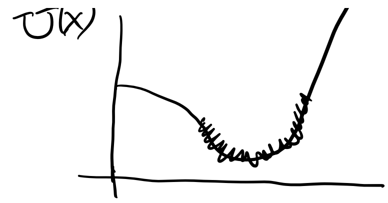
Inverted Pendulum:



$$m\ddot{x} = -\frac{\partial U}{\partial x} + f(x,t) \quad , \quad f(x,t) = f_0(x)\cos(-\Omega t + \phi)$$
$$\stackrel{!}{=} f_1(x)\cos\Omega t + f_2(x)\sin\Omega t$$

Here Ω is rapid, $\Omega \gg \omega_0 \sim \frac{1}{T} \sim \sqrt{\frac{g}{L}}$
So the Jitter is rapid.

Then let: $\quad x(t) = \bar{X}(t) + \xi(t)$

large scale slow motion $\qquad$ small scale fast motion.

$$\hookrightarrow m(\ddot{\bar{X}} + \ddot{\xi}) = \frac{\partial U}{\partial x}(\bar{X}+\xi) = \left.\frac{\partial U}{\partial X}\right|_{\bar{X}} + \left.\frac{\partial^2 U}{\partial x^2}\right|_{\bar{X}} \xi$$
$$\stackrel{!}{=} \frac{\partial U}{\partial \bar{X}} + \frac{\partial^2 U}{\partial \bar{X}^2}\xi$$

$$\hookrightarrow f(\bar{X}+\xi, t) = f(\bar{X},t) + \frac{\partial f(\bar{X},t)}{\partial \bar{X}}\xi$$

$$m\ddot{\bar{X}} + m\ddot{\xi} = -\frac{\partial U}{\partial \bar{X}} - \frac{\partial^2 U}{\partial \bar{X}^2}\xi + f(\bar{X},t) + \xi\frac{\partial f}{\partial \bar{X}}(\bar{X},t)$$

fast

but larger than $\xi\frac{\partial^2 U}{\partial \bar{X}^2}$ or $\xi\frac{\partial f}{\partial \bar{X}}$, since $\ddot{\xi} \sim \omega^2 \xi$

Fast Terms:

$$m\ddot{\xi} = f(\bar{X},t) = f_1(\bar{X})\cos\Omega t + f_2(\bar{X})\sin\Omega t$$

Find $\boxed{\xi = \frac{-f(\bar{x},t)}{m\Omega^2}}$

Slow Terms: Average over time Δt. : $\Omega^{-1} << \Delta t << T$

longer than jitter, but shorter compared to overall motion

$$m\ddot{\bar{x}} = -\frac{\partial U}{\partial \bar{x}}(\bar{x}) - \overline{\xi \frac{\partial^2 U}{\partial \bar{x}^2}} + \overline{\xi \frac{\partial f}{\partial \bar{x}}}$$

take average over ξ terms. since they are fast

Since $\xi = \frac{-1}{m\Omega^2}(f_1(x)\cos\Omega t + f_2(x)\sin\Omega t)$

then $\overline{\xi \frac{\partial^2 U}{\partial \bar{x}^2}} = 0$

For $\overline{\xi \frac{\partial f}{\partial \bar{x}}} = \overline{\frac{-f}{m\Omega^2} \frac{\partial f}{\partial \bar{x}}} = -\frac{\partial}{\partial \bar{x}}\left(\frac{\overline{f^2(\bar{x},t)}}{2m\Omega^2}\right)$

since $f(x,t) = f_1(x)\cos\Omega t + f_2(x)\sin\Omega t$

then $\overline{f^2} = \frac{f_1^2 + f_2^2}{2}$

$\hookrightarrow -\frac{\partial}{\partial \bar{x}}\left(\frac{\overline{f^2(\bar{x},t)}}{2m\Omega^2}\right) = -\frac{\partial}{\partial \bar{x}}\left(\frac{f_1^2 + f_2^2}{4m\Omega^2}\right)$

All together:

$$m\ddot{\bar{x}} = -\frac{\partial}{\partial x}\left(U + \frac{f_1^2 + f_2^2}{\ldots}\right) = -\frac{\partial}{\ldots}(U_{\ldots})$$

$$\cdot \quad \partial x \quad 4m\Omega^2) \quad \partial x (U_{eff})$$

For $|\dot{\xi}| = \Omega|\xi| = \left|\frac{f}{m\Omega}\right|$

then $\dfrac{\overline{f^2}}{2m\Omega^2} = \frac{1}{2}m\overline{|\dot{\xi}|^2}$

$$\boxed{\begin{aligned}
&\text{In general:}\\
&\quad L = \tfrac{1}{2}m_{eff}\dot{q}^2 - U(q) + L_{fast}(q,t)\\
&\quad L_{fast}(q,t) = L_1(q)\cos\omega t + L_2(q)\sin\omega t\\
&\text{Then}\\
&\quad m_{eff}\ddot{q} = -\frac{\partial U}{\partial q} + \frac{\partial L_1}{\partial q}\cos\omega t + \frac{\partial L_2}{\partial q}\sin\omega t\\
&\quad \overset{!}{=} -\frac{\partial}{\partial q}\left(U + \left(\frac{\partial L}{\partial q}\right)^2 \Big/ 4m_{eff}\Omega^2\right)
\end{aligned}}$$

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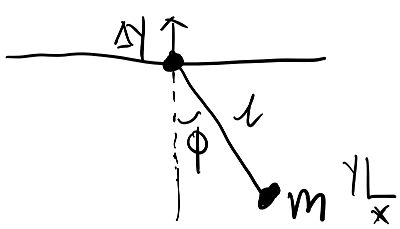


$y = \Delta y - l\cos\phi \qquad x = l\sin\phi$

$\Delta y = a\cos\Omega t$

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\overset{!}{=} \tfrac{1}{2}ml^2\dot{\phi}^2 + mla\Omega\sin\Omega t\,\frac{d}{dt}\cos\phi + \underbrace{\tfrac{1}{2}ma^2\Omega\sin^2\Omega t}$$

$$-mg(a\cos\Omega t - l\cos\phi)$$

total derivative   total derivative.

$$\doteq \frac{1}{2}ml^2\dot{\phi}^2 + \cancel{\frac{d}{dt}\left(mla\Omega\sin\Omega t\cos\phi\right)} - mla\Omega^2\cos\Omega t\cos\phi$$

$$+ mgl\cos\phi$$

$$L \doteq \frac{1}{2}ml^2\dot{\phi}^2 - mla\Omega^2\cos\Omega t\cos\phi + mgl\cos\phi$$

$$\underbrace{ml^2}_{m_{eff}}\ddot{\phi} = \underbrace{-mgl\sin\phi}_{-\frac{\partial U}{\partial\phi}} + \underbrace{lma\Omega^2\sin\phi}_{f_\phi(\phi)}\cos\Omega t$$

then $U_{eff} = -mgl\cos\phi + \dfrac{\overline{f_\phi^2(\phi)}}{4m_{eff}\Omega^2}$

Effective Ponderamotive Potential.

$$\doteq -mgl\cos\phi + \frac{(mal\Omega^2)^2}{4ml^2\Omega^2}\sin^2\phi$$

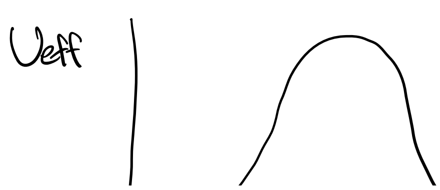
$$\doteq -mgl\cos\phi + \frac{ma^2\Omega^2}{4}\sin^2\phi$$

Notice: $\tau_{eff} = lma\Omega^2\sin\phi\cos\Omega t$

$\doteq F_{eff}\, l\sin\phi$

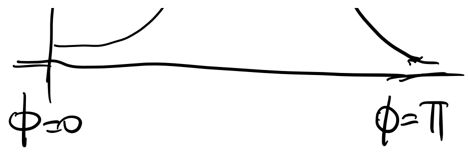
$(ma\Omega^2\cos\Omega t)$

Suppose: $ma^2\Omega^2 \gg mgl$



<u>loose stability:</u>

$ma^2\Omega^2 \sim mgl$



$$\Omega^2 \sim \frac{gl}{a^2}$$

How fast for Ω to see stable point.

Expand U_{eff} around $\phi = \pi$, $\cos\phi|_{\phi=\pi} \approx -1 + \frac{\phi^2}{2}$

$\vec{\text{Perturbation}}$ near U_{eff}.

$$U = -mgl\cos\phi + \frac{ma^2\Omega^4}{4}\sin^2\phi$$

$$U = -mgl\left(-1+\frac{\phi^2}{2}\right) + \frac{ma^2\Omega^2}{4}\phi^2$$

$$\overset{!}{=} mgl\left(1-\frac{\phi^2}{2} + \frac{a^2\Omega^2}{4gl}\phi^2\right)$$

$$U(\phi) \overset{!}{=} mgl\left(1+\left(\frac{-1}{2}+\frac{a^2\Omega^2}{4gl}\right)\phi^2\right)$$

In order for $U(\phi)$ to be stable $\frac{\partial^2}{\partial\phi^2}U > 0$, so we have concave up shape.

Thus :
$$\frac{-1}{2} + \frac{a^2\Omega^2}{4gl} > 0$$

$$\Omega^2 > \frac{2gl}{a^2} \quad \text{for stability}$$

Electron Ion Trap:

With only electrostatic force, there is no equilibrium.

let $\phi = \frac{1}{2} A \cos \Omega t \left(x^2 + y^2 - 2 z^2 \right)$