Introduction to the Finite Element Method

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Computational Fabrication (ISTD 01.110)

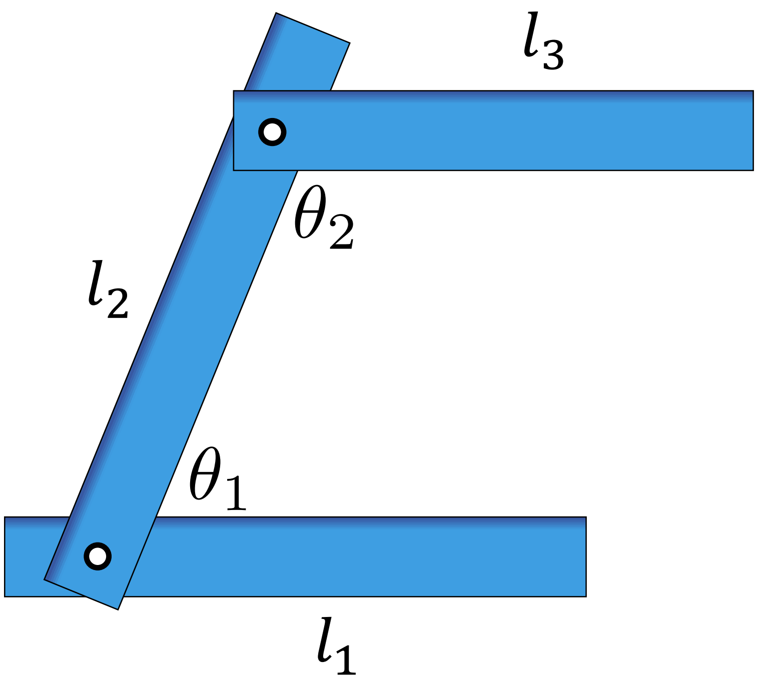
Monday, June 19, 2017

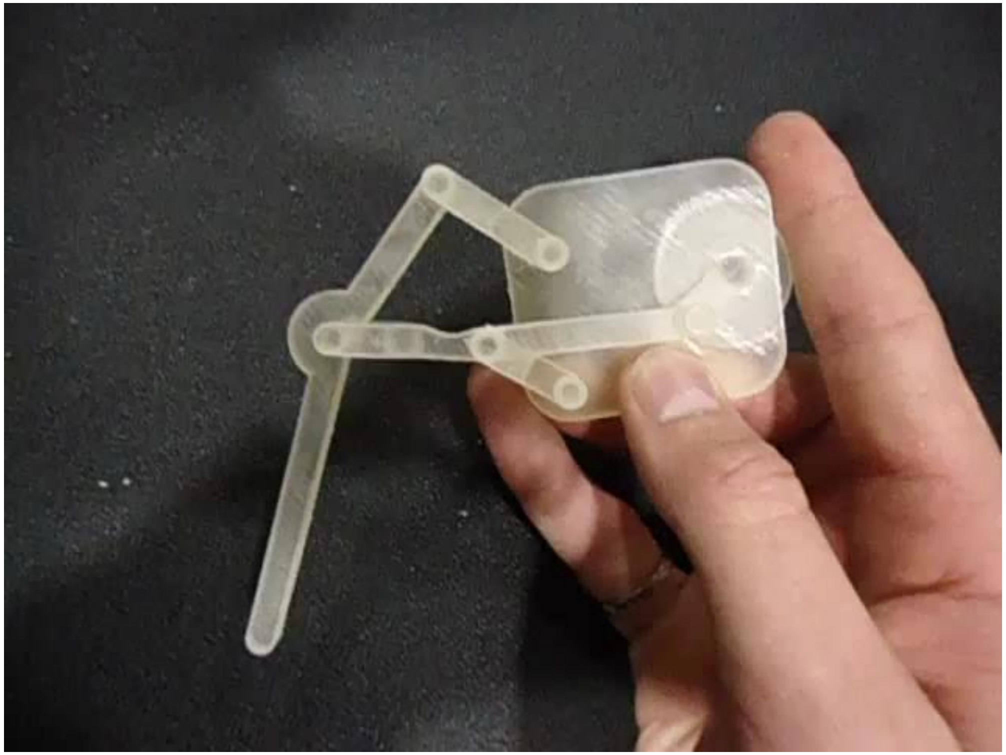
SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

SUTD Digital Manufacturing and Design Centre

Previous Lectures

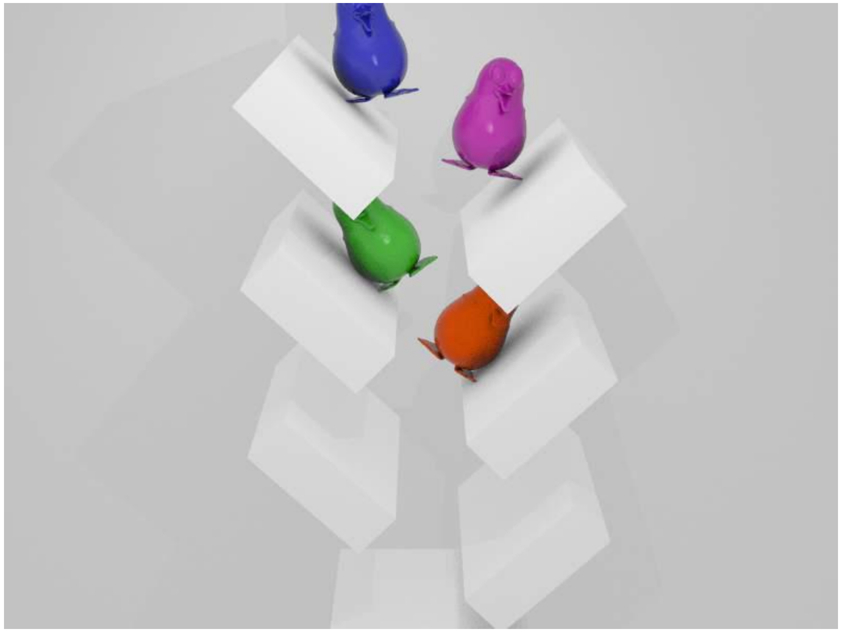
- Kinematics of Mechanisms
- Rigid body motions (links and joints)
- No forces and deformation





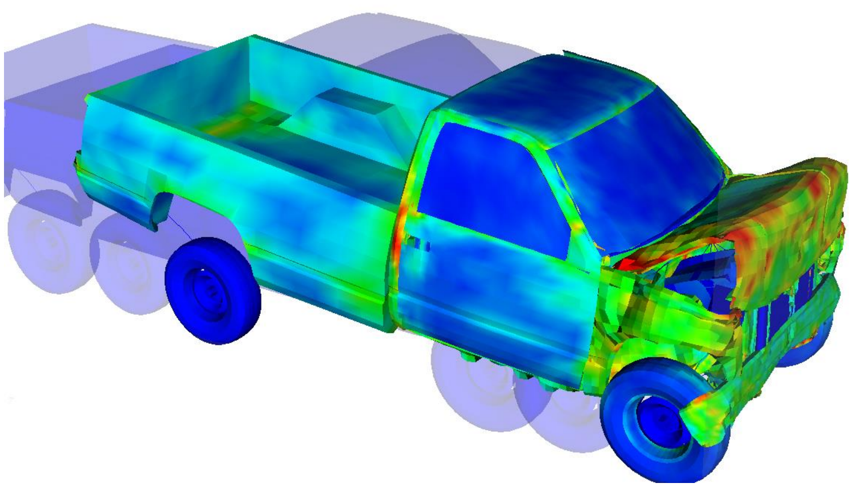
Today and next lectures

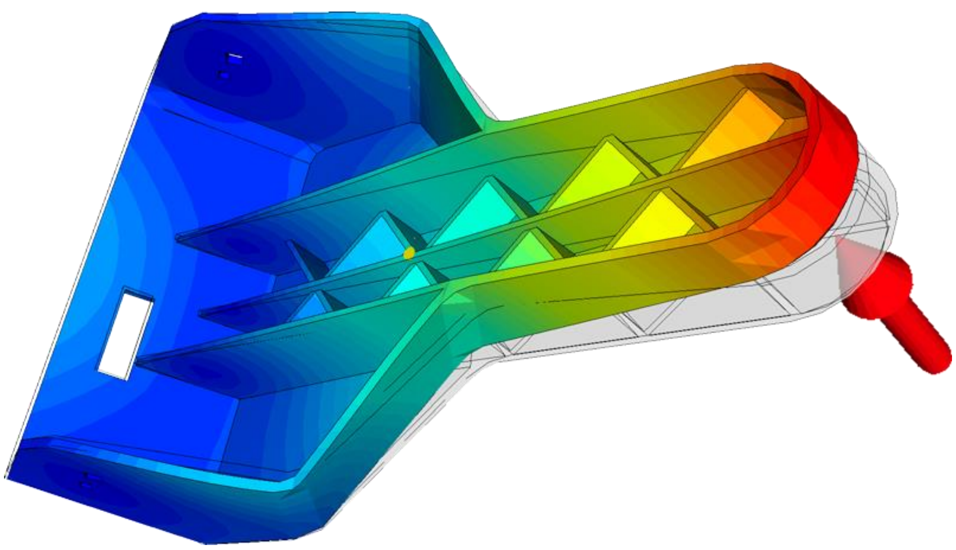
- Modelling of deformable objects and elastic deformations
- Basic introduction to structural and continuum mechanics
- Numerical discretization and computational simulation using the finite element method



Today’s agenda

- 1D mechanical modelling
  - Mechanics of a truss/bar
  - Finite element discretization of bar
  - Simple example calculations
- 3D continua
  - Continuum mechanics for infinitesimal deformations
  - Linear finite element method





1D mechanical modelling: truss/bar

- The truss or bar is the simplest structural component, characterized by two properties:
  - The bar has one preferred direction, the *longitudinal* or *axial* direction, which is much larger than the other two dimensions (*transverse* directions)
  - The bar resists an internal force along its longitudinal direction

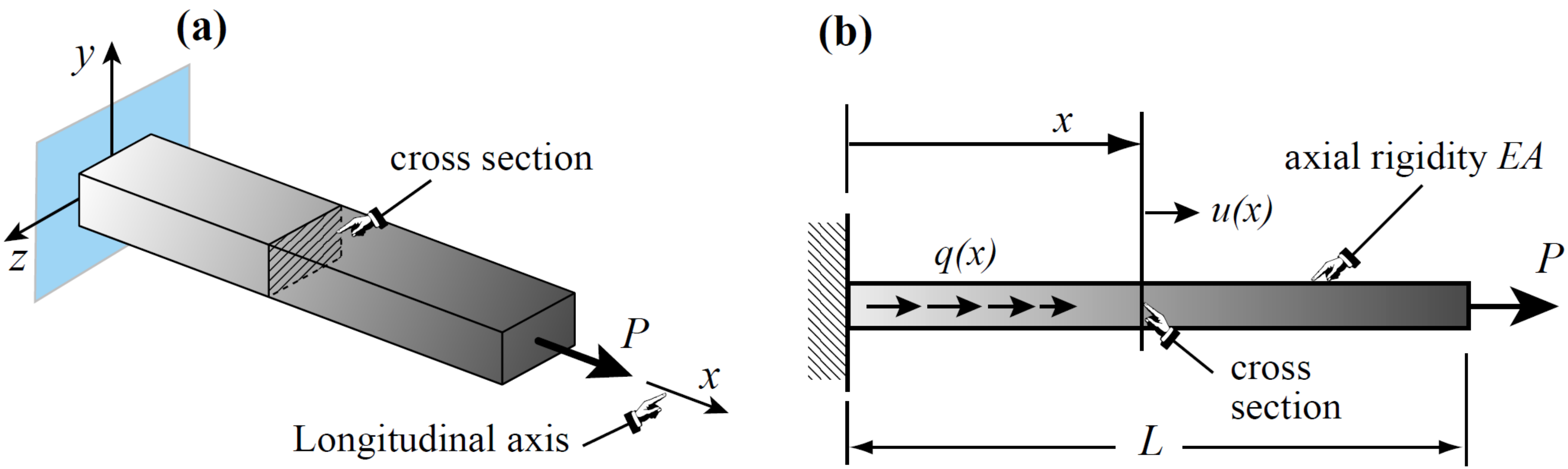
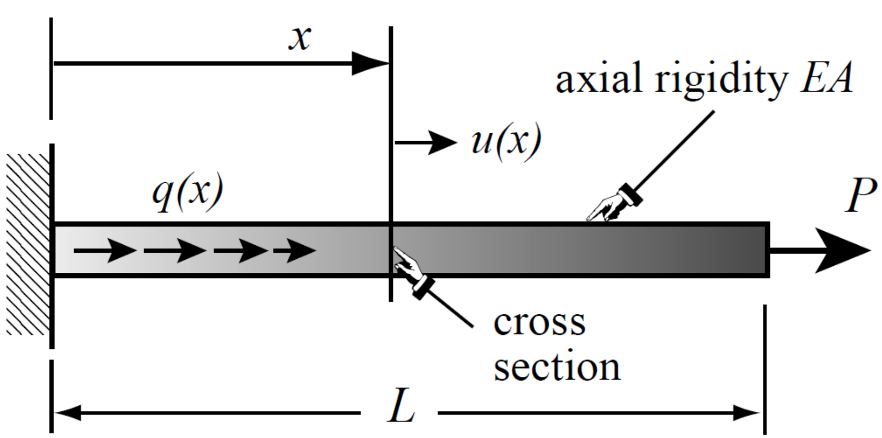


FIGURE 11.1. A fixed-free bar member: (a) 3D view showing reference frame; (b) 2D view on $\{x, y\}$ plane highlighting some quantities that are important in bar analysis.

Mechanical energy of bar

- Axial strain: $\varepsilon(x) = u'(x) = \lim\limits_{h \to 0} \frac{u(x+h) - u(x)}{h}$
- Axial stress: $\sigma(x) = E\,\varepsilon(x) = E\,u'(x)$
- Axial force: $f(x) = A\,\sigma(x) = EA\,u'(x)$
- Internal energy: $U = \frac{1}{2}\int_0^L A\,\sigma\,\varepsilon\,dx = \frac{1}{2}\int_o^L (EA\,u')\,u'dx$
- External energy: $V = -W = -\int_0^L q\,u\,dx - [P\,u]_0^L$
- Total potential energy: $\Pi[u] = U + V = U - W$



Parameters of bar:

L	Length of bar
E	Young's modulus (modulus of elasticity)
A	cross-section area
$q(x)$	External axial line load
P	External end force

Minimization of total potential energy

- Displacement function $u(x)$ must minimize the total potential energy $\Pi[u]$
- Variational calculus:

$$u \text{ minimizes } \Pi[u] \text{ if } \; d\Pi[\mathrm{u}] = 0$$

$$d\Pi = 0 \Leftrightarrow \; \frac{dF}{du} - \frac{d}{dx}\frac{dF}{du'} = 0 \text{ (Euler-Lagrange equation)}$$

where $F = \frac{1}{2} u' EA\, u' - qu$ is the integrand of $\Pi[u]$

$$\Leftrightarrow -q(x) - (EA\, u'(x))' = 0$$

- Strong (pointwise) differential equation of equilibrium of bar:

$$EA\, u''(x) + q(x) = 0 \quad \forall x \in (0, L)$$

Alternative derivation of equilibrium equation

- Balance of forces at an infinitesimal bar element Δx:

$$\sum F_x = 0 \quad \Leftrightarrow \quad A\sigma(x+\Delta x) - A\sigma(x) + q(x)\Delta x = 0$$

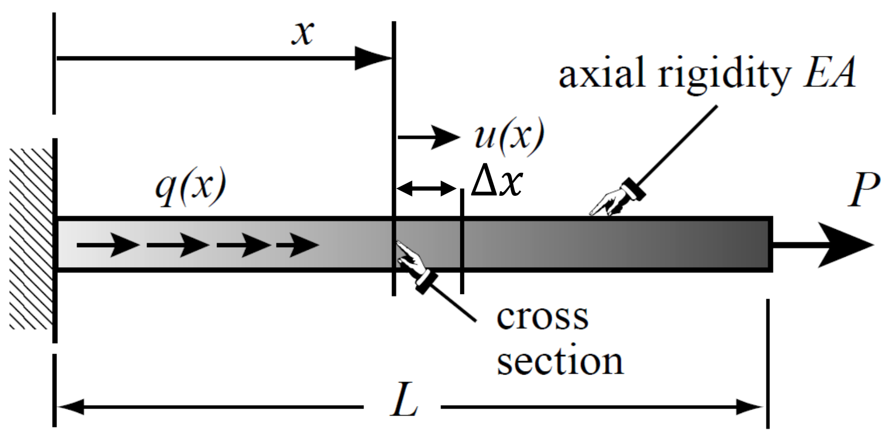
$$\Leftrightarrow \quad A\frac{\sigma(x+\Delta x) - \sigma(x)}{\Delta x} + q(x) = 0$$

- Limit for $\Delta x \rightarrow 0$:

$$A\sigma'(x) + q(x) = 0$$

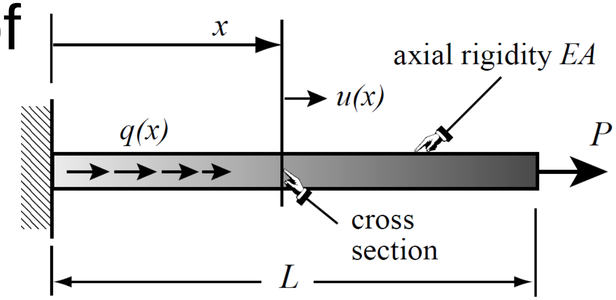
- With $\sigma = E\varepsilon = Eu'$:

$$EA\,u''(x) + q(x) = 0 \quad \forall x \in (0, L)$$



Analytical solution of equilibrium equation

- How does the deformation $u(x)$ of a bar of length $L = 1\ \mathrm{m}$ look like for $E = 0.5\ \mathrm{MPa}$, $A = 0.01\ \mathrm{m}^2$ and $q(x) = -1000\,\frac{\mathrm{N}}{\mathrm{m}}$?



- Integrate equilibrium equation:

$$EA\,u''(x) + q(x) = 0 \quad \Leftrightarrow \quad u''(x) = -q(x)/EA$$

$$\Rightarrow u'(x) = \int_0^x \frac{-q(s)}{EA}\,ds = \int_0^x \frac{1000\,\frac{\mathrm{N}}{\mathrm{m}}}{0.5 \cdot 10^6 \frac{\mathrm{N}}{\mathrm{m}^2} \cdot 0.01\ \mathrm{m}^2}\,ds = 0.2x\,\frac{1}{\mathrm{m}} + C_1$$

$$\Rightarrow u(x) = \int_0^x 0.2\,s\,\frac{1}{\mathrm{m}} + C_1\,ds = 0.1x^2\,\frac{1}{\mathrm{m}} + C_1 x + C_2$$

- We have 2 free variables ⇒ we need 2 <u>boundary conditions</u>:

- For instance:

$$u(0) = 0\ \mathrm{m} \Rightarrow C_2 = 0\ \mathrm{m}$$

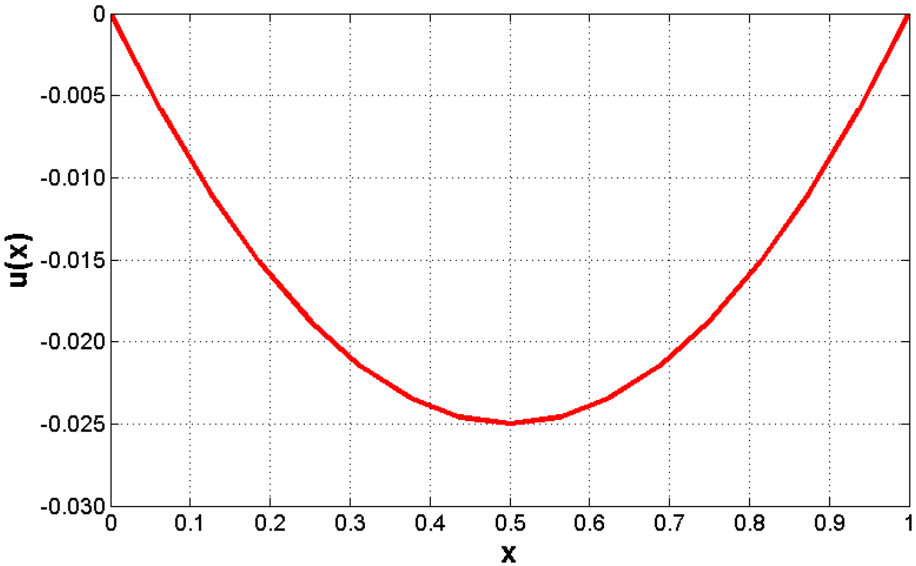
$$f(L) = P = 500\ \mathrm{N} = EA\,u'(L)$$

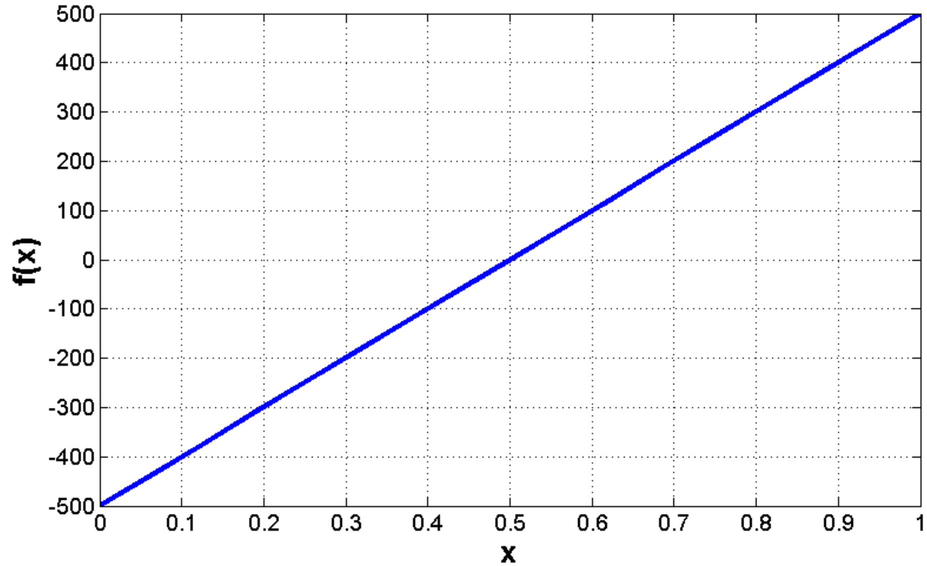
$$\Rightarrow C_1 = \frac{P}{EA} - 0.2 = -0.1$$

Analytical solution of equilibrium equation (cont'd)

- Altogether we have:

$$u(x) = 0.1x^2 \frac{1}{\mathrm{m}} - 0.1x$$
$$u'(x) = 0.2x\frac{1}{\mathrm{m}} - 0.1$$
$$u''(x) = 0.2\frac{1}{\mathrm{m}}$$
$$f(x) = EAu'(x) = 5000\ \mathrm{N}\left(0.2x\frac{1}{\mathrm{m}} - 0.1\right)$$





Stress-strain curve and stiffness

- Linear constitutive relationship:

$$\sigma = E\varepsilon$$

- Measurement of Young's modulus of a material or stiffness of a structure:
  - Steadily increase displacement ΔL which is applied to test sample/structure to increase strain:

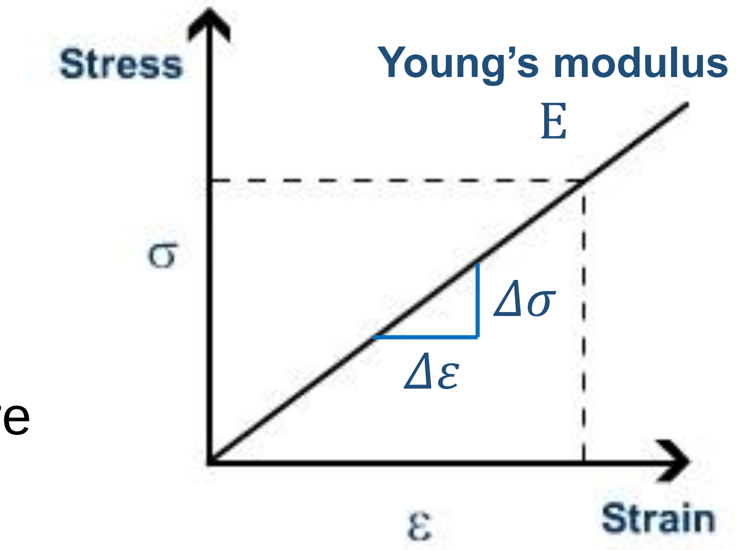
$$\varepsilon = u' \approx \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} = \frac{u}{L_0}$$

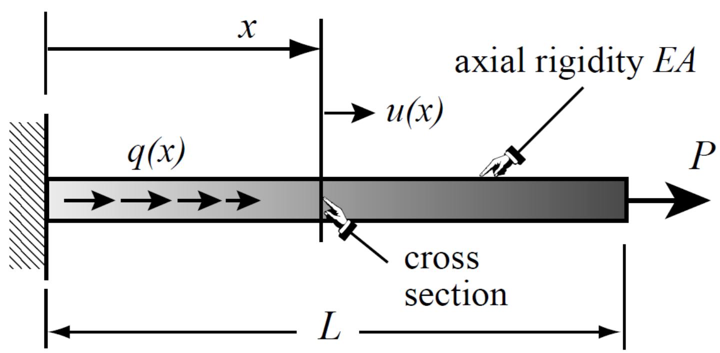
  - Measure response force $F(u)$ with increasing displacement
  - Use $F = A\sigma$ and $\sigma = E\varepsilon$ to determine E:

$$E = \frac{\sigma}{\varepsilon} = \frac{F}{A\varepsilon}$$

  - To avoid measurement errors:

$$E = \frac{\Delta\sigma}{\Delta\varepsilon}$$





Summary of bar problem

- The problem of analysing the mechanical deformation of a bar is given in terms of
  - a governing (ordinary) differential equation for the interior of the bar:
    $$EA\,u''(x) + q(x) = 0 \;\; \forall\, x \in (0, L)$$
  - and 2 boundary conditions for the ends of the bar:
    - Essential (Dirichlet) boundary conditions:
      $u(0) = \hat{u}_0$ or $u(L) = \hat{u}_L$
    - Natural (von Neumann) boundary conditions:
      $f(0) = P_0$ or $f(L) = P_L$
- The problem can be solved analytically by integrating the ODE and applying the boundary conditions
- Next, we want to determine an approximate, numerical solution using the finite element method

Numerical solution: method of weighted residuals

- We take the governing equation of the bar:

$$EA\, u''(x) + q(x) = 0 \quad \forall x \in (0, L),$$

- and multiply it with an arbitrary <u>test function</u> $w(x) \neq 0$, which fulfils the essential boundary conditions:

$$\Leftrightarrow \; w(x)\,(EA\, u''(x) + q(x)) \;\; \forall x \in (0, L).$$

- Then, we integrate the equation over the domain $(0, L)$:

$$\Leftrightarrow \; \int_0^L w(x)\,(EA\, u''(x) + q(x))dx = 0,$$

- and apply integration by parts to the first term:

$$\Leftrightarrow -\int_0^L w' EA\, u' dx + [w\; EA\; u']_0^L + \int_0^L w\; q\; dx = 0$$

$$\Leftrightarrow \int_0^L w' EA\, u' dx = \int_0^L w\; q\; dx + [w\; f]_0^L$$

Variational / weak formulation

Using the method of weighted residuals, from the strong form of the equilibrium equation:

$$EA\, u''(x) + q(x) = 0 \quad \forall x \in (0, L)$$

we have arrived at its equivalent weak or variational form:

$$\int_0^L w' EA\, u' dx = \int_0^L w\, q\, dx + [w\, P]_0^L \quad \forall w$$

Advantages of weak form:

- No second order differential term
- *Bilinear form* on left side is symmetric (w.r.t. u and w)
- Natural boundary conditions are already included

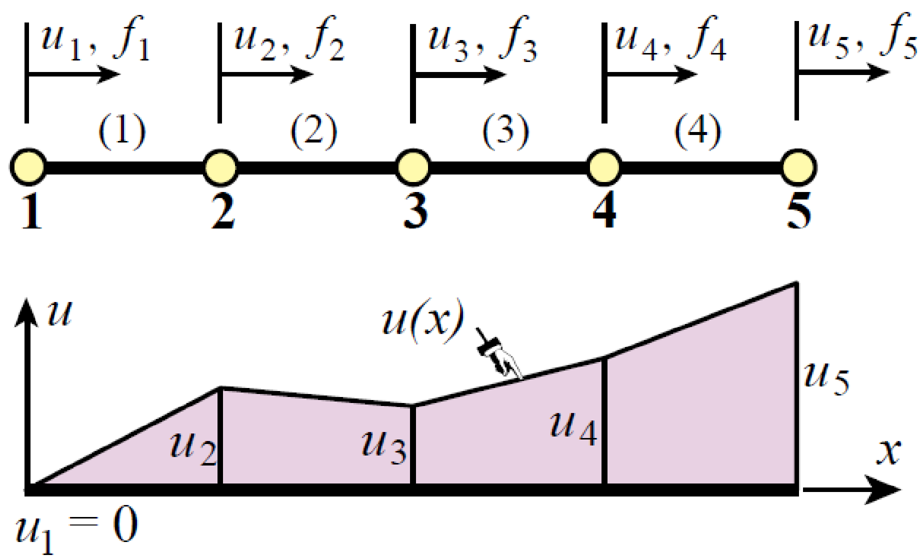
→ Use for numerical / finite element method

Finite element discretization of bar: idea

Idea: Discretize the bar into small elements and approximate the solution u of the weak form of the governing equation by piecewise linear functions u^h:

$$u(x) \approx u^h(x) = \sum_{i=1}^{n} N_i(x)\, u_i$$

u_i: nodal displacements
N_i: nodal basis functions

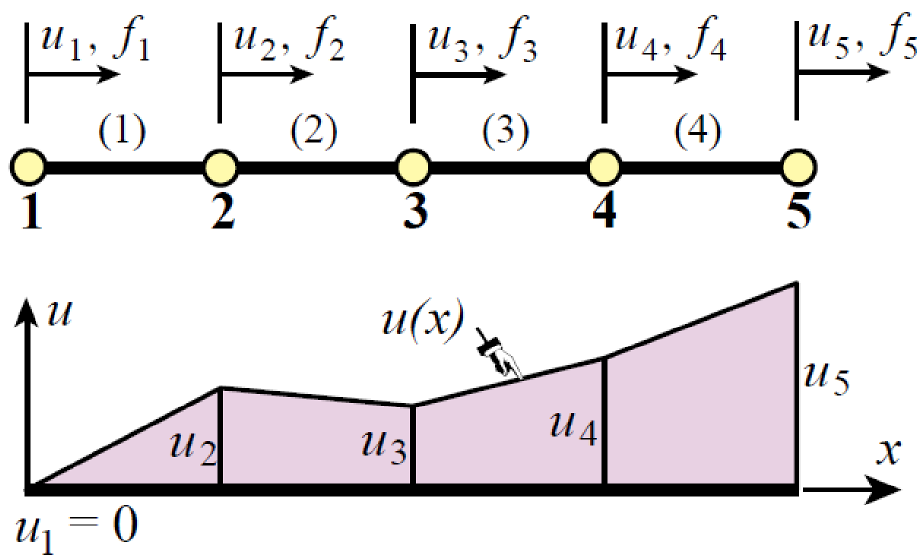


FEM for bar: discretization into finite elements

Discretization of domain:

$$\Omega = [0, L] = \bigcup_{e=1}^{\ell} \Omega^e = \bigcup_{e=1}^{\ell} [n_e, n_{e+1}],$$

$$\ell = n - 1, \qquad n_e = (e-1)\frac{L}{n}, \qquad L_e = n_{e+1} - n_e$$



FEM for bar: piecewise linear elements

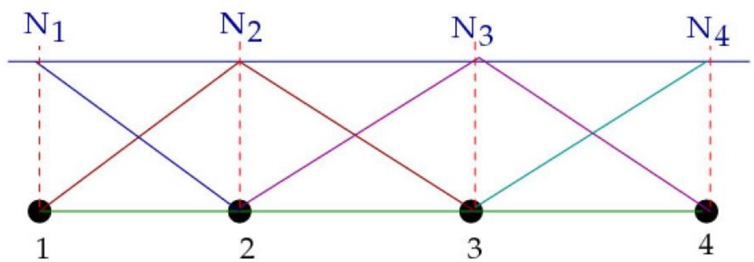
- Element-wise definition of solution:

$$u^e(x) = u^h(x)\Big|_{\Omega^e} = N_1^e(\bar{x}^e)\; u_e + N_2^e(\bar{x}^e)\; u_{e+1}$$

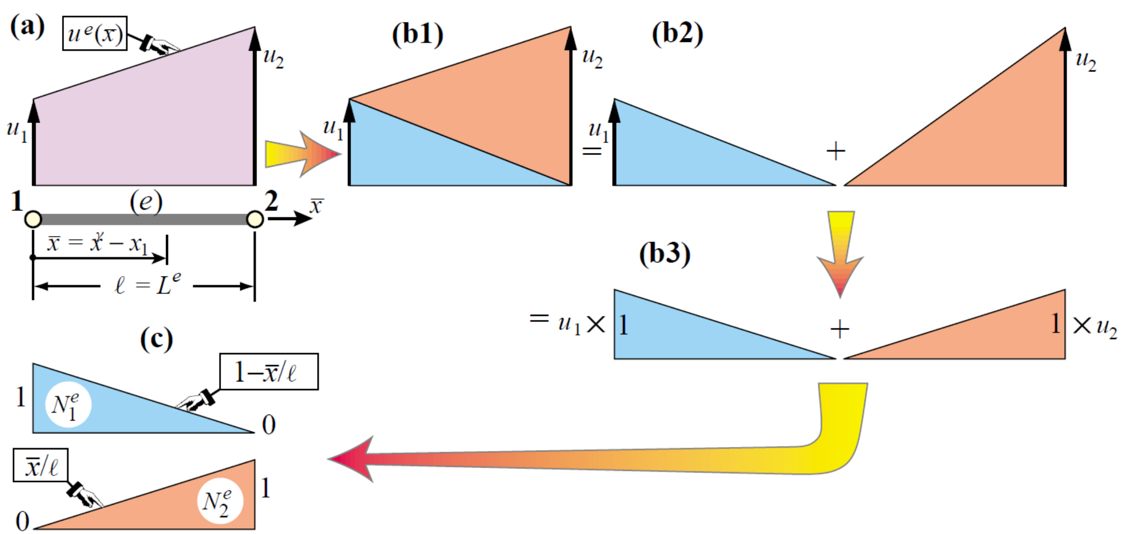
where $\bar{x}^e = \frac{x - n_e}{L_e}$ and $N_1^e(\bar{x}^e) = 1 - \bar{x}^e,\; N_2^e(\bar{x}^e) = \bar{x}^e$.

- Element-wise definition of basis functions:

$$N_i(x) = \begin{cases} N_1^e(\bar{x}^e), & x \in \Omega_{i-1} \\ N_2^e(\bar{x}^e), & x \in \Omega_i \\ 0, & else \end{cases}$$



- Analogous: element-wise def. of test functions $w^e(\bar{x}^e)$



FEM for bar: element-wise integration

- Element-wise partition of the weak form of the governing equation:

$$\int_0^L w' EA\, u' dx = \int_0^L w\, q\, dx + [w\, P]_0^L \quad \forall w$$

$$\Leftrightarrow \qquad \sum_{e=1}^{\ell} \int_{\Omega_e} w^{e\prime} EA\, u^{e\prime} dx = \sum_{e=1}^{\ell} \int_{\Omega_e} w^e\, q\, dx + [w\, P]_0^L$$

FEM for bar: element-wise integral contributions

- Evaluate the integrals separately for each element Ω^e:
  Right-hand side:

$$\int_{n_e}^{n_{e+1}} w^{e\prime} EA\, u^{e\prime} dx =$$

$$= EA \int_0^1 \frac{1}{L_e}(-w_e + w_{e+1}) \frac{1}{L_e}(-u_e + u_{e+1})\, L_e\, d\bar{x}$$

$$= \frac{EA}{L_e} \int_0^1 (w_e, \quad w_{e+1}) \begin{pmatrix} -1 \\ 1 \end{pmatrix} (-1, \quad 1) \begin{pmatrix} u_e \\ u_{e+1} \end{pmatrix} d\bar{x}$$

$$= (w_e, \quad w_{e+1}) \frac{EA}{L_e} \int_0^1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} d\bar{x} \begin{pmatrix} u_e \\ u_{e+1} \end{pmatrix}$$

$$= (w_e, \quad w_{e+1}) \frac{EA}{L_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_e \\ u_{e+1} \end{pmatrix}$$

$$= (w_e, \quad w_{e+1})\ \mathbf{K}_e \begin{pmatrix} u_e \\ u_{e+1} \end{pmatrix}$$

- Element stiffness matrix: $\mathbf{K}_e = \frac{EA}{L_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

FEM for bar: element-wise integral contributions

- Evaluate the integrals separately for each element Ω^e:
  Left-hand side:

$$\int_{n_e}^{n_{e+1}} w^e \; q \; dx =$$

$$= \int_0^1 ((1-\bar{x})w_e + \bar{x}\, w_{e+1})\,((1-\bar{x})q_e + \bar{x}\, q_{e+1})\, L_e \, d\bar{x}$$

$$= L_e \int_0^1 (w_e, \quad w_{e+1}) \begin{pmatrix} 1-\bar{x} \\ \bar{x} \end{pmatrix} (1-\bar{x}, \quad \bar{x}) \begin{pmatrix} q_e \\ q_{e+1} \end{pmatrix} d\bar{x}$$

$$= (w_e, \quad w_{e+1}) L_e \int_0^1 \begin{pmatrix} (1-\bar{x})^2 & (1-\bar{x})\bar{x} \\ (1-\bar{x})\bar{x} & \bar{x}^2 \end{pmatrix} d\bar{x} \begin{pmatrix} q_e \\ q_{e+1} \end{pmatrix}$$

$$= (w_e, \quad w_{e+1}) \frac{L_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_e \\ q_{e+1} \end{pmatrix}$$

$$= (w_e, \quad w_{e+1})\, \mathbf{b}_e$$

- Element force vector: $\mathbf{b}_e = \frac{L_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_e \\ q_{e+1} \end{pmatrix}$

FEM for bar: sum of element-wise contributions

- Element-wise partition of the weak form:

$$\sum_{e=1}^{\ell} \int_{\Omega_e} w^{e\prime} EA\, u^{e\prime} dx = \sum_{e=1}^{\ell} \int_{\Omega_e} w^{e}\, q\, dx + [w\, P]_0^L$$

$$\Leftrightarrow \quad \sum_{e=1}^{\ell} (w_e, \quad w_{e+1})\ \mathbf{K}_e \begin{pmatrix} u_e \\ u_{e+1} \end{pmatrix} =$$

$$= \quad \sum_{e=1}^{\ell} (w_e, \quad w_{e+1})\, \mathbf{b}_e - w_1 P_0 + w_n P_L$$

FEM for bar: assembly

- Assemble all element-wise contributions into matrix-vector notation:

$$
\begin{pmatrix} w_1 & w_2 & w_3 \cdots w_{n-1} & w_n \end{pmatrix} \frac{EA}{L_e} \begin{pmatrix} 1 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} =
$$

$\mathbf{K}_e$

$$
\begin{pmatrix} w_1 & w_2 & w_3 \cdots w_{n-1} & w_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}
$$

$\mathbf{b}_e$

FEM for bar: assembly

- Assemble all element-wise contributions into matrix-vector notation and apply essential boundary conditions:

$$u(0) = 0$$

$$\begin{pmatrix} w_1 & w_2 & w_3 \cdots w_{n-1} & w_n \end{pmatrix} \frac{EA}{L_e} \begin{pmatrix} 1 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} =$$

$$\mathbf{K}_e$$

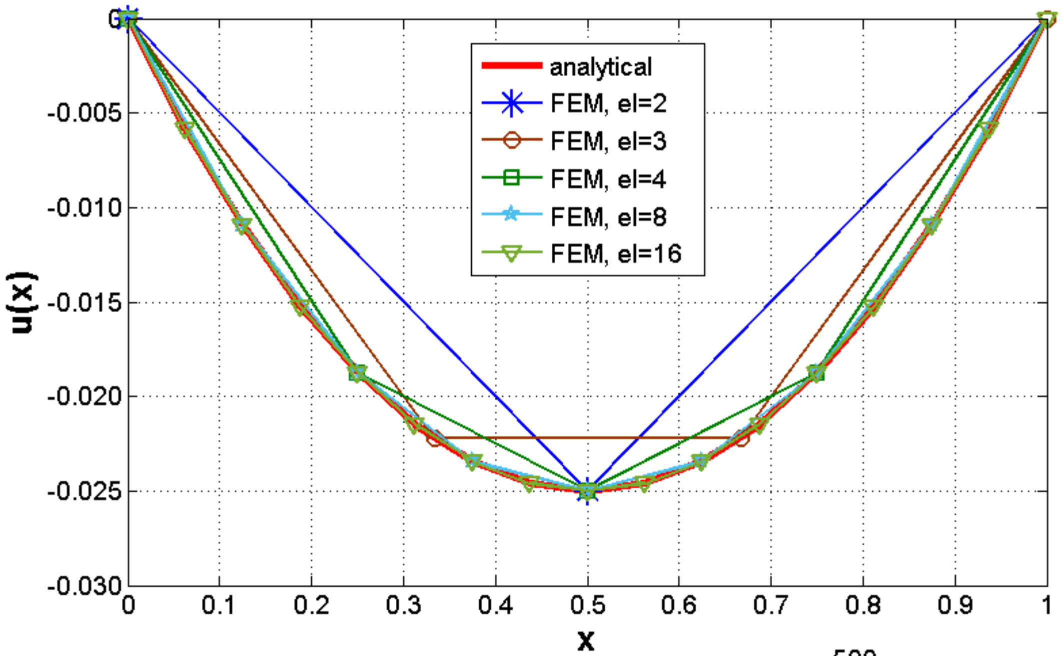
$$\begin{pmatrix} w_1 & w_2 & w_3 \cdots w_{n-1} & w_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} \quad \mathbf{b}_e$$

$$\Leftrightarrow \boxed{\mathbf{w}^{\mathrm{T}}\mathbf{K}\,\mathbf{u} = \mathbf{w}^{\mathrm{T}}\mathbf{b}}$$

FEM for bar: solution

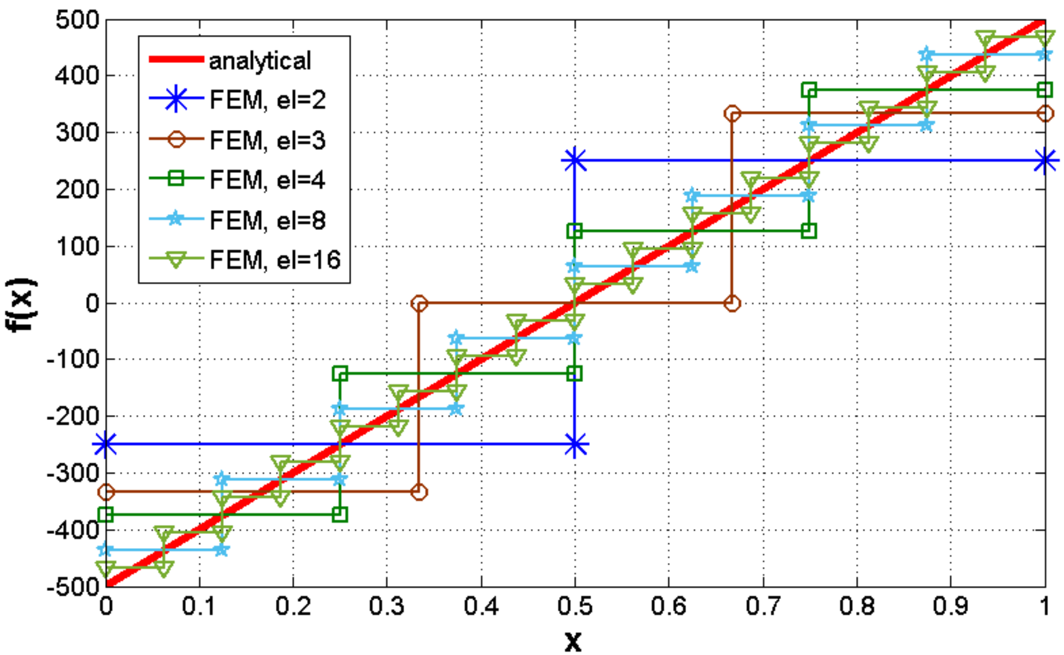
- Remember: the test function w was arbitrary and so are its coefficients $\mathbf{w}$: $\mathbf{w}^{\mathrm{T}}\mathbf{K}\,\mathbf{u} = \mathbf{w}^{\mathrm{T}}\mathbf{b}\ \ \forall\mathbf{w}$.
- Thus, it follows that $\mathbf{u}$ can be obtained as the solution of the $n^* \times n^*$-dimensional linear system of equations:
$$\mathbf{K}\,\mathbf{u} = \mathbf{b}.$$
- $u^h(x)$ is then given through the nodal displacement vector $\mathbf{u}$ and it holds that $u^h(n_i) = u_i$

Previous example: FEM solution



Displacements u^h

$$u(x) = 0.1x^2\,\frac{1}{\mathrm{m}} - 0.1x$$



Forces $f(u^h)$

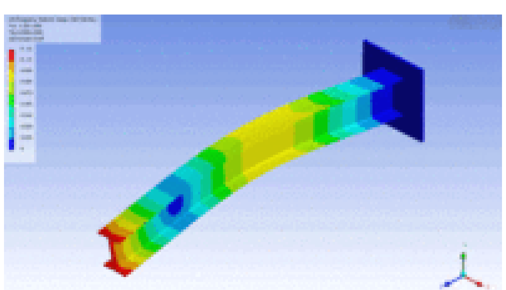
$$f(x) = 5000\ \mathrm{N}\left(0.2x\,\frac{1}{\mathrm{m}} - 0.1\right)$$

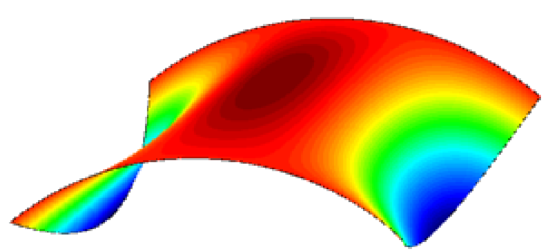
Important properties of Finite Element Method

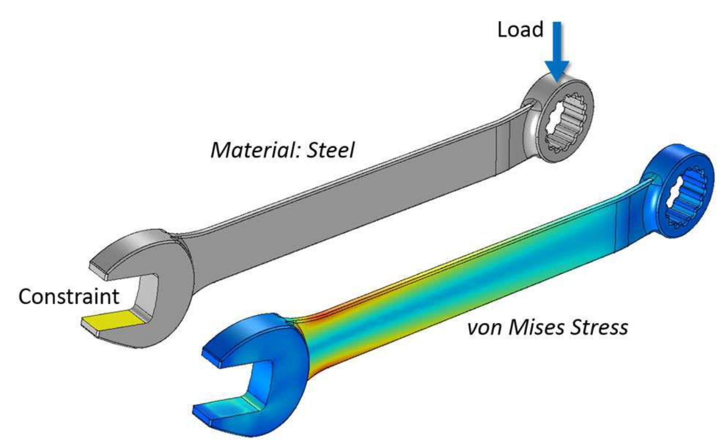
- **Convergence:** When the mesh is refined, the FEM solution converges towards the analytical solution:
$$u^h(x) \to u(x) \text{ for } h \to 0 \quad (h = L_e).$$
- **Strain energy:** The internal (strain) energy $U = \frac{1}{2}\int_o^L (EA\,u')\,u' dx$ is approximated as $U_h = \frac{1}{2}\mathbf{u}^{\mathbf{T}}\mathbf{K}\,\mathbf{u}$ and it holds $U_h \geq U$.
- **Linear system:** The stiffness matrix $\mathbf{K}$ is banded, sparse, symmetric and positive definite → $\mathbf{K}\,\mathbf{u} = \mathbf{b}$ can be solved efficiently by sparse and iterative linear solvers.

FEM in mechanics

- Mathematical models in terms of partial differential equations exist for many types of mechanical deformations:
  - Bars and trusses
  - Beams: 1D, 2D and 3D beam formulations
  - Plates, membranes and shells: 2D and 3D formulations
  - Solid bodies: 3D continuum mechanics
- Analytical solution of those problems in only possible in certain circumstances, for real-world problems we need to find approximate, numerical solutions
- For each of those models, finite element formulations can be derived and implemented

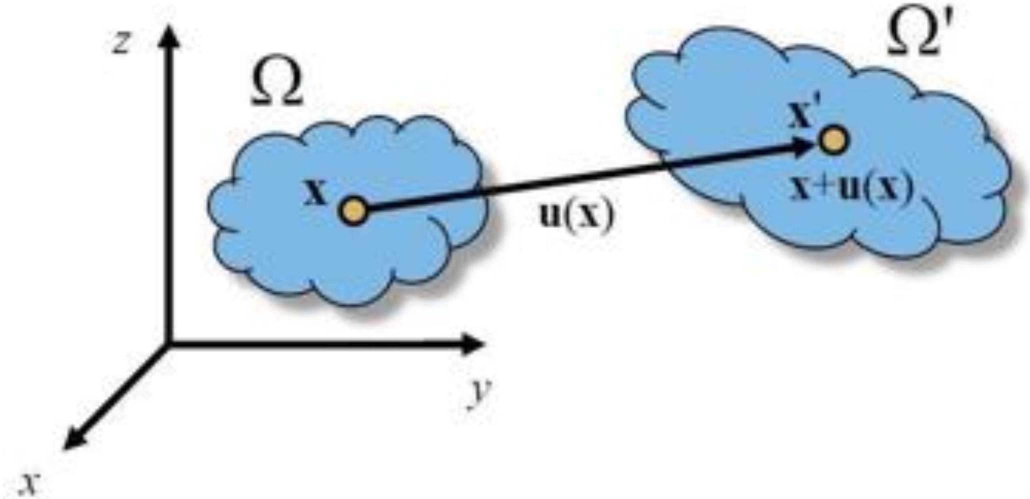






Linear continuum mechanics: deformation

- We study the small (infinitesimal), elastic deformation of a 3D continuum solid body under external and body loads
- The initial (material) configuration of the body is given in terms of the domain $\Omega \subset \mathbb{R}^3$
- The body deforms into a new (spatial) configuration, given by the deformed domain $\Omega' \subset \mathbb{R}^3$
- For every point $\boldsymbol{x} \in \Omega$, its corresponding deformed (spatial) coordinates $\boldsymbol{x}': \Omega \to \Omega', \boldsymbol{x}'(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ are given in terms of the deformation field $\boldsymbol{u}: \Omega \to \mathbb{R}^3$



Strain

- Similar to the bar case, the infinitesimal (linear) strain tensor $\boldsymbol{\varepsilon} \in \mathbb{R}^{3\times 3}$ is introduced as a measure of deformation:

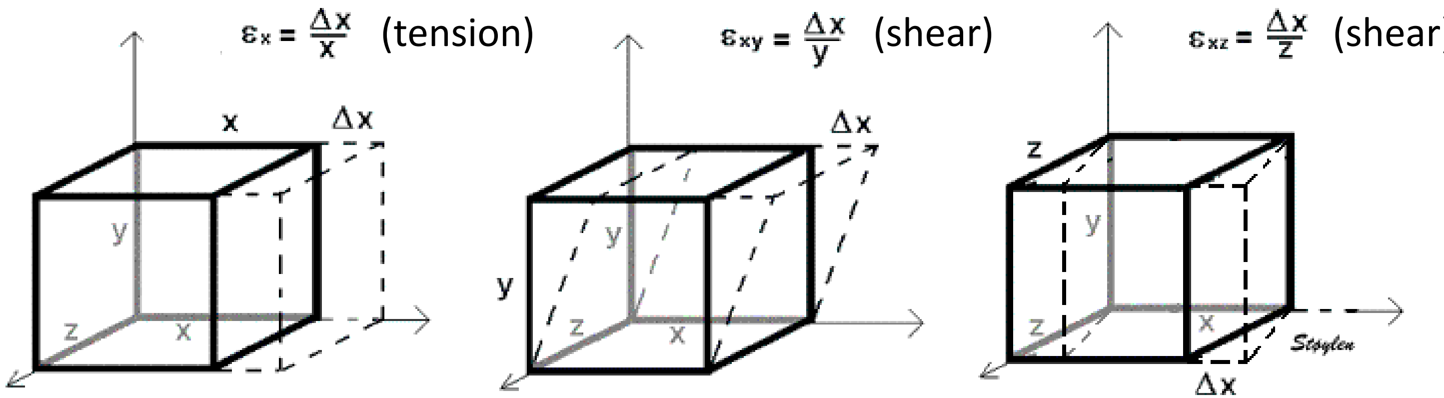
$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \frac{1}{2}\left(\nabla \boldsymbol{u}(\boldsymbol{x}) + \nabla \boldsymbol{u}^T(\boldsymbol{x})\right)$$

- Gradient of the deformation vector:

$$\nabla \boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix}, \qquad u_{i,j} := \frac{du_i}{dx_j}$$

- Detailed matrix notation of strain tensor:

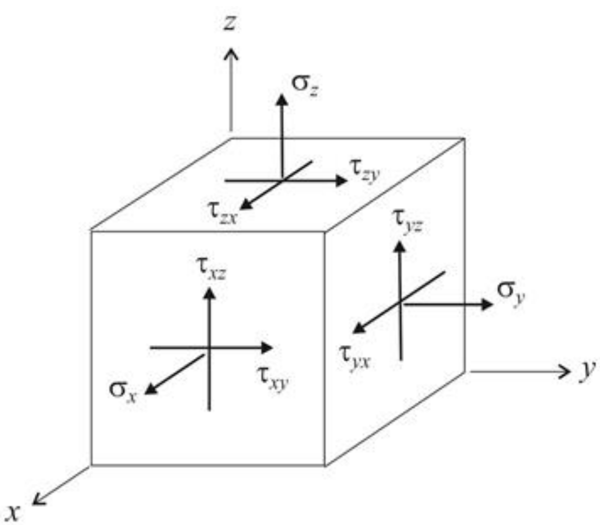
$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym. & & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ sym. & & u_{3,3} \end{pmatrix}$$

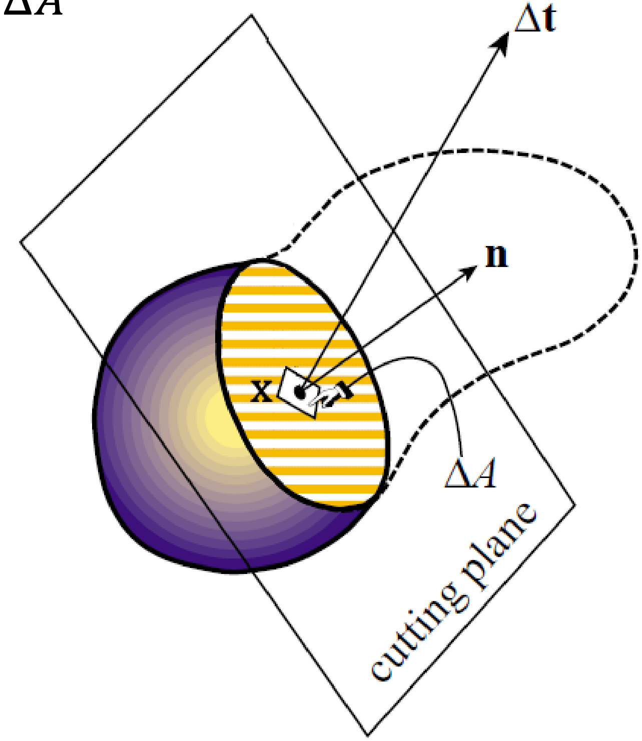


Stress

- Stress is a physical quantity that expresses the internal forces that neighbouring particles of a continuous material exert on each other
- Consider a point $\boldsymbol{x} \in \Omega$ and a cut through that point by a plane with normal $\boldsymbol{n} \in \mathbb{R}^3, \|\boldsymbol{n}\| = 1$. The stress/traction vector along that direction is defined as $\boldsymbol{t}_{\boldsymbol{n}}(\boldsymbol{x}) = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{t}}{\Delta A}$
- The Cauchy stress tensor $\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}$ is defined using:

$$\boldsymbol{t}_{\boldsymbol{n}}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}$$





Constitutive law

- The Cauchy stress tensor is related to the linear strain tensor through the linear elastic constitutive law (Hooke's law):

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{x})$$

- For an isotropic St. Venant-Kirchhoff material, the constitutive tensor $\boldsymbol{C} \in \mathbb{R}^{3\times3\times3\times3}$ depends on two material properties, Young's modulus E and Poisson ratio ν.

- With the Lamé constants $\mu = {}^{E}/_{2(1+\nu)}$ (shear modulus) and $\lambda = {}^{E\nu}/_{(1+\nu)(1-2\nu)}$ the constitutive law reads:

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I}$$

- In matrix-vector notation (so-called Voigt notation) it holds:

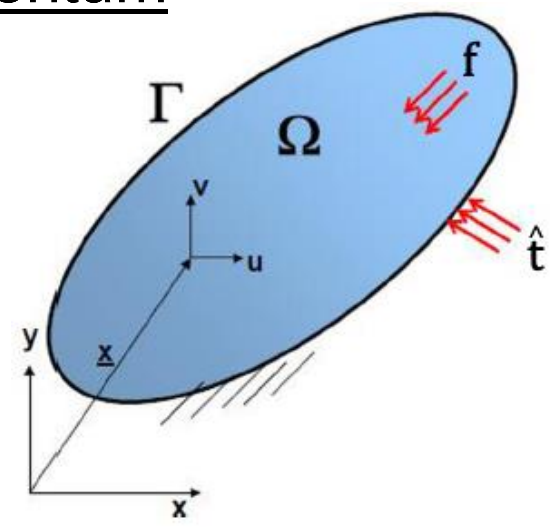
$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{13} \end{pmatrix}$$

$$\vec{\boldsymbol{\sigma}} = \boldsymbol{C}\,\vec{\boldsymbol{\varepsilon}}$$

Equilibrium equations

- Strong form of static balance of linear momentum (a partial differential equation, PDE):

$$\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}\ \forall \boldsymbol{x} \in \Omega$$



- Internal energy:

$$U = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}\colon \boldsymbol{\varepsilon}\, d\boldsymbol{x}$$

- External energy:

$$W = \frac{1}{2}\int_{\Omega} \boldsymbol{u}\cdot\boldsymbol{f}\, d\boldsymbol{x} + \frac{1}{2}\int_{\Gamma} \boldsymbol{u}\cdot\hat{\boldsymbol{t}}\, d\boldsymbol{s}$$

- Weak form:

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u})\colon \boldsymbol{\varepsilon}(\boldsymbol{w})d\boldsymbol{x} = \int_{\Omega} \boldsymbol{w}\cdot\boldsymbol{f}\, d\boldsymbol{x} + \int_{\Gamma_n} \boldsymbol{w}\cdot\hat{\boldsymbol{t}}\, d\boldsymbol{s}\ \forall \boldsymbol{w}$$

$\boldsymbol{f}$: external body forces (e.g. gravity)
$\hat{\boldsymbol{t}}$: external traction forces on the boundary

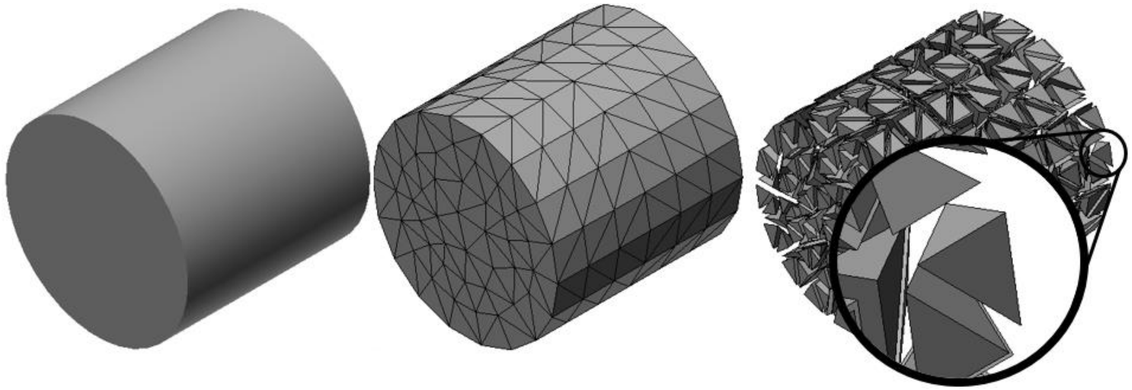
FEM for linear continuum mechanics

Apply the same concepts and steps as before for bar:

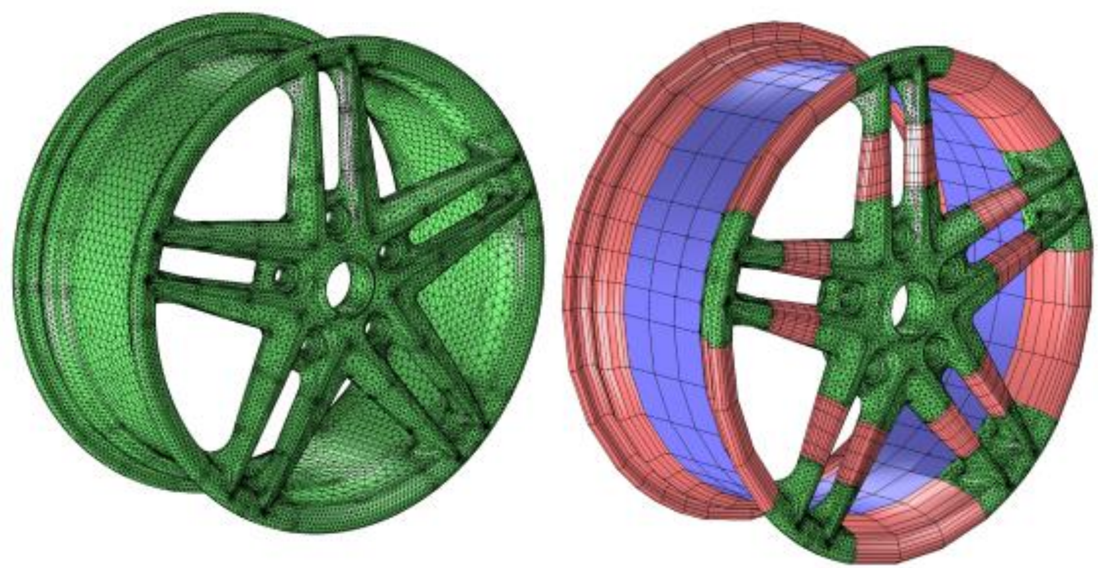
1. <u>Pre-processing</u> / meshing: discretization of domain into small elements
2. <u>Discretization</u> of displacement in each element using polynomial functions (linear, quadratic, …)
3. Evaluation of <u>weak form</u> of equilibrium equations leads to element-wise stiffness matrices and force vectors
4. <u>Assembly</u> of element stiffness matrices and force vectors into global stiffness matrix and force vector, application of boundary conditions
5. <u>Solution</u> of linear system for nodal displacement vector
6. <u>Post-processing</u>: evaluation of displacements, stresses etc.

Meshing

- Discretize domain into small volume elements,
  in 3D usually either tetrahedrons or hexahedrons



- For complex, engineering geometries meshing is not trivial!
- Elements should have similar size and not be skewed too much, more elements in areas of high stress and large deformation



Discretization of elements: geometry

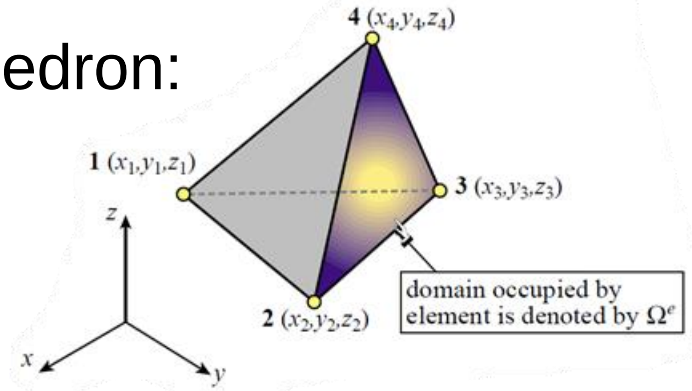
- For each element, discretize the geometry $\boldsymbol{x} \in \Omega^{\mathrm{e}}$ and the displacement $\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^h(\boldsymbol{x})$ using polynomial basis functions, e.g. tri-linear basis functions on tetrahedron
- Geometry of tetrahedron in terms of its 4 nodes:

$$\boldsymbol{x}(\boldsymbol{\xi})\Big|_{\Omega^{\mathrm{e}}} = N_1(\boldsymbol{\xi})\boldsymbol{x}_1 + N_2(\boldsymbol{\xi})\boldsymbol{x}_2 + N_3(\boldsymbol{\xi})\boldsymbol{x}_3 + N_4(\boldsymbol{\xi})\boldsymbol{x}_4$$

$$N_1(\boldsymbol{\xi}) = \xi_1, \qquad N_2(\boldsymbol{\xi}) = \xi_2,$$
$$N_3(\boldsymbol{\xi}) = \xi_3, \qquad N_4(\boldsymbol{\xi}) = \xi_4$$

- Using barycentric coordinates on tetrahedron:

$$\boldsymbol{\xi}:\ \xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$$



- In matrix form:

$$\boldsymbol{x}(\boldsymbol{\xi}) = \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \boldsymbol{J}^e \boldsymbol{\xi}$$

Discretization of elements: displacements

- Volume of tetrahedron:

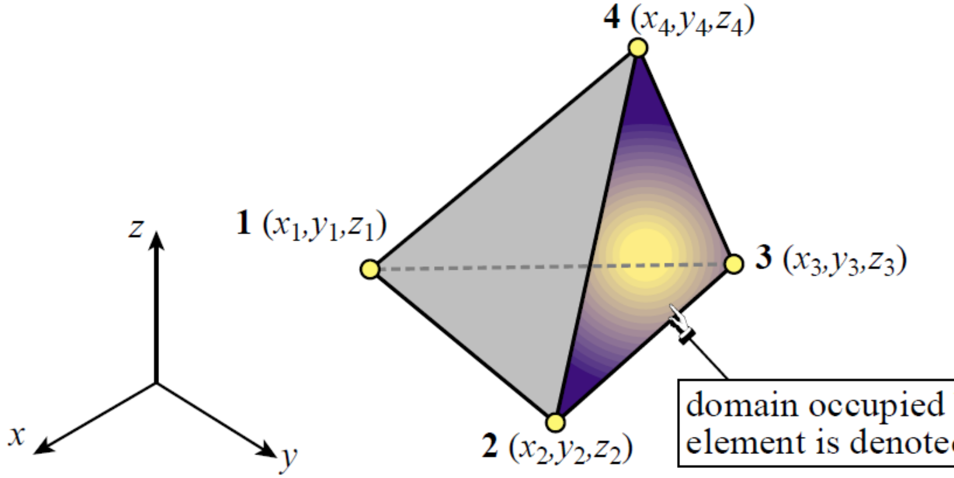
$$V^e = \frac{1}{6}\det \boldsymbol{J}^e = \frac{J}{6}$$

$\boldsymbol{J}^e$: Jacobian matrix, J: Jacobian determinant

- Displacements of tetrahedron in terms of its 4 <u>nodal displacements</u>:

$$\boldsymbol{u}(\boldsymbol{\xi})\Big|_{\Omega^e} = N_1(\boldsymbol{\xi})\boldsymbol{u}_1 + N_2(\boldsymbol{\xi})\boldsymbol{u}_2 + N_3(\boldsymbol{\xi})\boldsymbol{u}_3 + N_4(\boldsymbol{\xi})\boldsymbol{u}_4$$

$$N_1(\boldsymbol{\xi}) = \xi_1, \qquad N_2(\boldsymbol{\xi}) = \xi_2,$$
$$N_3(\boldsymbol{\xi}) = \xi_3, \qquad N_4(\boldsymbol{\xi}) = \xi_4,$$
$$\boldsymbol{\xi}:\ \xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$$



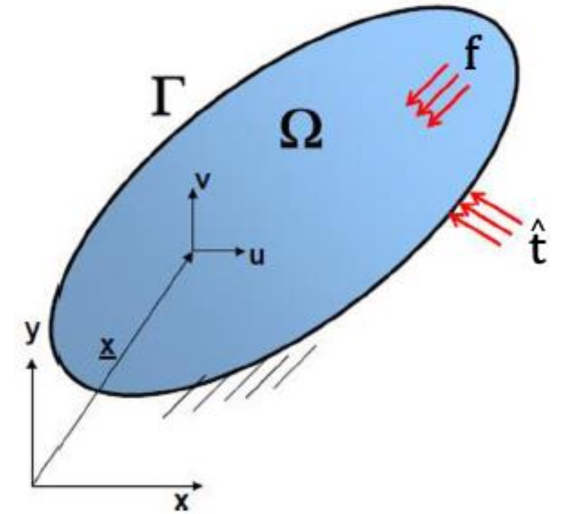
Discretization of elements: strain

- Matrix notation of displacements: $\boldsymbol{u}^h(\boldsymbol{\xi})|_{\Omega^{\mathrm{e}}} = \boldsymbol{N}(\boldsymbol{\xi})\,\boldsymbol{u}^e$,
  where $\boldsymbol{u}^e = \left(\boldsymbol{u}_1^{\mathrm{T}}, \boldsymbol{u}_2^{\mathrm{T}}, \boldsymbol{u}_3^{\mathrm{T}}, \boldsymbol{u}_4^{\mathrm{T}}\right)^{\mathrm{T}}$
  and $\boldsymbol{N}(\boldsymbol{\xi}) = (N_1(\boldsymbol{\xi})\, I_{3x3}, N_2(\boldsymbol{\xi})\, I_{3x3}, N_3(\boldsymbol{\xi})\, I_{3x3}, N_4(\boldsymbol{\xi})\, I_{3x3})$
- Definition of <u>strain tensor</u>: $\boldsymbol{\varepsilon}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{\nabla}\boldsymbol{u}^T(\boldsymbol{x})\right)$
- Strain in Voigt notation: $\vec{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \boldsymbol{D}\,\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{D}\,\boldsymbol{N}\,\boldsymbol{u}^e = \boldsymbol{B}\,\boldsymbol{u}^e$,

  where $\boldsymbol{D} = \begin{pmatrix} \partial/\partial x & & \\ & \partial/\partial y & \\ & & \partial/\partial z \\ \frac{1}{2}\partial/\partial y & \frac{1}{2}\partial/\partial x & \\ & \frac{1}{2}\partial/\partial z & \frac{1}{2}\partial/\partial y \\ \frac{1}{2}\partial/\partial z & & \frac{1}{2}\partial/\partial x \end{pmatrix}$

  and $\boldsymbol{B}$ is a 6x3 matrix that takes into account the partial differentiation w.r.t. $\boldsymbol{\xi}$, i.e. $\boldsymbol{B} \triangleq \frac{\boldsymbol{dN}}{\boldsymbol{d\xi}}\frac{\boldsymbol{d\xi}}{\boldsymbol{dx}}$.

Discretization of weak form



- Weak form on element Ω^e:

$$\int_{\Omega^e} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}) d\boldsymbol{x} =$$

$$= \int_{\Omega^e} \boldsymbol{w} \cdot \boldsymbol{f}\, d\boldsymbol{x} + \int_{\Gamma \cap \partial\Omega^e} \boldsymbol{w} \cdot \hat{\boldsymbol{t}}\, d\boldsymbol{s}$$

- <u>Stress</u> in Voigt notation: $\vec{\boldsymbol{\sigma}} = \boldsymbol{C}\,\vec{\boldsymbol{\varepsilon}} = \boldsymbol{C}\,\boldsymbol{B}\,\boldsymbol{u}^e$
- Discretized form:

$$\boldsymbol{w}^{e\,\mathrm{T}} \int_{\Omega^e} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{C}\,\boldsymbol{B}\, d\boldsymbol{x}\ \boldsymbol{u}^e = \boldsymbol{w}^{e\,\mathrm{T}} \int_{\Omega^e} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{f}\, d\boldsymbol{x} + \boldsymbol{w}^{e\,\mathrm{T}} \int_{\Gamma \cap \partial\Omega^e} \boldsymbol{N}^{\mathrm{T}}\, \hat{\boldsymbol{t}}\, d\boldsymbol{s}$$

- <u>Element stiffness matrix</u>: $\mathbf{K}^{\mathrm{e}} = \int_{\Omega^e} \mathbf{B}^{\mathrm{T}} \mathbf{C}\, \mathbf{B}\, d\boldsymbol{x}$
- <u>Element force vector</u>: $\mathbf{b}^{\mathrm{e}} = \int_{\Omega^e} \mathbf{N}^{\mathrm{T}} \boldsymbol{f}\, d\boldsymbol{x} + \int_{\Gamma \cap \partial\Omega^e} \mathbf{N}^{\mathrm{T}}\, \hat{\boldsymbol{t}}\, d\boldsymbol{s}$
- Use closed form / analytical evaluation or Gauss quadrature to evaluate the integrals for $\mathbf{K}^{\mathrm{e}}$ & $\mathbf{b}^{\mathrm{e}}$

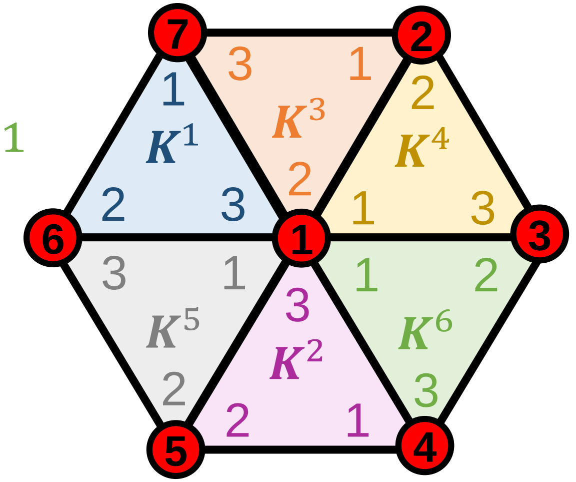
Assembly

- Assembly of element stiffness matrices and force vectors into global stiffness matrix and force vector:

$$\mathbf{K} = \bigvee_{e=1}^{\ell} \mathbf{K}^{e}, \qquad \mathbf{b} = \bigvee_{e=1}^{\ell} \mathbf{b}^{e}$$

Add contributions from each node of an element into global values of the node:

$$\mathbf{K}_{11} = \mathbf{K}^{1}_{33} + \mathbf{K}^{2}_{33} + \mathbf{K}^{3}_{22} + \mathbf{K}^{4}_{11} + \mathbf{K}^{5}_{11} + \mathbf{K}^{6}_{11}$$
$$\mathbf{K}_{12} = \mathbf{K}^{3}_{21} + \mathbf{K}^{4}_{12}$$
$$\mathbf{b}_{1} = \mathbf{b}^{1}_{3} + \mathbf{b}^{2}_{3} + \mathbf{b}^{3}_{2} + \mathbf{b}^{4}_{1} + \mathbf{b}^{5}_{1} + \mathbf{b}^{6}_{1}$$
...



- Apply essential boundary conditions, i.e. remove rows and columns for nodes/DOFs with Dirichlet B.C.

Solution of linear system

- Assembly leads to the n-dimensional linear system:
$$\mathbf{K}\,\mathbf{u} = \mathbf{b}$$
- Properties of the stiffness matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$:
  - It can be very large, e.g. $n > 10^6$
  - It is symmetric, i.e. $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$
  - It is sparse, i.e. in every row only a small no. of entries is non-zero
  - If no (or an in sufficient no. of) essential boundary conditions are applied, $\mathbf{K}$ is singular (i.e. it has 0-eigenvalues), which means that not all rigid body motions of the body are constrained
(in 3D there are 6 rigid body motions: 3 translations and 3 rotations)
- Solution methods for the <u>sparse linear system</u>:
  - Use sparse matrix representation of $\mathbf{K}$
  - Sparse direct solvers (factorization as $\mathbf{K} = \mathbf{LU}$ or $\mathbf{K} = \mathbf{QR}$, typically used when $n < 10^5$)
  - Sparse iterative solvers (iterative solution of linear system using matrix-vector multiplications)

Post-processing

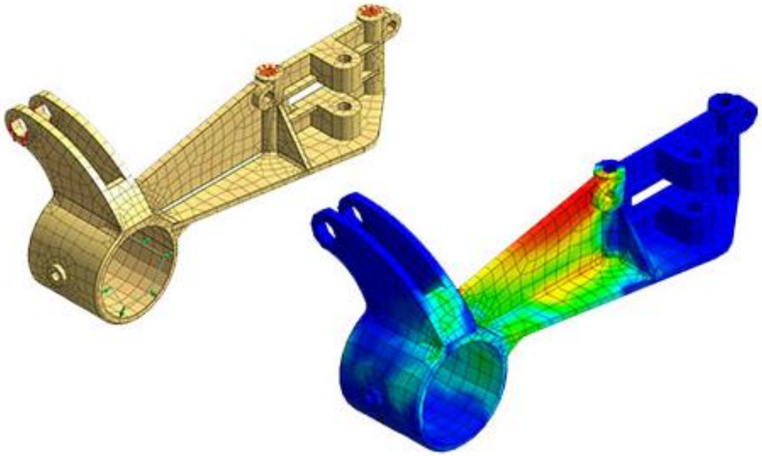
- Visualization of results of finite element analysis in terms of displaced body, strains, stresses etc.
- Displaced mesh can be visualized through its nodes:

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}$$

- Element-wise solution can be reconstructed through:

$$\boldsymbol{u}^h(\boldsymbol{\xi})\Big|_{\Omega^{\mathrm{e}}} = \boldsymbol{N}(\boldsymbol{\xi})\,\boldsymbol{u}^e$$

- Strains and stresses can be evaluated
- Strains and stresses are constant on an element for tri-linear finite elements



FEM for linear continuum mechanics

Summary of basic steps of finite element analysis (FEA):

1. Pre-processing / meshing: discretization of domain into small elements
2. Discretization of displacement in each element using polynomial functions (linear, quadratic, …)
3. Evaluation of weak form of equilibrium equations leads to element-wise stiffness matrices and force vectors
4. Assembly of element stiffness matrices and force vectors into global stiffness matrix and force vector, application of boundary conditions
5. Solution of linear system for nodal displacement vector
6. Post-processing: evaluation of displacements, stresses etc.

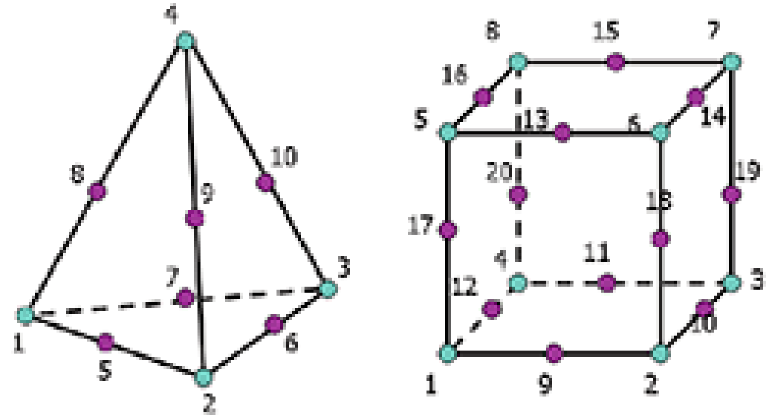
FEM pseudo-code

- Initialize $\mathbf{K}, \mathbf{b} = \mathbf{0}$
- For every element $e = 1, \ldots, \ell$
  - Local assembly of $\mathbf{K}^{\mathrm{e}}, \mathbf{b}^{\mathrm{e}}$ using weak form
  - Global assembly $\mathbf{K}^{\mathrm{e}} \rightarrow \mathbf{K}, \mathbf{b}^{\mathrm{e}} \rightarrow \mathbf{b}$ (incl. essential BC)
- Solve linear system $\mathbf{K}\,\mathbf{u} = \mathbf{b}$
- Use $\mathbf{u}$ to evaluate $\boldsymbol{u}^h, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \ldots$

Higher order elements

- So far: tetrahedral elements with tri-linear basis functions
- Choice of basis functions can also be higher order polynomials (quadratic, cubic, …) on tetrahedron or hexahedron:

Nodes of tetrahedral and hexahedral finite elements:
- nodes of linear elements
- additional node for quadratic elements



- Advantages: more accurate than linear elements, faster convergence
- Disadvantages: slower, since assembly time is longer (numerical quadrature) and linear solution

Elastodynamics

- So far only static forces and displacements:

$$-\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega$$

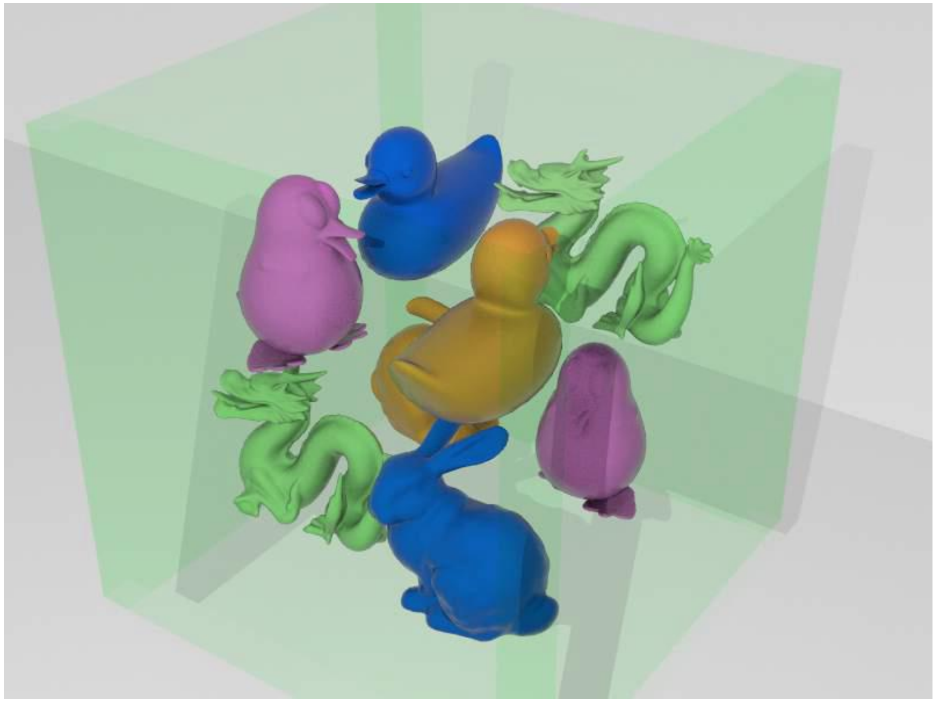
- Time-dependent, dynamic equilibrium equation:

$$\rho\,\ddot{\boldsymbol{u}}(\boldsymbol{x},t) - \mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t) \quad \forall \boldsymbol{x} \in \Omega, \mathrm{t} \in [T_0, T_1]$$

- Finite element semi-discretization:

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{b}(t) \quad \forall \mathrm{t} \in [T_0, T_1]$$

  - Solve using (standard) integrators for ODE systems
  - Post-processing of deformed mesh for every time step



Elastodynamics

- So far only static forces and displacements:

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- Finite element semi-discretization:

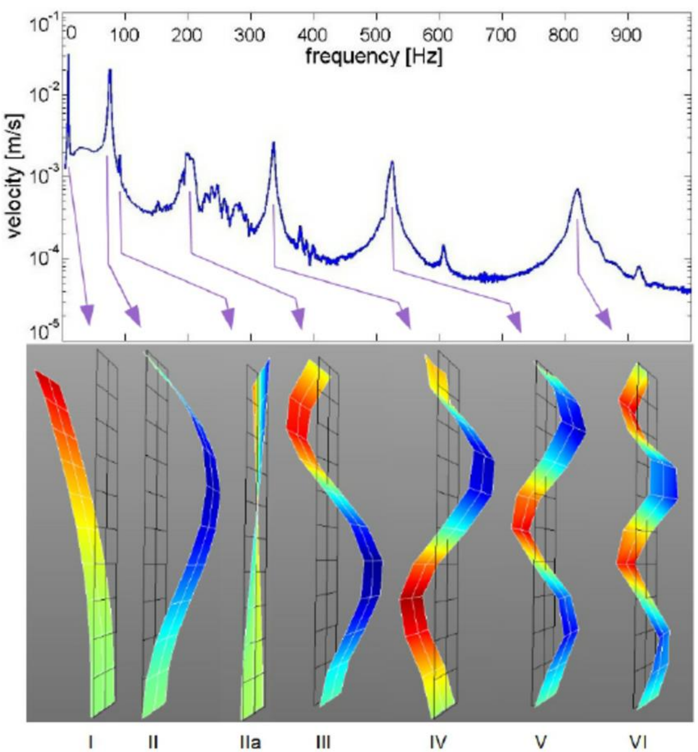
$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{b}(t) \;\;\; \forall \mathrm{t} \in [T_0, T_1]$$

  - Solve using (standard) integrators for ODE systems
  - Post-processing of deformed mesh for every time step

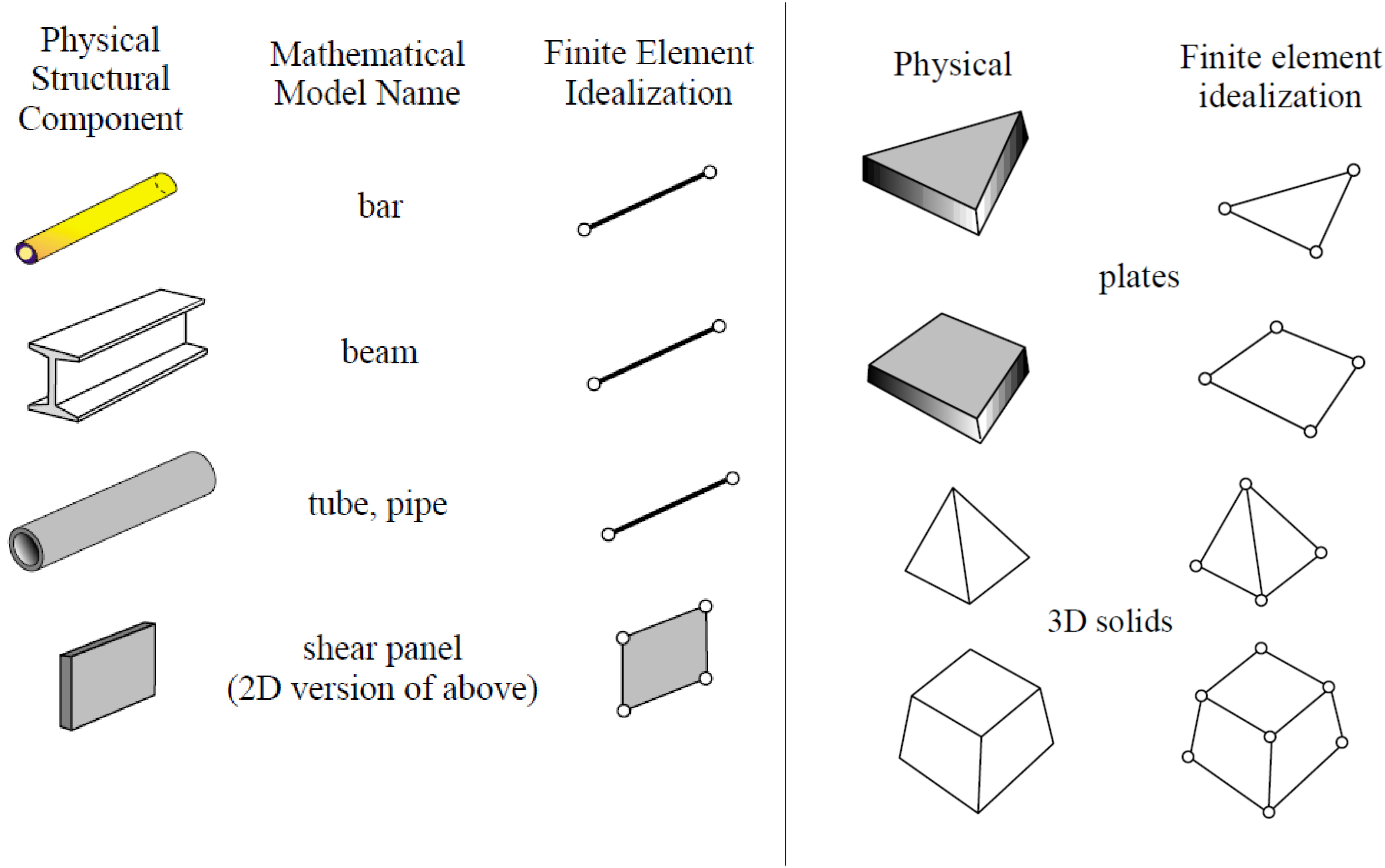
- Modal analysis:

$$-\omega^2 \mathbf{M}\boldsymbol{\phi} + \mathbf{K}\boldsymbol{\phi} = \mathbf{0}$$

  - Obtain eigenfrequencies and eigenmodes $(\omega_k, \boldsymbol{\phi}_k)_{k=1,\ldots,n}$
  - Lowest frequencies & modes characterize frequency response behaviour of structure



Other mechanical FEM models



Summary

- This lecture:
  - Bar as a simple 1D mechanical model
  - Introduction of Finite Element Method for bar
  - Short introduction to linear continuum mechanics
  - Overview of FEM for linear continuum mechanics
- Next lecture:
  - Nonlinear continuum mechanics
  - Nonlinear finite element method
  - Material and constitutive modelling

Further reading and acknowledgement

- Hughes, T. J. R: *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall Inc. (1987)
- https://en.wikiversity.org/wiki/Nonlinear_finite_elements
- Online lecture materials by Carlos Felippa (CU Boulder):
  http://www.colorado.edu/engineering/CAS/courses.d/IFEM.d/
  http://www.colorado.edu/engineering/CAS/courses.d/AFEM.d/
  http://www.colorado.edu/engineering/CAS/courses.d/NFEM.d/