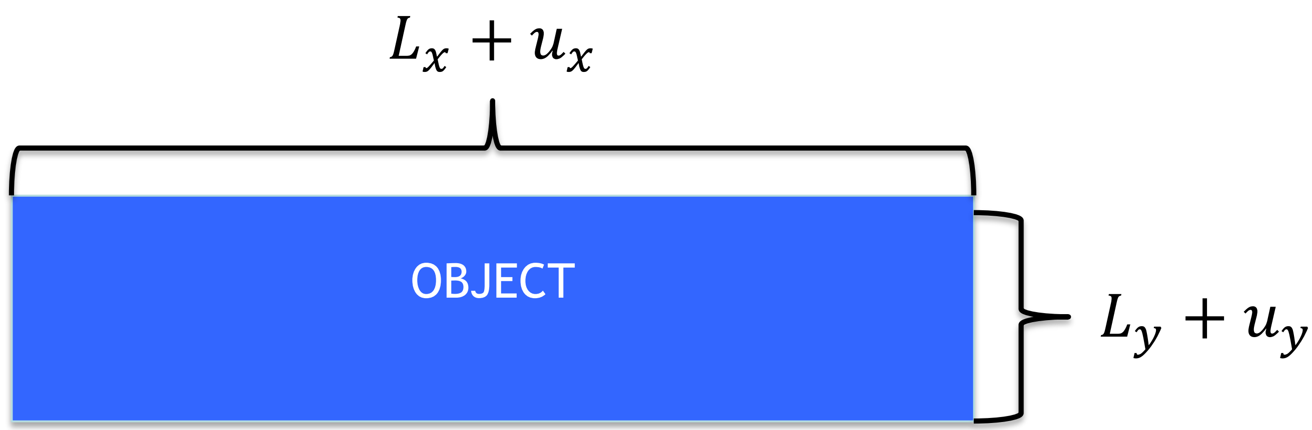
Simple Measurement: Poisson's Ratio

Compute changes in width and height

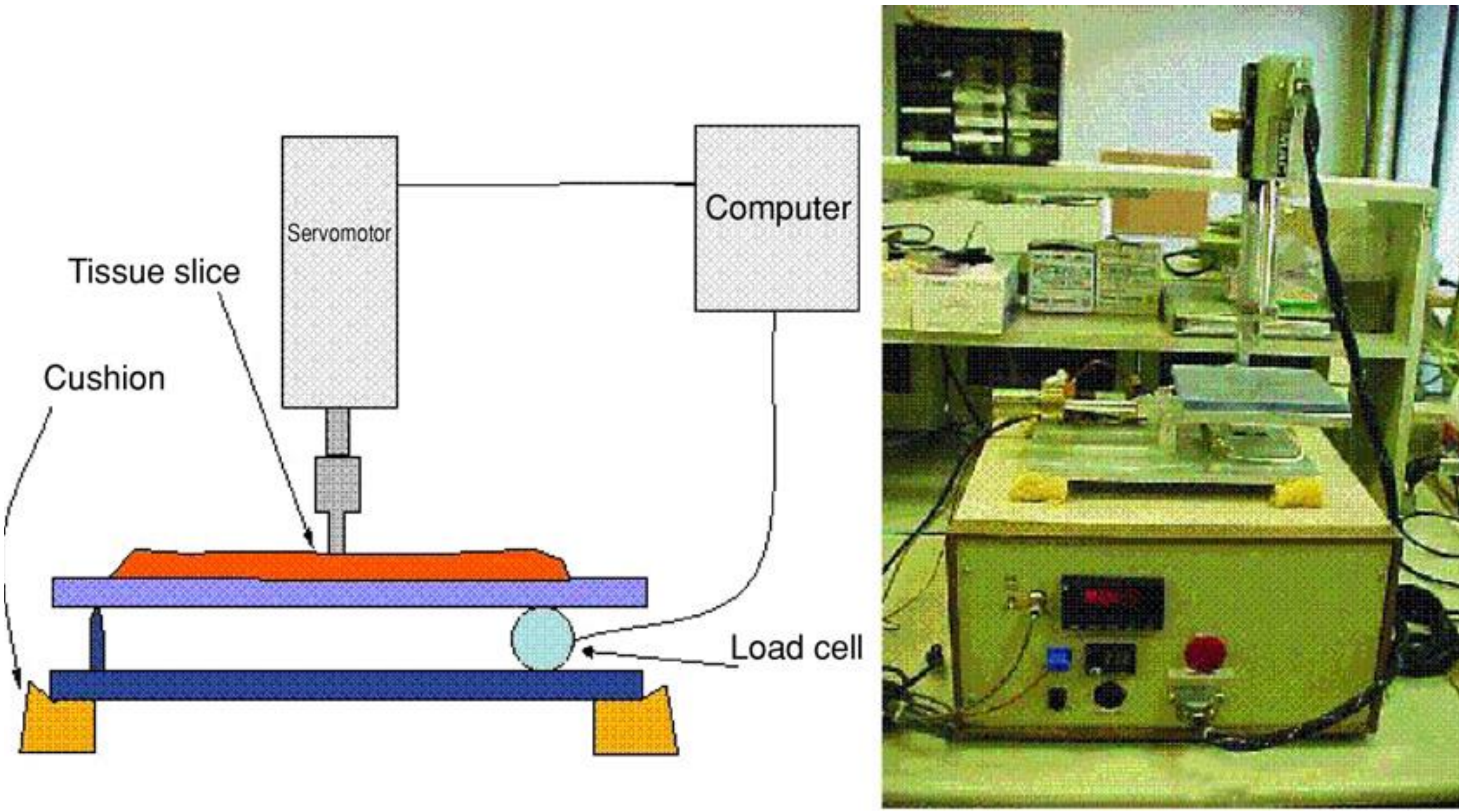


Simple Measurement: Poisson's Ratio

Poisson's Ratio $$\nu = -\frac{\varepsilon_y}{\varepsilon_x} \approx \frac{u_y / L_y}{u_x / L_x}$$



Fancier Measurement Setups



(a) (b)

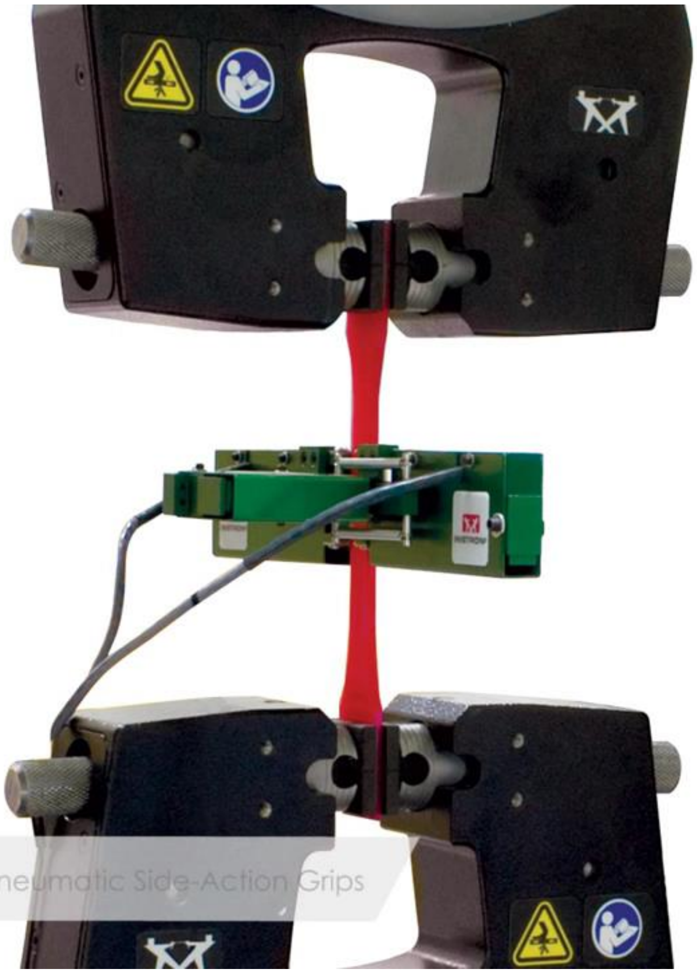
Samani, A. *et al. (2007)*
An inverse problem solution for measuring the elastic modulus of intact *ex vivo* breast tissue tumours

Measurement Devices

Preflex the foam before beginnng taking measurements

Compress the foam twice to 25% of its original thickness @ 4 mm/sec. Then, wait 6 minutes.

Measurement Devices



Bi-Axial Extensometer with 5 kN Pneumatic Side-Action Grips

Types of Materials

- There are many types of materials
  - Elastic ← Done
  - Plastic ← Now
  - Viscous
  - Composites
  - Cellular Materials
  - Lattice Structures
- Each one has different mechanical properties
- When we fabricate things we exploit these properties to achieve optimal results

Plastic Materials

- Defining Properties:
  - Object reference shape changes
  - Object does not always return to its original shape

Example: Crushing a Coke Can

Old Reference State

New Reference State

Example: Crushing a van

A Simple Model For Plasticity

- Recall our model for strain: $\frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$
- Let’s consider how to encode a change of reference shape into this metric
- We want to exchange $\mathbf{F}$ with ${}^w_p\mathbf{F}$, a deformation gradient that takes into account the new shape of our object

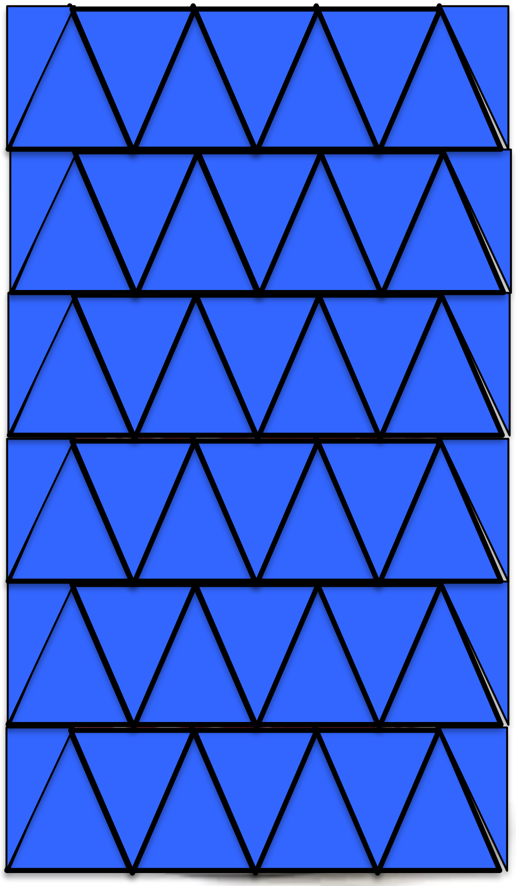
New Reference State

Example: Crushing a Coke Can

Old Reference State

New Reference State

Example: Crushing a Coke Can

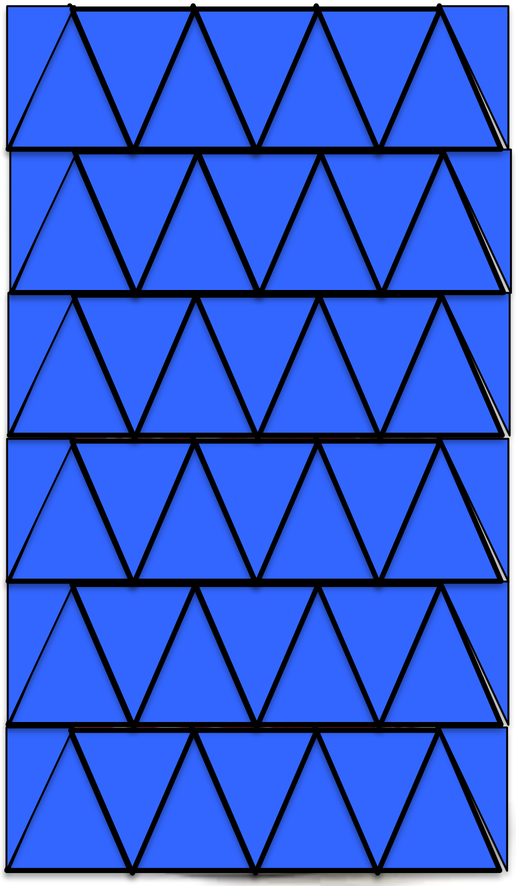


Mesh Lives Here!!!!

Old Reference State

New Reference State

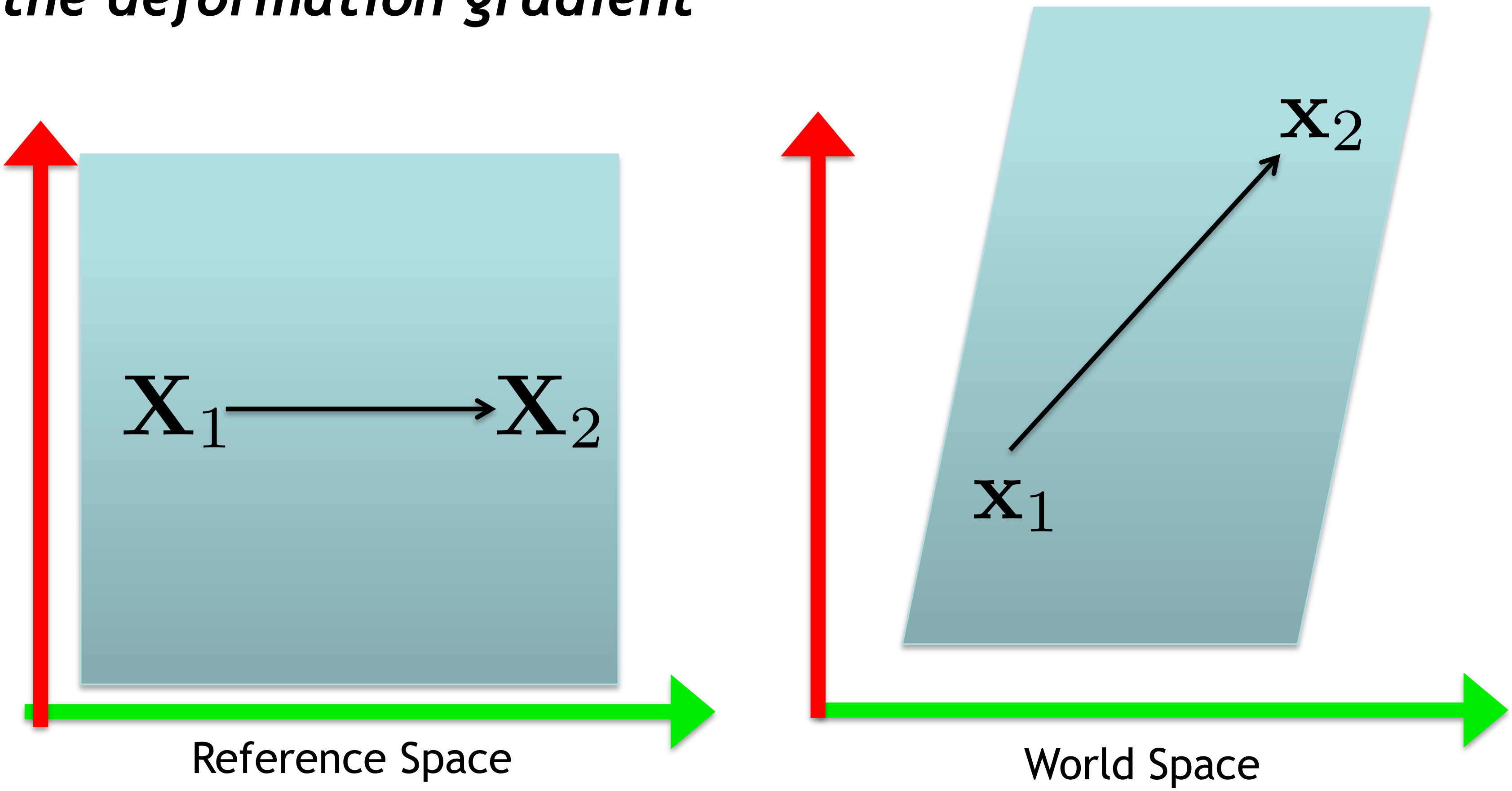
Example: Crushing a Coke Can



How can we encode shape change without changing the mesh

Continuum Mechanics: Deformation

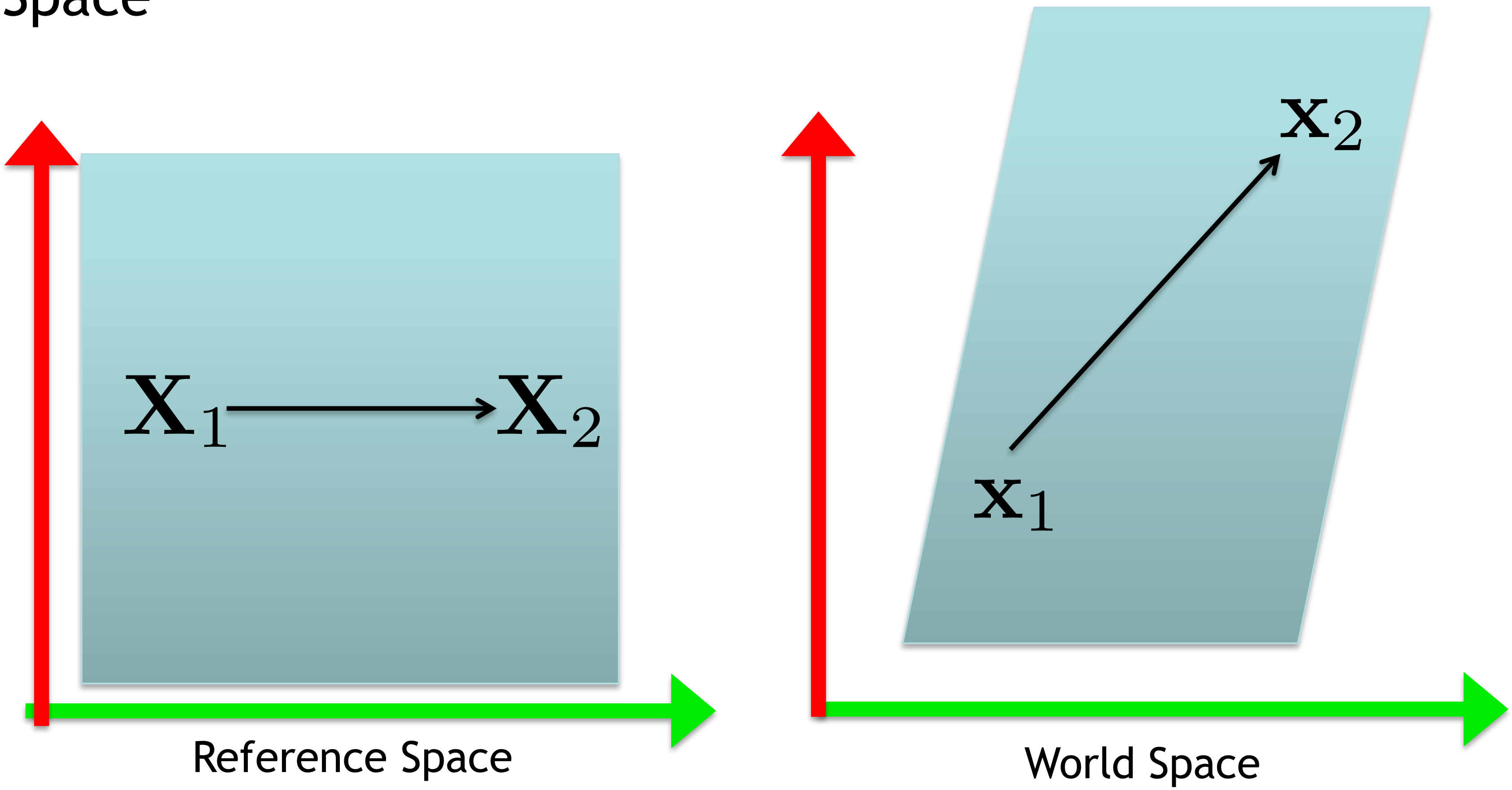
- $\mathbf{F}$ is our deformation measure called *the deformation gradient*



$$\mathbf{dx} \approx \mathbf{FdX}$$

Continuum Mechanics: Deformation

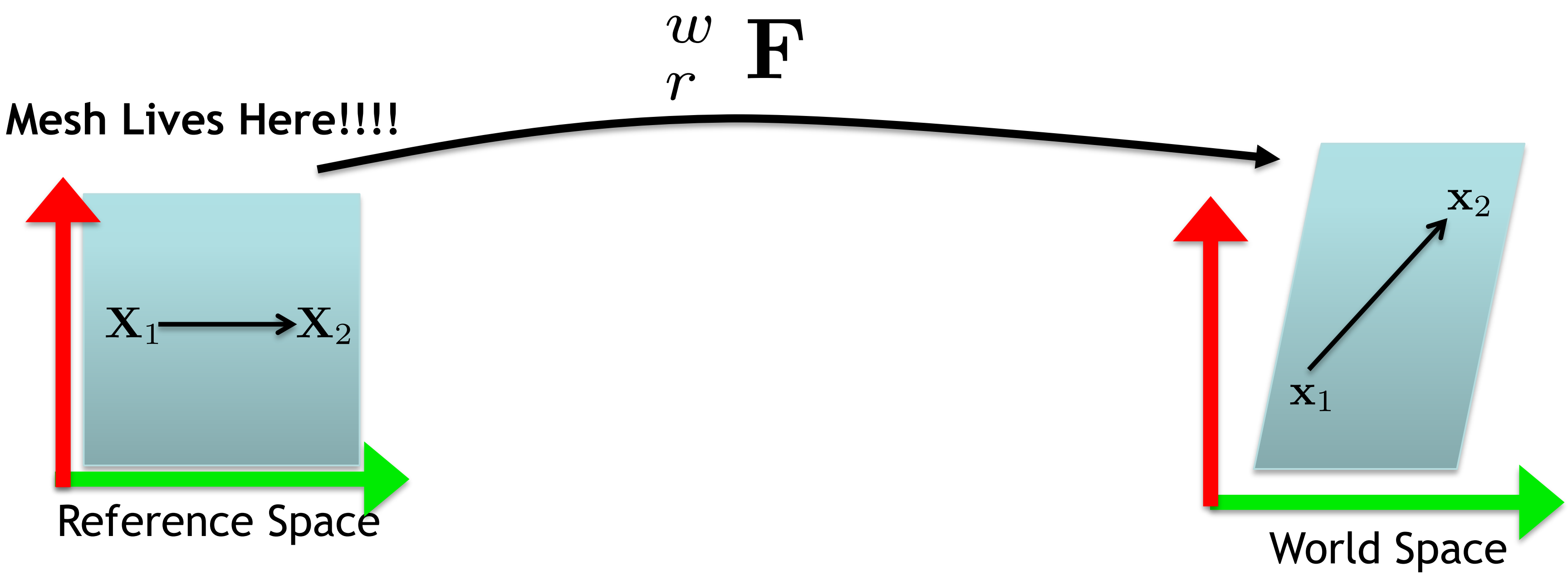
- **F** transforms a vector from Reference space to World Space



$$\mathbf{dx} \approx_r^w \mathbf{F}\mathbf{dX}$$

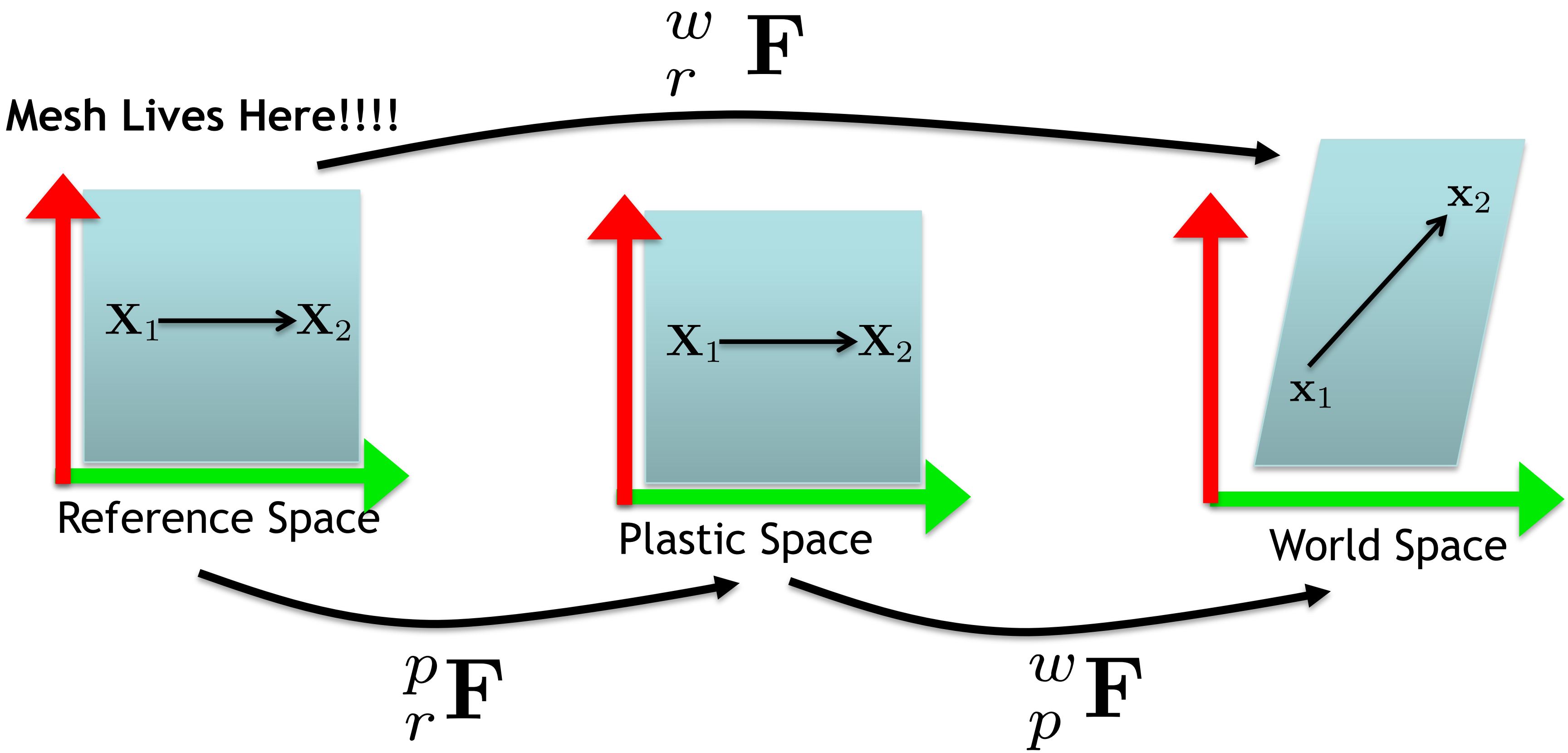
Continuum Mechanics: Deformation

- **F** transforms a vector from Reference space to World Space



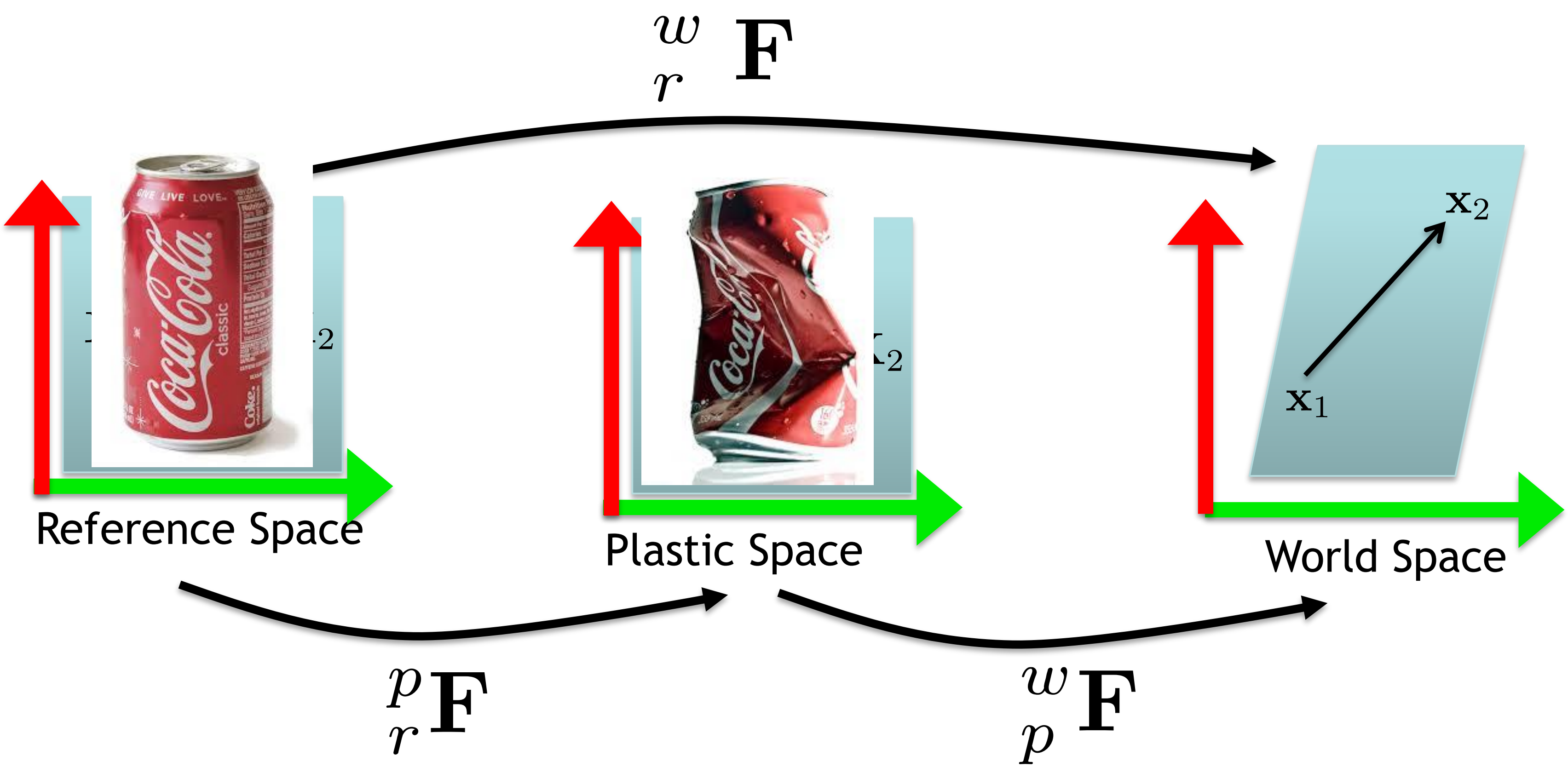
Continuum Mechanics: Deformation

- Introduce a new space



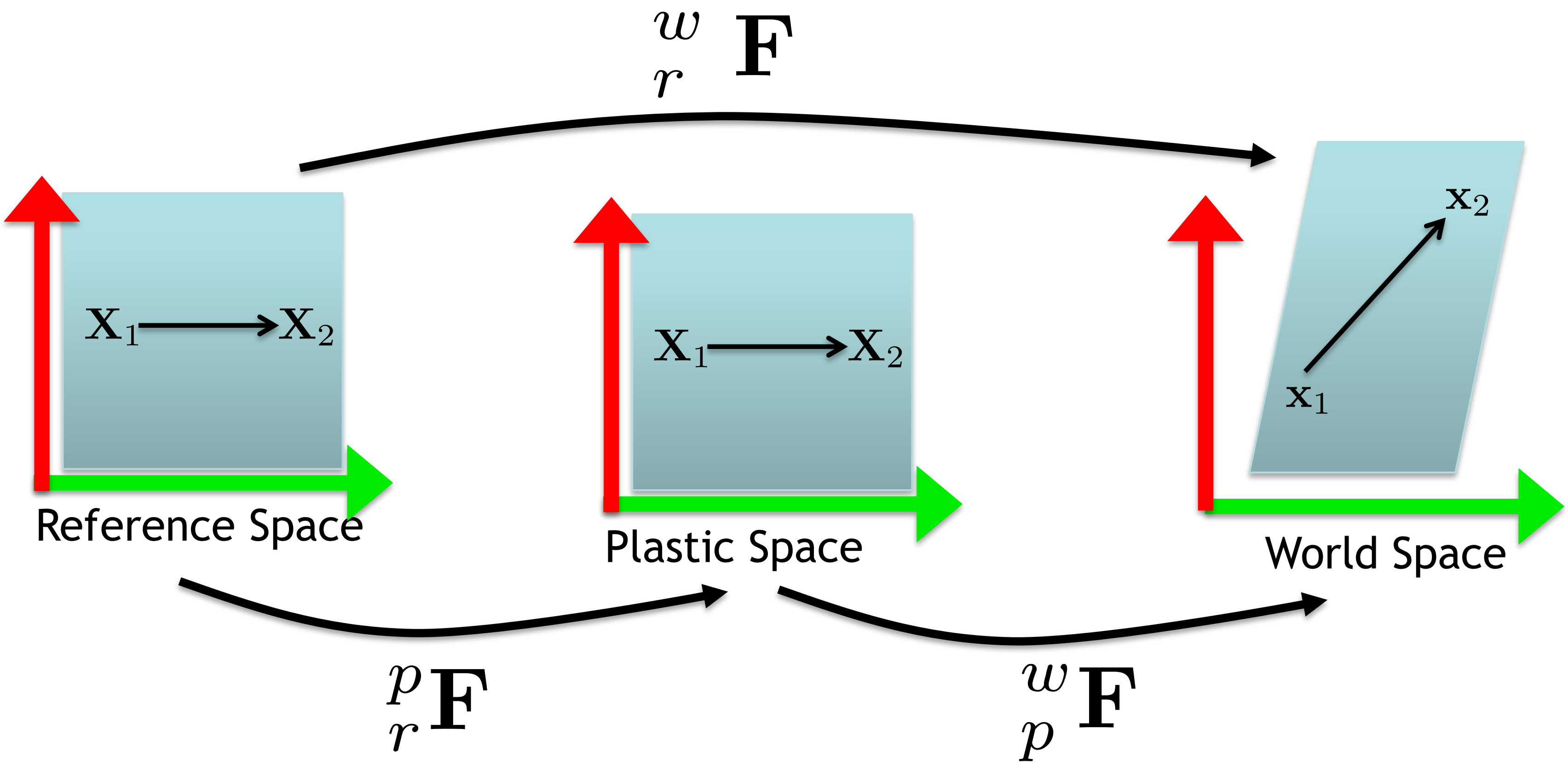
Continuum Mechanics: Deformation

- **Introduce a new space**



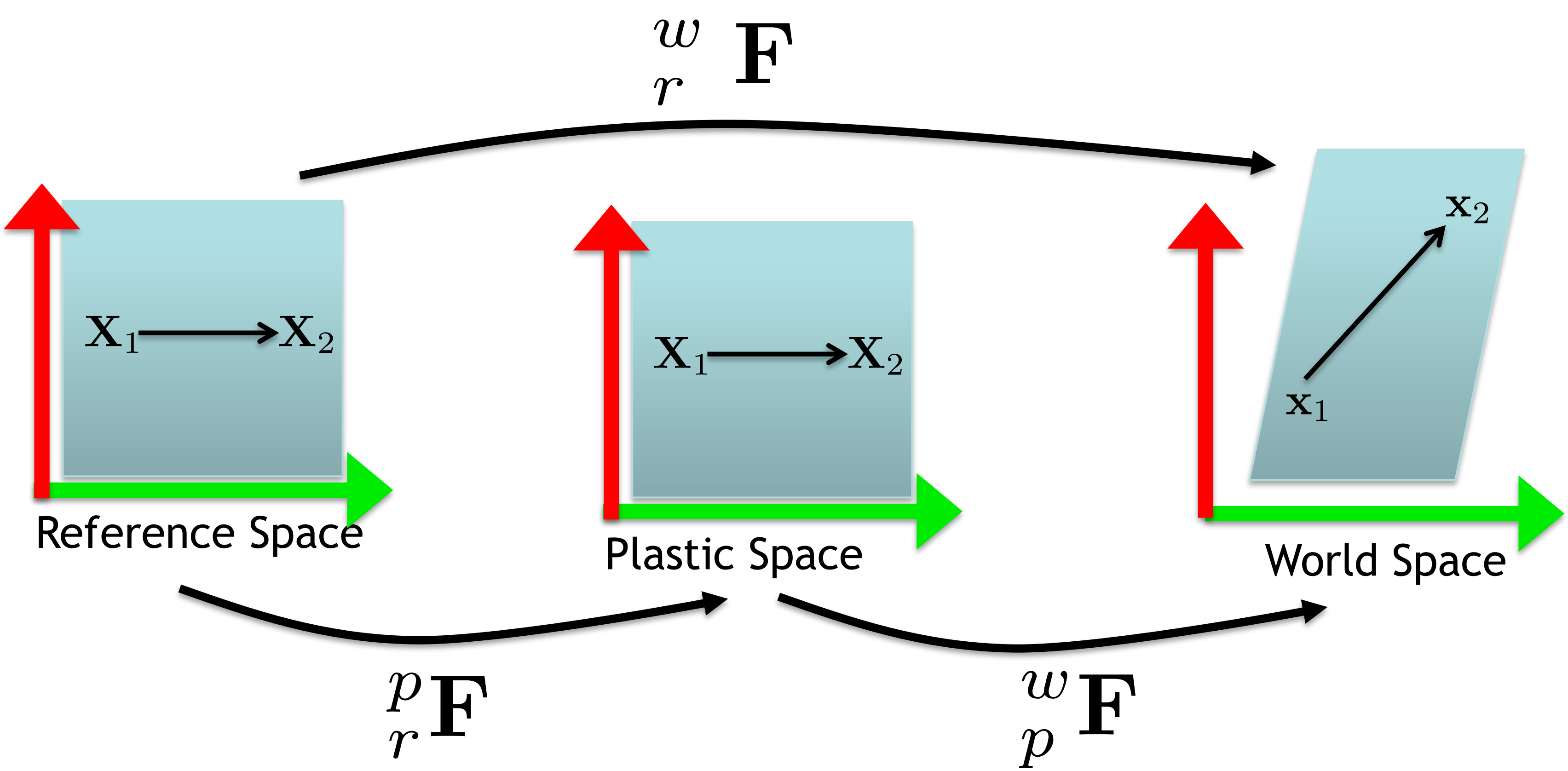
Continuum Mechanics: Deformation

- Our goal is to approximate ${}^{w}_{p}\mathbf{F}$ but we only have access to ${}^{w}_{r}\mathbf{F}$



Continuum Mechanics: Deformation

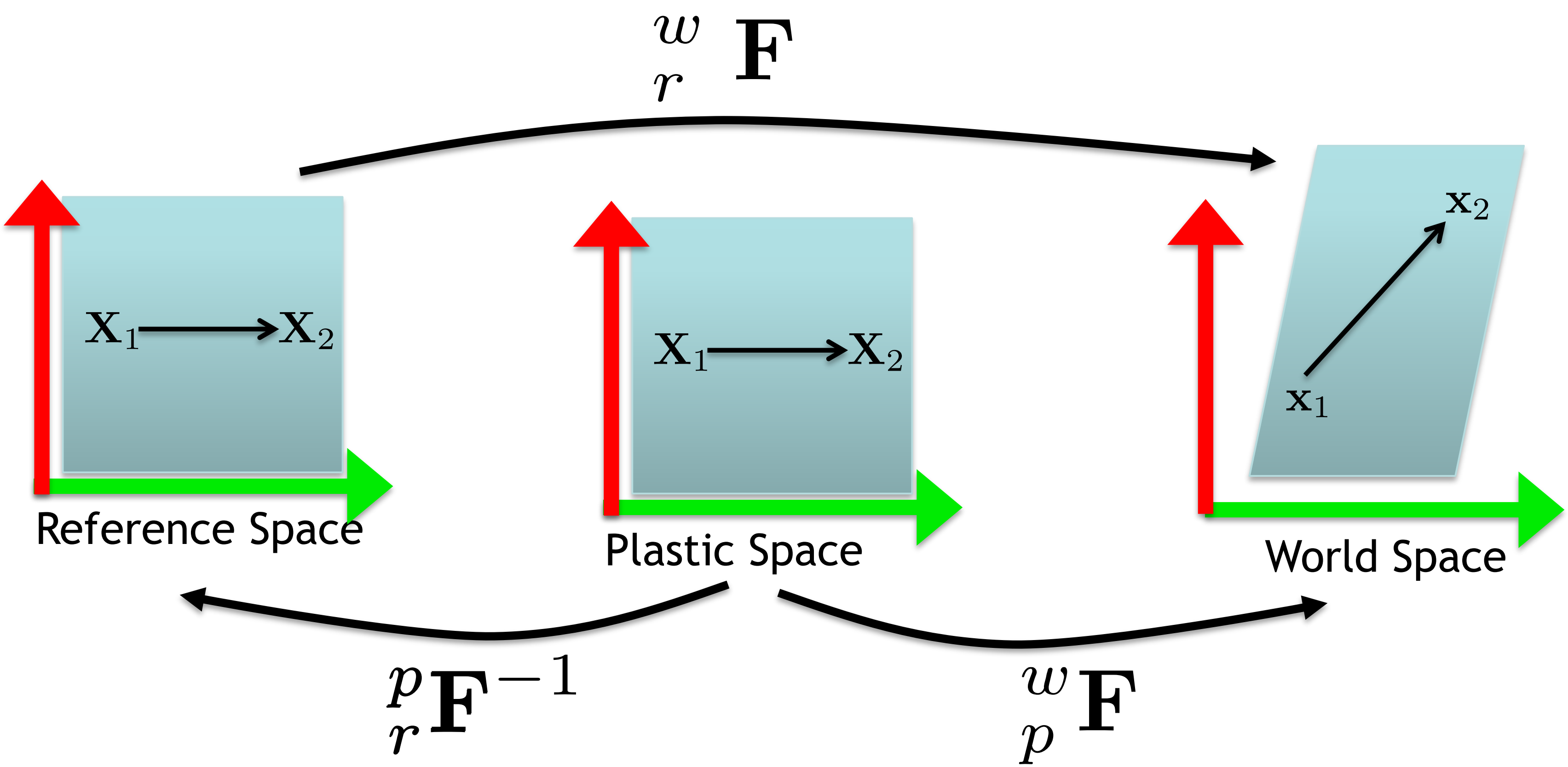
- Our goal is to approximate ${}^{w}_{p}\mathbf{F}$ but we only have access to ${}^{w}_{r}\mathbf{F}$



$$ {}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1} $$

Continuum Mechanics: Deformation

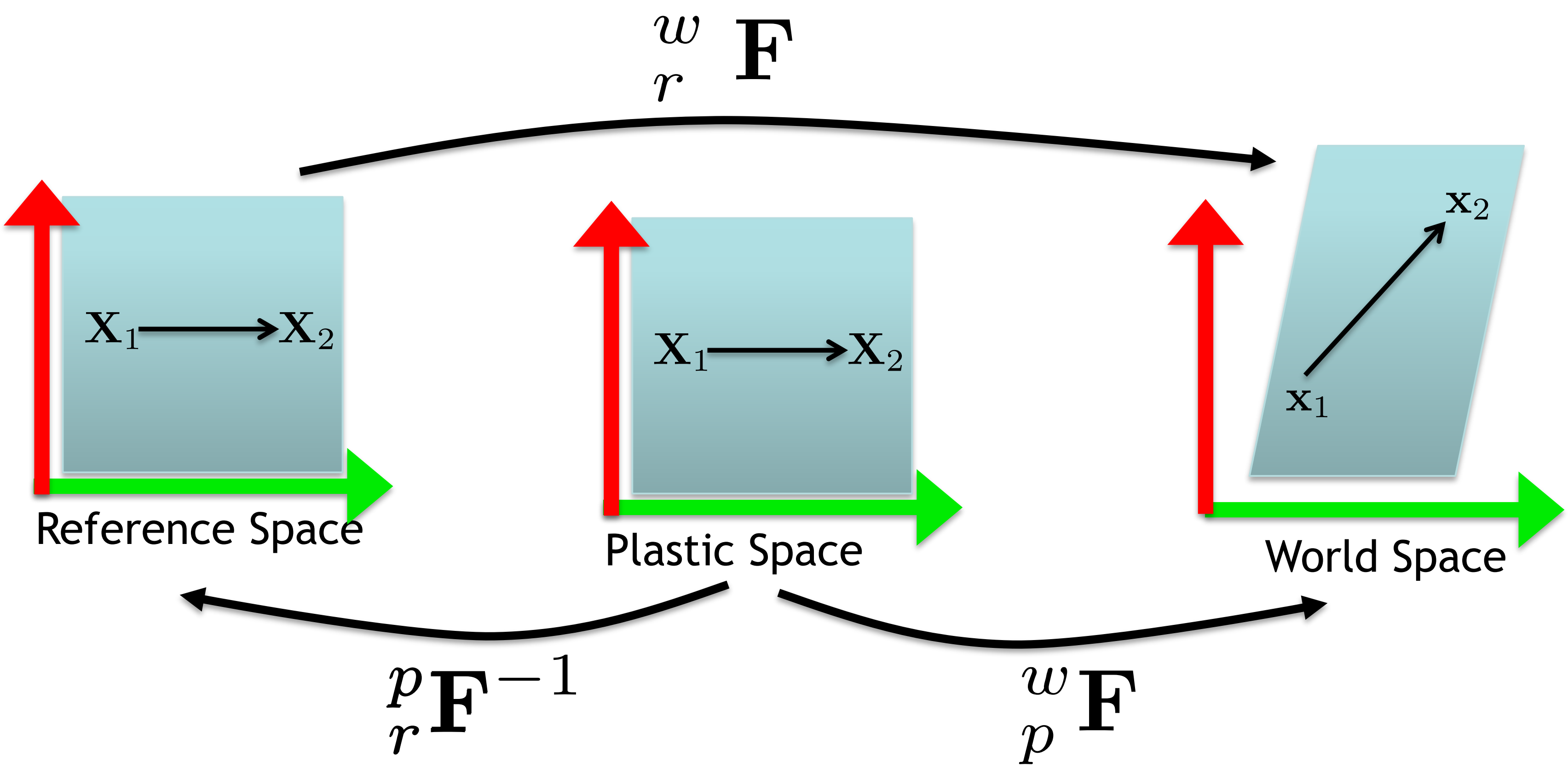
- Our goal is to approximate ${}^{w}_{p}\mathbf{F}$ but we only have access to ${}^{w}_{r}\mathbf{F}$



$$ {}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1} $$

Continuum Mechanics: Deformation

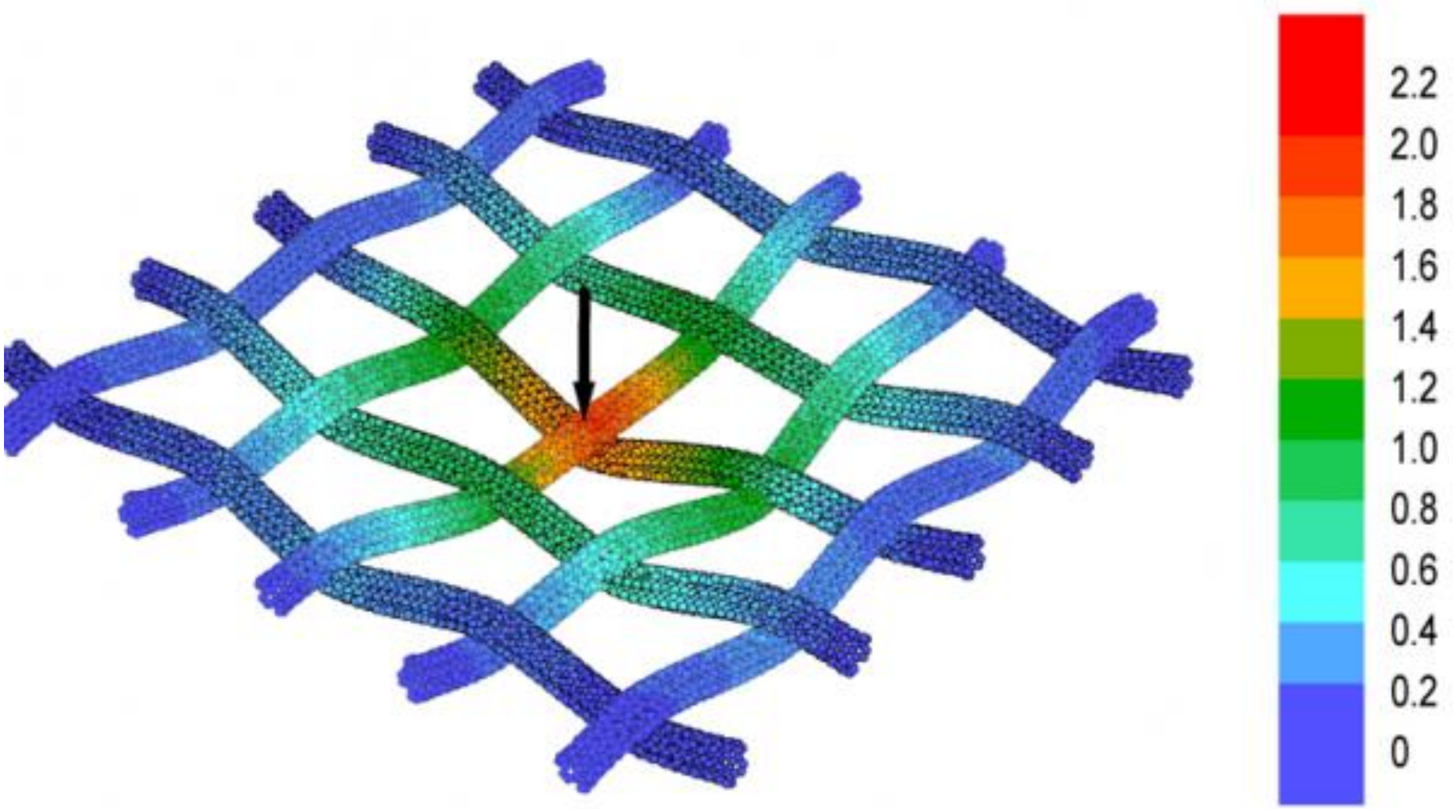
- We can store ${}^{p}_{r}\mathbf{F}^{-1}$ for each triangle in order to keep track of its plastic shape change



$$ {}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1} $$

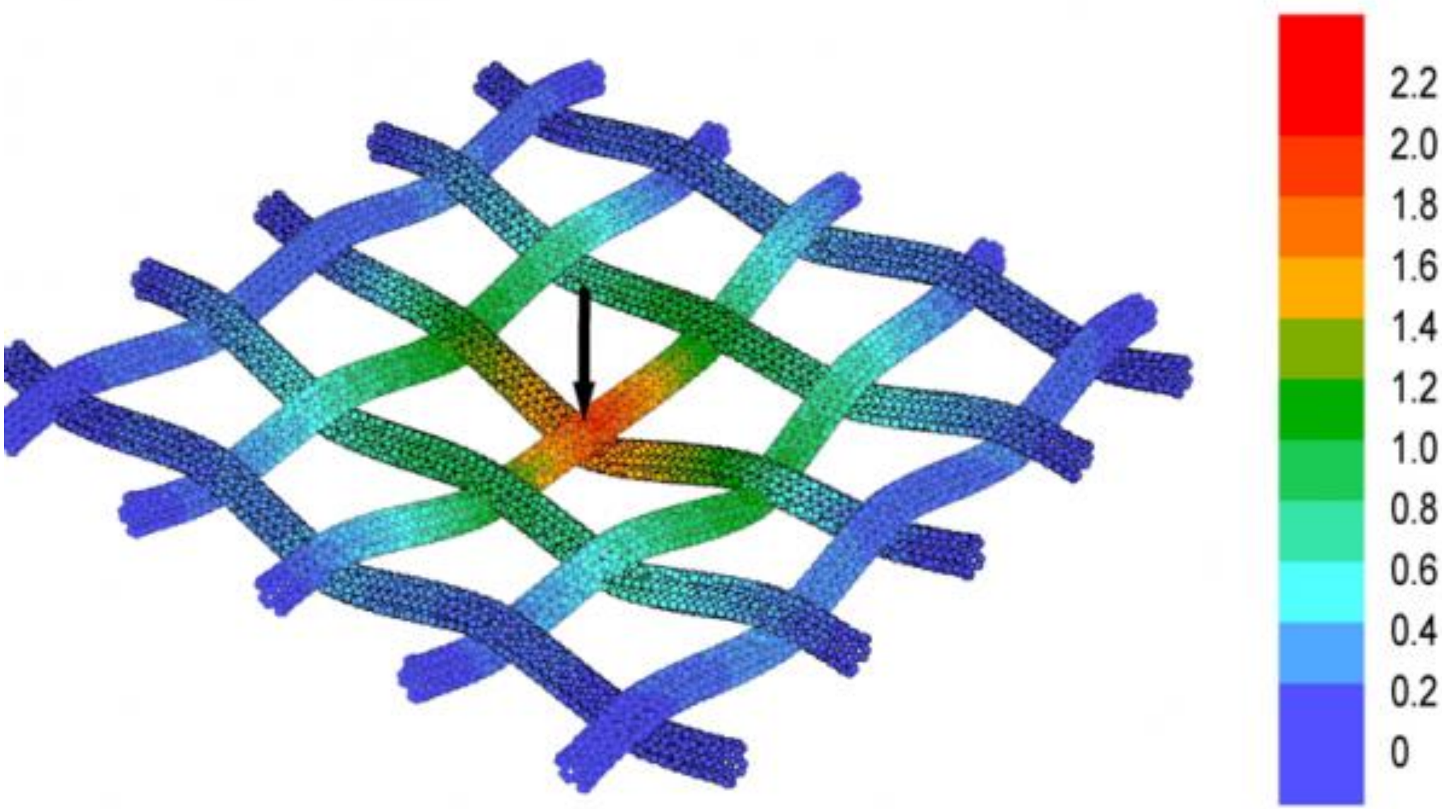
How to Compute the Plastic Deformation Gradient

- We compute the stress on each element during simulation



How to Compute the Plastic Deformation Gradient

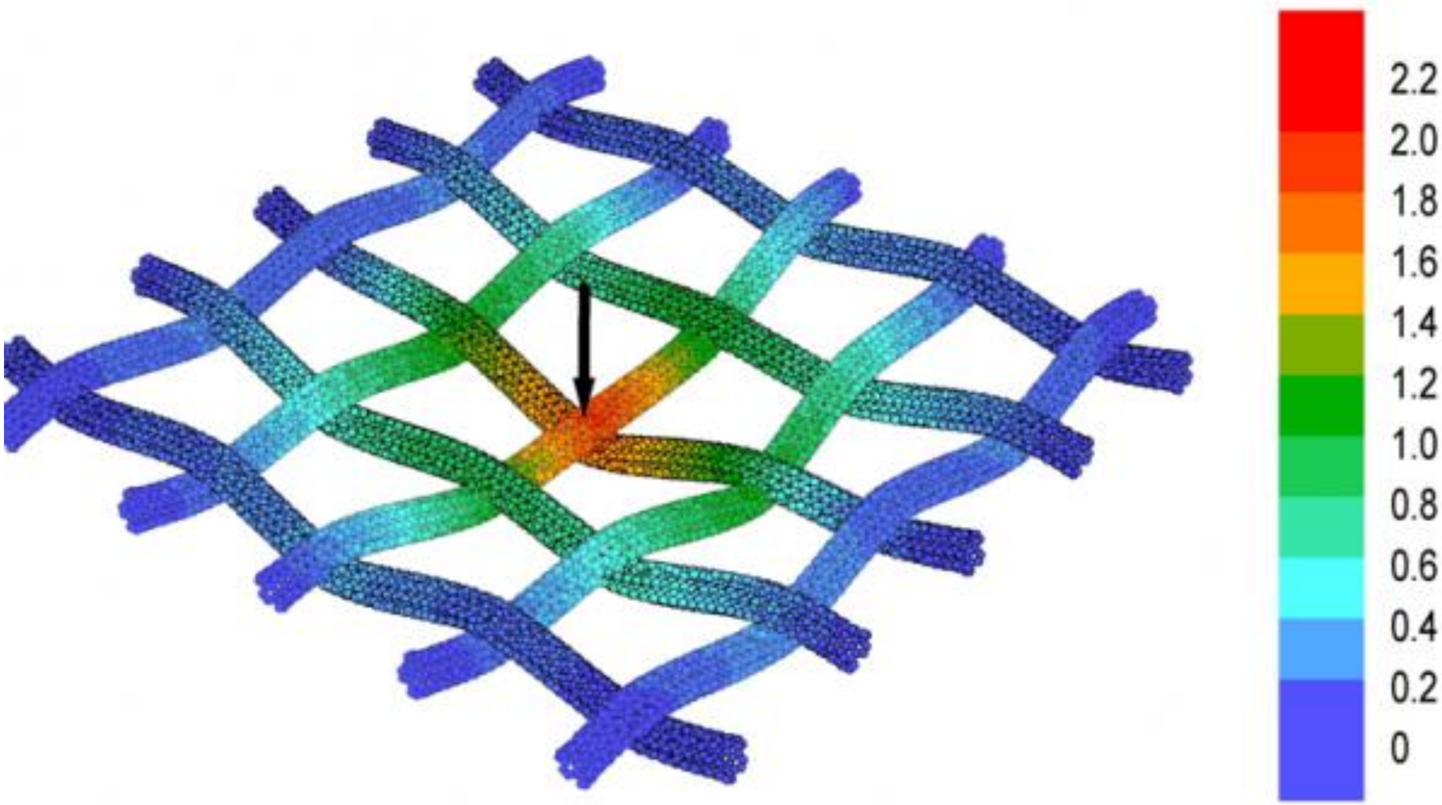
- When the stress in a triangles gets above a certain threshold we store **F** as ${}^{p}_{r}\mathbf{F}$



How to Compute the Plastic Deformation Gradient

- Each subsequent simulation step uses $\frac{1}{2}\left({}^{w}_{p}\mathbf{F}^{T}\,{}^{w}_{p}\mathbf{F} - \mathbf{I}\right)$

$${}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1}$$

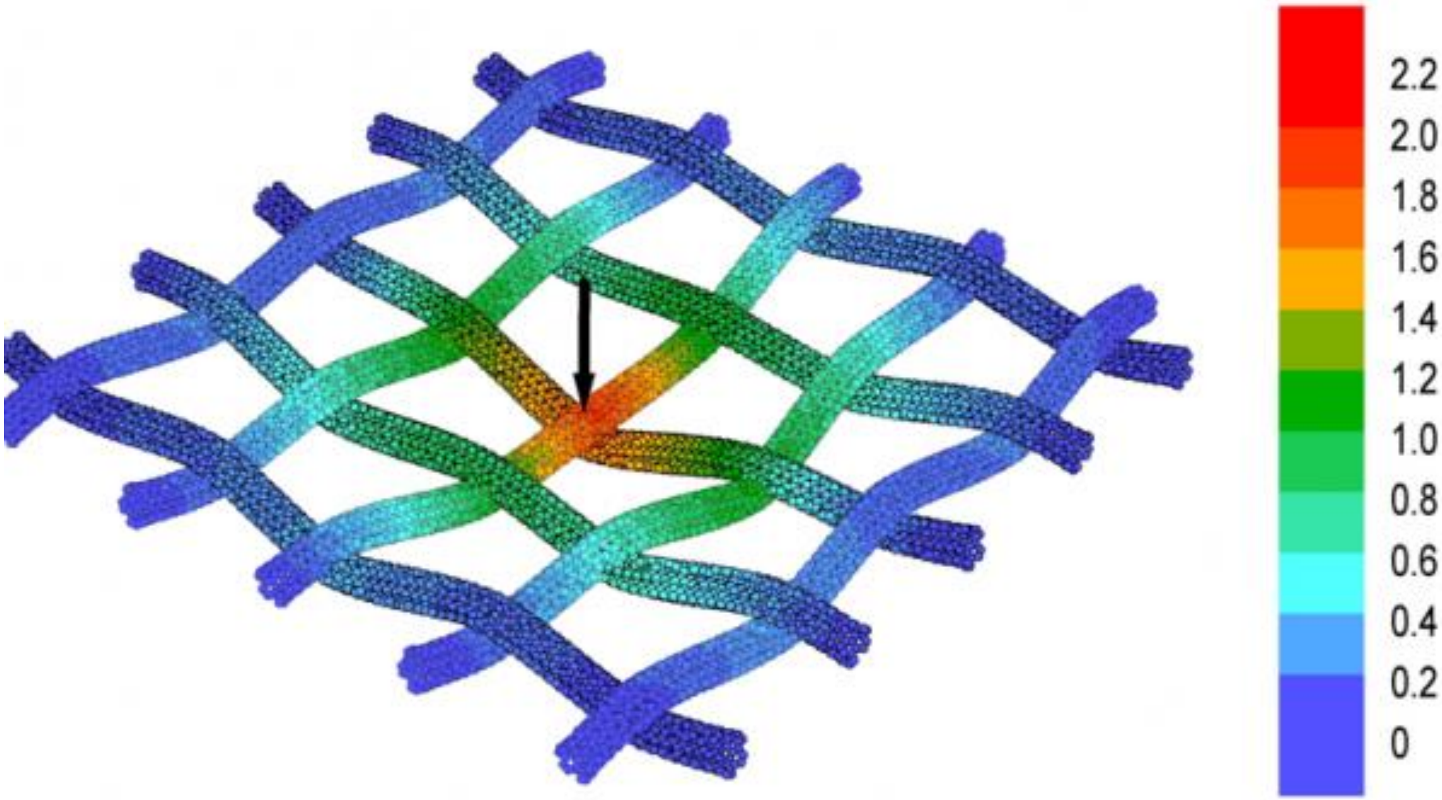


How to Compute the Plastic Deformation Gradient

- Each subsequent simulation step uses $\frac{1}{2}\left({}^{w}_{p}\mathbf{F}^{T}\,{}^{w}_{p}\mathbf{F} - \mathbf{I}\right)$

$${}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1}$$

- How do we decide on the threshold ?



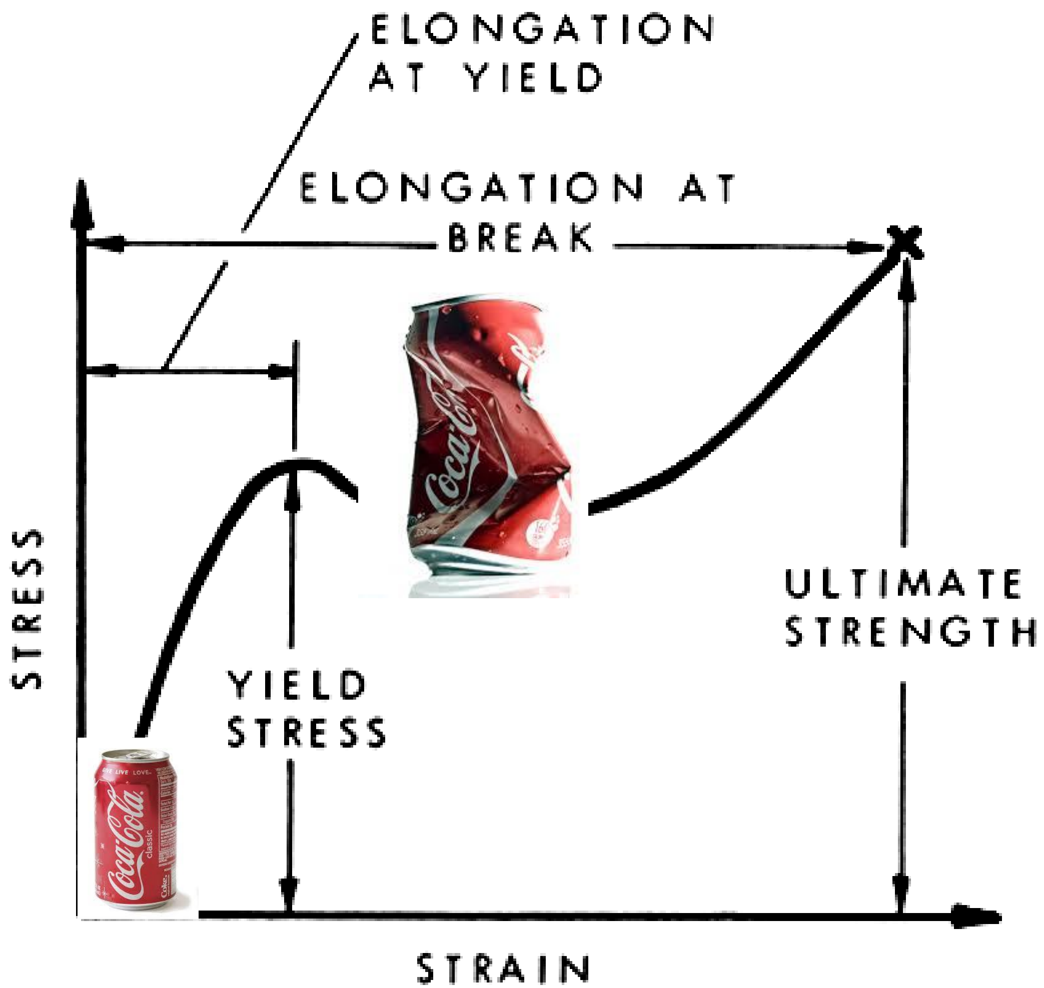
Measuring Plastic Materials

- We use a similar approach to elastic materials
- Except instead of a compression test, we use a tensile test
- We pull on the ends of the object then measure the strain induced

Measuring Plastic Materials

Other Interesting Material Properties

- Plasticity – Change in Reference State

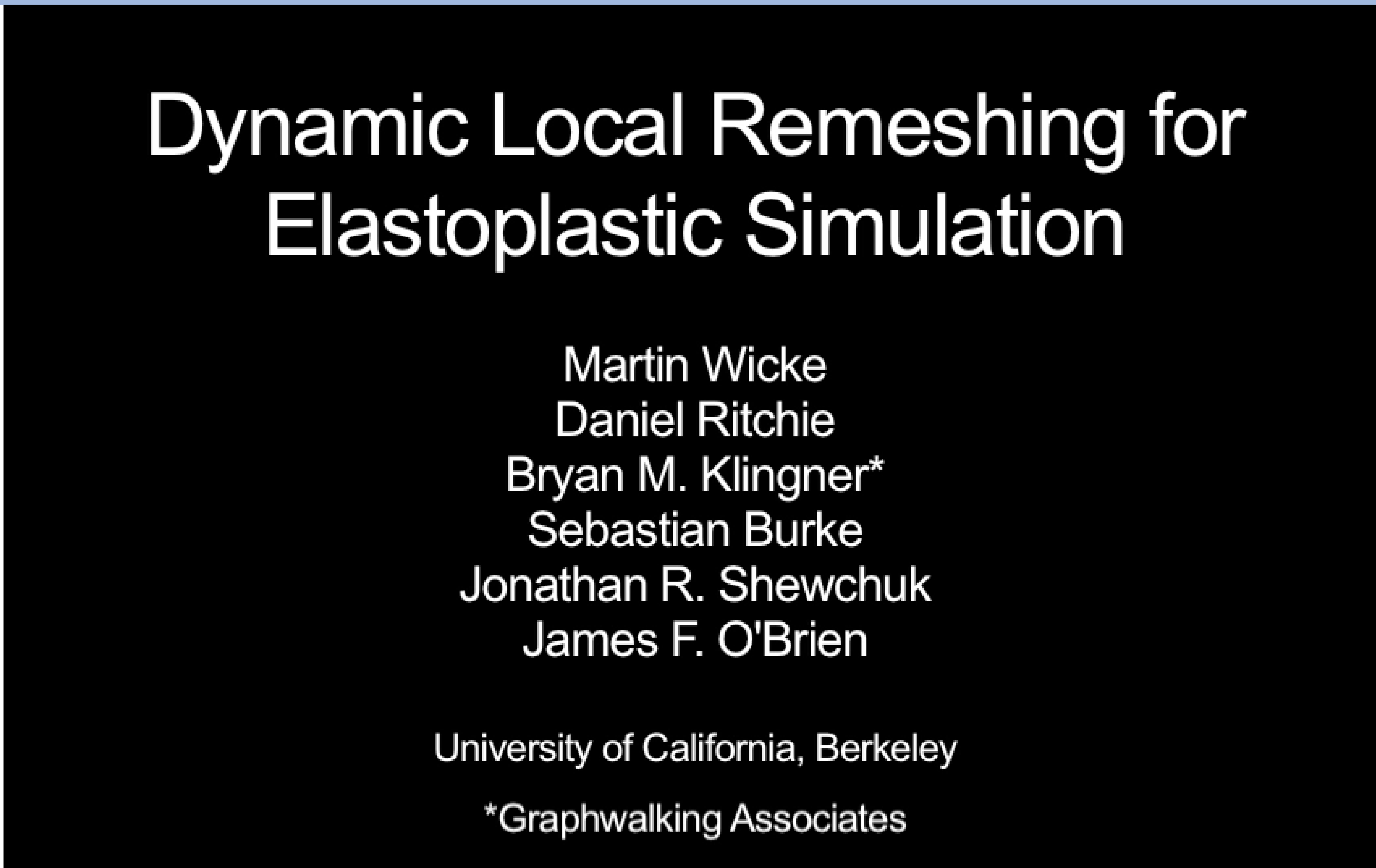


FEM with Plasticity

A Finite Element Method for
Animating Large Viscoplastic Flow

Adam W. Bargteil, CMU
Chris Wojtan, Georgia Tech
Jessica K. Hodgins, CMU
Greg Turk, Georgia Tech

Plasticity and Finite Elements



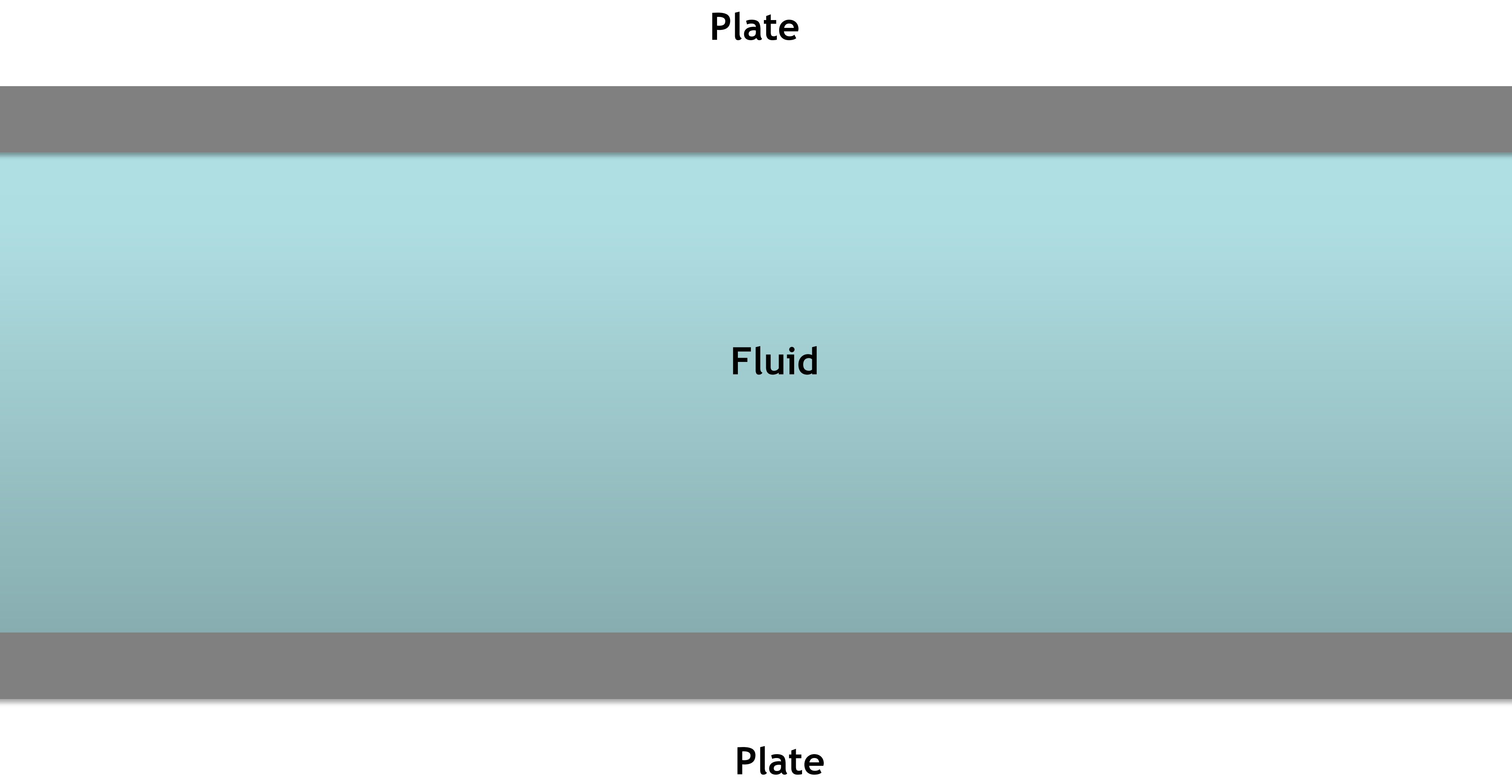
Types of Materials

- There are many types of materials
  - Elastic ← Done
  - Plastic ← Done
  - Viscous ← Briefly
  - Composites
  - Cellular Materials
  - Lattice Structures
- Each one has different mechanical properties
- When we fabricate things we exploit these properties to achieve optimal results

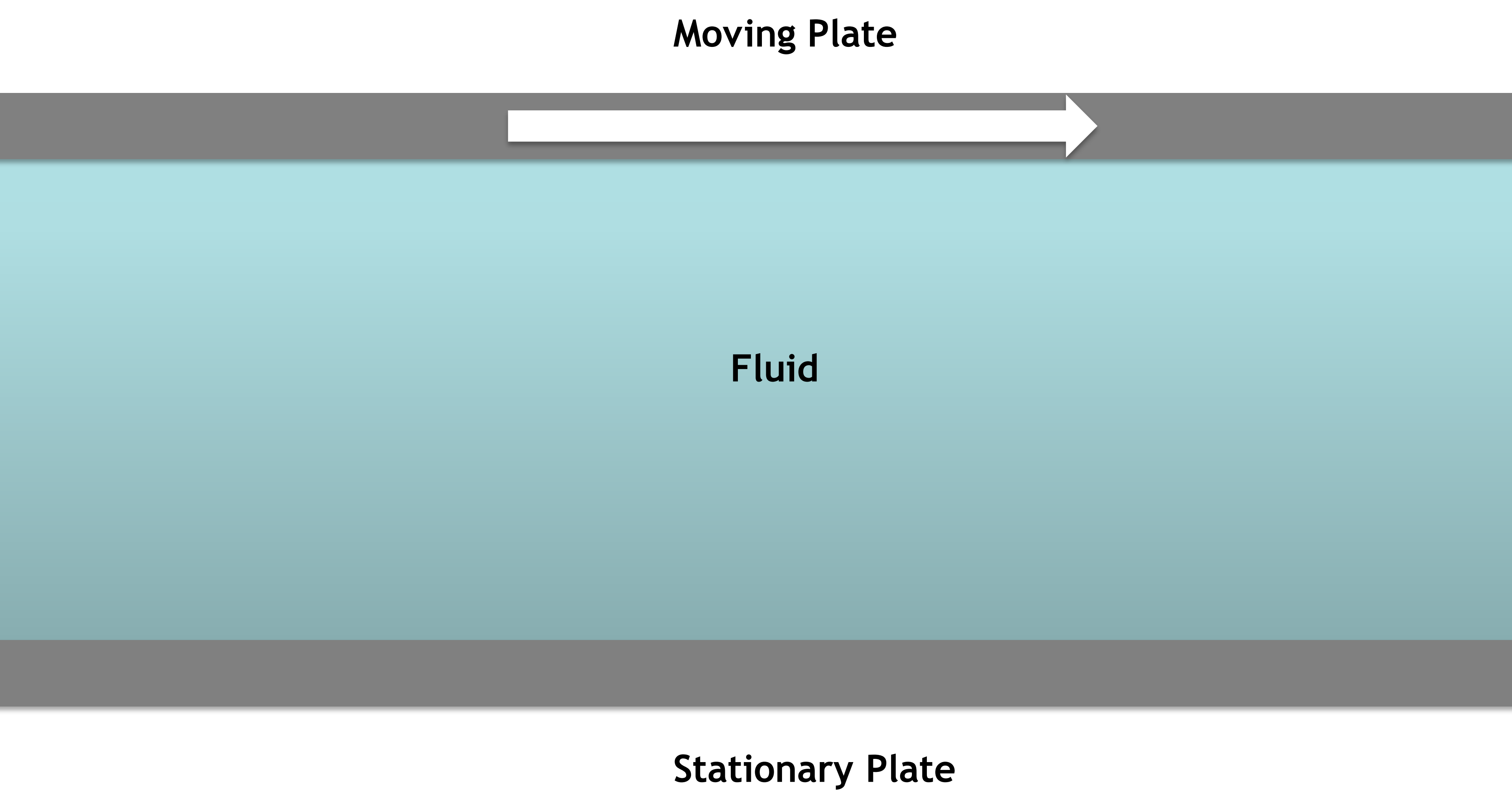
Viscous Materials

- Stress depends on strain rate (velocity) not strain
- Fluids are viscous materials
- The more viscous the material, the more it resists flowing

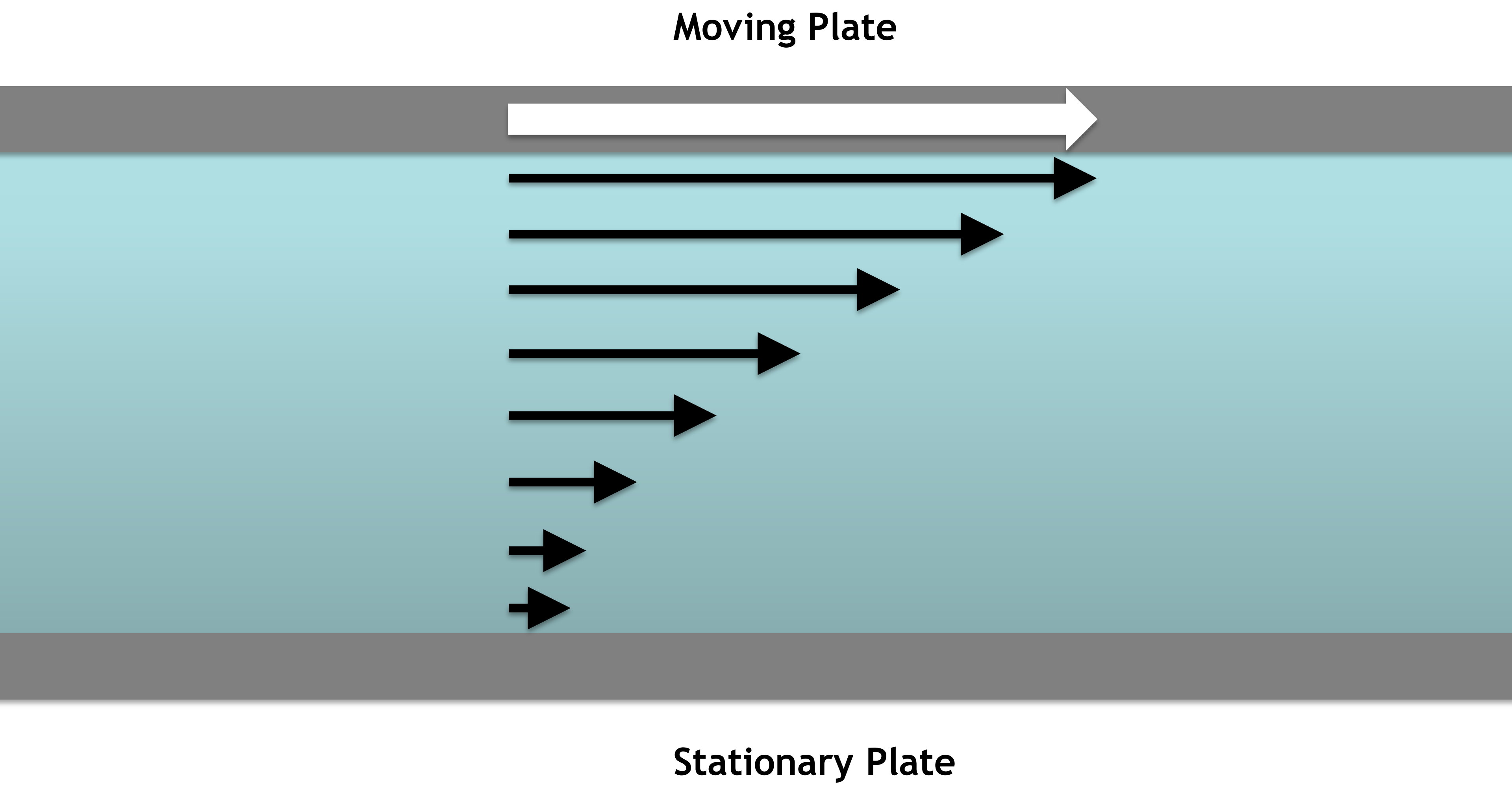
Shear Rate



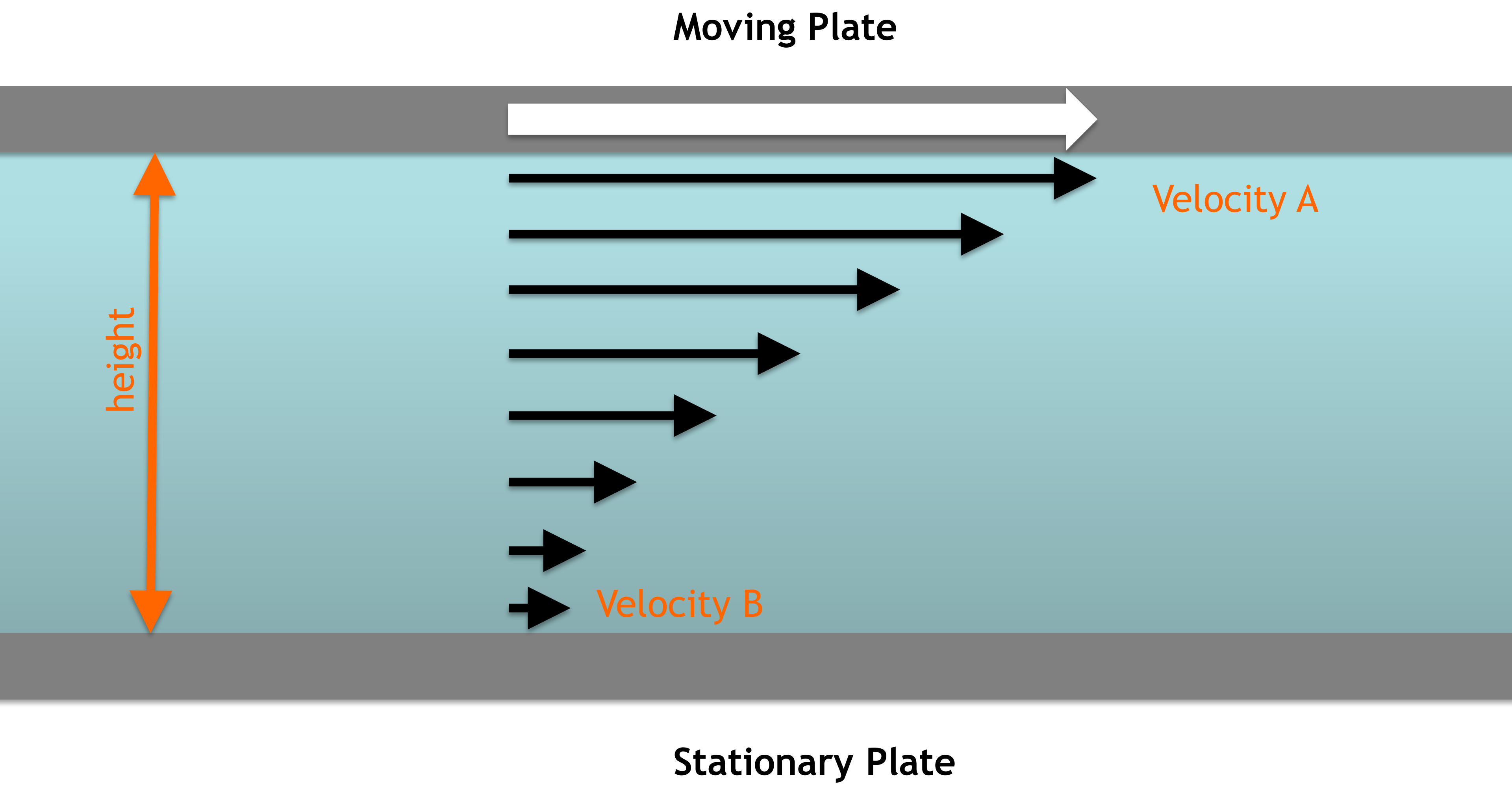
Shear Rate



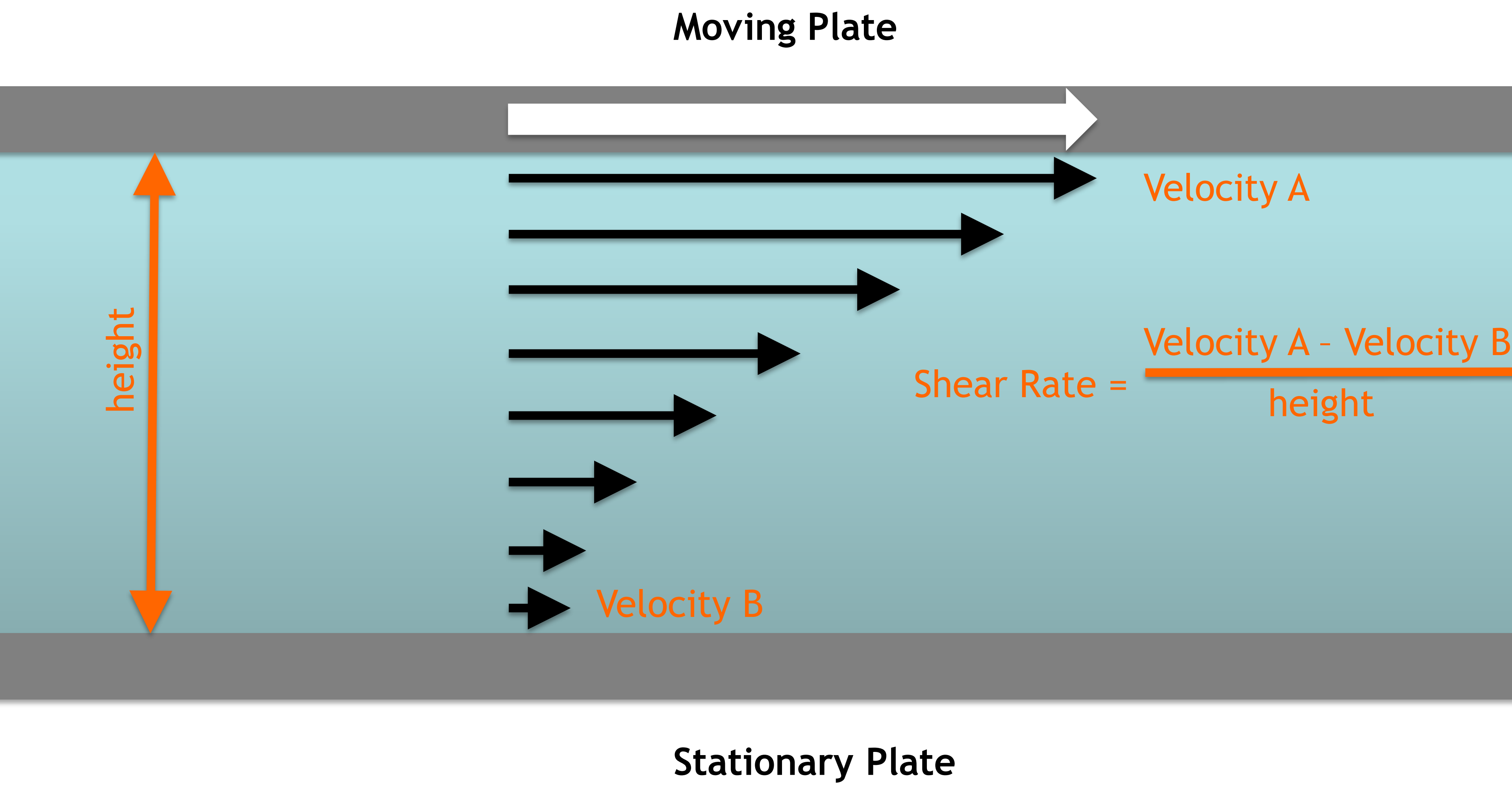
Shear Rate



Shear Rate


Moving Plate
Velocity A
height
Velocity B
Stationary Plate

Shear Rate



Moving Plate
Velocity A
height
Shear Rate = Velocity A - Velocity B / height
Velocity B
Stationary Plate

Viscous Materials

- Stress depends on strain rate (velocity) not strain
- Fluids are viscous materials
- The more viscous the material, the more it resists flowing

Viscous Materials

Not very viscous

Viscous

Modelling Viscous Materials

- Rayleigh Analogy: A viscous formulation derives from an elastic formulation when velocities replace positions and strain rates replace strains [Strutt 1945]

Material Models for Elastic Materials

- We've already seen one
- It's similar to a linear spring
- Linear Elasticity (Hooke's Law)

$$\sigma = \boxed{\mathbf{C}}\,\epsilon$$

Material
Tensor

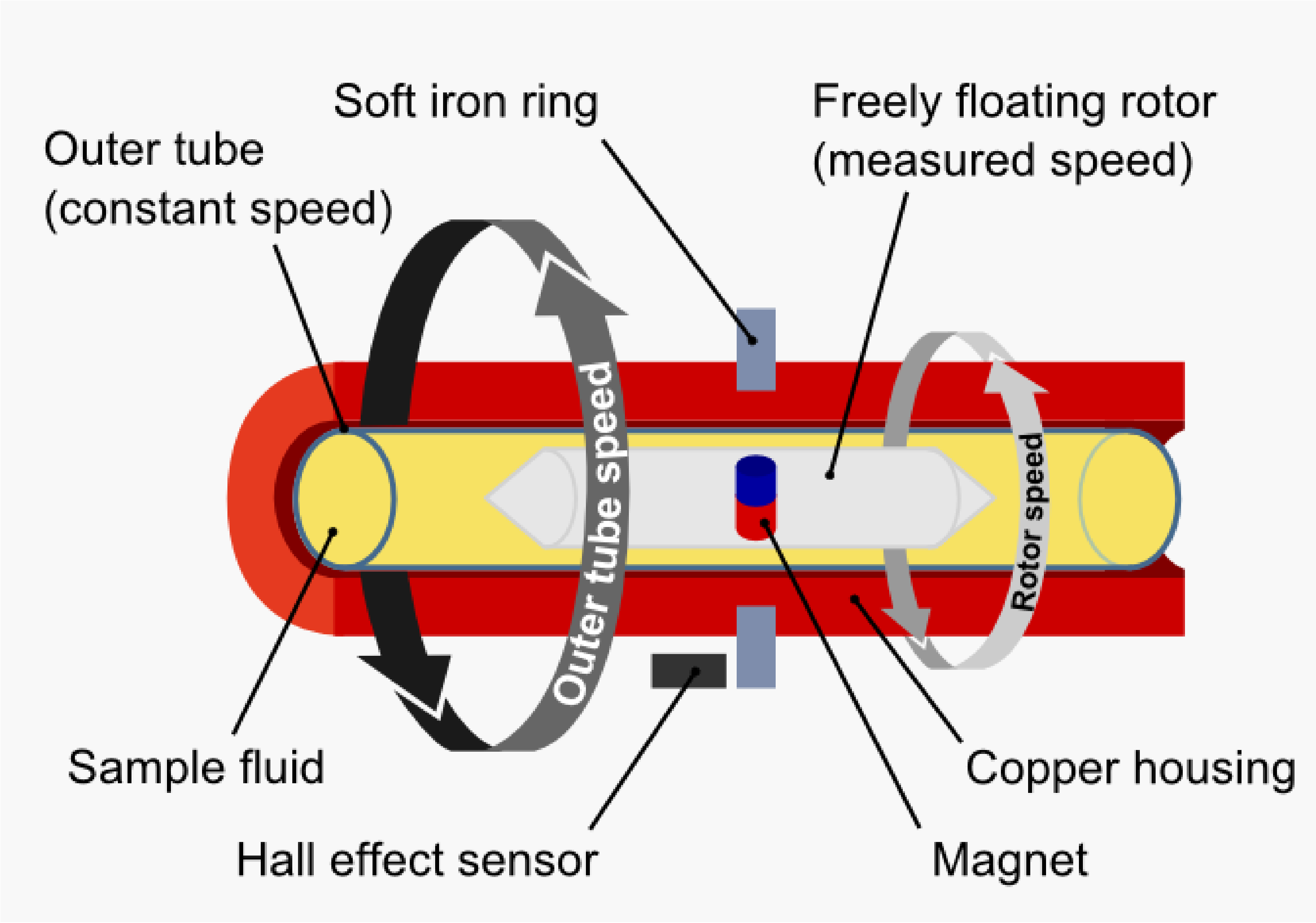
Material Models for Elastic Materials

- We've already seen one
- It's similar to a linear spring
- Linear Viscosity (Hooke's Law)

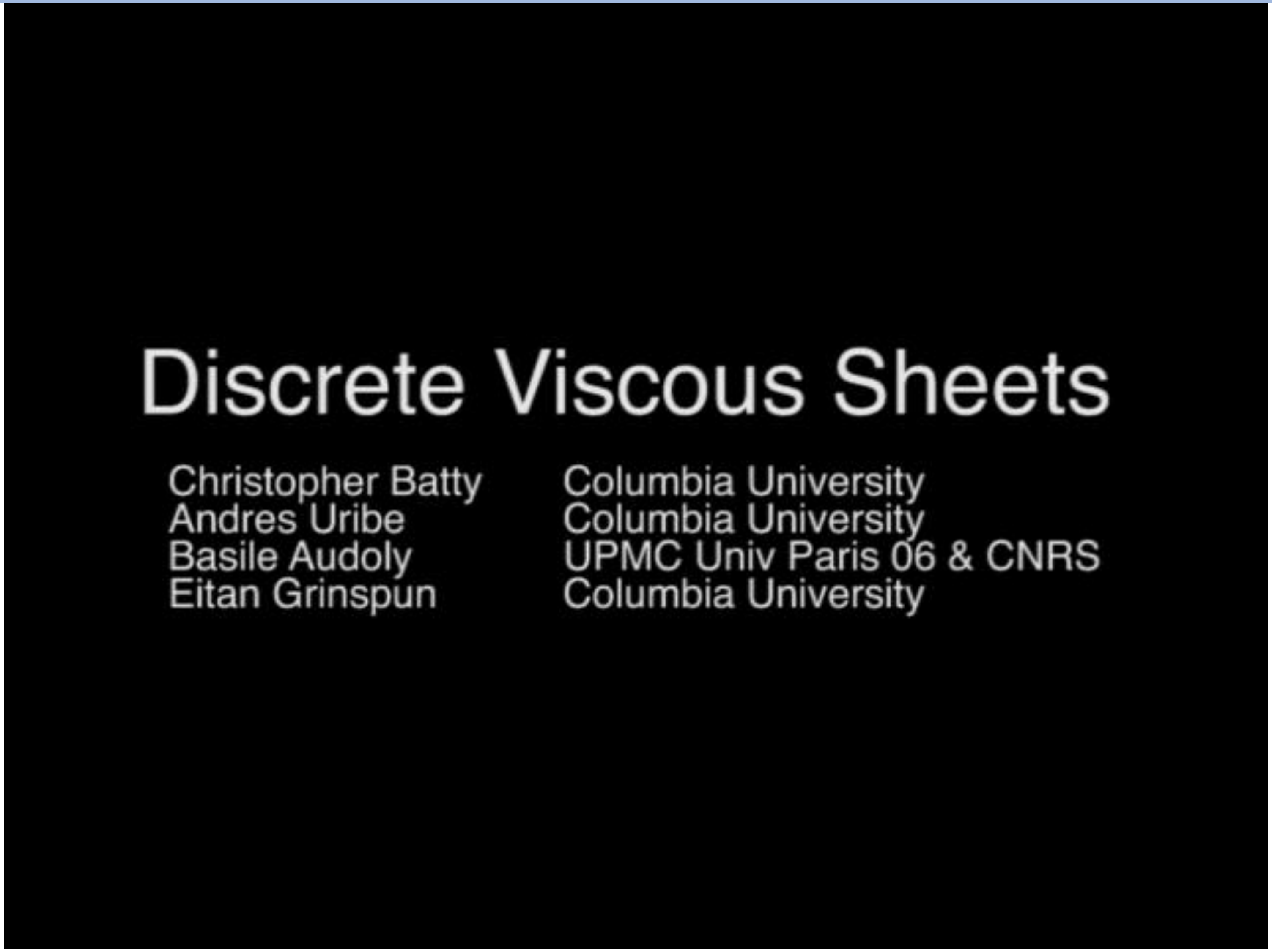
$$\sigma = \boxed{\mathbf{C}}\,\dot{\epsilon}$$

Material
Tensor

Measuring Viscosity



Finite Elements and Viscosity



Non-Newtonian Fluids

- Viscosity changes with shear rate

Non-Newtonian Fluids

Types of Materials

- There are many types of materials
  - Elastic ← Done
  - Plastic ← Done
  - Viscous ← Done
  - Composites
  - Cellular Materials
  - Lattice Structures

Next Lecture!

- Each one has different mechanical properties
- When we fabricate things we exploit these properties to achieve optimal results

Other intersting materials: Viscoelastic

"A Method for Animating Viscoelastic Fluids"

Tolga G. Goktekin

Adam W. Bargteil

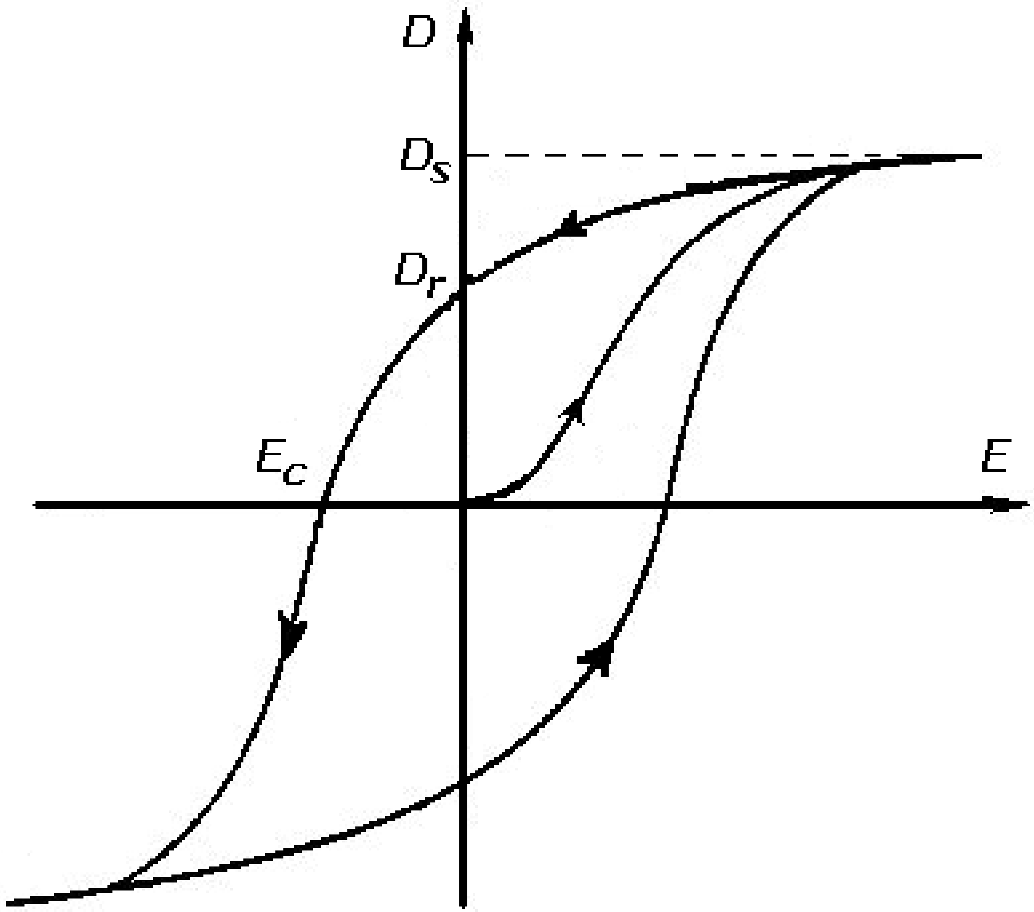
James F. O'Brien

ACM SIGGRAPH 2004

University of California, Berkeley

Other Interesting Material Properties

- Hysteresis



We're Done!

- You have now seen the following
  - Basic equations for continuum mechanics
  - The Finite Element Method
  - Different Material Models
  - How to Measure Parameters
  - How Typical FEM Software works

Additional Reading

- Continuum Mechanics
  - Mase and Mase
- Nonlinear Continuum Mechanics for Finite Element Analysis
  - Bonet and Wood