Coordinates and Transformations

50.017 Graphics and Visualization
Sai-Kit Yeung

Notes courtesy by Wojciech Matusik, many slides follow Steven Gortler’s book

Hierarchical modeling

- **Many coordinate systems:**
  - Camera
  - Static scene
  - car
  - driver
  - arm
  - hand
  - …

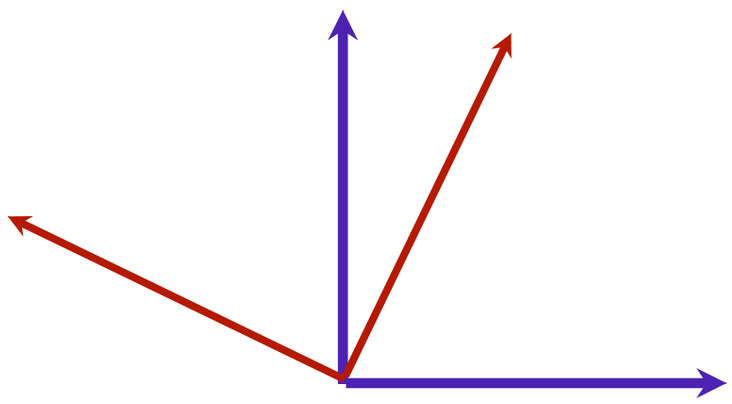
- **Makes it important to understand coordinate systems**

Coordinates

- We are used to represent points with tuples of coordinates such as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- But the tuples are meaningless without a clear coordinate system

could be this point in the red coordinate system

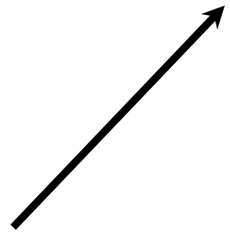
could be this point in the blue coordinate system



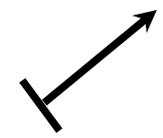
Different objects

- **Points**
  - represent locations

- **Vectors**
  - represent movement, force, displacement from A to B



- **Normals**
  - represent orientation, unit length



- **Coordinates**
  - Not vector
  - numerical representation of the above objects **in a given coordinate system**

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Points & vectors are different

- The 0 vector has a fundamental meaning: no movement, no force
- Why would there be a special 0 point?

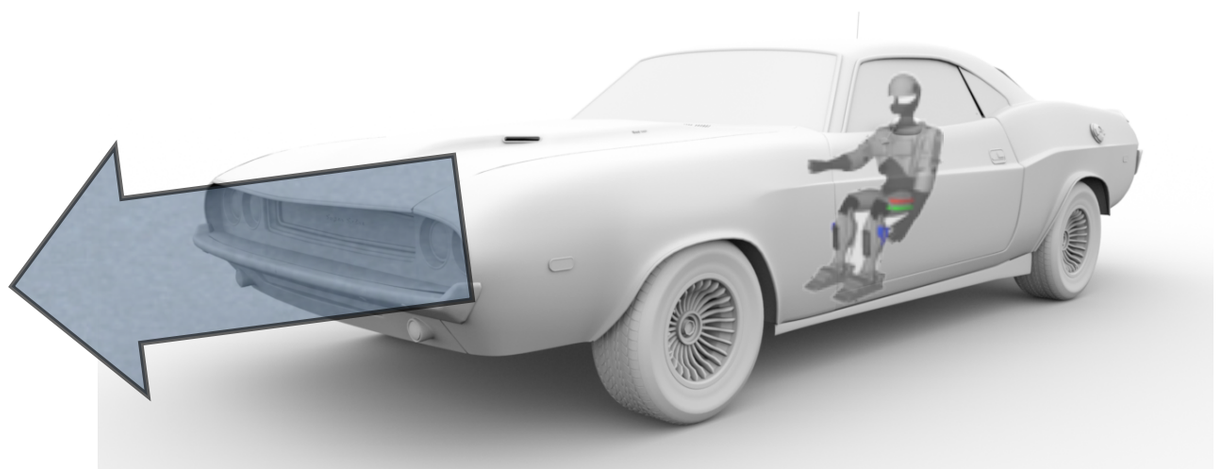
- **It's meaningful to add vectors, not points**
  - Boston location + Singapore location =?

+

= ?

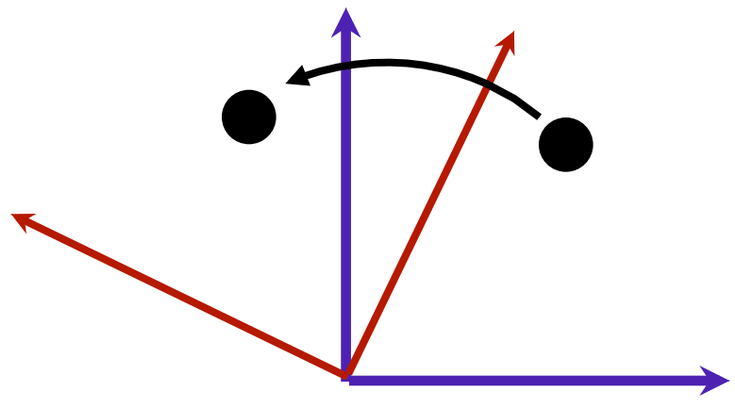
Points & vectors are different

- Moving car
  - points describe location of car elements
  - vectors describe velocity, distance between pairs of points
- **If I translate the moving car to a different road**
  - The points (location) change
  - The vectors (speed, distance between points) don't



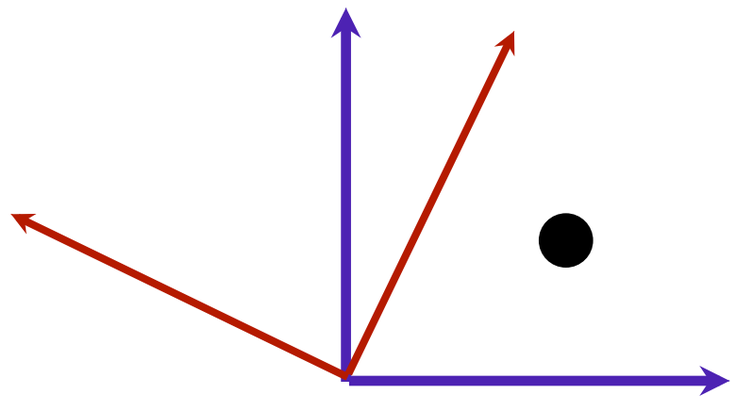
Matrices have two purposes

- (At least for geometry)
- Transform things
  - e.g. rotate the car from facing North to facing East



- Express coordinate system changes
  - e.g. given the driver's location in the coordinate system of the car, express it in the coordinate system of the world

same point in different coordinate system



Point doesn't get transformed, but different coordinates in another coordinate system

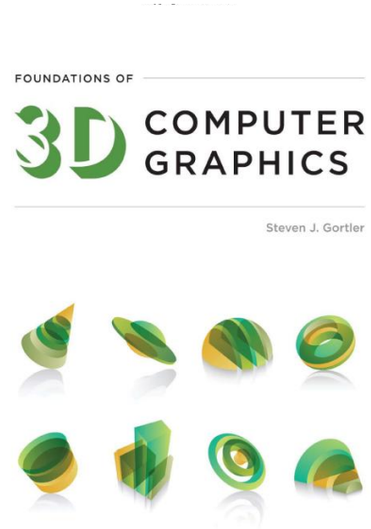
Goals for today

- Make it very explicit what coordinate system is used
- Understand how to change coordinate systems
- Understand how to transform objects
- Understand difference between points, vectors, normals and their coordinates
- More emphasis on keeping track of coordinate systems

Questions?

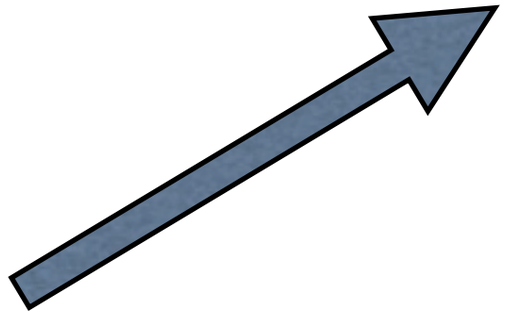
Reference

- This lecture follows the new book by Steven (Shlomo) Gortler from Harvard: Foundations of 3D Computer Graphics

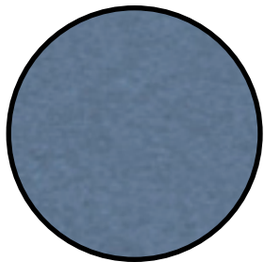


Plan

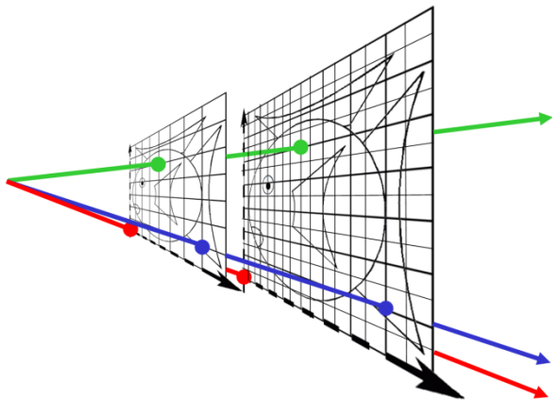
- Vectors



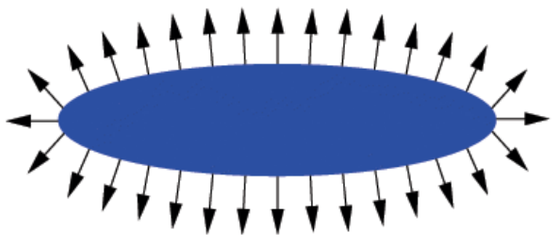
- Points



- Homogenous coordinates



- Normals (in the next lecture)



Vectors (linear space)

- Formally, a set of elements equipped with addition and scalar multiplication
  - plus other nice properties
- There is a special element, the zero vector
  - no displacement, no force

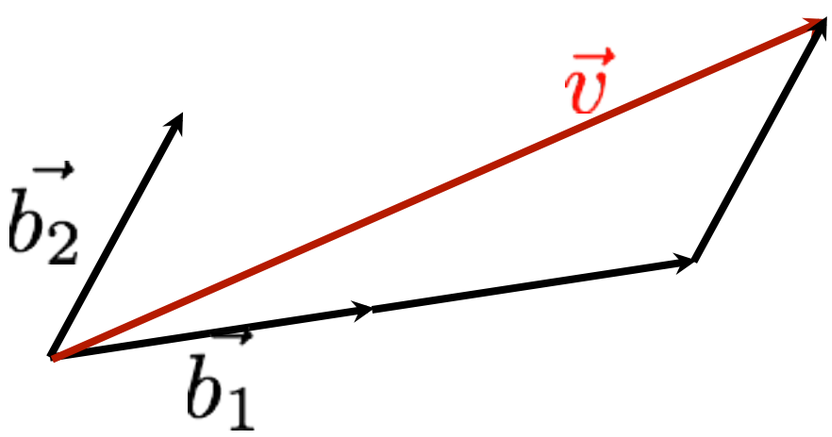
Vectors (linear space)

- We can use a ***basis*** to produce all the vectors in the space:
  - Given n basis vectors $\vec{b_i}$ any vector $\vec{v}$ can be written as $\vec{v} = \sum c_i \vec{b_i}$

i

scalars

Coordinates of this vector in the basis



here:

$\vec{v} = 2\vec{b_1} + \vec{b_2}$

Linear algebra notation

$$\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$$

- can be written as

$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Nice because it makes the basis (coordinate system) explicit
- Shorthand:

$$\vec{v} = \vec{\mathbf{b}}^t\mathbf{c}$$

Row vector, elements are the basis vectors

- where bold means triplet, t is transpose

Linear algebra notation

- Different from conventional linear algebra class
  - Keep the basis on the left
  - Calculations include the basis

=> Keep track of the coordinate system

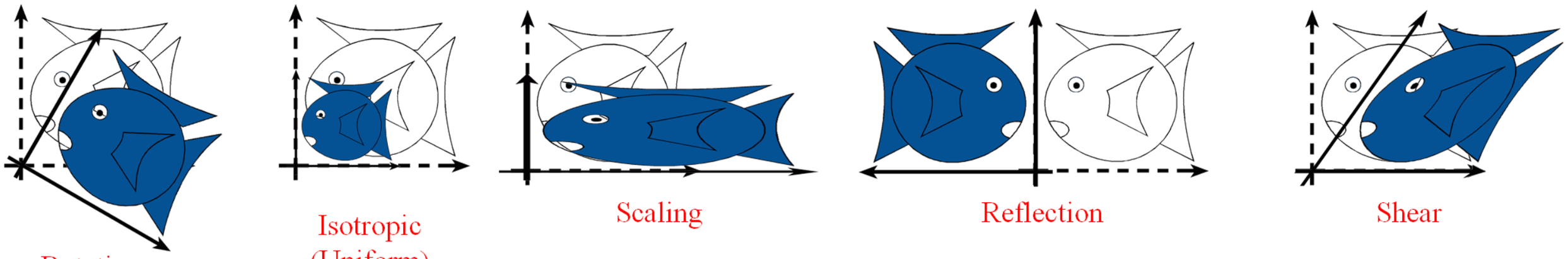
$$\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$$

$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{v} = \vec{\mathbf{b}}^t\mathbf{c}$$

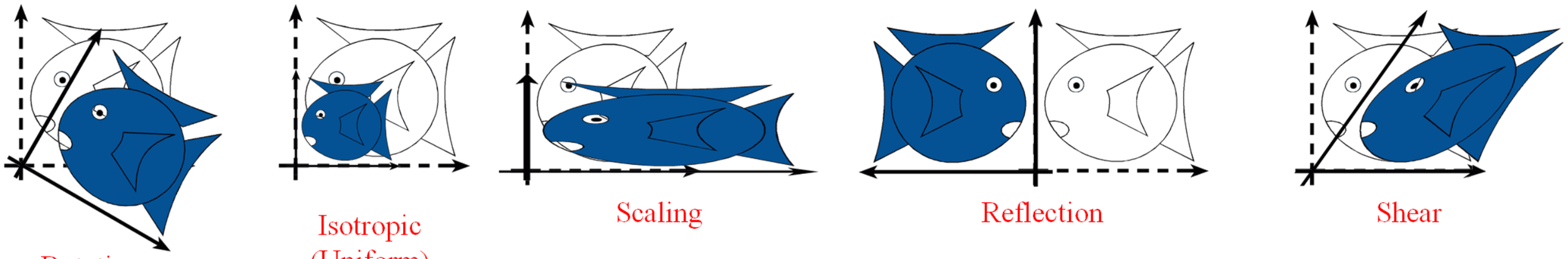
Questions?

Linear transformation



- Transformation $\mathcal{L}$ of the vector space
- What does it mean for the transformation being linear?
- What kind of mathematical property we need?

Linear transformation



- Transformation $\mathcal{L}$ of the vector space so that

$$\mathcal{L}(\vec{v}+\vec{u}) = \mathcal{L}(\vec{v}) + \mathcal{L}(\vec{u})$$

$$\mathcal{L}(\alpha\vec{v}) = \alpha\mathcal{L}(\vec{v})$$

- Note that it implies $\mathcal{L}(\vec{0}) = \vec{0}$
- Notation $\vec{v} \Rightarrow \mathcal{L}(\vec{v})$ for transformations

Matrix notation

- Linearity implies

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b_i}\right) = \quad ?$$

Matrix notation

- Linearity implies

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b_i}\right) = \sum_i c_i \mathcal{L}(\vec{b_i})$$

- i.e. we only need to know the basis transformation
- or in algebra notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Algebra notation

- The $\mathcal{L}(\vec{b_i})$ are also vectors of the space
- They can be expressed in the basis

...

Algebra notation

- The $\mathcal{L}(\vec{b_i})$ are also vectors of the space
- They can be expressed in the basis for example:

$$\mathcal{L}(\vec{b}_1) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ M_{3,1} \end{bmatrix}$$

coordinates

Algebra notation

- The $\mathcal{L}(\vec{b_i})$ are also vectors of the space
- They can be expressed in the basis for example:

$$\mathcal{L}(\vec{b}_1) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ M_{3,1} \end{bmatrix}$$

coordinates

- which gives us

$$\begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

Recap, matrix notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

New Basis
Same coordinates

New coordinates
Same Basis

- Given the coordinates ***c*** in basis $\vec{\mathrm{b}}$
  the transformed vector has coordinates Mc in $\vec{\mathrm{b}}$
- Dual perspective

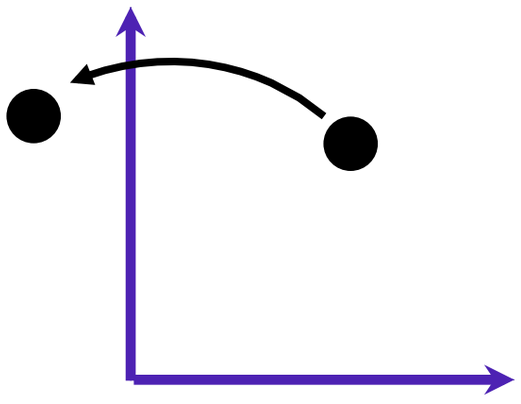
Recap, matrix notation

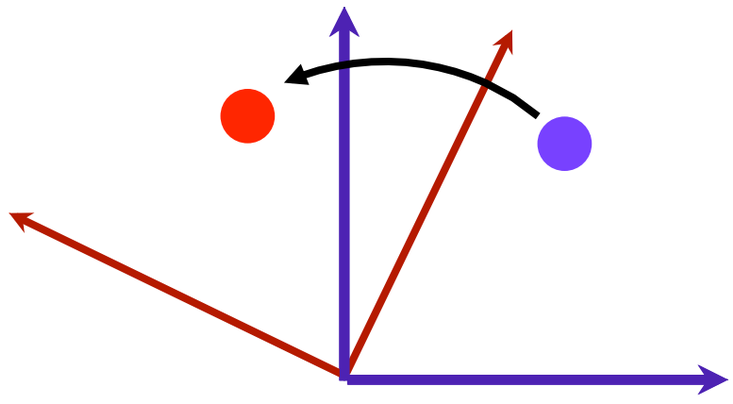
$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

New Basis
Same coordinates

New coordinates
Same Basis





Why do we care

- We like linear algebra
- It’s always good to get back to an abstraction that we know and for which smarter people have developed a lot of tools
- But we also need to keep track of what basis/ coordinate system we use

Questions?

Change of basis

- Critical in computer graphics
  - From world to car to arm to hand coordinate system
  - From Bezier splines to B splines and back

- problem with basis change:
  you never remember which is M or M^{-1}
  it’s hard to keep track of where you are

Change of basis

- Assume we have two bases $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$
- And we have the coordinates of $\vec{\mathrm{a}}$ in $\vec{\mathrm{b}}$
- e.g.

$$\vec{a_1} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix}$$

- i.e.

$$\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$$

- which implies

$$\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$$

What would be the coordinates of $\vec{\mathrm{b}}$ in $\vec{\mathrm{a}}$?

Change of basis

- We have $\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$ & $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$
- Given the coordinate of $\vec{v}$ in $\vec{\mathbf{b}}$: $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$

- What are the coordinates in $\vec{\mathbf{a}}$?

Change of basis

- We have $\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$ & $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$
- Given the coordinate of $\vec{v}$ in $\vec{\mathbf{b}}$: $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$

- Replace $\vec{\mathbf{b}}$ by its expression in $\vec{\mathbf{a}}$

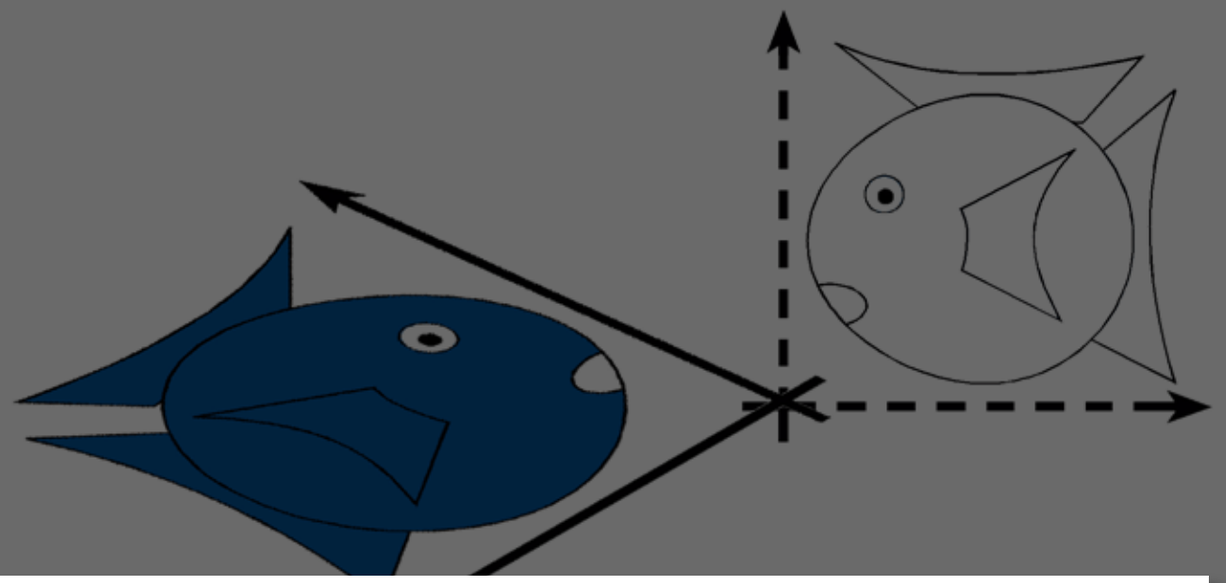
$$\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$$

- $\vec{v}$ has coordinates $M^{-1}\mathbf{c}$ in $\vec{\mathbf{a}}$
- Note how we keep track of the coordinate system by having the basis on the left

Questions?

Linear Transformations

- $L(p + q) = L(p) + L(q)$
- $L(ap) = a\ L(p)$



Translation is not linear:

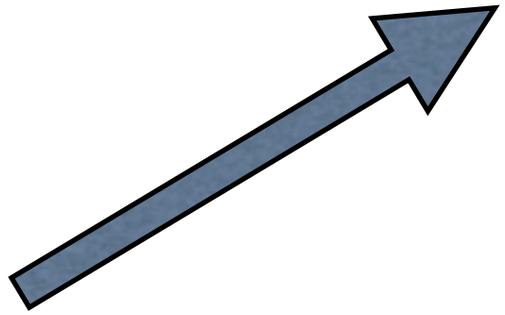
$$f(p) = p+t$$

$$f(ap) = ap+t \neq a(p+t) = a\ f(p)$$

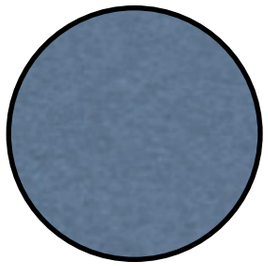
$$f(p+q) = p+q+t \neq (p+t)+(q+t) = f(p) + f(q)$$

Plan

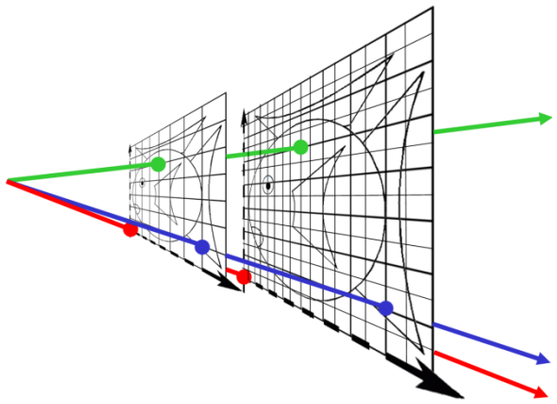
- Vectors



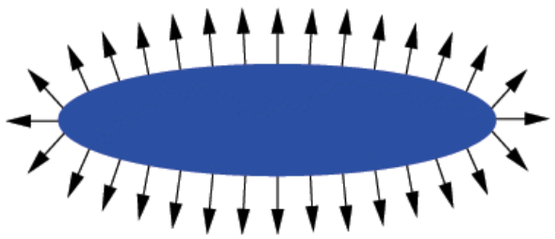
- Points



- Homogenous coordinates



- Normals



Points vs. Vectors

- A point is a location
- A vector is a motion between two points
- Adding vectors is meaningful
  - going 3km North + 4km East = going 5km North-East
- Adding points is not meaningful
  - Boston location + Singapore location = ?
- Multiplying a point by a scalar?
- The zero vector is meaningful (no movement)
- Zero point ?

Affine space

- Points are elements of an affine space
- We denote them with a tilde $\tilde{p}$

- Affine spaces are an extension of vector spaces

Point-vector operations

- Subtracting points gives a vector

$$\tilde{p} - \tilde{q} = \vec{v}$$

- Adding a vector to a point gives a point

$$\tilde{q} + \vec{v} = \tilde{p}$$

Frames

- A frame is an origin $\tilde{o}$ plus a basis $\vec{\mathrm{b}}$
- We can obtain any point in the space by adding a vector to the origin

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i$$

- using the coordinates ***c*** of the vector in $\vec{\mathrm{b}}$

Algebra notation

- We like matrix-vector expressions
- We want to keep track of the frame
- We're going to cheat a little for elegance and decide that 1 times a point is the point

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

- $\tilde{p}$ is represented in $\vec{\mathbf{f}}$ by 4 coordinate, where the extra dummy coordinate is always 1 (for now)

For now, notation convenience

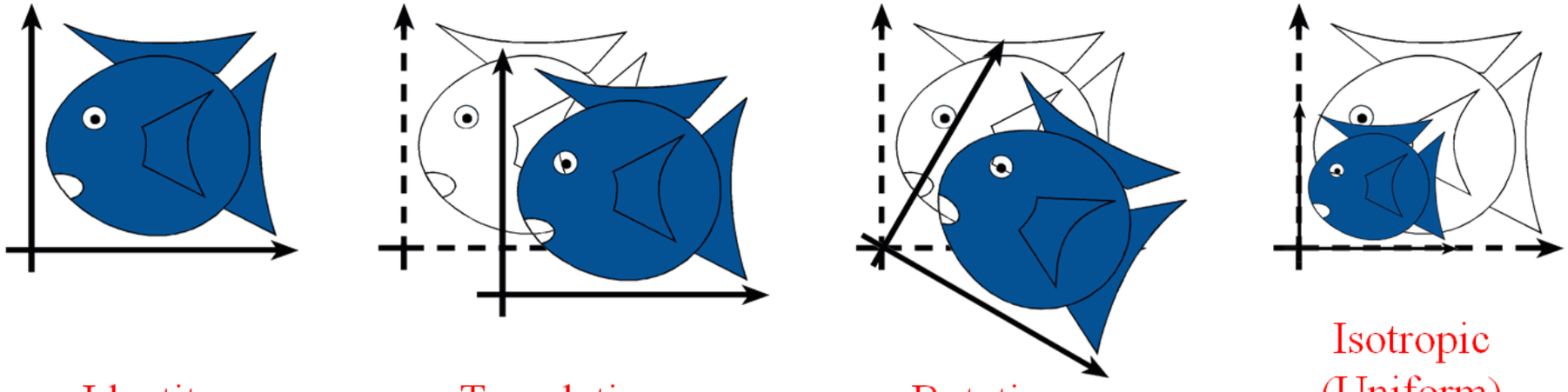
Recap

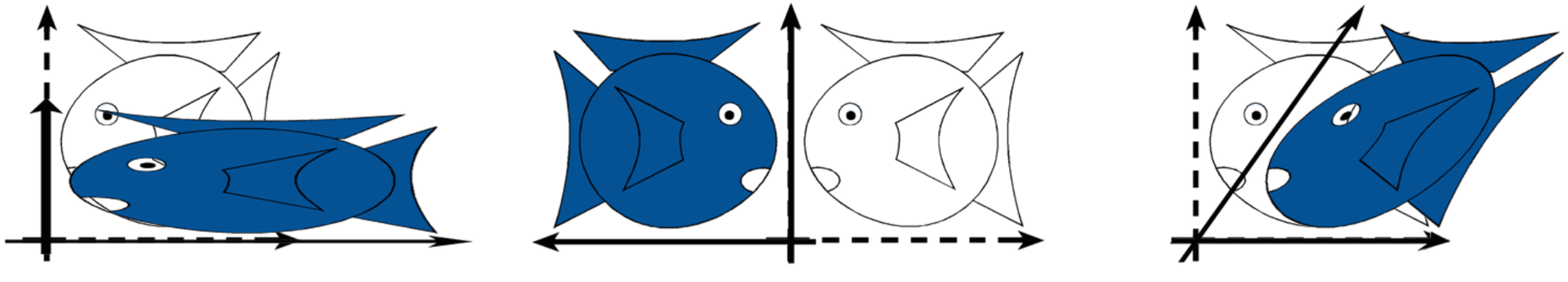
- **Vectors can be expressed in a basis**
  - Keep track of basis with left notation $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
  - Change basis $\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$
- **Points can be expressed in a frame (origin+basis)**
  - Keep track of frame with left notation
  - adds a dummy 4th coordinate always 1

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

Affine transformations

- Include all linear transformations
  - Applied to the vector basis
- Plus translation





Matrix notation

- We know how to transform the vector basis

$$\begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

- We will soon add translation by a vector $\vec{t}$

$$\tilde{p} \Rightarrow \tilde{p} + \vec{t}$$

Linear component

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\tilde{o} + \sum_i c_i \mathcal{L}(\vec{b_i}) = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

- Note how we leave the fourth component alone

Translation component

$$\tilde{p} \Rightarrow \tilde{p} + \vec{t}$$

- Express translation vector t in the basis

$$\vec{t} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} M_{14} \\ M_{24} \\ M_{34} \end{bmatrix}$$

Translation

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\tilde{o} + \vec{t} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & M_{14} \\ 0 & 1 & 0 & M_{24} \\ 0 & 0 & 1 & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Full affine expression

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\tilde{o} + \vec{t} + \sum_i c_i \mathcal{L}(\vec{b_i}) = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Which tells us both how to get a new frame f^tM
or how to get the coordinates Mc after transformation

Questions?

More notation properties

- If the fourth coordinate is zero, we get a vector
- Subtracting two points:

$$\tilde{p} = \vec{f}^t \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \qquad \tilde{p'} = \vec{f}^t \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ 1 \end{bmatrix}$$

- Gives us $\tilde{p} - \tilde{p'} = \vec{f}^t \begin{bmatrix} c_1 - c'_1 \\ c_2 - c'_2 \\ c_3 - c'_3 \\ 0 \end{bmatrix}$

  a vector (last coordinate = 0)

More notation properties

- Adding a point

$$\tilde{p} = \vec{f^t} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

to a vector

$$\vec{v} = \vec{f^t} \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ 0 \end{bmatrix}$$

- Gives us

$$\tilde{p} + \vec{v} = \vec{f^t} \begin{bmatrix} c_1 + c'_1 \\ c_2 + c'_2 \\ c_3 + c'_3 \\ 1 \end{bmatrix}$$

a point (4th coordinate=1)

More notation properties

- vectors are not affected by the translation part

$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$$

- because their 4th coordinate is 0
- If I rotate my moving car in the world, I want its motion to rotate
- If I translate it, motion should be unaffected

Questions?

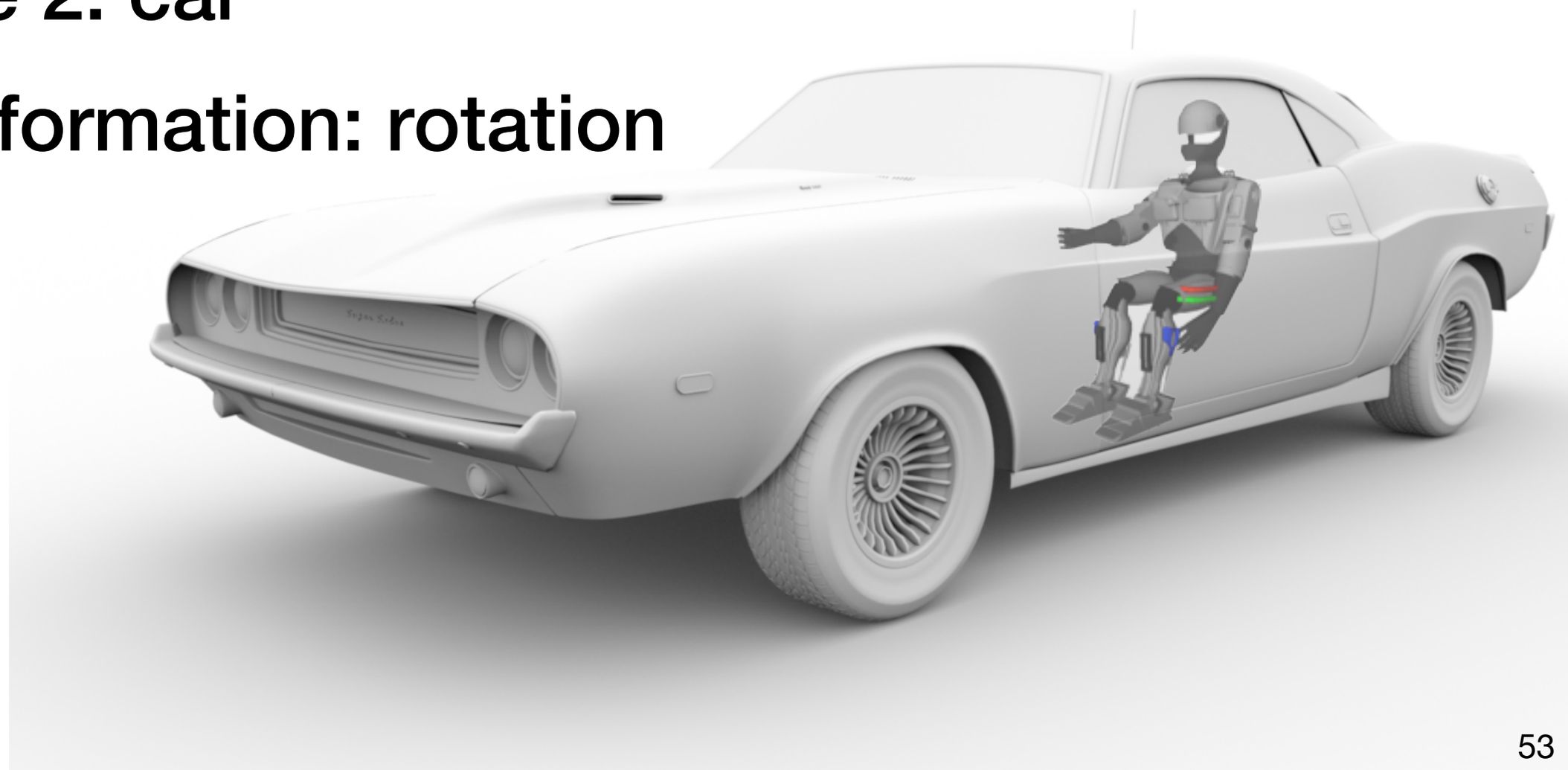
Frames & hierarchical modeling

- Many coordinate systems (frames):
  - Camera
  - Static scene
  - car
  - driver
  - arm
  - hand
  - ...

- Need to understand nested transformations

Frames & hierarchical modeling

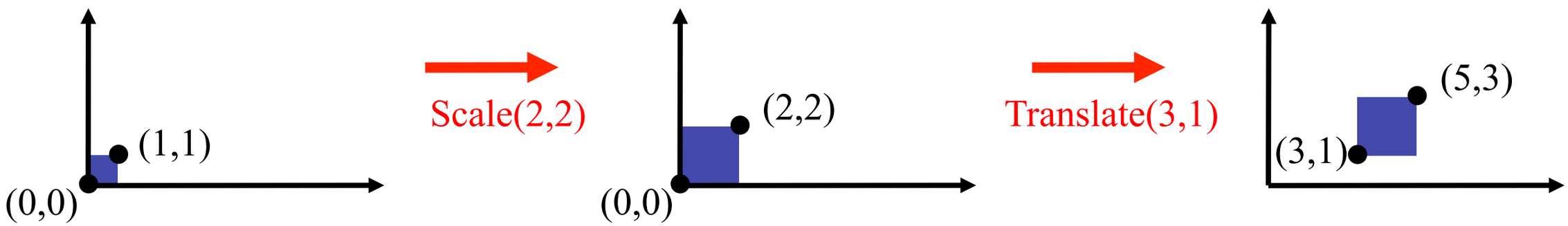
- Example: what if I rotate the wheel of the moving car:
- frame 1: world
- frame 2: car
- transformation: rotation



Questions?

How are transforms combined?

Scale then Translate



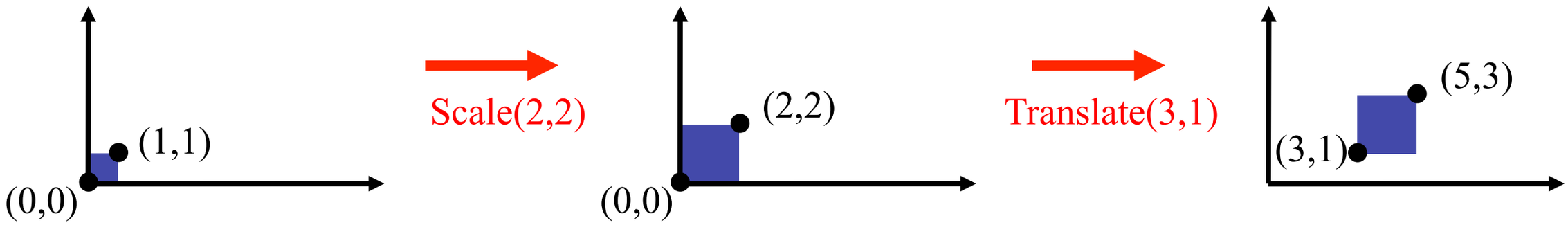
Use matrix multiplication: $p' = T(Sp) = TSp$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

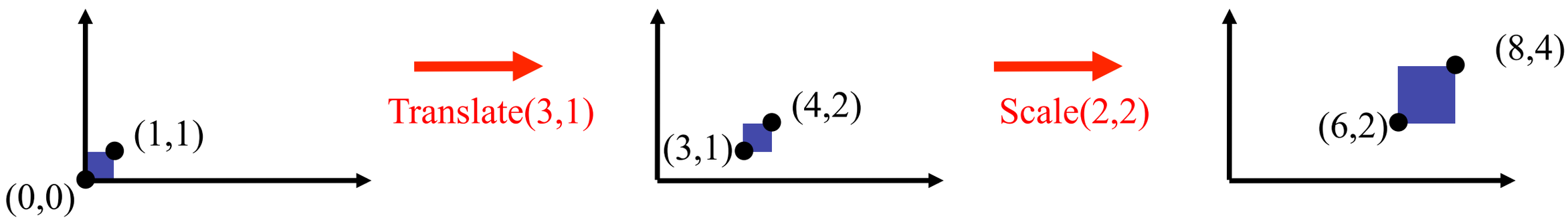
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$



Translate then Scale: $p' = S(T\,p) = ST\,p$



Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

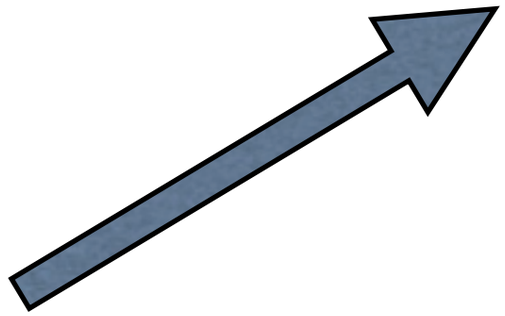
Translate then Scale: $p' = S(T\,p) = ST\,p$

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

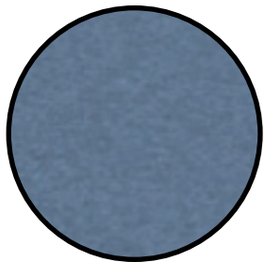
Questions?

Plan

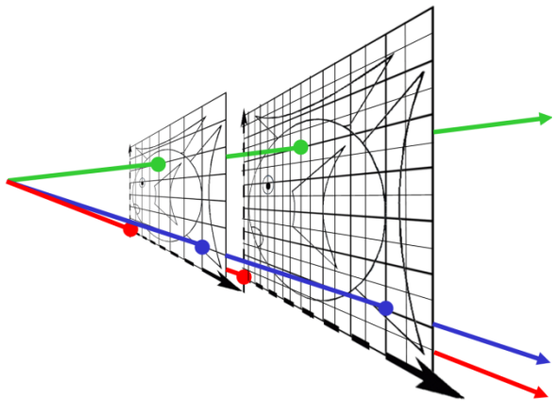
- Vectors



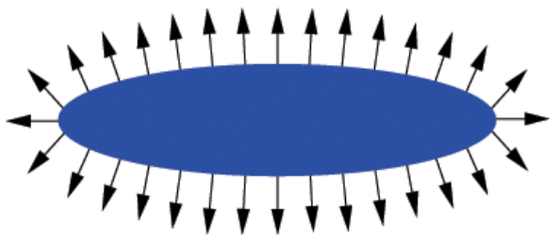
- Points



- **Homogenous coordinates**



- Normals



Forward reference and eye

- The fourth coordinate is useful for perspective projection
- Called homogenous coordinates

Homogeneous Coordinates

- **Add an extra dimension (same as frames)**
  - in 2D, we use 3-vectors and 3 x 3 matrices
  - In 3D, we use 4-vectors and 4 x 4 matrices
- **The extra coordinate is now an arbitrary value,** w
  - You can think of it as “scale,” or “weight”
  - For all transformations except perspective, you can just set *w*=1 and not worry about it

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Projective Equivalence

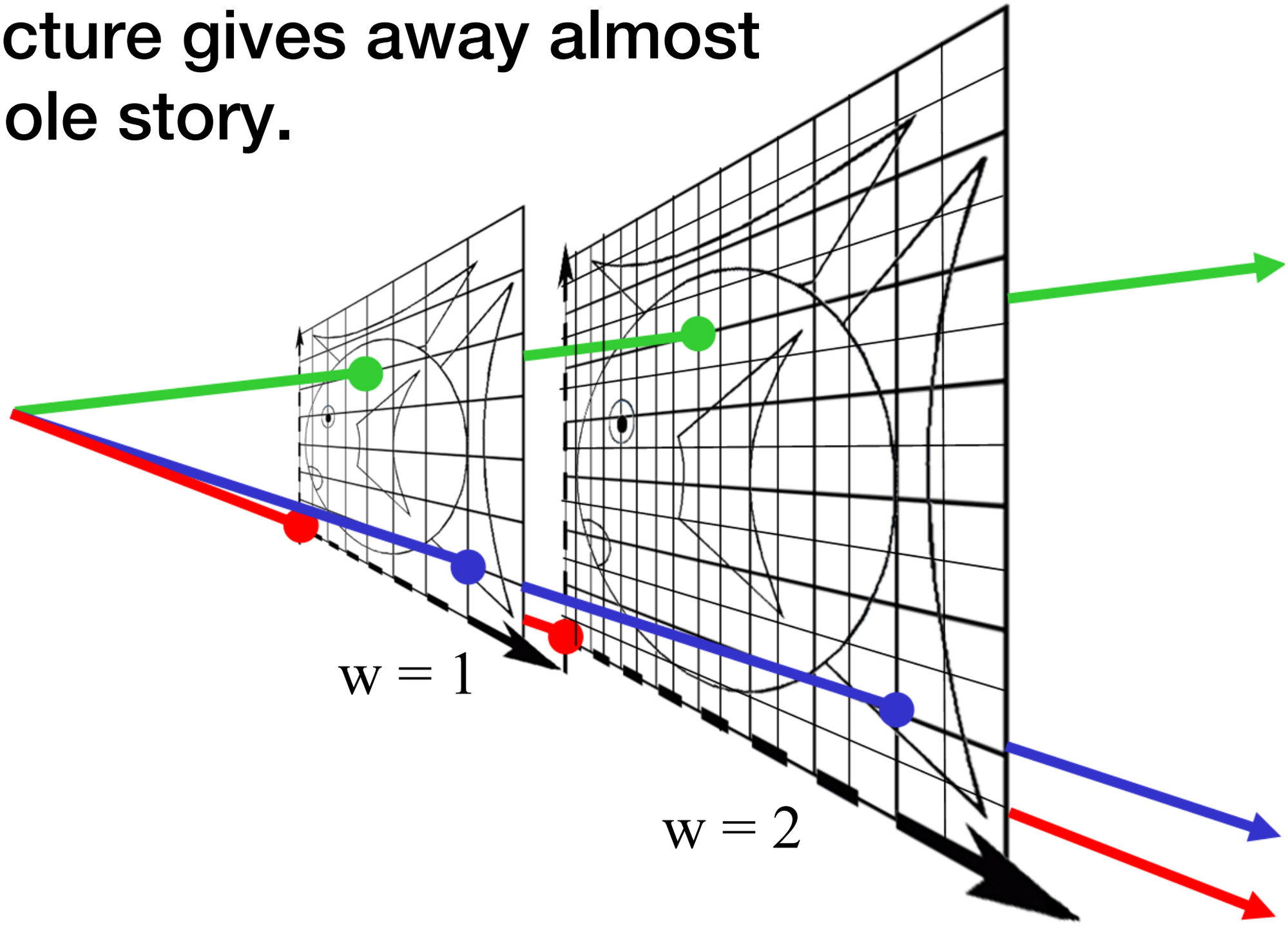
- All non-zero scalar multiples of a point are considered identical
- to get the equivalent Euclidean point, divide by the last coordinate (Homogenization)

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ az \\ aw \end{pmatrix} \overset{w \neq 0}{=} \begin{pmatrix} x/w \\ y/w \\ z/w \\ 1 \end{pmatrix}$$

$$a \neq 0$$

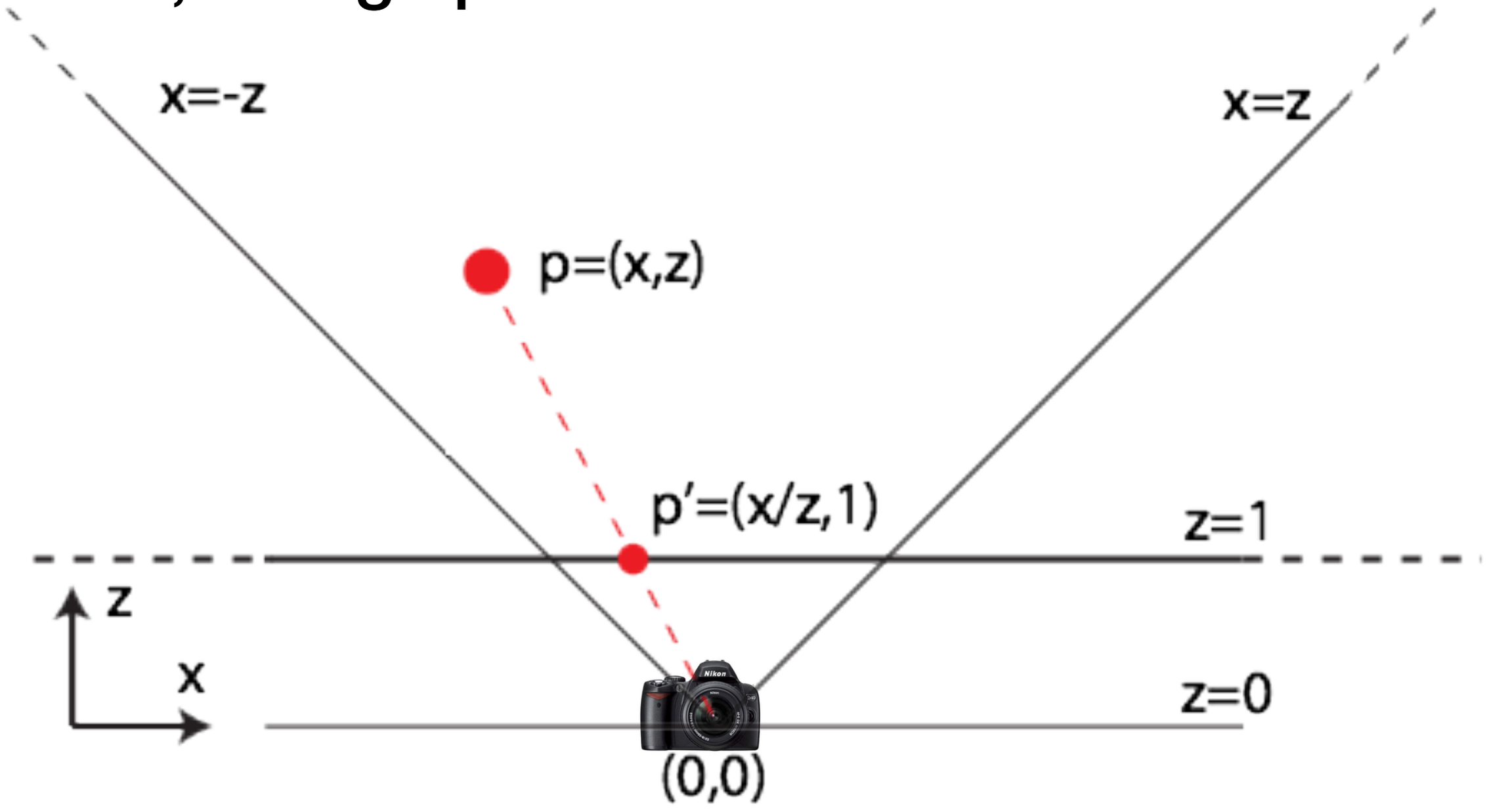
Why bother with extra coord?

- This picture gives away almost the whole story.



Perspective in 2D

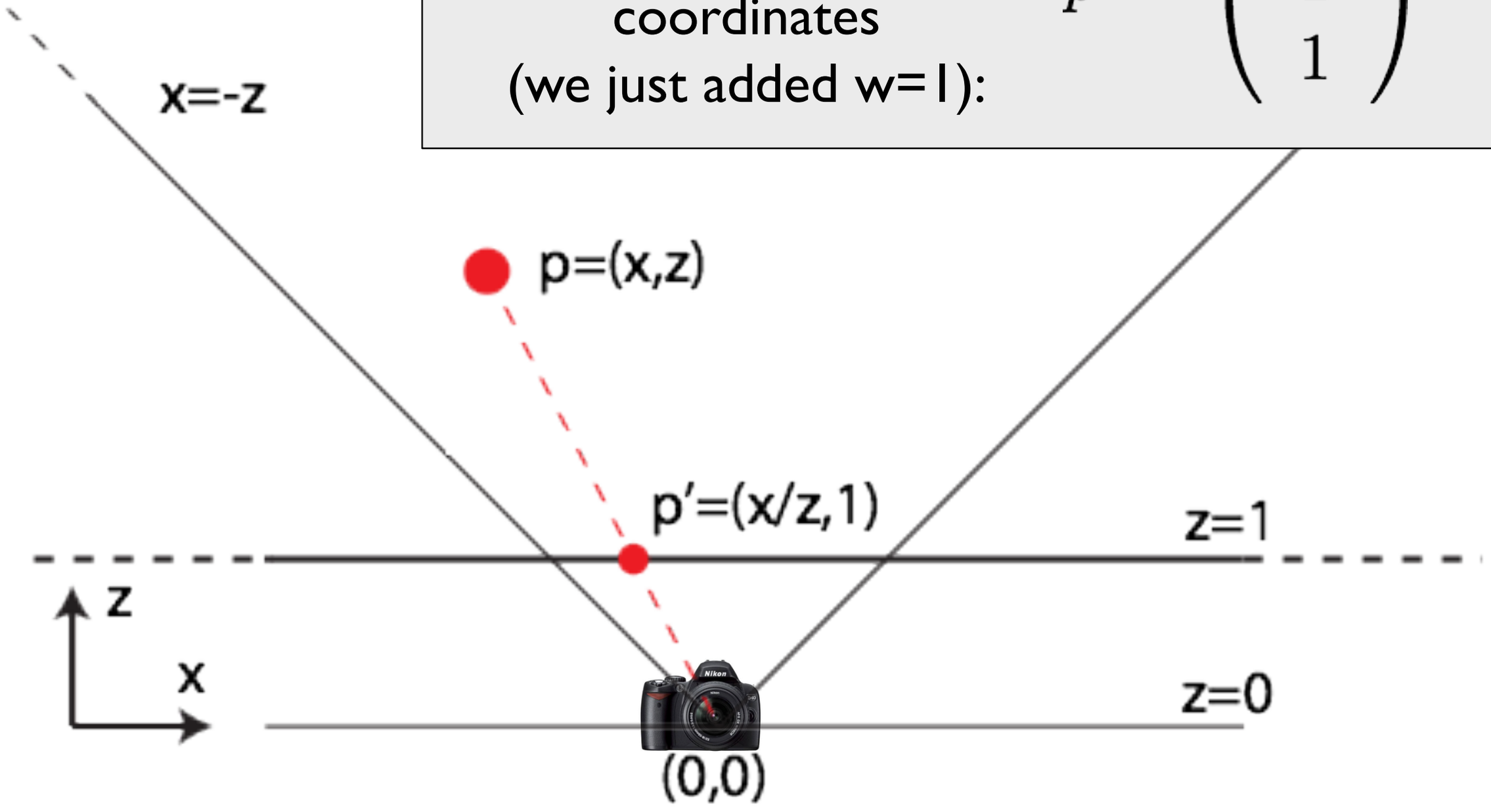
- Camera at origin, looking along *z*, 90 degree f.o.v., “image plane” at *z*=1



Perspective in 2D

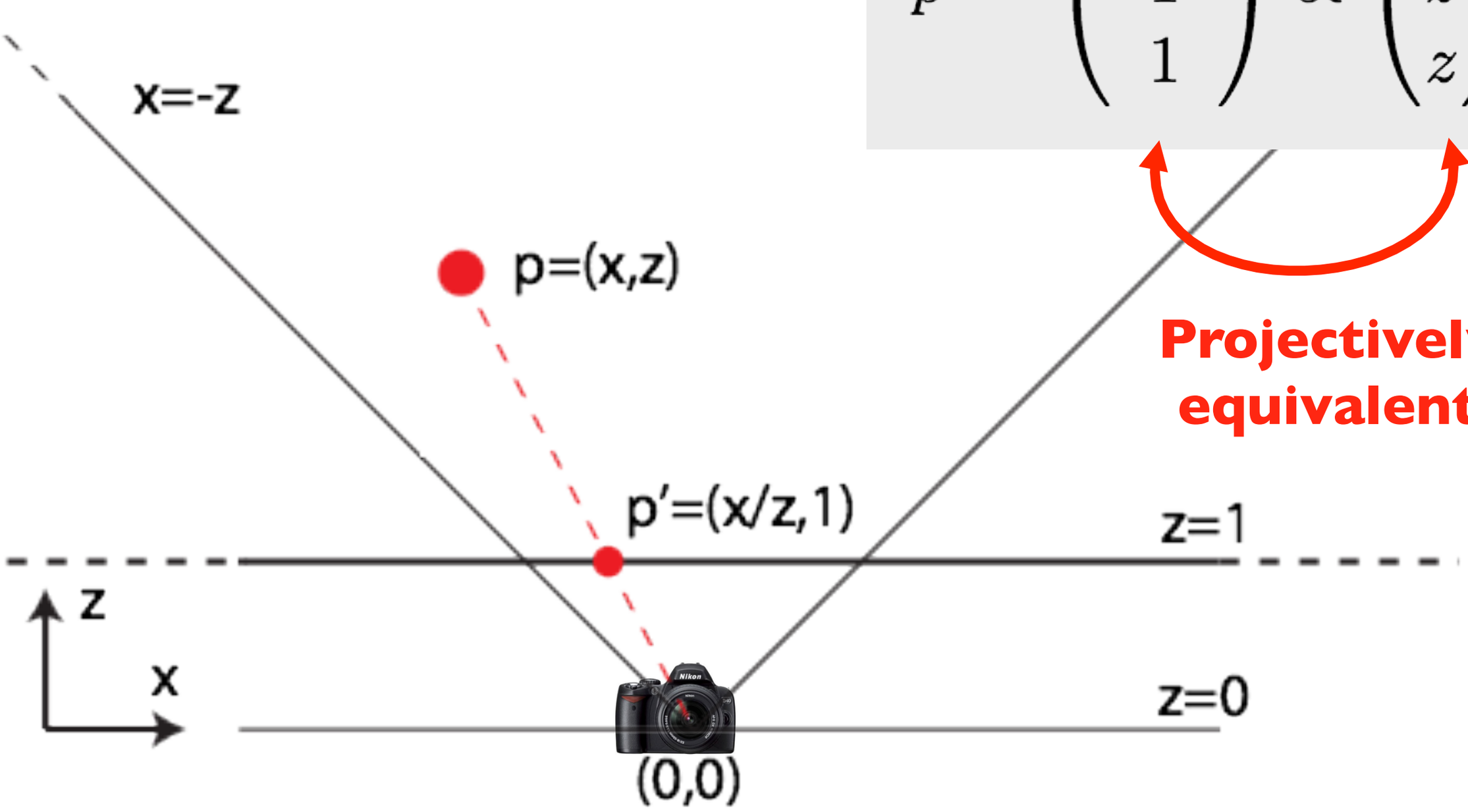
The projected point in homogeneous coordinates (we just added w=1):

$$p' = \begin{pmatrix} x/z \\ 1 \\ 1 \end{pmatrix}$$



Perspective in 2D

$$p' = \begin{pmatrix} x/z \\ 1 \\ 1 \end{pmatrix} \propto \begin{pmatrix} x \\ z \\ z \end{pmatrix}$$

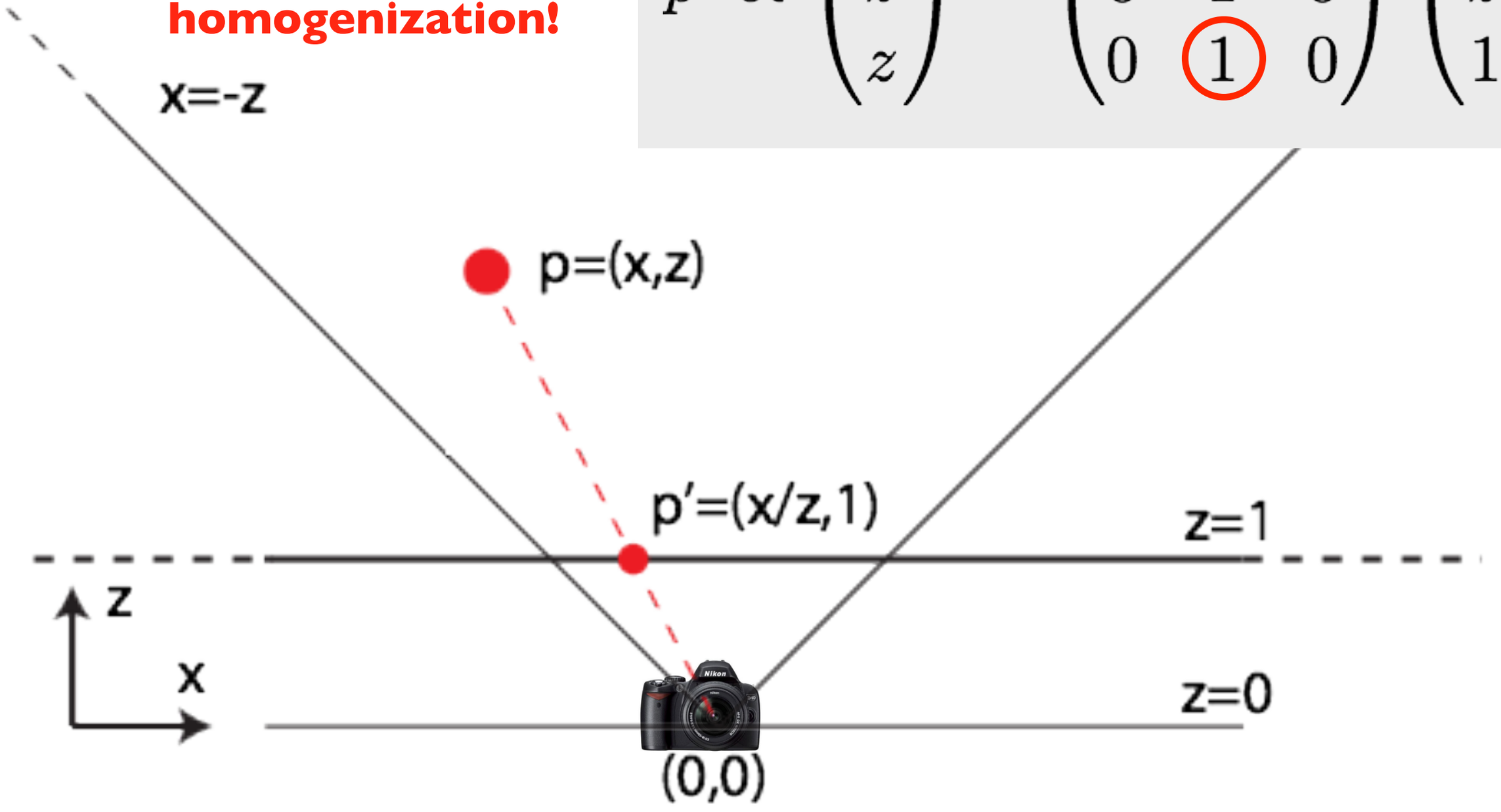


Projectively equivalent

Perspective in 2D

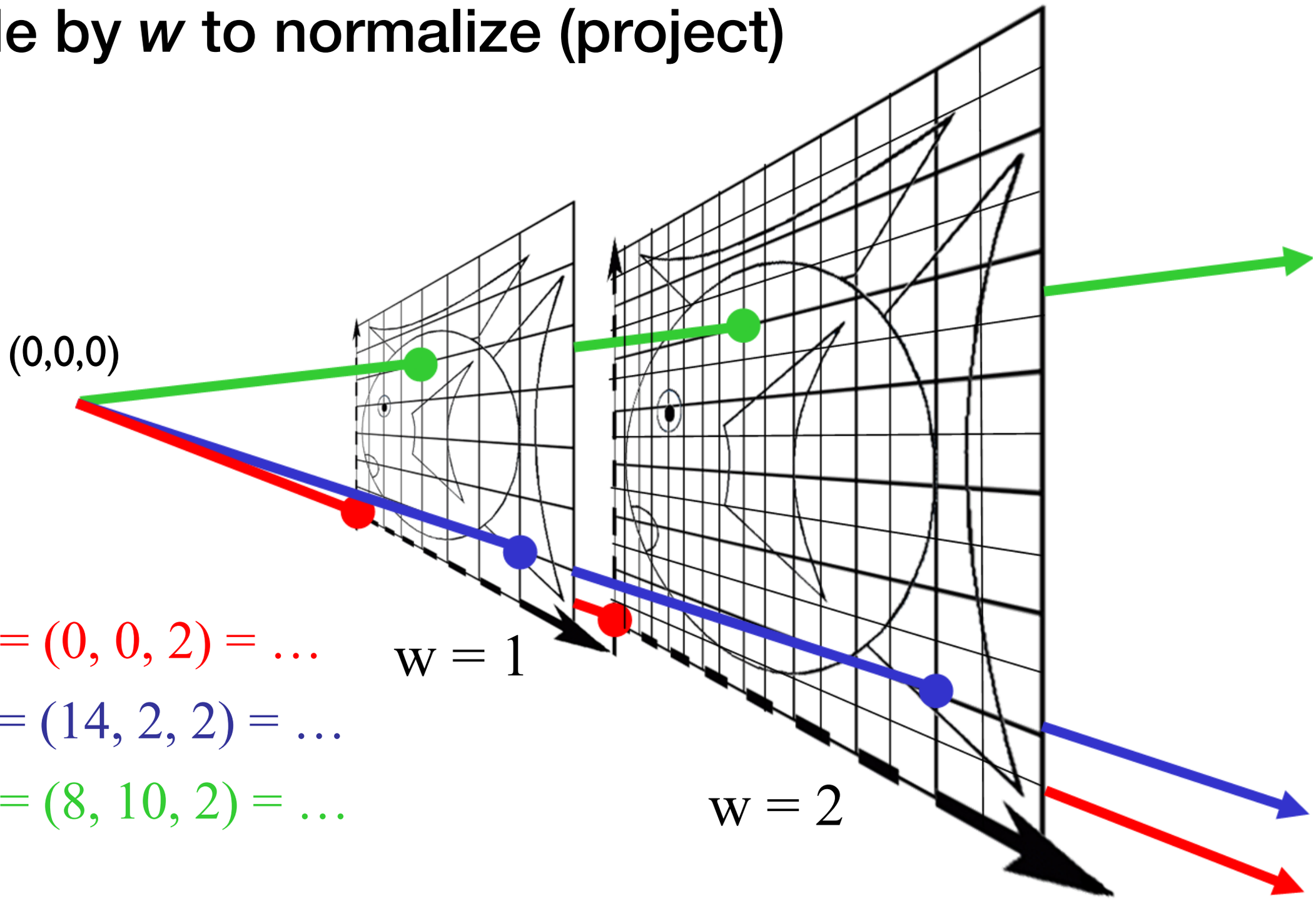
We'll just copy z to w, and get the projected point after homogenization!

$$p' \propto \begin{pmatrix} x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \\ 1 \end{pmatrix}$$



Homogeneous Visualization

- Divide by *w* to normalize (project)



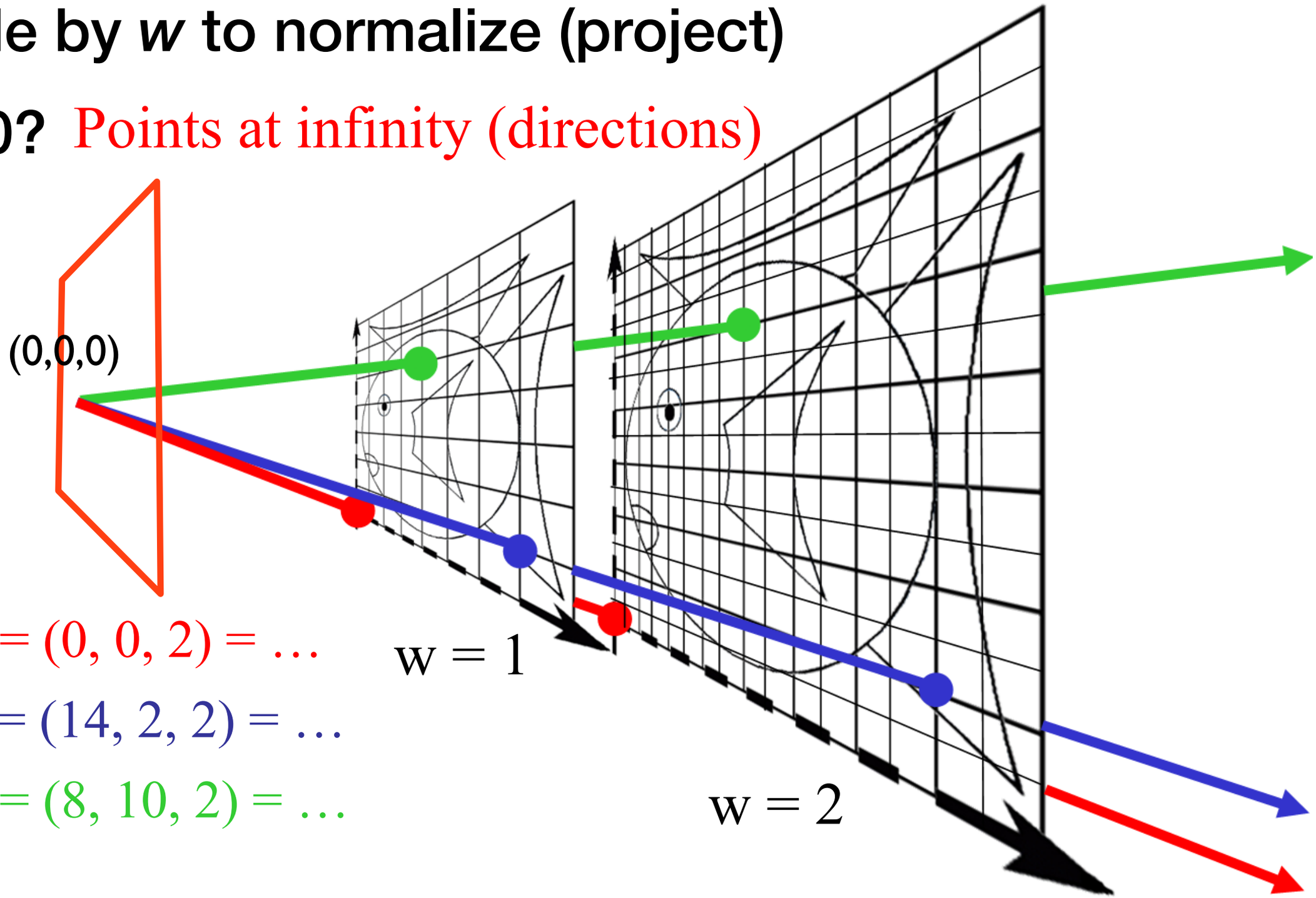
(0, 0, 1) = (0, 0, 2) = …

(7, 1, 1) = (14, 2, 2) = …

(4, 5, 1) = (8, 10, 2) = …

Homogeneous Visualization

- Divide by *w* to normalize (project)
- *w* = 0? Points at infinity (directions)



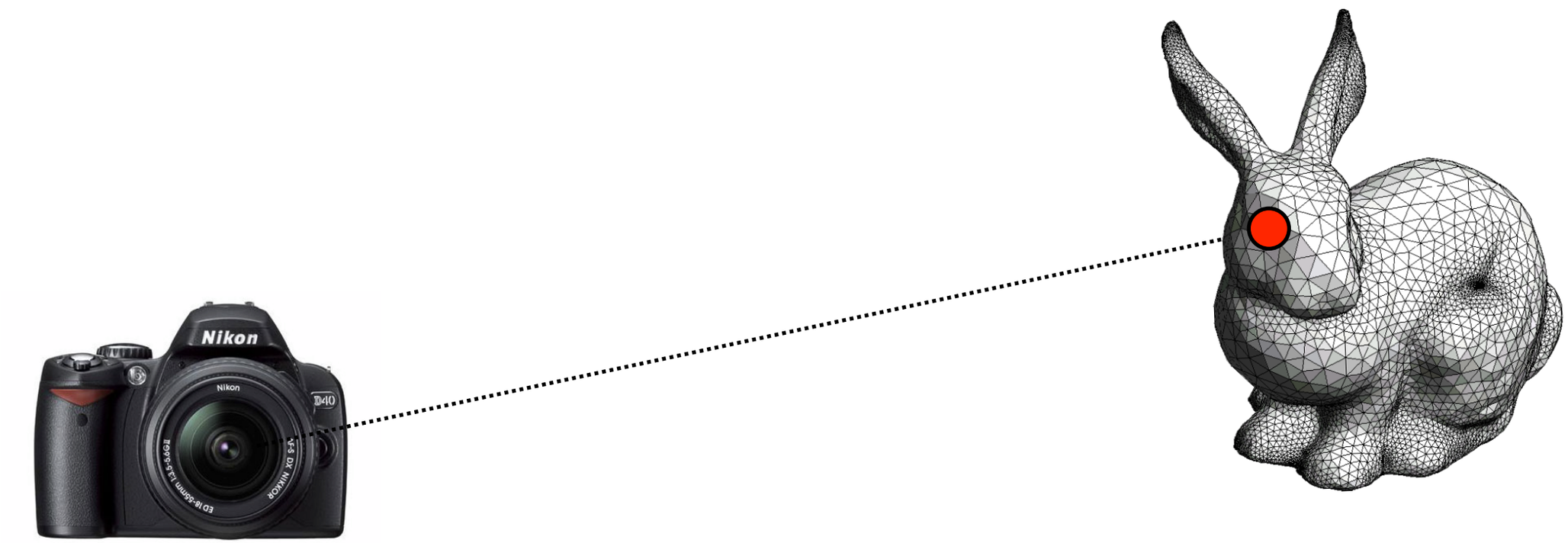
(0, 0, 1) = (0, 0, 2) = …

(7, 1, 1) = (14, 2, 2) = …

(4, 5, 1) = (8, 10, 2) = …

Projective Equivalence – Why?

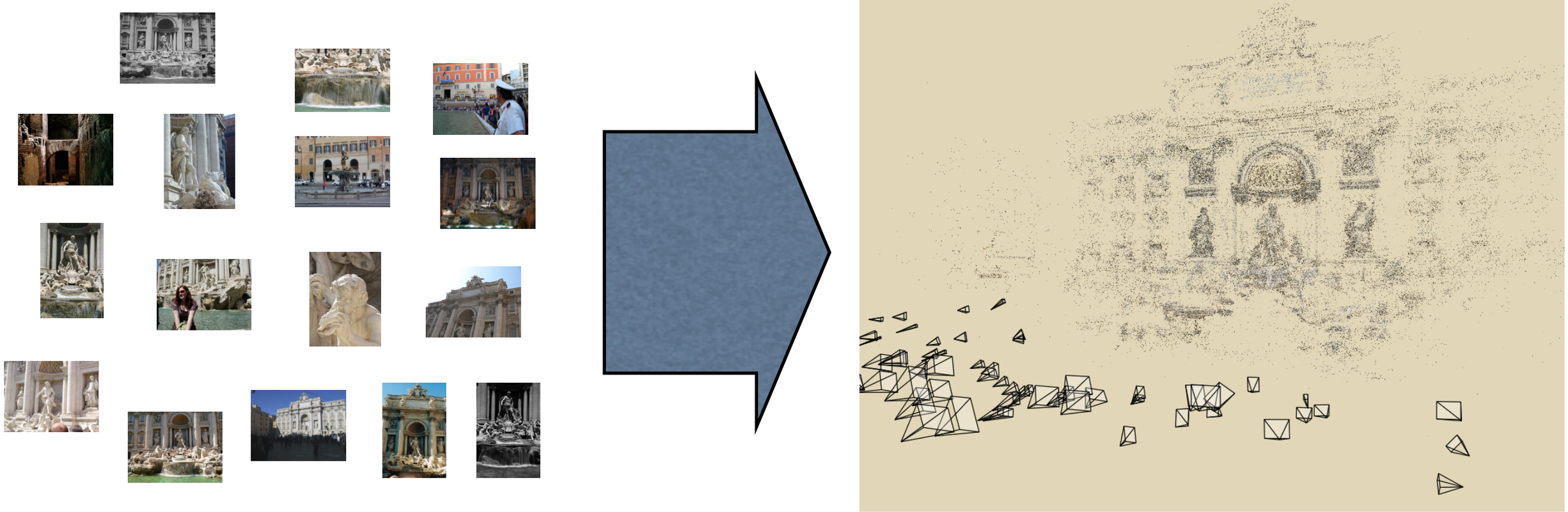
- For affine transformations, adding w=1 in the end proved to be convenient.
- The real showpiece is **perspective.**



Questions?

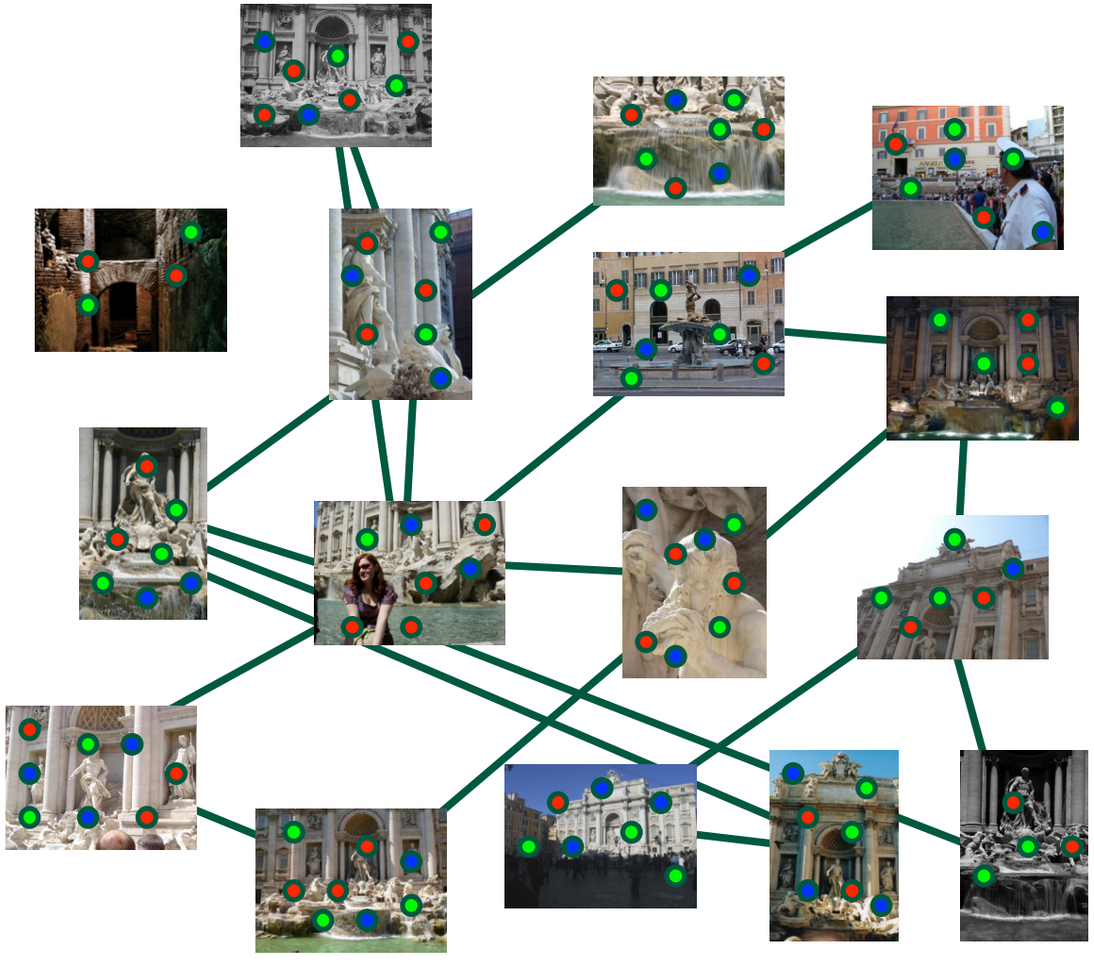
Eye candy: photo tourism

- Application of homogenous coordinates
- Goal: given N photos of a scene
  - find where they were taken
  - get 3D geometry for points in the scene



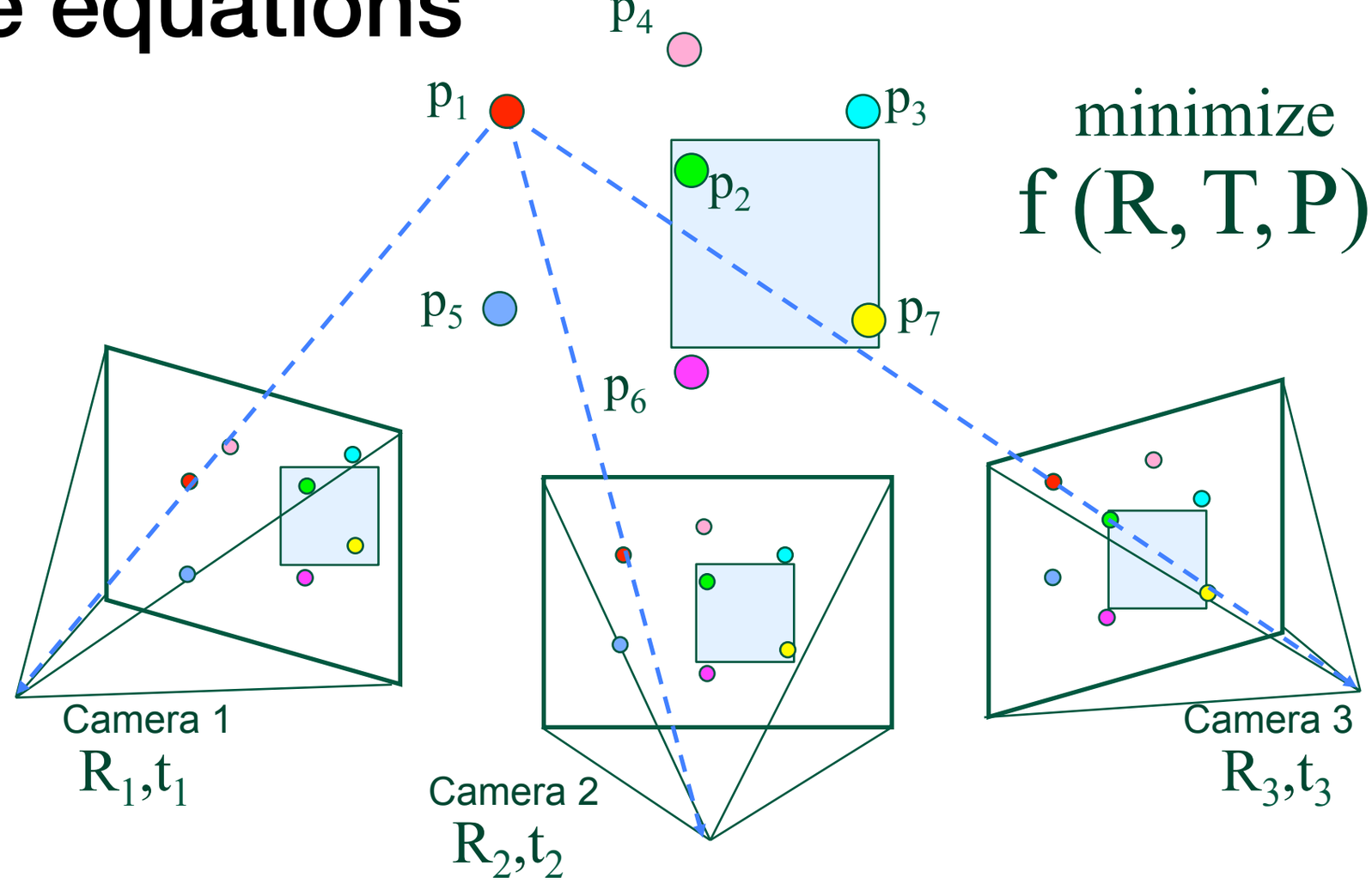
Step 1: point correspondences

- Extract salient points (corners) from images
- Find the same scene point in other images



Structure from motion

- Given point correspondences
- Unknowns: 3D point location, camera poses
- For each point in each image, write perspective equations



Eye candy: photo tourism

Photo Tourism

Exploring photo collections in 3D

Noah Snavely Steven M. Seitz Richard Szeliski

University of Washington *Microsoft Research*

SIGGRAPH 2006

That's All for Today

- Other Cool Stuff
  - Algebraic Groups
  - http://phototour.cs.washington.edu/
  - http://phototour.cs.washington.edu/findingpaths/
  - Free-form deformation of solid objects
  - Harmonic coordinates for character articulation