Hamiltonian of Phase Space:

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi \qquad \leftarrow \text{Lagrangian of pendulum.}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m l^2 \dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{m l^2}$$

$$\mathcal{H} = p_\phi \dot{\phi} - L$$

$$\stackrel{!}{=} \frac{p_\phi^2}{2 m l^2} - mgl\cos\phi$$

so $\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi}$ and $\dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi}$

For small θ, $\cos\theta = 1 - \frac{\theta^2}{2}$

$$E = \frac{p_\theta^2}{2} + \frac{\theta^2}{2} + \text{const}$$

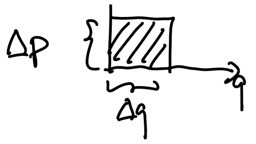
circle. in phase space

$$\left.\begin{aligned} q &\to Q(t) = \frac{1}{2} a t^2 + \frac{p}{m} t + q \\ p &\Rightarrow P(t) = mat + p \end{aligned}\right\} (q,p) \to (Q(t,q,p),\ P(t,q,p))$$

Liouville Theorem:

$\Rightarrow$ The area in phase-space is constant in time.

Proof: $p \uparrow$ $\qquad p \Rightarrow P(t) = p - \frac{\partial \mathcal{H}}{\partial q} \delta t = p + \dot{p}\,\delta t$



$$q \to Q(t) = q + \frac{\partial H}{\partial p}\delta t = q + \dot q\,\delta t$$

$$d(\Delta p) = \left(\dot p|_{top} - \dot p|_{bot}\right)\delta t$$
$$\overset{!}{=} \frac{\partial \dot p}{\partial p}\Delta p\,\delta t$$
$$\overset{!}{=} -\frac{\partial^2 H}{\partial p\,\partial q}\Delta p\,\delta t \qquad \dot p = -\frac{\partial H}{\partial q}$$

$$d(\Delta q) = \left(\dot q|_{right} - \dot q|_{left}\right)\delta t$$
$$\overset{!}{=} \frac{\partial \dot q}{\partial q}\Delta q\,\delta t$$
$$\overset{!}{=} \frac{\partial^2 H}{\partial p\,\partial q}\Delta q\,\delta t$$

Then:

$$d(\Delta p\Delta q) = d(\Delta p)\Delta q + \Delta p\, d(\Delta q)$$
$$\overset{!}{=} \left(-\frac{\partial^2 H}{\partial p\,\partial q} + \frac{\partial^2 H}{\partial p\,\partial q}\right)\Delta p\,\Delta q\,\delta t$$
$$\cong 0$$

Second proof:

$$\text{Area} = \int_{R(t)} dQ\,dP = \int_{R(t_0)} \left|\frac{\partial(Q,P)}{\partial(q,p)}\right| dq\,dp$$

$$q \to Q(\delta t, p, q) = q + \frac{\partial H}{\partial p}\delta t = q + \dot Q(q,p)\,\delta t$$

$$p \to P(\delta t, p, q) = p - \frac{\partial H}{\partial q}\delta t = p + \dot{P}(q,p)\,\delta t$$

Jacobian: $$\frac{\partial(Q,P)}{\partial(q,p)} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$

$\hookrightarrow$ use:

$$Q(t) = q + \frac{\partial H}{\partial p}\delta t$$
$$P(t) = p - \frac{\partial H}{\partial q}\delta t$$

$$\hookrightarrow = \begin{vmatrix} 1 + \delta t \frac{\partial^2 H}{\partial p \partial q} & \delta t \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial q^2}\delta t & 1 - \delta t \frac{\partial^2 H}{\partial p \partial q} \end{vmatrix}$$

$$\hookrightarrow = 1 + \delta t^2 \left\{ -\left(\frac{\partial^2 H}{\partial p \partial q}\right)^2 + \frac{\partial^2 H}{\partial p^2}\frac{\partial^2 H}{\partial q^2} \right\}$$

$$\downarrow \simeq 1 + \mathcal{O}(\delta t^2)$$

So $$A = \int_{R(t)} dQ\, dP = \int_{R(t_0)} \left| \frac{\partial(Q,P)}{\partial(q,p)} \right| dq\, dp \simeq \int_{R(t_0)} dq\, dp$$

<u>For multiple coordinates:</u>

$$q^i \to Q^i(q,p) = q^i + \frac{\partial H}{\partial p_i}\delta t$$

$$p^i \to P^i(q,p) = p_i - \frac{\partial H}{\partial q^i}\delta t$$

$$A(t) = \int_{R(t)} dQ^i\, dP_i = \int_{R(t_0)} \left| \frac{\partial(Q,P)}{\partial(q,p)} \right| dq^i\, dp_i$$

$$J = \frac{\partial(Q,P)}{\partial(q,p)} = \begin{pmatrix} \frac{\partial Q^i}{\partial q^j} & \frac{\partial Q^i}{\partial p_j} \\ \frac{\partial P_i}{\partial q^j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}$$

$$\hookrightarrow = \begin{vmatrix} \delta^i_j + \delta t \frac{\partial^2 H}{\partial p_i \partial q^j} & \frac{\partial^2 H}{\partial p_i \partial p_j} \delta t \\ \frac{-\partial^2 H}{\partial q^i \partial q^j} & \delta^i_j - \delta t \frac{\partial^2 H}{\partial p_i \partial q^j} \end{vmatrix}$$

natural log.

$$\det A = \exp\left\{ \mathrm{Tr}(\log A) \right\}$$

$$= e^{\log \lambda_1 + \log \lambda_2 + \cdots}$$

$$\doteq (\lambda_1)(\lambda_2) \cdots (\lambda_n)$$

terms with H.

Since $A \cong 1 + \delta t M$

$$\log A = \log(1 + \delta t M)$$

$$\log A \doteq \delta t\, M \quad \text{by Taylor}$$

$\hookrightarrow$ $\mathrm{Tr} \log A = \mathrm{Tr}(\delta t M)$

and $\mathrm{Tr}(\delta t M) = \sum_i \frac{\partial^2 H}{\partial q^i \partial p_j} - \frac{\partial^2 H}{\partial q^i \partial p_j} = 0.$

then $\exp(\mathrm{Tr}(\delta t M)) = 1 + \mathrm{Tr}(\delta t M)$ by Taylor.

Then

$$\exp\{\mathrm{Tr}\log A\} = 1 + \mathrm{Tr}(\delta t M) \simeq 1$$

If $V(t) = \int_{R(t)} dQ\,dP = V(t_0) = \int_{R(t_0)} dq\,dp$ ← volume.

then $f(t,q,p) = \dfrac{dN}{dp\,dq}$ ← # density = phase space density.

then $N = \int dq\,dp\, f(t,q,p)$ is constant since # of particles are constant.

If dN, $dp\,dq$ are both constant, f : phase-space density is also constant.

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} = 0$$

$$\hookrightarrow \boxed{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} = 0}$$ Liouville Equation.

Define Poisson bracket:

$$\{f, H\} = \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i}$$

then Liouville Eq: f is P.S. density

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} = 0.}$$

For arbitrary $O(t,q,p)$

$$\boxed{\frac{dO}{dt} = \frac{\partial O}{\partial t} + \{O, H\}}$$

properties of poisson bracket:

1) $\{f, g\} = -\{g, f\}$

2) $\{f_1 + f_2\ g\} = \{f_1, g\} + \{f_2, g\}$

3) $\{fg, h\} = \{f, h\}g + f\{g, h\}$ ← Leibniz rule.

4) $\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$

Jacobian Identity.

5) $\{q^i, q^j\} = 0$

$\{p_i, p_j\} = 0$

$\{q^i, p_j\} = \delta^i_j$

6) $\{f, p_i\} = \frac{\partial f}{\partial q^i}$

$\{f, q^i\} = -\frac{\partial f}{\partial p_i}$

7) $\dot{q} = \{q, H\} = -\{H, q\} = \frac{\partial H}{\partial p}$

$\dot{p} = \{p, H\} = -\{H, p\} = -\frac{\partial H}{\partial q}$

Poisson Theorem:

If I and J are constant in time, so $I(q,p)$ and $J(q,p)$

or $\dot{I} = \{I, H\} = 0$ and $\dot{J} = \{J, H\} = 0$

then $\{I, J\}$ is also constant.

Proof:

$$\frac{d\{I,J\}}{dt} = \{\{I,J\}, H\}$$

$$= -\{H, \{I,J\}\}$$

$$\stackrel{!}{=} \{J, \underbrace{\{H, I\}}_{-\frac{dI}{dt} = -\cancel{\frac{\partial I}{\partial t}} + \{H,I\}}\} + \{I, \underbrace{\{J, H\}}_{\dot{J}}\} = 0$$

Ex: $\vec{L} = \vec{r} \times \vec{p}$

$L_1 = r_2 p_3 - r_3 p_2$

$L_2 = r_3 p_1 - r_1 p_3$

$$\{L_1, L_2\} = \{r_2 p_3 - r_3 p_2, r_3 p_1 - r_1 p_3\}$$

$$\stackrel{!}{=} \{r_2 p_3, r_3 p_1\} + \{r_3 p_2, r_1 p_3\} - \cancel{\{r_3 p_2, r_3 p_1\}} - \cancel{\{r_2 p_3, r_1 p_3\}}$$

$$\doteq \cancel{\{r_2, r_3 p_1\}} p_3 + r_2 \{p_3, r_3 p_1\}$$

$$+ \cancel{\{r_3, r_1 p_3\}} p_2 + r_3 \{p_2, r_1 p_3\}$$

$$L_3 \stackrel{!}{=} r_1 p_2 - r_2 p_1$$