1) <u>A constrained oscillator:</u>

A particle of mass m, moving in 3-D. is bound to origin O. by a spring, k. The particle is constrained to lie in a plane with normal vector $\vec{N}(t)$. Neglect gravity. let $\vec{R}(t)$ denote position of particle.

a) Write down the Lagrangian with Lagrange multiplier λ to enforce constraint, $\vec{R}\cdot\vec{N}(t)=0$

$$T=\frac{1}{2}m\left(\dot{\vec{R}}\cdot\dot{\vec{R}}\right)=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)=\frac{1}{2}m\,\dot{r}^{i\,2}$$

$$V=\frac{1}{2}k\left(\vec{R}-\vec{R}_{eq}\right)^2=\frac{1}{2}k\left(x^2+y^2+z^2\right)=\frac{1}{2}k\left(r^i-r_{eq}^i\right)^2$$

$$\text{Constraint: } \lambda\left(\vec{R}\cdot\vec{N}(t)\right)=\lambda\left(r^i n^i\right)=0$$

$$L=\frac{1}{2}m\dot{r}^{i\,2}-\frac{1}{2}k(r^i)^2-\lambda(r^i n^i)$$

$$\frac{\partial L}{\partial \dot{r}^i}=P_i=m\dot{r}^i$$

$$\frac{\partial L}{\partial r^i}=\frac{d}{dt}(P_i)=-k(r^i)-\lambda n^i=m\ddot{r}^i$$

$$m\ddot{r}^i n^i=-kr^in^i-\lambda n^i n^i$$

$$m\ddot{r}^i n^i=-\lambda$$ ← Need to determine λ then plug into EOM.

$$m\ddot{r}^i=-kr^i+m\ddot{r}^i n^i n^i$$

$$\boxed{\text{EOM:} \quad \ddot{r}^i + \omega_0^2 r^i = \ddot{r}^j n_j n^i}$$

b) Assume a slow variation in $N(t)$.

Assume $R(t) = A^i(t)\sin\omega t$.

Then $\frac{\dot{N}(t)}{N(t)} \ll \omega$ to be slow.

Show: $$\frac{1}{G}\frac{dA(t)}{dt} = \left(N(t)\times\dot{N}(t)\right)\times A(t) = -A(t)\times(N\times\dot{N})$$

$$\stackrel{!}{=} N(-A\cdot\dot{N}) - (-A\cdot N)\dot{N}$$

$$\stackrel{!}{=} -N(A\cdot\dot{N}) + (A\cdot N)\dot{N}$$

$$R = A(t)\sin\omega t$$

$$\dot{R} = \dot{A}\sin\omega t + A\omega\cos\omega t$$

$$\ddot{R} = \ddot{A}\sin\omega t + \dot{A}\omega\cos\omega t + \dot{A}\omega\cos\omega t - A\omega^2\sin\omega t.$$

$$\ddot{R} = \ddot{A}\sin\omega t + 2\dot{A}\omega\cos\omega t - \omega^2 R$$

$$m\ddot{R} + kR = mN(N\cdot\ddot{R})$$

$$\ddot{A}^i\sin\omega t + 2\dot{A}^i\omega\cos\omega t - \omega^2 R + \omega_0^2 R = N^i\left(N_j\cdot\left(\ddot{A}^j\sin\omega t + 2\dot{A}^j\omega\cos\omega t - \omega^2 R\right)\right)$$

$$\ddot{A}^i\sin\omega t + 2\dot{A}^i\omega\cos\omega t - (\omega^2 + \omega_0^2)R = N^i\left(N_j\cdot\left(\ddot{A}^j\sin\omega t + 2\dot{A}^j\omega\cos\omega t\right)\right)$$

$$R\cdot N = 0$$

or $\dot{R}\cdot N + \dot{N}\cdot R = 0$

$$\dot{R}\cdot N = -\dot{N}\cdot R$$

$$N\cdot(\dot{A}\sin\omega t + \vec{A}\omega\cos\omega t) = -(\dot{N}\cdot A)\sin\omega t.$$

$(N\cdot A + N\cdot \dot{A})\sin wt + \underbrace{\cancel{(N\cdot A)w\cos wt}}_{=0} = 0$

So $\boxed{N\cdot\dot{A} + \dot{N}\cdot A = 0}$

$\ddot{A}\cdot N + 2\dot{N}\cdot\dot{A} + \underbrace{\ddot{N}\cdot A}_{=0} = 0$

$=0$, since N vary slowly

$$\boxed{\ddot{A}\cdot N = -2\dot{N}\cdot\dot{A}}$$

then:

$\cancel{\ddot{A}\sin wt} + 2\dot{A}w\cos wt - (w^2 - w_0^2)R = N(N\cdot(\cancel{\ddot{A}\sin wt} + 2\dot{A}w\cos wt))$

Since N is slow, then $\ddot{A}$ is also slow.

$2\dot{A}w\cos wt - (w^2 - w_0^2)A\sin wt = 2N(N\cdot\dot{A})w\cos wt$

since $N\cdot\dot{A} = -\dot{N}\cdot A$

$2\dot{A}w\cos wt - (w^2 - w_0^2)A\sin wt = -2N(\dot{N}\cdot A)w\cos wt$

$\dot{A} = \underbrace{\cancel{\frac{w^2 - w_0^2}{2w}A\tan wt}}_{\text{let } w = w_0} - N(\dot{N}\cdot A)$

$\dot{A} = -N(\dot{N}\cdot A)$

Since $(\underbrace{A\cdot N}_{0})\dot{N}$ is 0

$\dot{A} = (N\cdot A)\dot{N} - N(\dot{N}\cdot A) = (N\times\dot{N})\times A$

c) Suppose $N(t)$ evolves as follows:

$$N(t) = \begin{cases} e_z \cos\left(\frac{\pi t}{2\tau}\right) + e_x \sin\left(\frac{\pi t}{2\tau}\right) & 0 \le t \le \tau \\ e_x \cos\left(\frac{\pi(t-\tau)}{2\tau}\right) + e_y \sin\left(\frac{\pi(t-\tau)}{2\tau}\right) & \tau \le t \le 2\tau \\ e_y \cos\left(\frac{\pi(t-2\tau)}{2\tau}\right) + e_z \sin\left(\frac{\pi(t-2\tau)}{2\tau}\right) & 2\tau \le t \le 3\tau \end{cases}$$

Suppose $A(t=0) = e_x$, determine $A(3\tau)$

Compare angle through which $A(t)$ has rotated after internal 3τ. compared to solid angle swept out by $N(t)$.

Solid angle: $\frac{\pi}{2} = \frac{r^2 \sin\theta d\theta d\phi}{r^2} = \sin\theta d\theta d\phi$

$$\frac{dA}{dt} = (N \times \dot{N}) \times A$$

$\dot{N} = -\frac{\pi}{2\tau}\sin\left(\frac{\pi t}{2\tau}\right)e_z + \frac{\pi}{2\tau}\cos\left(\frac{\pi t}{2\tau}\right)e_x \qquad 0 \le t \le \tau$

$\dot{N} = -\frac{\pi}{2\tau}\sin\left(\frac{\pi(t-\tau)}{2\tau}\right)e_x + \frac{\pi}{2\tau}\cos\left(\frac{\pi(t-\tau)}{2\tau}\right)e_y \qquad \tau \le t \le 2\tau$

$\dot{N} = -\frac{\pi}{2\tau}\sin\left(\frac{\pi(t-\tau)}{2\tau}\right)e_y + \frac{\pi}{2\tau}\cos\left(\frac{\pi(t-\tau)}{2\tau}\right)e_z \qquad 2\tau \le t \le 3\tau$

$$N \times \dot{N} \begin{cases} \left(\frac{\pi}{2\tau}\cos^2\left(\frac{\pi t}{2\tau}\right) + \frac{2\pi}{\tau}\sin^2\left(\frac{\pi t}{2\tau}\right)\right)e_y = \frac{\pi}{2\tau}e_y & 0 \le t \le \tau \\ \left(\frac{\pi}{2\tau}\cos^2\left(\frac{\pi(t-\tau)}{2\tau}\right) + \frac{\pi}{2\tau}\sin\left(\frac{\pi(t-\tau)}{2\tau}\right)\right)e_z = \frac{\pi}{2\tau}e_z & \tau \le t \le 2\tau \\ \text{similarly} \quad \frac{\pi}{2\tau}e_x & 2\tau \le t < 3\tau \end{cases}$$

$$\dot{A} = (N \times \dot{N}) \times A$$

$$\dot{A} \begin{cases} \frac{\pi}{2T}(A_z \hat{x} - A_x \hat{z}) & 0 \le t < T \\ \frac{\pi}{2T}(A_x \hat{y} - A_y \hat{x}) & T \le t \le 2T \\ \frac{\pi}{2T}(A_y \hat{z} - A_z \hat{y}) & 2T \le t \le 3T \end{cases}$$

$0 \le t < T.$

$\dot{A}_x = \frac{\pi}{2T} A_z$ $\qquad A_z = -\sin\left(\frac{\pi}{2T}t\right) \Rightarrow \dot{A}_z = -\left(\frac{\pi}{2T}\right)\cos\left(\frac{\pi}{2T}t\right) = -\frac{\pi}{2T}A_x$

$\dot{A}_y = 0$ $\qquad A_x = \cos\left(\frac{\pi}{2T}t\right). \Rightarrow \dot{A}_x = \frac{\pi}{2T}\sin\left(\frac{\pi}{2T}t\right) = \frac{\pi}{2T}A_z$

$\dot{A}_z = \frac{-\pi}{2T} A_x$

From $0 \to T$.

$$A = -\cos\left(\frac{\pi}{2T}T\right)\hat{x} - \sin\left(\frac{\pi}{2T}T\right)\hat{z}$$

$A(t=T) \stackrel{!}{=} -\hat{e}_z$

$T \le t \le 2T$:

replace above with $A_z \to A_x$ $\hat{x} \to \hat{y}$ $A_x \to A_y$ $\hat{z} \to \hat{x}$

then $\quad A_x = \sin\left(\frac{\pi}{2T}t\right) \quad A_y = -\cos\left(\frac{\pi}{2T}t\right) \quad A_z = -1$

$$A = \sin\left(\frac{\pi}{2T}t\right)\hat{x} - \cos\left(\frac{\pi}{2T}t\right)\hat{y} - \hat{z}$$

$\vec{\text{precession}}$ around $\hat{z}$.

$2T \le t \le 3T.$

$A_z \to A_y$ $\hat{x} \to \hat{z}$ $A_x \to A_z$ $\hat{z} \to \hat{y}$

$A = +\sin\left(\frac{\pi}{2\tau}(t-2\tau)\right)\hat{y} - \cos\left(\frac{\pi}{2\tau}(t-2\tau)\right)\hat{z}$

at $t = 2\tau$

$A = -1\,\hat{z}$

at $t = 3\tau$

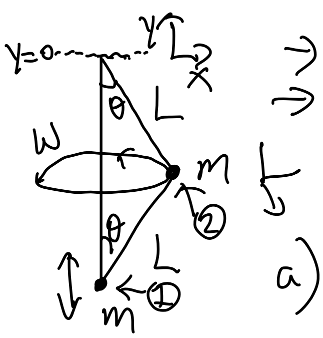
$A = \hat{y}$

From $e_x \to e_y$ over $0 \le t \le 3\tau$

Which is $\frac{\pi}{2}$.

solid angle of N is also $\frac{\pi}{2}$.

2) Masses on a rotating thin rod:



→ upper mass moves in circle.
→ lower mass moves up & down.

a) $T = \frac{1}{2}m\dot{z}_1^2 + \frac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right)$

$V = mgz + mgz_2$

$z_1 = -2L\cos\theta \qquad = 2L\dot{\theta}\sin\theta$

$z_2 = -L\cos\theta \qquad = L\dot{\theta}\sin\theta$

$$x_2 = L\sin\theta\cos\phi = -L\sin\theta\,\dot\phi\sin\phi + L\cos\theta\cos\phi\,\dot\theta$$
$$y_2 = L\sin\theta\sin\phi = L\sin\theta\,\dot\phi\cos\phi + L\cos\theta\sin\phi\,\dot\theta$$

$$T = \frac{1}{2}m(2L\dot\theta\sin\theta)^2 + \frac{1}{2}m\left\{(L\dot\theta\sin\theta)^2 + (L\sin\theta\,\dot\phi)^2 + (L\cos\theta\,\dot\theta)^2\right\}$$
$$\stackrel{!}{=} \frac{1}{2}m(2L\dot\theta\sin\theta)^2 + \frac{1}{2}m(L^2\dot\theta^2 + L^2\sin^2\theta\,\dot\phi^2)$$

$$V = -3mgL\cos\theta$$

$$\dot\phi = \omega$$

$$\mathcal{L} = \frac{1}{2}m(2L\dot\theta\sin\theta)^2 + \frac{1}{2}m(L^2\dot\theta^2 + L^2\sin^2\theta\,\dot\phi^2) + 3mgL\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = 4mL^2\sin^2\theta\,\dot\theta + mL^2\dot\theta = mL^2(4\sin^2\theta+1)\dot\theta = P_\theta$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = 4mL^2\dot\theta^2\sin\theta\cos\theta + mL^2\sin\theta\cos\theta\,\omega^2 - 3mgL\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \frac{\partial\mathcal{L}}{\partial\theta}$$

$$\boxed{mL^2\frac{d}{dt}\left((4\sin^2\theta+1)\dot\theta\right) = mL^2(4\dot\theta^2+\omega^2)\sin\theta\cos\theta - 3mgL\sin\theta}$$

$$\ddot\theta(1+4\sin^2\theta) = (\omega^2-4\dot\theta^2)\sin\theta\cos\theta - \frac{3g}{L}\sin\theta$$

b) No explicit dependence on time, so E is conserved

$$h = P_\theta\dot\theta - \mathcal{L}$$
$$\stackrel{!}{=} mL^2(4\sin^2\theta+1)\dot\theta^2 - \left\{\frac{mL^2}{2}\left((4\sin^2\theta+1)\dot\theta^2 + \sin^2\theta\,\omega^2\right)\right\} - 3mgL\cos\theta$$

$$h \stackrel{!}{=} \frac{1}{2}mL^2\left\{(4\sin^2\theta+1)\dot\theta^2 - \sin^2\theta\,\omega^2\right\} - 3mgL\cos\theta$$

when $\theta = 0$, $\dot{\theta} = \Omega$, and ignore gravity

$$E = \frac{1}{2} m L^2 \Omega^2$$

when $\theta = \frac{\pi}{2}$

$$\cancel{\frac{1}{2} m L^2} \Omega^2 = \cancel{\frac{1}{2} m L^2} \left(\left(4 \sin^2\left(\tfrac{\pi}{2}\right) + 1\right) \dot{\theta}^2 - \sin^2\left(\tfrac{\pi}{2}\right) w^2 \right)$$

$$\Omega^2 = 5 \dot{\theta}^2 - w^2$$

$$\dot{\theta}^2 = \frac{\Omega^2 + w^2}{5}$$

$$\dot{\theta}\left(\theta = \tfrac{\pi}{2}\right) = \sqrt{\frac{\Omega^2 + w^2}{5}}$$

c) $\dot{\theta} = 0 \qquad \ddot{\theta} = 0$

look at effective potential: $V_{eff}(\theta)$

$$h = \frac{1}{2} m L^2 \left\{ \left(4 \sin^2\theta + 1\right) \dot{\theta}^2 - \sin^2\theta \, w^2 \right\}$$

$$V_{eff}(\theta) = -\frac{1}{2} m L^2 \sin^2\theta \, w^2$$

$$\frac{\partial V_{eff}}{\partial \theta} = 0 = -m L^2 \sin\theta \cos\theta \, w^2$$

when $\theta = \frac{\pi}{2}$, potential is at minimum so θ is fixed

d) Now consider gravity, find the critical frequency at which solution exists where θ is fixed.
What is value of θ?

$$V_{eff}(\theta) = -\frac{1}{2} mL^2 \sin^2\theta\, w^2 - 3mgL\cos\theta$$

$$\frac{\partial V_{eff}}{\partial \theta} = -mL^2 \sin\theta\cos\theta\, w^2 + 3mgL\sin\theta$$

$$= \sin\theta\left(-mL^2 w^2 \cos\theta + 3mgL\right)$$

$$\overset{!}{=} 0$$

$$\theta = 0, \pi \quad \text{or} \quad -\cos\theta\, w^2 + 3g/L = 0$$

$$w^2 = \frac{3g/L}{\cos\theta} \leftarrow \text{max of } \cos\theta \text{ leads to min of } w^2$$

$\cos\theta = 1$ at max

$$\text{So} \quad w^2 \geq 3\frac{g}{L} = 3w_0^2 = W_{crit}^2$$

$$\boxed{W_{crit}^2 = 3W_0^2}$$

If $W^2 > W_{crit}^2$

$$\theta = \cos^{-1}\left(\frac{3w_0^2}{\quad}\right)$$

$$\theta = \cos^{-1}\left(\frac{\quad}{w^2}\right)$$

for $g=0$, $w^2=0$

$$\theta = \cos^{-1}\left(\frac{3w_0^2}{0}\right) = \frac{\pi}{2}$$

e) Now consider small oscillations about equilibrium angle.

$$\theta_{eq} = \cos^{-1}\left(\frac{3w_0^2}{w^2}\right)$$

$$V_{eff}(\theta) = -\frac{1}{2}mL^2\sin^2\theta\, w^2 - 3mgL\cos\theta$$

$$\ddot{\theta}(1+4\sin^2\theta) = (w^2 - 4\dot{\theta}^2)\sin\theta\cos\theta - \frac{3g}{L}\sin\theta$$

$$\theta = \theta_{eq} + \delta\theta$$

$$\ddot{\theta} = \delta\ddot{\theta} \qquad \dot{\theta} = \delta\dot{\theta}$$

$$\sin\theta_{eq} = \sqrt{1-\cos^2\theta_{eq}} = \sqrt{1-\left(\frac{3w_0^2}{w^2}\right)^2} = \sqrt{\frac{w^4 - 9w_0^4}{w^4}}$$

$$\cos\theta_{eq} = \frac{3w_0^2}{w^2}$$

$$\sin^2\theta = 1 - \cos^2(\theta_{eq} + \delta\theta)$$
$$\stackrel{!}{=} 1 - \cos^2\theta_{eq} - 2\sin\theta_{eq}\cos\theta_{eq}\,\delta\theta$$
$$\stackrel{!}{=} 1 - \left(\frac{3w_0^2}{w^2}\right)^2 - 2\sqrt{1-\left(\frac{3w_0^2}{w^2}\right)^2}\,\frac{3w_0^2}{w^2}\,\delta\theta$$

$$\sin\theta = \sin(\theta_{eq} + \delta\theta)$$
$$\stackrel{!}{=} \sin(\theta_{eq}) + \cos\theta_{eq}\,\delta\theta$$

$$\cos\theta = \cos(\theta_{eq} + \delta\theta)$$
$$\stackrel{!}{=} \cos\theta_{eq} - \sin\theta_{eq}\,\delta\theta$$

$$\sin\theta\cos\theta = \sin\theta_{eq}\cos\theta_{eq} + \cos^2\theta_{eq}\,\delta\theta - \sin^2\theta_{eq}\,\delta\theta$$
$$\stackrel{!}{=} \sin\theta_{eq}\cos\theta_{eq} + (-1 + 2\cos^2\theta_{eq})\,\delta\theta$$

$$\ddot\theta(1+4\sin^2\theta) = (w^2 - 4\dot\theta^2)\sin\theta\cos\theta - \frac{3g}{L}\sin\theta$$

Zeroth order:

$$0 = w^2\sin\theta\cos\theta - \frac{3g}{L}\sin\theta$$
$$\sin\theta\left(w^2\cos\theta - \frac{3g}{L}\right) = 0.$$
$$\cos\theta = \frac{3w_0^2}{w^2}$$

First order:

$$\delta\ddot{\theta}\left(1+4\sin^2\theta_{eq}\right) = w^2\left(\cos^2\theta_{eq}-\sin^2\theta_{eq}\right)\delta\theta$$
$$-4\dot{\theta}^2\sin\theta_{eq}\cos\theta_{eq}$$
$$-\frac{3g}{L}\cos\theta_{eq}\,\delta\theta$$

$$\delta\ddot{\theta}\left(5-4\cos^2\theta_{eq}\right) = w^2\left(-1+2\cos^2\theta_{eq}\right)\delta\theta$$
$$-3w_0^2\cos\theta_{eq}\,\delta\theta$$
$$-4\dot{\theta}^2\sin\theta_{eq}\cos\theta_{eq}$$

$$\delta\ddot{\theta}\left(5-4\left(\frac{3w_0^2}{w^2}\right)^2\right) = w^2\left(-1+2\left(\frac{3w_0^2}{w^2}\right)^2\right)\delta\theta$$
$$-3w_0^2\left(\frac{3w_0^2}{w^2}\right)\delta\theta$$
$$-4\dot{\theta}^2\sqrt{1-\left(\frac{3w_0^2}{w^2}\right)^2}\left(\frac{3w_0^2}{w^2}\right)$$

$$\delta\ddot{\theta}\left(5-4\left(\frac{3w_0^2}{w^2}\right)^2\right) = \left(-w^2+\frac{9w_0^4}{w^2}\right)\delta\theta$$
$$-4\delta\dot{\theta}^2\left(\underbrace{\sqrt{\frac{w^4}{w^4}-\left(\frac{3w_0^2}{w^2}\right)^2}\left(\frac{3w_0^2}{w^2}\right)}\right)$$

since $(\delta\theta)^2$ ← drop second order.

$$\delta\ddot{\theta}+\left(w^2-\frac{9w_0^4}{w^2}\right)\delta\theta$$

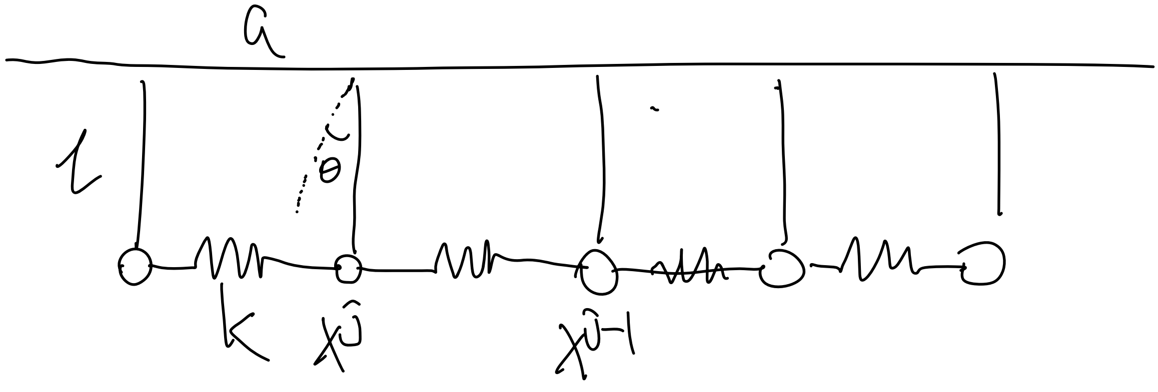
$$\frac{(\;\;\;\;\;\;w^2\;)}{5-4\left(\frac{3w_0^2}{w^2}\right)^2}=0$$

$$\ddot{\delta\theta}+\left(\frac{w^4-9w_0^4}{w^2}\right)\left(\frac{w^4}{5w^4-36w_0^4}\right)\delta\theta$$

$$\ddot{\delta\theta}+\underbrace{\left(\frac{w^4-9w_0^4}{5w^4-36w_0^4}\right)w^2}_{\text{oscillation frequency squared.}}\delta\theta=0$$

$$\sqrt{w^2\left(\frac{1-9\left(\frac{w_0}{w}\right)^4}{5-36\left(\frac{w_0}{w}\right)^4}\right)}=\text{oscillation frequency.}$$

3) Coupled chain harmonic oscillator:



$$L=\sum_j \frac{1}{2}m\dot{x}_j^2+\frac{1}{2}m\dot{y}_j^2$$
$$-\frac{1}{2}k(\Delta l_{j-1,j})^2-\frac{1}{2}k(\Delta l_{j,j+1})^2-mgy_j$$

$x = l\sin\theta \qquad y = -l\cos\theta .$

$$T = \frac{1}{2} m l^2 \dot{\theta}^{j\,2}$$

$$\Delta l_{j-1,j} = \sqrt{(\vec{r}^{\,j-1} - \vec{r}^{\,j})^2} - \sqrt{(\vec{r}^{\,j-1}_{eq} - \vec{r}^{\,j}_{eq})^2}$$

$$\doteq \sqrt{\left[((j-1)a + \Delta x^{j-1}) - (ja + \Delta x^j)\right]^2 + \left[l + \underbrace{\Delta y^{j-1}}_{l\cos\theta \simeq l.} - (l + \Delta y^j)\right]^2} - ((j-1)a - ja)$$

$$\doteq \cancel{-a} + \Delta x^{j-1} - \Delta x^j \cancel{+a}$$

$$\doteq \Delta x^{j-1} - \Delta x^j$$

$$L = \sum_j \frac{1}{2} m l^2 \dot{\theta}^{j\,2} - \frac{1}{2} k (\Delta x^{j-1} - \Delta x^j)^2 - \frac{1}{2} k (\Delta x^j - \Delta x^{j+1})^2 + mgl\cos\theta^j$$

$$L = \sum_j \frac{1}{2} m l^2 \dot{\theta}^{j\,2} - \frac{1}{2} k l^2 (\theta^{j-1} - \theta^j)^2$$

$$-\frac{1}{2}kl^2(\theta^j - \theta^{j+1})$$

$$+ mgl\left(1 - \frac{\theta^{j2}}{2}\right)$$

$$\frac{\partial L}{\partial \dot\theta} = ml^2 \dot\theta^j = P_{\theta^j}$$

$$\frac{\partial L}{\partial \theta^j} = kl^2(\theta^{j-1} - \theta^j) - kl^2(\theta^j - \theta^{j+1})$$

$$- mgl\theta^j$$

$$\boxed{ml^2 \ddot\theta^j = kl^2(\theta^{j-1} - 2\theta^j + \theta^{j+1}) - mgl\theta^j}$$

b) let $\theta^j = Ae^{i(kx_j - wt)}$

$$\ddot\theta^j = -w^2 Ae^{i(kx_j - wt)}$$

$$-ml^2w^2Ae^{i(kx_j - wt)} = kl^2 \{$$

$$Ae^{i(k(j-1)a - wt)}$$

$$-2Ae^{i(kja - wt)}$$

$$+ Ae^{i(k(j+1)a - wt)}\}$$

$$- mgl\, Ae^{i(kja - wt)}$$

$$-ml^2w^2 = kl^2\{e^{-ika} - 2 + e^{ika}\}$$
$$- mgl$$

$$-w^2 = \frac{2k}{m}\{\cos(ka) - 1\} - g/l$$

$$w^2 = \frac{k}{m}\{2 - 2\cos ka\} + g/l$$

$$w^2 = \frac{k}{m} 4\sin^2\left(\frac{ka}{2}\right) + g/l$$

$$w = \sqrt{4\frac{k}{m}\sin^2\left(\frac{ka}{2}\right) + g/l}$$

For $ka \ll 1$

$$\sin^2\left(\frac{ka}{2}\right) = \left(\frac{ka}{2}\right)^2 - \frac{\left(\frac{ka}{2}\right)^4}{3} + \frac{2\left(\frac{ka}{2}\right)^6}{45} \ldots$$

$$w = \sqrt{\frac{4k}{m}\left(\frac{ka}{2}\right)^2 + g/l}$$

$$w = \pm\sqrt{\Omega^2 + w_0^2 k^2 a^2}$$

$$W \stackrel{!}{=} \pm\sqrt{\Omega^2 + v_0^2 k^2}$$

↙ let $v_0^2 = w_0^2 a^2$

...

$$\frac{dw}{dk} = \cancel{\frac{1}{2}}\left(\Omega^2 + v_0^2 k^2\right)^{-\frac{1}{2}} \left(\cancel{2} k v_0^2\right)$$

$$\frac{dw}{dk} \stackrel{!}{=} \pm\frac{v_0^2 k}{\sqrt{\Omega^2 + v_0^2 k^2}}$$

c) i) consider

$$S[q(t,x)] = \int dt dx \, \frac{1}{2}\mu\left(\partial_t q(t,x)\right)^2 - \frac{1}{2} Y\left(\partial_x q(t,x)\right)^2 - \frac{1}{2}\gamma^2 q^2(t,x)$$

<u>determine EOM:</u>

$$S[q+\delta q] = \int dt dx \, \mathcal{L}(q+\delta q, \partial_a q + \partial_a \delta q)$$

$$\stackrel{!}{=} \iint dt dx \, \mathcal{L} + \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial_a q}\partial_a \delta q$$

$$\delta S = \iint dt dx \, \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial_a q}\partial_a \delta q$$

$$= \iint dt\,dx \; \frac{\partial \mathcal{L}}{\partial q}\delta q + \underbrace{\partial_a\left(\frac{\partial \mathcal{L}}{\partial_a q}\delta q\right)}_{\substack{=0 \\ \text{by boundary} \\ \text{term}}} - \partial_a\left(\frac{\partial \mathcal{L}}{\partial_a q}\right)\delta q$$

$$\stackrel{!}{=} \iint dt\,dx \left[\frac{\partial \mathcal{L}}{\partial q} - \partial_a\left(\frac{\partial \mathcal{L}}{\partial_a q}\right)\right]\delta q$$

$$\partial_a\left(\frac{\partial \mathcal{L}}{\partial_a q}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

For $\mathcal{L} = \frac{1}{2}u(\partial_t q)^2 - \frac{1}{2}\Upsilon(\partial_x q)^2 - \frac{1}{2}\gamma^2 q^2$

$$-\gamma^2 q = \frac{\partial}{\partial t}\left(u\frac{\partial}{\partial t}q\right) - \frac{\partial}{\partial x}\left(\Upsilon\frac{\partial}{\partial x}q\right)$$

$$\boxed{u\frac{\partial^2 q}{\partial t^2} - \Upsilon\frac{\partial^2}{\partial x^2}q + \gamma^2 q = 0}$$

ii) let $q = Ae^{ikx - i\omega t}$

$$\frac{\partial^2 q}{\partial t^2} = -\omega^2 Ae^{ikx - i\omega t}$$

$$\frac{\partial^2 q}{\partial x^2} = -k^2 Ae^{ikx - i\omega t}$$

$$-\mu\omega^2 \cancel{Ae^{ikx-i\omega t}} + \Upsilon k^2 \cancel{Ae^{ikx-i\omega t}} + \gamma^2 A \cancel{e^{ikx-i\omega t}} = 0$$

$$-\mu\omega^2 + \Upsilon k^2 + \gamma^2 = 0$$

$$\omega^2 = \frac{\Upsilon k^2 + \gamma^2}{\mu}$$

$$\omega = \pm\sqrt{\frac{\Upsilon k^2 + \gamma^2}{\mu}}$$

Compare with $\omega = \pm\sqrt{\Omega^2 + v_0^2 k^2}$

$$\frac{\Upsilon}{\mu} = v_0^2 \quad \text{and} \quad \frac{\gamma^2}{\mu} = \Omega^2$$

$$\frac{\Upsilon}{\mu} = (\omega_0 a)^2 = \frac{ka^2}{m}$$

$$\int dx\, \mu\left(\frac{\partial}{\partial t} q\right)^2 = \frac{1}{2} m \ell^2 \dot{\theta}_j^2$$

$$\mu = \frac{1}{2} m \ell^2$$

d) Consider a rightmoving planewave of wave number k and Amplitude A_0

What is the average work done per unit

time by pendulum A on B.

$q = A_0 e^{ikx_j - i\omega t}$

The torque on j-th oscillator on j+1 oscillator is $\tau = -kl(\theta^j - \theta^{j+1})$

Power: $\frac{dW}{dt} = \tau \dot{\theta}$

$\tau = -klA_0\left(e^{ikx_j - i\omega t} - e^{ikx_{j+1} - i\omega t}\right)$

$\stackrel{!}{=} -klA_0 e^{ikx_j - i\omega t}\left(1 - e^{ika}\right)$

$\dot{\theta} = -i\omega A_0 e^{ikx_j - i\omega t}$

$\frac{dW}{dt} = klA_0^2 \omega i e^{2(ikx_j - i\omega t)}\left(1 - e^{ika}\right)$

use $\boxed{\overline{A(t)B(t)} = \frac{1}{2}\mathrm{Re}\left[A_\omega B_\omega^*\right]}$

$\overline{\dot{W}} = \frac{1}{2}kl|A_0|^2 \omega \mathrm{Re}\left[\qquad (1 - e^{ika})i\right]$

$$\mathrm{Re}\Big(\underbrace{\big(1-\cos ka - i\sin ka\big)\, i}_{=\ \sin ka}\Big)$$

$$\overline{\dot{W}} = \frac{1}{2} k t |A_0|^2 \omega \sin ka$$