1) Parametric Resonance with Damping:

Consider oscillator with small damping η,
a time dependent mass, $m(t) = m_0(1 + \mu\cos\Omega t)$
small

The frequency $\Omega \cong 2\omega_0 + \epsilon$, ϵ is small

Have EOM:

$$m_0 \frac{d}{dt}\{(1+\mu\cos\Omega t)\dot{q}\} + m_0\omega_0^2 q + m_0\eta\dot{q} = 0$$

a) Determine the regions in ϵ, μ, where the oscillations are stable and unstable.

$$m_0 \frac{d}{dt}\{(1+\mu\cos\Omega t)\dot{q}\} + m_0\omega_0^2 q + m_0\eta\dot{q} = 0$$

$$\hookrightarrow m_0\left[(1+\mu\cos\Omega t)\ddot{q} - \mu\Omega\sin\Omega t\,\dot{q}\right] + m_0\omega_0^2 q + m_0\eta\dot{q} = 0$$

$$\hookrightarrow \cancel{m_0}(1+\mu\cos\Omega t)\ddot{q} + \cancel{m_0}\omega_0^2 q + \cancel{m_0}(\eta - \mu\Omega\sin\Omega t)\dot{q} = 0$$

let $\Omega = 2\omega = 2\omega_0 + \epsilon$

let $q(t) = q^{(0)} + q^{(1)}$

$q^{(0)} = a(t)\cos\omega t + b(t)\sin\omega t$

$$\dot q^{(0)} = \dot a\cos\omega t - a\omega\sin\omega t + \dot b\sin\omega t + b\omega\cos\omega t$$

$$\ddot q^{(0)} = \cancel{\ddot a\cos\omega t} - \dot a\omega\sin\omega t - \dot a\omega\sin\omega t - a\omega^2\cos\omega t$$

$$\cancel{\ddot b\sin\omega t} + \dot b\omega\cos\omega t + \dot b\omega\cos\omega t - b\omega^2\sin\omega t$$

$$\ddot q^{(0)} + \ddot q^{(1)} + \ddot q^{(0)} u\cos\Omega t + \omega_0^2 q^{(0)} + \omega_0^2 q^{(1)} + (\eta - u\Omega\sin\Omega t)\dot q^{(0)} = 0$$

To zeroth order:

$$\ddot q^{(0)} + \omega_0^2 q^{(0)} = 0.$$

$$-\omega^2(a\cos\omega t + b\sin\omega t) + \omega_0^2(a\cos\omega t + b\sin\omega t)$$

$$\omega = \omega_0 + \frac{\epsilon}{2}$$

$$\omega^2 = \omega_0^2 + \epsilon\omega_0$$

↳ Additional first order term:

$$-\epsilon\omega_0(a\cos\omega t + b\sin\omega t) - 2\dot a\omega\sin\omega t + 2\dot b\omega\cos\omega t$$

To First order.

$$\ddot q^{(1)} + \omega_0^2 q^{(1)} + \ddot q^{(0)} u\cos 2\omega t + (\eta - u\Omega\sin 2\omega t)\dot q^{(0)} - \epsilon\omega_0 q^{(0)}$$

$$-2\dot a\omega\sin\omega t + 2\dot b\omega\cos\omega t = 0$$

↳ $$\ddot q^{(1)} + \omega_0^2 q^{(1)} - \omega^2 u\cos 2\omega t(a\cos\omega t + b\sin\omega t)$$

$$+ (\eta - u2\omega\sin 2\omega t)\omega(-a\sin\omega t + b\cos\omega t)$$

$$-\epsilon\omega_0(a\cos\omega t + b\sin\omega t)$$

$$-2\dot a\omega\sin\omega t + 2\dot b\omega\cos\omega t = 0$$

$$\cos(2\omega t)\cos\omega t = \frac{\left(e^{2i\omega t}+e^{-2i\omega t}\right)\left(e^{i\omega t}+e^{-i\omega t}\right)}{4}$$

$$\stackrel{!}{=} \frac{e^{3i\omega t}+e^{-3i\omega t}+e^{i\omega t}+e^{-i\omega t}}{4}$$

$$\stackrel{!}{=} \frac{1}{2}\left(\cos 3\omega t+\cos \omega t\right)$$

$$\cos 2\omega t \sin \omega t = \left(\frac{e^{2i\omega t}+e^{-2i\omega t}}{2}\right)\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)$$

$$\stackrel{!}{=} \frac{e^{3i\omega t}-e^{-3i\omega t}-e^{i\omega t}+e^{-i\omega t}}{4i}$$

$$\stackrel{!}{=} \frac{1}{2}\left(\sin 3\omega t-\sin \omega t\right)$$

$$\sin 2\omega t \cos \omega t = \left(\frac{e^{2i\omega t}-e^{-2i\omega t}}{2i}\right)\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)$$

$$\stackrel{!}{=} \frac{e^{3i\omega t}-e^{-3i\omega t}+e^{i\omega t}-e^{-i\omega t}}{4i}$$

$$\stackrel{!}{=} \frac{1}{2}\left(\sin 3\omega t+\sin \omega t\right)$$

$$\sin 2\omega t \sin \omega t = \left(\frac{e^{2i\omega t}-e^{-2i\omega t}}{2i}\right)\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)$$

$$\stackrel{!}{=} -\frac{1}{2}\left(\frac{e^{3i\omega t}+e^{-3i\omega t}-e^{i\omega t}-e^{-i\omega t}}{2}\right)$$

$$\stackrel{!}{=} -\frac{1}{2}\left(\cos 3\omega t-\cos \omega t\right)$$

$$\ddot{q}^{(1)} + \omega_0^2 q^{(1)} - \omega^2 u \cos 2\omega t\,(a\cos\omega t + b\sin\omega t)$$
$$+ (\eta - u 2\omega \sin 2\omega t)\,\omega(-a\sin\omega t + b\cos\omega t)$$
$$- \epsilon\omega_0 (a\cos\omega t + b\sin\omega t)$$
$$- 2\mathring{a}\omega\sin\omega t + 2\mathring{b}\,\omega\cos\omega t = 0$$

$$\hookrightarrow\ \ddot{q}^{(1)} + \omega_0^2 q^{(1)} - \frac{\omega^2 u}{2}\{a(\cos 3\omega t + \cos\omega t) + b(\sin 3\omega t - \sin\omega t)\}$$
$$(-u2\omega)\frac{\omega}{2}\{a(\cos 3\omega t - \cos\omega t) + b(\sin 3\omega t + \sin\omega t)\}$$
$$- \epsilon\omega_0 (a\cos\omega t + b\sin\omega t)$$
$$- 2\mathring{a}\omega\sin\omega t + 2\mathring{b}\,\omega\cos\omega t$$
$$+ \omega\eta(-a\sin\omega t + b\cos\omega t) = 0$$

Now remove secular terms, group by $\cos\omega t$, $\sin\omega t$.

$\underline{\cos\omega t:}$

$$2\mathring{b}\omega - \frac{\omega^2 u}{2}a + u\omega^{\cancel{2}}a - \epsilon\cancel{\omega_0}a + \frac{\eta}{2}\omega b = 0$$
$$\mathring{b} - \left(\frac{\omega u}{4} - \frac{u\omega}{2} + \frac{\epsilon}{2}\right)a + \left(\frac{\eta}{2}\right)b = 0$$
$$\mathring{b} - \left(-\frac{\omega_0 u}{4} + \frac{\epsilon}{2}\right)a + \left(\frac{\eta}{2}\right)b = 0$$

$\underline{\sin\omega t:}$

$$-2\mathring{a}\omega + \frac{\omega^2 u}{2}b - u\omega^2 b - \epsilon\omega_0 b - \omega\eta a = 0$$

$$\dot{a} - \left(\frac{wu}{4} - \frac{uw}{2} - \frac{\epsilon}{2}\right) b + \frac{\eta}{2} a = 0.$$

$$\dot{a} - \left(-\frac{wu}{4} - \frac{\epsilon}{2}\right) b + \frac{\eta}{2} a = 0$$

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{\eta}{2} & -\left(\frac{uw_0}{4} + \frac{\epsilon}{2}\right) \\ -\left(\frac{uw_0}{4} - \frac{\epsilon}{2}\right) & -\frac{\eta}{2} \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = C_+ e^{\lambda_+ t} E_+ + C e^{\lambda_- t} E_-$$

$$\left(-\frac{\eta}{2} - \lambda\right)^2 - \left(\left(\frac{uw_0}{4}\right)^2 - \left(\frac{\epsilon}{2}\right)^2\right) = 0$$

$$\lambda^2 + \lambda\eta + \left(\frac{\eta}{2}\right)^2 - \left(\frac{uw_0}{4}\right)^2 + \left(\frac{\epsilon}{2}\right)^2 = 0$$

$$\lambda = \frac{-\eta}{2} \pm \sqrt{\left(\frac{\eta}{2}\right)^2 - \left[\left(\frac{\eta}{2}\right)^2 - \left(\frac{uw_0}{4}\right)^2 + \left(\frac{\epsilon}{2}\right)^2\right]}$$

$$= \frac{-\eta}{2} \pm \sqrt{\left(\frac{uw_0}{4}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}$$

$$e^{-\frac{\eta}{2}t}\, e^{\pm\sqrt{\left(\frac{uw_0}{4}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}\, t}$$

Stable when $\left(\frac{u w_0}{4}\right)^2 - \left(\frac{\epsilon}{2}\right)^2 < 0.$ to have oscillations.

$$\frac{u w_0}{4} < \frac{\epsilon}{2}$$

$$\left|\frac{u w_0}{2\eta}\right| < \left|\frac{\epsilon}{\eta}\right|$$

Also stable when

$$-\frac{\eta}{2} \pm \sqrt{\left(\frac{u w_0}{4}\right)^2 - \left(\frac{\epsilon}{2}\right)^2} < 0$$

to have decay amplitude.

$$\left(\sqrt{\left(\frac{u w_0}{4}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}\right)^2 < \left(\frac{\eta}{2}\right)^2$$

$$\left(\frac{u w_0}{2\eta}\right)^2 - \left(\frac{\epsilon}{\eta}\right)^2 < 1$$

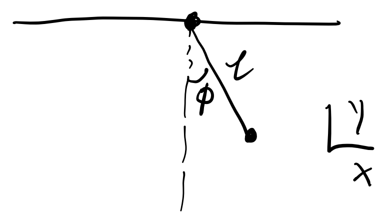
$\hookrightarrow$ $1 + \left(\frac{\epsilon}{\eta}\right)^2 > \left(\frac{u w_0}{2\eta}\right)^2$ for stable.

Problem 2: A pendulum in a harmonic electric field.

A simple pendulum consists of a particle of mass, m, at the end of weightless rod of length, l. The particle has charge, q, and sits in an electric field of amplitude E_0 directed into the horizontal direction, which oscillates rapidly with frequency, Ω, $\Omega \gg \sqrt{g/l}$

a) Determine the Lagrangian of this system.

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - q\phi - U(r)$$



$$E = -\vec{\nabla}\phi$$

know Horizontal $\rightarrow$ $E(t) = E_0 \cos \Omega t \, \hat{x}$

then $\phi = -E_0 x \cos \Omega t$

$x = l \sin\phi \qquad y = -l\cos\phi$

$$\Rightarrow L = \frac{1}{2} m l^2 \dot{\phi}^2 + qE_0 l \sin\phi \cos\Omega t + mgl\cos\phi$$

$$U = -qE_0 l \sin\phi \cos\Omega t - mgl\cos\phi$$

b) Above a critical field strength, E_c, the position $\phi = 0$ becomes unstable. Determine E_c, and determine the stability for $E > E_c$.

Sketch the potential for $E > E_c$ and $E < E_c$.

Since lagrangian composed of fast and slow components,

let $\phi(t) = \varphi(t) + \theta(t)$

Slow component $\quad$ fast component

$$\hookrightarrow \; ml^2(\ddot{\varphi} + \ddot{\theta}) = -\frac{\partial}{\partial\phi}\Big(\underbrace{-mgl\cos\phi}_{U(\phi)}\Big) - \frac{\partial}{\partial\phi}\Big(\underbrace{-qE_0 l\sin\phi\cos\Omega t}_{L_{fast}(\phi,t)}\Big)$$

Expand $\frac{\partial}{\partial\phi}U(\phi=\varphi+\theta) = \frac{\partial}{\partial\phi}U\Big|_{\phi=\varphi} + \frac{\partial^2}{\partial\phi^2}U\Big|_{\phi=\varphi}\theta$

$$\frac{\partial}{\partial\phi}L_{fast}(\phi=\varphi+\theta,t) = \frac{\partial}{\partial\phi}L_{fast}\Big|_{\phi=\varphi} + \frac{\partial^2}{\partial\phi^2}L_{fast}\Big|_{\phi=\varphi}\theta$$

We have $ml^2\ddot{\theta} = -\frac{\partial}{\partial\phi}(-qE_0 l\sin\phi\cos\Omega t) = -\frac{\partial}{\partial\phi}L_{fast}$

$$ml^2\ddot{\theta} = qE_0 l\cos\phi\cos\Omega t$$

let $\theta = a\cos\Omega t.$

$$\ddot{\theta} = -\Omega^2 a\cos\Omega t$$

$$-\Omega^2 ml^2 a\cos\cancel{\Omega t} = qE_0 l\cos\phi\cos\cancel{\Omega t}$$

$$a = \frac{-qE_0 l\cos\phi}{\Omega^2 ml^2}$$

$$\boxed{\theta(\phi,t) = \frac{-qE_0 l\cos\phi}{\Omega^2 ml^2}\cos\Omega t}$$

$$ml^2\ddot{\varphi} = -\frac{\partial}{\partial\phi}U\Big|_{\phi=\varphi} - \overline{\underbrace{\frac{\partial^2}{\partial\phi^2}U\Big|_{\phi=\varphi}\theta}_{\simeq 0}} - \overline{\frac{\partial^2}{\partial\phi^2}L_{fast}\Big|_{\phi=\varphi}\theta}$$

constant in time $\quad$ harmonic.

$$-\frac{\partial}{\partial\phi^2}L_{fast}\Big|_{\phi=\varphi}\ \theta = \frac{\partial}{\partial\phi}\left(qE_0 l\cos\phi\cos\Omega t\right)\left(\frac{-qE_0 l\cos\varphi}{\Omega^2 ml^2}\cos\Omega t\right)$$

$$\overset{!}{=} -\frac{\partial}{\partial\phi}\left(\frac{q^2E_0^2l^2}{2\Omega^2 l^2 m}\cos^2\Omega t\ \ \cos^2\phi\right)$$

$$ml^2\ddot{\varphi} = -\frac{\partial}{\partial\phi}\left[U\Big|_{\phi=\varphi} + \frac{q^2E_0^2l^2}{2\Omega^2 l^2 m}\cos\Omega t\cos^2\phi\Big|_{\phi=\varphi}\right]$$

$$\overset{!}{=} -\frac{\partial}{\partial\phi}\left[mgl\cos\varphi + \frac{q^2E_0^2}{2\Omega^2 m}\cos^2\Omega t\ \cos^2\varphi\right]$$

$$\text{average}: \frac{1}{2\pi-0}\int_0^{2\pi}\cos^2\Omega t = \frac{1}{2\pi}\pi = \frac{1}{2}$$

$$U_{eff}\ \overset{!}{=} -\frac{\partial}{\partial\phi}\underbrace{\left[-mgl\cos\varphi + \frac{q^2E_0^2}{4\Omega^2 m}\cos^2\varphi\right]}$$

$$\frac{U_{eff}}{mgl} = \frac{\partial}{\partial\phi}\left[-\cos\varphi + \left(\frac{qE_0}{2mg}\frac{\omega_0}{\Omega}\right)^2\cos^2\varphi\right]$$

To find unstable point around $\phi=0$, expand around $\phi=0$.

$$\cos\varphi \cong 1-\frac{\varphi^2}{2} \qquad \cos^2\varphi \cong 1-\varphi^2$$

$$-\left(1-\frac{\varphi^2}{2}\right) + \left(\frac{qE_0}{2mg}\frac{\omega_0}{\Omega}\right)^2\left(1-\varphi^2\right) = 0.$$

$$-1 + \frac{1}{2}\varphi^2 + \left(\frac{qE_0}{2mg}\frac{\omega_0}{\Omega}\right)^2 - \left(\frac{qE_0}{2mg}\frac{\omega_0}{\Omega}\right)^2\varphi^2 = 0$$

$$\left(\frac{1}{2}-\left(\frac{qE_0}{2mg}\frac{w_0}{\Omega}\right)^2\right)\varphi^2 \quad -1+\left(\frac{qE_0}{2mg}\frac{w_0}{\Omega}\right)^2 > 0$$

To have unstable potential, need φ^2 to have negative coefficient.

$$\frac{1}{2}-\left(\frac{qE_0}{2mg}\frac{w_0}{\Omega}\right)^2 < 0.$$

$$E_0^2 > 2\left(\frac{mg}{q}\right)^2\left(\frac{\Omega}{w_0}\right)^2$$

What is the new point of stability for $E_0 > E_c$?

Stability happens at minimum:

$$\frac{\partial}{\partial\phi}\left(-\cos\varphi+\left(\frac{qE_0}{2mg}\frac{w_0}{\Omega}\right)^2\cos^2\varphi\right)=0.$$

$$\sin\phi-\left(\frac{qE_0}{2mg}\frac{w_0}{\Omega}\right)^2 2\cos\varphi\sin\varphi=0.$$

$$\sin\varphi\left(1-\left(\frac{qE_0 w_0}{2mg\Omega}\right)^2 2\cos\varphi\right)=0$$

Either $\sin\varphi=0$ or

$$1-2\left(\frac{qE_0 w_0}{2mg\Omega}\right)^2\cos\varphi=0.$$

Thus stable when

$$\varphi_{stable} = \cos^{-1}\left(\frac{1}{2}\left(\frac{qE_0\omega_0}{2mg\Omega}\right)^{-2}\right)$$

c) Is E_c large or small compared to $\frac{mg}{q}$.

$$E_c^2 = 2\left(\frac{mg}{q}\right)^2\left(\frac{\Omega}{\omega_0}\right)^2$$

$$E_c = \sqrt{2}\,\frac{mg}{q}\,\frac{\Omega}{\omega_0}$$

$$\frac{E_c}{\frac{mg}{q}} = \sqrt{2}\,\frac{\Omega}{\omega_0}$$

Know $\Omega \gg \omega_0$

so $E \gg \frac{mg}{q}$

3) A driven set of oscillators

Consider a set of coupled harmonic oscillators with external time dependent forces.

The oscillator Lagrangian without the forces reads:

$$L_0 = \sum_{ij} \frac{1}{2} M_{ij}\, \dot{q}^i \dot{q}^j - \frac{1}{2} K_{ij}\, q^i q^j$$

The Lagrangian for the forces driving the systems?

$$L_{int} = \sum_i F_i(t)\, q^i$$

Thus:

$$L_{tot} = L_0 + L_{int}$$

Switch to eigen basis:

$$q^i = \sum_a E^i_a\, Q^a$$

Here E^i_a is the a-th eigen-vector of

$$K\vec{E}_a = \lambda_a M \vec{E}_a \qquad \text{where } \lambda_a = \omega_a^2$$

with property:

$$\sum_{ij} E^i_a M_{ij} E^j_b = \delta_{ab}.$$

a) Determine the Lagrangian for coordinates Q^a, and show

$$\ddot{Q}_a + \omega_a^2 Q^a = F_a$$

where $F_a = \sum_i F_i E^i_a$, is the projection of the force vector $\vec{F}$ onto the a-th normal mode.

or $\quad F_a = \vec{F} \cdot \vec{E}_a$

$$L_{int} = \sum_{ij} \frac{1}{2} M_{ij}\, \dot{q}^i \dot{q}^j - \frac{1}{2} k_{ij}\, q^i q^j + \sum_i F_i(t)\, q^i$$

$$L_{tot} = \sum_{ij} \frac{1}{2} M_{ij}\, E^i_a Q^a E^j_b Q^b - \frac{1}{2} \underbrace{K_{ij}\, E^i_a}_{W_a^2 M_{ij} E^i_a} Q^a E^j_b Q^b$$

$$+ F_i E^i_a Q^a$$

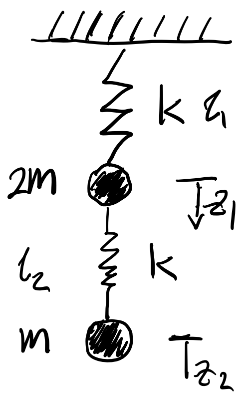
$$\overset{!}{=} \frac{1}{2} \delta_{ab} \dot{Q}^a \dot{Q}^b - \frac{1}{2} W_a^2 M_{ij} E^i_a E^j_a Q^a Q^b$$

$$+ F_a Q^a$$

$$L_{tot} \overset{!}{=} \frac{1}{2} \dot{Q}^{a^2} - \frac{1}{2} W_a^2 Q^{a^2} + F_a Q^a$$

$$\hookrightarrow \ddot{Q}^a = -W_a^2 Q^a + F_a$$

$$\ddot{Q}^a + W_a^2 Q^a = F_a$$

b) 

⇒ Consider mass 2m, m. with spring constant K

⇒ let z_1 and z_2 to represent the displacement.

⇒ An external force, $F(t)$ is applied to the lower mass, m. $F > 0$, meaning downward.

⇒ Assume $F(t) \to 0$ as $t \to \pm\infty$

⇒ system initially at rest.

⇒ let $F(w)$ to represent fourier mode.

b) 1) Find Lagrangian without force, find frequency and Normal modes.
2) Include the external force, and find EOM.

Without Force terms:

$$y_1 = z_1 + l_1 \qquad y_2 = z_2 + l_1 + l_2$$

$$L = \frac{1}{2}(2m)\dot{z}_1^{\,2} + \frac{1}{2}m\dot{z}_2^{\,2} - \left(\frac{1}{2}kz_1^2 + \frac{1}{2}k(z_2 - z_1)^2\right)$$

There is no gravity since gravity puts spring into equilibrium.

$$\stackrel{!}{=} m\dot{z}_1^{\,2} + \frac{1}{2}m\dot{z}_2^{\,2} - kz_1^2 + kz_1z_2 - \frac{1}{2}kz_2^2$$

$$2m\ddot{z}_1 = -2kz_1 + kz_2$$

$$m\ddot{z}_2 = kz_1 - kz_2$$

$$z^i = \sum A e^{i\omega_a t} E^i{}_a = \sum_a Q^a E^i{}_a$$

$$\hookrightarrow -\omega^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E^0 \\ E^1 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E^0 \\ E^1 \end{pmatrix}$$

$$\begin{vmatrix} \left(-2\omega^2 + \frac{2k}{m}\right) & \frac{-k}{m} \\ \frac{-k}{m} & -\omega^2 + \frac{k}{m} \end{vmatrix}$$

$$\stackrel{!}{=} \left(-2\omega^2 + \frac{2k}{m}\right)\left(-\omega^2 + \frac{k}{m}\right) - \left(\frac{k}{m}\right)^2$$

$$\stackrel{!}{=} 2\omega^4 - 2\omega^2\frac{k}{m} - 2\omega^2\frac{k}{m} + 2\left(\frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2$$

$$\stackrel{!}{=} \omega^4 - 2\omega^2\frac{k}{m} + \frac{1}{2}\left(\frac{k}{m}\right)^2$$

$$\omega^2 = 2\frac{k}{m} \pm \sqrt{4\left(\frac{k}{m}\right)^2 - \frac{4}{2}(k/m)^2}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$2$$

$$\stackrel{!}{=} \frac{k}{m} \pm \frac{k}{m}\sqrt{\frac{1}{2}}$$

$$\omega_\pm^2 \stackrel{!}{=} \frac{k}{m}\left(\frac{2\pm\sqrt{2}}{2}\right)$$

For $\omega^2 = \omega_+^2 = \frac{k}{m}\left(\frac{2+\sqrt{2}}{2}\right)$

$$\begin{pmatrix} \left(-2\omega^2 + \frac{2k}{m}\right) & \frac{-k}{m} \\ \frac{-k}{m} & -\omega^2 + \frac{k}{m} \end{pmatrix} \begin{pmatrix} E_+^0 \\ E_+^1 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} \left[-(2+\sqrt{2})+2\right] & -1 \\ -1 & -\left(\frac{2+\sqrt{2}}{2}\right)+1 \end{pmatrix} \begin{pmatrix} E_+^0 \\ E_+^1 \end{pmatrix}$$

$$\hookrightarrow \frac{k}{m}\begin{pmatrix} -\sqrt{2} & -1 \\ -1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} E_+^0 \\ E_+^1 \end{pmatrix}$$

$$\sqrt{2}\, E_+^0 + E_+^1 = 0$$

$$E_+ = (1, -\sqrt{2})$$

For $\omega^2 = \omega_-^2 = \frac{k}{m}\left(\frac{2-\sqrt{2}}{2}\right)$

$$\begin{pmatrix} \left(-2\omega^2 + \frac{2k}{m}\right) & \frac{-k}{m} \\ \frac{-k}{m} & -\omega^2 + \frac{k}{m} \end{pmatrix} \begin{pmatrix} E_+^0 \\ E_+^1 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} \left[-(2-\sqrt{2})+2\right] & -1 \\ -1 & -\left(\frac{2-\sqrt{2}}{2}\right)+1 \end{pmatrix} \begin{pmatrix} E_+^0 \\ E_+^1 \end{pmatrix}$$

$$\hookrightarrow \frac{k}{m}\begin{pmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} E_+^0 \\ E_+^1 \end{pmatrix}$$

$$\sqrt{2}\,E_+^0 - E_+^1 = 0$$

$$E_- = (1, \sqrt{2})$$

$$\omega_\pm^2 = \frac{k}{m}\left(\frac{2\pm\sqrt{2}}{2}\right) \qquad E_\pm = (1, \mp\sqrt{2})$$

Then:

$$z^i = A_+ E_+^i \cos(-\omega_+ t + \phi_-) + A_- E_-^i \cos(-\omega_- t + \phi_-)$$

$$z^i \overset{!}{=} Q^+ E_+^i + Q^- E_-^i$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = Q_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} + Q_- \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

New basis vectors composed using original basis.

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} Q_+ + Q_- \\ \sqrt{2}(Q_- - Q_+) \end{pmatrix}$$

original basis

Normal mode or eigen basis

<u>Now we include external force term</u>

$$L = \frac{1}{2}(2m)\dot{z}_1^2 + \frac{1}{2}m\dot{z}_2^2 - \left(\frac{1}{2}kz_1^2 + \frac{1}{2}k(z_2 - z_1)^2\right) + F(t)\,z_2$$

Convert to eigen basis.

$$\mathcal{L} = \frac{1}{2}(2m)(\dot{Q}_+ + \dot{Q}_-)^2 + \frac{1}{2}m\,2(\dot{Q}_- - \dot{Q}_+)^2$$
$$- \frac{1}{2}k\left\{(Q_+ + Q_-)^2 + \left(\sqrt{2}(Q_- - Q_+) - Q_+ - Q_-\right)^2\right\} + F(t)\sqrt{2}(Q_- - Q_+)$$

$$\stackrel{!}{=} 2m(\dot{Q}_+^2 + Q_-^2) - \frac{1}{2}k\left(Q_+^2 + 2Q_+Q_- + Q_-^2 + \left[(\sqrt{2}-1)Q_- + Q_+(-\sqrt{2}-1)\right]^2\right)$$
$$+ \sqrt{2}F(t)(Q_- - Q_+)$$

$$\stackrel{!}{=} 2m(\dot{Q}_+^2 + Q_-^2) - \frac{1}{2}k\left(Q_+^2 + 2Q_+Q_- + Q_-^2 + (3-2\sqrt{2})Q_-^2 + (3+2\sqrt{2})Q_+^2 - 2Q_-Q_+\right) + \sqrt{2}F(t)(Q_- - Q_+)$$

$$\stackrel{!}{=} 2m(\dot{Q}_+^2 + Q_-^2) - \frac{1}{2}k\left[(4+2\sqrt{2})Q_+^2 + (4-2\sqrt{2})Q_-^2\right]$$
$$+ \sqrt{2}F(t)(Q_- - Q_+)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_+}\right) = 4m\ddot{Q}_+ = \frac{\partial L}{\partial Q_+} = -(4+2\sqrt{2})kQ_+ - \sqrt{2}F(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_-}\right) = 4m\ddot{Q}_- = \frac{\partial L}{\partial Q_-} = -(4-2\sqrt{2})kQ_- + \sqrt{2}F(t)$$

$$\left.\begin{aligned}
\ddot{Q}_+ &= -\left(\frac{2+\sqrt{2}}{2}\right)\frac{k}{m}Q_+ + \frac{\sqrt{2}}{4m}F(t) \\
&\stackrel{!}{=} -\omega_+^2 Q_+ - \frac{\sqrt{2}}{4m}F(t)
\end{aligned}\right\}\ \text{decoupled EOM}$$

$$\ddot{Q}_- = -\left(\frac{2-\sqrt{2}}{2}\right)\frac{k}{m}Q + \frac{\sqrt{2}}{4m}F(t)$$
$$\stackrel{!}{=} -\omega_-^2 Q_- + \frac{\sqrt{2}}{4m}F(t)$$

c) Solve for the motion of both masses.

If EOM involves with external force $F(t)$, then we can use retarded green function to solve.

Remember: $\mathcal{L}_t X(t) = F(t)$.

$t=t_0$ $\quad \mathcal{L}_t G_R(t,t_0) = \delta(t-t_0)$. $\leftarrow$ $G_R(t,t_0)$ is the solution when $F(t) = \delta(t-t_0)$.

$t<t_0$ $\quad G_R(t,t_0) = 0$

$t>t_0$ $\quad \mathcal{L}_t G_R(t,t_0) = 0$ $\leftarrow$ Homogeneous solution.

Then $X_\delta(t) = \int_{-\infty}^{\infty} dt_0 \, G_R(t,t_0) F(t_0)$

$$\ddot{G}_R^{+} + \omega_+^2 G_R = \delta(t-t_0)$$

$$\ddot{G}_R^{-} + \omega_-^2 G_R = \delta(t-t_0)$$

Find G_R: by setting $F(t)=0$,

$$\ddot{G}_R^{\pm} + \omega_\pm^2 G_R = 0$$

then $G_R^{\pm} = A_\pm \cos(-\omega_\pm t + \phi_\pm)$

Now find conditions to solve for $A_\pm$ and $\phi_\pm$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{d^2}{dt^2} G_R^{\pm} + \underbrace{\omega_\pm^2 G_R^{\pm}}_{=0} \, dt = \underbrace{\int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0)\, dt}_{=1}$$

$$\frac{d}{dt} G_R^{\pm} \Big|_{\cancel{t_0-\epsilon}}^{t_0+\epsilon} = 1$$

① $\boxed{\frac{d}{dt} G_R^{\pm} \Big|_{t_0+\epsilon} = 1}$

$t_0+\epsilon$

② second condition:

$$G_R^{\pm} = 0 \qquad \text{for} \quad t < t_0$$

so by continuity, $\boxed{G^{\pm}\big|_{t_0+\epsilon} = 0}$ as well

$$G^{\pm}(t=t_0) = A_{\pm}\cos(\omega_{\pm} t_0 + \phi_{\pm}) = 0$$

then $\phi_{\pm} = -\omega_{\pm} t_0 + \frac{\pi}{2}$.

so $G_R^{\pm}(t) = A_{\pm}\sin(\omega_{\pm}(t-t_0))$

$$\frac{dG^{\pm}}{dt}\bigg|_{t_0+\epsilon} = \omega_{\pm} A_{\pm}\cos(\omega_{\pm}(t_0-t_0)) = 1.$$

$$A_{\pm} = \frac{1}{\omega_{\pm}}.$$

Then

$$G_R^{\pm} = \begin{cases} \dfrac{\sin(\omega_{\pm}(t-t_0))}{\omega_{\pm}} & \text{for} \quad t > t_0 \\ 0 & \text{for} \quad t < t_0 \end{cases}$$

or $G_R^{\pm} = \dfrac{\theta(t-t_0)\sin(\omega_{\pm}(t-t_0))}{\omega_{\pm}}$

Then

$$\boxed{Q^{\pm} = \mp\int_{-\infty}^{\infty} dt_0 \, \frac{F_2 F(t_0)}{4m} \, \frac{\theta(t-t_0)\sin(\omega_{\pm}(t-t_0))}{\omega_{\pm}}}$$

$$\vec{z}^{i} = Q^{+}E^{i}_{+} + Q^{-}E^{i}_{-}$$

$$z_1 = Q^+ + Q^-$$
$$z_2 = \sqrt{2}(Q^- - Q^+)$$

d) Show Total work done by external force is

$$W = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) |F(\omega)|^2$$

$$\Delta E = W = \int F(t)\, dQ$$
$$\stackrel{!}{=} \int F(t)\, \dot{Q}\, dt$$

Find $\dot{Q}$:

$$\dot{Q}_+(t) = \frac{d}{dt} \int_{-\infty}^{t} -dt'\, G_R(t-t') \frac{F(t')}{4m} \sqrt{2}$$

$$\stackrel{!}{=} \frac{d}{dt} \int_{-\infty}^{t} -dt' \frac{\theta(t-t') \sin(\omega_+(t-t'))}{\omega_+} \frac{F(t')}{4m} \sqrt{2}$$

$$\stackrel{!}{=} \int_{-\infty}^{t} -dt' \Big[\underbrace{\cancel{\delta(t-t') \sin(\omega_+(t-t'))}}_{\substack{\text{due to } \delta(t-t'), \\ t=t' \\ \text{then } \sin(\omega_+(t-t'))=0}} + \theta(t-t') \omega_+ \cos(\omega_+(t-t')) \Big] \frac{F(t')}{4m\omega_+} \sqrt{2}$$

$$\stackrel{!}{=} -\int_{-\infty}^{t} dt'\, \theta(t-t') \cos(\omega_+(t-t')) \frac{\sqrt{2} F(t')}{4m}$$

Then $W_+ = \int dt\, F(t)\, \dot{z}_2(t)$

$$= \int dt\, F(t)\, \sqrt{2}\, (\dot{Q}_- - \dot{Q}_+)$$