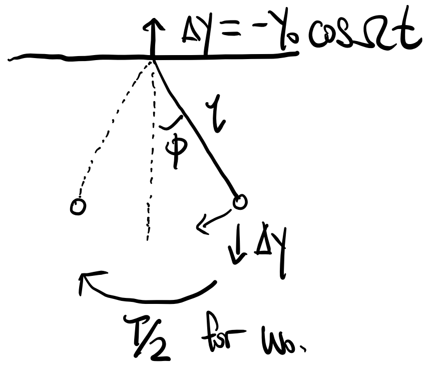
Parametric Resonance: (resonance frequency changes)

So:

$$\frac{d^2q}{dt^2} + \omega_0^2\left(1 + \underbrace{h}_{\text{small drive amplitude}} \cos \overbrace{\Omega}^{\text{choose } \Omega = 2\omega_0 + \epsilon} t\right) q = 0$$

$$\frac{d^2q}{dt^2} + \omega_0^2 q = -\omega_0^2 h \cos \Omega t \; q$$



Here drop at top by Δy
pull up at bottom by Δy

so let $\Delta y = -y_0 \cos \Omega t$

here $\Omega = 2\omega_0$ since we complete 1 cycle of pulling and dropping as it swings over half the period

Then $\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$

$= \frac{1}{2} m(\dot{l}^2 + l^2\dot{\phi}^2) + mgl\cos\phi$

$l(t) = L - y_0 \cos \Omega t$

$\dot{l} = \Omega y_0 \sin \Omega t$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{d}{dt}(m l^2 \dot{\phi}) = m(\ddot{\phi} l^2 + \dot{\phi}\, 2l\dot{l})$$

$\partial \mathcal{L}$ ... $l \sin\phi$

$$\frac{\partial}{\partial\phi} = -mgl\sin\phi$$

$$\ddot{\phi} + \dot{\phi}\frac{2\dot{l}}{ml} = -\frac{mg}{l}\sin\phi = -\frac{g}{l}\sin\phi.$$

$$\ddot{\phi} + \frac{2}{m}\dot{\phi}\left(\frac{\Omega y_0 \sin\Omega t}{L - y_0\cos\Omega t}\right) = \frac{-g}{L - y_0\cos\Omega t}\sin\phi.$$

$$\stackrel{!}{=} \frac{-g}{L}\left(1+\frac{y_0}{L}\cos\Omega t\right)\sin\phi.$$

$$\ddot{\phi} + \frac{2}{m}\frac{y_0}{L}\Omega\left(1+\frac{y_0}{L}\cos\Omega t\right)\sin\Omega t\,\dot{\phi} \stackrel{!}{=} -w_0^2\left(1+\frac{y_0}{L}\cos\Omega t\right)\phi \quad \text{for } \phi \ll 1$$

↳ ignore y_0^2 terms.

$$\ddot{\phi} + \underbrace{\frac{2}{m}\frac{y_0}{L}\Omega\sin\Omega t\,\dot{\phi}}_{\frac{d}{dt}(\sin\Omega t\,\phi) - \Omega\cos\Omega t\,\phi} = \ddot{\phi} - \frac{2}{m}\frac{y_0}{L}\Omega^2\cos\Omega t\,\phi.$$

$$\Rightarrow \quad \ddot{\phi} + w_0^2\left(1+\frac{y_0}{L}\cos\Omega t\right)\phi = 0.$$

<u>Consider unperturbed oscillator:</u>

take Δy very small, so motion over short times is unperturbed

Tension at top: $T^y_{min} = mg\cos\phi_{max}$

$$\stackrel{!}{=} mg\left(1 - \frac{\phi_{max}^2}{2}\right)$$

potential.

$$\stackrel{!}{=} mg - E/L \qquad \Rightarrow E \propto \phi_{max}^2 = \frac{1}{2}mgL\,\phi_{max}^2$$

$\stackrel{!}{=}$ ...

$= mgy.$

Tension at bottom:

$$T_{max} = mg + \frac{mv^2}{2} = mg + \frac{2E}{L}$$

Then work done by hand:

$$W = T_{max}\,\Delta y - T_{min}\,\Delta y$$

$$\stackrel{!}{=} \frac{3E}{2}\Delta y$$ ← over half cycle. or $\Delta T = \frac{T}{2}$

Then

$$\dot{E} = \frac{\Delta E}{\Delta T} = \frac{\Delta E}{T/2} = \frac{6E\Delta y}{TL}$$

$$\hookrightarrow E(t) = E_0 e^{\frac{6\Delta y}{TL}t}$$ ← energy grows exponentially in time, which causes runaway.

If $\Omega = 2\omega_0$, then we're at resonance. as we saw previously.

So we need to detune Ω by ϵ,

$$\Omega = 2\omega_0 + \epsilon.$$

Again, have equation:

$$\ddot{q} + \omega_0^2(1 + h\cos\Omega t)q = 0$$

$$\Rightarrow \ddot{q} + \omega_0^2 q = -h\cos\Omega t\, q$$

Here $q \propto e^{\pm i\omega_0 t}$ and $\cos\Omega t \propto e^{\pm i\Omega t}$

in order for

$$\cos\Omega t \, q \propto e^{i\omega_0 t}$$

need $\Omega = 2\omega_0$ ← resonance condition.

To detune:

let $\Omega = 2\omega_0 + \epsilon$ and $2\omega = 2\omega_0 + \epsilon$

let $q(t) = q^{(0)} + q^{(1)} + \cdots$

let $q^{(0)} = a(t)\cos(-\omega t + \phi(t))$

$$\stackrel{!}{=} a(t)\left[\cos(-\omega t)\cos(\phi) - \sin(-\omega t)\sin\phi\right]$$

$$\stackrel{!}{=} \underset{\uparrow}{a(t)}\cos(\omega t) + \underset{\uparrow}{b(t)}\sin\omega t$$

incorporate $\phi(t)$ into amplitude.

$$\dot{q}^{(0)} = \dot{a}\cos\omega t - a\omega\sin\omega t + \dot{b}\sin\omega t + b\omega\cos\omega t$$

$$\ddot{q}^{(0)} = -\omega\dot{a}\sin\omega t - \dot{a}\omega\sin\omega t - a\omega^2\cos\omega t + \ddot{a}\cos\omega t$$
$$+ \dot{b}\omega\cos\omega t + \dot{b}\omega\cos\omega t - b\omega^2\sin\omega t + \ddot{b}\sin\omega t$$

$$\stackrel{!}{=} -2\dot{a}\omega\sin\omega t + 2\dot{b}\omega\cos\omega t - \omega^2(a\cos\omega t + b\sin\omega t) + \mathcal{O}(\ddot{a}, \ddot{b})$$

Then

$$\ddot{q}^{(0)} + \omega_0^2 q^{(0)} + \ddot{q}^{(1)} + \omega_0^2 q^{(1)} = -h\cos 2\omega t \, q^{(0)}$$

$$\hookrightarrow \ddot{q}^{(1)} + \omega_0^2 q^{(1)} + \underbrace{(\omega_0^2 - \omega^2)}_{-\epsilon\omega_0} q^{(0)} = 2\dot{a}\omega\sin\omega t - 2\dot{b}\omega\cos\omega t$$
$$- h\cos 2\omega t\left[a\cos\omega t + b\sin\omega t\right]$$

since $\omega = \omega_0 + \frac{\epsilon}{2}$

$\omega^2 = \omega_0^2 + \epsilon\omega_0$

Find $-h\cos 2\omega t[a\cos\omega t + b\sin\omega t]$

$$= -ha\left[\frac{e^{i2\omega t}+e^{-i2\omega t}}{2}\right]\left[\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right] - hb\left[\frac{e^{i2\omega t}+e^{-i2\omega t}}{2}\right]\left[\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right]$$

$$= \frac{-ha}{4}\left\{e^{i3\omega t}+e^{i\omega t}+e^{-i\omega t}+e^{-i3\omega t}\right\} - \frac{hb}{4i}\left[e^{i3\omega t}-e^{i\omega t}+e^{-i\omega t}-e^{-3i\omega t}\right]$$

$$\stackrel{!}{=} -\frac{ha}{2}\left(\cos 3\omega t + \cos\omega t\right) - \frac{hb}{2}\left[\sin 3\omega t - \sin\omega t\right]$$

then:

$$\ddot{q}^{(1)} + \underbrace{\omega_0^2}_{\approx\omega^2} q^{(1)} = 2\dot{a}\omega\sin\omega t - 2\dot{b}\omega\cos\omega t + \epsilon\omega_0[a\cos\omega t + b\sin\omega t]$$

$$-\frac{ha}{2}\left(\cos 3\omega t + \cos\omega t\right) - \frac{hb}{2}\left[\sin 3\omega t - \sin\omega t\right]$$

Remove secular terms:

let $\omega_0 \cong \omega$ to first order.

$$\left(-2\dot{b}\omega + \epsilon\omega a - \frac{ha}{2}\right)\cos\omega t = 0$$

$$\left(2\dot{a}\omega + \epsilon\omega b + \frac{hb}{2}\right)\sin\omega t = 0$$

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & \frac{h}{4\omega}+\frac{\epsilon}{2} \\ \frac{h}{4\omega}-\frac{\epsilon}{2} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_+ E_+ + \lambda_- E_- = 0$$

In general, these have a solution form of:

$$\begin{pmatrix} a \\ b \end{pmatrix} = C_+ e^{\lambda_+ t}(E_+) + C_- e^{\lambda_- t}(E_-)$$

Recall: If $\vec{x}(t) = \vec{\eta} e^{\lambda t} \Rightarrow \dot{\vec{x}} = \lambda \vec{\eta} e^{\lambda t}$
and $\dot{\vec{x}} = A\vec{x}$
$\lambda \vec{\eta} e^{\lambda t} = A \vec{\eta} e^{\lambda t}$
$(A - \lambda I)\vec{\eta} = 0.$ $\therefore$ λ and $\vec{\eta}$ are eigenvalues and vectors of A.

First find eigenvalues λ:

$$\begin{pmatrix} 0 & \frac{h}{4w} + \frac{\epsilon}{2} \\ \frac{h}{4w} - \frac{\epsilon}{2} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

$$\det \begin{pmatrix} -\lambda & \frac{h}{4w} + \frac{\epsilon}{2} \\ \frac{h}{4w} - \frac{\epsilon}{2} & -\lambda \end{pmatrix} = \lambda^2 - \left[\left(\frac{h}{4w}\right)^2 - \left(\frac{\epsilon}{2}\right)^2 \right] = 0.$$

$$\hookrightarrow \lambda = \pm \sqrt{\left(\frac{h}{4w}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}$$

Note: Since solution goes like $e^{\lambda t}$

If $\frac{\epsilon}{2} < \frac{h}{4w}$, then we have real roots, which causes exponential growth

If $\frac{\epsilon}{2} > \frac{h}{4w}$, then we have imaginary roots, causes oscillatory behavior.

Now solve for eigenvectors by plug into $(A - \lambda I)\vec{\eta} = 0$ and solve for $\vec{\eta}$.

For $\lambda = -\sqrt{\left(\frac{h}{4w}\right)^2 + \left(\frac{\epsilon}{2}\right)^2}$

$$\begin{pmatrix} \sqrt{\left(\frac{h}{4w}\right)^2 - \left(\frac{\epsilon}{2}\right)^2} & \frac{h}{4w} + \frac{\epsilon}{2} \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} c_1^{+} \\ \cdots \end{pmatrix}$$

$$\begin{pmatrix} \frac{h}{4w} - \frac{\epsilon}{2} & \sqrt{\left(\frac{h}{4w}\right)^2 - \left(\frac{\epsilon}{2}\right)^2} \end{pmatrix} \begin{pmatrix} e'_2 \end{pmatrix}$$

$$\sqrt{\left(\frac{h}{4w}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}\, e_1^\dagger + \left(\frac{h}{4w} + \frac{\epsilon}{2}\right) e_2^\dagger = 0$$

$$\left(\frac{h}{4w} - \frac{\epsilon}{2}\right) e_1^\dagger + \sqrt{\left(\frac{h}{4w}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}\, e_2^\dagger = 0$$