Side Notes: Change Coordinate: $q^A \to \bar{q}^A$

$$dq^A \Rightarrow d\bar{q}^A = \frac{d\bar{q}^A}{dq^B}\, dq^B$$

$$\bar{P}_A = \frac{\partial L}{\partial \dot{\bar{q}}^A} = \frac{\partial q^B}{d\bar{q}^A}\frac{\partial L}{\partial \dot{q}^B} = \frac{\partial q^B}{\partial \bar{q}^A} P_B = (M^{-1})^B{}_A\, P_B$$

$$M^A{}_B = \frac{\partial \bar{q}^A}{\partial q^B}$$

Hamiltonian:

First Integral or hamiltonian function:

$$\boxed{\begin{array}{ll} h(q,\dot{q},t) = P_A \dot{q}^A - L(q,\dot{q},t) & \text{where } P_A = \frac{\partial L}{\partial \dot{q}}(q,\dot{q},t) \\ h(q,\dot{q},t) \overset{!}{=} \frac{\partial L}{\partial \dot{q}}(q,\dot{q},t)\,\dot{q} - L(q,\dot{q},t) & \text{here } P_A \text{ is a function of } q,\dot{q},t \end{array}}$$

* If L is independent of t, or $L(q,\dot{q})$, then $h = \text{const}$

Proof:

$$\frac{dh}{dt} = \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}}(q,\dot{q},t)\right\}\dot{q} + \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial q}\dot{q}$$

$$- \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial t}$$

$$= \underbrace{\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}}(q,\dot{q},t)\right\} - \frac{\partial L}{\partial q}\dot{q}}_{\text{Due to Euler-Lagrange, } =0} - \frac{\partial L}{\partial t}$$

$$\boxed{\frac{dh}{dt} = -\frac{\partial L}{\partial t}}$$

$\Leftarrow$ If $L(q,\dot{q})$, not explicitly a function of t, then $\frac{dh}{dt}=0$, or h is constant. meaning Energy is conserved.

Ex: $L = \frac{1}{2}a_{ij}(q)\dot{q}^i\dot{q}^j + b_i(q)\dot{q}^i - V(q,t)$

$$p_l = \frac{\partial L}{\partial \dot{q}^l} = \frac{1}{2}a_{ij}\frac{\partial \dot{q}^i}{\partial \dot{q}^l}\dot{q}^j + \frac{1}{2}a_{ij}\dot{q}^i\frac{\partial \dot{q}^j}{\partial \dot{q}^l} + b_i\frac{\partial \dot{q}^i}{\partial \dot{q}^l}$$

$$\overset{!}{=} \frac{1}{2}a_{ij}\delta^i_l\dot{q}^j + \frac{1}{2}a_{ij}\delta^j_l\dot{q}^i + b_i\delta^i_l$$

Note: $\frac{\partial \dot{q}^i}{\partial \dot{q}^l} = \delta^i_l$

$$p_l \overset{!}{=} \frac{1}{2}a_{jl}\dot{q}^j + \frac{1}{2}a_{il}\dot{q}^i + b_l$$

$$p_l \overset{!}{=} a_{jl}\dot{q}^j + b_l$$

$$h = (a_{jl}\dot{q}^j + \cancel{b_l})\dot{q}^l - \left(\frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j + \cancel{b_i\dot{q}^i} - V(q,t)\right)$$

$$h \overset{!}{=} \frac{1}{2}a_{jl}\dot{q}^j\dot{q}^l + V(q,t)$$

since L is not explicitly function of t, then energy is conserved.

1D particle example:

$$\mathcal{L} = \frac{1}{2} m(q)\dot{q}^2 - V_{eff}(q)$$

$$h = \frac{1}{2} m(q)\dot{q}^2 + V_{eff}(q) = E \quad \leftarrow \text{Total energy is constant.}$$

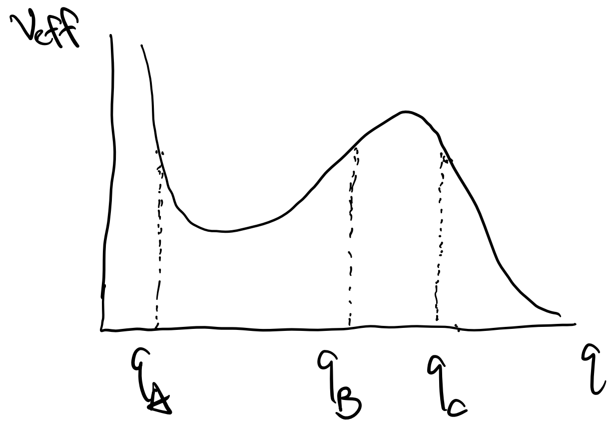
$$\hookrightarrow \frac{1}{2} m(q)\dot{q}^2 = E - V_{eff}(q)$$

$$\frac{dq}{dt} = \sqrt{\frac{2\left[E - V_{eff}(q)\right]}{m(q)}}$$

$$\hookrightarrow \int_{q_0}^{q} \frac{dq \;\; m(q)}{\pm\sqrt{2\left(E - V_{eff}(q)\right)}} = \int_{t_0}^{t} dt$$

$$\hookrightarrow I(q, q_0) = t - t_0$$

Suppose $V_{eff}(q)$ takes form:

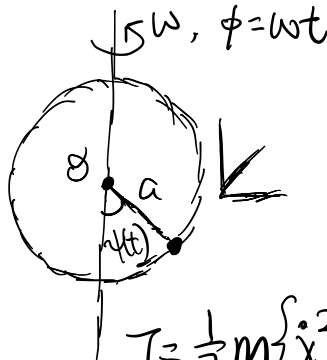


oscillation if $q_A < q < q_B$

Unbounded motion if $q > q_c$

If oscillation, then $t_B - t_A = \frac{T(E)}{2}$ for const E.

Ex: Bead on the Hoop:



$$y = a\sin\phi\sin\psi \Rightarrow \dot{y} = a\{\cos\phi\sin\psi\,\dot\phi + \sin\phi\cos\psi\,\dot\psi\}$$
$$x = a\cos\phi\sin\psi \Rightarrow \dot{x} = a\{-\sin\phi\sin\psi\,\dot\phi + \cos\phi\cos\psi\,\dot\psi\}$$
$$z = -a\cos\psi \Rightarrow \dot{z} = a\sin\psi\,\dot\psi$$

$$T = \tfrac{1}{2}m\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}$$
$$\overset{!}{=} \tfrac{1}{2}ma^2\{\cos^2\phi\sin^2\psi\,\dot\phi^2 + \sin^2\phi\cos^2\psi\,\dot\psi^2 + \cancel{2\cos\phi\sin\phi\cos\psi\sin\psi\,\dot\phi\dot\psi}$$
$$+ \sin^2\phi\sin^2\psi\,\dot\phi^2 + \cos^2\phi\cos^2\psi\,\dot\psi^2 - \cancel{2\cos\phi\sin\phi\cos\psi\sin\psi\,\dot\phi\dot\psi}$$
$$+ \sin^2\psi\,\dot\psi^2\}$$
$$\overset{!}{=} \tfrac{1}{2}ma^2\{\sin^2\psi\,\dot\phi^2 + \cos^2\psi\,\dot\psi^2 + \sin^2\psi\,\dot\psi^2\}$$
$$\overset{!}{=} \tfrac{1}{2}ma^2\{\sin^2\psi\,\dot\phi^2 + \dot\psi^2\}$$

$$V = mgz = -mga\cos\psi$$

$$L = \tfrac{1}{2}ma^2\{\sin^2\psi\,\dot\phi^2 + \dot\psi^2\} + mga\cos\psi \quad \text{where } \dot\phi = \omega$$
$$\overset{!}{=} \tfrac{1}{2}\underbrace{ma^2}_{\hat{m}}\dot\psi^2 - \underbrace{\{-\tfrac{1}{2}ma^2\omega^2\sin^2\psi - mga\cos\psi\}}_{V_{eff}(\psi)}$$

$$h = \frac{\partial L}{\partial \dot q}\dot q - L$$
$$\overset{!}{=} \tfrac{1}{2}ma^2\dot\psi^2 + \{-\tfrac{1}{2}ma^2\omega^2\sin^2\psi - mga\cos\psi\} = \text{const } E$$

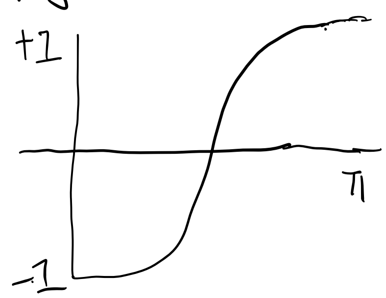
Here we note that $h \neq T + V$ since $L(\psi, \dot{\psi})$

Good idea to make dimensionless parameter:

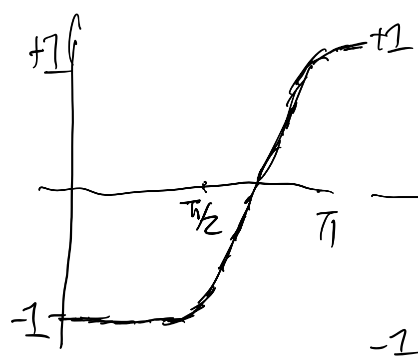
$$\frac{V_{eff}}{mga} = -\frac{1}{2}\frac{a}{g}\omega^2 \sin^2\psi - \cos\psi$$

Case I: $\omega = 0$

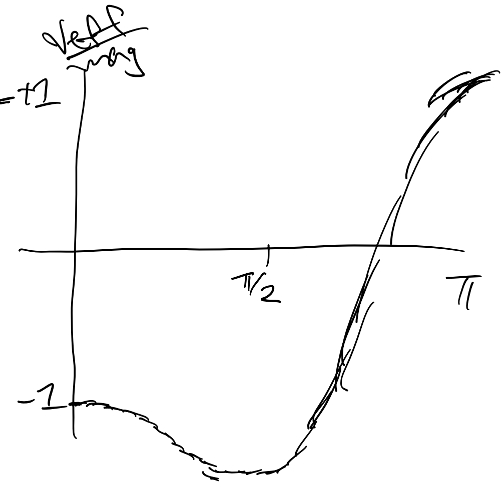
$\frac{V_{eff}}{mga} = -\cos\psi$



Case II: $\omega = \omega_{crit}$



Case III: ω = large



$$\frac{V_{eff}}{mga} = -\frac{1}{2}\frac{a}{g}\omega^2 \sin^2\psi - \cos\psi$$

if ψ is small

$$\cong -\left\{\frac{a\omega^2}{2g}\psi^2 + 1 - \frac{\psi^2}{2}\right\}$$

$$\cong -1 + \underbrace{\left(-\frac{a\omega^2}{2g} + \frac{1}{2}\right)}\psi^2$$

note:

$\cos\psi \approx 1 - \frac{\psi^2}{2!}$

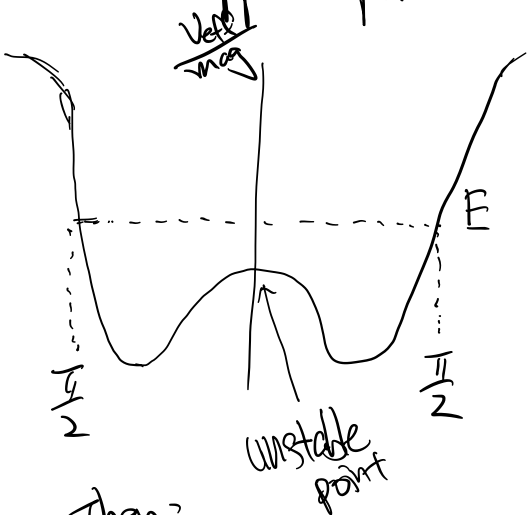
$\sin\psi \approx \psi - \frac{\psi^3}{3!}$

set $\frac{-a\omega^2}{2g} + \frac{1}{2} = 0$

or $\omega = \sqrt{\frac{g}{a}}$

$\therefore$ case 1 if $\omega^2 < g/a$
case 2 if $\omega^2 = g/a$
case 3 if $\omega^2 > g/a$

Now suppose particle initially placed at $\psi = \frac{\pi}{2}$



$E_{tot} = V_{eff}(\psi = \frac{\pi}{2})$ initially.
$\quad \stackrel{!}{=} -\frac{1}{2} m a^2 \omega^2$

always look at Veff.

Then:

$$h = -\frac{1}{2} m a^2 \omega^2 = \frac{1}{2} m a^2 \dot{\psi}^2 + V_{eff}(\psi)$$

$$\stackrel{!}{=} \frac{1}{2} m a^2 \dot{\psi}^2 + \left\{ -\frac{1}{2} m a^2 \omega^2 \sin^2\psi - mga\cos\psi \right\}$$

$$E - V_{eff} = \frac{1}{2} m a^2 \dot{\psi}^2$$

$$\sqrt{\frac{2(E - V_{eff})}{ma^2}} = \frac{d\psi}{dt}$$

$$\int_{t_{-\frac{\pi}{2}}}^{t_{\frac{\pi}{2}}} dt = \int \sqrt{\frac{ma^2}{2(E - V_{eff})}}\, d\psi$$

$$t_{\frac{\pi}{2}} - t_{-\frac{\pi}{2}} = \frac{T}{2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{ma^2}{2\left(-\frac{1}{2}ma^2\omega^2 + \frac{1}{2}ma^2\omega^2\sin^2\psi - mga\cos\psi\right)}}\, d\psi$$

$$\frac{T}{2} = \sqrt{\frac{ma^2}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1}{\frac{1}{2}ma^2\omega^2\left(-1+\sin^2\psi\right) - mga\cos\psi}}\, d\psi$$

Hamiltonian:

$\mathcal{H}(q, p, t) \leftarrow$ Eliminate $\dot{q}$ in $\mathcal{L}(q, \dot{q}, t)$

with $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} + \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial t} dt$$

Define $\mathcal{H} = p\dot{q} - \mathcal{L}$

$$\begin{aligned}
d\mathcal{H} &= d(p\dot{q}) - d\mathcal{L} \\
&= \dot{q}\,dp + p\,d\dot{q} - d\mathcal{L} \\
&= \dot{q}\,dp + \cancel{p\,d\dot{q}} - \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q}} - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial t} dt \\
d\mathcal{H} &= \dot{q}\,\underline{\underline{dp}} - \frac{\partial \mathcal{L}}{\partial q} \underline{\underline{dq}} - \frac{\partial \mathcal{L}}{\partial t} \underline{\underline{dt}}
\end{aligned}$$

$\therefore$ $\mathcal{H}$ is a function of $p, \dot{q}, t$

$$\boxed{\begin{aligned} &\frac{\partial \mathcal{H}}{\partial p} = \dot{q} \\ &-\frac{\partial H}{\partial q} = \frac{\partial \mathcal{L}}{\partial q} = \frac{dp}{dt} \\ &\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \end{aligned}}$$

Simple Ex: $\mathcal{L} = \frac{1}{2} m \dot{q}^2 - V(q)$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \quad \Rightarrow \quad \dot{q} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{q}(q,p) - \mathcal{L}(q, \dot{q}(q,p), t)$$

$$\stackrel{!}{=} p\frac{p}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + V(q)$$

$$\mathcal{H} \stackrel{!}{=} \frac{p^2}{2m} + V(q)$$

EOM:

$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial V}{\partial q}$$

And $\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$

Summary:

$$\boxed{\begin{aligned} &p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad \frac{dp}{dt} = \frac{\partial \mathcal{L}}{\partial q} \\ &\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \\ &\qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t} \end{aligned}}$$

General Expression:

$$\mathcal{L} = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + b_i \dot{q}^i - V(q)$$

Find p_i:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{1}{2} a_{ij} \left\{ \frac{\partial \dot{q}^i}{\partial \dot{q}^i} \dot{q}^j + \frac{\partial \dot{q}^j}{\partial \dot{q}^i} \dot{q}^i \right\} + b_i$$

$$\overset{!}{=} \frac{1}{2} a_{ij} \left\{ \dot{q}^j + \delta_i^j \dot{q}^i \right\} + b_i$$

$$p_i = a_{ij} \dot{q}^j + b_i$$

Invert to find $\dot{q}^i(p_i)$:

$$\dot{q}^j = (a_{ij})^{-1} (p_i - b_i)$$

$$\dot{q}^j \overset{!}{=} (a^{-1})^{ij} (p_i - b_i)$$

Now write $\mathcal{H}(q, p, t)$:

$$\mathcal{H} = p_i \dot{q}^i - \mathcal{L}$$

$$\overset{!}{=} p_i \dot{q}^i - \left\{ \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + b_i \dot{q}^i - V(q) \right\}$$

$$\overset{!}{=} (p_i - b_i) \dot{q}^i - \left\{ \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j - V(q) \right\}$$

$$\overset{!}{=} (p_i - b_i) (a^{-1})^{ij} (p_j - b_j) - \left\{ \frac{1}{2} a_{ij} \left[(a^{-1})^{ij} (p_j - b_j) \right] \left[(a^{-1})^{ij} (p_i - b_i) \right] - V(q) \right\}$$

$$\mathcal{H} = \frac{1}{2} a_{ij} (p_i - b_i)(p_j - b_j) + V(q)$$

$\uparrow$

Here $\mathcal{H}$ is a function of b_i
unlike h, hamiltonian function.

Note: $\mathcal{H} = p\dot{q}(q,p) - \mathcal{L}(q, \dot{q}(q,p), t) \leftarrow$ in terms of p, q, t

$h = p(q,\dot q)\,\dot q - \mathcal{L}(q,\dot q,t) \leftarrow$ in terms of $q, \dot q, t$

<u>Hamiltonian from Action</u>: (Derive EOM from action)

$$S[q,p] = \int dt\, \mathcal{L}(q,p,t)$$

$$S[q+\delta q, p+\delta p] = \int dt \left\{ p\dot q - \mathcal{H}(q,p,t) \right\}$$

$$\overset{!}{=} \int dt\, (p+\delta p)\frac{d}{dt}(q+\delta q) - \mathcal{H}(q+\delta q, p+\delta p, t)$$

$$\overset{!}{=} \int dt\, (p+\delta p)(\dot q + \delta \dot q) - \left\{ \mathcal{H} + \frac{\partial \mathcal{H}}{\partial q}\delta q + \frac{\partial \mathcal{H}}{\partial p}\delta p \right\}$$

$$\cancel{S} + \delta S \overset{!}{=} \cancel{\int dt\, (p\dot q - \mathcal{H})} + \int dt \left(p\frac{d}{dt}\delta q - \frac{\partial \mathcal{H}}{\partial q}\delta q \right) + \int dt \left(\dot q - \frac{\partial \mathcal{H}}{\partial p} \right)\delta p$$

$$\delta S = \int dt \left\{ \left(p\frac{d}{dt}\delta q - \frac{\partial \mathcal{H}}{\partial q}\delta q \right) + \left(\dot q - \frac{\partial \mathcal{H}}{\partial p} \right)\delta p \right\}$$

$$\overset{!}{=} \int dt \left\{ \frac{d}{dt}(p\,\delta q) - \frac{dp}{dt}\delta q - \frac{\partial \mathcal{H}}{\partial q}\delta q + \left(\dot q - \frac{\partial \mathcal{H}}{\partial p} \right)\delta p \right\}$$

$$= \cancel{P\delta q}\Big|_{q_0} + \int dt \left(-\frac{dP}{dt} - \frac{\partial H}{\partial q}\right)\delta q + \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p$$

$$\boxed{\frac{dP}{dt} = -\frac{\partial H}{\partial q}} \qquad \boxed{\dot{q} = \frac{\partial H}{\partial p}}$$

vs. $\boxed{\frac{dP}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}} \qquad \boxed{P = \frac{\partial L}{\partial \dot{q}}}$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$