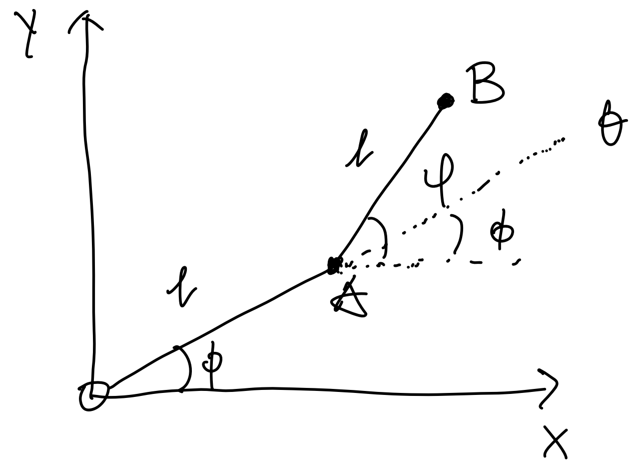
1)



$x_A = l\cos\phi$ $\qquad$ $y_A = l\sin\phi$

$x_B = l\cos\phi + l\cos(\theta)$ $\qquad$ $y_B = l\sin\phi + l\sin\theta$

$\dot{x}_A = -l\sin\phi\,\dot{\phi}$ $\qquad$ $\dot{y}_A = l\cos\phi\,\dot{\phi}$

$\dot{x}_B = -l\sin\phi\,\dot{\phi} - l\sin\theta\,\dot{\theta}$ $\qquad$ $\dot{y}_B = l\cos\phi\,\dot{\phi} + l\cos\theta\,\dot{\theta}$

$$L = \frac{1}{2}M(\dot{x}_A^2 + \dot{y}_A^2) + \frac{1}{2}M(\dot{x}_B^2 + \dot{y}_B^2)$$

$$= \frac{1}{2}Ml^2\dot{\phi}^2 + \frac{1}{2}Ml^2\Big([\sin\phi\,\dot{\phi} - \sin\theta\,\dot{\theta}]^2 + [\cos\phi\,\dot{\phi} + \cos\theta\,\dot{\theta}]^2\Big)$$

$$= \frac{1}{2}Ml^2\dot{\phi}^2 + \frac{1}{2}Ml^2\big(\sin^2\phi\,\dot{\phi}^2 + 2\sin\phi\sin\theta\,\dot{\phi}\dot{\theta} + \sin^2\theta\,\dot{\theta}^2 + \cos^2\phi\,\dot{\phi}^2 + \cos^2\theta\,\dot{\theta}^2$$

$$+2\cos\phi\cos\theta\,\dot\phi\dot\theta)$$

$$= Ml^2\dot\phi^2 + \frac{1}{2}Ml^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\cos(\phi-\theta)\right)$$

$$L = \frac{1}{2}Ml^2\left(2\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\cos(\phi-\theta)\right)$$

let $\theta = \varphi + \phi$

$$L = \frac{1}{2}Ml^2\left(2\dot\phi^2 + (\dot\varphi+\dot\phi)^2 + 2\dot\phi(\dot\varphi+\dot\phi)\cos\varphi\right)$$

$$L = \frac{1}{2}Ml^2\left(2\dot\phi^2 + \dot\phi^2 + 2\dot\phi\dot\varphi + \dot\varphi^2 + \left[2\dot\phi^2 + 2\dot\phi\dot\varphi\right]\cos\varphi\right)$$

$$L = \frac{1}{2}Ml^2\left(3\dot\phi^2 + 2\dot\phi^2\cos\varphi + \dot\varphi^2 + 2\dot\phi\dot\varphi(1+\cos\varphi)\right)$$

$$L = \frac{1}{2}Ml^2\dot\phi^2(3+2\cos\varphi) + \frac{1}{2}ml^2\dot\varphi^2 + Ml^2\dot\phi\dot\varphi(1+\cos\varphi)$$

$$\boxed{\frac{\partial L}{\partial\dot\phi} = Ml^2\dot\phi(3+2\cos\varphi) + Ml^2\dot\varphi(1+\cos\varphi) = P_\phi = \text{constant}}$$

$$\frac{\partial L}{\partial\dot\varphi} = ml^2\dot\varphi + ml^2\dot\phi(1+\cos\varphi) = ml^2\left(\dot\varphi + \dot\phi(1+\cos\varphi)\right)$$

$$h = P_\phi\dot\phi + P_\varphi\dot\varphi - L$$

$$\doteq \frac{1}{2}Ml^2\phi(3+2\cos\varphi) + \frac{1}{2}ml^2\dot{\varphi}^2 + Ml^2\dot{\phi}\dot{\varphi}(1+\cos\varphi)$$

constant energy.

$$R = p_\phi\dot{\phi} - L$$

$$\doteq p_\phi\left(\frac{p_\phi}{ml^2(3+2\cos\varphi)} - \frac{\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi}\right) -$$

$$\left\{\frac{1}{2}ml^2(3+2\cos\varphi)\left(\frac{p_\phi}{ml^2(3+2\cos\varphi)} - \frac{\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi}\right)^2 \right.$$

$$\left. + \frac{1}{2}ml^2\dot{\varphi} + ml^2\dot{\varphi}(1+\cos\varphi)\left(\frac{p_\phi}{ml^2(3+2\cos\varphi)} - \frac{\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi}\right)\right\}$$

$$R = \frac{p_\phi^2}{ml^2(3+2\cos\varphi)} - \frac{p_\phi\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi}$$

$$- \frac{1}{2}ml^2(3+2\cos\varphi)\left[\cancel{\left(\frac{p_\phi}{ml^2(3+2\cos\varphi)}\right)^2} - \cancel{\frac{2\frac{p_\phi}{ml^2}\dot{\varphi}(1+\cos\varphi)}{(3+2\cos\varphi)^2}}\right.$$

$$\left. + \left(\frac{\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi}\right)^2\right]$$

$$- \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{p_\phi\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi} + \frac{ml^2\dot{\varphi}^2(1+\cos\varphi)^2}{3+2\cos\varphi}$$

$$R = \frac{p_\phi^2}{2ml^2(3+2\cos\varphi)} + \frac{ml^2\dot{\varphi}^2(1+\cos\varphi)^2}{2(3+2\cos\varphi)} - \frac{1}{2}ml^2\dot{\varphi}^2$$

$$- \frac{p_\phi\dot{\varphi}(1+\cos\varphi)}{3+2\cos\varphi}$$

$$L_{eff} = -R$$

$$= \frac{1}{2}ml^2\dot{\varphi}^2\left(1 - \frac{(1+\cos\varphi)^2}{3+2\cos\varphi}\right) + \frac{p_\phi(1+\cos\varphi)}{3+2\cos\varphi}\dot{\varphi}$$

$$- \frac{p_\phi^2}{2ml^2(3+2\cos\varphi)}$$

b) i)

If B is moving faster than A, then there will be a centrifugal force, aligning A with B. Therefore

$\dot{\phi}$ is large, but φ and $\dot{\varphi}$ is small.

ii) so expand φ around 0..

since $\dot{\varphi}$ is small, keep only 0th order

$$L_{eff} = \frac{1}{2}ml^2\dot{\varphi}^2\left(1 - \frac{4}{5}\right) + \frac{p_\phi 2}{5}\dot{\varphi}$$

$$- \frac{p_\phi^2}{2ml^2}\left(3+2\left(1-\frac{\varphi^2}{2}\right)\right)^{-1}$$

$$= \frac{1}{2}ml^2\dot{\varphi}^2\left(\frac{1}{5}\right) + \frac{2}{5}p_\phi\dot{\varphi} - \frac{p_\phi^2}{2ml^2}\frac{1}{(5-\varphi^2)}$$

$$\doteq \frac{1}{10} m l \varphi + \frac{2}{5} P_\phi \varphi - \frac{P_\phi}{10 m l^2}\left(1 - \frac{\varphi}{5}\right)'$$

$$\doteq \frac{1}{10} m l^2 \dot{\varphi}^2 + \frac{2}{5} P_\phi \dot{\varphi} - \frac{P_\phi^2}{10 m l^2}\left(1 + \frac{\varphi^2}{5}\right)$$

$$\doteq \frac{1}{10} m l^2 \dot{\varphi}^2 + \frac{2}{5} P_\phi \dot{\varphi} - \frac{P_\phi^2}{50 m l^2}\varphi^2 - \cancel{\frac{P_\phi^2}{10 m l^2}} \quad \text{const}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{5} m l^2 \dot{\varphi} + \frac{2}{5} P_\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{1}{5} m l^2 \ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = -\frac{1}{25}\frac{P_\phi^2}{m l^2}\varphi$$

$$\frac{1}{5} m l^2 \ddot{\varphi} = -\frac{1}{25}\frac{P_\phi^2}{m l^2}\varphi$$

$$\ddot{\varphi} = -\underbrace{\frac{1}{5}\frac{P_\phi^2}{(m l^2)^2}}_{= \omega_0^2}\varphi$$

$$\boxed{\omega_0^2 \simeq \frac{P_\phi^2}{5(m l^2)^2}}$$

$$\varphi = A\cos(\omega_0 t + \Phi)$$

at $t=0$, $\varphi=0$

$$\varphi=\varphi_0 \sin(\omega_0 t)$$

$$Ml^2\dot{\phi}(3+2\cos\varphi)+Ml^2\dot{\varphi}(1+\cos\varphi)=P_\phi$$

$$\hookrightarrow \quad 5ml^2\dot{\phi} \cong P_\phi$$

$$5ml^2\left(\frac{2V_0}{2l}\right)=P_\phi$$

$$5mlV_0 \cong P_\phi$$

Also know

$$5ml^2\dot{\phi}+2ml^2\dot{\varphi}\cong P_\phi$$

$$\dot{\phi}=\frac{P_\phi}{5ml^2}-\frac{2}{5}\dot{\varphi}$$

$$\phi-\cancel{\phi_0}^{\,0}=\frac{P_\phi}{5ml^2}t-\frac{2}{5}\varphi$$

$$\phi=\frac{P_\phi}{5ml^2}t-\frac{2}{5}\varphi_0\sin\omega_0 t$$

$$\dot{\varphi}=\omega_0\varphi_0\cos\omega_0 t$$

$$\dot{\varphi}(t=0)=\frac{\Delta v}{l}=\omega_0\varphi_0$$

$$\frac{\Delta v}{\omega_0 l}=\varphi_0$$

$$\varphi=\frac{\Delta v}{\omega_0 l}\sin\omega_0 t$$

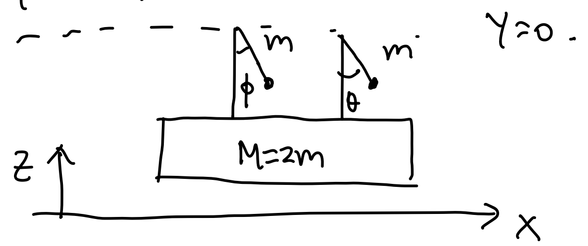
$$\phi = \frac{P_\phi}{5ml^2}t - \frac{2}{5}\varphi$$

$$\dot{\phi} = \frac{R v_0}{R l} = \frac{P_\phi}{5ml^2} - \frac{2}{5}\frac{\Delta v}{l}$$

$$\frac{P_\phi}{5ml^2} = \frac{v_0}{l} + \frac{2}{5}\frac{\Delta v}{l}$$

$$\boxed{\begin{aligned} \phi(t) &= \left(\frac{v_0}{l} + \frac{2}{5}\frac{\Delta v}{l}\right)t - \frac{2}{5}\varphi \\ \varphi(t) &= \frac{\Delta v}{w_0 l}\sin w_0 t \end{aligned}}$$

2) Coupled pendulum:



a)

$x_1 = l\sin\phi + X + \text{const}$ $\quad \dot{x}_1 = l\cos\phi\,\dot{\phi} + \dot{X}$

$y_1 = -l\cos\phi$ $\quad \dot{y}_1 = l\sin\phi\,\dot{\phi}$

$x_2 = l\sin\theta + X + \text{const}$ $\quad \dot{x}_2 = l\cos\theta\,\dot{\theta} + \dot{X}$

$y_2 = -l\cos\theta$ $\quad \dot{y}_2 = l\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}m\left((l\cos\phi\,\dot{\phi} + \dot{x})^2 + l^2\sin^2\phi\,\dot{\phi}^2\right)$$

$$+\frac{1}{2}m\left((l\cos\theta\,\dot{\theta} + \dot{x})^2 + l^2\sin^2\theta\,\dot{\theta}^2\right)$$

$$\stackrel{!}{=} m\dot{x}^2 + \frac{1}{2}m\left(l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{\phi}\dot{x} + \dot{x}^2 + l^2\sin^2\phi\,\dot{\phi}^2\right.$$
$$\left. + l^2\cos^2\theta\,\dot{\theta}^2 + 2l\cos\theta\,\dot{\theta}\dot{x} + \dot{x}^2 + l^2\sin^2\theta\,\dot{\theta}^2\right)$$

$$\stackrel{!}{=} 2m\dot{x}^2 + \frac{1}{2}m\left(l^2\dot{\phi}^2 + l^2\dot{\theta}^2 + 2l\cancel{\cos\phi}\,\dot{\phi}\dot{x} + 2l\cancel{\cos\theta}\,\dot{\theta}\dot{x}\right)$$

$$V = mgy_1 + mgy_2$$

$$\stackrel{!}{=} -mgl(\cos\phi + \cos\theta) = -mgl\left(1 - \frac{\phi^2}{2} + 1 - \frac{\theta^2}{2}\right)$$

$$\stackrel{!}{=} mgl\left(\frac{\phi^2}{2} + \frac{\theta^2}{2}\right) + \text{const.}$$

$$L = 2m\dot{x}^2 + \frac{1}{2}m\left(l^2\dot{\phi}^2 + l^2\dot{\theta}^2 + 2l\left(\dot{\phi}\dot{x} + \dot{\theta}\dot{x}\right)\right)$$
$$- mgl\left(\frac{\phi^2}{2} + \frac{\theta^2}{2}\right)$$

$$\frac{\partial L}{\partial \dot{x}} = 4m\dot{x} + ml\left(\dot{\phi} + \dot{\theta}\right) = \text{constant.}$$

$$\dot{x} = \frac{P_x}{4m} - \frac{l(\dot{\phi} + \dot{\theta})}{4}$$

$$R = P_x\dot{x} - L$$

$$\stackrel{!}{=} \frac{P_x^2}{4m} - \frac{P_x l(\phi+\theta)}{4}$$

$$-\Big\{2m\left(\frac{P_x}{4m} - \frac{l(\dot\phi+\dot\theta)}{4}\right)^2 + \frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}ml^2\dot\phi^2$$

$$-mgl\left(\frac{\phi^2}{2}+\frac{\theta^2}{2}\right)$$

$$+ml(\dot\phi+\dot\theta)\left[\frac{P_x}{4m} - \frac{l(\dot\phi+\dot\theta)}{4}\right]\Big\}$$

$$\stackrel{!}{=} \frac{P_x^2}{8m} - \frac{ml^2(\dot\phi+\dot\theta)^2}{8} - \frac{1}{2}ml^2\dot\theta^2 - \frac{1}{2}ml^2\dot\phi^2$$

$$+mgl\left(\frac{\phi^2}{2}+\frac{\theta^2}{2}\right) - \frac{P_x l(\dot\phi+\dot\theta)}{4} + \frac{ml^2(\dot\phi+\dot\theta)^2}{4}$$

$$\stackrel{!}{=} \cancel{\frac{P_x^2}{8m}} + \frac{ml^2(\dot\phi+\dot\theta)^2}{8} - \frac{1}{2}ml^2\dot\theta^2 - \frac{1}{2}ml^2\dot\phi^2$$

$$+\frac{mgl}{2}(\phi^2+\theta^2) - \frac{P_x l}{4}(\dot\phi+\dot\theta)$$

$$L = \frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}ml^2\dot\phi^2 - \frac{ml^2}{8}(\dot\phi+\dot\theta)^2 - \frac{mgl}{2}(\phi^2+\theta^2)$$

$$+\frac{P_x l}{4}(\dot\phi+\dot\theta)$$

$$\frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta + \frac{P_x l}{4} - \frac{ml^2}{4}(\dot\phi+\dot\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = ml^2\ddot\theta - \frac{ml^2}{4}(\ddot\phi+\ddot\theta) = \frac{3ml^2}{4}\ddot\theta - \frac{ml^2}{4}\ddot\phi$$

$$\frac{\partial L}{\partial\theta} = -mgl\theta$$

$$\frac{\partial L}{\partial \dot\phi} = ml^2\dot\phi + \frac{P_x l}{4} - \frac{ml^2}{4}(\dot\phi + \dot\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = ml^2\ddot\phi - \frac{ml^2}{4}(\ddot\phi + \ddot\theta) = \frac{3}{4}ml^2\ddot\phi - \frac{ml^2}{4}\ddot\theta$$

$$\frac{\partial L}{\partial \phi} = -mgl\phi$$

$$\left.\begin{aligned} &\frac{3}{4}ml^2\ddot\phi - \frac{ml^2}{4}\ddot\theta + mgl\phi = 0 \\ &\frac{3}{4}ml^2\ddot\theta - \frac{ml^2}{4}\ddot\phi + mgl\theta = 0 \end{aligned}\right\} \begin{aligned} &\frac{3}{4}\ddot\phi - \frac{1}{4}\ddot\theta + \frac{g}{l}\phi = 0 \\ &\frac{3}{4}\ddot\theta - \frac{1}{4}\ddot\phi + \frac{g}{l}\theta = 0 \end{aligned}$$

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} \ddot\phi \\ \ddot\theta \end{pmatrix} + \begin{pmatrix} 4\omega_0^2 & 0 \\ 0 & 4\omega_0^2 \end{pmatrix}\begin{pmatrix} \phi \\ \theta \end{pmatrix} = 0$$

$$\begin{pmatrix} -3\omega^2 & \omega^2 \\ \omega^2 & -3\omega^2 \end{pmatrix} + \begin{pmatrix} 4\omega_0^2 & 0 \\ 0 & 4\omega_0^2 \end{pmatrix} = 0$$

$$\begin{vmatrix} -3\omega^2 + 4\omega_0^2 & \omega^2 \\ \omega^2 & -3\omega^2 + 4\omega_0^2 \end{vmatrix} = (-3\omega^2 + 4\omega_0^2)^2 - \omega^4 = 0$$

$$-\omega^4 + 9\omega^4 - 24\omega^2\omega_0^2 + 16\omega_0^2 = 0$$

$$8\omega^4 - 24\omega^2\omega_0^2 + 16\omega_0^2 = 0$$

$$\omega^4 - 3\omega^2\omega_0^2 + 2\omega_0^2 = 0$$

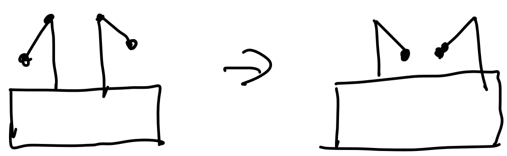
$$(\omega^2-\omega_0^2)(\omega^2-2\omega_0^2)=0$$

$$\omega^2=\omega_0^2 \text{ or } 2\omega_0^2$$

let $\omega^2=\omega_0^2$

$$\begin{pmatrix}-3\omega_0^2+4\omega_0^2 & \omega_0^2\\ \omega_0^2 & -3\omega_0^2+4\omega_0^2\end{pmatrix}=\begin{pmatrix}\omega_0^2 & \omega_0^2\\ \omega_0^2 & \omega_0^2\end{pmatrix}\begin{pmatrix}E_1^-\\ E_2^-\end{pmatrix}$$

$$E^-=\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{with } \boxed{\omega_-^2=\omega_0^2=g/l}$$

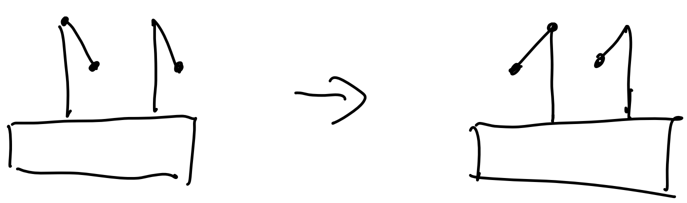


let $\omega^2=2\omega_0^2$

$$\begin{pmatrix}-6\omega_0^2+4\omega_0^2 & 2\omega_0^2\\ 2\omega_0^2 & -6\omega_0^2+4\omega_0^2\end{pmatrix}=\begin{pmatrix}-2\omega_0^2 & 2\omega_0^2\\ 2\omega_0^2 & -2\omega_0^2\end{pmatrix}\begin{pmatrix}E_1^+\\ E_2^+\end{pmatrix}$$

$$E^+=\begin{pmatrix}1\\1\end{pmatrix} \quad \text{with } \boxed{\omega_+^2=2\omega_0^2=2g/l}$$

zero mode, moving at COM frame.



b) suppose base $2m$, pushed by $F(t) = P_0 \delta(t)$

Determine subsequent oscillation.

$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} = Q^+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Q^- \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\ddot{Q}^+ + \omega_+^2 Q^+ = 0$
$\ddot{Q}^- + \omega_-^2 Q^- = 0$

$$L_{ext} = F(t) X$$

$$L = 2m\dot{x}^2 + \frac{1}{2}m\left(l^2\dot{\phi}^2 + l^2\dot{\theta}^2 + 2l\left(\dot{\phi}\dot{x} + \dot{\theta}\dot{x}\right)\right)$$
$$- mgl\left(\frac{\phi^2}{2} + \frac{\theta^2}{2}\right)$$

$$\stackrel{!}{=} 2m\dot{x}^2 + \frac{1}{2}ml^2\left((\dot{Q}^+ + \dot{Q}^-)^2 + (\dot{Q}^+ - \dot{Q}^-)^2\right)$$
$$+ml\left((\dot{Q}^+ + \dot{Q}^-)\dot{x} + (\dot{Q}^+ - \dot{Q}^-)\dot{x}\right)$$
$$-\frac{mgl}{2}\left((Q^+ + Q^-)^2 + (Q^+ - Q^-)^2\right)$$

$$\stackrel{!}{=} 2m\dot{x}^2 + ml^2\left(\dot{Q}^{+2} + \dot{Q}^{-2}\right) + 2ml\dot{Q}^+\dot{x}$$

$$-mgl\left(Q^{+2}+Q^{-2}\right)+F(t)\,X$$

$$\ddot{Q}^{+}+\omega_{+}^{2}Q^{+}=0 \Rightarrow Q^{+}=A_{+}\cos\omega_{+}t+\phi_{+}$$

$$\ddot{Q}^{-}+\omega_{-}^{2}Q^{-}=0 \Rightarrow Q^{-}=A_{-}\cos(\omega_{-}t+\phi_{-})$$

$$\frac{\partial L}{\partial \dot{X}}=4m\dot{X}+2ml\dot{Q}^{+}$$

$$\frac{\partial L}{\partial X}=P_0\delta(t)$$

First solve for X without external force to get Greens function, then use Green function to solve.

$$4m\ddot{X}+2ml\ddot{Q}^{+}=0$$

$$4m\ddot{X}-2ml\,\omega_{+}^{2}A_{+}\cos(\omega_{+}t+\phi_{+})=0$$

$$\ddot{X}-\omega_{+}^{2}\underbrace{\tfrac{1}{2}A_{+}l}_{X_0}\cos(\omega_{+}t+\phi_{+})=0$$

For $G(t=t_0,t_0)=0$

require

$$X(t,t_0) = \frac{1}{2} A_+ i \sin(\omega_+(t-t_0)) = \frac{1}{2} i\, Q^+(t)$$

$$4m\ddot{X} + 4m\omega_+^2 X = \delta(t-t_0)$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} 4m\ddot{G} + 4m\omega_+^2 G \, dt = \int \delta(t-t_0) = 1$$

$$\left.\frac{dG}{dt}\right|_{t_0+\epsilon} = 1$$

For $\frac{dG}{dt}(t=t_0,t_0) = \frac{1}{4m}$

$$\frac{1}{2} A_+ i\omega_+ = \frac{1}{4m}$$

$$A_+ = \frac{1}{2mi\omega_+}$$

$$G(t,t_0) = \frac{\theta(t-t_0)}{4m\omega_+} \sin(\omega_+(t-t_0))$$

$$X = \int_{-\infty}^{\infty} dt_0\, P_0\, \delta(t_0) \frac{1}{4m\omega_+} \sin(\omega_+(t-t_0))$$

$$\stackrel{!}{=} \frac{P_0}{4m\omega_+} \sin(\omega_+ t)$$

but we also know:

$$4m x + 2ml Q^+ = P_x$$
$$4m\dot{x} + 2ml\dot{Q}^+ - P_x = 0$$

$$4m\dot{x} + 2ml\omega_+ A_+ \cos\omega_+(t-t_0) = P_x = P_0$$

So require additional terms:

$$x(t) = \frac{P_0}{4m\omega_+}\sin(\omega_+ t) + \frac{P_0}{4m}t$$

$$Q^+(t) = \frac{-P_0}{2ml\omega_+}\sin(\omega_+ t)$$

$$Q^-(t) = 0$$

We find the overall Greens function for force $\delta(t-t_0)$

$$G(t,t_0) = \theta(t-t_0)\frac{\sin(\omega_+(t-t_0))}{4m\omega_+} + \frac{1}{4m}(t-t_0)$$

For force $\quad F(t) = F_0 e^{-|t|/\tau}$

$$x = \int_{-\infty}^{\infty} dt_0\, F(t_0)\, G(t,t_0)$$

$$= \int^{\infty} dt_0\, F_0 e^{-|t_0|/\tau}\frac{1}{4m}\left(\frac{\sin(\omega_+(t-t_0))}{\omega_+} + (t-t_0)\right)$$

$$W = \int dt\, F(t)\, \dot{X}(t)$$

$$\dot{X}(t) = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dt_0\, F(t_0)\, G(t,t_0)$$

$$= \int_{-\infty}^{\infty} dt_0\, F(t_0)\, \frac{\partial G}{\partial t}(t,t_0)$$

$$W = \int_{-\infty}^{\infty} dt\, F_0\, e^{-|t|/\tau} \int_{-\infty}^{\infty} dt_0\, F_0\, e^{-|t_0|/\tau} \left(\cos(\omega_+(t-t_0)) + \frac{1}{4m} \right)$$

$$\doteq \frac{1}{4m} \left(\int_{-\infty}^{\infty} dt\, F(t) \right)^2 + \frac{1}{4m} \iint dt\, dt_0\, F(t) \cos(\omega_+(t-t_0)) F(t_0)$$

let $\tilde{F}(\omega) = \int dt\, e^{i\omega t} F(t)$

$$\cos(\omega_+(t-t_0)) = \frac{1}{2}\left[e^{i\omega_+(t-t_0)} + e^{-i\omega_+(t-t_0)} \right]$$

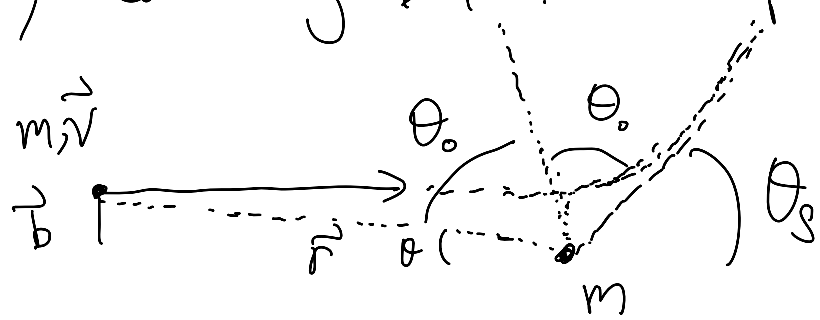
$$= \frac{1}{4m}|F(w=0)| + \frac{1}{4m}\iint dt\,dt_0\, \frac{1}{2}\Big[e^{iw_f t} e^{-iw_f t_0} + e^{-iw_f t} e^{iw_f t_0}\Big] F(t)F(t_0)$$

$$= \frac{1}{4m}|F(w=0)|^2 + \frac{1}{4m}\int dt\, \frac{1}{2}\Big(F(w_f)^* e^{iw_f t} + e^{-iw_f t} F(w_f)\Big) F(t)$$

$$= \frac{1}{4m}|F(w=0)|^2 + \frac{1}{4m}\,\frac{1}{2}\Big[|F(w^t)|^2 + |F(w^t)|^2\Big]$$

$$= \frac{1}{4m}|F(w=0)|^2 + \frac{1}{4m}|F(w^t)|^2$$

3) Scattering between two particles



$b = r\sin\theta$

$$U(r) = \frac{h}{r^2}, \quad h > 0$$

$$\vec{R}_{cm} = \frac{m\vec{r}_A + m\vec{r}_B}{m+m} = \frac{\vec{r}_A + \vec{r}_B}{2}$$

$$\Delta\vec{r}_A = \vec{r}_A - \vec{R}_{cm} = \vec{r}_A - \frac{\vec{r}_A+\vec{r}_B}{2} = \frac{\vec{r}_A-\vec{r}_B}{2} = \frac{1}{2}\vec{r}$$

$$\Delta\vec{r}_B = \vec{r}_B - \vec{R}_{cm} = -\frac{(\vec{r}_A-\vec{r}_B)}{2} = -\frac{\vec{r}}{2}$$

$$T = \frac{1}{2}m\left(\dot{\vec{R}}_{cm}+\frac{1}{2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m\left(\dot{\vec{R}}_{cm}-\frac{1}{2}\dot{\vec{r}}\right)^2$$

$$= \frac{1}{2}(2m)\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{8}m\dot{\vec{r}}^{\,2} + \frac{1}{8}m\dot{\vec{r}}^{\,2}$$

$$= m\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{4}m\dot{\vec{r}}^{\,2}$$

$$\mathcal{L} = m\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{4}m\dot{\vec{r}}^{\,2} - \frac{h}{r^2}$$

since $\frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}} = m\dot{\vec{R}}_{cm}$ = constant, ignore.

$$\mathcal{L} = \frac{1}{4}m\dot{\vec{r}}^{\,2} - \frac{h}{r^2}$$

$b = r\sin\theta$

$$\vec{r} = -r\sin\theta\,\hat{y} - r\cos\theta\,\hat{z}$$

$$\dot{\vec{r}}^{\,2} = \left(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\right)^2 + \left(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\right)^2$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

$$\mathcal{L} = \frac{1}{4}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{h}{r^2}$$

$$\frac{\partial L}{\partial \dot\theta} = P_\theta = l = \frac{1}{2} m r^2 \dot\theta = \vec{r} \times \vec{p}_r = r \sin\theta \, \frac{1}{2} m v = \boxed{\frac{1}{2} b m v = l}$$

$$R = P_\theta \dot\theta - L$$

$$= \frac{2P_\theta^2}{mr^2} - \frac{1}{4} m \dot r^2 - \frac{1}{4} m r^2 \left(\frac{2P_\theta}{mr^2}\right)^2 + \frac{h}{r^2}$$

$$= \frac{P_\theta^2}{mr^2} - \frac{1}{4} m \dot r^2 + \frac{h}{r^2}$$

$$L_{eff} = -R$$

$$= \frac{1}{4} m \dot r^2 - \frac{1}{r^2}\left(\frac{P_\theta^2}{m} + h\right)$$

$$P_r = \frac{\partial L}{\partial \dot r} = \frac{1}{2} m \dot r$$

$$\frac{\partial L}{\partial r} = \left(\frac{P_\theta^2}{mr^2} + h\right) \frac{-1}{r^3}$$

$$E = P_\theta \dot\theta + P_r \dot r - L$$

$$= \frac{1}{2} m r^2 \dot\theta^2 + \frac{1}{2} m \dot r^2 - \frac{1}{4} m \dot r^2 - \frac{1}{4} m r^2 \dot\theta^2 + \frac{h}{r^2}$$

$$E = \frac{1}{4} m (\dot r^2 + r^2 \dot\theta^2) + \frac{h}{r^2}$$

$\boxed{E = \frac{1}{4} m v^2}$ $\leftarrow$ initial, when $r \to \infty$

$$v = \sqrt{\frac{4E}{m}}$$

$$b = \frac{2P_\phi}{mv} \qquad \cancel{\tfrac{1}{2}} \cancel{m} r^2 \dot\theta = \cancel{\tfrac{1}{2}} b \cancel{m} v$$

$$\dot\theta = \frac{bv}{r^2}$$

b) $$E = \frac{1}{4} m(\dot r^2 + r^2 \dot\theta^2) + \frac{h}{r^2}$$

$$\sqrt{\left[E - \left(\frac{1}{4} m r^2 \dot\theta^2 + \frac{h}{r^2}\right)\right]\frac{4}{m}} = \dot r$$

$$\int \sqrt{\frac{m}{4}} \frac{dr}{\sqrt{E - \left(\frac{1}{4} m r^2 \dot\theta^2 + \frac{h}{r^2}\right)}} = \int dt = \int d\theta \frac{dt}{d\theta}$$

$$\frac{1}{\dot\theta} = \frac{mr^2}{2l}$$

$$\int \sqrt{\frac{m}{4}} \frac{dr}{\sqrt{E - \left(\frac{1}{4} m r^2 \dot\theta^2 + \frac{h}{r^2}\right)}} = \int d\theta$$

$$\int_{r_{min}}^{r} \sqrt{\frac{m}{4}} \frac{2l}{m} \frac{dr}{r^2 \sqrt{E - \left(\frac{1}{4} m r^2 \dot\theta^2 + \frac{h}{r^2}\right)}} = \int_0^\theta d\theta$$

$$\frac{1}{4} m r^2 \dot\theta^2 = \frac{l^2}{m}$$

$$\int^{r} \frac{l}{\sqrt{m}} \frac{dr/r^2}{\sqrt{E - \left(\frac{l^2}{m} + h\right)/r^2}} = \theta$$

let $u = \frac{1}{r}$

$du = \frac{-1}{r^2} dr.$

$$\int^{\frac{1}{r}} \frac{l}{\sqrt{m}} \frac{-du}{\sqrt{E - (\frac{l^2}{m} + h) u^2}} = \theta$$

$$\int^{1/r} \frac{-du}{\sqrt{\frac{mE}{l^2} - (1 + \frac{mh}{l^2}) u^2}} = \theta$$

use $\frac{d}{dx} \arccos\left(\frac{x}{\sqrt{a}}\right) = \frac{-1}{\sqrt{a - x^2}}$

$$\int^{1/r} \frac{1}{\sqrt{1 + \frac{mh}{l^2}}} \frac{-du}{\sqrt{\left(\frac{mE}{l^2}\right) \frac{l^2}{(l^2 + mh)} - u^2}}$$

$$\Rightarrow = \sqrt{\frac{1}{1 + \frac{mh}{l^2}}} \arccos\left(\frac{u}{\sqrt{\frac{mE}{l^2 + mh}}}\right) = \theta$$

$$\boxed{\sqrt{\frac{l^2 + mh}{mE}} \cos\left(\sqrt{1 + \frac{mh}{l^2}}\, \theta\right) = \frac{1}{r}}$$

c) $\theta = \sqrt{\frac{1}{1 + \frac{mh}{l^2}}} \arccos\left(\frac{1}{r} \sqrt{\frac{l^2 + mh}{mE}}\right)$

For $r \to \infty$

$$\theta = \sqrt{\frac{1}{1+\frac{mh}{l^2}}}\ \arccos(\ 0\ )$$

$$\theta_0 \doteq \frac{\pi/2}{\sqrt{1+\frac{mh}{l^2}}}$$

$$2\theta_0 = \frac{\pi}{\sqrt{1+\frac{mh}{l^2}}}$$

$$\theta_s = \pi - 2\theta_0 = \pi\left(1 - \frac{1}{\sqrt{1+\frac{mh}{l^2}}}\right)$$

$$l = \tfrac{1}{2}bmv \qquad E = \tfrac{1}{4}mv^2$$

$$\theta_s = \pi\left(1 - \frac{1}{\sqrt{1+\frac{mh}{\frac{1}{4}b^2m^2v^2}}}\right)$$ ✓ rewrite θ_s in b.

$$\doteq \pi\left(1 - \frac{1}{\sqrt{1+\frac{h}{b^2E}}}\right)$$

$$1 - \frac{\theta_s}{\pi} = \frac{1}{\sqrt{1+\frac{h}{b^2E}}}$$

$$\left(\frac{\pi}{\pi-\theta_s}\right)^2 = 1 + \frac{h}{b^2E}$$

$$\sqrt{\left[\left(\frac{\pi}{\pi-\theta_s}\right)^2 - 1\right]\frac{E}{h}} = \frac{1}{b}$$

$$\frac{1}{b} = \sqrt{\frac{E}{h}}\sqrt{\left(\frac{\pi^2-(\pi-\theta_s)^2}{(\pi-\theta_s)^2}\right)} = \sqrt{\frac{E}{h}}\sqrt{\frac{2\pi\theta_s-\theta_s^2}{(\pi-\theta_s)^2}}$$

$$b = \sqrt{\frac{h}{E}}\frac{\pi-\theta_s}{\sqrt{2\pi\theta_s-\theta_s^2}}$$

then

$$\frac{d\sigma}{d\theta_s} = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|$$

$$\frac{db}{d\theta_s} = \sqrt{\frac{h}{E}}\,\frac{-\sqrt{2\pi\theta_s-\theta_s^2}-(\pi-\theta_s)\frac{1}{2}(2\pi\theta_s-\theta_s^2)^{-1/2}(2\pi-2\theta_s)}{2\pi\theta_s-\theta_s^2}$$

$$\stackrel{!}{=} \sqrt{\frac{h}{E}}\left[\frac{-2\pi\theta_s+\theta_s^2-(\pi-\theta_s)^2}{(2\pi\theta_s-\theta_s^2)^{3/2}}\right]$$

$$\stackrel{!}{=} \sqrt{\frac{h}{E}}\left[\frac{\cancel{-2\pi\theta_s}\cancel{+\theta_s^2}-\pi^2\cancel{+2\pi\theta_s}\cancel{-\theta_s^2}}{(2\pi\theta_s-\theta_s^2)^{3/2}}\right]$$

$$\stackrel{!}{=} \sqrt{\frac{h}{E}}\left[\frac{-\pi^2}{(2\pi\theta_s-\theta_s^2)^{3/2}}\right]$$

$$\frac{d\sigma}{d\theta_s} = \frac{h}{E}\,\frac{-\pi^2}{(2\pi\theta_s-\theta_s^2)^{3/2}}\,\frac{\pi-\theta_s}{(2\pi\theta_s-\theta_s^2)^{1/2}}\,\frac{1}{\sin\theta_s}$$

$$\stackrel{!}{=} \frac{h}{E}\,\frac{\pi^2(\theta_s-\pi)}{\theta_s^2(\pi-\theta_s)^2}\,\frac{1}{\sin\theta_s}$$

$$\theta_s \; (2\pi - \theta_s)$$

for $E = \frac{1}{4}mv^2$

$$\frac{1}{=} \frac{4h}{mv^2} \frac{\pi^2(\theta_s - \pi)}{\theta_s^2(2\pi-\theta_s)^2} \frac{1}{\sin\theta_s}$$

d) Now convert to laboratory frame.

Find $\frac{d\sigma(\phi)}{d\phi}$

To convert to lab frame:

$$\vec{r}_A = \vec{R}_{cm} + \Delta\vec{r}_A = \vec{R}_{cm} + \frac{1}{2}\vec{r}$$

$$\dot{\vec{r}}_A = \dot{\vec{R}}_{cm} + \frac{1}{2}\dot{\vec{r}}$$

we had $\mathcal{L} = m\dot{\vec{R}}_{cm}^2$

& $P_{cm,R} = 2m\dot{\vec{R}}_{cm} = mv$

initial momentum of particle A, particle B is at rest.

so $\dot{\vec{R}}_{cm} = \frac{\vec{v}}{2}$

$$\dot{\vec{r}}_A = \frac{\vec{v}}{2} + \frac{1}{2}\dot{\vec{r}} = \frac{1}{2}(\dot{\vec{r}} + \vec{v}) \Rightarrow \boxed{\phi = \theta/2}$$

substitute this to $\dot{\vec{r}}$, there is a factor of $\frac{1}{2}$

Since # of scattered particles are:

$$\frac{d\sigma}{d\theta}\sin\theta\, d\theta = \frac{d\sigma}{d\phi}\sin\phi\, d\phi$$

$$d\phi = \frac{d\theta}{2}$$

$$\frac{d\sigma}{d\phi} = \frac{d\sigma(\theta)}{d\theta}\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}$$

$$\frac{d\sigma}{d\phi} \stackrel{!}{=} \frac{d\sigma}{d\theta}\frac{\sin 2\phi}{\sin(\phi)}\, 2\frac{d\phi}{d\phi}$$

$$\sin 2\phi = 2\sin\phi\cos\phi$$

$$\frac{d\sigma}{d\phi} = \left.\frac{d\sigma}{d\theta}\right|_{\theta=2\phi} 4\cos\phi$$