① A non-linear oscillator:

→ an oscillator of mass m and resonant frequency w_0, has damping force $F_D = -Bv^3$ with $B > 0$,

→ The motion initialized with amplitude a_0 and $v=0$ at $t=0$

a) Define suitable dimensionless variable so that a dimensionless version of the equation is:

$$\frac{d^2\bar{x}}{dt^2} + \bar{x} + \epsilon\left(\frac{d\bar{x}}{d\bar{t}}\right)^3 = 0$$

↳ $\frac{d^2x}{dt^2} + w_0^2 x + \frac{B}{m}\left(\frac{dx}{dt}\right)^3 = 0$

↳ $\frac{1}{w_0^2}\frac{d^2x}{dt^2} + x + \frac{B}{w_0^2 m}\left(\frac{dx}{dt}\right)^3 = 0$

let $\bar{t} = w_0 t$

↳ $\frac{d^2x}{d\bar{t}^2} + x + \frac{B}{w_0^2 m}\left(\frac{dx}{d\bar{t}}\right)^3 = 0$

since $x \sim a_0 \cos 4$

$x = \bar{x} a_0$

↳ $\cancel{a_0}\frac{d^2\bar{x}}{d\bar{t}^2} + \cancel{a_0}\bar{x} + \frac{B}{w_0^2 m} a_0^{\cancel{3}\,2}\left(\frac{d\bar{x}}{d\bar{t}}\right)^3 = 0$

$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \underbrace{\frac{Ba_0^2}{w_0^2 m}}_{\epsilon}\left(\frac{d\bar{x}}{d\bar{t}}\right)^3$

require $\epsilon = \frac{Ba_0^2}{\omega_0^2 m} << 1$ to consider nonlinear terms small.

b) Work with secular perturbation theory to determine $\bar{x}(\bar{t})$.

At $\bar{t}=0$, $\bar{x}=1$, $\frac{d\bar{x}}{d\bar{t}}=0$

let $\bar{x}^{(0)}(\bar{t}) = a(\bar{t})\cos(-\bar{t}+\phi(\bar{t}))$

$$\frac{d\bar{x}}{d\bar{t}} = \dot{a}\cos(-\bar{t}+\phi) + a\omega\sin(-t+\phi) - a\dot{\phi}\sin(-t+\phi)$$

$$\frac{d^2\bar{x}}{dt^2} = \cancel{\ddot{a}\cos(-\omega\bar{t}+\phi)} - \dot{a}\sin(-\omega t+\phi)[-\omega+\dot{\phi}]$$
$$+\dot{a}\omega\sin(-\omega t+\phi) + a\omega\cos(-\omega t+\phi)[-\omega+\dot{\phi}]$$
$$-\dot{a}\dot{\phi}\sin(-\omega t+\phi) - a\ddot{\phi}\sin(-\omega t+\phi) - a\dot{\phi}\cos(-\omega t+\phi)[-\omega+\dot{\phi}]$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} \overset{!}{=} -a\omega^2\cos(-\omega t+\phi) + 2\dot{a}\omega\sin(-\omega t+\phi) + 2a\dot{\phi}\cos(-\omega t+\phi) + \mathcal{O}.$$

$$\left(\frac{d\bar{x}}{dt}\right)^3 = \left[\dot{a}\cos(-\omega\bar{t}+\phi) + a\omega\sin(-\omega\bar{t}+\phi)\right]^3$$

$$\overset{!}{=} \dot{a}^3\cos^3(-\omega\bar{t}+\phi) + a^3\omega^3\sin^3(-\omega\bar{t}+\phi)$$
$$+3\dot{a}^2\cos^2(-\omega\bar{t}+\phi)\,a\omega\sin(-\omega\bar{t}+\phi)$$
$$+3\dot{a}\cos(-\omega\bar{t}+\phi)\,a^2\omega^2\sin^2(-\omega\bar{t}+\phi)$$

$$\overset{!}{\simeq} a^3\omega^3\sin^3(-\omega\bar{t}+\phi)$$ ← only care about zeroth order since ϵ is already first order.

$$\ldots\; a^3\omega^3\left(\frac{e^{i(-\omega\bar{t}+\phi)} - e^{-i(-\omega\bar{t}+\phi)}}{\ldots}\right)^3$$

$\hookrightarrow a^3 w^3 \left(\frac{\quad}{2i} \right)$

$$= \frac{a^3 w^3}{-4} \left(\frac{e^{3i(-w\bar{t}+\phi)} - e^{-3i(-w\bar{t}+\phi)} - 3e^{i(-w\bar{t}+\phi)} + 3e^{-i(-w\bar{t}+\phi)}}{2i} \right)$$

$$= \frac{a^3 w^3}{-4} \left\{ \sin(3(-w\bar{t}+\phi)) - 3\sin(-w\bar{t}+\phi) \right\}$$

Then $\ddot{\bar{x}} + \bar{x} + \epsilon(\dot{\bar{x}})^3 = 0$

$$\ddot{\bar{x}}^{(0)} + \ddot{\bar{x}}^{(1)} + \bar{x}^{(0)} + \bar{x}^{(1)} + \epsilon\left(\dot{\bar{x}}^{(0)}\right)^3 = 0$$

know $w = 1$

$\hookrightarrow 2\dot{a}\sin(-\bar{t}+\phi) + 2a\dot{\phi}\cos(-\bar{t}+\phi) - \frac{a^3}{4}\left\{\sin(3(-\bar{t}+\phi)) - 3\sin(-\bar{t}+\phi)\right\}$
$+ \ddot{\bar{x}}^{(1)} + \bar{x}^{(1)} = 0$

let secular terms go zero by adjusting $a(\bar{t})$ and $\phi(\bar{t})$

$$\left[2\dot{a} + \frac{3a^3}{4}\right]\sin(-\bar{t}+\phi) = 0 \Rightarrow \dot{a} = -\frac{3a^3}{8} \Rightarrow \int \frac{da}{a^3} = \frac{-3}{8}(\bar{t} - \bar{t}_0)$$

$$\left.\frac{1}{-2a^2}\right|_{\bar{t}_0}^{\bar{t}} = \left.-\frac{3}{8}\bar{t}\right|_{\bar{t}=0}^{\bar{t}}$$

$$\frac{-1}{2a^2} + \frac{1}{2a_0^2} = \frac{-3}{8}\bar{t}$$

$$\frac{3}{4}\bar{t} + \frac{1}{a_0^2} = \frac{1}{a^2}$$

or $a = \left(\frac{3}{4}\bar{t} + \frac{1}{a_0^2}\right)^{-1/2}$

$$2a\dot{\phi}\cos(-\bar{t}+\phi) = 0 \Rightarrow \phi = \phi_0$$

$\hookrightarrow \ddot{\bar{x}}^{(1)} + \bar{x}^{(1)} + -\frac{a^3}{4}\sin(3(-\bar{t}+\phi)) = 0$

then $\bar{x}^{(1)} = A\sin(3(-\bar{t}+\phi))$

$\ddot{\bar{x}}^{(1)} = -9A\sin(3(-\bar{t}+\phi))$

$$(-9+1)A\sin(3(-\bar{t}+\phi)) - \frac{a^3}{4}\sin(3(-\bar{t}+\phi)) = 0$$

$$A = \frac{1}{-32}\left[\frac{3}{4}\bar{t} + \frac{1}{a_0^2}\right]^{-3/2}$$

$\bar{x} = \bar{x}^0 + \bar{x}^{(1)}$

$$\frac{1}{-}\left[\frac{3}{4}\bar{t} + \frac{1}{a_0^2}\right]^{-1/2}\cos(-\bar{t}+\phi_0) - \frac{1}{32}\left[\frac{3}{4}\bar{t} + \frac{1}{a_0^2}\right]^{-3/2}\sin(3(-\bar{t}+\phi_0))$$

at $\bar{t}=0$, $\dot{\bar{x}} = 0$. , so $\underline{\underline{\phi_0 = 0}}$.

at $\bar{t}=0$ $\bar{x}=1$

$\hookrightarrow \bar{x}(\bar{t}=0) = a_0 = 1$

$$\bar{x} = \left(\frac{3}{4}\bar{t}+1\right)^{-1/2}\cos(-\bar{t}) - \frac{1}{32}\left[\frac{3}{4}\bar{t}+1\right]^{-3/2}\sin(-3\bar{t})$$

2) Consider a lagrangian:

$$\mathcal{L} = \frac{1}{2} m (1+\epsilon q^2)\dot{q}^2 - \frac{1}{2} m \omega_0^2 q^2$$

Calculate the frequency of oscillations using secular perturbation theory.

Do the same using integral given in class for the period of 1D system.

Assume $\epsilon A^2 \ll 1$. where A is the amplitude of oscillations.

Since no explicit time dependence, energy, h, is conserved.

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m(1+\epsilon q^2)\dot{q}$$

$$h = m(1+\epsilon q^2)\dot{q}^2 - \left\{ \frac{1}{2} m (1+\epsilon q^2)\dot{q}^2 - \frac{1}{2} m \omega_0^2 q^2 \right\}$$

$$h = \frac{1}{2} m (1+\epsilon q^2)\dot{q}^2 + \frac{1}{2} m \omega_0^2 q^2$$

$$\frac{1}{2} m (1+\epsilon q^2)\dot{q}^2 = E - \frac{1}{2} m \omega_0^2 q^2$$

$$\dot{q} = \sqrt{\frac{2}{m(1+\epsilon q^2)} \left\{ E - \frac{1}{2} m \omega_0^2 q^2 \right\}}$$

$$\int_{-A}^{A} \sqrt{\frac{m}{2}} \frac{\sqrt{(1+\epsilon q^2)}}{\sqrt{E - \frac{1}{2} m \omega_0^2 q^2}}\, dq = \int_0^{T/2} dt = \frac{T}{2} = \qquad T = \frac{2\pi}{\omega}$$

$$\int^{A} \sqrt{(1+\epsilon q^2)} \qquad \sqrt{1+\epsilon q^2} \approx 1 + \frac{1}{2}\epsilon q^2$$

$$\int_0 2\sqrt{\frac{m}{2}}\frac{\quad}{\sqrt{E-\frac{1}{2}m\omega_0^2q^2}}\,dq = \frac{T}{2}$$

$$\hookrightarrow \underbrace{\int_0^A \frac{\sqrt{2m}}{\sqrt{E-\frac{1}{2}m\omega_0^2q^2}}\,dq}_{\frac{T_0}{2}=\frac{\pi}{\omega_0}} + \underbrace{\int_0^A \frac{\sqrt{2m}\,\frac{1}{2}\epsilon q^2}{\sqrt{E-\frac{1}{2}m\omega_0^2q^2}}\,dq} = T/2 = \frac{\pi}{\omega}$$

$$\hookrightarrow = \sqrt{\frac{2m}{\frac{1}{2}m\omega_0^2}}\arcsin\left(\sqrt{\frac{\frac{1}{2}m\omega_0^2A^2}{E}}\right) \leftarrow \arcsin(1) = \frac{\pi}{2}$$

$$+\frac{\sqrt{2m}}{2}\epsilon\frac{E}{2(\frac{1}{2}m\omega_0^2)^{3/2}}\left[\underbrace{\arcsin\left(\sqrt{\frac{\frac{1}{2}m\omega_0^2A^2}{E}}\right)}_{=\frac{\pi}{2}} - \frac{1}{2}\sin\left(2\arcsin\left(\sqrt{\frac{\frac{1}{2}m\omega_0^2A^2}{E}}\right)\right)\right]$$

for $E=\frac{1}{2}m\omega_0^2A^2$

$$\hookrightarrow = \frac{\pi}{\omega_0} + \frac{\epsilon A^2}{2\omega_0}\left(\frac{\pi}{2} - \frac{1}{2}\underbrace{\sin\left(2\,\frac{\pi}{2}\right)}_{=0}\right)$$

$$\frac{\pi}{\omega} \overset{!}{=} \frac{\pi}{\omega_0} + \frac{\pi\epsilon A^2}{4\omega_0}$$

$$\frac{1}{\omega} = \frac{1}{\omega_0} + \frac{\epsilon A^2}{4\omega_0}$$

$$\frac{1}{\omega} = \frac{1}{\omega_0}\left(1+\frac{\epsilon A^2}{4}\right)$$

$$\hookrightarrow \boxed{\omega = \omega_0\left(1-\frac{\epsilon A^2}{4}\right)}$$ ← From integration.

Now consider secular perturbation.

$$L = \frac{1}{2}m(1+\epsilon q^2)\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(m(1+\epsilon q^2)\dot{q}\right)$$

$$\overset{!}{=} m(1+\epsilon q^2)\ddot{q} + 2m\epsilon q\dot{q}^2$$

$$\frac{\partial L}{\partial q} = m\epsilon q\dot{q}^2 - m\omega_0^2 q$$

$$m(1+\epsilon q^2)\ddot{q} + 2m\epsilon q\dot{q}^2 - m\epsilon q\dot{q}^2 + m\omega_0^2 q = 0$$

$$\hookrightarrow \quad m(1+\epsilon q^2)\ddot{q} + m\omega_0^2 q + m\epsilon q\dot{q}^2 = 0$$

$$\hookrightarrow \quad m\ddot{q} + m\omega_0^2 q + m\epsilon[q^2\ddot{q} + q\dot{q}^2] = 0$$

$$\hookrightarrow \quad \ddot{q} + \omega_0^2 q + \epsilon[q^2\ddot{q} + q\dot{q}^2] = 0$$

$$q^{(0)} = a(t)\cos(-\omega t + \phi)$$

$$\dot{q}^{(0)} = \dot{a}\cos(-\omega t + \phi) - a\sin(-\omega t+\phi)(-\omega + \dot{\phi})$$

$$\ddot{q}^{(0)} = \ddot{a}\cos(-\omega t+\phi) - \dot{a}\sin(-\omega t+\phi)[-\omega + \dot{\phi}]$$
$$+\dot{a}\omega\sin(-\omega t+\phi) + a\omega\cos(-\omega t+\phi)[-\omega+\dot{\phi}]$$
$$-\dot{a}\dot{\phi}\sin(-\omega t+\phi) - a\ddot{\phi}\sin(-\omega t+\phi) - a\dot{\phi}\cos(-\omega t+\phi)[-\omega+\dot{\phi}]$$

$$\doteq 2\dot{a}\omega\sin(-\omega t+\phi) + 2a\omega\dot{\phi}\cos(-\omega t+\phi)$$
$$- a\omega^2\cos(-\omega t+\phi) + \mathcal{O}$$

$$q^2\ddot{q} = a^2\cos^2(-\omega t+\phi)\left[-a\omega^2\cos(-\omega t+\phi)\right]$$
$$\doteq -a^3\omega^2\cos^3(-\omega t+\phi)$$
$$\doteq -a^3\omega^2\left[\frac{e^{i(-\omega t+\phi)}+e^{-i(-\omega t+\phi)}}{2}\right]^3$$
$$\doteq \frac{-a^3\omega^2}{4}\left[\frac{e^{3i(-\omega t+\phi)}+e^{-3i(-\omega t+\phi)}+3e^{i(-\omega t+\phi)}+3e^{-i(-\omega t+\phi)}}{2}\right]$$
$$\doteq \frac{-a^3\omega^2}{4}\left[\cos(3(-\omega t+\phi)) + 3\cos(-\omega t+\phi)\right]$$

$$q\dot{q}^2 = a\cos(-\omega t+\phi)\left[a\omega\sin(-\omega t+\phi)\right]^2$$
$$\doteq a^3\omega^2\left\{\cos(-\omega t+\phi)\left[1-\cos^2(-\omega t+\phi)\right]\right\}$$
$$\doteq a^3\omega^2\left\{\cos(-\omega t+\phi) - \cos^3(-\omega t+\phi)\right\}$$
$$\doteq a^3\omega^2\left\{\cos(-\omega t+\phi) - \frac{1}{4}\left[\cos(3(-\omega t+\phi)) + 3\cos(-\omega t+\phi)\right]\right\}$$
$$\doteq a^3\omega^2\left\{\frac{1}{4}\cos(-\omega t+\phi) - \frac{1}{4}\cos(3(-\omega t+\phi))\right\}$$

Put all pieces together:

$$q + \omega_0^2 q + \epsilon\left[q^2 q + q q^2\right] = 0$$

$$\hookrightarrow \ddot{q}^{(0)} + \ddot{q}^{(1)} + \omega_0^2 q^{(0)} + \omega_0^2 q^{(1)} + \epsilon\left[q^{2(0)}\ddot{q}^{(0)} + q^{(0)}\dot{q}^{2(0)}\right] = 0$$

$$\hookrightarrow \left(-\omega^2 + \omega_0^2\right) a\cos(-\omega t + \phi) + 2\dot{a}\omega\sin(-\omega t + \phi)$$
$$+ 2a\omega\dot{\phi}\cos(-\omega t + \phi) + \ddot{q}^{(1)} + \omega_0^2 q^{(1)}$$
$$+ \epsilon\left\{\frac{-a^3\omega^2}{4}\left[\cos(3(-\omega t + \phi)) + 3\cos(-\omega t + \phi)\right]\right.$$
$$\left. + a^3\omega^2\left[\frac{1}{4}\cos(-\omega t + \phi) - \frac{1}{4}\cos(3(-\omega t + \phi))\right]\right\} = 0$$

$$\hookrightarrow \cancel{\left(-\omega^2 + \omega_0^2\right)} a\cos(-\omega t + \phi) + 2\dot{a}\omega\sin(-\omega t + \phi) + 2a\omega\dot{\phi}\cos(-\omega t + \phi)$$
$$- \frac{\epsilon a^3\omega^2}{2}\left[\cos(-\omega t + \phi) + \cos(3(-\omega t + \phi))\right\}$$
$$+ \ddot{q}^{(1)} + \omega_0^2 q^{(1)} = 0$$

↳ Remove secular terms:

$$2\dot{a}\omega\sin(-\omega t + \phi) = 0 \quad \text{or} \quad a = A$$

$$\left[\cancel{\left(-\omega^2 + \omega_0^2\right)} A + 2A\dot{\phi}\omega - \frac{\epsilon A^3\omega^2}{2}\right]\cos(-\omega_0 t + \phi) = 0$$

let $\omega = \omega_0$.

then $$2\dot{\phi}A\omega_0 = \frac{\epsilon A^3\omega_0^2}{2}$$

$$\varphi = \frac{w_0 \epsilon A}{4}$$

$$\phi = \frac{w_0 \epsilon A^2}{4} t$$

$$q^0 = A\cos\left(-w_0 t + \frac{w_0 \epsilon A^2}{4} t\right)$$

$$\stackrel{!}{=} A\cos\left(-w_0\left(1-\frac{\epsilon A^2}{4}\right)t\right)$$

$$w = w_0\left(1 - \frac{\epsilon A^2}{4}\right)$$

3) Anharmonic Oscillations to quadratic order.

Consider $U = \frac{1}{2} m w_0^2 q^2 + \frac{c}{3} q^3$

let's approximate $q(t)$ to second order:

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)}$$

Then $L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m w_0^2 q^2 - \frac{c}{3} q^3$

then $m\ddot{q} + m w_0^2 q + c q^2 = 0$

a) choose units so that

$$\frac{d^2 \bar{q}}{d\bar{t}^2} + \bar{q} + \bar{c}\bar{q}^2 = 0$$

let $\bar{t} = w_0 t$

$q \sim a_0 \cos 4$

$$\dot{\bar{q}} = q/a_0$$

then $\ddot{q} + \omega_0^2 q + \frac{c}{m} q^2 = 0$

$\hookrightarrow \frac{d^2\bar{q}}{d\bar{t}^2} \omega_0^2 a_0 + \omega_0^2 a_0 \bar{q} + \frac{c a_0^2}{m} \bar{q}^2 = 0$

$\hookrightarrow \frac{d^2\bar{q}}{d\bar{t}^2} + \bar{q} + \frac{a_0 c}{\omega_0^2 m} \bar{q}^2 = 0$

$$\bar{c} = \frac{a_0 c}{\omega_0^2 m}$$

If $\bar{c}$ is small, then $\bar{c} \ll 1$

or $c \ll \frac{\omega_0^2 m}{a_0}$

b) let $q^{(0)} = A \cos \omega t = A \cos((\omega_0 + \Delta\omega)t)$

$\uparrow$ constant

$\dot{q}^{(0)} = -\omega A \sin \omega t$

$\ddot{q}^{(0)} = -\omega^2 A \cos \omega t$

$q^2_{(0)} = A^2 \cos^2(\omega t)$

$\frac{d^2\bar{q}}{d\bar{t}^2} + \bar{q} + \bar{c}\,\bar{q}^2 = 0$ keep only 0th and 1st to achieve second order

first order

↳ $\ddot q^{(0)} + \ddot q^{(1)} + \ddot q^{(2)} + q^{(0)} + q^{(1)} + q^{(2)} + \bar c\left(q^{(0)} + q^{(1)} + q^{(2)}\right)^{-} = 0$

$$\left(q^{(0)} + q^{(1)} + q^{(2)}\right)^2 = q^{(0)2} + 2q^{(0)}q^{(1)} + \cancel{2q^{(0)}q^{(2)}} + \cancel{2q^{(1)}q^{(2)}} + \cancel{q^{(1)2}}$$
$$+ \cancel{q^{(2)2}}$$
$$\overset{!}{=} q^{(0)2} + 2q^{(0)}q^{(1)} + \mathcal{O}(\epsilon^2, \epsilon^s)$$

group by order:

(0) $\ddot q^{(0)} + q^{(0)} = 0$

then $(-w^2+1)A\cos wt = 0$.

$w = 1 + w_1 + w_2 \Rightarrow w^2 = 1 + 2w_1 + 2w_2 + 2w_1w_2 +$
$w_1^2 + w_2^2$

↳ $(-1 - 2w_1 - 2w_2 - w_1^2 + 1)A\cos wt = 0$.

↳ $(-2w_1 - 2w_2 - w_1^2)A\cos wt = 0$

(1) $\ddot q^{(1)} + q^{(1)} + \bar c\, q^{(0)2} = 0$

$\ddot q^{(1)} + q^{(1)} + \bar c A^2\cos^2(wt) - 2w_1 A\cos wt = 0$

$$\cos^2(wt) = \left(\frac{e^{iwt} + e^{-iwt}}{2}\right)^2 = \frac{1}{2}\left(\frac{e^{2iwt} + e^{-2iwt} + 2}{2}\right)$$

$$\doteq \frac{1}{2}(\cos 2wt + 1)$$

$$\hookrightarrow \ddot{q}^{(1)} + q^{(1)} + \frac{\bar{c}A^2}{2}(1+\cos 2wt) - 2w_1 A\cos wt = 0$$

let $q^{(1)} = B\cos 2wt - \frac{\bar{c}A^2}{2}$

$$\ddot{q}^{(1)} = -4w^2 B\cos 2wt$$

$$-4w^2 B\cos 2wt + B\cos 2wt - \frac{\bar{c}A^2}{2} + \frac{\bar{c}A^2}{2} + \frac{\bar{c}A^2}{2}\cos 2wt$$
$$\cancel{-2w_1 A\cos wt} = 0$$

let $\boxed{w_1 = 0}$ to get rid of secular term.

$$-4w^2B + B + \frac{\bar{c}A^2}{2} = 0.$$

$$B(1-4w^2) + \frac{\bar{c}A^2}{2} = 0$$

$$B = \frac{\bar{c}A^2}{2(4w^2-1)}$$

$$\doteq \frac{\bar{c}A^2}{2(4(1+w_1+w_2)^2-1)}$$

only keep terms to zeroth order to make B first order.

$$\boxed{B \doteq \frac{\bar{c}A^2}{6}}$$

then $\boxed{q^{(1)} = \frac{\bar{c}A^2}{6}\cos 2wt - \frac{\bar{c}A^2}{2}}$

$\omega_1 = 0$

(2) $\ddot{q}^{(2)} + q^{(2)} + 2c\,q^{(0)} q^{(1)} - \left(2\omega_2 + \cancel{\omega_1^2}\right) A\cos\omega t = 0$

$\hookrightarrow \ddot{q}^{(2)} + q^{(2)} + 2c\left[A\cos\omega t\right]\left[\frac{cA^2}{6}\cos 2\omega t - \frac{cA^2}{2}\right] - 2\omega_2 A\cos\omega t = 0.$

$$\cos\omega t\cos 2\omega t = \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right)\left(\frac{e^{2i\omega t} + e^{-2i\omega t}}{2}\right)$$

$$= \left(\frac{e^{3i\omega t} + e^{-3i\omega t} + e^{i\omega t} + e^{-i\omega t}}{4}\right)$$

$$= \frac{1}{2}\left(\cos 3\omega t + \cos\omega t\right)$$

$\hookrightarrow \ddot{q}^{(2)} + q^{(2)} + c^2A^3\left[\frac{1}{6}\left(\cos 3\omega t + \cos\omega t\right) - \cos\omega t\right]$
$- 2\omega_2 A\cos\omega t = 0$

$\hookrightarrow \ddot{q}^{(2)} + q^{(2)} + c^2A^3\left[\frac{1}{6}\cos 3\omega t - \frac{5}{6}\cos\omega t\right]$
$- 2\omega_2 A\cos\omega t. = 0$

let $q^{(2)} = D\cos 3wt$

$$\ddot{q} = -D9w^2 \cos 3wt$$

$$\simeq -D9w_0^2 \cos 3wt$$

$$= -D9\cos 3wt \quad w_0 = 1$$

$$\hookrightarrow (-9+1)D\cos 3wt + \frac{c^2A^3}{6}\cos 3wt$$
$$-\frac{5}{6}c^2A^3\cos wt - 2w_2 A\cos wt = 0.$$

let secular terms disappear.

$$-\frac{5}{6}c^2A^3 - 2W_2 A = 0$$

$$\boxed{W_2 = -\frac{5}{12}c^2A^2}$$

$$\hookrightarrow -8D\cos 3wt + \frac{c^2A^3}{6}\cos 3wt = 0$$

$$8D = c^2A^3$$

$$8D = \frac{\quad}{6}$$

$$D = -\frac{c^2A^3}{48}$$

$$\boxed{q^{(2)} = \frac{c^2A^3}{48}\cos 3wt}$$

All together:

$$q = q^{(0)} + q^{(1)} + q^{(2)}$$

$$q \doteq \underbrace{A\cos wt}_{q^{(0)}} \underbrace{- \frac{cA^2}{2} + \frac{cA^2}{6}\cos 2wt}_{q^{(1)}} + \underbrace{\frac{c^2A^3}{48}\cos 3wt}_{q^{(2)}}$$

$$w = w_0 + w_1 + w_2 + \cdots$$

$$\doteq 1 + 0 - \frac{5}{12}c^2A^2$$

$\underline{w \doteq 1 - \frac{5}{12}c^2}$ and $A = 1$ to zeroth order

For $q(t=0) = 1$.

$$1 = A - \frac{cA^2}{\quad} + \frac{cA^2}{\quad} + \frac{c^2A^3}{\quad}$$

$\quad 2 \quad 6 \quad 48$

c) Explain why perturbation fails when $c = 0.55$

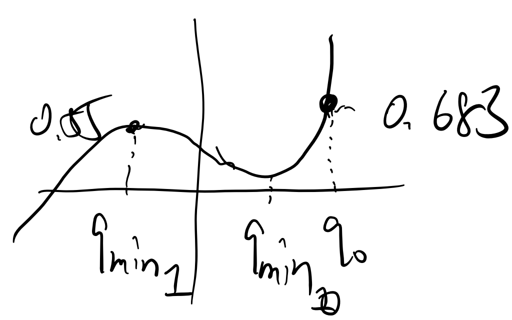
$$U = \frac{1}{2} m\omega_0^2 q^2 + \frac{c}{3} q^3$$

$$\frac{U}{m\omega_0^2 a_0^2} = \frac{1}{2}\bar{q}^2 + \frac{c\, a_0}{m\omega_0^2}\frac{1}{3}\bar{q}^3$$

$$\bar{U} = \frac{1}{2}\bar{q}^2 + \bar{c}\,\frac{1}{3}\bar{q}^3$$

For $\bar{c} = 0.55$

$$\bar{U}(\bar{q} = q_0 = 1) = \frac{1}{2} + \frac{0.55}{3} = 0.683$$



$$\frac{\partial \bar{U}}{\partial \bar{q}} = \bar{q} + \bar{c}\,\bar{q}^2 = 0$$

$$\frac{\partial \bar U}{\partial \bar q} = \bar q + c\bar q^2 = 0.$$

$$\bar q_{min} = 0 \quad \text{or} \quad 1 + c\bar q = 0$$

$$\frac{1}{-c} = \bar q$$

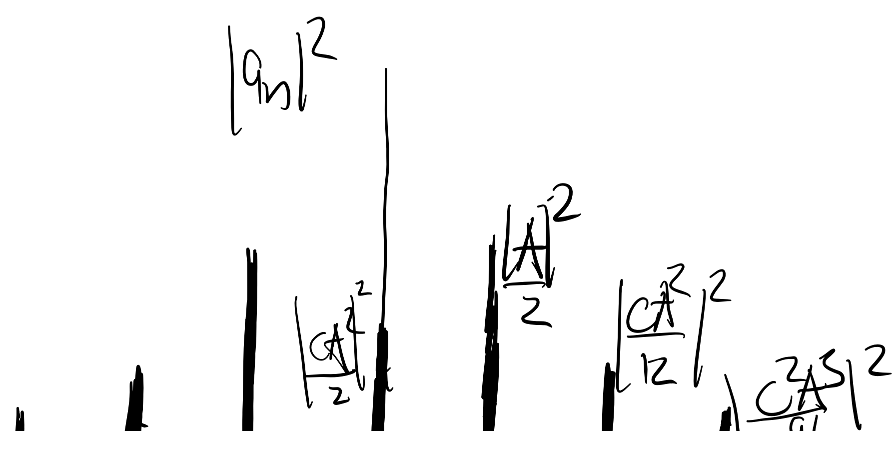
$$\frac{1}{-0.55} = \bar q_{min}$$

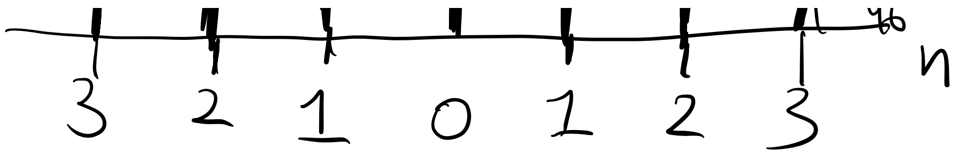
$$\bar U\left(\bar q = \frac{1}{-0.55}\right) = \frac{1}{2}\left(\frac{1}{-0.55}\right)^2 + \frac{1}{3}\left(\frac{1}{-0.55}\right)^3 0.55$$

$$= 0.55.$$

When E starts off as $E = \bar U(q = q_0, c = 0.55) = 0.683$ but there is another slope on the other side of potential, which is bound by 0.55, so there is no oscillation.

d)





$$q(t) = \sum_n q_n e^{-i2\pi n t/T}$$

$$q \doteq \underbrace{A\cos\omega t}_{q^{(0)}} \underbrace{- \frac{cA^2}{2} + \frac{cA^2}{6}\cos 2\omega t}_{q^{(1)}} + \underbrace{\frac{c^2A^3}{48}\cos 3\omega t}_{q^{(2)}}$$

$$q \doteq A\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right) - \frac{cA^2}{2} + \frac{cA^2}{6}\left(\frac{e^{2i\omega t}+e^{-2i\omega t}}{2}\right)$$
$$+ \frac{c^2A^3}{48}\left(\frac{e^{3i\omega t}+e^{-3i\omega t}}{2}\right)$$