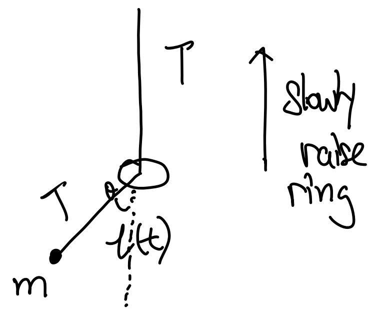
Adiabatic Invariance: (Slow is the key word).



→ There is upward force by string on ring.

→ Displacement Δz is up, so string does work, so string's energy goes down

Since $\omega = \sqrt{\frac{g}{l(t)}}$ and if ring increase height, then $l(t)$ increases.

so $\omega(t)$ goes down.

By Newton:

$$F_z = T - T\cos\theta = T\frac{\theta^2}{2} \propto E$$

$$\overline{F_z} = T\overline{\frac{\theta^2}{2}} = \frac{1}{2} mg\,\overline{\theta}^2 = \frac{\overline{E}(t)}{2\,l(t)}$$

$$\overline{W} = -\overline{\frac{dE}{dt}} = \overline{F_z}\frac{dl}{dt} = \frac{\overline{E}}{2l}\frac{dl}{dt}$$

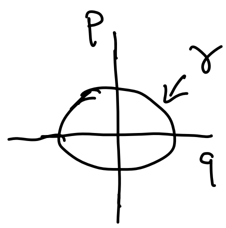
then $\frac{d(E\sqrt{l})}{dt} = 0$ so $\frac{d}{dt}\left(\frac{E}{\omega}\right) = 0$

In this problem:

Hamiltonian → $H(q, p, \lambda(t))$

slow function of time.

If we have fixed energy, we have a closed orbit in phase space.



Then $p_\gamma = p_\gamma(q, E, \lambda)$

Determined by

$$E = H(q, p_\gamma, \lambda)$$

Define a quantity: I, adiabatic invariant.

★

$$\boxed{\begin{aligned} I(E,\lambda) &= \int dq\, dp \\ I &\overset{!}{=} \oint_\gamma p_\gamma\, dq \end{aligned}}$$

← Area is constant due to Liouville Theorem.

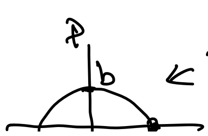
← Stokes theorem, convert from area to line integral.

If $E(t)$ and $\lambda(t)$, both function of time

then
$$\frac{d}{dt}\left(\oint_\gamma p_\gamma\, dq \right) = 0$$

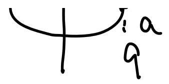
Ex: For SHO

$$E = \frac{p^2}{2} + \frac{w^2}{2} q^2$$ ← Determines $p_\gamma(E)$



← Ellipse

$b = \sqrt{2E}$ when $q = 0$



$a = \sqrt{\frac{2E}{w^2}}$ when $p=0$

then $I = \int dq\, dp = \pi ab = \pi \frac{2E(t)}{w(t)}$

Proof :

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \left(\frac{\partial H}{\partial \lambda}\right)_p \frac{d\lambda}{dt}$$

Need an average rate of change. with slow varying $\lambda(t)$

$$\overline{\frac{dE}{dt}} = \overline{\left(\frac{\partial H}{\partial \lambda}\right)_p} \frac{d\lambda}{dt}$$ ← average over 1-cycle of orbit.

since $\frac{d\lambda}{dt}$ is small, only work with first order so $\frac{d\lambda}{dt} \simeq \frac{\Delta\lambda}{\Delta t}$

$$\hookrightarrow \frac{\Delta E}{\Delta t} = \overline{\left(\frac{\partial H}{\partial \lambda}\right)_p} \frac{\Delta\lambda}{\Delta t}$$

$$\overline{\left(\frac{\partial H}{\partial \lambda}\right)_p} = \frac{1}{T} \oint \left(\frac{\partial H}{\partial \lambda}\right)_p dt$$

For $dt = \frac{dq}{\dot{q}} = \frac{dq}{\frac{\partial H}{\partial p}}$

$$\frac{\Delta E}{\Delta t} = \frac{\oint \left(\frac{\partial H}{\partial \lambda}\right)_p \left(\frac{\partial p}{\partial H}\right)_\lambda dq \; \frac{\Delta\lambda}{\Delta t}}{\oint \left(\frac{\partial p}{\partial H}\right)_\lambda dq}$$

Since $\left(\frac{\partial E}{\partial \lambda}\right) = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\ }$ $= 0$

$$\left(\frac{\partial H}{\partial \lambda}\right)_E \quad \frac{}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0$$

then $$\frac{\partial P}{\partial \lambda} = \frac{-\,\partial H/\partial \lambda}{\partial H/\partial p}$$

then $$\frac{\Delta E}{\Delta t} = \frac{-\oint \frac{\partial P}{\partial \lambda}\, dq}{\oint \left(\frac{\partial P}{\partial E}\right) dq} \; \frac{\Delta \lambda}{\Delta t}$$

$$\oint \underbrace{\left\{ \left(\frac{\partial P}{\partial E}\right)\frac{\Delta E}{\Delta t} + \left(\frac{\partial P}{\partial \lambda}\right)\frac{\Delta \lambda}{\Delta t} \right\}}_{\oint \frac{\Delta P}{\Delta t}\, dq} dq = 0.$$

So $I = \oint p\, dq$ is invariant.

Ex 2:

A slowly change magnetic field:

Consider the circular orbits in xy plane with $x>0$ of a particle mass m and charge q, in a constant field B. in $\hat{z}$,

a) Use Hamilton formulation to determine the radius and angular frequency of circular orbit.

use gauge: $\vec{A} = B(0, x, 0)$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{q}{c}\dot{\vec{r}}\cdot\vec{A}$$
$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{q}{c}Bx\dot{y}$$

$$P_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x}$$
$$P_y = \frac{\partial\mathcal{L}}{\partial\dot{y}} = m\dot{y} + m\omega_B x \quad \Rightarrow \quad \dot{y} = \frac{P_y}{m} - \omega_B x$$

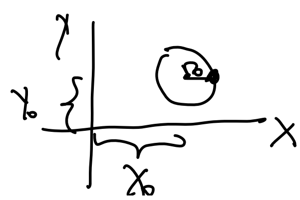
$$H = P_x\dot{x} + P_y\dot{y} - \mathcal{L}$$
$$= \frac{P_x^2}{m} + P_y\left(\frac{P_y}{m} - \omega_B x\right) - \left(\frac{P_x^2}{2m} + \frac{m}{2}\left(\frac{P_y}{m} - \omega_B x\right)^2\right) - m\omega_B x\left(\frac{P_y}{m} - \omega_B x\right)$$

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} \cancel{- \omega_B x P_y + P_y \omega_B x} - \frac{m}{2}\omega_B^2 x^2 - \omega_B x P_y + m\omega_B^2 x^2$$
$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} - \omega_B x P_y + \frac{1}{2}m\omega_B^2 x^2$$

$$-\frac{\partial H}{\partial x} = \dot{P}_x = m\ddot{x} = \omega_B P_y - m\omega_B^2 x$$

$$-\frac{\partial H}{\partial y} = \dot{P}_y = 0 \quad \Rightarrow \quad P_y \text{ is constant.}$$

$$\frac{\partial H}{\partial P_x} = \dot{x} = \frac{P_x}{m}$$
$$\frac{\partial H}{\partial P_y} = \dot{y} = \frac{P_y}{m}$$



$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} - \omega_B x P_y + \frac{1}{2} m \omega_B^2 x^2$$

$$= \frac{P_x^3}{2m} + \frac{1}{2} m \omega_B^2 \left(x^2 - \frac{2}{m\omega_B} x P_y + \frac{P_y^2}{m^2 \omega_B^2} \right)$$

$$H = \frac{P_x^3}{2m} + \frac{1}{2} m \omega_B^2 \Big(\underbrace{x - \frac{P_y}{m\omega_B}}_{\text{a shift of origin in potential.}} \Big)^2$$

$$P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + m\omega_B x = \text{constant}$$

when $x = x_0$, $\dot{y} = 0$

So $P_y = m\omega_B x_0 \Rightarrow \frac{P_y}{m\omega_B} = x_0$

We know at $x = x_0 + r_0$, $P_x = 0$

so $E_{tot} = H(x = x_0 + r_0) = \frac{1}{2} m \omega_B^2 r_0^2$

so $r_0 = \sqrt{\frac{2E_{tot}}{m\omega_B^2}}$

$$m\ddot{x} = \omega_B P_y - m\omega_B^2 x$$

$$\ddot{x} + \omega_B^2 x - \frac{\omega_B}{m} P_y = 0$$

$\frac{\omega_B}{m} P_y$ ↓ constant.

so $x = A\cos(\omega_B t) + \frac{\omega_B}{m} P_y$.

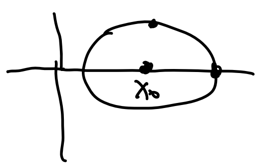
Clearly angular frequency is ω_B

b) How does radius and the center of the circular orbit changes as $B(t)$ slowly increased

$$I = \oint p_x \, dq .$$

know $p_y = m\dot{y} + m\omega_B x$ is conserved.

know $H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega_B^2 \left(x - \frac{p_y}{m\omega_B} \right)^2 = \frac{1}{2} m\omega_B^2 r_0^2 = E$



when $x = x_0$

$$E = \frac{1}{2} m\omega_B^2 {}_0^2 = \frac{p_{x_0}^2}{2m}$$

$$p_{x_0}^2 = m^2 \omega_B^2 r_0^2$$

$$p_{x_0} = m\omega_B r_0 = \sqrt{2mE}$$

When $x = x_0 + r_0$

$$E = \frac{1}{2} m\omega_B^2 r_0^2$$

$$\sqrt{\frac{2E}{m\omega_B^2}} = r_0$$

Then

$$I = \int dq\,dp \, \frac{1}{2\pi}$$

$$= \cancel{\pi} \sqrt{2mE} \sqrt{\frac{2E}{m\omega_B^2}} \, \frac{1}{2}$$

$$I = \frac{1}{2} \frac{\cancel{2\pi} E}{\omega_B}$$

$$I = \frac{\frac{1}{2} m\omega_B^{\cancel{2}} r^2}{\cancel{\omega_B}} \propto \omega_B r^2 \propto B r^2$$

$$\sqrt{\frac{2I}{m\omega_B}} = r(t)$$

$$I_0 = B_0 r_0^2 = B r^2 \Rightarrow r = \sqrt{\frac{B_0}{B}}\, r_0$$

When B increase r_0^2 should decrease proportionally
to let I remain constant.

Also we $P_y = m w_0 X_0$ to be constant

or $X = \frac{P_y}{m w_B}$

So if B increase X_0 also decrease proportionally.
to remain constant.