Expert Systems with Applications 40 (2013) 7468–7477

Contents lists available at SciVerse ScienceDirect

Expert Systems with Applications

journal homepage: www.elsevier.com/locate/eswa

A Max–Min Ant System for the split delivery weighted vehicle routing problem

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ARTICLE INFO

Keywords:
SDVRP
WVRP
Max–Min Ant system
Benchmarking test
Meta-heuristics

ABSTRACT

In real-word cargo transportation practice, such as the deliveries of perishable food and hazardous materials, neglecting the cargo weight in a typical vehicle routing problem (VRP) may prevent the routes from being the most cost effective. Thus, this paper proposes the split-delivery weighted vehicle routing problem (SDWVRP), which consists of constructing the optimal routes, with respect to the constraints on vehicle capacity and cargo weight, to serve a given set of customers with the minimum cost. A Max–Min Ant System (MMAS-SD) algorithm to solve the SDWVRP is developed and a set of theorems and corollaries are proposed to provide an easy approach for route splitting in a typical Weighted VRP (WVRP). The benefit of Split-Delivery for WVRP, as compared to that of SDVRP, primarily lies in its impact on the geographic position and loading weight feature. Large sets of benchmark instances, which are classified into cluster, random and mix types of the three different distribution types, are calculated to demonstrate the effectiveness of the SDWVRP modeling. In addition, the comparison between SDWVRP and WVRP is also carried out via analysis of vehicle numbers, total cost-savings, and the impact of weight variance and mean weight on the ratio of cost-savings and related vehicle numbers of SDWVRP over WVRP to demonstrate the superiority of SDWVRP in determining optimal routes and resulting in substantial cost savings.

1. Introduction

Traditionally, the capacitated vehicle routing problem (VRP) (Agra, Christiansen, Figueiredo, et al., 2013; Li, Leung, & Tian, 2012; Çatay, 2010) is concerned with constructing a minimum cost set of routes with a fleet of homogeneous vehicles to serve the demand of a set of customers. The demands on a route cannot exceed a vehicle's capacity constraints, and all vehicles must visit all customers exactly once. However, once the single-visit assumption is relaxed, cost savings can be substantial by allowing split deliveries. Thus, the Split Delivery Vehicle Routing Problem (SDVRP) is proposed by Dror and Trudeau (1989, 1990). Some structural properties of the problem and complexity results were proved, and a dedicated local search algorithm was proposed. As a variant of VRP (Agra et al., 2013; Archetti & Speranza, 2012; Marinakis, 2012), SDVRP relaxes the assumption into the case that each customer can be visited several times rather than once, which coincides with practical scenarios. Consequently the SDVRP has received increasing attention from researchers since the 1990s (Belenguer & Martinez, 2000; Lee, 2002; Jin, Liu, & Eksioglu, 2008; Moreno, Poggi de Aragao, & Uchoa, 2010).

Several researchers focused on an exact approach to 2006, SDVRP (Belenguer & Martinez, 2000; Dror, Laporte, & Trudeau, 1994; Lee, 2002; Archetti, Savelsbergh, & Speranza, 2006, 2008; Jin et al., 2008; Moreno et al., 2010). Recently, a new branch-and-price-and-cut method proposed by Archetti, Feillet, Gendreau, and Speranza (2011) has successfully reduced the optimality gap on most of the benchmark instances and found new best solutions in those cases in which the fleet of vehicles is limited or unlimited to the minimum possible number of vehicles. Heuristic solution approaches for the SDVRP have also been proposed by a large number of researchers, e.g., construction heuristics by Frizzell and Giffin (1992, 1995), a tabu search algorithm by Archetti, Mansini, and Speranza (2005), an optimization-based heuristic by Archetti & Speranza (2008), a local search-based meta-heuristic by Derigs, Li, and Vogel (2009), an adaptive memory algorithm by Aleman, Zhang, and Hill (2010), and a mimetic algorithm by Belenguer, Benavent, Labadi, Prins, and Reghioui (2010). To address the practical problems, the variants of SDVRP have been highly inspired in recent years. Gendreau, Dejax, Feillet, and Gueguen (2006) addressed the problem of a time window (SDVRPTW) by using column generation and branch-and-bound techniques. Mitra, 2008 proposed a parallel clustering technique to solve the

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http://dx.doi.org/10.1016/j.eswa.2013.06.068

pick-up delivery SDVRP. Tavakkoli-Moghaddam, Safaei, Kah, and Rabbani (2007) considered a heterogeneous fleet SDVRP.

On the other hand, the classic VRP and its variants assume that the weight loaded in the vehicle remains unchanged throughout the route, which results in a minimum cost solution being equivalent to a minimum distance solution. That is, the costs associated with the loads in the vehicles are neglected in the objective function of most VRP models; as a result, the optimal route may not be the minimum cost route. However, in real-world cargo transportation practice, the cost is dependent not only on the distance traveled but also on the weight loaded. For example, a toll-by-weight measure for China Expressway was proposed in which the cargo-truck are charged based on their loading quantity (the sum of the truck weight and cargo weight) and the traveling distance rather than only by the distance. In the practical delivery of perishable/fruit food, a dedicated container/vehicle is equipped to monitor the status of the food for minimum delivery loss rather than traveling distance when optimizing routes. The delivery loss risk for perishables is measured not only on traversing distance but also on the quantity of the perished or hazardous products. To model the above problems precisely for general logistics management, the weight loaded in the vehicle should be considered as a variable part of the objective when optimizing the vehicle routine rather than as a constant from one customer to another in a routine. Consequently, a new type of VRP modeling approach, the Weighted Vehicle Routing Problem (WVRP), was recently proposed to take the loading cost into consideration (Kuo & Wang, 2012; Tang, Zhang, & Pan, 2010; Zhang, Qin, Zhu, & Lim, 2012; Zhang, Tang, & Fung, 2011). The WVRP considers the load weight in the routine as a variable rather than as a constant of the objective. Here, the "weight" is a general term that could be extended to represent the number or values of objects (cargo/goods/passenger) to be delivered or the importance/priority of customers (or points). Likewise, the terminology of "cost", apart from financial costs, also provides broad meanings – the minimum emission/petrol consumption when transporting general goods, the minimum risk of loss incurred when transporting perishable foods, livestock or dangerous goods, the maximum utility, degree of satisfaction, values or benefits to delivery customers, etc. Through these means, the classical VRP could be absorbed in this broad-sense definition when the loading costs coefficient is zero. Recently, the multi-depot vehicle routing problem with loading cost (MDVRPLC) was proposed by Kuo and Wang (2012) using a variable neighborhood search (VNS). A single vehicle routing problem that takes the vehicle's weight into account (SVRPTBW) is formulated under the background of the toll-by-weight scheme of China's Expressway, and the exact approach is reached by developing a branch and bound algorithm (Zhang et al., 2012).

This paper aims to consider VRP by taking the loading weight and split deliveries into consideration, which is an extension of the SDVRP to consider the effect of loading weight as an objective function. We refer to the Split Delivery Weighted Vehicle Routine Problem (SDWVRP) hereafter in the paper. The formulation of the SDWVRP's objective function and its interpretation are demonstrated. To effectively solve the SDWVRP, a Max–Min Ant System for SDWVRP, shortened to MMAS-SD∗∗∗, is conducted and a large set of benchmark instances are calculated to prove the effectiveness of the SDWVRP formulation. The experimental results clearly show which types of customer priority and distribution for which the SDWVRP formulation is superior to SDVRP solutions. When compared with WVRP, the effectiveness of the SDWVRP formulation is significantly influenced by the number of vehicles and customer priority; therefore, the optimization of the above factors can generate substantial cost savings.

Following the introduction, Section 2 presents the formulation of the SDWVRP. Section 3 introduces the basic idea and scheme of the MMAS-SD algorithm. Computational experiments of the MMAS-SD in diverse types of VRP benchmark instances are displayed in Sections 4 and 5 to demonstrate the effectiveness and applicability of the SDWVRP model, and the performance of the algorithm. The conclusion and discussion are given in the last section.

2. The Formulation of the SDWVRP and properties on the solution

As we have mentioned, the loaded quantity plays a crucial role in real-world cargo transportation practices, such as the delivery of fruit or other perishable food, the delivery of dangerous goods, and carbon emission. Thus, it is beneficial for us to take the weight of the cargo into consideration. With respect to these considerations, we propose a new model, called the split delivery weighted vehicle routing problem (SDWVRP), with the objective of highlighting the transportation costs generated by the cargo weight. Before presenting the formulation of the SDWVRP, we give a brief summary of the SDVRP in the following subsection.

2.1. The formulation of the SDWVRP

As indicated in Section 1, the WVRP represents a modeling approach to the VRP that takes into account the effect of variable weights of the customers (points) into the objective function when optimizing the vehicle routes.

The WVRP can be defined on a graph $G = (V,E)$, where V is the vertex set and E is the arc set. The vertex set V includes the depot v_0 and customers $V_c = \{v_1, v_2, \ldots, v_n\}$. $E = \{(v_i, v_j) | v_i, v_j \in V, \ i \neq j\}$ is the arc set. A distance matrix $\mathbf{D} = (d_{ij})$ is defined on E, each element of which denotes the distance between vertex v_i, and v_j. The customers are indexed from 1 to n, in which each customer i has a non-negative demand (i.e., weight) q_i for the delivery from the depot. When compared with the classical VRP, the weight is a general term that may represent not only the quantity in absolute way but also the priority of the customers in a relative way. For example, it may represent the quantity or values of objects (cargo/goods) to be delivered, the importance or priority of customers (points), or the number of passengers at a point. In addition to the parameters, we assume the following:

(1) The distance matrix $\mathbf{D}$ is symmetric and satisfies the triangle inequality.
(2) The depot has enough homogeneous vehicles and the number of vehicles is assumed as K.
(3) Each vehicle has a weight M with capacity Q.
(4) Each customer's demand is less than the vehicle capacity.

The following variables are used to formulate the model:

x_{ijk} is a binary variable, which is equal to 1 when the arc(i,j) is traveled by vehicle k and otherwise is equal to 0, for i, $j = 1,2,\ldots,n$.
y_{ijk} is a real number variable, denoting the weight loaded on the arc(i,j), short for the arc-load weight. It is a nonnegative real number, and it is less than Q if the arc(i,j) is traveled by vehicle k and otherwise is equal to 0, for $i, j = 1,2,\ldots,n$.
w_{ki} is a nonnegative real number variable, which represents the weight loading for customer i by vehicle k ($w_{k0} = 0$), in which the value is less than Q and less than the q_i of each customer's demand.

In comparison with the WVRP, when modeling SDWVRP, an important factor is that the vehicle weight should be taken into

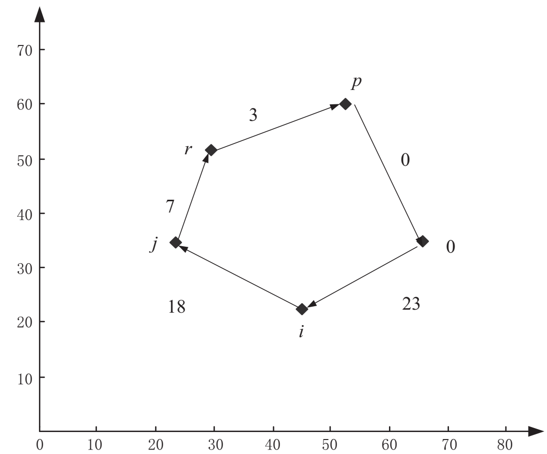

Fig. 1. An example of arc-load weight.

consideration. If we neglect the vehicle weight, the traveling cost of the SDWVRP is equal to that of the WVRP on their return to the depot, which is obviously unreasonable. For a feasible route, such as that shown in Fig. 1and $0 \to i \to j \to r \to p \to 0$, the vehicle k passes four customers on the given route, and the vehicle capacity is $Q = 25$. The weight of each customer is $q_i = 5$, $q_j = 11$, $q_r = 4$, and $q_p = 3$. Thus, we find $y_{0ik} = 23 + M$, $y_{ijk} = 18 + M$, $y_{jrk} = 7 + M$, $y_{rpk} = 3 + M$, $y_{p0k} = M$.

The split delivery vehicle routing problem satisfies the following constraints:

(1) The total demand served by any vehicle does not exceed its capacity.
(2) The length of the dispatching route does not exceed the maximum traveling distance of each vehicle.
(3) The demand of each customer is totally satisfied and can be served by more than one vehicle.
(4) Diversified cargo weight can lead to different transportation costs.

The objective of the SDWVRP is to minimize the total cost, which takes both the load quantity and the distance traveled into consideration. Without a loss of generality, we can simplify the model by setting the cost coefficient to be 1, which leads to the following model:

$$Min\cos t = \sum_{i\in V}\sum_{j\in V}\sum_{k\in K} d_{ij}\cdot x_{ijk}\cdot y_{ijk} \quad (1)$$

$$\sum_{j\in V}\sum_{k\in K} x_{ijk} \geqslant 1, \quad \forall i \in V_c \quad (2)$$

$$\sum_{i\in V} x_{irk} - \sum_{j\in V} x_{rjk} = 0, \quad \forall r \in V, \quad \forall k \in K \quad (3)$$

$$\sum_{j\in V_c} x_{0jk} - \sum_{j\in V_c} x_{j0k} = 0, \quad \forall k \in K \quad (4)$$

$$\sum_{i\in V} w_{ki} + M = \sum_{i\in V} y_{0ik}, \quad \forall k \in K \quad (5)$$

$$\sum_{k\in K} w_{ki}\left(\sum_{j} x_{jik}\right) = q_i, \quad \forall i \in V \quad (6)$$

$$\sum_{i\in S}\sum_{j\in S} x_{ijk} \leqslant |S| - 1, \quad \forall S \subseteq V_c, \quad 2 \leqslant |S| \leqslant n, \quad \forall k \in K \quad (7)$$

$$\sum_{j\in V} x_{jik}y_{jik} - \sum_{j\in V} x_{ijk}y_{ijk} = w_{ki}, \quad \forall i \in V_c, \quad k \in K \quad (8)$$

$$0 \leqslant (w_{ki} + M)x_{ijk} \leqslant y_{ijk} \leqslant (Q + M)x_{ijk}, \quad \forall i,j \in V, \quad k \in K \quad (9)$$

$$x_{ijk} \in \{0,1\}, \quad \forall, i,j \in V, \quad k \in K. \quad (10)$$

$$0 \leqslant w_{ki} \leqslant q_i, w_{ki} = 1,2\ldots Q, \quad \forall i \in V, \quad k \in K \quad (11)$$

The objective function aims to find the minimum total travel costs for all vehicles. Constraint (2) represents that at least one vehicle serves the demand of customer i. Formula (3) is the conservation equality of flow, which ensures the number of vehicles entering or leaving the customer depot is unchanged. Eq. (4) is the depot constraint that each vehicle starts from and returns to the depot. Constraint (5) shows that the arc-load weight is composed of vehicle weight (M) and total demand of customers to be served. Function (6) is a constraint that ensures all the customers' demands are well served. Constraint (7) is the sub-tour elimination constraint. Eq. (8) shows the relationship between the priority of customer i and the vehicle loads on the two arcs linking customers i and j, and constraints (9) and (10) are the value constraints of the decision variables. Formula (11) shows that the loading quantity of vehicle k in i does not exceed the demand of customer i, and w_{ki} is restricted to be an integer. In comparison with SDVRP, apart from the objective function, the additional variables y_{ijk} are introduced in the formulation of SDWVRP, and consequently, the additional constraints (5), (8), and (9) are imposed.

Proposition 1 (*SDWVRP is a NP-hard problem.*). *It is well known that the classic SDVRP was proved to be an NP-hard problem* (Archetti et al., 2005, 2011), *and because the SDWVRP takes the cargo weight into consideration, it is more complicated than the SDVRP, and it should also be an NP-hard problem.*

Corollary 1 (*The optimal solution of the SDVRP is not guaranteed to be the optimal solution to the SDWVRP.*). *Corollary 1 is easily proved because $y = F(x,w) \in \mathbf{Y}$ is not a constant, but a quadratic or polynomial function of (x,w); thus, the optimum for the SDVRP may not be the optimum for the SDWVRP.*

2.2. The properties of the solution of SDWVRP

The theorems and properties of the solution of SDVRP are given in Dror and Trudeau (1990). In this section, we will first analyze under which conditions the demands of a customer should be split when modeling the SDWVRP. Assuming that the demand of customer B, which does not exceed the vehicle loading capacity Q, can be split into two partitions, w_{k_1B} and $w_{k_2B}(w_{k_1B} + w_{k_2B} < Q)$, and is satisfied by vehicles k_1 and k_2, then we put customers on both routes together and represent them as points A and C, with their demand represented as $w_{k_1A}(w_{k_1A} \leqslant Q - w_{k_2B})$, $w_{k_2C}(w_{k_2C} \leqslant Q - w_{k_1B})$; therefore, the combination of points A and C is

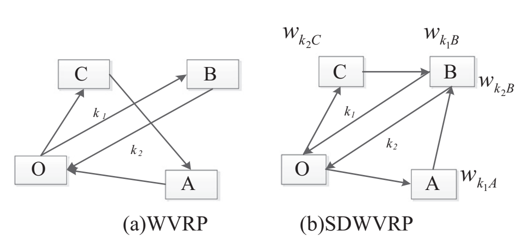

Fig. 2. An example of customer points to be split.

substantial. The routes of the SDWVRP and the WVRP are shown in Fig. 2, where (a) vehicle k_1 travels following the sequence of OC → CA → AO, while vehicle k_2 follows the sequence of OB → BO, (b) vehicle k_1 travels following the sequence of OC → CB → BO, while vehicle k_2 follows the sequence of OA → AB → BO. The objective costs for the split and un-split cases are calculated below.

(a) For split routes:

$$C_{spilt} = (w_{k_2c} + w_{k_1B}) \cdot d_{OC} + w_{k_1B} \cdot d_{CB} + (w_{k_1A} + w_{k_2B}) \cdot d_{OA} + w_{k_2B} \cdot d_{AB} \tag{12}$$

(b) For un-split routes:

$$C_{unspilt} = (w_{k_1A} + w_{k_2c}) \cdot d_{OC} + w_{k_1A} \cdot d_{CA} + (w_{k_1B} + w_{k_2B}) \cdot d_{OB} \tag{13}$$

By subtracting (12) from (13), we have

$$C_{unspilt} - C_{spilt} = (d_{OB} - d_{CB} - d_{OC}) * w_{k_1B} + (d_{OB} - d_{OA} - d_{AB}) * w_{k_2B} + (d_{OC} + d_{CA} - d_{OA}) * w_{k_1A} \tag{14}$$

When split routes are more beneficial, there is an inequality $C_{unspilt} - C_{spilt} > 0$, which indicates that

$$(d_{OC} + d_{CA} - d_{OA}) * w_{k_1A} > (d_{OC} + d_{CB} - d_{OB}) * w_{k_1B} + (d_{OA} + d_{AB} - d_{OB}) * w_{k_2B} \tag{15}$$

Let point a denote the point before B in a routine; that is, a vehicle starts at the depot O and reaches point a after visiting several customers. When it reaches point B, we check its capacity constraint to know whether point B needs to be split and then determine the quantity on the route. Then, the car continues to visit several customers, the set of which is denoted by S, and returns to the depot O. As shown in Fig. 3, starting from a point O to point B, when passing point a, we define a penalty function $l_B(a) = d_{aO} + d_{aB} - d_{BO}$ to denote the additional cost associated with the point; it is illustrated as a common partition between the red line and blue line. Of course, this cost is zero if point a is in the segment OB. In this sense, formula (15) is equivalent to (16). That means point B is split if and only if (16) holds.

$$\begin{cases} l_B(C) \cdot w_{k_1B} + l_B(A) \cdot w_{k_2B} < (d_{OC} + d_{CA} - d_{OA}) \cdot w_{k_1A} \\ l_B(C) \cdot w_{k_1B} + l_B(A) \cdot w_{k_2B} < (d_{OA} + d_{AC} - d_{CO}) \cdot w_{k_2C} \end{cases} \tag{16}$$

For any two partial routes in which a and b connect with the split point B and satisfy the capacity constraint, we can have

$$\begin{cases} l_B(a) \cdot w_{Ba} + l_B(b) \cdot w_{Bb} < (d_{oa} + d_{ab} - d_{bo}) \cdot w_b \\ l_B(a) \cdot w_{Ba} + l_B(b) \cdot w_{Bb} < (d_{ob} + d_{ba} - d_{ao}) \cdot w_a \, , \forall a, b \in V_B \end{cases} \tag{17}$$

where V_B is defined as a set of points who satisfy the capacity constraint and can form a new route with point B. Let

$$\varphi(a,b) = \min\{(d_{oa} + d_{ab} - d_{bo}) \cdot w_b, (d_{ob} + d_{ba} - d_{ao}) \cdot w_a\}, \quad \forall a, b \in V_B. \tag{18}$$

To summarize the above analysis, one can construct a property and corollary on the split of the customer point B in the optimal solution as follows.

Property 1. *In the SDWVRP, point B can be split in the optimal solution when it satisfies the formula (19):*

$$l_B(a) \cdot w_{Ba} + l_B(b) \cdot w_{Bb} < \varphi(a,b), \quad \exists a, b \in V_B \tag{19}$$

Corollary 2. *For any two partial routes a and b, connected at point B, the optimal split strategy for a split customer B is achieved such that* $l_B(a) \cdot w_{Ba} = l_B(b) \cdot w_{Bb}$ *holds.*

3. The Max–Min Ant System for solving the SDWVRP

As stated in Section 2, the SDWVRP is an NP-hard problem, and its solution is not guaranteed to be equivalent to that of the SDVRP. To solve the SDWVRP effectively an improved Max–Min Ant System algorithm combining the features of split delivery (MMAS-SD) is proposed in this paper.

3.1. The fundamental concepts of the MMAS-SD

Ant algorithm optimization (ACO), a type of meta-heuristics optimization method proposed by Dorigo (1996), simulates the food-seeking behaviors of ant colonies in nature. Following the ACO principle, Stutzle and Hoos (2000) presented a Max–Min Ant System (MMAS) from the original Ant System based on the probabilistic construction of the solutions. It has been successfully applied to solve various types of the VRP and other combinatorial optimization problems (Bell & McMullen, 2004; Çatay, 2010).

As for the SDWVRP, the demand of customers can be served by two or more vehicles; thus, in the ACO algorithm, the demand of customers should be split when the ants generate the first iterant solutions. In a classic VRP/WVRP, if a customer's demand exceeds the capacity of a vehicle, a new vehicle is dispatched from the depot to serve the need of this customer. However, in the SDWVRP, instead of sending a new vehicle, we choose to split the demand of the customer and let this specific vehicle serve part of the customer's demand while another vehicle serves the remainder of it. By combining the special characteristics of the SDWVRP, we propose an improved Max–Min Ant System for the SDWVRP (MMAS-SD) based on the classic Max–Min Ant System algorithm. It follows the basic framework of the ACO; however, the critical components are designed as follows.

To describe the procedures of the MMAS–SD algorithm, the notations related to the algorithm are first given as follows:

n = the number of the customers (nodes); m = the colony size; d_{ij} = the distance from node i to j, where $i, j = 1, 2, \ldots, n$; and q_i = the weight of customer i.

3.2. The generation of solutions

An individual ant simulates a vehicle, and its route is constructed by incrementally selecting customers until all customers are visited. In this paper, we use a greedy algorithm to generate the initial route. Assume that ***P*** is a partial solution. Those

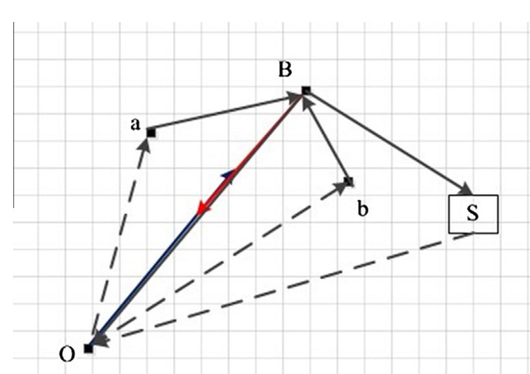

Fig. 3. An illustration of customer points to be spilt.

customers who were already visited by an ant or violated its capacity constraints, are stored in the infeasible customer list $N(\boldsymbol{P})$.

The ant $k(k = 1, 2, \ldots, m)$ decided to combine costumers based on both the visibility and the pheromone information. First, a random number q, which is uniformly distributed within $[0,1]$, is generated to compare with a given parameter $q_0 \in (0,1)$. If $q \leqslant q_0$, the next customer j for the kth ant at the ith node is determined to have the highest product of pheromone and heuristic information:

$$j = \operatorname{argmax}_{k \in N - N(P)} \{\tau_{ik}^{\alpha} \bullet \eta_{ik}^{\beta}\} \tag{20}$$

τ_{ij} is the pheromone density of the arc (i,j), α and β are the relative influences of the pheromone trails and the visibility values, respectively, and $\eta_{ij} = q_j/d_{ij}$ is the heuristic information for the transition from state i to j. It represents the priority is given to customers with higher weight per distance.

Otherwise, the ant uses the following probabilistic formula to choose the next customer:

$$p_{ij} = \frac{\tau_{ij}^{\alpha} \bullet \eta_{ij}^{\beta}}{\sum_{k \in N(P)} \tau_{ik}^{\alpha} \bullet \eta_{ik}^{\beta}} \tag{21}$$

3.3. The choice rule of a division point

Unlike normal VRP models, the SDWVRP will split a customer's demand instead of going back to the depot and starting a new route when the demand of the customer node is beyond the vehicle's capacity. To consider that a split delivery may lead to a lower delivery cost, during the search process, an ant needs to follow a rule to choose customers and split the demand. Combining the properties proposed in the Section 2.2, we present the rule to choose the customers for split delivery in this section.

If the tabu list of the ACO does not absorb all the customers and the demand of the customer does not exceed the capacity of the vehicle, the customer depots are chosen based on the transfer probability p_{ij}. In this way, the tabu list is updated when the demand of the customer is satisfied. The vehicle continuously visits the customers following the sequence with the minimum cost until the demands of all the customers exceed the vehicle capacity.

However, if the demand of the unvisited customers exceeds the available capacity of the current vehicle, some splitting measures should be taken to ensure the minimum delivery cost. Assuming that a vehicle starts from customer i and heads to a customer s, depot j is chosen for the minimum cost as per formula (22):

$$j = \arg\{\min\{(Q_{left} + M)^* d_{is} + M^* d_{s0}\} | s \in N - N(P)\} \tag{22}$$

where, Q_{left} is the existing load of the vehicle, M is the vehicle weight, and d_{is} is the distance between nodes i and s.

As is explained in the following example, we assume that depots n_1 to n_r represent the depots that can be selected by probability p_{ij}, while s_1, s_2 to s_c represent the depots that exceed the vehicle capacity. If we set $Q = 50$, in city i, the capacity of the vehicle is $Q_{left} = 20$, and the demand of s_1 and s_2 are equal to 25 and 30, respectively. Likewise, the demands of s_c all exceed the Q_{left}. In this way, the split point is chosen based on formula (22). If s_1 is selected to be split, then the demand of s_1 is reset to 5, which is equal to the demand of s_1 minus Q_{left}. A new route is finally established to continue satisfying the demand of the remaining customers, as shown in Fig. 4.

When all the vehicle have reached their maximum capacity, a new vehicle will be sent from the depot until all the demands of the customers are satisfied. The rule enables the minimum cost of deliveries between the selected customers and the depot and ensures the current demand of customers after they have been selected in an efficient way.

3.4. Update of pheromone information

After each ant has built a completed route, the pheromone trails are updated only by the best ant. To avoid early convergence, the value of pheromone trails on each solution are limited to an interval $[\tau_{\min}, \tau_{\max}]$ (Stutzle & Hoos, 2000), where

$$\tau_{\max} = 1/C^{best} \tag{23}$$

$$\tau_{\min} = \tau_{\max}(1 - \sqrt[n]{0.05})/((n-2)\sqrt[n]{0.05}/2) \tag{24}$$

After all the ants finish their tours, the pheromone trails are updated based on a pheromone update rule. The rule of the pheromone trails is performed by the following equation:

$$\tau_{ij}(t+n) = (1-\rho)\tau_{ij}(t) + \Delta\tau_{ij}^{best}(t) \tag{25}$$

Here, $0 < \rho \leqslant 1$ is the rate of pheromone evaporation, which is applied to avoid the unlimited accumulation of pheromones and the resulting possible premature convergence toward a suboptimal solution region. The notations $\tau_{ij}(t+n)$ and $\tau_{ij}(t)$ are the pheromones on the arc (i,j) after and before updating, respectively, and $\Delta\tau_{ij}^{best}(t)$ is defined as the ant cycle model (Zhao et al., 2010):

$$\Delta\tau_{ij}^{best}(t) = \begin{cases} \frac{1}{C^{best}}, & \text{if}(i,j) \text{ belongs to the best tour} \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

where C^{best} denotes the cost of either the local best or the global best solution. It is different from the classic VRP model.

3.5. The overall procedures of the MMAS-SD

To describe the MMAS-SD procedures, let nc be the index of iteration number, and *IterNum* be the maximum iteration number as the stop rule. $k = \{1, 2, \ldots, m\}$ is the index of the ant, $tabu_k$ is the set of customers that ant k has passed, and correspondingly, $allowed_k = V_c - tabu_k$ represents the set of customers available for the next step. The route set constructed by ant k is denoted by R_k; r_k is the number of vehicles in set R_k, and $Route_{r_k}$ represents the r_k^{th} route. The optimal objective function is denoted by f. The analysis of the algorithm is summarized in the following procedures:

Step 1: Initialization: Set $nc = 0$; $Q_{left} = Q$, $R_k = \varnothing$, $k = 1$. $f = 0$; $r_k = 1$, $Route_{r_k} = \emptyset$.$tabu_k = \varnothing$, $allowed_k = V_c$.
Step 2: Use a greedy algorithm to generate the initial routes. Calculate the optimal solution f and the pheromone information $\tau_{ij}(t)$ based on formula (23).
Step 3: **If** $nc > IterNum$, go to **Step8**;
Step 4: **for**(k=1; k<m; k++) {The m ants are set randomly on n city nodes.}

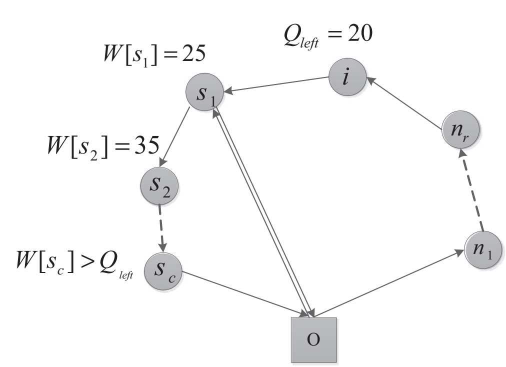

Fig. 4. An example of choice rule of division point.

Step 5: **for** (k=1; k<m; k++){**while**($allowed_k \neq \emptyset$){**If** $q_j < Q_{left}$ and $q_j > 0$, **then** choose j from $allowed_k$ following the formula (20) and (21); $tabu_k \leftarrow j$, $allowed_k = allowed_k - \{j\}$, $Route_{r_k} \leftarrow j$; $q_j = 0; Q_{left} = Q_{left} - q_j$; **Else if** $q_j > Q_{left}$ and $q_j > 0$, **then** choose j from $allowed_k$ following the formula (22) using split strategy, here $q_j = q_j - Q_{left}$, $Q_{left} = Q$, $Route_{r_k} \leftarrow j, r_k$++;} Calculate the optimal solutionf_k of ant $k, R_k \leftarrow Route_{r_k}$}

Step 6: Update the historical optimal solution $f = \min(f_k)$; $k = 1, 2, \ldots, m$.

Step 7: Calculate the pheromone information,**If** $\tau_{ij}(t) < \tau_{\min}$, **then** $\tau_{ij}(t) = \tau_{\min}$; **If** $\tau_{ij}(t) > \tau_{\max}$, **then** $\tau_{ij}(t) = \tau_{\max}$; update the $\tau_{ij}(t)$, nc++, go to **Step2**.

Step 8: Output the results, stop the process.

4. Efficiency analysis of the SDWVRP model

In this section, we will conduct computational experiments to analyze the efficiency and adaptability of the SDWVRP model. For this purpose, the experiments were composed of two parts. One is the comparison with the SDVRP model to illustrate the necessity of taking the cargo weight into consideration. Another is the comparison with the WVRP model to prove the importance of splitting the demand of customer depots in terms of the criteria of vehicle numbers and expenses under different types of customer distributions and weight features.

4.1. The comparison between the SDWVRP and SDVRP

The testing problems are sourced from benchmark data sets on the webpage of http://neo.lcc.uma.es/radi-aeb/WebVRP/index.html?/Problem_Instances/instances.html of VRP instances. The testing problems are classified into seven types from the perspectives of customers' geographical distribution and weight distribution. The distribution of customer locations is classified as a random distribution (RD), a several clusters distribution (CD), a mix of cluster and random distribution (RC), a regular distribution with the same abscissas (RSA), and depot-centered customers' distribution like a magnetic field (DCMF). The weight feature is characterized in terms of the ratio of the standard deviation (STDEV) and the average (AVG), short for the RATIO, of the weights of the customers involved in an instance. From this point of view, three classes are given: LTO (STDEV/AVG less than 1.0), GTO (STDEV/AVG greater than 1.0), ZERO (the same weight, i.e., STDEV = 0). These testing problems are of sizes from 22 to 442 and have different vehicle capacities that vary from less than 10 to larger than 10,000. Detailed information about the benchmarking instances is shown in Table 1, of which the column 'instance' includes the name given in the benchmarking problem.

Hereafter, the comparison between the objective value of SDWVRP and SDVRP is demonstrated as below in Table 2. Here, the C_{SDVRP} represents the objective function value of the SDWVRP model corresponding to the best known route of the SDVRP model. Meanwhile, the C_{SDWVRP} is hereafter the objective function value of the SDWVRP found by MMAS-SD. The column "GAP1" is defined as $(C_{\mathrm{SDVRP}} - C_{\mathrm{SDWVRP}})/C_{\mathrm{SDWVRP}} * 100\%$ in this paper.

It can be seen from Table 2 that the SDWVRP is more efficient than the SDVRP. Another observation is that for any type of customers' geographical distribution, the instances with weight features of GTO resulted in more cost savings than the other instances. That means the SDWVRP modeling is more effective in dealing with the instances with a larger variance ratio of the customer weight. However, the computations show that for the instances with the same distribution features, the more scattered customers are, the more effective the SDWVRP modeling will be. We could not obtain a similar conclusion for those instances with tight distributions.

4.2. The comparison of the model SDWVRP and WVRP in terms of vehicle numbers and cost-savings

When satisfying the demand of all customers, delivery splitting makes the optimization of transfer routes possible. Denote $r(\mathrm{WVRP})$ and $r(\mathrm{SDWVRP})$ as the least vehicle number obtained by the WVRP and SDWVRP models, respectively. We use $\frac{r(\mathrm{SDWVRP})}{r(\mathrm{WVRP})}$ to measure the improvement of the SDWVRP from the WVRP from the performance of the vehicle numbers, short for the vehicle ratio. The smaller the ratio was, the more improvement was demonstrated.

When all customers have an identical weight (demands), the WVRP model is equivalent to VRP, and vice versa. The SDWVRP can be equivalently transferred to the SDVRP, and it has been proved that $\frac{1}{2} \leqslant \frac{r(\mathrm{SDWVRP})}{r(\mathrm{WVRP})} \leqslant 1$.

Let C_{WVRP} represent the optimal solution of an objective function for the WVRP model solved by BEAM-MMAS (Guan, 2012). The cost-saving ratio is defined as $(C_{\mathrm{WVRP}} - C_{\mathrm{SDWVRP}})/C_{\mathrm{SDWVRP}} * 100\%$ to measure the cost-savings of the SDWVRP model over the WVRP model.

In the following section, we will illustrate how the cost-savings and vehicle ratio vary with the geographical distribution and weight features. To this end, we take the benchmarking instances described by the geographical distribution and weight features.

Table 1
Description and features of the testing problems.

No.	Instance	Size		Weight		Type no.	Distribution feature	Weight feature
		n	Capacity	AVG	STDEV			
P1	A-n33-k6	33	100	16.9	11.2	1	RD	LTO
P2	A-n65-k9	65	100	13.7	7.28			
P3	E-n76-k7	76	220	18.2	7.9			
P4	B-n41-k6	41	100	14.2	5.6	2	CD	LTO
P5	B-n63-k10	63	100	14.9	8.22			
P6	E-n30-k3	30	4500	439.7	600	3	RD	GTO
P7	F-n45-k4	45	2010	164.1	255.6			
P8	ulysses-n16-k3	16	5	1	0	4	RD	ZERO
P9	att-n48-k4	48	15	1	0			
P10	F-n135-k7	134	2210	109.1	186.6	5	RC	GTO
P11	ta-n101-k11b	101	1842	195	263			
P12	F-n72-k4	71	30000	1617.5	2987.8	6	RSA	GTO
P13	kelly01	240	550	20	10	7	DCMF	LTO
P14	Kelly02	320	700	20	10			
P15	Kelly03	400	900	20	10			

Table 2
The comparison between the objective values of SDWVRP and SDVRP.

Type of instance	Instance number	Instance	C_{SDVRP}	C_{SDWVRP}	Runtime (Sec.)	GAP1 (%)
RD-LTO	P1	A-n33-k6	83,597	81,532	3.9	2.47
	P2	A-n65-k9	114,504	111,007	34	3.05
	P3	E-n76-k7	18,8923	188,555	43	0.19
CD-LTO	P4	B-n41-k6	137,133	136,049	7	0.79
	P5	B-n63-k10	167,474	167,127	30	0.21
RD-GTO	P6	E-n30-k3	2,695,337	2,505,084	3.3	7.06
	P7	F-n45-k4	1,629,594	1,534,857	7.8	5.81
RD-ZERO	P8	ulysses-n16-k3	388	365	0.4	5.93
	P9	att-n48-k4	638,757	584,290	10	8.53
RC-GTO	P10	F-n135-k7	2,472,658	2,388,835	49	3.39
	P11	ta-n101-k11b	4,785,623	4,518,946	119	5.57
RSA-GTO	P12	F-n72-k4	10,952,482	10,331,621	362	5.67
CMF-LTO	P13	kelly01	3,797,285	3,669,419	475	3.37
	P14	kelly02	7,030,276	6,878,197	935	2.16
	P15	kelly03	11,947,031	11,525,875	2364	3.5

Table 3
The cost-savings ratio of the model of the SDWVRP to the WVRP under instances of type Cpr101.

Instance mean weight	Variance of weight						
	0	16	64	144	196	400	900
10	1.000000	1.000000	1.000000	–	–	–	–
30	0.991699	1.000000	1.000000	0.998432	0.999305	–	–
50	1.000000	0.999127	0.999275	0.998073	0.996945	0.999962	0.997163
70	0.880974	0.9600100	0.955119	0.989321	0.995821	0.988401	0.979615
80	0.939593	0.935254	0.953284	0.973735	0.980431	0.989464	0.999355
90	0.980323	0.982782	0.993516	0.989467	0.995901	0.987617	0.991475
95	0.999685	0.99911	0.999879	0.993552	0.977979	0.975986	0.953516
100	1.000000	0.949306	0.979576	0.912362	0.945152	0.912631	0.962192
105	0.782176	0.793357	0.836915	0.839405	0.847452	0.904675	0.916505
110	0.778389	0.777935	0.813287	0.839948	0.844027	0.873615	0.890412
120	0.797730	0.810786	0.811289	0.824292	0.840019	0.842782	0.888628
130	0.836668	0.842791	0.843969	0.845110	0.850303	0.858608	0.889872
150	0.893000	0.901676	0.896554	0.905099	0.899745	0.899682	0.900249
170	0.952447	0.95659	0.95432	0.947388	0.954143	0.951986	0.958303
190	0.996568	1.000628	0.995874	0.994872	0.995135	0.999397	0.999858

The data sets are derived in three types from the perspective of the geographic distribution of customers: random (RD), cluster (CD) and mixed random, and cluster (RC), each contains only 100 customer points. They are Cpr101, Rber101, RCta101, of which the capacity of vehicles is set to $Q = 200$, and their dead weight is ($M = 70$). For each type of geographical distribution of instances, a combination of the average and variance values is given to represent the weight of customers in a given instance. The mean weights of the customers are 10, 30, 50, 70, 80, 90, 95, 100, 105, 110, 120, 130, 150, 170, 190, and the variance weights of the customers are 0, 16, 64, 144, 196, 400, 900. In this case, for each type of geographical distribution, we can generate 105 testing instances, each of which has 100 customer points, and the weight of customers is obtained with a random Beta distribution. For example, Table 3 shows several of the 105 testing instances generated from Cpr101, where each pair of columns and rows corresponds to an instance.

Each element of the matrix in Table 3 represents the ratio of the cost-savings of the model with the SDWVRP compared to the WVRP corresponding to that instance; e.g., if the variance of the weight equals 64 and the instance mean of the weight equals 50, then the corresponding ratio of cost-saving is 0.999275. That means the cost of the SDWVRP only is nearly half of the modeled WVRP. Table 4 shows the ratio of the vehicle number of the model of the SDWVRP to the WVRP for the instances under the type Cpr101. Given to the large variance, we cannot successfully generate proper instances; thus, some ratios are represented as "–".

Fig. 5 plots the distribution of the cost-savings of the model SDWVRP under three types of instances (CD, RD and RC), where the horizontal axis represents the cost-savings, and the vertical axis is the number of instances. Similarly, Fig. 6 depicts the distribution of the vehicle ratio of the SDWVRP and WVRP models under three types of instances. From Fig. 5, one can observe that there are 55 instances that are in the cluster distributions between the interval (0.95, 1.0], and from Fig. 6, it is easy to see that there are 16 instances for each distribution that reach the optimal vehicle number ratio of 0.5. Moreover, it is obvious that of all types of instances, more than 50% of them can bring about a 5% cost savings by using the SDWVRP model than the WVRP, and approximately 10% of them can even lead to a 20% cost savings. From the figure above, it also indicates that the geographic distributions have only a limited impact on the cost savings and vehicle ratios.

4.3. The impact of weight features on the cost savings and vehicle ratios of the model of the SDWVRP to the WVRP

In this subsection, we will illustrate how the mean and variance of customers' weight influence the cost-savings and vehicle ratio of the model of the SDWVRP as compared to the WVRP. To this end, we take instance Cpr101 as an example, and the capacity and other parameters of the model are the same as described in Section 4.2.

From the perspectives of the mean weight of customers, we set two values of the customers' mean weights; they are 95 and 105 to represent just under and just over the vehicle's capacity. The

Table 4
Ratio of vehicle numbers between the SDWVRP and the WVRP model under type of Cpr101.

Instance Mean weight	Variance of weight						
	0	16	64	144	196	400	900
10	1.000000	1.000000	1.000000	–	–	–	–
30	0.882353	0.941177	1.000000	0.937500	1.000000	–	–
50	1.000000	0.928577	0.961539	0.923077	0.962963	0.960000	1.000000
70	0.660000	0.795454	0.897435	0.921052	0.923076	0.925000	0.942857
80	0.800000	0.800000	0.800000	0.854166	0.891304	0.888889	0.953488
90	0.900000	0.900000	0.920000	0.897959	0.920000	0.938776	0.938776
95	0.960000	0.960000	0.96000	0.960000	0.923077	0.941177	0.907408
100	0.500000	0.510000	0.500000	0.520000	0.510000	0.520000	0.510000
105	0.530000	0.530000	0.530000	0.520000	0.550000	0.540000	0.560000
110	0.550000	0.550000	0.560000	0.560000	0.560000	0.550000	0.560000
120	0.600000	0.600000	0.600000	0.610000	0.600000	0.610000	0.600000
130	0.650000	0.650000	0.660000	0.660000	0.660000	0.650000	0.650000
150	0.750000	0.760000	0.760000	0.760000	0.750000	0.750000	0.750000
170	0.850000	0.860000	0.850000	0.850000	0.850000	0.870000	0.850000
190	0.950000	0.960000	0.950000	0.940000	0.950000	0.960000	0.979798

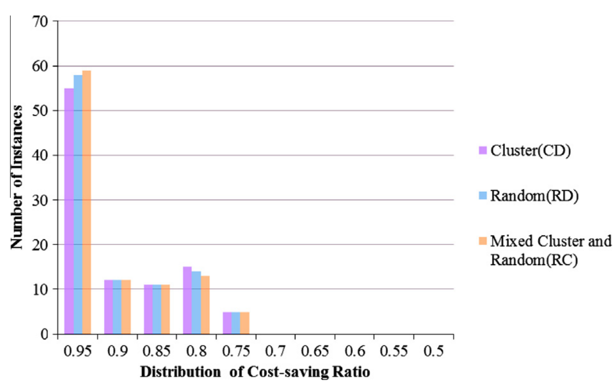

Fig. 5. The distribution of cost-savings of SDWVRP model.

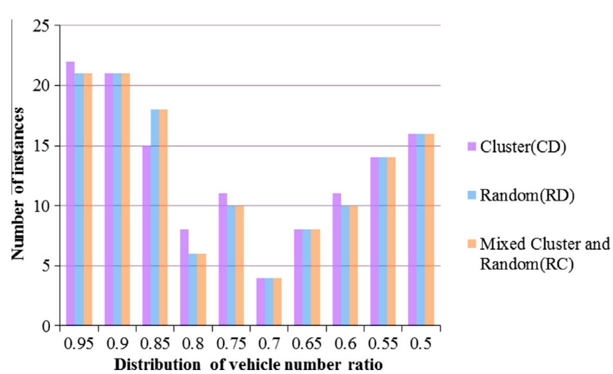

Fig. 6. The distribution of the vehicle ratio of SDWVRP and WVRP models.

cost-savings of the SDWVRP model and the vehicle ratio of the DSWVRP and WVRP models vary with the variance of the customers' weights are depicted in Figs. 7 and 8, respectively. One can observe that when the mean weight is just below the vehicle capacity, i.e., 95, an increase in the weight variance will cause more customers' weights to exceed half of the vehicle capacity. Therefore, these customers should be served by a single vehicle in the WVRP model, which subsequently leads to an increase in the transportation cost C_{WVRP} and results in a decline of the cost-savings ratio. Meanwhile, when the mean weight is just above the vehicle capacity, the decrease in the WVRP transportation cost will contribute positively to cost-savings; i.e., for the mean weight of 105, as the variance varies from 0 to 900, the ratio of the cost-savings increases from 0.78 to 0.92. The trend of variance effects on the ratio of vehicle numbers is similar; however, the ascent and descent of the vehicle number ratio is not as obvious.

Similarly, we select a set of values for the variance of customers' weights, i.e., 0, 196 and 900, in the testing experiments to analyze their effects on the cost-savings. The mean values of customers' weights of 10, 30, 50, 70, 80, 90, 95, 100, 105, 110, 120, 130, 150, 170, and 190 are tested and analyzed. The impact of the mean of the customers' weights on cost-savings and vehicle ratios are plotted as shown in Figs. 6 and 7, respectively.

From Figs. 9 and 10, we can observe several features of the SDWVRP model:

(1) For the case that the variance equals 0, when the mean weight is just above the vehicle's capacity, i.e., the points 105 and 110, the cost-savings reaches its summit; moreover, when the mean weight varies from 105 to 190, the cost-savings declines. It can be concluded that vehicle numbers are reduced by inviting splitting methods. This curve coincides with the features when the mean weight is just below and just above the vehicle capacity.
(2) In general, the variance of the mean weight has negative effects on the cost-savings, and lower variance can bring a larger cost-savings with the SDWVRP model. That is, a large variance potentially reduces the split cost savings.
(3) For a specified value of variance greater than 0, the cost-savings first increase with the values of the mean of customers' weight until just above half of capacity, e.g., 105, 110, 120 and 130, and then begins to decrease.
(4) The trend of the cost-savings and vehicle ratio are approximately identical. This indicates that the cost-savings and vehicle number increase or decrease together, which means

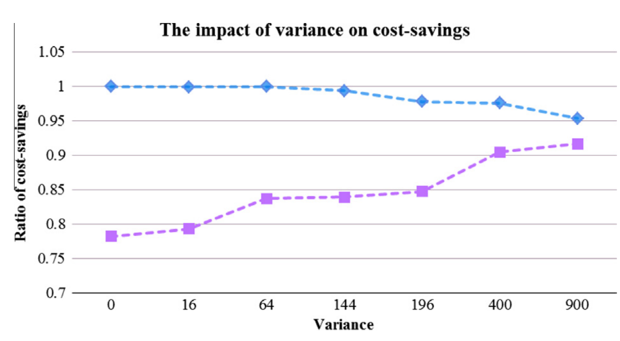

Fig. 7. The impact of the variance of customers' weight on the cost-savings of the model of the SDWVRP.

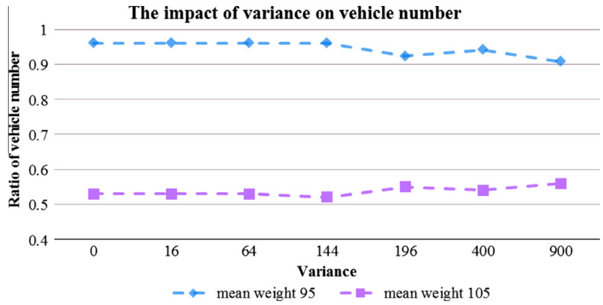

Fig. 8. The impact of variance of customers' weight on vehicle ratios of the model SDWVRP and VRP.

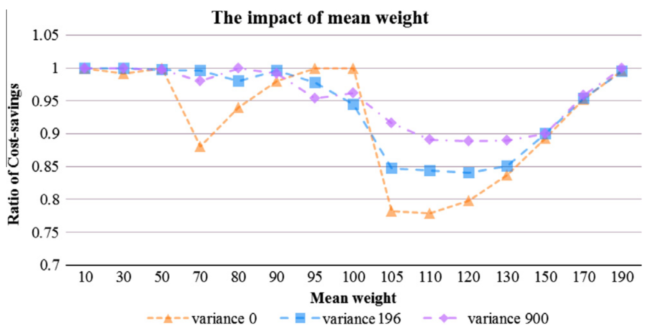

Fig. 9. The impact of mean of customers' weight on cost-savings of SDWVRP.

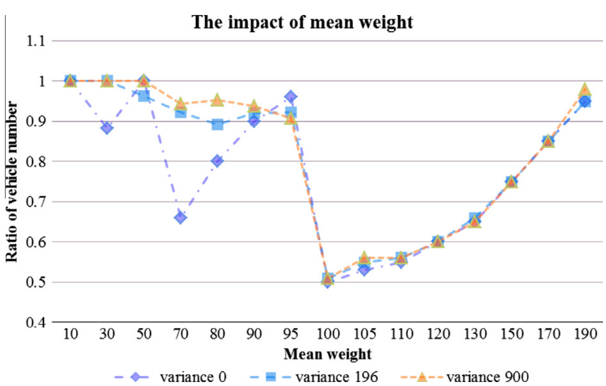

Fig. 10. The impact of mean of customers' weights on the vehicle ratios of SDWVRP to WVRP.

that allowing the split of customer demand can reduce the vehicle numbers, and further leads to substantial cost savings.

(5) The testing experiments on instance Rber101 and RCta101 have the similar results.

4.4. Summary of the experiments

According to above testing calculation and analysis, when dealing with specific customers' demand and distribution, considering the weight and splitting of customers can bring about potential benefits, which in turn lead to substantial cost savings. In particular, we found are following insights:

(1) When customers' demands and distribution are greatly diversified, the SDWVRP model proves to be more beneficial and can bring about desirable cost savings.
(2) When splitting the demand of customers, a decrease in the transportation cost contributes to the reduction of vehicle numbers.
(3) When the mean customers' weight is between $\frac{1}{2}$ and $\frac{3}{4}$ of the vehicle capacity, the total cost savings can reach its maximum.
(4) Compared with the WVRP model, the cost savings from the SDWVRP model is significantly influenced by the mean and variance of the customers' demands and vehicle capacity but is only slightly influenced by the customer geographical distributions.

5. Conclusions

The SDVRP is considered a new branch of the Vehicle Routing Problem (VRP), which has been described rarely in the literature and reports and is seldom solved by heuristic algorithm. This paper proposes a new vehicle routing problem, motivated by real-world transportation practices, which take the cargo weight as well as split deliveries into consideration. The Max–Min Ant System algorithm is contributed to apply to the SDWVRP. The computational results on a large set of benchmark instances display the necessity and effectiveness of including the cargo weight in the formulation of the vehicle routing problem and therefore prove that the SDWVRP is more beneficial when compared to the SDVRP. Moreover, the SDWVRP maintains its superiority relative to the WVRP because by significantly diminishing the vehicle numbers, cost savings proves to be substantial when also allowing split deliveries.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (71021061, 61273204), the Fundamental Research Funds for the Central Universities (N090204001, N110404021), and the Funding for the PhD program from MOE of China (20120042110023).

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