Simple Harmonic Oscillator (SHO):

$$\underbrace{m\frac{d^2x}{dt^2} + m\eta\frac{dx}{dt} + m\omega_0^2 x}_{\mathcal{L}_t\, x(t) = F(t)} = F(t)$$

A general solution has form:

$$x(t) = x_h(t) + x_s(t)$$

where the Homogeneous Solution satisfies:

$$\mathcal{L}_t\, x_h(t) = 0 = F(t)$$

and the specific solution satisfies:

$$\mathcal{L}_t\, x_s(t) = F(t)$$

Try homogeneous solution: $x_h = Ae^{-i\omega t}$

$$m\frac{d^2}{dt^2}x + m\eta\frac{dx}{dt} + m\omega_0^2 x = 0$$

$$\left(-\omega^2 m + -i\omega m\eta + m\omega_0^2\right)Ae^{-i\omega t} = 0$$

solve for ω:

$$\omega = \frac{i\cancel{m}\eta \pm \sqrt{-\cancel{m}^2\eta^2 + 4\cancel{m}^2\omega_0^2}}{-2\cancel{m}}$$

$$i\eta \pm \sqrt{4\omega_0^2 - \eta^2}$$

$$= \frac{\cdots}{-2}$$

$$\omega_\pm \overset{!}{=} \pm\sqrt{\omega_0^2 - \left(\frac{\eta}{2}\right)^2} - i\frac{\eta}{2}$$

$$\overset{!}{=} \pm\omega' - \frac{i\eta}{2}$$

Then:

$$x_h(t) = Ae^{-i\omega_\pm t}$$

$$\overset{!}{=} Ae^{-i\omega_+ t} + Be^{-i\omega_- t}$$

For real solutions, $B = A^*$ for $A = |A|e^{i\phi}$

$$x_h(t) = \mathrm{Re}\left[A e^{-i\omega_+ t}\right]$$

$$\overset{!}{=} |A|\,\mathrm{Re}\left[e^{-i(\omega_+ t + \phi)}\right]$$

$$\boxed{x_h(t) \overset{!}{=} |A|\, e^{-\frac{\eta}{2}t} \cos(-\omega' t + \phi)}$$

Now determine the specific solution by adding the external force.

In general $F(t) = \sum_\omega F_\omega e^{-i\omega t}$ $\Leftarrow$ A superposition of Fourier Modes.

Assume there is only one mode:

then:

$$\left(m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right) x_s(t) = F_\omega e^{-i\omega t}$$

Try solution: $x_s(t) = x_\omega e^{-i\omega t}$

then $\left(-m\omega^2 - im\eta\omega + m\omega_0^2\right) x_\omega \cancel{e^{-i\omega t}} = F_\omega \cancel{e^{-i\omega t}}$

$$x_\omega = \underbrace{\frac{1/m}{-\omega^2 - i\omega\eta + \omega_0^2}} F_\omega$$

$$x_\omega = G_R(\omega) F_\omega$$

↑ Retarded green func.

So $$G_R(\omega) = \frac{1/m}{-\omega^2+\omega_0^2 - i\omega\eta}$$

Then the amplitude $|A| = |G_R(\omega)|$: amplitude of oscillation.

$$G_R(\omega) = \frac{1/m}{-\omega^2+\omega_0^2-i\omega\eta} \frac{-\omega^2+\omega_0^2+i\omega\eta}{-\omega^2+\omega_0^2+i\omega\eta}$$

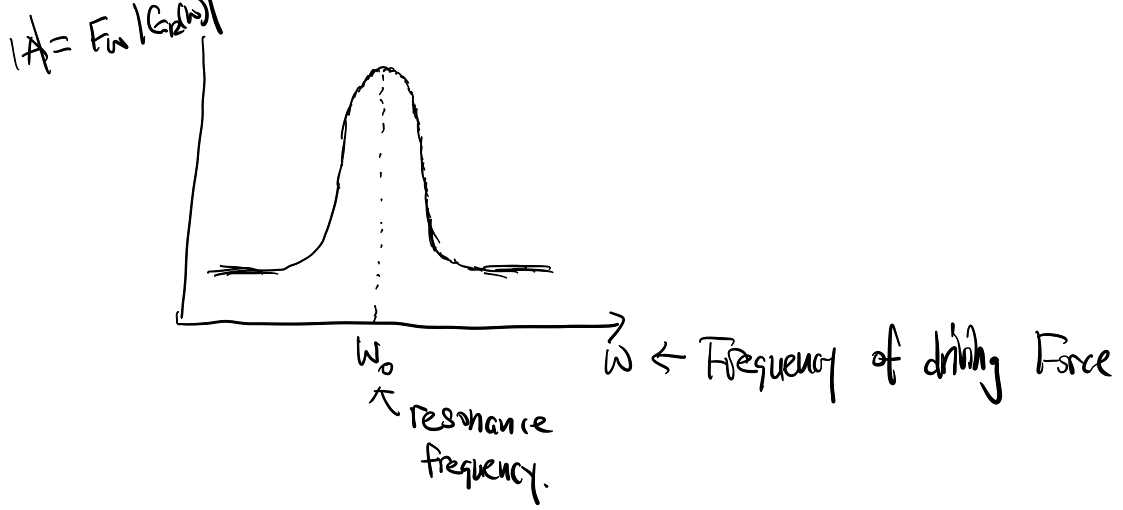
$$= \frac{\left(-\omega^2+\omega_0^2+i\omega\eta\right)/m}{\left(-\omega^2+\omega_0^2\right)^2+(\omega\eta)^2}$$

$$|G_R(\omega)| = \sqrt{G_R(\omega)\, G^*(\omega)}$$

$$= \frac{\sqrt{\left(-\omega^2+\omega_0^2+i\omega\eta\right)\left(-\omega^2+\omega_0^2-i\omega\eta\right)}/m}{\left(-\omega^2+\omega_0^2\right)^2+(\omega\eta)^2}$$

$$= \frac{\sqrt{\left(-\omega^2+\omega_0^2\right)^2+(\omega\eta)^2}\,/m}{\left(-\omega^2+\omega_0^2\right)^2+(\omega\eta)^2}$$

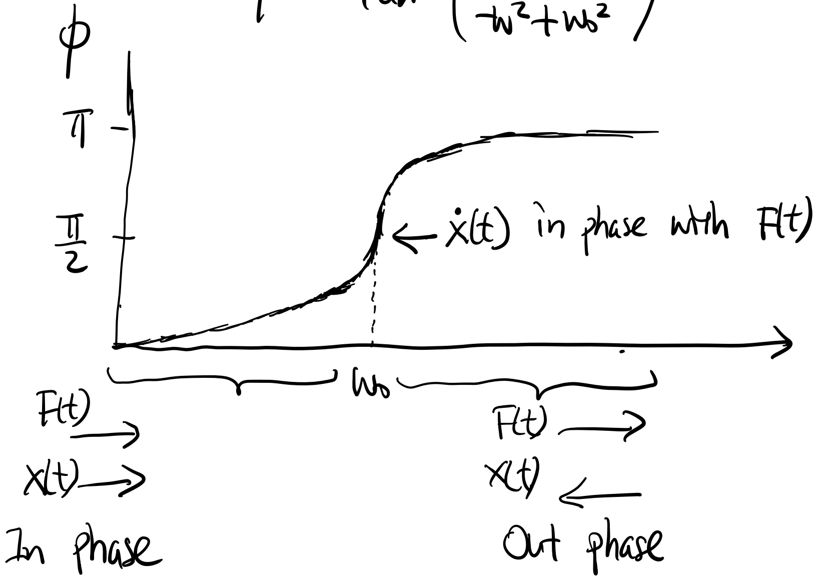
$$\left| G_R(\omega) \right| = \frac{1/m}{\sqrt{(-\omega^2+\omega_0^2)^2+(\omega\eta)^2}} \Leftarrow \text{Amplitude of oscillation}$$



To get the phase:

$$\tan\phi = \frac{\mathrm{Im}\, G_R(\omega)}{\mathrm{Re}\, G_R(\omega)} = \frac{\omega\eta}{-\omega^2+\omega_0^2}$$

$$\phi = \tan^{-1}\left(\frac{\omega\eta}{-\omega^2+\omega_0^2}\right)$$



Resonance Behavior for $\eta = 0$, or no damping:

we then have:

$$x_s(t) = Re\left[\frac{F_0/m}{-w^2+w_0^2} e^{-iwt}\right] \leftarrow \text{goes } \infty \text{ if } w = w_0$$

so is not a solution when $w = w_0$

In general:

$$x_s(t) = \underbrace{A\cos w_0 t + B\sin w_0 t}_{\text{Homogeneous solution}} + \underbrace{\frac{F_0/m}{-w^2+w_0^2}\cos wt}_{\text{specific solution}}$$

Now let $A \to A'$:

then $$x(t) = A'\cos w_0 t + B\sin w_0 t + \frac{F_0}{m}\frac{(\cos wt - \cos w_0 t)}{(-w^2 + w_0^2)}$$

Now if we take limit $w \to w_0$

then $w = w_0 + \Delta w$ on resonance.

then

$$\cos wt = \cos(w_0 + \Delta w)t$$
$$\stackrel{!}{=} \cos(w_0 t)\underbrace{\cos(\Delta w t)}_{\simeq 1} - \sin(w_0 t)\underbrace{\sin(\Delta w t)}_{\Delta w t}$$
$$\stackrel{!}{=} \cos(w_0 t) - \sin(w_0 t)\Delta w t.$$

And $\quad -w^2+w_0^2 = -(w_0+\Delta w)^2 + w_0^2 \overset{\sim}{=} -2\Delta w\, w_0$

Now plug it into $X(t)$:

$$x(t) = A'\cos w_0 t + B\sin w_0 t + \frac{F_0}{m}\frac{(\cos wt - \cos w_0 t)}{(-w^2+w_0^2)}$$

$$\overset{!}{=} A'\cos w_0 t + B\sin w_0 t + \frac{F_0}{m}\frac{-\sin(w_0 t)\cancel{\Delta w}\, t}{-2\cancel{\Delta w}\, w_0}$$

$$X(t) \overset{!}{=} A'\cos w_0 t + B\sin w_0 t + \underbrace{\frac{F_0}{2mw_0}}_{\text{growing magnitude.}} t\sin(w_0 t)$$

↖ 90° out of phase at resonance