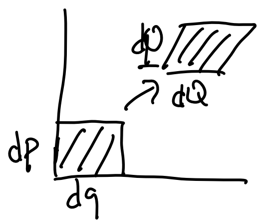
$$\text{Hamiltonian Evolution from } \overset{\text{Initial}}{(q,p)} \rightarrow \overset{\text{Final}}{(Q,P)}$$



$$q \rightarrow Q(t) = q + \Delta q \quad , \Delta q = \dot{q}\,\delta t = \frac{\partial H}{\partial p}\delta t$$

$$p \rightarrow P(t) = p + \Delta p \quad , \Delta p = \dot{p}\,\delta t = \frac{\partial H}{\partial q}\delta t$$

$$\frac{d(\Delta p \Delta q)}{dt} = \frac{d(\Delta p)}{dt}\Delta q + \Delta p \frac{d(\Delta q)}{dt}$$

$$\overset{!}{=} \left(\frac{\partial \dot{p}}{\partial p} \Delta p \Delta q + \frac{\partial \dot{q}}{\partial q} \Delta p \Delta q \right) \delta t$$

$$\overset{!}{=} \left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial q \partial p} \right) \Delta p \Delta q \, \delta t$$

$$\overset{!}{=} 0$$

← so volume is conserved.
for any general transformation.

Special Transformation: Canonical Transformation :

$$\left.\begin{aligned} q &\rightarrow Q(q,p) \\ p &\rightarrow P(q,p) \\ H(q,p) &\rightarrow \tilde{H}(Q,P) \end{aligned}\right\} \quad \begin{aligned} &\text{preserve structure:} \\ &\dot{P} = \frac{-\partial \tilde{H}}{\partial Q} \\ &\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \end{aligned}$$

For small transforms: small λ.

Need generator, $G(q,p)$, such

$$q \to Q(q,p) = q + \frac{\partial G}{\partial p}\lambda \qquad \text{so } \Delta q = \frac{\partial G}{\partial p}$$

$$p \to P(q,p) = p - \frac{\partial G}{\partial q}\lambda \qquad \Delta p = -\frac{\partial G}{\partial q}$$

The canonical transform preserves symplectic form:

$$\frac{dx^i}{dt} = J^{ij}\frac{\partial H}{\partial x^j} \qquad \text{where } x^i = \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ p_1 \\ p_2 \end{pmatrix}$$

$$\text{and } J^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then a canonical transform

$$x^i \to Y^i \qquad \text{and} \qquad \frac{dY^i}{dt} = J^{ij}\frac{\partial \tilde{H}}{\partial Y^i}$$

only if:

canonical map condition.

$$\left(MJM^T\right) = J^{ij} \qquad \text{for } M = \frac{\partial Y^i}{\partial x^j}$$

We also see that the area is again conserved:

$d(\Delta q \Delta p) \qquad d(\Delta q) \ldots \quad d(\Delta p)$

$$\frac{d}{dt}\cdots = \frac{d}{dt}\Delta p + \Delta q \frac{d}{dt}$$

$$\overset{!}{=} \frac{\partial \dot q}{\partial q}\Delta q \Delta p\, \delta t + \frac{\partial \dot p}{\partial p}\Delta p \Delta q\, \delta t$$

$$\overset{!}{=} \frac{\partial^2 G}{\partial p \partial q}\Delta q \Delta p\, \delta t - \frac{\partial^2 G}{\partial p \partial q}\Delta p \Delta q\, \delta t$$

$$\overset{!}{=} 0$$

<u>Observable</u> $O(q,p)$ over the map.

$$O(q,p) \to O(Q,P)$$

So $$\delta O = O(Q,P) - O(q,p)$$

$$\overset{!}{=} O\left(q + \frac{\partial G}{\partial p}\lambda,\ p - \frac{\partial G}{\partial q}\lambda\right) - O(q,p)$$

$$\overset{!}{=} O(q,p) + \frac{\partial G}{\partial p}\lambda \frac{\partial O}{\partial q}(q,p) - \frac{\partial G}{\partial q}\lambda \frac{\partial O}{\partial p}(q,p) - O(q,p)$$

$$\delta O \overset{!}{=} \{O, G\}\lambda$$

if G leaves H invariant, or $\delta H = 0$

or $$\{G, H\} = 0 = \frac{dG}{dt}$$

For $G = \vec{n} \cdot \vec{p}$

$$\left.\begin{aligned} \vec{q} &\Rightarrow \vec{Q} = \vec{q} + \vec{n}\lambda \\ \vec{p} &\Rightarrow \vec{P} = \vec{p} \end{aligned}\right\} \Rightarrow \text{Momentum generates translation}$$

For $G = \vec{n}\cdot(\vec{r}\times\vec{p}) = \vec{p}\cdot(\vec{n}\times\vec{r}) = \vec{r}\cdot(\vec{p}\times\vec{n})$

$$\vec{r} \Rightarrow \vec{r} + \frac{\partial G}{\partial \vec{p}}\lambda = \vec{r} + (\vec{n}\times\vec{r})\lambda = \vec{r} + \delta\vec{\theta}\times\vec{r}$$

$$\vec{p} \Rightarrow \vec{p} - \frac{\partial G}{\partial \vec{r}}\lambda = \vec{p} - (\vec{p}\times\vec{n})\lambda = \vec{p} - \vec{p}\times\delta\vec{\theta}$$

$$\stackrel{!}{=} \vec{p} + \delta\vec{\theta}\times\vec{p}$$

Poisson Brackets:

We have $f(q,p)$ and $g(q,p)$

$$\{f,g\} = \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right)$$

For canonical transform: $q \to Q$, $p \to P$

$$F(Q,P) = f(q,p)$$
$$G(Q,P) = g(q,p)$$

then $\{F,G\}_{P,Q} = \{f,g\}_{p,q}$

Since F is arbitrary, this must be a property of poisson bracket.

So:

$$\{Q^i, P^j\} = \{Q^i, P^j\} = 1$$

$\{\ \}_{pq} \quad \{\ \}_{PQ}$

another property of canonical transform. →

$$\boxed{\text{and } \{Q^i, Q^j\}_{p,q} = \{P^i, P^j\}_{p,q} = 0}$$? or 1

Different proofs for canonical Transform.

Transformed variable.

① $MJM = J$ for $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $M = \frac{dT^i}{dx^j}$

② There is a G, so that

$$\Delta q = \frac{\partial G}{\partial p} \quad \text{and} \quad \Delta p = \frac{-\partial G}{\partial q}$$

③ Curl free condition:

$$\frac{\partial(\Delta q)}{\partial q} + \frac{\partial(\Delta p)}{\partial p} = 0$$

④ Poisson Bracket Condition

$$\{Q^i, P^j\}_{pq} = \{Q^i, P^j\}_{PQ} = 1$$

$$\{Q^i, Q^j\}_{p,q} = \{P^i, P^j\}_{p,q} = 1$$

General (Not infinitesimal)

For EOM and action: $\quad H = p\dot{q} - L$ or $L = p\dot{q} - H$

$$S_1 = \int p\,dq - H\,dt$$
$$S_2 = \int p\,dq - \tilde{H}\,dt$$

$\Big\}$ $S_1 - S_2 = \int \frac{dF}{dt}\,dt$

otherwise action changes.

Then

$$= \int p\,dq - \int P\,dQ - \int (H - \tilde{H})\,dt$$

$$\stackrel{!}{=} \underbrace{\int \frac{\partial F}{\partial q}\,dq + \frac{\partial F}{\partial Q}\,dQ + \frac{\partial F}{\partial t}\,dt}_{\frac{dF(q,Q,t)}{dt}\,dt}$$

← assume $F(q, Q, t)$

then $p = \frac{\partial F}{\partial q}$ $\qquad P = \frac{-\partial F}{\partial Q}$

$$H - \tilde{H} = \frac{-\partial F}{\partial t} \quad \text{or} \quad \tilde{H} = H + \frac{\partial F}{\partial t}$$

use $p = \frac{\partial F}{\partial q}$ to solve for $Q(q, p, t)$

then use $P = \frac{-\partial F}{\partial Q}$ after know Q.

Similarly: Given $\Phi = \Phi(q, P, t)$

$$S_1 - S_2 = \int p\,dq + Q\,dP - (H - \tilde{H})\,dt$$

$$\stackrel{!}{=} \int \frac{\partial \Phi}{\partial q}\,dq + \frac{\partial \Phi}{\partial P}\,dP + \frac{\partial \Phi}{\partial t}\,dt$$

$$p = \frac{\partial \Phi}{\partial q}(q, P, t) \qquad Q = \frac{\partial \Phi}{\partial P}(q, P, t)$$

$$\tilde{H} = H + \frac{\partial \Phi}{\partial t}(q, P, t)$$