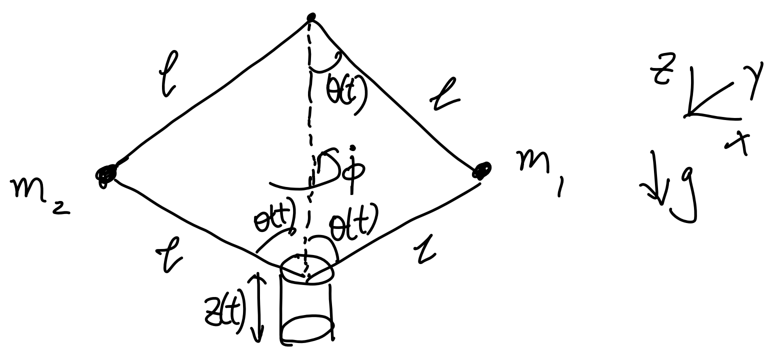
1) Oscillations of a rotating system



m_1: $x_1 = \ell\sin\theta\cos\phi$ $\quad y_1 = \ell\sin\theta\sin\phi$ $\quad z_1 = -\ell\cos\theta$

$\dot{x}_1 = \ell(\cos\theta\cos\phi\,\dot\theta - \sin\theta\sin\phi\,\dot\phi)$ $\quad \dot{y}_1 = \ell(\cos\theta\sin\phi\,\dot\theta + \sin\theta\cos\phi\,\dot\phi)$ $\quad \dot{z}_1 = \ell\sin\theta\,\dot\theta$

m_2: $x_2 = -x_1$ $\quad y_2 = -y_1$ $\quad z_1 = z_2$

cylinder: $x_{cy,cm} = 0$ $\quad y_{cy,cm} = 0$ $\quad z_{cm} = -2\ell\cos\theta$ + const.

$\dot{z}_{cm} = 2\ell\sin\theta\,\dot\theta$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right) + \frac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right)$$
$$+ \frac{1}{2}M\dot{z}_{cm}^2 + \frac{1}{2}I\dot\phi^2 - mgz_1 - mgz_2 - Mgz_{cm}$$

$$\mathcal{L} = m\ell^2\big(\cos^2\theta\cos^2\phi\,\dot\theta^2 - 2\sin\theta\cancel{\sin\phi\cos\theta\cos\phi}\,\dot\theta\dot\phi + \sin^2\theta\sin^2\phi\,\dot\phi^2$$
$$+ \cos^2\theta\sin^2\phi\,\dot\theta^2 + 2\cos\theta\cancel{\sin\phi\sin\theta\cos\phi}\,\dot\phi\dot\theta + \sin^2\theta\cos^2\phi\,\dot\phi^2$$
$$+ \sin^2\theta\,\dot\theta^2\big)$$
$$+ \frac{1}{2}M\left(2\ell\sin\theta\,\dot\theta\right)^2 + \frac{1}{2}I\dot\phi^2 + 2mg\ell\cos\theta + 2Mg\ell\cos\theta$$

$$\mathcal{L} = m\ell^2\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2}I\dot\phi^2 + (m+M)2g\ell\cos\theta$$
$$+ \frac{1}{2}M\left(4\ell^2\sin^2\theta\,\dot\theta^2\right)$$

b) $$\frac{\partial L}{\partial \dot\phi} = P_\phi = 2ml^2\sin^2\theta\,\dot\phi + I\dot\phi$$ ⇐ conservation of angular momentum

$$\frac{\partial L}{\partial \phi} = 0 \qquad \hookrightarrow \dot\phi = \frac{P_\phi}{I + 2ml^2\sin^2\theta}$$

$$\frac{\partial L}{\partial \dot\theta} = 2ml^2\dot\theta + 4Ml^2\sin^2\theta\,\dot\theta$$

$$\frac{\partial L}{\partial \theta} = 2ml^2\sin\theta\cos\theta\,\dot\phi^2 - (m+M)2gl\sin\theta$$
$$4Ml^2\sin\theta\cos\theta\,\dot\theta^2$$

Energy = h is also constant since L not explicit function of time

$$h = P_{\dot\phi}\dot\phi + P_{\dot\theta}\dot\theta - L$$

$$\overset{!}{=} 2ml^2\sin^2\theta\,\dot\phi^2 + I\dot\phi^2 + 2ml^2\dot\theta^2 + 4Ml^2\sin^2\theta\,\dot\theta^2$$
$$- \Big\{ml^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \tfrac{1}{2}I\dot\phi^2 + (m+M)2gl\cos\theta$$
$$+ \tfrac{1}{2}M(4l^2\sin^2\theta\,\dot\theta^2)\Big\}$$

$$\overset{!}{=} ml^2\dot\theta^2 + ml^2\sin^2\theta\,\dot\phi^2 + \tfrac{1}{2}I\dot\phi^2 + 2Ml^2\sin^2\theta\,\dot\theta^2$$
$$-(m+M)2gl\cos\theta$$

$$P_\phi^2$$

$$= (ml^2 + 2Ml^2\sin^2\theta)\dot\theta^2 + \underbrace{\frac{P_\phi^2}{2I + 4ml^2\sin^2\theta} - (m+M)2gl\cos\theta}_{V_{eff}(\theta)}$$

c) Consider stable configurations with $\dot\phi = \omega$, $\theta = 0$

at $t=0$, cylinder is given an upward kick performing work, W, over a short time and infinitesimal displacement. Find equation that determines the maximum deflection angle, θ_{max}

$$E(t=0) = \frac{P_\phi^2}{2I} - (m+M)2gl + ml^2\dot\theta^2$$

$$\stackrel{!}{=} \frac{(I\omega)^2}{2I} - (m+M)2gl + \underbrace{ml^2\dot\theta^2}_{W}$$

$$E(t=0) \stackrel{!}{=} \frac{1}{2}I\omega^2 - (m+M)2gl + W$$

After Δt, system reach θ_{max} (turning point), so $\dot\theta = 0$

$$E = \frac{1}{2}I\omega^2 - (m+M)2gl + W = \frac{P_\phi^2}{2I + 4ml^2\sin^2\theta_{max}} - (m+M)2gl\cos\theta_{max}$$

Solve to get θ_{max}

d) To show unstable: $\frac{\partial^2 V_{eff}}{\partial\theta} < 0$

$$\frac{\partial^2 V_{eff}}{\partial\theta^2} = \frac{\partial^2}{\partial\theta^2}\left(\frac{P_\phi^2}{2I + 4ml^2\sin^2\theta} - (m+M)2gl\cos\theta\right)$$

Expand around $\theta = 0$.

$$\sin^2\theta \cong \theta^2 \qquad \cos\theta = 1 - \frac{\theta^2}{2}$$

$$V_{eff} = \frac{P_\phi^2}{2I} \frac{1}{1 + \frac{2ml^2}{I}\theta^2} - (m+M)2gl\left(1 - \frac{\theta^2}{2}\right)$$

$$\cong \frac{P_\phi^2}{2I}\left(1 - \frac{2ml^2}{I}\theta^2\right) - (m+M)2gl\left(1 - \frac{\theta^2}{2}\right)$$

$$\cong \frac{P_\phi^2}{2I} - 2gl(m+M) + \left[(m+M)gl - \frac{P_\phi^2 ml^2}{I^2}\right]\theta^2$$

$$\frac{\partial^2 V_{eff}}{\partial\theta^2} = (m+M)gl - \frac{P_\phi^2 ml^2}{I^2}$$

For unstable

$$(m+M)gl < \frac{P_\phi^2 ml^2}{I^2}$$

$\omega_0^2 = g/l$

$$\frac{(m+M)\omega_0^2}{m} < \frac{P_\phi^2}{I^2}$$

$$P_\phi > \sqrt{\left(\frac{m+M}{m}\right)} I\omega_0$$

or if $P_\phi(\theta=0) \cong I\omega$

$$\omega^2 > (m+M)\omega_0^2$$

$$\omega > \sqrt{\left(\frac{m+M}{m}\right)}\,\omega_0 = \omega_c$$

2) Energy loss in a classical collision

$b = r\sin\theta$

$(x, y, 0)$. $U = \frac{1}{2}m\omega_0^2(x^2+y^2)$

initially at rest.

constrained to move in x,y direction.

$$V(|\vec{r}-\vec{r}_p|) = U_0 e^{-k(\vec{r}-\vec{r}_p)^2}$$

$$\vec{r} = (x, y, 0)$$

$$\vec{r}_p = (b, 0, v_0 t)$$

a) $E = \frac{1}{2}\mu v_0^2$ for $r \to \infty$ (far away)

$$\frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu \dot{\vec{r}}_p^{\,2} + U_0 e^{-k|\vec{r}-\vec{r}_p|^2}$$

$$L = \frac{1}{2}\mu \dot{\vec{r}}_p^{\,2} - U_0 e^{-k(\vec{r}-\vec{r}_p)^2}$$

$$\frac{\partial L}{\partial \vec{r}_p} = -U_0 e^{-k(\vec{r}-\vec{r}_p)^2}(-k)\,2|\vec{r}-\vec{r}_p|(-)$$

$$\mu \ddot{\vec{r}}_p = -2kU_0|\vec{r}-\vec{r}_p|\,e^{-k|\vec{r}-\vec{r}_p|^2}$$

$$U\ddot{r}_x = -2kU_0 \, |x - x_p| \, e^{-k\left[(x-x_p)^2 + (z-V_0t)^2 + y^2\right]}$$

$$\stackrel{!}{=} -2kU_0 \, |x-b| \, e^{-k\left[(x-b)^2 + (z-V_0t)^2 + y^2\right]}$$

for $|r| \ll b$

$$= -2kU_0 \, |x-b| \, e^{-kb^2\left(\left(\frac{x}{b}-1\right)^2 + \cancel{\left(\frac{y}{b}\right)^2} + \frac{(z-V_0t)^2}{b^2}\right)}.$$

$$= -2kU_0 \, |x-b| \, e^{-kb^2\left(1 + \left(\frac{z}{b} - \frac{V_0t}{b}\right)^2\right)}$$

$$\stackrel{!}{=} -2kU_0 \, |\cancel{x}-b| \, e^{-kb^2} \, e^{-k(V_0t)^2}$$

$$F_x \stackrel{!}{=} -2kU_0 b \, e^{-kb^2} \, e^{-k(V_0t)^2} \qquad V_0 = \sqrt{2mE}$$

b) Determine the displacement and velocity of target during collision.

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}m\omega_0^2\left(x^2+y^2\right) - U_0 e^{-k(\vec{r} - \vec{r}_p)^2}$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}m\omega_0^2\left(x^2+y^2\right) - U_0 e^{-k\left((x-b)^2 + (z-V_0t)^2 + y^2\right)}$$

$$m\ddot{x} + m\omega_0^2 x = -\frac{\partial}{\partial x} U_0 e^{-k\left((x-b)^2 + (z-V_0t)^2 + y^2\right)}$$

$$m\ddot{y} + m\omega_0^2 y = -\frac{\partial}{\partial y} U_0 e^{-k\left((x-b)^2 + (z-V_0t)^2 + y^2\right)}$$

$$m\ddot{z} = -\frac{\partial}{\partial z} U_0 e^{-k\left((x-b)^2 + y^2 + (z-V_0t)^2\right)}$$

$\partial\tau$

$$F_x = -U_0 e^{-k[(x-b)^2 + (z-v_0 t)^2 + y^2]}(-2k)(x-b)$$

$$\overset{!}{=} -2U_0 k b e^{-k[(x-b)^2 + (z-v_0 t)^2 + y^2]}$$

$$\overset{!}{=} -2U_0 k b e^{-kb^2[(\frac{x}{b}-1)^2 + (\frac{z-v_0 t}{b})^2 + (\frac{y}{b})^2]}$$

$$F_x \overset{!}{=} -2U_0 k b\, e^{-kb^2} e^{-k(v_0 t)^2}$$

$$F_y = -U_0 e^{-k[(x-b)^2 + y^2 + (z-v_0 t)^2]}(-2ky) \approx 0.$$

$r \ll b$

$$F_z = -U_0 e^{-k[(x-b)^2 + y^2 + (z-v_0 t)^2]}(-2k)(z-v_0 t)$$

$$\overset{!}{=} -2v_0 t U_0 e^{-kb^2} e^{-k(v_0 t)^2}$$

We know

$$m\ddot{x} + m w_0^2 x = 0$$

has solution $x = A\cos(-w_0 t + \phi)$

$$G_R(t, t_0) = A\cos(-w_0(t-t_0)) \quad \text{for } t > t_0$$

$$G_R(t, t_0) = 0 \quad \text{for } t < t_0$$

due to continuity

$$G(t=t_0, t_0) = A\cos(0) = 0$$

let $G_R(t, t_0) = \theta(t-t_0) A \sin(w_0(t-t_0))$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} m\frac{d^2G_R}{dt^2} + w_0^2 m G_R \, dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0) = 1$$

$$\left.\frac{dG_R}{dt}\right|_{t_0+\epsilon} = 1.$$

$$\left. A w_0 \cos(w_0(t-t_0)) \right|_{t_0+\epsilon} = 1.$$

$$A = \frac{1}{w_0 m}$$

$$G_R(t, t_0) = \frac{\theta(t-t_0)}{w_0 m} \sin(w_0(t-t_0))$$

To find specific sol.

$$\frac{d^2G}{dt^2} + w_0^2 G_R = \frac{F_x(t, t_0)}{m}$$

$$x_p = \int^{\infty} dt_0 \, G(t, t_0) \, F(t_0)$$

since

$$\mathcal{L} X_\delta = \int_{-\infty}^{\infty} dt_0 \, \underbrace{\mathcal{L} G(t, t_0)}_{\delta(t-t_0)} F(t_0))$$

$$X_\delta = \int_{t_0}^{t} dt_0 \, \frac{\theta(t-t_0)}{w_0 m} \sin(w_0(t-t_0)) \left(-2U_0 kb\, e^{-kb^2} e^{-k(v_0 t_0)^2}\right)$$

Then velocity is $\frac{dX_s}{dt}$

$$\frac{dX_\delta}{dt} = \frac{d}{dt} \int_{t_0}^{t} dt_0 \, \frac{1}{w_0 m} \sin(w_0(t-t_0)) \left(-2U_0 kb\, e^{-kb^2} e^{-k(v_0 t_0)^2}\right)$$

$$V_x \doteq \int_{t_0}^{t} dt_0 \, \frac{1}{m} \cos(w_0(t-t_0)) \left[-2U_0 kb e^{-kb^2} e^{-k(v_0 t_0)^2}\right]$$

c) Determine total energy absorbed after a collision at impact parameter b.

If define:

$$a(t) = V_x(t) + i w_0 x$$ position of the target.

then

$$E = \frac{1}{2} m v_x^2 + \frac{1}{2} m \omega_0^2 x^2$$

$$E(t) \overset{!}{=} \frac{1}{2} m \left| a a^* \right|^2 \leftarrow$$

$$E(t) \overset{!}{=} \frac{1}{2} m \left(v_x^2 + \omega_0^2 x^2 \right)$$

$$v_x(t) = \int_{t_0}^{t} dt_0 \frac{1}{m} \cos(\omega_0 (t - t_0)) f e^{-k(v_0 t_0)^2}$$

$$i\omega_0 x_0 = \int_{t_0}^{t} dt_0 \frac{i}{m} \sin(\omega_0 (t - t_0)) f e^{-k(v_0 t_0)^2}$$

$$a(t) = \int_{t_0}^{t} dt_0 \frac{1}{m} f e^{-k(v_0 t_0)^2} \left(\underbrace{\cos(\omega_0 (t - t_0)) + i \sin(\omega_0 (t - t_0))}_{e^{i\omega_0 (t - t_0)}} \right)$$

$$\overset{!}{=} \int_{t_0}^{t} dt_0 \frac{1}{m} f e^{-k(v_0 t_0)^2} e^{i\omega_0 (t - t_0)}$$

$$= \int_{t_0}^{t} dt_0 \frac{f}{m} e^{-k v_0^2 t_0^2} e^{i\omega_0 (t - t_0)}$$

$$a(t) = \frac{f}{m} \sqrt{\frac{\pi}{k v_0^2}} e^{\frac{\omega_0}{4 k v_0^2}} e^{i\omega_0 t}$$

Then $E(b) = \frac{1}{2} m \left| a a^* \right|$

$$= \frac{f^2}{m^2} \frac{\pi}{k v_0^2} e^{\frac{w_0}{2kv_0^2}}$$

for $f = -2U_0 k b e^{-kb^2}$

Total energy of target.

$$E(b) = \frac{(2U_0 k)^2}{k^2 m^2 v_0^2} \pi \, b^2 e^{-2kb^2} e^{\frac{w_0}{2kv_0^2}} \frac{1}{2} m$$

d) Now consider a dilute infinite medium consists of n targets per volume, what is the energy lost per length by the projectile.

$\longrightarrow$ b $\uparrow$ z.

$$\underset{\substack{\uparrow \\ \text{\# of collisions} \\ \text{per length}}}{dN} = \underset{\substack{\uparrow \\ \text{\# density} \\ = \frac{N}{V} \leftarrow \text{\# of targets}}}{n} \underbrace{2\pi b \, db}_{dA}$$

energy absorbed per collision

$$\underbrace{\int E(t)\, dN}_{\text{energy absorbed per length}}$$

$$\hookrightarrow = \int_0^\infty \frac{2\pi U_0^2}{m v_0^2} e^{\frac{w_0}{2kv_0^2}} 2\pi b^3 e^{-2kb^2} db$$

$$db = \frac{d(2kb^2)}{4kb}$$

$$\hookrightarrow = \int_0^\infty \frac{4\pi^2 \tau_0^2}{m V_0^2} e^{\frac{W_0}{2kV_0^2}} b^3 \frac{e^{-2kb^2}}{4kb} d(2kb^2)$$

$$= \int_0^\infty \frac{\pi^2 \tau_0^2}{m V_0^2} \frac{e^{\frac{W_0}{2kV_0^2}}}{2k^2} (2kb^2) e^{-2kb^2} d(2kb^2)$$

$$\stackrel{!}{=} \frac{\pi^2 \tau_0^2}{2mk^2 V_0^2} n$$

3) A) Consider a free particle of mass m in 1D, and a straight-line path from (t_0, x_0) to (t, x)

i) Evaluate action,

$$S[x(t')] = \int_{t_0}^{t} dt' \frac{1}{2} m \dot{x}^2 = S_{free}(t, x, t_0, x_0)$$

$$\stackrel{!}{=} \int_{t_0}^{t} dt' \frac{1}{2} m \left(\frac{dx}{dt'}\right)^2$$

$$\stackrel{!}{=} \frac{1}{2} m v^2 (t - t_0)$$

constant since its a free particle.

$$\stackrel{!}{=} \frac{1}{2} m \left(\frac{x - x_0}{t - t_0}\right)^2 (t - t_0)$$

$$\stackrel{!}{=} \frac{1}{2} m \frac{(x - x_0)^2}{t_0 - t}$$

ii) $\frac{-\partial S}{\partial t} = \frac{1}{2}m\frac{(x-x_0)^2}{(t-t_0)^2} = -\frac{1}{2}mv^2 =$ kinetic energy.

$$\frac{\partial S}{\partial x} = \frac{m(x-x_0)}{t-t_0} = mv = p$$

iii) for $t_0 \to -\infty$ $\qquad t_0, x_0$ large.

$$S = \frac{1}{2}m\frac{(x-x_0)^2}{t-t_0}$$

$$\overset{!}{=} \frac{1}{2}m\, x_0^2\left(\frac{x}{x_0}-1\right)^2 \frac{1}{t_0}\ \frac{-1}{1-\frac{t}{t_0}}$$

$$\overset{!}{=} -\frac{1}{2}m\, x_0^2\left(1-2\frac{x}{x_0}\right)\frac{1}{t_0}\left(1+\frac{t}{t_0}\right)$$

$$\overset{!}{=} -\frac{1}{2}m\left(x_0^2-2xx_0\right)\left(\frac{1}{t_0}+\frac{t}{t_0^2}\right)$$

$$\overset{!}{=} -\frac{1}{2}mv_0^2t_0 - \frac{1}{2}mv_0^2t + mxv_0$$

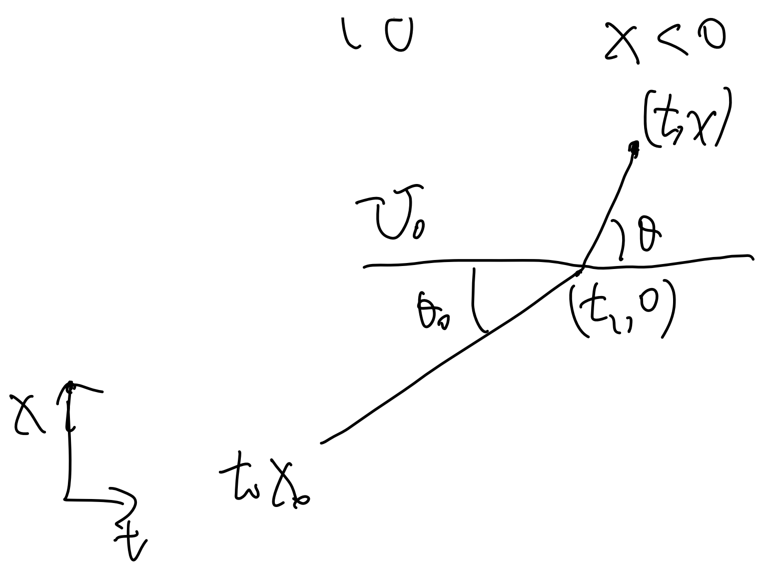
$$\overset{!}{=} -E_0t_0 - E_0t + mxv_0$$

$+$

$$\overset{!}{=} E_0t_0 - E_0t + P_0x$$

B) Now consider the particle in potential:

$$U(x) = \begin{cases} U_0 & x>0 \end{cases}$$



1) Evaluate the action $S(t, x, E_0, t_0, t_1)$

Extremize this action to find the relation

between velocities v, v_0

and angles θ, θ_0

$$S = \int_{t_0}^{t} dt' \, \frac{1}{2} m \dot{x}^2 - \theta(x) V_0$$

$$S = \int_{t_0}^{t_1} dt' \, \frac{1}{2} m \dot{x}^2 + \int_{t_1}^{t} dt' \, \frac{1}{2} m \dot{x}^2 - V_0$$

$$= \left(\cancel{P_0 x} - E t_1\right)\Big|_{x=0} + E t_0 + \left(\frac{1}{2} m v^2 - V_0\right)(t - t_1)$$

$$= -E(t_1 - t_0) + \left[\frac{1}{2} m \frac{(x-0)^2}{(t-t_1)^2} - V_0\right] t - t_1$$

$$= -E(t_1 - t_0) + \frac{1}{2}m\frac{x^2}{t-t_1} - U_0(t-t_1)$$

$$\frac{\partial S}{\partial t_1} = -E_0 + \frac{1}{2}m\frac{x^2}{(t-t_1)^2} + U_0 = 0.$$

$$\frac{1}{2}mv^2 = E_0 - U_0$$

$$v = \sqrt{\frac{2}{m}(E_0 - U_0)} \qquad v_0 = \sqrt{\frac{2}{m}E_0}$$

$$v = \sqrt{v_0 - \frac{2}{m}U_0}$$

$$v = \tan\theta = \frac{x}{t-t_1} = \sqrt{\frac{2}{m}(E_0 - U_0)}$$

$$v_0 = \tan\theta_0 = \frac{0-x}{t_1 - t_0} = \sqrt{\frac{2}{m}E_0}$$

$$\tan\theta = \sqrt{\tan^2\theta_0 - \frac{2}{m}U_0}$$

(i) For x above $\quad x > 0 \quad t < t_1$

$$S[t, x, E_0, t_0, t_1] = -E_0(t_1 - t_0) + \frac{1}{2}m\frac{x^2}{t-t_1} - U_0(t-t_1)$$

$$S[t, x, E_0, t_0] = \text{Extremize}[S(t, x, E_0, t_0)]\Big|_{t_1}$$

$$\frac{\partial S}{\partial t_1} = -E_0 + \frac{1}{2}m\frac{x^2}{(t-t_1)^2} + U_0 = 0$$

$$\frac{1}{2}m\frac{x^2}{\ldots} = E_0 - U_0$$

$$\frac{1}{2}m\frac{x^2}{(t-t_1)^2} = E_0 - U_0$$

$$(t-t_1) = \sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}$$

$$t_1 = t - \sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}$$

$$S[t,x,t_0,E_0] = -E_0\left(t-\sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}-t_0\right)+\frac{1}{2}m\frac{x^2}{\sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}}$$

$$-U\left(\sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}\right)$$

$$= -E_0(t-t_0)+E_0\sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}+\sqrt{\frac{1}{2}m(E_0-U_0)}\;x$$

$$-U_0\left(\sqrt{\frac{1}{2}m\frac{x^2}{E_0-U_0}}\right)$$

$$= -E_0(t-t_0)+\sqrt{\frac{1}{2}m(E_0-U_0)}\;2x$$

$$\stackrel{!}{=} -E(t-t_0)+\underbrace{\sqrt{2m(E_0-U_0)}}_{p}\;x$$

For $\underline{x<0}$?

$$S = -E_0(t_1-t_0)+\frac{1}{2}m\frac{x^2}{t-t_1}-\cancel{U_0(t-t_1)}$$

$$\frac{\partial S}{\partial t_1} = -E_0+\frac{1}{2}m\frac{x^2}{(t-t_1)^2} = 0$$

$$t_1 = t-\sqrt{\frac{1}{2}m\frac{x^2}{E}}$$

$$S = -E_0\left(t - \sqrt{\tfrac{1}{2} m \tfrac{x^2}{E_0}} - t_0\right) + \tfrac{1}{2} m \frac{x^3}{\sqrt{\tfrac{1}{2} m \tfrac{x^3}{E_0}}}$$

$$\stackrel{!}{=} -E_0(t-t_0) + \sqrt{\tfrac{1}{2} m E_0}\; x + E_0 \sqrt{\tfrac{1}{2} m \tfrac{x^2}{E_0}}$$

$$\stackrel{!}{=} \sqrt{2mE_0}\; x - E_0(t-t_0)$$

$$\stackrel{!}{=} P_0 x - E_0(t-t_0)$$

(ii) $\left.\frac{\partial S}{\partial t}\right|_{x=0^+} - \left.\frac{\partial S}{\partial t}\right|_{x=0^-} = -E_0 + E_0 = 0$

$\left.\frac{\partial S}{\partial x}\right|_{x=0^+} - \left.\frac{\partial S}{\partial x}\right|_{x=0^-} = P - P_0$

c) Now consider potential:

$$U(x) = \begin{cases} 0 & x < 0 \\ U_0 & 0 < x < a \\ 0 & x > a \end{cases}$$

i) For classical path and $x > a$

determine the extremized action $S(t, x; E_0, t_0)$

To evaluate the action for the classical path with E_0.

use Hamilton Formalism with path γ

for $x > a$: $S_\gamma = \int_\gamma dt' \mathcal{L}$

$\stackrel{!}{=} \int_\gamma dt' \, p\dot{x} - H$

$\stackrel{!}{=} \int p dx' - \int_{t_1}^{t} H dt'$

If particle arrives at the barrier $x=0$ at $t=t_1$, then the subsequent change in action, ΔS, is found by integrating from $(t_1, 0) \to (t, x)$

Since $H = E_0$ is constant

Action from $(t_1, 0) \to (t, x)$ $\to \Delta S_\gamma \stackrel{!}{=} -E_0(t-t_1) + \int_0^a p\, dx' + \int_a^x P_0\, dx'$

$\Delta S_\gamma \stackrel{!}{=} -E_0(t-t_1) + pa + P_0(x-a)$

Then S_{free} is from $(-\infty, x_0) \to (t_1, 0)$

$S_{free} = \int_{t_0}^{t_1} \frac{1}{2} m \dot{x}^2 \, dt'$

$\stackrel{!}{=} \frac{1}{2} m \frac{x^2}{(t_1-t_0)^2}(t_1 - t_0)$

$\stackrel{!}{=} \frac{1}{2} m \frac{x^2}{t_1 - t_0}$

$\stackrel{!}{=} E_0 t_0 + (P_0 x - E_0 t_1)$

$x=0$ $\stackrel{!}{=} -E_0(t_1 - t_0)$

Then total Action is from $(-\infty, x_0) \to (t_1, 0)$

$S = S_{free} + \Delta S$

$$= -E_0(t_1 - t_0) - E_0(t - t_1) + Pa + P_0(x-a)$$

$$= -E_0(t - t_0) + Pa + P_0(x-a)$$

compared to $S_{free} = E_0 t_0 + P_0 x - E_0 t$

Difference is

$$\delta = Pa - P_0 a$$

ii) $$\frac{\partial \delta}{\partial E_0} = \frac{\partial}{\partial E_0}\left(\sqrt{2m(E_0 - U_0)} - \sqrt{2mE_0}\right) a$$

$$= \frac{1}{2}\left(\frac{2m}{\sqrt{2m(E_0 - U_0)}} - \frac{2m}{\sqrt{2mE_0}}\right) a$$

$$= \left(\sqrt{\frac{m}{2(E_0 - U_0)}} - \sqrt{\frac{m}{2E_0}}\right) a$$