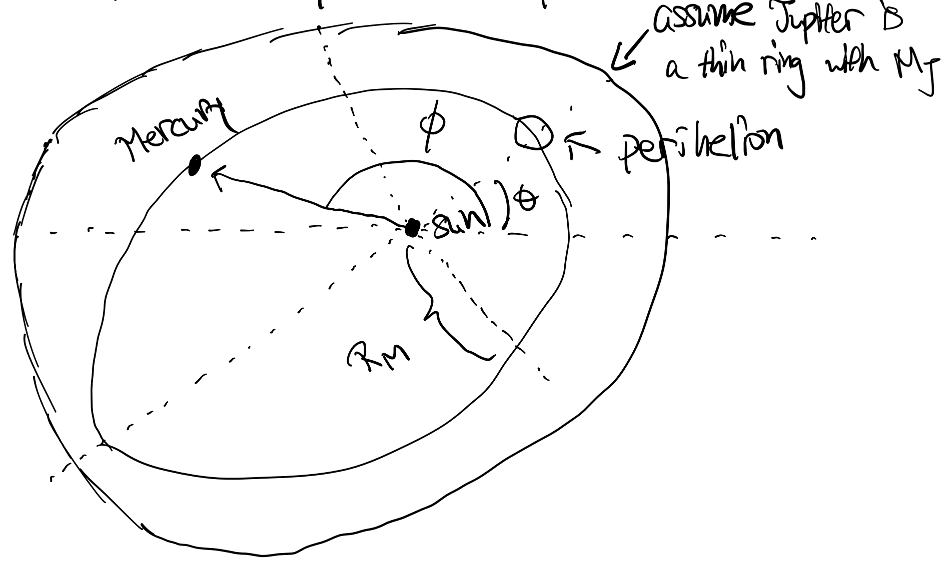
1 The precession of Mercury due to Jupiter.



a) Show that for $R_J \gg R_M$ the lagrangian of Mercury, interacting with the sun of mass $M_\odot$ and a ring of mass M_J with orbital radius R_J.

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2 \quad \text{for} \quad \alpha = \frac{GmM_J}{4R_J^3}$$

know:

$$T = \frac{1}{2}m\{\dot{r}^2 + (r\dot{\phi})^2\}$$

$$U = U(|\vec{r}-\vec{R}_J|) + \left(-\frac{GmM_\odot}{r}\right)$$

$$U = -Gm\int\frac{dM_J}{|\vec{r}-\vec{R}_J|} - \frac{GmM_\odot}{r}$$

Assume $\vec{r} \ll \vec{R}_J$:

$$\frac{1}{|\vec{r}-\vec{R}|} = \frac{1}{\sqrt{(\vec{r}-\vec{R})\cdot(\vec{r}-\vec{R})}}$$

$$\sqrt{(\vec r - \vec R_J)\cdot(\vec r - \vec R_J)}$$

$$\doteq \frac{1}{\sqrt{r^2 - 2\vec r\cdot\vec R_J + R_J^2}}$$

$$\doteq \frac{1}{\sqrt{r^2 + R_J^2 - 2rR_J\cos\psi}}$$

$$\doteq \frac{1}{R_J\sqrt{1+\left(\frac{r}{R_J}\right)^2 - 2\frac{r}{R_J}\cos\psi}}$$

know :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$$

$$\doteq \frac{1}{R_J}\left(1 - \frac{1}{2}\left\{-\frac{2r}{R_J}\cos\psi + \left(\frac{r}{R_J}\right)^2\right\} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\left\{\left(\frac{r}{R_J}\right)^2 - \frac{2r}{R_J}\cos\psi\right\}^2\right)$$

$$= \frac{1}{R_J}\left(1 + \frac{r}{R_J}\cos\psi - \frac{1}{2}\left(\frac{r}{R_J}\right)^2 + \frac{3}{8}\left\{\left(\frac{r}{R_J}\right)^4 - 4\left(\frac{r}{R_J}\right)^3\cos\psi + 4\left(\frac{r}{R_J}\right)^2\cos^2\psi\right\}\right)$$

$$O\left(\left(\frac{r}{R_J}\right)^3, \left(\frac{r}{R_J}\right)^4\right)$$

$$\frac{1}{|\vec r - \vec R_J|} \doteq \frac{1}{R_J}\left(1 + \frac{r}{R_J}\cos\psi + \frac{1}{2}\left(\frac{r}{R_J}\right)^2\left\{3\cos^2\psi - 1\right\}\right)$$

$$U(|\vec r - \vec R_J|) = -Gm\int\frac{dM_J}{|\vec r - \vec R_J|}$$

$$\doteq \frac{-Gm}{R_J}\int_0^{2\pi} 1 + \frac{r}{R_J}\cos\psi + \frac{1}{2}\left(\frac{r}{R_J}\right)^2\left\{3\cos^2\psi - 1\right\}\frac{dM_J}{dl}R_J\,d\psi$$

$$\doteq -Gm\int^{2\pi}$$

$$= \frac{\cdot}{R_J}\int_0 1 + \frac{1}{R_J}\cos\psi + \frac{1}{2}\left(\frac{\cdot}{R_J}\right)\{3\cos^2\psi - 1\}\frac{\cdot}{2\pi R_J} R_J \, d\psi$$

$$\stackrel{!}{=} \frac{-GmM_J}{2\pi R_J}\left(2\pi + 0 + \frac{1}{2}\left(\frac{r}{R_J}\right)^2\{3\pi - 2\pi\}\right)$$

$$\stackrel{!}{=} \frac{-GmM_J}{R_J} - \frac{GmM_J r^2}{4R_J^3}$$

Constant value ignorable.

Hence : $\mathcal{L} = T - V$

$$\mathcal{L} \stackrel{!}{=} \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\phi})^2 + \frac{GmM_\odot}{r} + \frac{GmM_J}{4R_J^3}r^2$$

b) i) Since mass, time, space are arbitrary, then we can set three parameters, l, m, k to be unity.

Construct unit of Length, R_0, time, τ_0, and E_0 with $l, m, k \equiv GM_\odot m$

$$l = m^2 kg\, s^{-1} \quad k = m^3 kg\, s^{-2}$$

$$E = m^2 kg\, s^{-2}$$

$$R_0 = \frac{l^2}{mk}$$

$$\tau_0 = \frac{l^3}{mk^2}$$

$$E_0 = \frac{mk^2}{l^2}$$

ii) let $\underline{r} = r/R_0 \quad \underline{t} = t/\tau_0$

original: $\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r\dot{\phi})^2 + \frac{G m M_\odot}{r} + \frac{G m M_J}{4 R_J^3} r^2$

$$\frac{dr}{dt} = \frac{d\Gamma}{dt}\frac{R_0}{\tau_0} = \frac{\frac{l^2}{mk}}{\frac{l^3}{mk^2}} = \frac{k}{l}$$

$$\mathcal{L} = \frac{1}{2} m \left(\frac{R_0}{\tau_0}\right)^2 \left(\frac{d\Gamma}{dt}\right)^2 + \frac{1}{2} m \frac{R_0^2}{\tau_0^2} \Gamma^2 \left(\frac{d\phi}{dt}\right)^2 + \frac{k}{\Gamma R_0} + \frac{G m M_J}{4 R_J^3} R_0^2 \Gamma^2$$

$$= \frac{1}{2} \frac{m k^2}{l^2} \left(\frac{d\Gamma}{dt}\right)^2 + \frac{1}{2} \frac{m k^2}{l^2} \Gamma^2 \left(\frac{d\phi}{dt}\right)^2 + \frac{m k^2}{l^2} \frac{1}{\Gamma} + \frac{G m M_J}{4 R_J^3} \left(\frac{l^2}{mk}\right)^2 \Gamma^2$$

$$\frac{\mathcal{L}}{\frac{mk^2}{l^2}} = \frac{\mathcal{L}}{E_0} = \frac{1}{2}\left(\frac{d\Gamma}{dt}\right)^2 + \frac{1}{2}\Gamma^2\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{\Gamma} + \underbrace{\frac{M_J}{4 R_J^3} \frac{\cancel{k}}{M_\odot} R_0^2 \frac{l^3}{mk^2}}_{\frac{M_J}{4 M_\odot}\left(\frac{R_0}{R_J}\right)^3 = \alpha} \Gamma^2$$

$$\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\phi}^2 + \frac{1}{r} + \alpha r^2$$

iii)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt}\left(r^2 \dot{\phi}\right) = 0$$

$P_\phi = r^2 \dot{\phi} =$ constant.

$\dot{\phi} = \frac{P_\phi}{r^2}$ if let $P_\phi = 1$

$$\dot{\phi} \stackrel{!}{=} \frac{1}{r^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial}{\partial r} L$$

$$\frac{d}{dt}(\dot{r}) = \ddot{r} \stackrel{!}{=} r\dot{\phi}^2 - \frac{1}{r^2} + 2\alpha r$$

$$\ddot{r} = r\left(\frac{p_\phi}{r^2}\right)^2 - \frac{1}{r^2} + 2\alpha r$$

$$\stackrel{!}{=} \frac{p_\phi^3}{r^3} - \frac{1}{r^2} + 2\alpha r$$

$$-\frac{\partial}{\partial r} V_{eff} \stackrel{!}{=} \frac{1}{r^3} - \frac{1}{r^2} + 2\alpha r$$

$$V_{eff}(r) = -\left\{\frac{-1}{2r^2} + \frac{1}{r} + \alpha r^2\right\}$$

$$V_{eff}(r) \stackrel{!}{=} \frac{1}{2r^2} - \frac{1}{r} - \alpha r^2$$

$$\ddot{r} = -\frac{\partial}{\partial r} V_{eff}(r)$$

c) determine the radius $r_{min}(\alpha)$ for the circular orbit to first order in α.

Find r_{min}:

$$\frac{\partial V_{eff}}{\partial r} = \frac{-1}{r^3} + \frac{1}{r^2} - 2\alpha r = 0$$

$$\hookrightarrow \underbrace{2\alpha r^4}_{\text{small}} - r + 1 = 0$$

here α is small to first order.

0^{th} order, ignore α since α is first order.

$$-r+1=0$$
$$r=1$$

$$r = r_0 + \delta r = 1 + \delta r + \mathcal{O}(\delta r^2)$$

$$2\alpha(1+\delta r)^4 - (1+\delta r) + 1 = 0.$$
$$\hookrightarrow \quad 2\alpha(1+2\delta r)^2 - \delta r = 0$$
$$2\alpha(1+4\delta r) - \delta r = 0.$$
$$2\alpha + \underbrace{\cancel{8\alpha\delta r}}_{\mathcal{O}(2)} - \delta r = 0$$

$$2\alpha = \delta r$$

then $r_{min}(\alpha) = r_0 + \delta r = \underset{r_0}{1} + \underset{\delta r}{2\alpha}$

d) Find radial oscillation for slight disturbance from circular orbit of the previous part

$$\ddot{r} = \frac{1}{r^3} - \frac{1}{r^2} + 2\alpha r$$

let $r(t) = r_{min}(\alpha) + \delta r(t)$. ← perturbation.

$$\frac{d^2}{dt^2} r = \frac{d^2}{dt^2} r_{min} + \frac{d^2}{dt^2} \delta r(t) = \ddot{\delta r}(t)$$

$$\ddot{\delta r} = \frac{1}{(1+2\alpha+\delta r)^3} - \frac{1}{(1+2\alpha+\delta r)^2} + 2\alpha(1+2\alpha+\delta r)$$

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}$$

$$(1+2\alpha+\delta r)^{-3} = 1-3(2\alpha+\delta r)+\frac{-3(-3-1)}{2}(2\alpha+\delta r)^2$$

$$\doteq 1-6\alpha-3\delta r+6(4\alpha^2+\delta r^2+4\alpha\delta r)$$

$$\doteq 1-6\alpha-3\delta r+24\alpha\delta r$$

$$(1+2\alpha+\delta r)^{-2} = 1-2(2\alpha+\delta r)+\frac{-2(-2-1)}{2}(2\alpha+\delta r)^2$$

$$\doteq 1-4\alpha-2\delta r+3(2\alpha^2+\delta r^2+4\alpha\delta r)$$

$$\doteq 1-4\alpha-2\delta r+12\alpha\delta r$$

$$\ddot{\delta r} = \cancel{1}\cancel{-6\alpha}-3\delta r+24\alpha\delta r\cancel{-1}\cancel{+4\alpha}+2\delta r-12\alpha\delta r\cancel{+2\alpha}+2\alpha\delta r$$

$$\doteq -\delta r+14\alpha\delta r$$

$$\ddot{\delta} \doteq \underbrace{(-1+14\alpha)}_{=-\omega^2}\delta r \leftarrow \text{SHO}$$

$$-\omega^2 = -1+14\alpha$$

$$-\ \omega = \sqrt{1-14\alpha} \cong 1-7\alpha$$

$$\omega = 2\pi f = \frac{2\pi}{T} = 1-7\alpha$$

$$\boxed{T = \frac{2\pi}{1-7\alpha} \cong 2\pi(1+7\alpha)}$$

e) Show that the angle of perihelion of ellipse θ changes by $\Delta\theta = 6\pi\alpha$ every time the $\underline{\text{particle reaches}}$ $\underbrace{\text{the closest approach.}}_{\text{perihelion.}}$ after one period τ.

how to relate θ with r, ϕ.

θ is $\phi(r = r_{closest})$

$P_\phi = r^2\dot{\phi}$ ← constant. or $\dot{\phi} = \frac{1}{r^2}$

$$\dot{\phi} = r^{-2}$$

$$\int_{\phi(0)}^{\phi(T)} d\phi = \int_0^T r^{-2}\, dt$$

↳ $$2\pi + \Delta\theta = \int_0^T (1+2\alpha+\delta r)^{-2}\, dt$$

$$\stackrel{!}{=} \int 1 - 2(2\alpha+\delta r) + \frac{-2(-2-1)}{2}(2\alpha+\delta r)^2\, dt$$

$$\stackrel{!}{=} \int 1 - 4\alpha - 2\delta r + 3(4\alpha\delta r)\, dt$$

$$\stackrel{!}{=} (1-4\alpha)T + \int_0^T -2\delta r + 12\alpha\delta r\, dt$$

From part d) $\delta r(t) = A\cos\omega t + B\sin\omega t$

$$\stackrel{!}{=} (1-4\alpha)\tau + \int_0^T (-2+12\alpha)\, A\cos\omega t + B\sin\omega t\, dt$$

$$\stackrel{!}{=} (1-4\alpha)T + (-2+12\alpha)\left[\frac{A}{\omega}\sin\omega t\Big|_0^T - \frac{B}{\omega}\cos\omega t\Big|_0^T\right]$$

$$\underbrace{\qquad}_{=0} \qquad \underbrace{\qquad}_{=0}$$

$$\stackrel{!}{=} (1-4\alpha)T$$

$$\stackrel{!}{=} (1-4\alpha)2\pi(1+7\alpha)$$

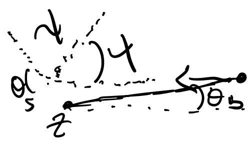
$$\stackrel{!}{=} 2\pi(1+7\alpha-4\alpha-\cancel{28\alpha^2})$$

$$\cancel{2\pi}+\Delta\theta \stackrel{!}{=} \cancel{2\pi}+6\pi\alpha$$

$$\boxed{\Delta\theta = 6\pi\alpha}$$

2) A particle of mass μ moves in the repulsive $1/r^2$ potential.

$$U(r) = \frac{h}{r^2}, \quad h > 0$$



a) Find equation for $r(\phi)$ with energy E and angular momentum l.

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu(r\dot{\phi})^2 - \frac{h}{r^2}$$

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$\frac{d}{dt}(\mu\dot{r}) = \frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 + \frac{h}{2r^3}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \mu r^2\dot{\phi} = p_\phi \qquad \dot{\phi} = \frac{p_\phi}{\mu r^2}$$

$$H = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{p_\phi^2}{2\mu r^2} + \frac{h}{r^2}}_{V_{eff}}$$

$$2\left\{E - \left(\frac{p_\phi^2}{\quad} + \frac{h}{r^2}\right)\right\} = \quad dr$$

$$\sqrt{\frac{2}{\mu}\left(E - \frac{\quad}{2\mu r^2} \quad\right)} = \frac{\quad}{dt}$$

$$\sqrt{\frac{\mu}{2}}\int\frac{dr}{\sqrt{E-\frac{p_\phi^2}{2\mu r^2}-\frac{h}{r^2}}} = \int dt = \int \frac{dt}{d\phi}\,d\phi$$

$$\sqrt{\frac{\mu}{2}}\int\frac{dr}{\sqrt{E-\frac{p_\phi^2}{2\mu r^2}-\frac{h}{r^2}}} = \int_{\phi_0}^{\phi}\frac{\mu r^2}{p_\phi}\,d\phi$$

$$\frac{l}{\sqrt{2\mu}}\int_{\infty}^{r}\frac{dr}{r^2\sqrt{E-\left(\frac{l^2}{2\mu}+h\right)\frac{1}{r^2}}} = \phi$$

let $u = \frac{1}{r}$

$$\frac{du}{dr} = -\frac{1}{r^2}$$

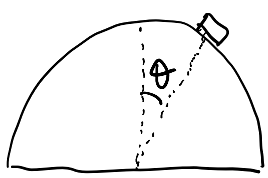
$$dr = -r^2 du$$

$$\hookrightarrow \frac{l}{\sqrt{2\mu}}\int_{\frac{1}{r}}^{\infty}\frac{du}{\sqrt{E-\lambda u^2}} = \phi$$

let $\lambda = \frac{l^2}{2\mu}+h$ $\quad\hookrightarrow$

3) A hoop on a cylinder.

a)



$$\psi = r - R = 0$$

$$x = r\sin\theta \qquad y = r\cos\theta$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m r^2\dot{\theta}^2$$

$$U = mgy = mgr\cos\theta$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta + \lambda(r-R)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{d}{dt}(m\dot r) = \frac{\partial L}{\partial r} = mr\dot\theta^2 - mg\cos\theta + \lambda \quad ①$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{d}{dt}(mr^2\dot\theta) = \frac{\partial L}{\partial \theta} = mgr\sin\theta \quad ②$$

$$\frac{\partial L}{\partial \lambda} = 0 = r - R \Rightarrow r = R \quad ③$$

$$\hookrightarrow \dot r = 0$$

From ①: $\frac{d}{dt}(m\dot r) = 0 = mR\dot\theta^2 - mg\cos\theta + \lambda$

$$\lambda = -mR\dot\theta^2 + mg\cos\theta$$

Initial at rest $E = mgR\cos(\theta=0)$

$$E \stackrel{!}{=} mgR$$

$$mgR = \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 + mgr\cos\theta - \lambda(r-R)$$

if $r = R$

$$mgR = \frac{1}{2}mR^2\dot\theta^2 + mgR\cos\theta$$

$$\dot\theta^2 = mgR(1-\cos\theta)\frac{2}{mR^2} \xrightarrow{} 2\frac{g}{R}(1-\cos\theta) \leftarrow \dot\theta^2 \text{ when block doesn't fall off}$$

$$\lambda = -mR\left(\frac{2g}{R}(1-\cos\theta)\right) + mg\cos\theta$$

$$\stackrel{!}{=} -2mg(1-\cos\theta) + mg\cos\theta$$

$\stackrel{!}{=} -2mg + 2mg\cos\theta + mg\cos\theta$

$\lambda \stackrel{!}{=} -2mg + 3mg\cos\theta$

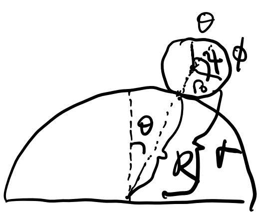
$\lambda \stackrel{!}{=} mg(\underbrace{-2+3\cos\theta})$

block fall off when supporting force goes zero
which means there is no constraint.

$-2+3\cos\theta = 0$

$\boxed{\cos\theta = 2/3}$

b) Now consider a hoop of mass m and radius, r_0, on a fixed cylinder with radius R.



i) Determine the relaxation between X and Y of a point on the rim of the loop in r, θ, ψ

$X = r\sin\theta + r_0 \sin\psi$

$Y = r\cos\theta + r_0\cos\psi$

ii) $\dot{x} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} + r_0\cos\psi\,\dot{\psi}$

$\dot{y} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} - r_0\sin\psi\,\dot{\psi}$

$\dot{x}^2 + \dot{y}^2 = \dot{r}^2\cancel{\sin^2\theta} + \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}} + 2\dot{r}r_0\cos\psi\sin\theta\,\dot{\psi}$
$+ r^2\cancel{\cos^2\theta}\,\dot{\theta}^2 + 2rr_0\cos\theta\cos\psi\,\dot{\theta}\dot{\psi} + r_0^2\cancel{\cos^2\psi}\,\dot{\psi}^2$
$+ \dot{r}^2\cancel{\cos^2\theta} - \cancel{2r\dot{r}\cos\theta\sin\theta\,\dot{\theta}} - 2\dot{r}r_0\cos\theta\sin\psi\,\dot{\psi}$
$+ r^2\cancel{\sin^2\theta}\,\dot{\theta}^2 + 2rr_0\sin\theta\sin\psi\,\dot{\theta}\dot{\psi} + r_0^2\cancel{\sin^2\psi}\,\dot{\psi}^2$

$$T = \frac{1}{2}\int dm\,(\dot{x}^2 + \dot{y}^2)$$

$$\doteq \frac{1}{2}\int \frac{dm}{dl}\, r_0\, d\psi \Big[\dot r^2 + r^2\dot\theta^2 + r_0^2\dot\psi^2 + \cancel{2r_0\,\dot r\dot\psi(\cos\psi\sin\theta)}$$

$$+\cancel{2 r r_0 \cos\theta\cos\psi\,\dot\theta\dot\psi} - \cancel{2\dot r r_0\cos\theta\sin\psi\,\dot\psi}$$

$$+\cancel{2 r r_0 \sin\theta \sin\psi\,\dot\theta\dot\psi}\Big]$$

$$= \frac{1}{2}\left(\frac{m}{2\pi r_0}\right) 2\pi r_0 \left[\dot r^2 + r^2\dot\theta^2 + r_0^2\dot\psi^2\right]$$

$$T \doteq \frac{1}{2} m\dot r^2 + \frac{1}{2} m r^2\dot\theta^2 + \frac{1}{2} m r_0^2\dot\psi^2$$

$$U = mgy$$

$$\mathcal{L} = \frac{1}{2}m\dot r^2 + \frac{1}{2}m r^2\dot\theta^2 + \frac{1}{2}m r_0^2\dot\psi^2 - mg[r\cos\theta + r_0\cos\psi] + \lambda(r - R - r_0)$$

iii) Determine relation between $d\theta$ and $d\psi$ if non-slip

$$\psi = \theta + \phi$$

$$r_0\phi = R\theta$$

$$r_0(\psi - \theta) = R\theta$$

$$r_0\psi = (R + r_0)\theta$$

$$r_0\psi = r\theta$$

$$d\psi = \frac{R + r_0}{r_0}\, d\theta \quad \text{or} \quad \dot\psi = \frac{R + r_0}{r_0}\,\dot\theta = \frac{r}{r_0}\,\dot\theta$$

iv)

$$\mathcal{L} = \frac{1}{2}m\dot r^2 + \frac{1}{2}m r^2\dot\theta^2 + \frac{1}{2}m r_0^2\dot\psi^2 - mg[r\cos\theta] + \lambda_1(r - R - r_0)$$
$$+ \lambda_2\big((R + r_0)\theta - r_0\psi\big)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) = \frac{d}{dt}(m\dot r) = \frac{\partial \mathcal{L}}{\partial r} = m r\dot\theta^2 \ldots$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{d}{dt}(m\dot r) = \frac{\partial L}{\partial r} = mr\dot\theta^2 - mg\cos\theta + \lambda_1$ ①

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{d}{dt}(2mr^2\dot\theta) = \frac{\partial L}{\partial \theta} = mgr\sin\theta + \lambda_2(R+r_0)$ ②

$r = R + r_0$ ③ $\quad (R+r_0)\theta = r_0\psi$ ④

$E_{init} = mgr = mg(R+r_0)$ with constraint $r = R+r_0$

$mgr = \frac{1}{2}mr^2\dot\theta^2 + \frac{1}{2}mr_0^2\left(\frac{R+r_0}{r_0}\right)^2\dot\theta^2 + mgr\cos\theta$

$mgr = mr^2\dot\theta^2 + mgr\cos\theta$

$\dot\theta^2 = \frac{mgr}{mr^2}(1-\cos\theta)$

$\dot\theta^2 = \frac{g}{r}(1-\cos\theta) \leftarrow \dot\theta^2$ when constraints apply

If hoop falls off, then $\lambda = 0$, $\dot r = \ddot r = 0$ with $r = R + r_0$

Use ①: $\frac{d}{dt}(m\dot r) = mr\dot\theta^2 - mg\cos\theta + \lambda_1 = 0$

$\lambda_1 = -mr\dot\theta^2 + mg\cos\theta$

$\stackrel{1}{=} -mr\frac{g}{r}(1-\cos\theta) + mg\cos\theta$

$\stackrel{1}{=} mg(-1 + \cos\theta + \cos\theta)$

$\stackrel{1}{=} mg(\underbrace{-1 + 2\cos\theta}_{=0})$

Need to let $\lambda_1 = 0$

$-1 + 2\cos\theta = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$