50.017 Graphics and Visualization

Bézier Curves and Splines

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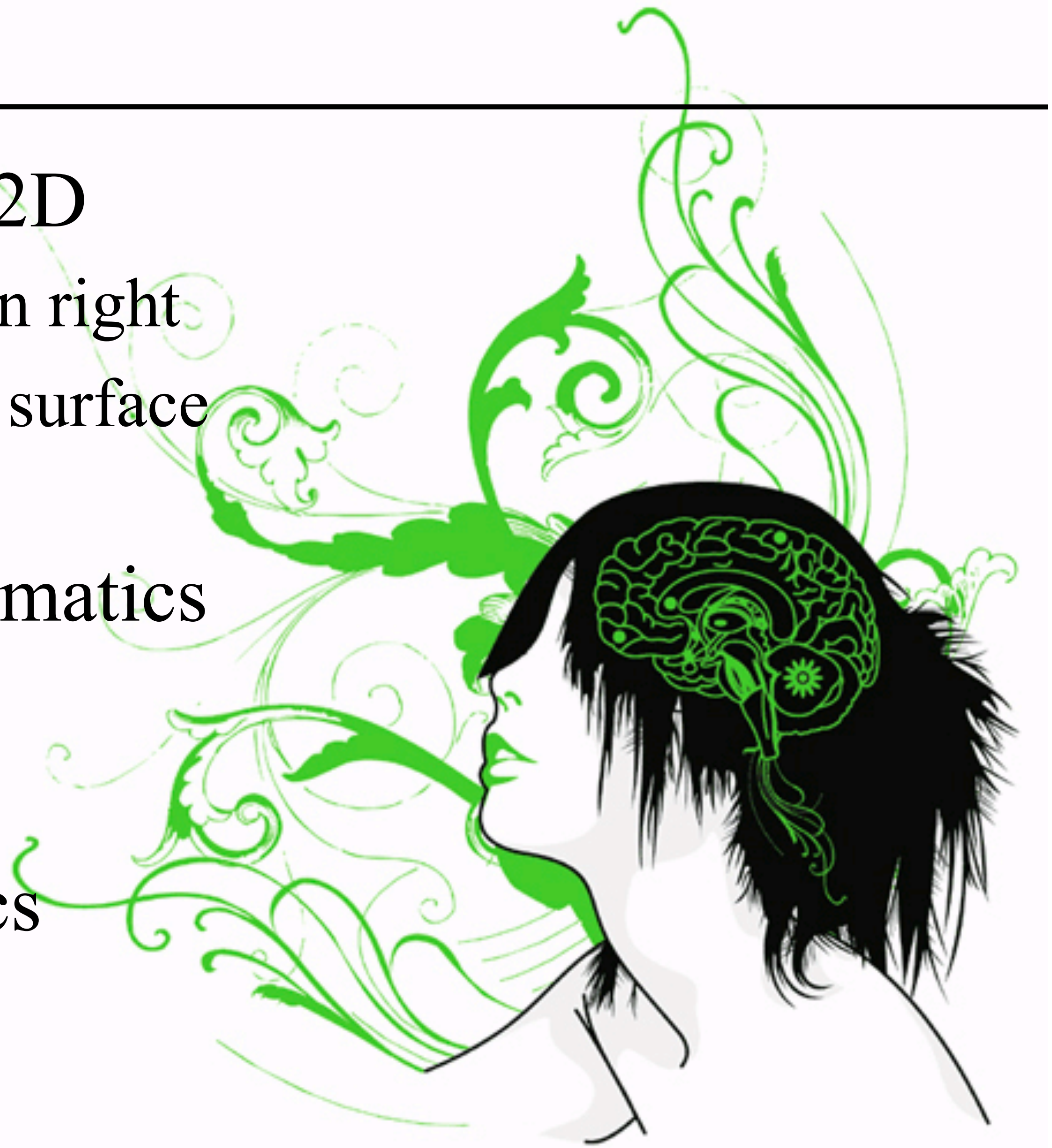
Notes courtesy by
Prof. Wojciech Matusik

Before We Begin

- Anything on your mind concerning Assignment 0?
- Any questions about the course?

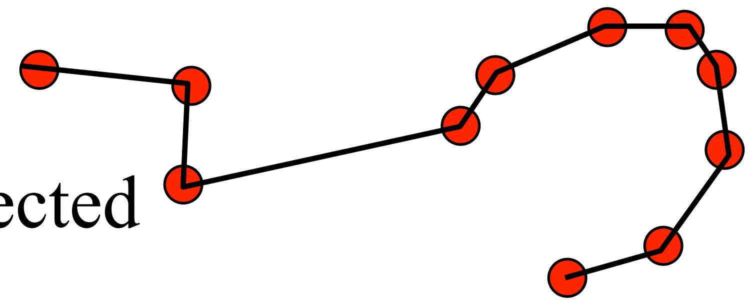
Today

- Smooth curves in 2D
  - Useful in their own right
  - Provides basis for surface editing
- Theoretical mathematics
  - Charles Hermite
  - Sergei Bernstein
- Popular to graphics
  - Pierre Bézier
  - Paul de Casteljau

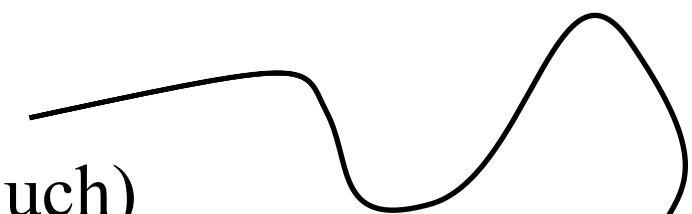


Guilherme Marconi

Modeling 1D Curves in 2D

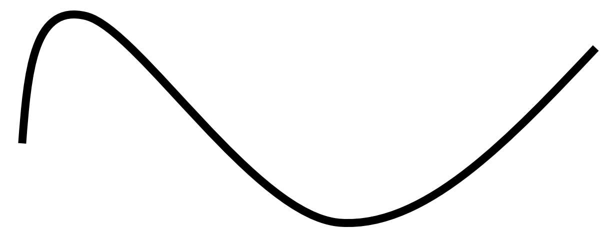


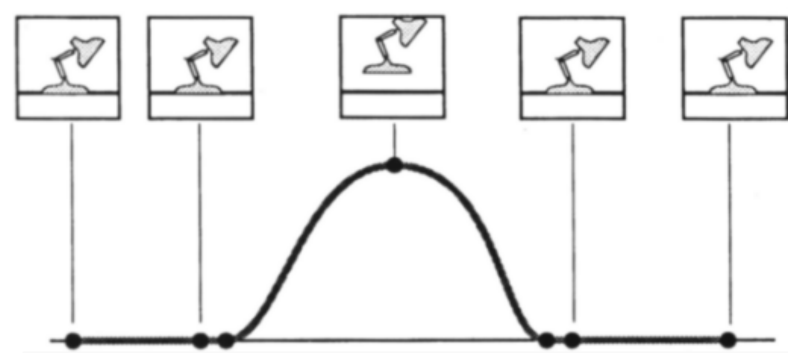
- Polylines
  - Sequence of vertices connected by straight line segments
  - Useful, but not for smooth curves
  - This is the representation that usually gets drawn in the end (a curve is converted into a polyline)
- Smooth curves
  - How do we specify them?
  - A little harder (but not too much)



Splines

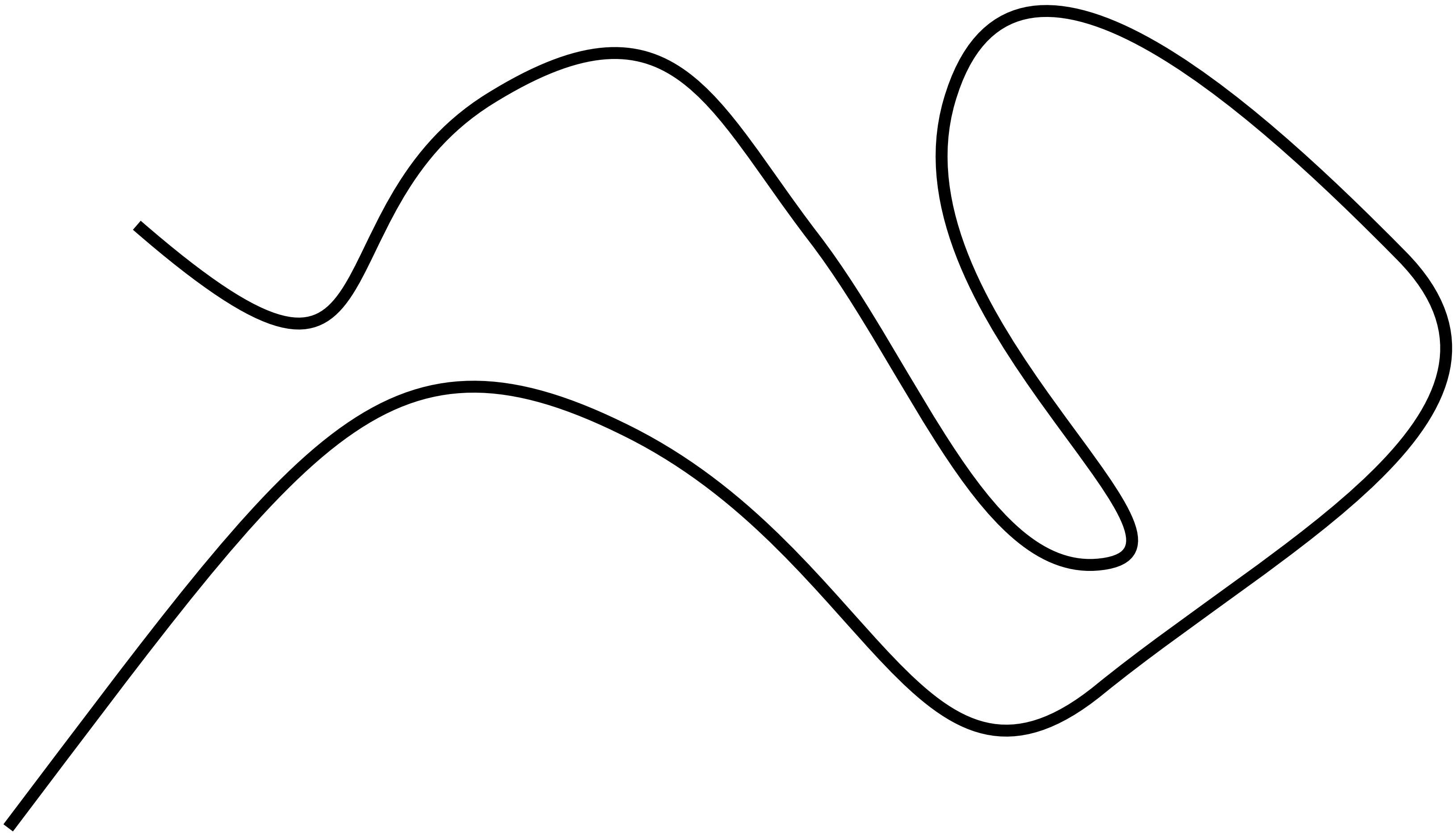
- A type of smooth curve in 2D/3D
  - Defined by a polynomial
  - Controlled by certain “control points”
- Many different uses
  - 2D illustration (e.g., Adobe Illustrator)
  - Fonts (e.g., PostScript, MS TrueType)
  - 3D modeling
  - Animation: trajectories
- Important concepts
  - Interpolate points
  - Maintain smoothness





Demo

How Many Dimensions?



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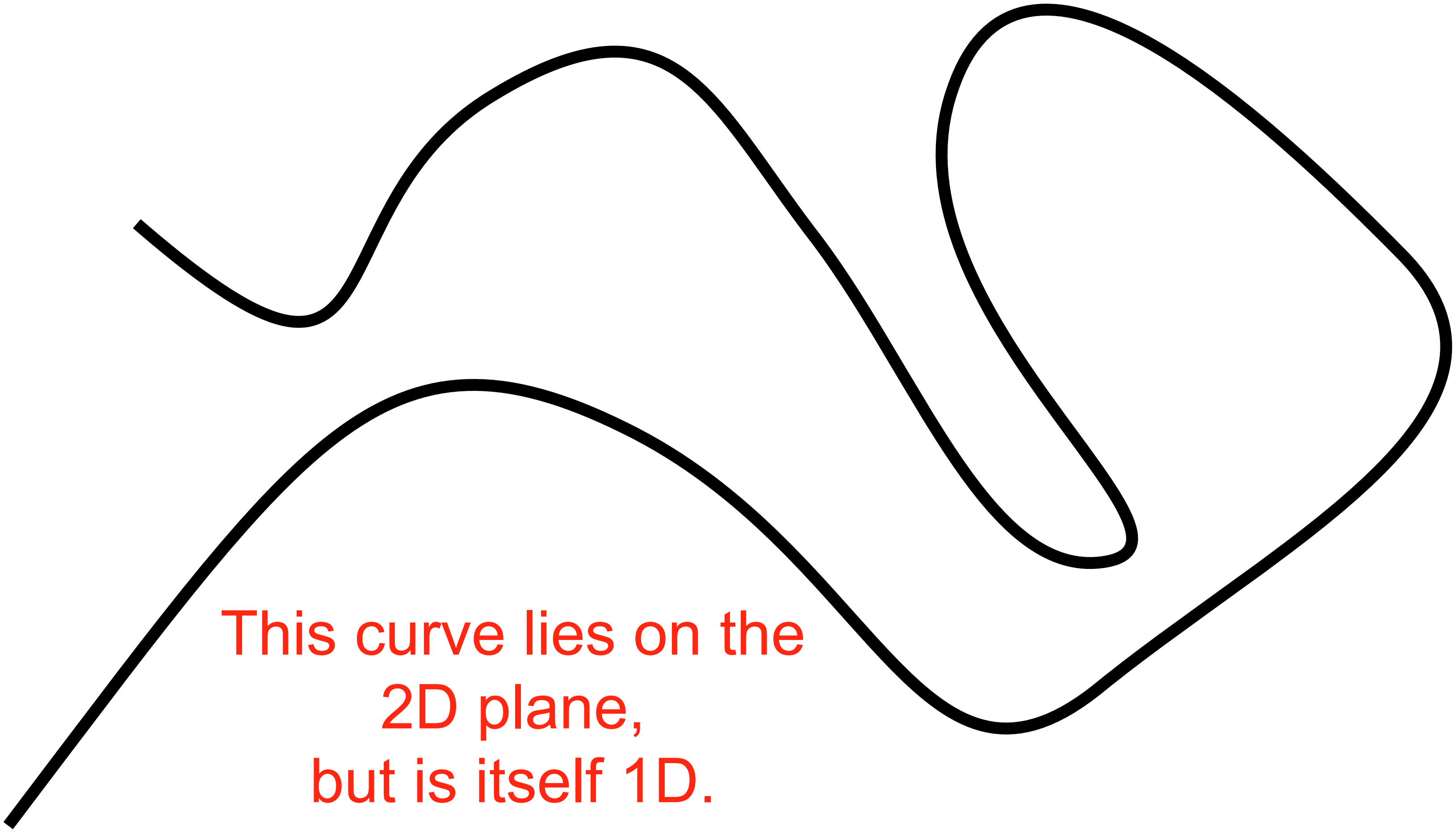
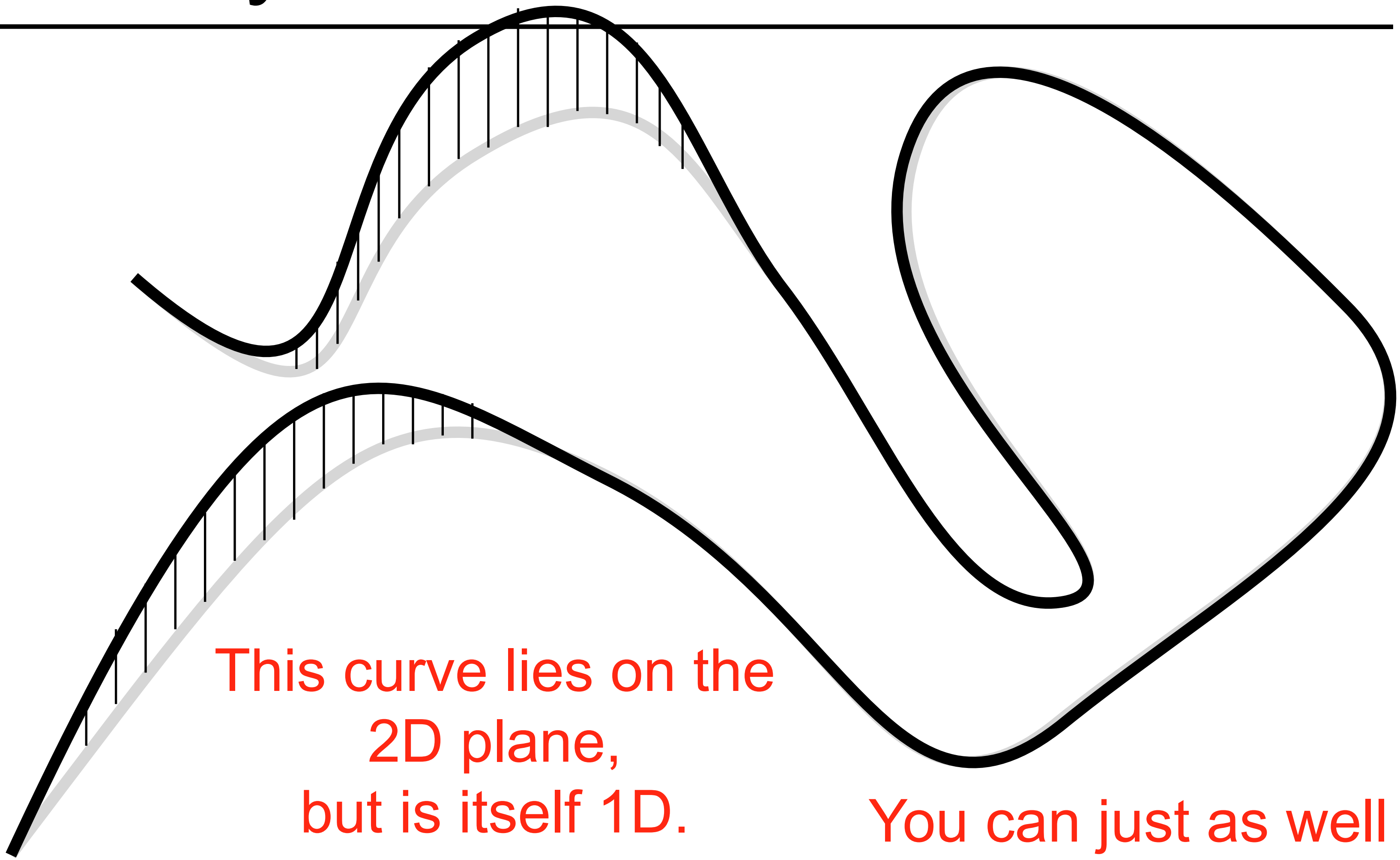


This curve lies on the
2D plane,
but is itself 1D.

How Many Dimensions?



This curve lies on the 2D plane, but is itself 1D.

You can just as well define 1D curves in 3D space.

Two Definitions of a Curve

1) A continuous 1D set of points in 2D (or 3D)

2) A mapping from an interval S onto the plane

   – That is, *P(t)* is the point of the curve at parameter *t*

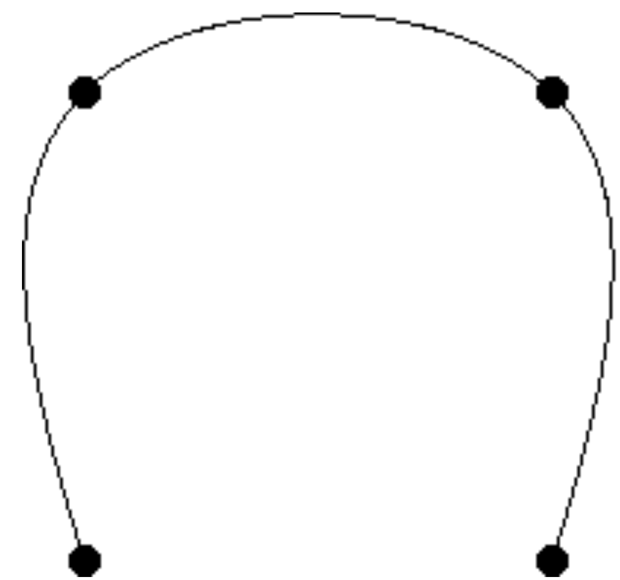
$$P : \mathbb{R} \ni S \mapsto \mathbb{R}^2, \qquad P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Parametric representation

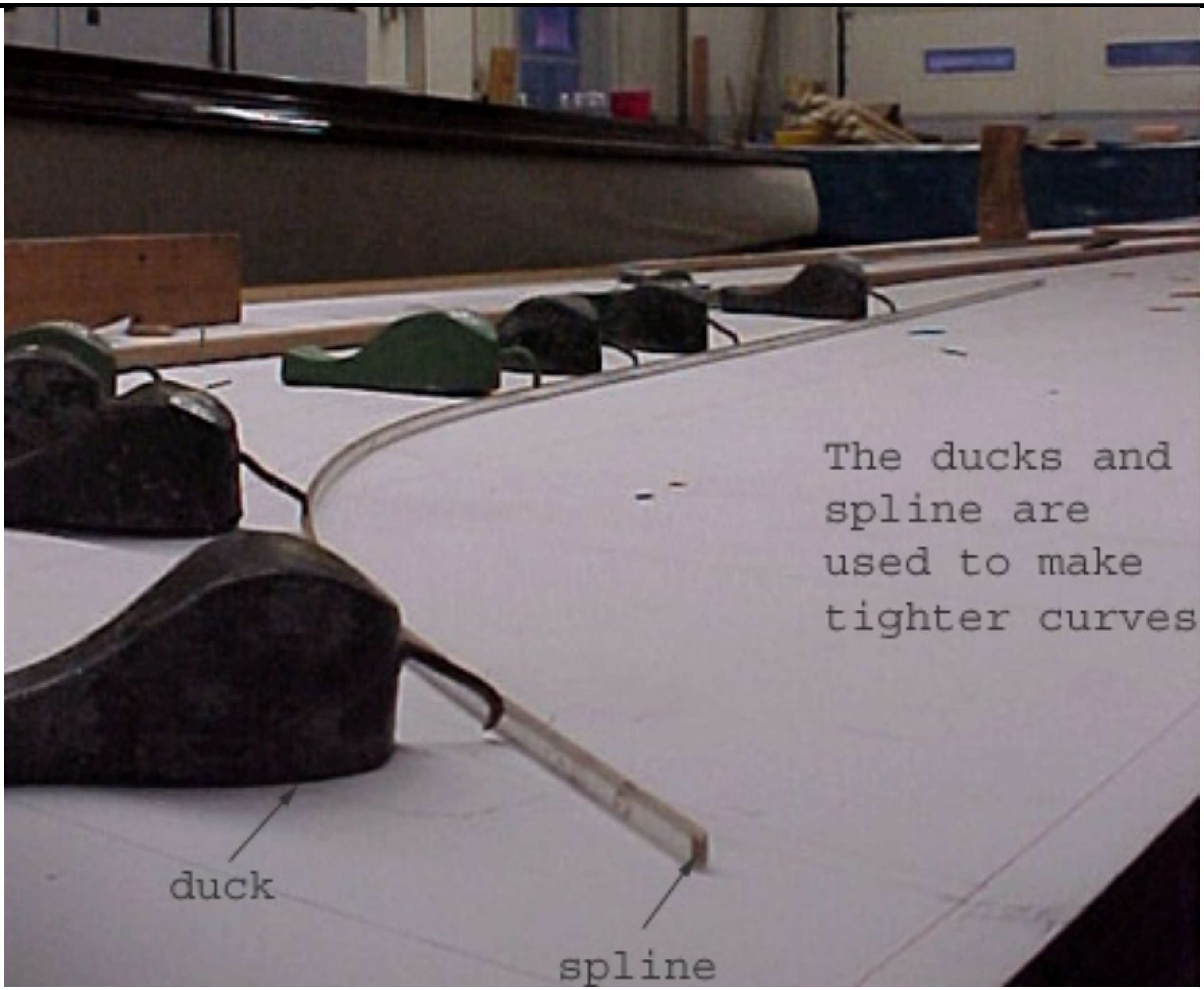
- Big differences
  - It is easy to generate points on the curve from the 2nd
  - The second definition can describe trajectories, the speed at which we move on the curve

General Principle of Splines

- Curves specified by controls points
  - Usually by user
- We will interpolate the control points by a smooth curve
  - The curve is completely determined by the control points.
  - Parametric represenation



Physical Splines



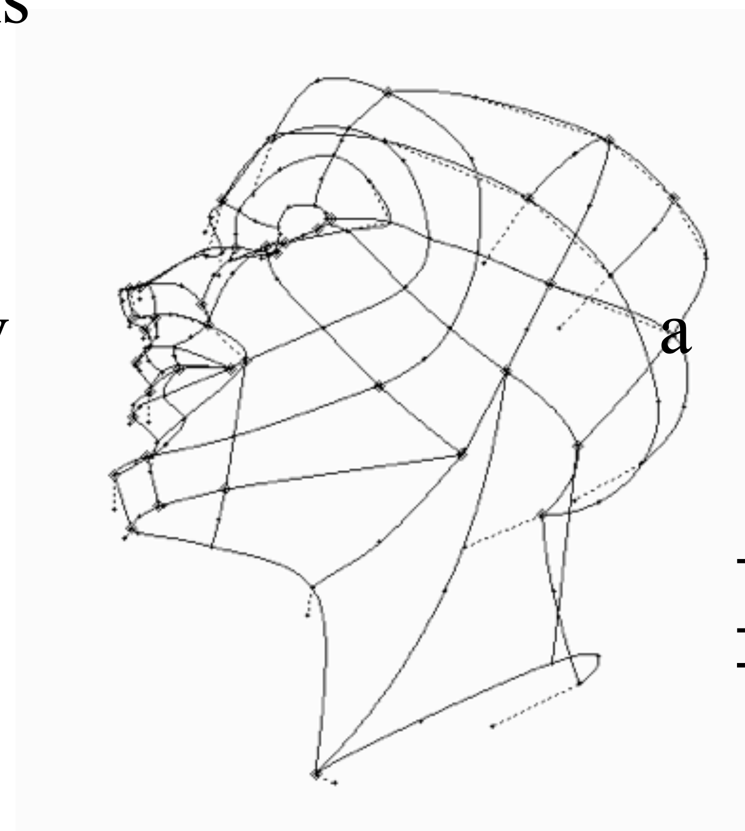
www.abm.org

See http://en.wikipedia.org/wiki/Flat_spline

Two Application Scenarios

1) Approximation/interpolation
- We have “data points”, how can we interpolate?
- Important in many applications

2) User interface/modeling
- What is an easy way to specify a smooth curve?
- Our main perspective today.



celshader.com

vectorportal.com

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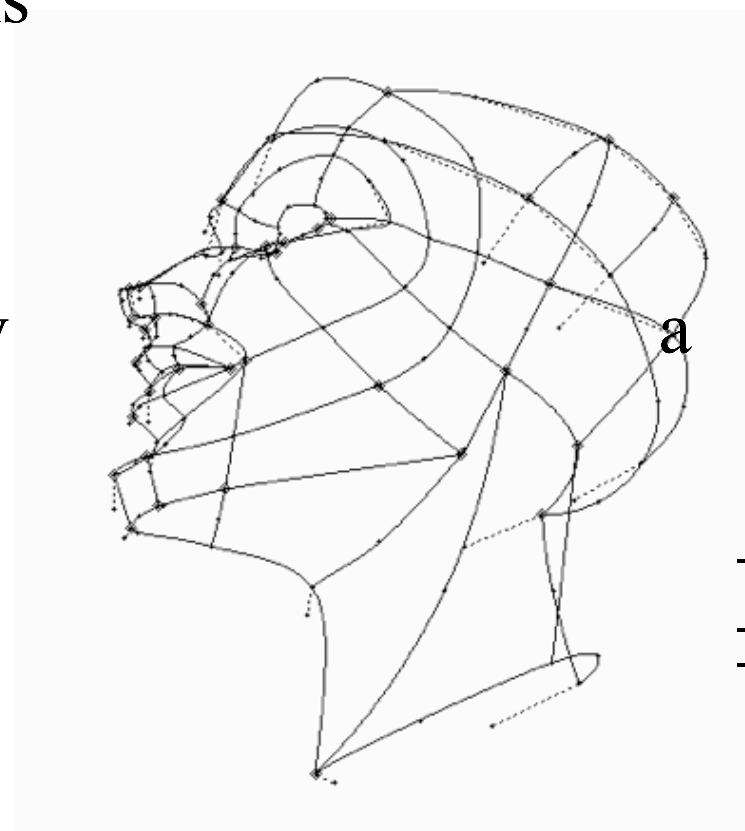


celshader.com

Questions?

vectorportal.com

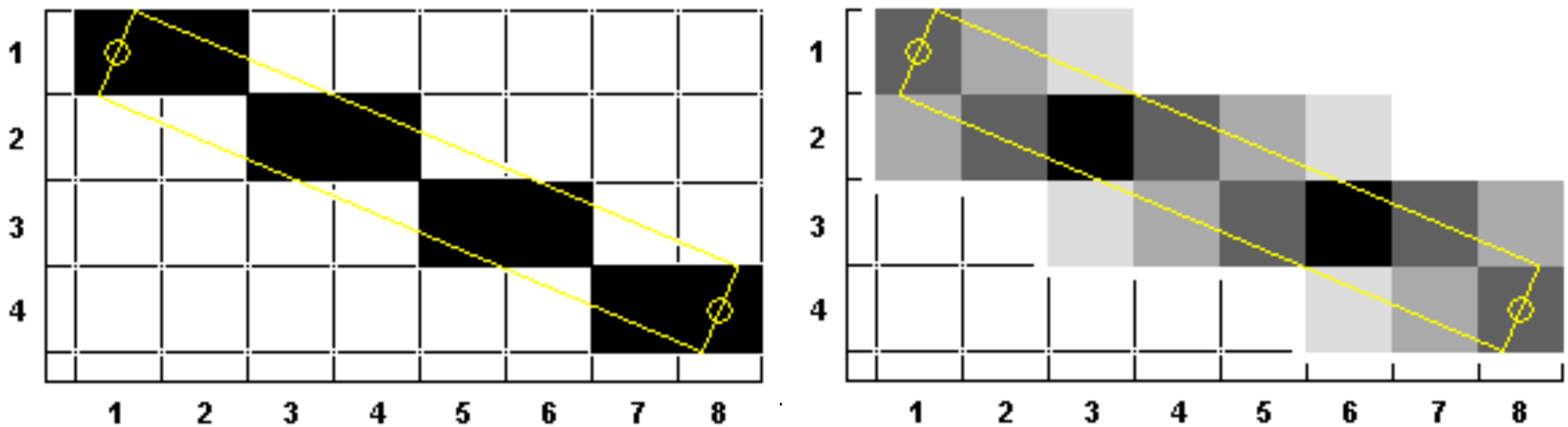
Splines: Recap

- Specified by a few control points
  - Good for UI
  - Good for storage

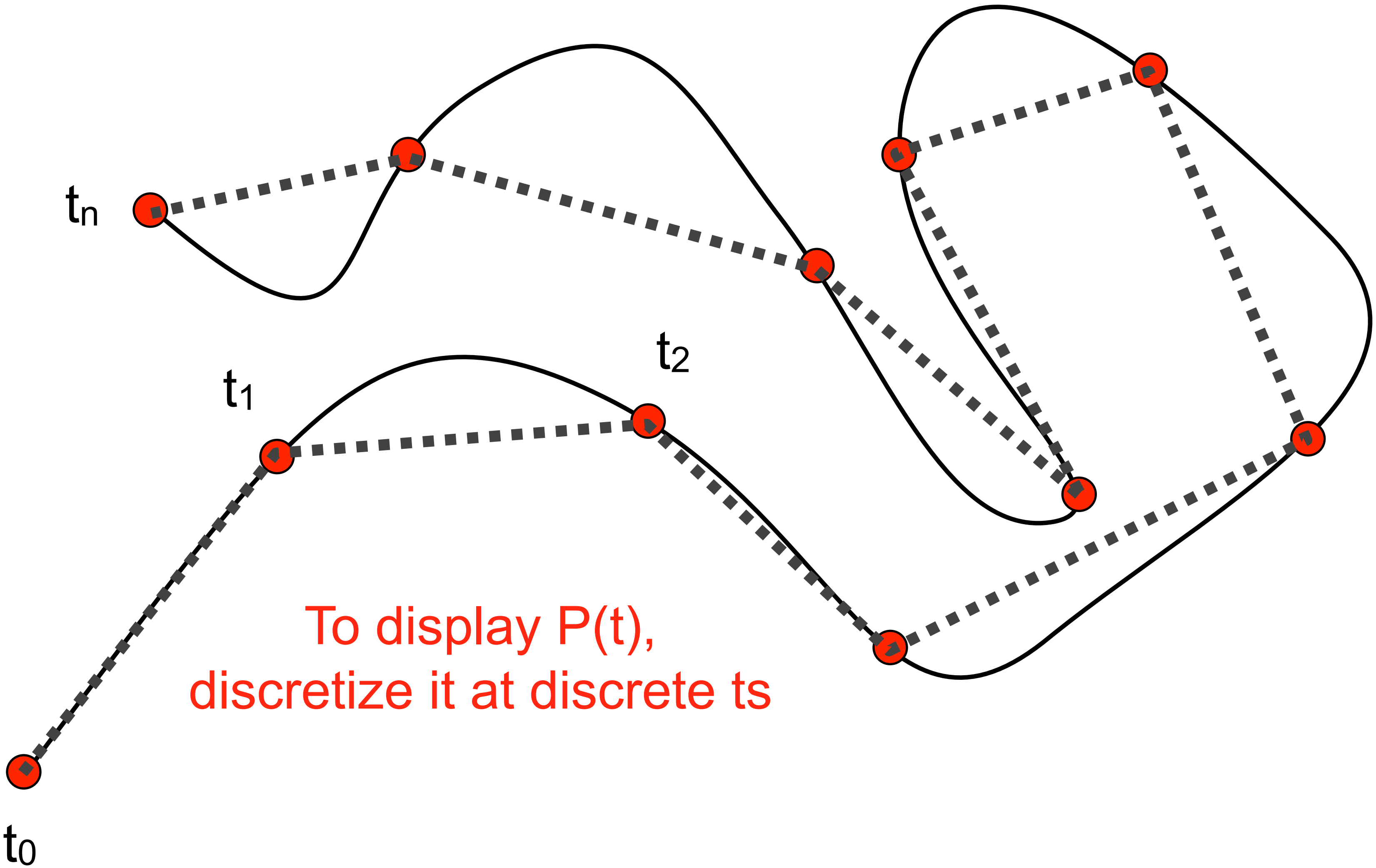
- Results in a smooth parametric curve P(t)
  - Just means that we specify x(t) and y(t)
  - In practice: low-order polynomials, chained together
  - Convenient for animation, where t is time
  - Convenient for tessellation because we can discretize t and approximate the curve with a polyline

Tessellation

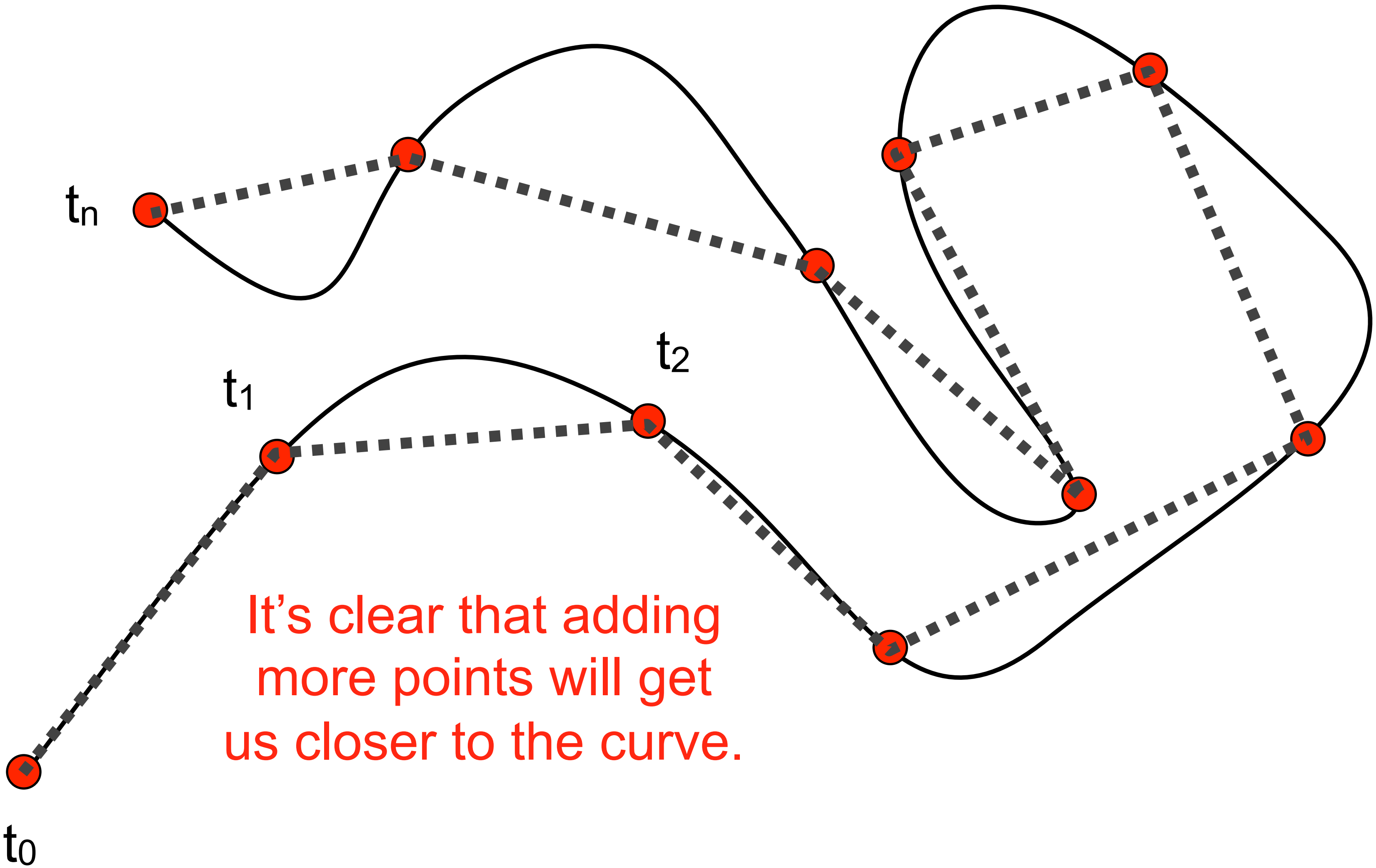
- It is easy to rasterize mathematical line segments into pixels
  - OpenGL and the graphics hardware can do it for you
- But polynomials and other parametric functions are harder



Tessellation

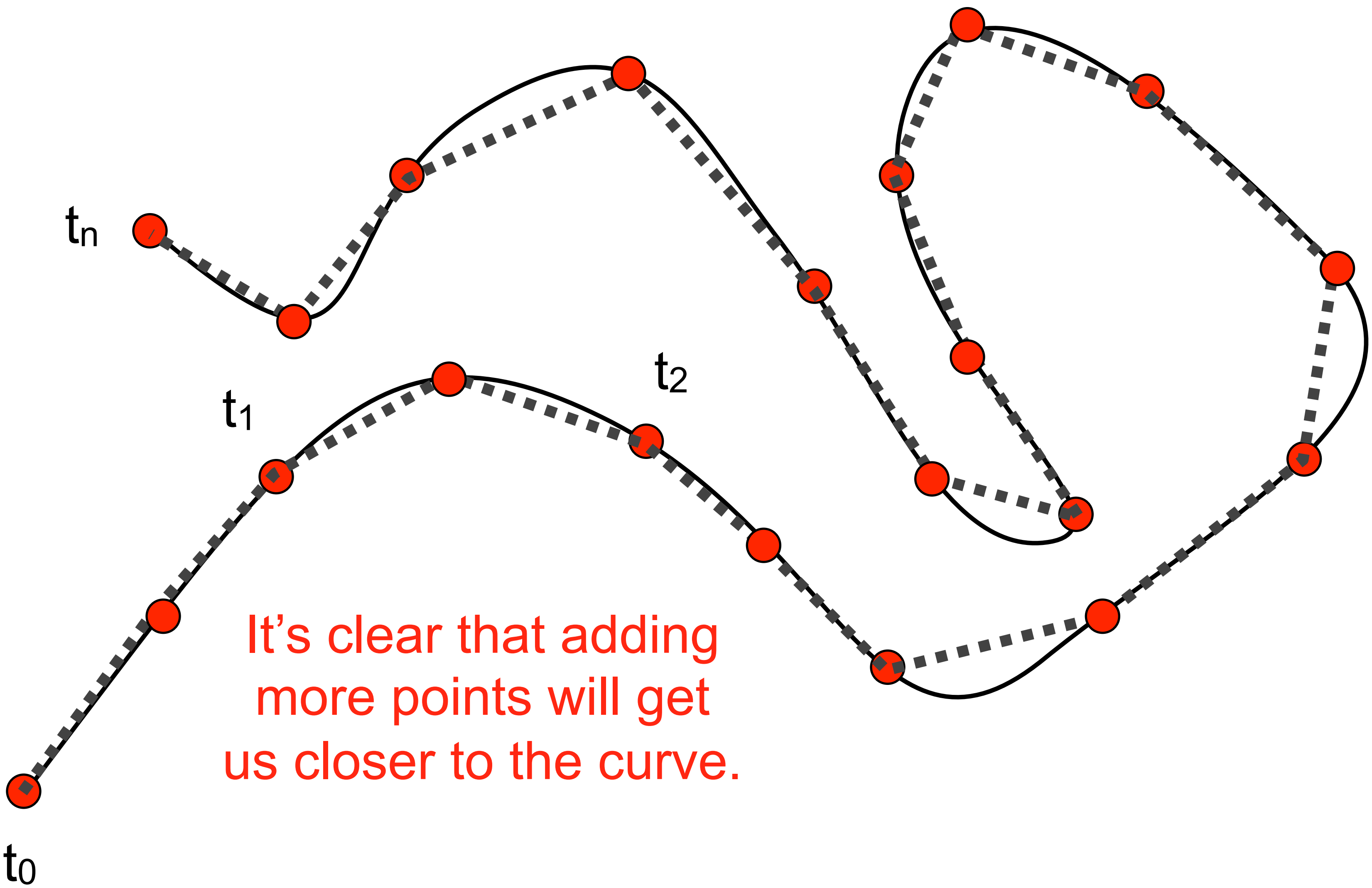


Tessellation



It's clear that adding more points will get us closer to the curve.

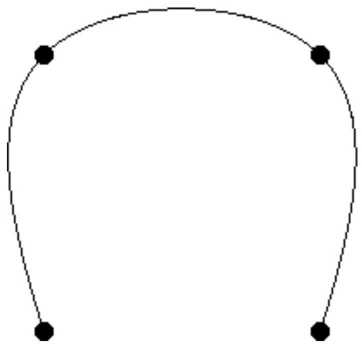
Tessellation



It's clear that adding more points will get us closer to the curve.

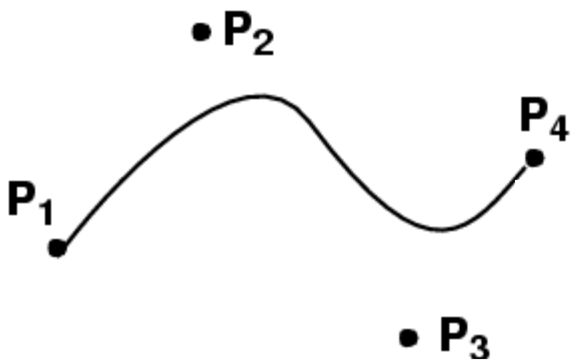
Interpolation vs. Approximation

- Interpolation
  - Goes through all specified points
  - Sounds more logical



Interpolation

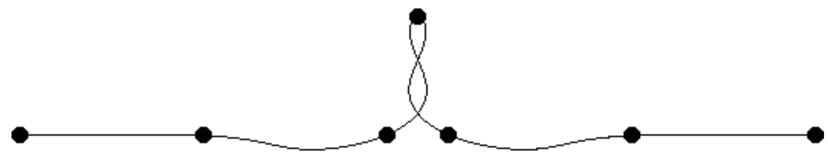
- Approximation
  - Does not go through all points



Approximation

Interpolation vs. Approximation

- Interpolation
  - Goes through all specified points
  - Sounds more logical
  - But can be more unstable
- Approximation
  - Does not go through all points
  - Turns out to be convenient
- We will do something



Interpolation

Approximation

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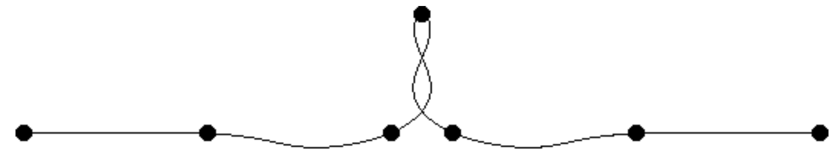
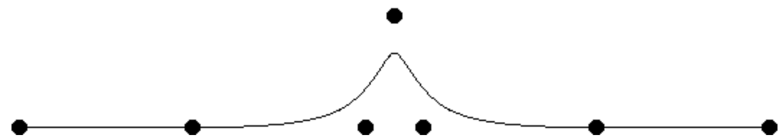


Interpolation

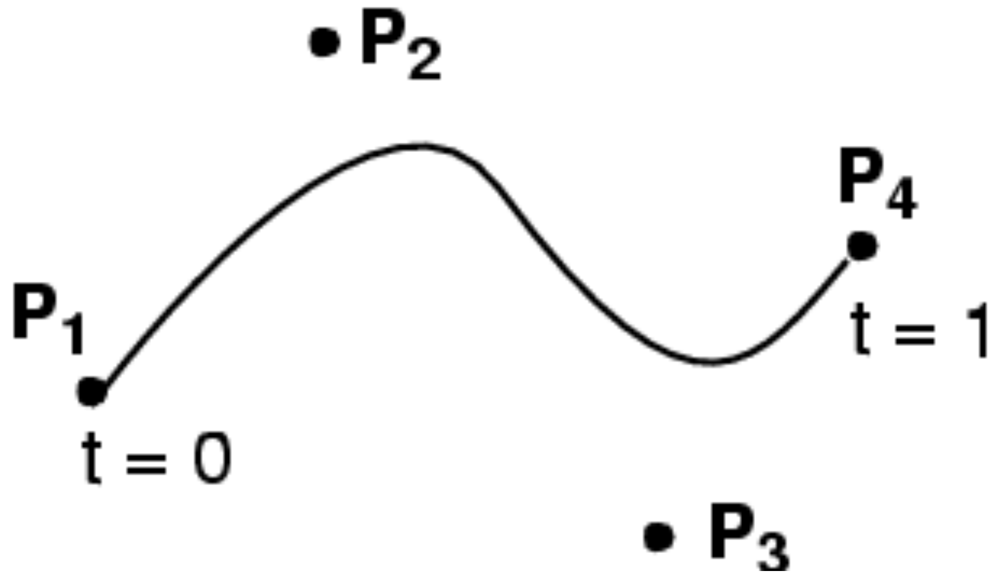


Approximation

Questions?

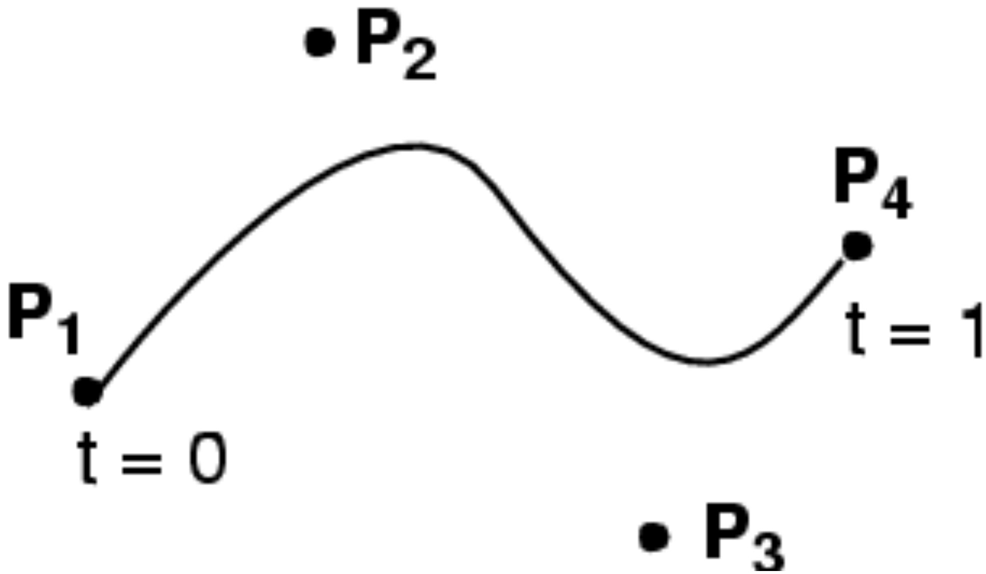
Cubic Bézier Curve

- User specifies 4 control points P_1 ... P_4
- Curve goes through (interpolates) the ends P_1, P_4
- Approximates the two other ones
- Cubic polynomial



Cubic Bézier Curve

- $P(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$



That is,

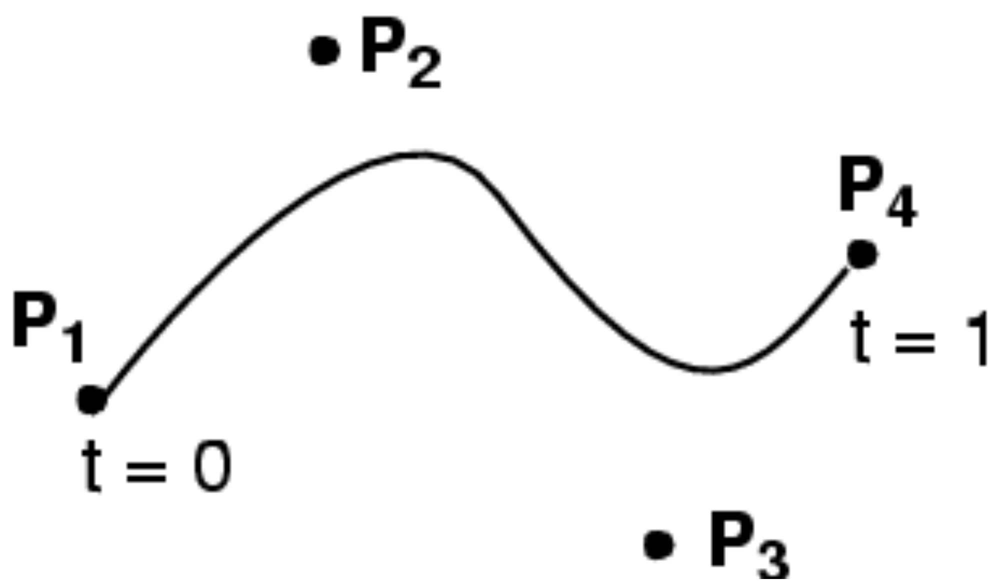
$$x(t) = (1-t)^3\ x_1 + 3t(1-t)^2\ x_2 + 3t^2(1-t)\ x_3 + t^3\ x_4$$

$$y(t) = (1-t)^3\ y_1 + 3t(1-t)^2\ y_2 + 3t^2(1-t)\ y_3 + t^3\ y_4$$

Cubic Bézier Curve

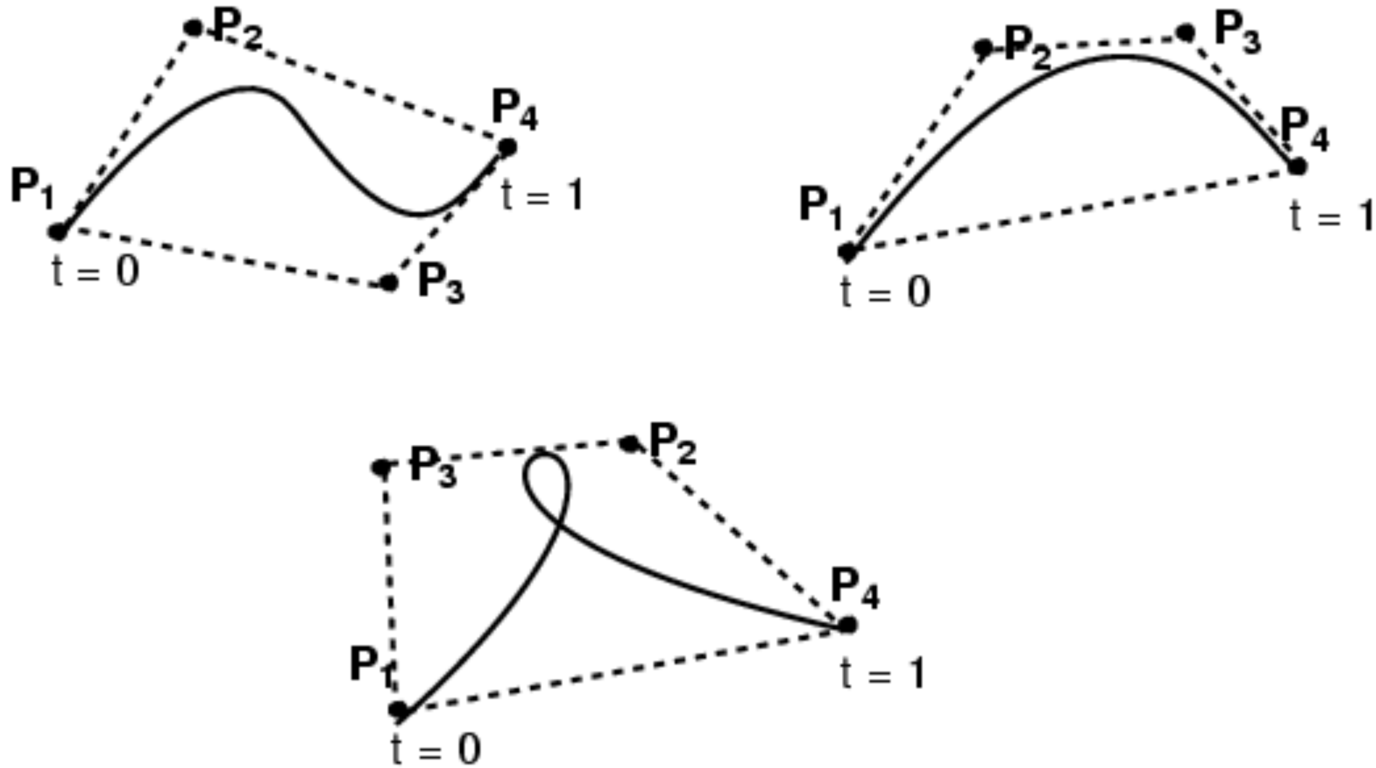
- $$P(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

Verify what happens for t=0 and t=1



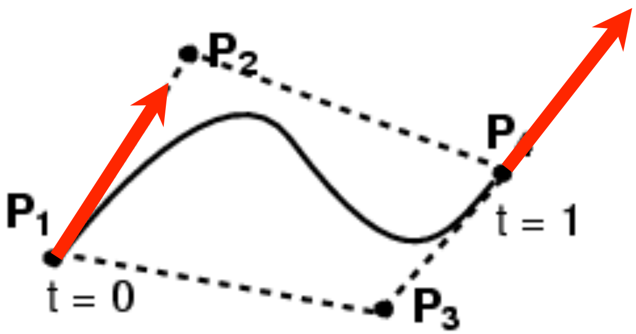
Cubic Bézier Curve

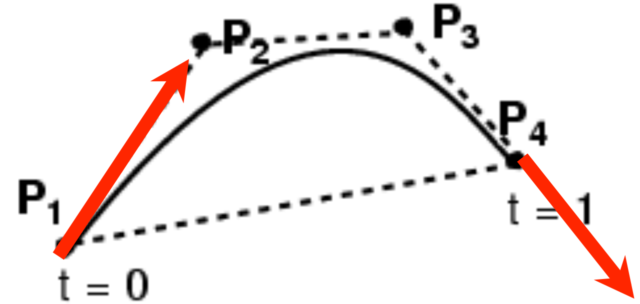
- 4 control points
- Curve passes through first & last control point

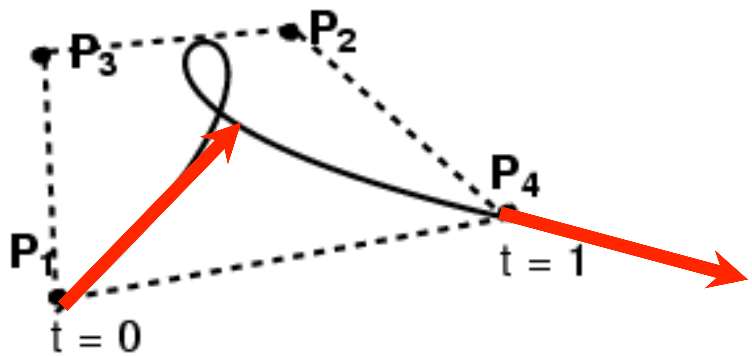


Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at P_1 to $(P_1 - P_2)$ and at P_4 to $(P_4 - P_3)$







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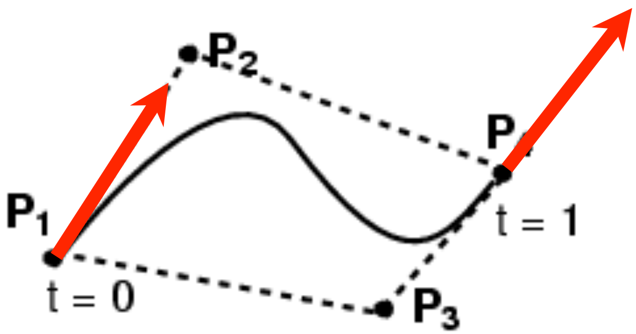

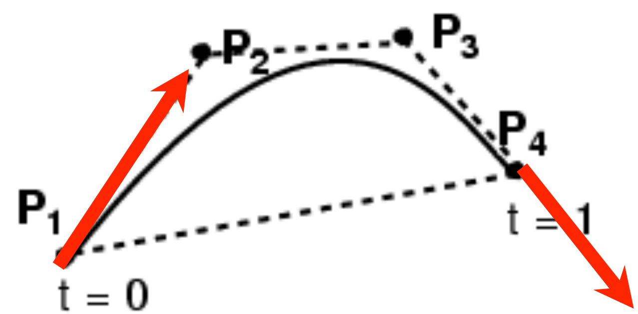

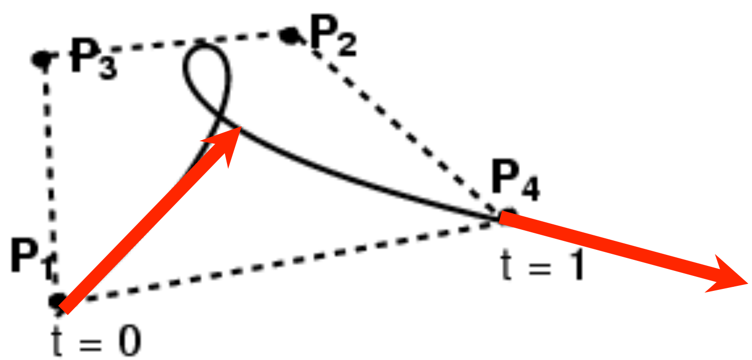

A Bézier curve is bounded by the convex hull of its control points.

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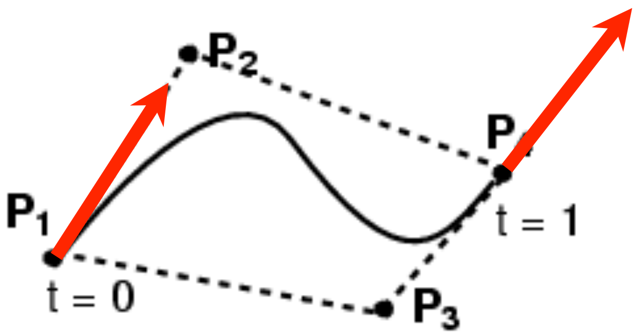
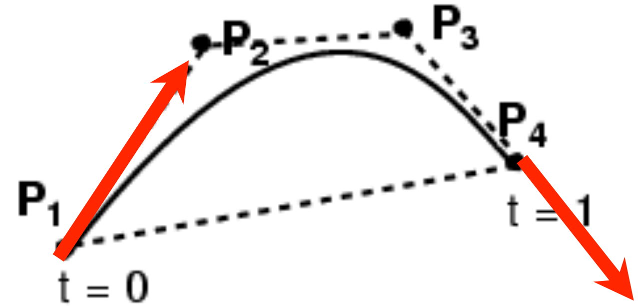
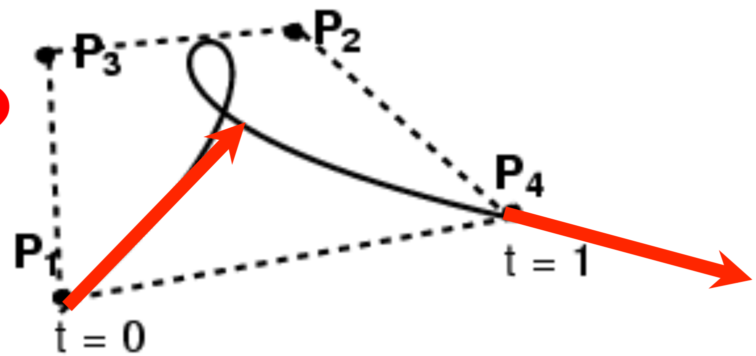




Questions?



A Bézier curve is bounded by the convex hull of its control points.

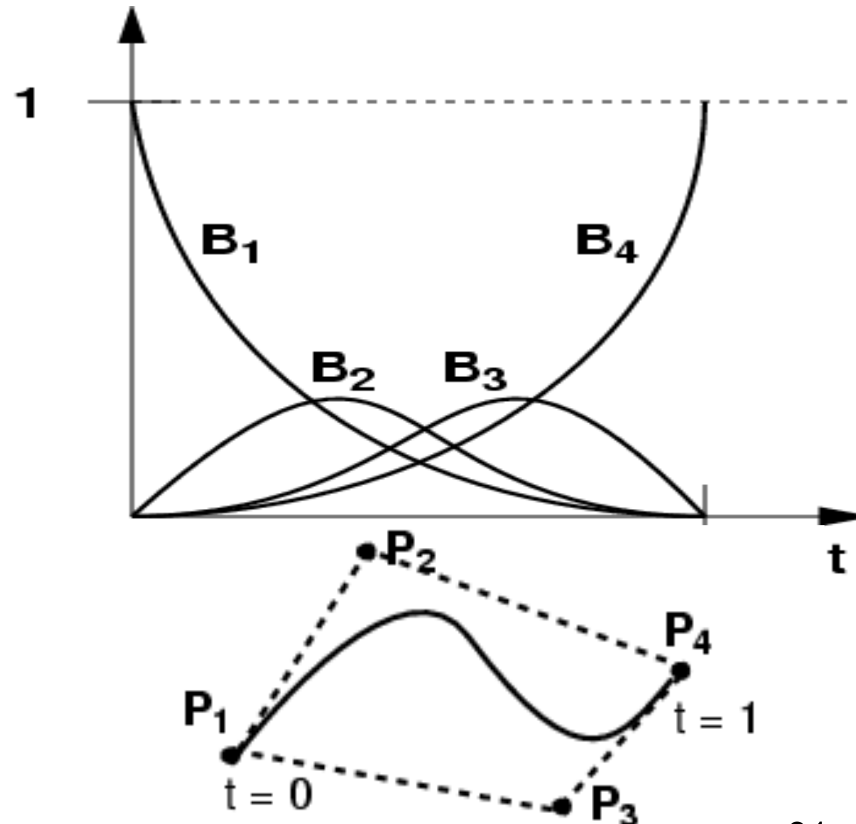
Why Does the Formula Work?

- Explanation 1:
  - It is all magic.
- Explanation 2:
  - These are smart weights that describe the influence of each control point.
- Explanation 3:
  - It is a linear combination of basis polynomials.

Weights

$$P(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

- P(t) is a weighted combination of the 4 control points with weights:
  - $B_1(t)=(1-t)^3$
  - $B_2(t)=3t(1-t)^2$
  - $B_3(t)=3t^2(1-t)$
  - $B_4(t)=t^3$
- First, P_1 is the most influential point, then P_2, P_3, and P_4



Weights

- P_2 and P_3 never have full influence
  - Not interpolated!

$$
\begin{aligned}
P(t) = \quad & (1-t)^3 && P_1 \\
+ \quad & 3t(1-t)^2 && P_2 \\
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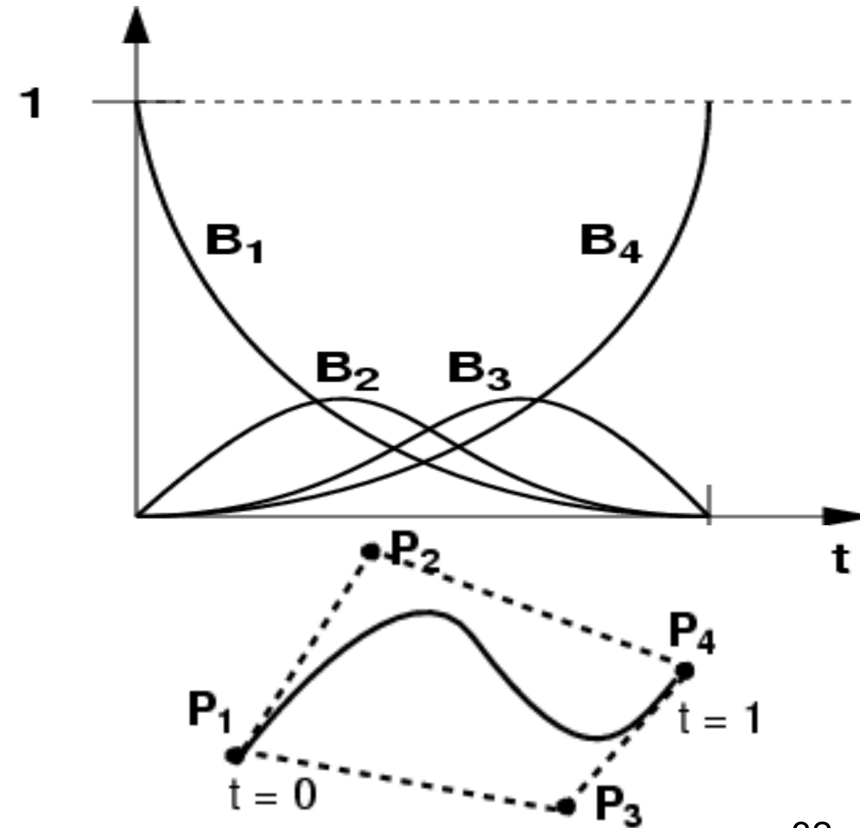
\end{aligned}
$$



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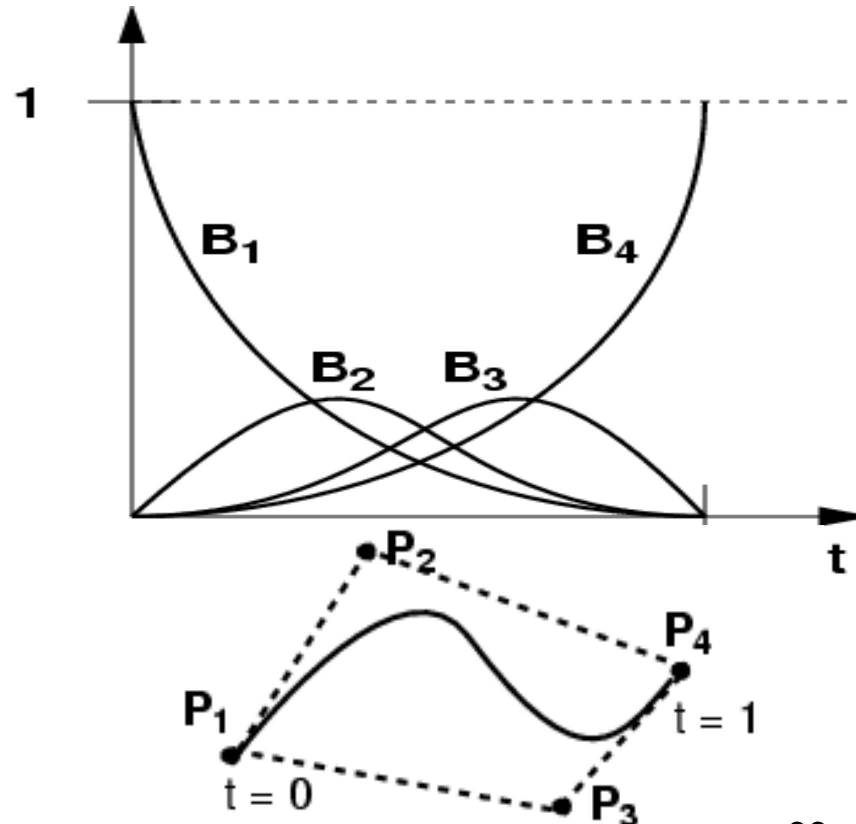


Questions?

Why Does the Formula Work?

- Explanation 1:
  - It is all magic.
- Explanation 2:
  - These are smart weights that describe the influence of each control point
- Explanation 3:
  - **It is a linear combination of basis polynomials.**
  - **The opposite perspective:**
    control points are the weights of polynomials!!!

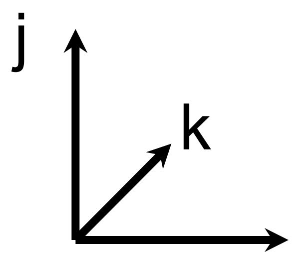
Why Study Splines as Vector Space?

- Understand relationships between types of splines
  - Conversion
- Express what happens when a spline curve is transformed by an affine transform (rotation, translation, etc.)
- Cool simple example of non-trivial vector space
- Important to understand for advanced methods such as finite elements

Usual Vector Spaces

- In 3D, each vector has three components x, y, z
- But geometrically, each vector is actually the sum

$$v = x\,\vec{\boldsymbol{i}} + y\,\vec{\boldsymbol{j}} + z\,\vec{\boldsymbol{k}}$$



- i, j, k are basis vectors

- Vector addition: just add components
- Scalar multiplication: just multiply components

Polynomials as a Vector Space

- Monomials – polynomials with one term
- Polynomials $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$
- Can be added: just add the coefficients

$$
\begin{aligned}
(y + z)(t) = {} & (a_0 + b_0) + (a_1 + b_1)t + \\
& (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n
\end{aligned}
$$

- Can be multiplied by a scalar: multiply the coefficients

$$
s \cdot y(t) =
$$

$$
(s \cdot a_0) + (s \cdot a_1)t + (s \cdot a_2)t^2 + \ldots + (s \cdot a_n)t^n
$$

Polynomials as a Vector Space

- Polynomials $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$

- In the polynomial vector space, $\{1, t, \ldots, t^n\}$ are the basis vectors, $a_0, a_1, \ldots, a_n$ are the components

Polynomials as a Vector Space

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Questions?

Subset of Polynomials: Cubic

$$y(t) = a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3$$

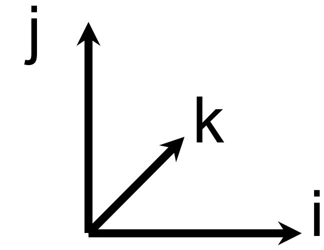
- Closed under addition & scalar multiplication
  - Means the result is still a cubic polynomial (verify!)
- Cubic polynomials also compose a vector space
  - A 4D subspace of the full space of polynomials
- The x and y coordinates of cubic Bézier curves belong to this subspace as functions of t.

Basis for Cubic Polynomials

More precisely:
What is a basis?

$$\vec{v} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

In 3D

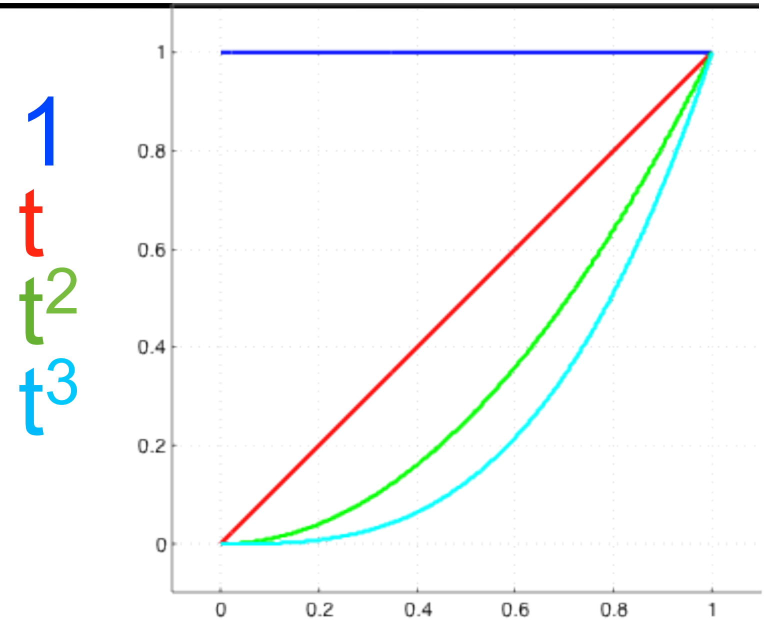


- A set of "atomic" vectors
  - Called basis vectors
  - Linear combinations of basis vectors span the space
    - i.e. any cubic polynomial is a sum of those basis cubics
- Linearly independent
  - Means that no basis vector can be obtained from the others by linear combination
    - Example: i, j, i+j do not form a basis (missing k direction!)

Canonical Basis for Cubics

- Definition
  - Basis given by monomials

$$\{1, t, t^2, t^3\}$$



- Any cubic polynomial is a linear combination of these:

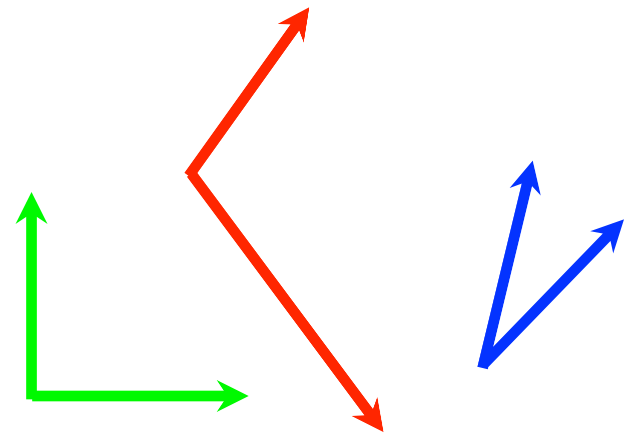
$$a_0+a_1t+a_2t^2+a_3t^3 = a_0*1+a_1*t+a_2*t^2+a_3*t^3$$

- They are linearly independent
  - Means you cannot write any of the four monomials as a linear combination of the others. (You can try.)

Different Basis

- For example:
  - $\{1,\ 1+t,\ 1+t+t^2,\ 1+t-t^2+t^3\}$
  - $\{t^3,\ t^3+t^2,\ t^3+t,\ t^3+1\}$

2D examples



- These can all be obtained from $1, t, t^2, t^3$ by linear combination
- Infinite number of possibilities, just like you have an infinite number of bases to span R^2

Matrix-Vector Notation

- For example:

  1, 1+t, 1+t+t^2, 1+t-t^2+t^3

  t^3, t^3+t^2, t^3+t, t^3+1

Change-of-basis matrix

"Canonical" monomial basis

These relationships hold for each value of t

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Matrix-Vector Notation

- For example:

  1, 1+t, 1+t+t^2, 1+t-t^2+t^3

  t^3, t^3+t^2, t^3+t, t^3+1

Change-of-basis matrix

"Canonical" monomial basis

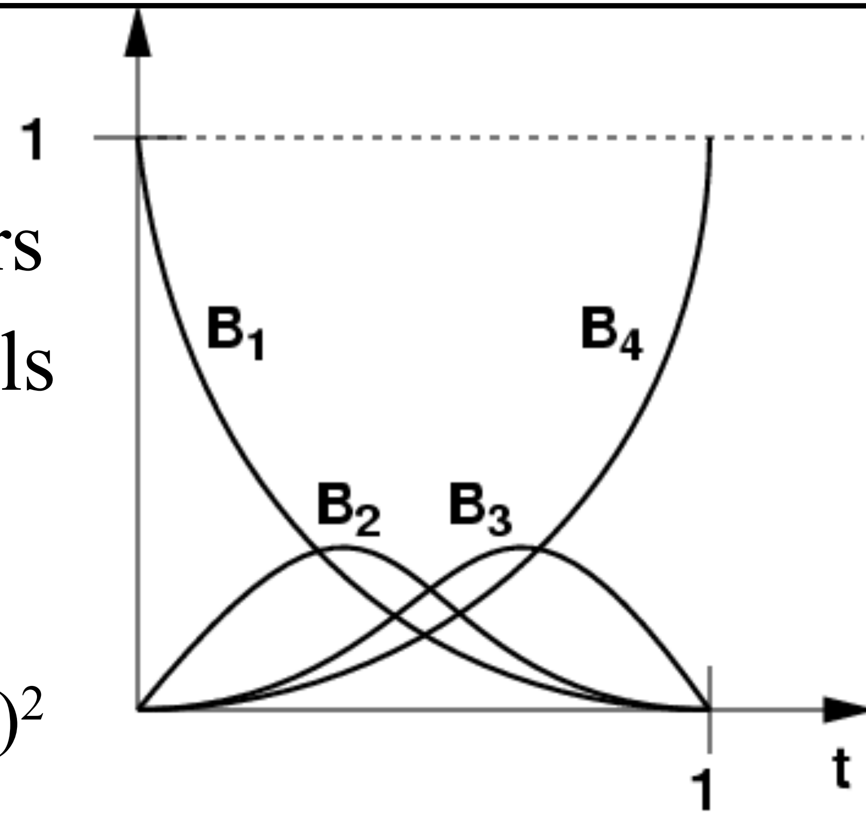
$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Not any matrix will do!
If it's singular, the basis set will be linearly dependent, i.e., redundant and incomplete.

Bernstein Polynomials

- For Bézier curves, the basis polynomials/vectors are Bernstein polynomials



- For cubic Bezier curve:

  $B_1(t)=(1-t)^3$ $B_2(t)=3t(1-t)^2$

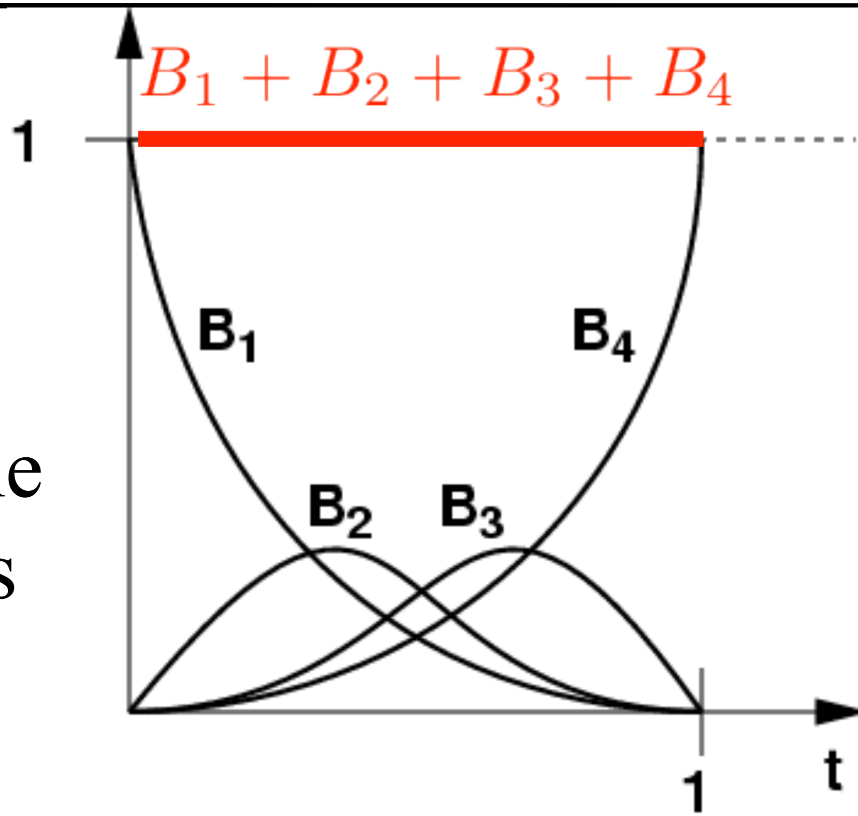
  $B_3(t)=3t^2(1-t)$ $B_4(t)=t^3$

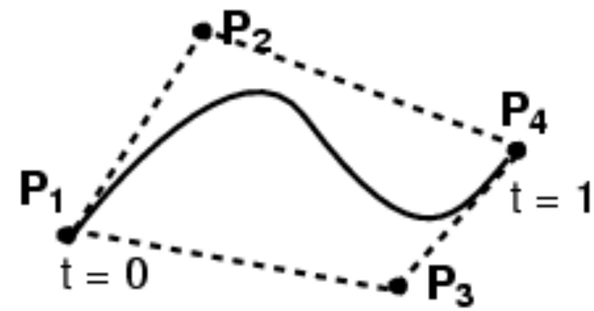
  (careful with indices, many authors start at 0)

- Defined for any degree

Properties of Bernstein Polynomials

- ≥ 0 for all $0 \leq t \leq 1$
- Sum to 1 for every t
  - called partition of unity
- These two together are the reason why Bézier curves lie within convex hull
- $B_1(0) = 1$
  - Bezier curve interpolates P_1
- $B_4(1) = 1$
  - Bezier curve interpolates P_4





Bézier Curves in Bernstein Basis

- $P(t) = P_1B_1(t) + P_2B_2(t) + P_3B_3(t) + P_4B_4(t)$
  - P_i are 2D points (x_i, y_i)
- $P(t)$ is a linear combination of the control points with weights equal to Bernstein polynomials at t
- But at the same time, the control points (P_1, P_2, P_3, P_4) are the "coordinates" of the curve in the Bernstein basis
  - In this sense, specifying a Bézier curve with control points is exactly like specifying a 2D point with its x and y coordinates.

Two Different Vector Spaces!!!

- The plane where the curve lies, a 2D vector space
- The space of cubic polynomials, a 4D space
- Don't be confused!
- The 2D control points can be replaced by 3D points – this yields space curves.
  - The math stays the same, just add z(t).
- The cubic basis can be extended to higher-order polynomials
  - Higher-dimensional vector space
  - More control points

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  - Higher-dimensional vector space
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Questions?

Change of Basis

- How do we go from Bernstein basis to the canonical monomial basis 1, t, t^2, t^3 and back?
  - With a matrix!

- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

New basis vectors

How You Get the Matrix

Cubic Bernstein:

- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$

Expand these out and collect powers of t.
The coefficients are the entries in the matrix B!

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Change of Basis, Other Direction

- Given B1...B4, how to get back to canonical 1, t, t^2, t^3 ?

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Change of Basis, Other Direction

That's right, with the inverse matrix!

- Given B1...B4, how to get back to canonical 1, t, t^2, t^3 ?

$$\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}^{-1}} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

Recap

- Cubic polynomials form a 4D vector space.
- Bernstein basis is canonical for Bézier.
  - Can be seen as influence function of data points
  - Or data points are coordinates of the curve in the Bernstein basis
- We can change between basis with matrices.

Recap

- Cubic polynomials form a 4D vector space.
- Bernstein basis is canonical for Bézier.
  - Can be seen as influence function of data points
  - Or data points are coordinates of the curve in the Bernstein basis
- We can change between basis with matrices.

Questions?

More Matrix-Vector Notation

$$P(t) = \sum_{i=1}^{4} P_i \; B_i(t) = \sum_{i=1}^{4} \left[\begin{pmatrix} x_i \\ y_i \end{pmatrix} B_i(t) \right]$$

Bernstein polynomials
(4x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

point on curve
(2x1 vector)

matrix of
control points (2 x 4)

Flashback

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Cubic Bézier in Matrix Notation

point on curve
(2x1 vector)

Canonical
monomial basis

$$\mathrm{P(t)} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} =$$

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

"Geometry matrix"
of control points $P_1..P_4$
(2 x 4)

"Spline matrix"
(Bernstein)

General Spline Formulation

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

- Geometry: control points coordinates assembled into a matrix (P_1, P_2, …, P_{n+1})
- Spline matrix: defines the type of spline
  – Bernstein for Bézier
- Power basis: the monomials ($1, t, ..., t^n$)
- Advantage of general formulation
  – Compact expression
  – Easy to convert between types of splines
  – Dimensionality (plane or space) does not really matter

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Questions?

A Cubic Only Gets You So Far

- What if you want more control?

Higher-Order Bézier Curves

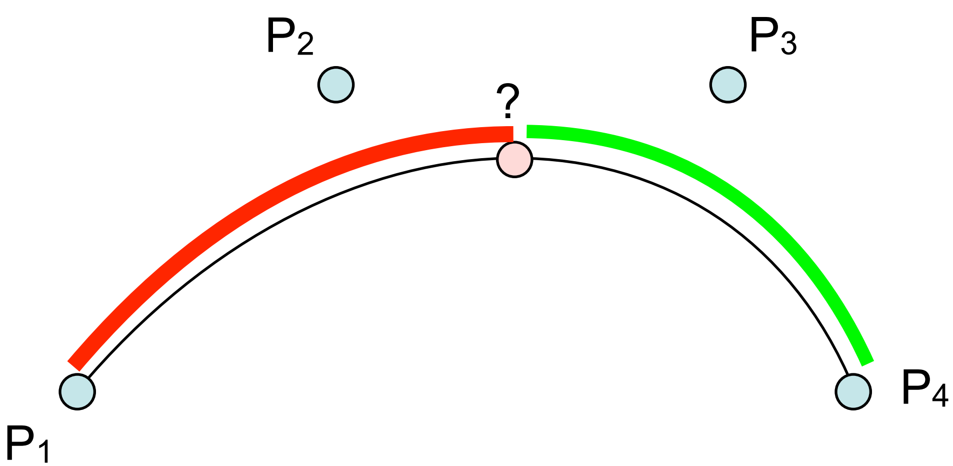
- > 4 control points
- Bernstein Polynomials as the basis functions
  - For polynomial of order *n*, the i^{th} basis function is

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling
- You will not need this in this class

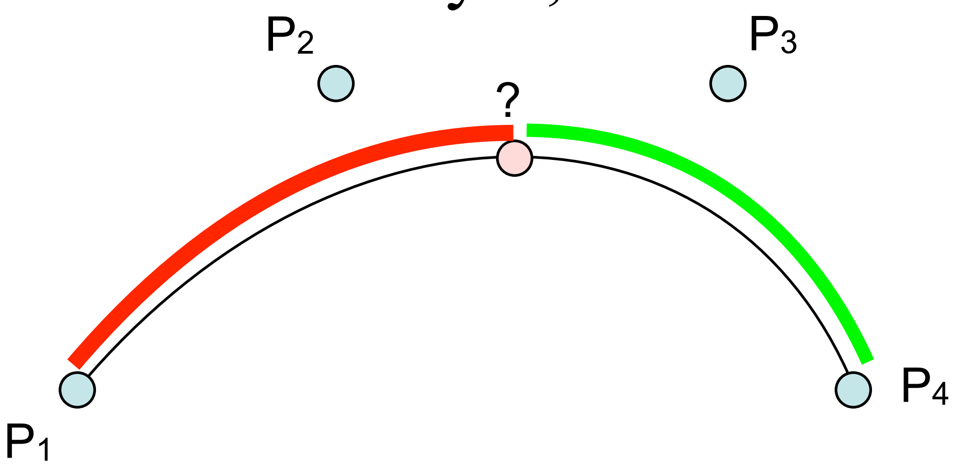
Subdivision of a Bezier Curve

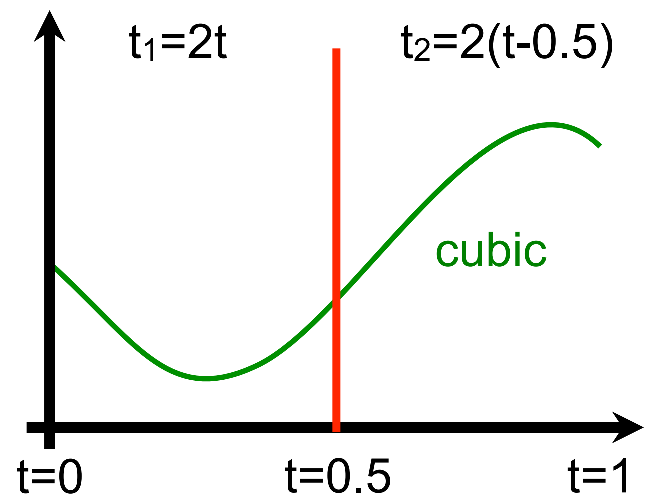
- Can we split a Bezier curve in the middle into two Bézier curves?
  - This is useful for adding detail
  - It avoids using nasty higher-order curves



Subdivision of a Bezier Curve

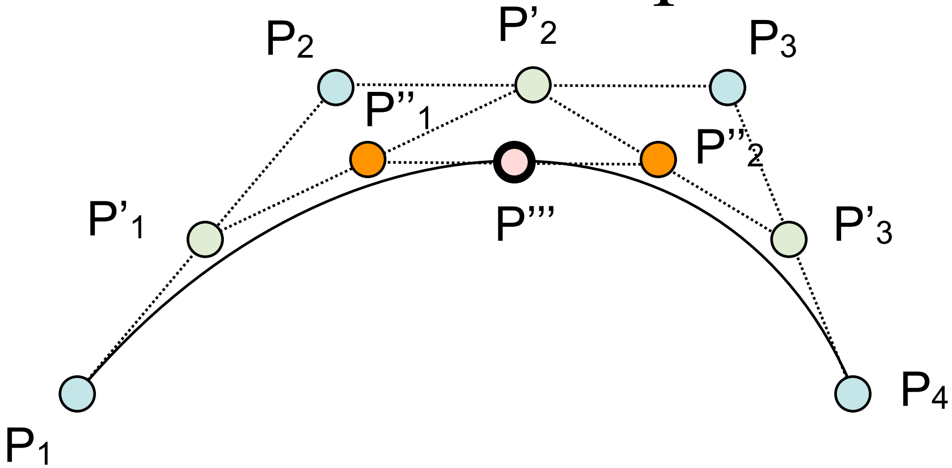
- Can we split a Bezier curve in the middle into two Bézier curves?
  - The resulting curves are again a cubic (Why? A cubic in t is also a cubic in 2t)
  - Hence it must be representable using the Bernstein basis. So yes, we can!





De Casteljau Construction

- Take the middle point of each of the 3 segments
- Construct the two segments joining them
- Take the middle of those two new segments
- Join them
- Take the middle point P'''



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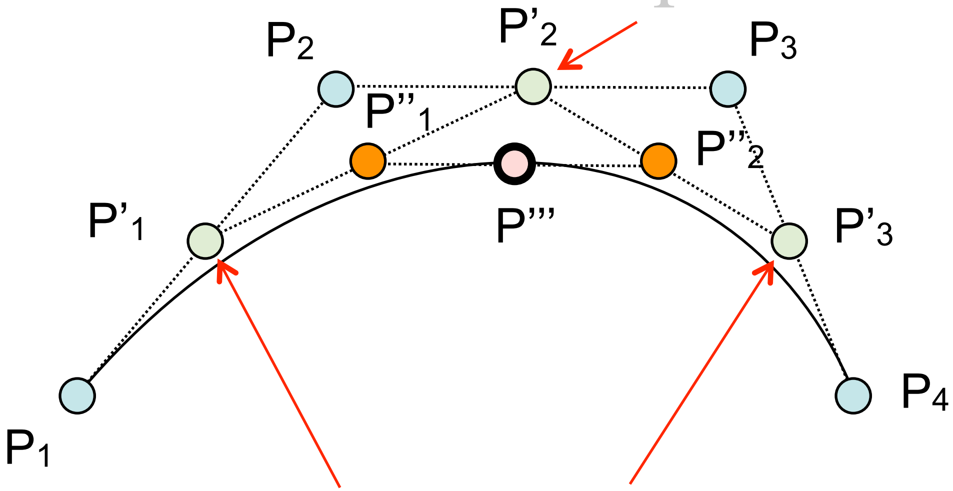

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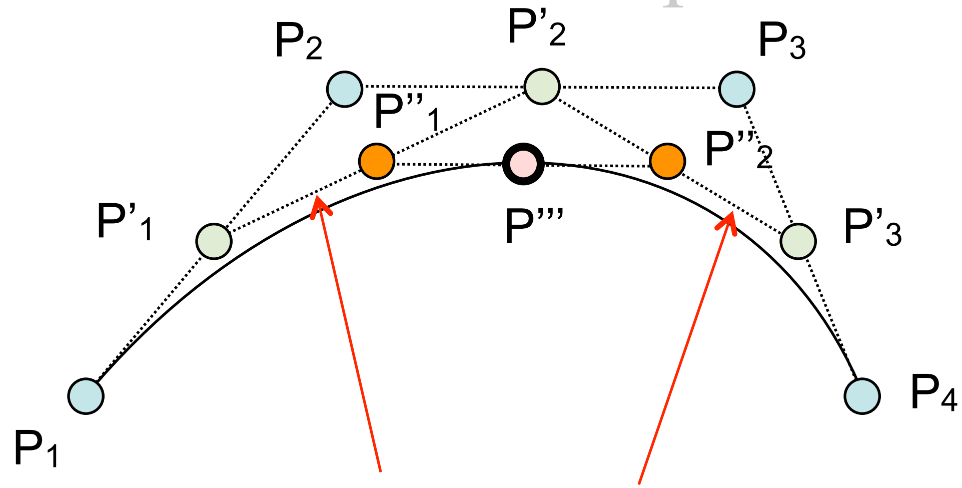

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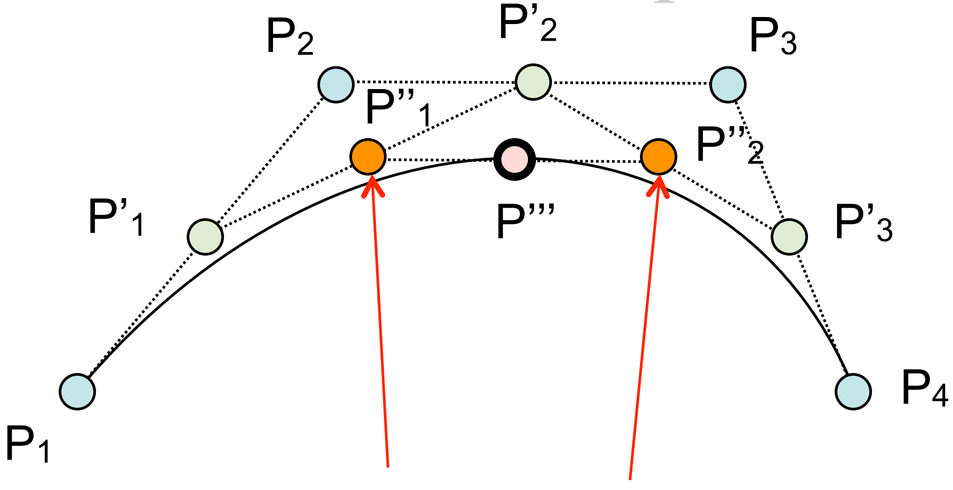
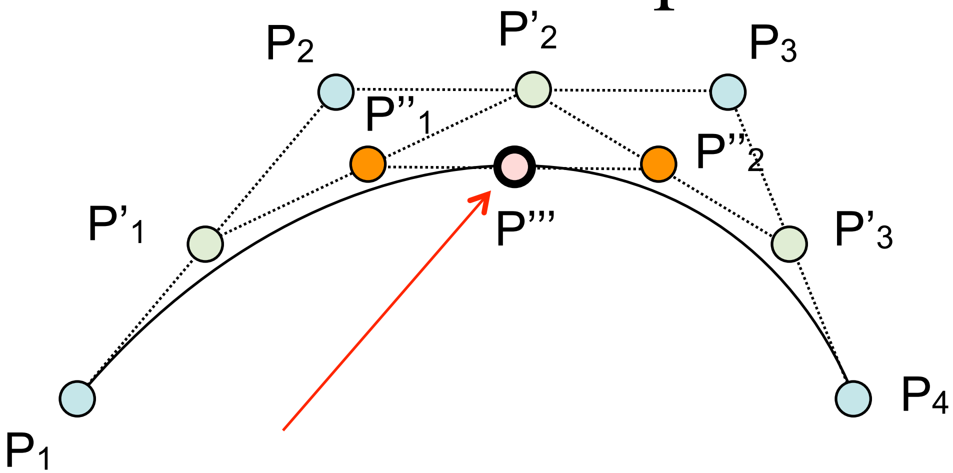


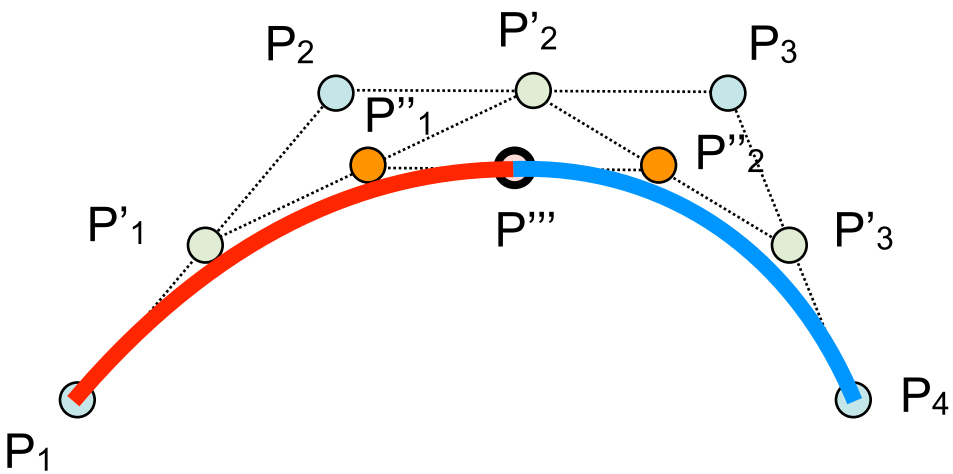
De Casteljau Construction

- Take the middle point of each of the 3 segments
- Construct the two segments joining them
- Take the middle of those two new segments
- Join them
- Take the middle point P’’’



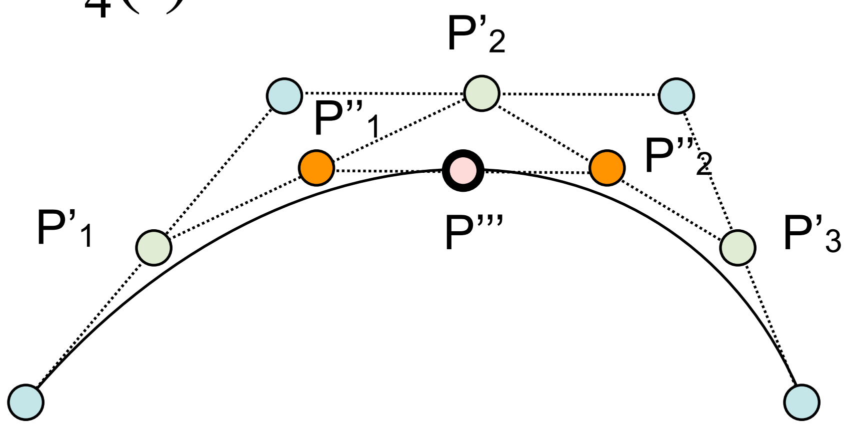
Result of Split in Middle

- The two new curves are defined by
  - P_1, P'_1, P''_1, and P'''
  - P''', P''_2, P'_3, and P_4
- Together they exactly replicate the original curve!
  - Originally 4 control points, now 7 (more control)



Sanity Check

- Do we actually get the middle point?
- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$



$$
\begin{aligned}
P_1' &= 0.5(P_1 + P_2) \\
P_2' &= 0.5(P_2 + P_3) \\
P_3' &= 0.5(P_3 + P_4)
\end{aligned}
$$

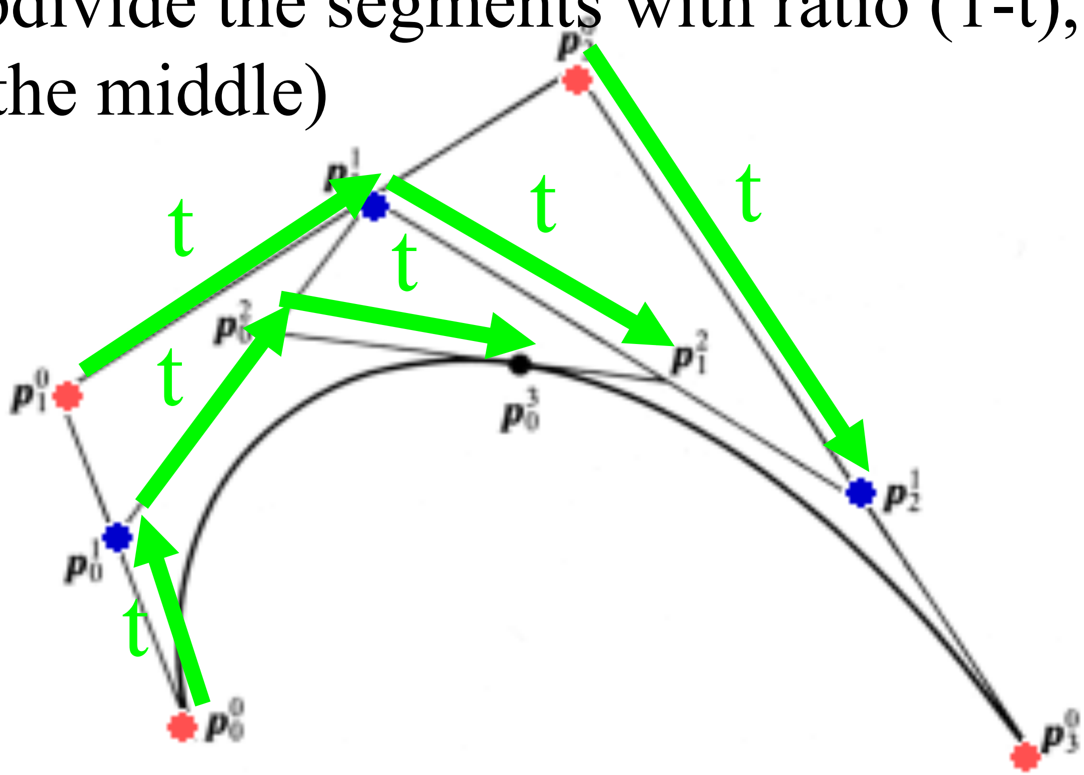
$$
\begin{aligned}
P_1'' &= 0.5(P_1' + P_2') \\
P_2'' &= 0.5(P_2' + P_3')
\end{aligned}
$$

$$
\begin{aligned}
P''' &= 0.5(P_1'' + P_2'') \\
&= 0.5\left(0.5(P_1' + P_2') + 0.5(P_2' + P_3')\right) \\
&= 0.5\left(0.5\left[0.5(P_1 + P_2) + 0.5(P_2 + P_3)\right] + \right. \\
&\quad \left. 0.5\left[0.5(P_2 + P_3) + 0.5(P_3 + P_4)\right]\right) \\
&= 1/8P_1 + 3/8P_2 + 3/8P_3 + 1/8P_4
\end{aligned}
$$

✔

De Casteljau Construction

- Actually works to construct a point at any t, not just 0.5
- Just subdivide the segments with ratio (1-t), t (not in the middle)



Recap

- Bezier curves: piecewise polynomials
- Bernstein polynomials
- Linear combination of basis functions
  - Basis: control points   weights: polynomials
  - Basis: polynomials   weights: control points
- Subdivision by de Casteljau algorithm
- All linear, matrix algebra

That's All for Today, Folks

- Further reading
  - Buss, Chapters 7 and 8

  - Fun stuff to know about function/vector spaces
    - http://en.wikipedia.org/wiki/Vector_space
    - http://en.wikipedia.org/wiki/Functional_analysis
    - http://en.wikipedia.org/wiki/Function_space

- Inkscape is an open source vector drawing program for Mac/Windows. Try it out!