50.017 Linear Algebra Review

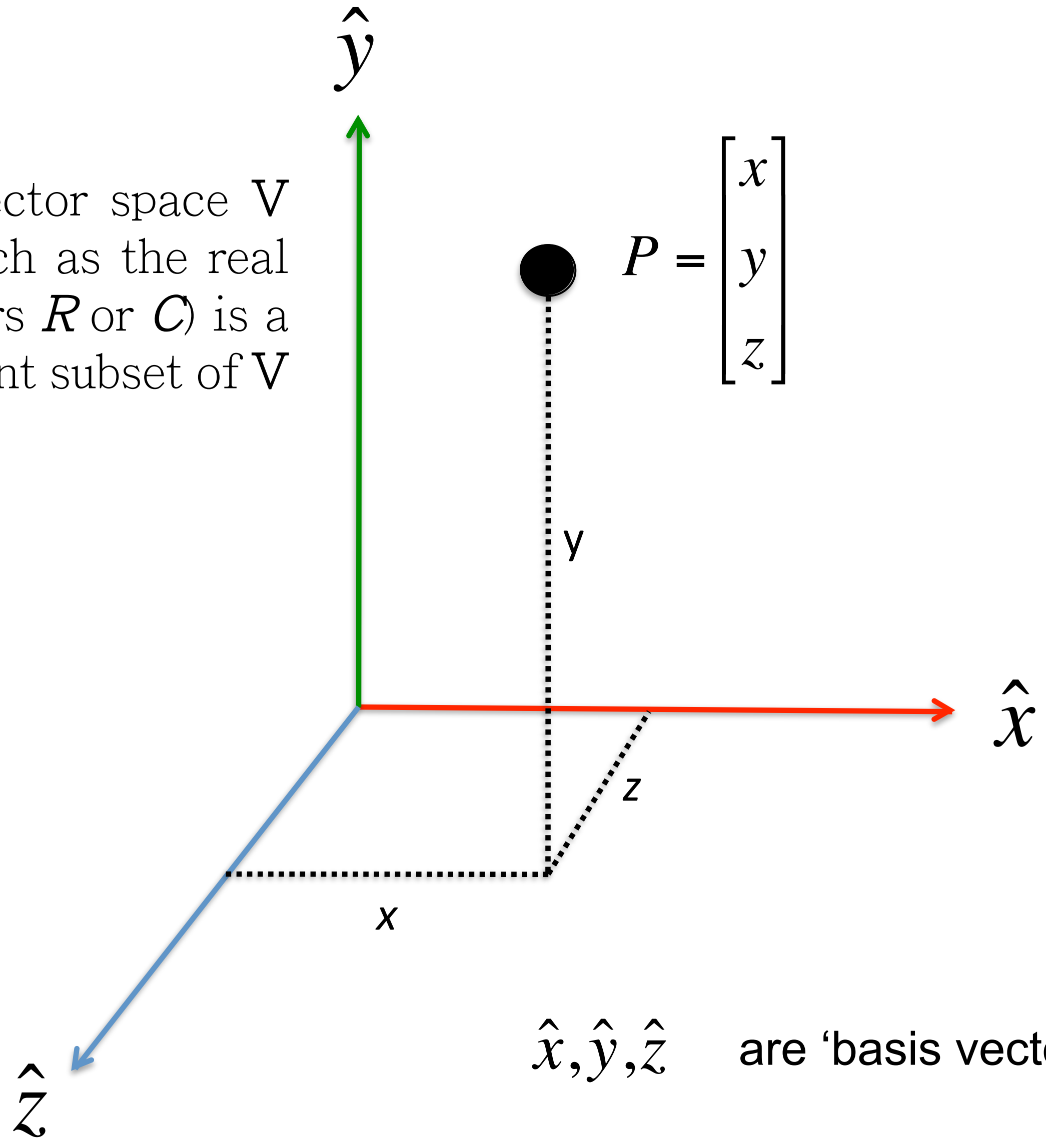
Zhipeng Mo

Overview

- Linear algebra
  - Points, Vectors in R^3
  - Operations (dot product, norm, cross-product)
- Geometry
  - lines, planes
- Matrices
  - Transformations

A point

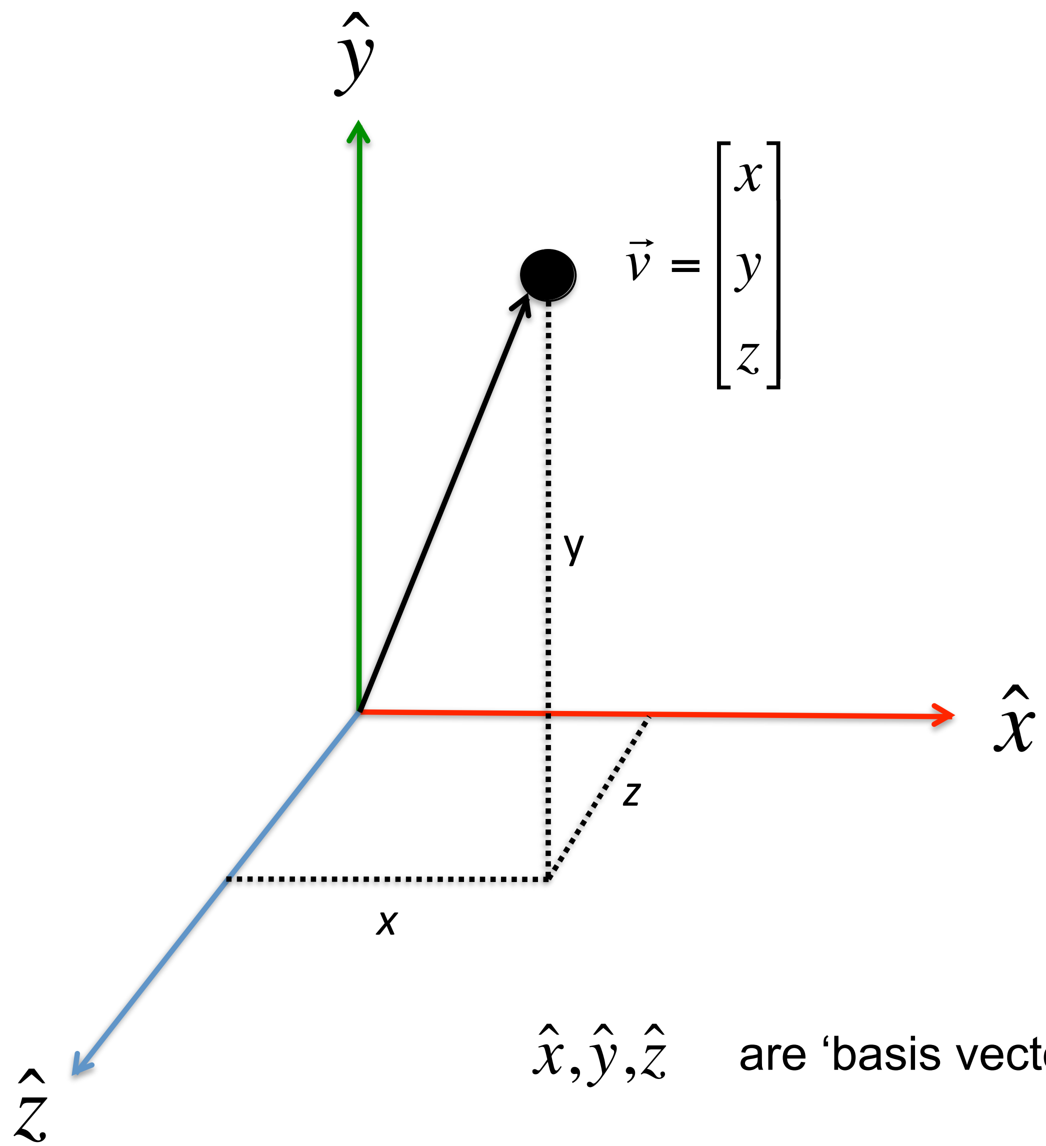
A basis B of a vector space V over a field F (such as the real or complex numbers R or C) is a linearly independent subset of V that spans V.



$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\hat{x}, \hat{y}, \hat{z}$ are ‘basis vectors’

A vector



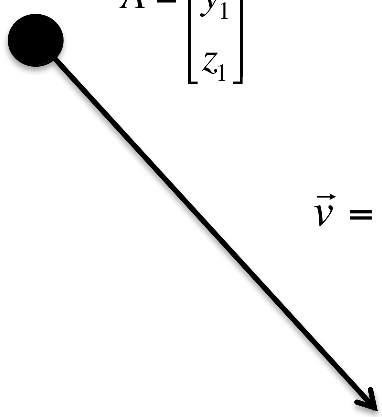
$\hat{x}, \hat{y}, \hat{z}$ are 'basis vectors'

Points vs. vectors

- Same form: $[x,y,z]^T$
- Same class in vecmath library: `Vector3f`
- Both are specified as numbers:
  - coordinates w.r.t. some a basis
- But, conceptually:
  - points are positions
  - vectors have *direction* and *length*
- In homogeneous coordinates, you can think of
  - points as [x,y,z,1] (projected to [x/1,y/1,z/1])
  - vectors as [x,y,z,0] (representing a 3D "point at infinity" in the direction of [x,y,z])

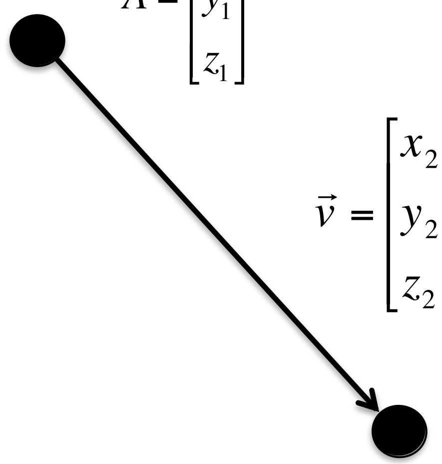
point + vector = _____

$$A = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$



$$\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

point + vector = point



$$A = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

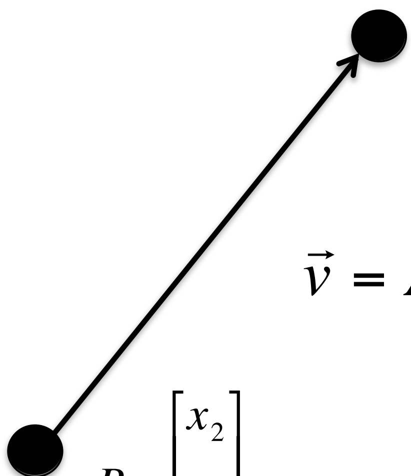
$$\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$B = A + \vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

$$1 \qquad 0 \qquad 1$$

point – point = vector

$$A = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

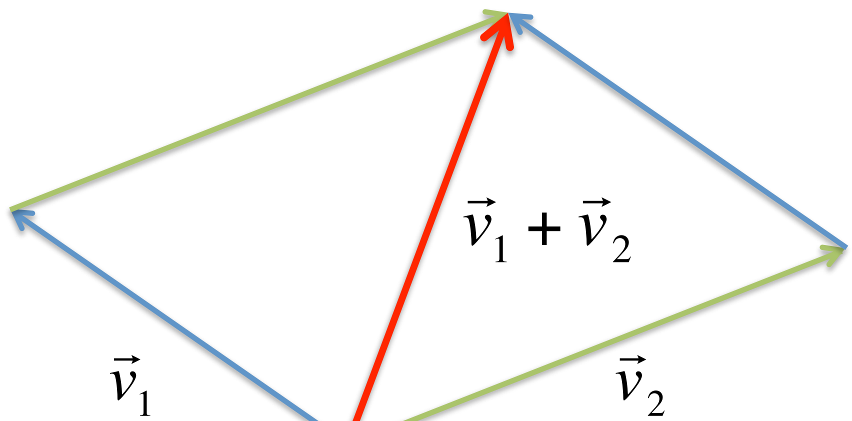


$$\vec{v} = A - B$$

$$B = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$\begin{array}{cccccccc} \vec{v} = A - B = & \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} & - & \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} & = & \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix} \\ & 1 & & 1 & & 0 \end{array}$$

vector + vector = vector

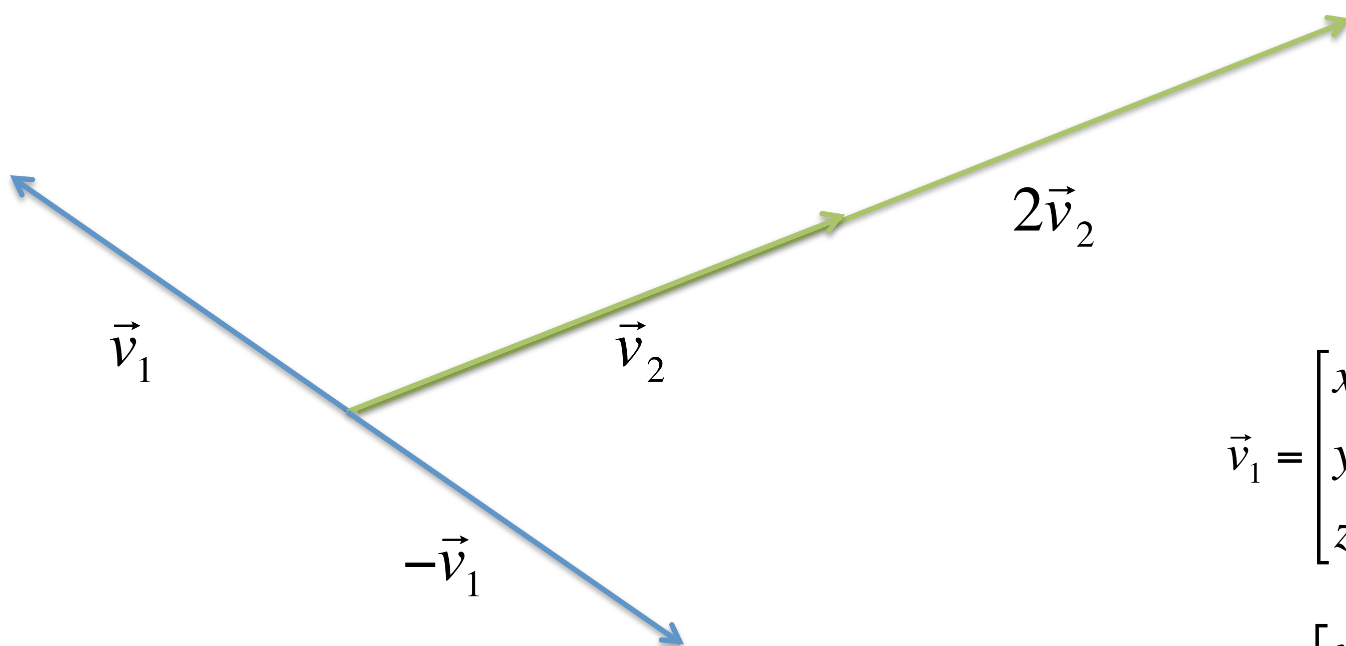


$$\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \vec{v}_2 + \vec{v}_1$$

$0 \qquad 0 \qquad 0$

Vector operations



$$\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad -\vec{v}_1 = \begin{bmatrix} -x_1 \\ -y_1 \\ -z_1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad 2\vec{v}_2 = \begin{bmatrix} 2x_2 \\ 2y_2 \\ 2z_2 \end{bmatrix}$$

Vector space axioms

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$$

$$\vec{v} + 0 = \vec{v}$$

$$(a + b)\vec{v} = a\vec{v} + b\vec{v}$$

$$\vec{v} + \vec{w} = 0 \longrightarrow \vec{w} = -\vec{v}$$

$$a(b\vec{v}) = (ab)\vec{v}$$

$$1\vec{v} = \vec{v}$$

$$\vec{v} - \vec{w} = \vec{w} + (-\vec{v})$$

$$\frac{\vec{v}}{a} = \left(\frac{1}{a}\right)\vec{v}$$

More vector operations

$$\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Dot product $\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = \vec{v}_1^{\,T} \vec{v}_2$

Norm (length) $\|\vec{v}_1\| = |\vec{v}_1| = \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \sqrt{x_1^{\,2} + y_1^{\,2} + z_1^{\,2}}$

Normalization $\hat{v}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} \longrightarrow \|\hat{v}_1\| = 1$

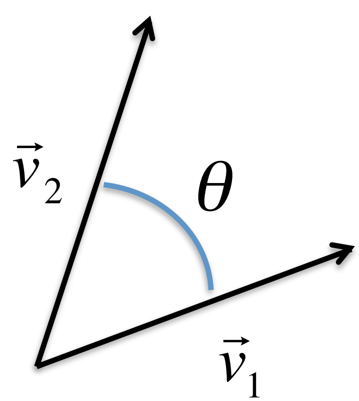
Properties of the dot product

commutative $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1$

distributive $\vec{v}_1 \cdot \left(\vec{v}_2 + \vec{v}_3\right) = \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{v}_3$

$$\left(a\vec{v}_1\right) \cdot \left(b\vec{v}_2\right) = \left(ab\right)\left(\vec{v}_1 \cdot \vec{v}_2\right)$$

Angle between two vectors

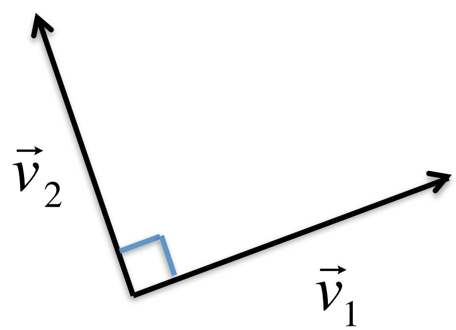


$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|}$$

Orthogonal vectors

- Two vectors are *orthogonal* if:

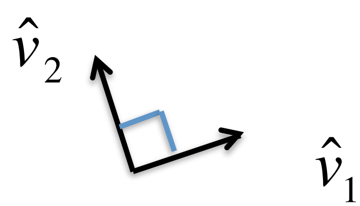
$$\vec{v}_1 \cdot \vec{v}_2 = 0$$



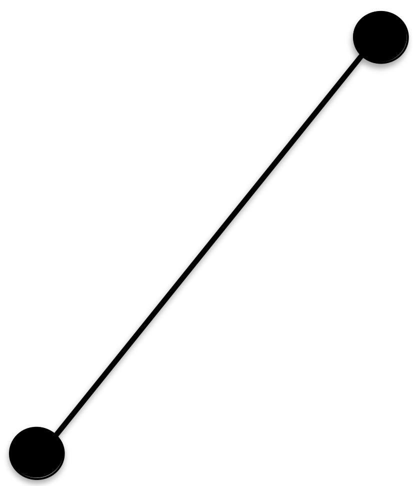
Orthonormal vectors

- Two vectors are *orthonormal* if:

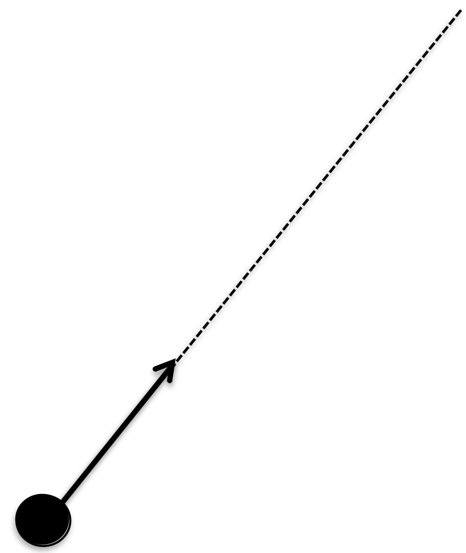
$$\vec{v}_1 \cdot \vec{v}_2 = 0 \qquad \left\|\vec{v}_1\right\| = 1 \qquad \left\|\vec{v}_2\right\| = 1$$



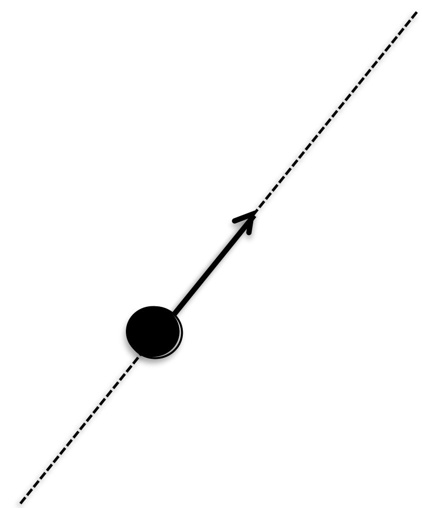
Lines



Line segment
origin + end points

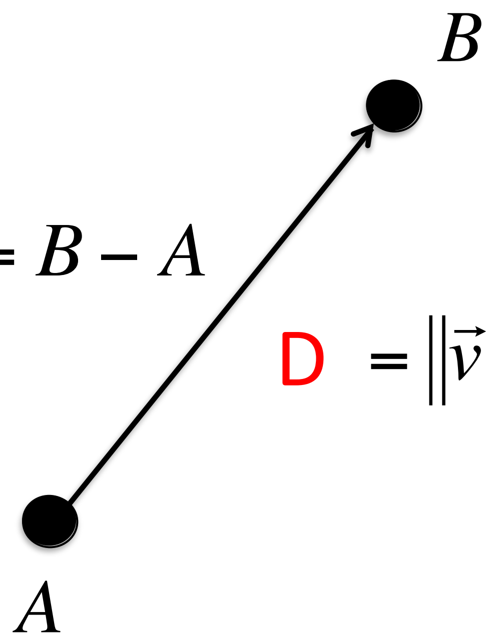


Semi-infinite line (ray)
origin + vector
$P = O + t*d, t > 0$



Infinite line
Any point on the line +
direction vector
$P = O + t*d$

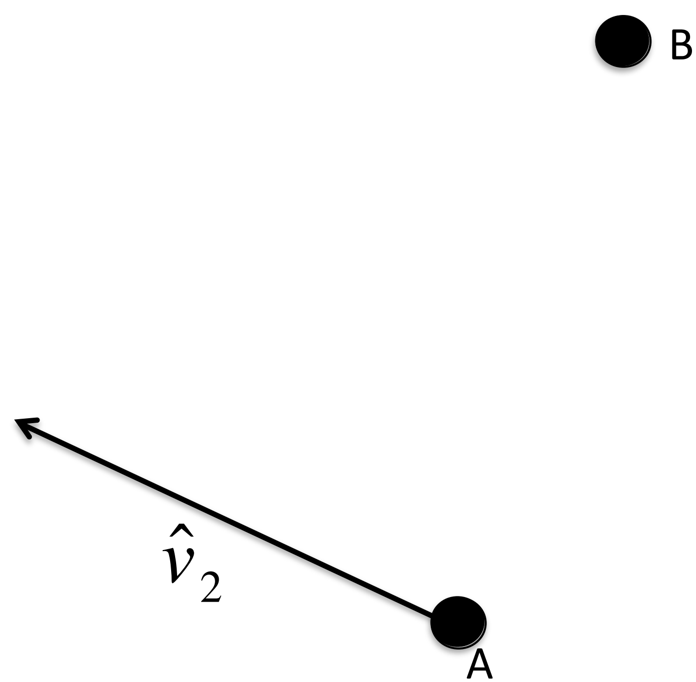
Distance between points



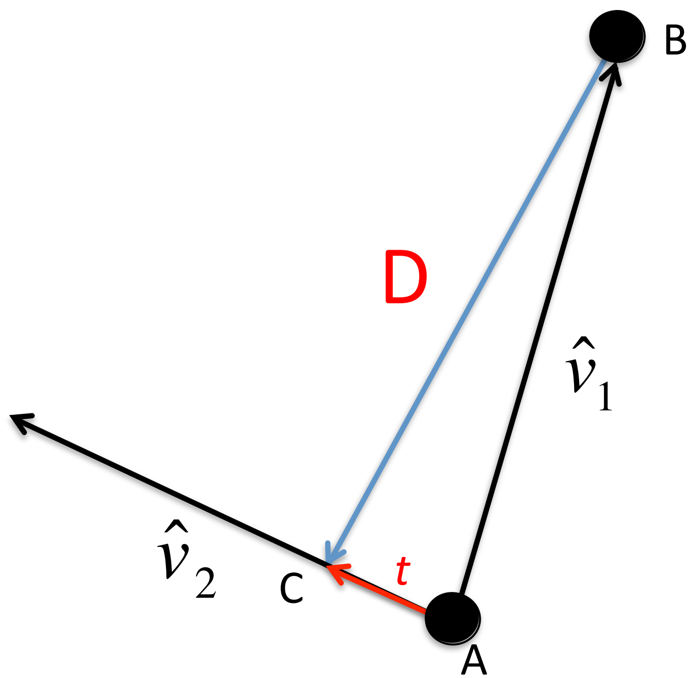
$$\vec{v} = B - A$$

$$D = \left\|\vec{v}\right\|$$

closest distance from point to line



closest distance from point to line



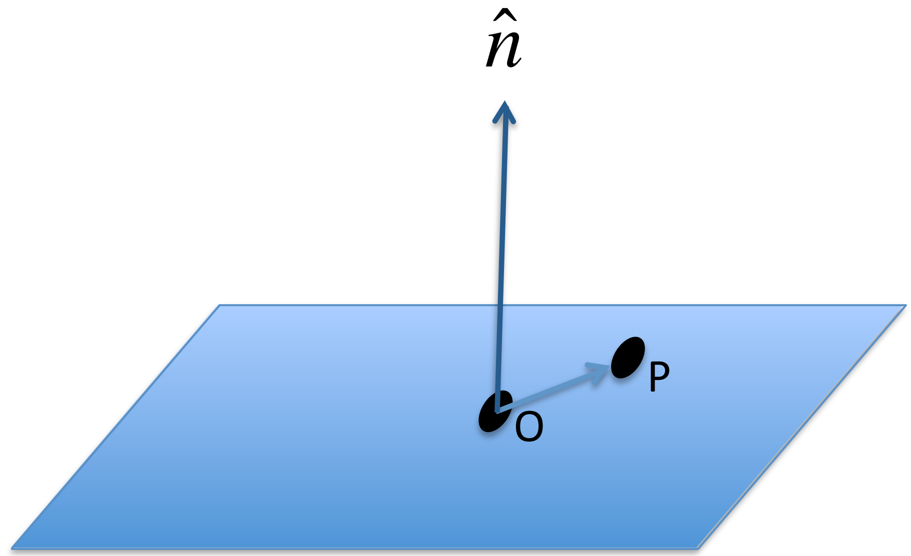
$$\hat{v}_1 = \frac{B - A}{|B - A|}$$

$$t = \vec{v}_1 \cdot \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$C = A + t\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

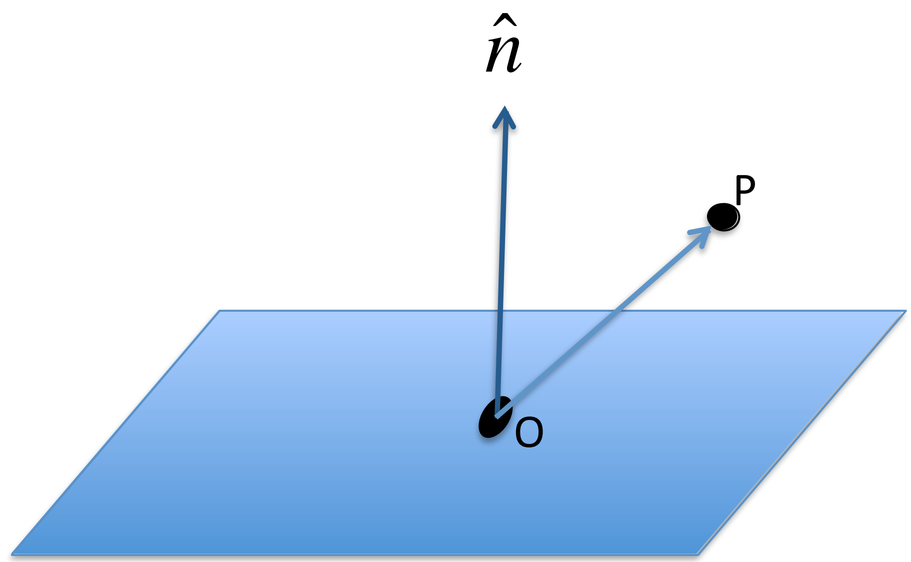
$$D = \|C - B\|$$

A plane



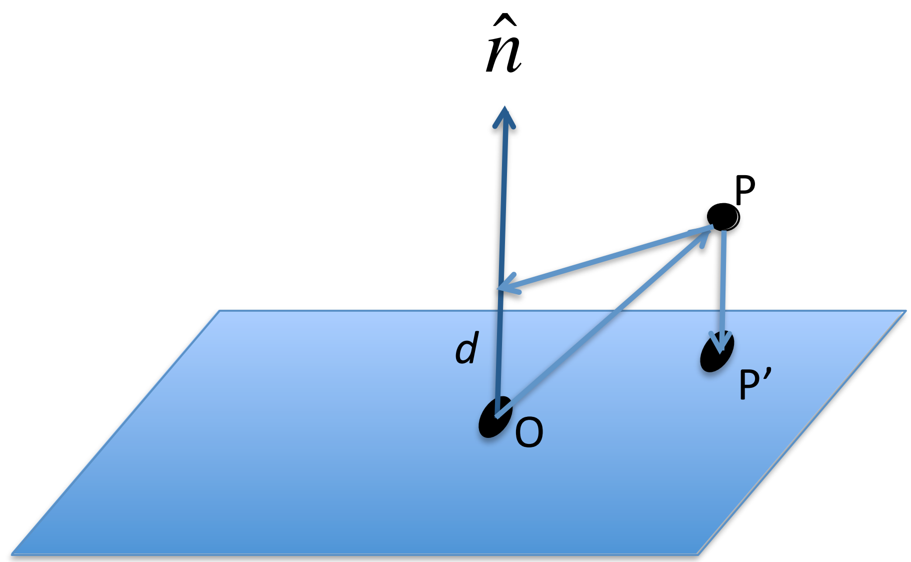
$$(P - O) \cdot \hat{n} = 0$$

closest distance from point to plane



$$\left(P - O\right) \cdot \hat{n} = ?$$

closest distance from point to plane

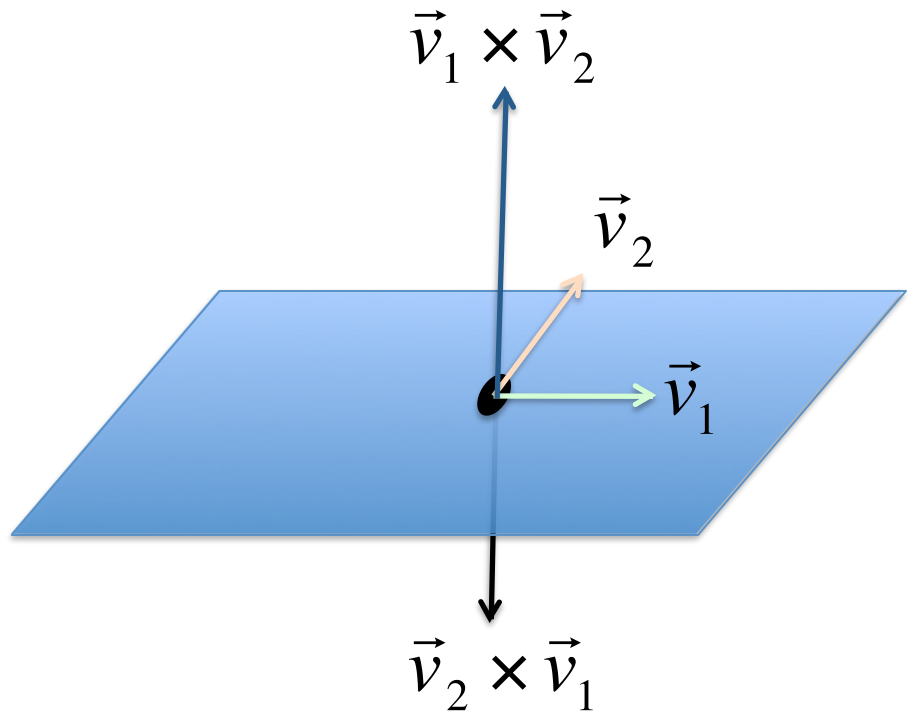


$$\left(P - O\right) \cdot \hat{n} = d$$

$$P' = P - d\hat{n}$$

$$\left(P' - O\right) \cdot \hat{n} = \left(P - d\hat{n} - O\right) \cdot \hat{n} = \left(P - O\right) \cdot \hat{n} - d\hat{n} \cdot \hat{n} = d - d = 0$$

The cross product



$$\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \qquad \vec{v}_1 \times \vec{v}_2 = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & x_1 \\ y_1 & x_1 & \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Properties of the cross product

anti-commutative $$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

distributive over:

addition $$\vec{u} \times \left(\vec{v} + \vec{w}\right) = \left(\vec{u} \times \vec{v}\right) + \left(\vec{u} \times \vec{w}\right)$$

scalar multiplication $$\left(a\vec{u}\right) \times \vec{v} = \vec{u} \times \left(a\vec{v}\right) = a\left(\vec{u} \times \vec{v}\right)$$

Vector operations in vecmath

```
Vector3f pt1( 1, 1, 1 );
Vector3f pt2( 4, 7, 6 );
Vector3f v1 = pt2 - pt1;

cout << “pt1: ”; pt1.print();
cout << “pt2: ”; pt2.print();
cout << “v1: ”; v1.print();

Vector3f v2( -3, 4, 5 );
cout << “v2: ”; v2.print();

Vector3f v1_cross_v2 = Vector3f::cross( v1, v2 );
cout << “v1 x v2: ”; v1_cross_v2.print();
cout << “|v1 x v2|: ” << v1_cross_v2.abs() << endl;

v1_cross_v2.normalize();
cout << “v1 x v2: ”; v1_cross_v2.print();
cout << “|v1 x v2|: ” << v1_cross_v2.abs() << endl;

float v1_dot_v2 = Vector3f::dot( v1, v2 );
cout << “v1 . v2: ” << v1_dot_v2 << endl;

float v1_cross_v2__dot_v2 = Vector3f::dot( Vector3f::cross( v1, v2 ), v2 );
cout << “( v1 x v2 ) . v2: ” << v1_cross_v2__dot_v2 << endl;
```

pt1 : <1 1 1>
pt2 : <4 7 6>
v1 : <3 6 5>
v2 : <-3 4 5>

v1 x v2 : <10 -30 30>
|v1 x v2| : <43.589>

v1 x v2 : <0.229 -0.688 0.688>
|v1 x v2| : 1

v1 . v2 : 40

v1 x v2 . v2 : 0

Matrix operations

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$AA = A^2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Not commutative $AB \neq BA$

$$\det(AB) = \det(BA)$$

Associative $A(BC) = (AB)C$

$$c(AB) = (cA)B$$

$$(Ac)B = A(cB)$$

$$(AB)c = A(Bc)$$

(where *c* is a scalar)

3x3 matrix inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^{-1} = 1/\mathrm{DET} * \begin{vmatrix} a_{33}a_{22}-a_{32}a_{23} & -(a_{33}a_{12}-a_{32}a_{13}) & a_{23}a_{12}-a_{22}a_{13} \\ -(a_{33}a_{21}-a_{31}a_{23}) & a_{33}a_{11}-a_{31}a_{13} & -(a_{23}a_{11}-a_{21}a_{13}) \\ a_{32}a_{21}-a_{31}a_{22} & -(a_{32}a_{11}-a_{31}a_{12}) & a_{22}a_{11}-a_{21}a_{12} \end{vmatrix}$$

$$\mathrm{DET} = a_{11}(a_{33}a_{22}-a_{32}a_{23})-a_{21}(a_{33}a_{12}-a_{32}a_{13})+a_{31}(a_{23}a_{12}-a_{22}a_{13})$$

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix is *singular* (not invertible) when det = 0.

In vecmath:

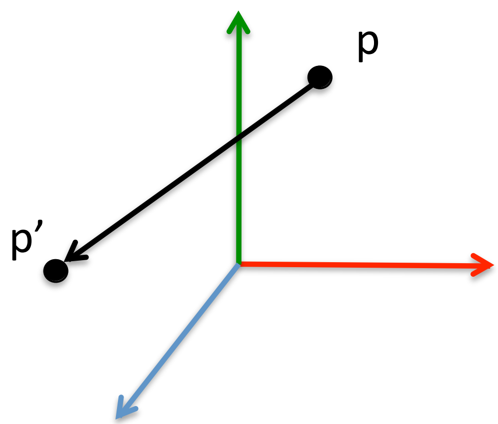
```
Matrix3f A( … );
bool isSingular;
Matrix3f invA = A.inverse( &isSingular, 0.001f );
```

Matrices and Vectors

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{bmatrix}$$

Transform Matrices



$$\vec{p}' = M\vec{p} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$M = TR \neq RT$$

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation or translation first?

What point are we rotating around?

In vecmath

```
Matrix4f T = Matrix4f::translation( Vector3f( 1, 2, 3 ) );
Matrix4f R = Matrix4f::rotation
     ( Vector3f( 1, 0, 0 ), M_PI / 4.0f );

cout << "T = " << endl; T.print();
cout << "R = " << endl; R.print();
cout << "T*R = " << endl; ( T * R ).print();
cout << "R*T = " << endl; ( R * T ).print();

Vector4f p( 0, 0, 0, 1 );

Vector4f transformed_p = T * R * p;
cout << "T*R: "; transformed_p.print();

transformed_p = R * T * p;
cout << "R*T: "; transformed_p.print();
```

```
T =
[[1 0 0 1]
[0 1 0 2]
[0 0 1 3]
[0 0 0 1]]

R =
[[1 0 0 0]
[0 0.707107 -0.707107 0]
[0 0.707107 0.707107 0]
[0 0 0 1]]

T*R =
[[1 0 0 1]
[0 0.707107 -0.707107 2]
[0 0.707107 0.707107 3]
[0 0 0 1]]

R*T =
[[1 0 0 1]
[0 0.707107 -0.707107 -0.707107]
[0 0.707107 0.707107 3.53553]
[0 0 0 1]]

T*R : [0 0 0] --> [1 2 3]
R*T : [0 0 0] --> [1 -0.707107 3.53553]
```

In OpenGL

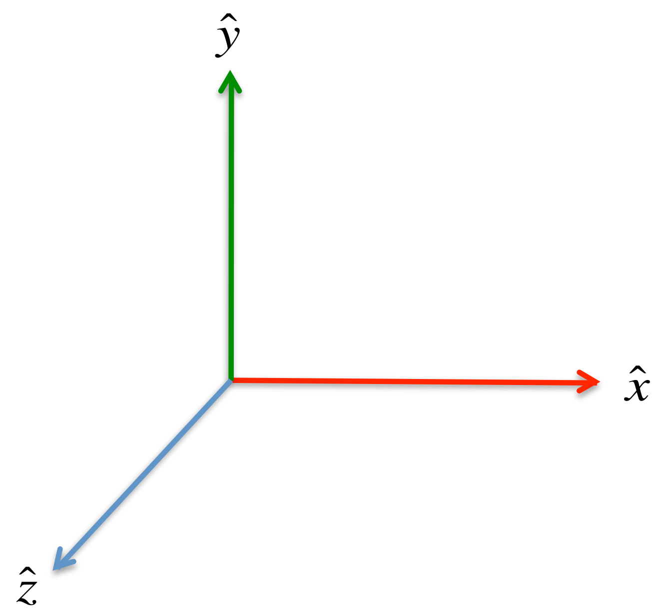
```
glMatrixMode( GL_MODELVIEW );  // Current matrix affects objects positions
glLoadIdentity();              // Initialize to the identity

// Position the camera at [0,0,5], looking at [0,0,0],
// with [0,1,0] as the up direction.
gluLookAt(0.0, 0.0, 5.0,
          0.0, 0.0, 0.0,
          0.0, 1.0, 0.0);

glTranslated ( 1, 2, 3 ); // the translation is applied second
glRotated ( 45.0, 1, 0, 0 ); // the rotation is applied first

// draw object
```

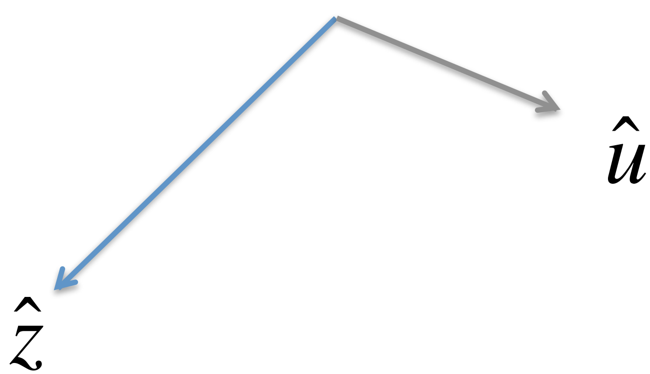
Orthonormal basis



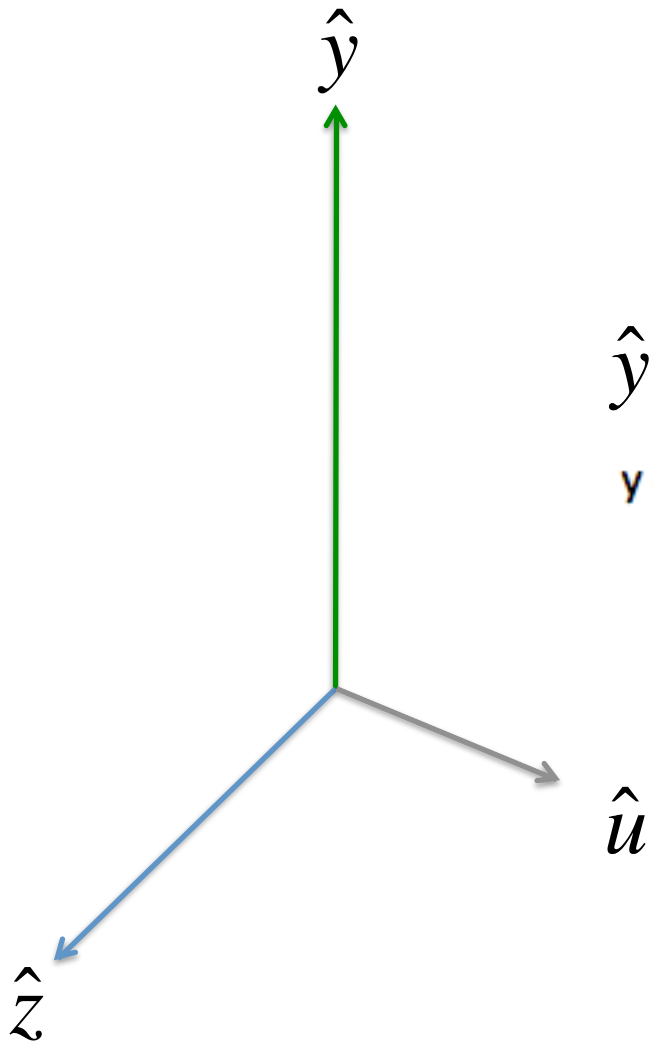
$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

How to create orthonormal basis

Given vectors u and z,
u not orthogonal to z



How to create orthonormal basis



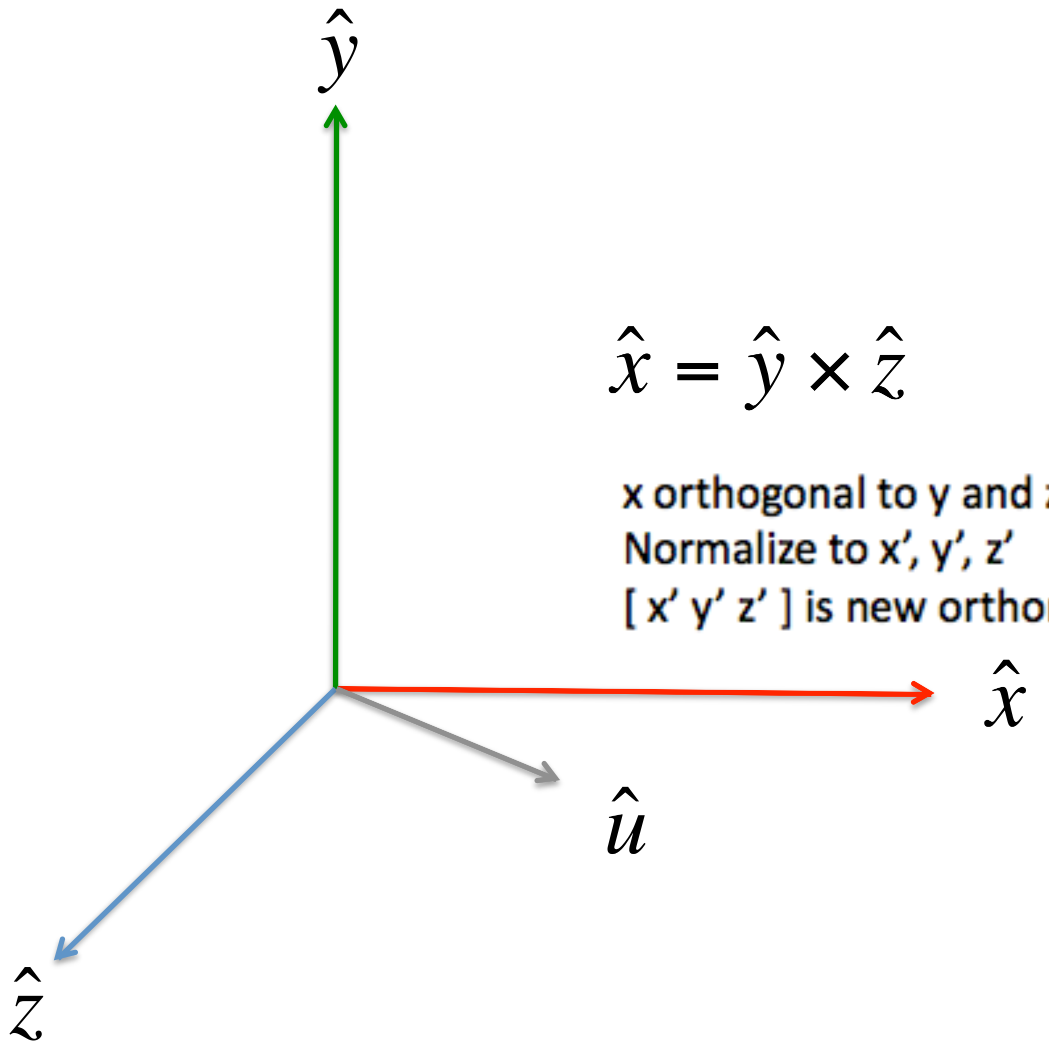
$$\hat{y} = \hat{z} \times \hat{u}$$

y orthogonal to both z and u

How to create orthonormal basis

$$\hat{x} = \hat{y} \times \hat{z}$$

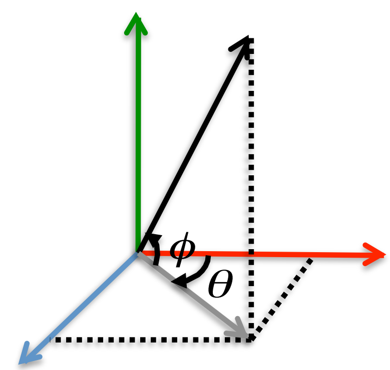
x orthogonal to y and z
Normalize to x', y', z'
[x' y' z'] is new orthonormal basis



More on rotation matrices

- Rotation matrices
  - 9 variables
  - 3 degrees of freedom
    - rotation axis – 2 numbers
      - given x,y, $z^2=1-x^2-y^2$
      - two angles
    - rotation angle
    - `Matrix4f::rotation(axis,angle)`

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$



- Problems
  - After accumulating many rotations by matrix multiplication, due to limited precision ("drift")
    - We have to renormalize (not straight-forward)
- Solution: quaternions (to be discussed in lecture)

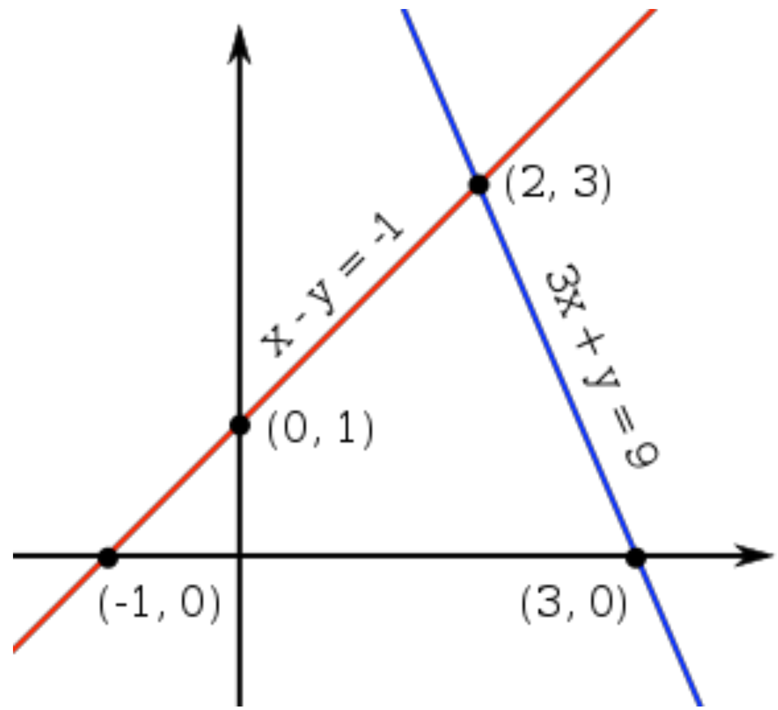
Linear systems of equations

$$x - 3y = 1$$
$$3x + y = 9$$

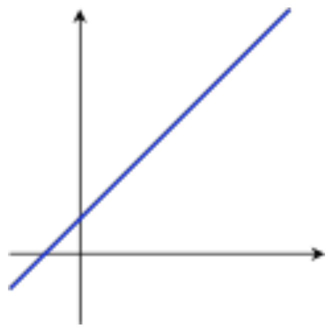
$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

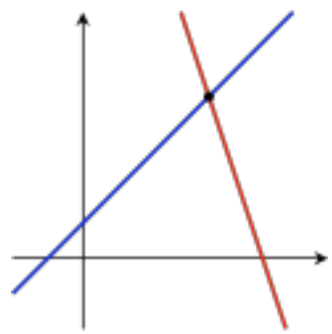


Solutions of linear systems



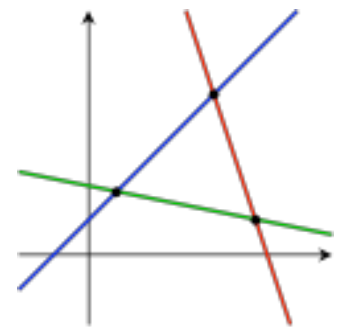
$$x - 3y = 1$$

- One equation, two variables
- Underdetermined
- Infinite solutions



$$x - 3y = 1$$
$$3x + y = 9$$

- Two equations, two variables
- Unique solution if equations are independent
- Infinite solutions if equations are dependent



$$x - 3y = 1$$
$$3x + y = 9$$
$$-x + 2y = 3$$

- Three equations, two variables
- Overdetermined
- No solution if equations are independent
- Unique solution if any two of the equations are dependent
- Infinite solutions if all equations are dependent