Retarded Green Function:

$$\underbrace{\left(m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right)}_{L_t} X = \delta(t-t_0) = F(t)$$

Here force is a delta function symbolizing an impulsive kick.

Now suppose the solution to this equation is $\underline{X(t) = G_R(t,t_0)}$ ⇐ call $G_R(t,t_0)$ ⇔ Green's func.

depends on t_0, or when you kick the system.

Or

$$\boxed{L_t\, G_R(t,t_0) = \delta(t-t_0)}$$

Then we can use $G_R(t,t_0)$ to find the solution of the original equation.

Then the specific solution is

$$\boxed{X_S(t) = \int_{-\infty}^{\infty} dt_0\, G_R(t,t_0)\, F(t_0)}$$

after we determined the Green function.

To prove:

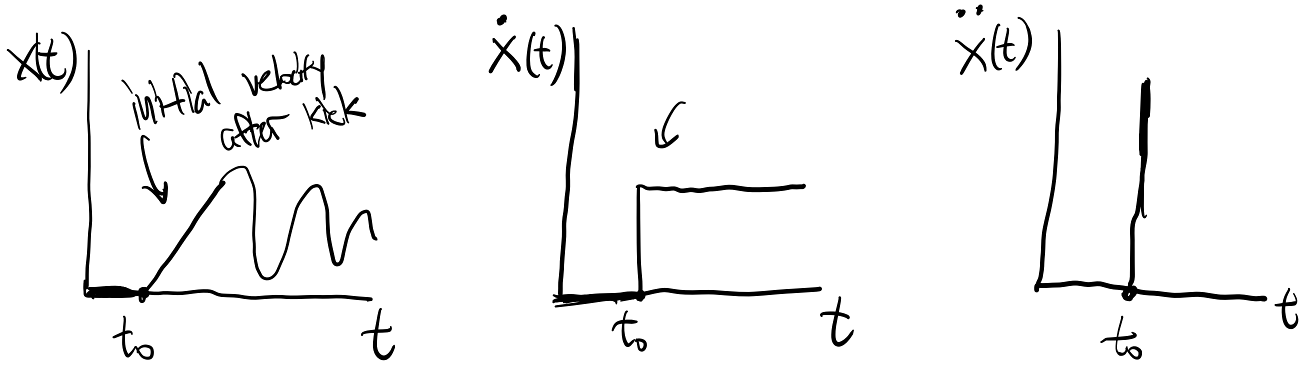
$$\mathcal{L}_t\, X_s(t) = \int_{-\infty} dt_0\, \underbrace{\mathcal{L}_t\, G_R(t,t_0)}_{\delta(t-t_0)} F(t_0)$$

$$\mathcal{L}_t\, X_s(t) \overset{!}{=} F(t)$$

Green Function:

A function that satisfy the following equation:

$$\left(m\frac{d^2}{dt^2} + m\gamma\frac{d}{dt} + m\omega_0^2\right) G_R(t,t_0) = \delta(t-t_0) = F(t)$$



Need few conditions:

No movement before the kick.

① $G_R(t,t_0) = 0$ for $t < t_0$ ⇐ Retarded condition.

② $t > t_0$ $\quad \mathcal{L}_t G_R(t,t_0) = 0$ ⇐ G_R satisfies Homogeneous solution.

Boundary Conditions:

① Continuity: $\lim_{t \to t_0^+} G_R(t, t_0) = 0$

② Determine G by integrate equation of motion.

$$\int_{t_0-\epsilon}^{t_0+\epsilon} dt \left[m \frac{d^2G}{dt^2} + m\eta \frac{dG}{dt} + \underbrace{m\omega_0^2 G}_{} \right] = \int_{t_0-\epsilon}^{t_0+\epsilon} dt\, \delta(t-t_0) = 1$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} dt\, m\omega_0^2 G(t, t_0) = 0$$

since $\lim_{t \to t_0^-} G_R(t, t_0) = 0$, condition ①

then due to continuity, $\lim_{t \to t_0^+} G_R(t, t_0) = 0$.

So overall $\int_{t_0-\epsilon}^{t_0+\epsilon} dt\, G(t, t_0)\, m\omega_0^2 = 0$

$$\hookrightarrow \; m \frac{dG}{dt}\Big|_{t_0-\epsilon}^{t_0+\epsilon} + \underbrace{m\eta G \Big|_{t_0-\epsilon}^{t_0+\epsilon}}_{=0 \text{ since } G \text{ is continuous.}} = 1$$

$$m \frac{dG}{dt}\Big|_{t_0-\epsilon}^{t_0+\epsilon} = m \frac{dG}{dt}(t_0+\epsilon, t_0) - \underbrace{m \frac{dG}{dt}(t_0-\epsilon, t_0)}_{} = 1$$

$\uparrow$

$= 0$ since

$\lim_{t \to t_0^-} \frac{dG}{dt} = 0$

$\hookrightarrow \boxed{m \frac{dG}{dt}\Big|_{t_0+\epsilon} = 1}$

$\uparrow$ use it to find constant amplitude and phase for $t > t_0$.

$\rightarrow$ used to reproduce initial condition.

For $t > t_0$, we have $\delta(t-t_0) = 0$, so G_R take the form of homogeneous solution.

$$\begin{cases} G_R(t, t_0) = x_h(t) = A\cos(-\omega'(t-t_0) + \phi)\, e^{-\frac{\eta}{2}(t-t_0)} & \text{for } t > t_0 \\ G_R(t, t_0) = 0 & \text{for } t < t_0 \end{cases}$$

Now use boundary conditions to determine amplitude A, and ~~ρ~~ ϕ.

① $\lim_{t \to t_0^+} G_R(t, t_0) = 0$

② $m \frac{dG_R}{dt}(t,t_0)\Big|_{t_0+\epsilon} = 1$

Then the condition ① implies $\phi = \pi/2$

$$G_R = A\cos\left(-w'(t-t_0) + \pi/2\right) e^{-\frac{\eta}{2}(t-t_0)}$$

$$= -A\sin\left(-w'(t-t_0)\right) e^{-\frac{\eta}{2}(t-t_0)}$$

Condition ②

$$m\frac{dG}{dt}\Big|_{t_0+\epsilon} = -w'mA\cos\left(-w'(t-t_0)\right)e^{-\frac{\eta}{2}(t-t_0)}\Big|_{t_0+\epsilon} - \frac{\eta}{2}Am\,\cancel{\sin\left(-w'(t-t_0)\right)}\,e^{-\frac{\eta}{2}(t-t_0)}\Big|_{t_0+\epsilon} = 1$$

↳ $-w'mA = 1$

$\Rightarrow A = \frac{-1}{w'm}$

Then

$$G_R(t,t_0) = \frac{-1}{w'm}\sin\left(-w'(t-t_0)\right)e^{-\frac{\eta}{2}(t-t_0)} \quad \text{for } t > t_0$$

$$G_R(t,t_0) = 0 \quad \text{for } t < t_0$$

Combine both by using $\Theta(t-t_0) = \begin{cases} 1 & \text{for } t>t_0 \\ 0 & \text{otherwise.} \end{cases}$

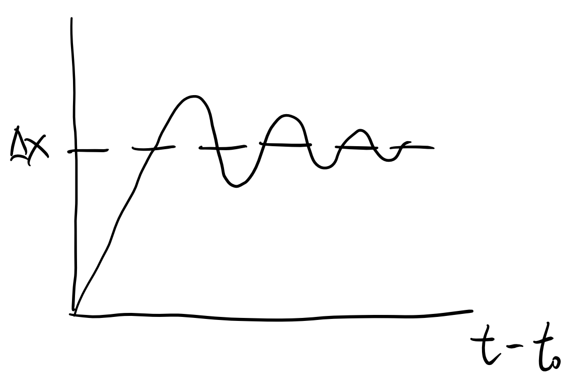
$$\hookrightarrow \quad G_R(t,t_0) = \frac{-\Theta(t-t_0)}{w' m} \sin(-w'(t-t_0))\, e^{\frac{-\eta}{2}(t-t_0)}$$

Examples:

Consider

$$f(t) = \begin{cases} 0 & t<0 \\ f_0 & t>0 \end{cases}$$

expect $t \to \infty \qquad \Delta x = f_0/m\omega_0^2$



Know

$$X_S(t) = \int_{-\infty}^{\infty} dt_0\, G_R(t,t_0)\, F(t_0)$$

and recall $G_R(t,t_0) = 0$ when $t < t_0$

or $t_0 > t$

We also know $F(t_0) = \begin{cases} 0 & t_0 < 0 \\ f_0 & t_0 > 0 \end{cases}$

$$X_S(t) = \underbrace{\cancel{\int_{-\infty}^{0}}}_{0} \int_0^t \underbrace{\cancel{\int_t^{\infty}}}_{0} dt_0 \, G_R(t,t_0) F(t_0)$$

$$\overset{!}{=} \int_0^t dt_0 \, G_R(t,t_0) F(t_0)$$

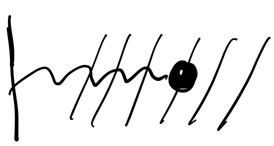
$$\overset{!}{=} \int_0^t dt_0 \, G_R(t,t_0) f_0$$

$$X_S(t) \overset{!}{=} \int_0^t e^{-\frac{\eta}{2}(t-t_0)} \frac{\sin(w_0(t-t_0))}{m w_0} f_0 \, dt_0$$

Finds the specific solution.

Example 2:

Brownian Motion:



$\leftarrow$ SHO in medium at temp T

Force by medium

$$\left(m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m w_0^2\right) X = F(t)$$

Consider random force since the timescale of the medium exerting the force is very short.

For random force:

$$\langle F(t) \rangle = 0$$

$$\langle F(t) F(t') \rangle = A\,\delta(t-t')$$ ← squared force is only correlated when $t = t'$

⇒ at late times,

$$P(x,p) = C e^{-H/T} \qquad H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2$$

$$E = \langle H \rangle = \frac{\int dx\,dp\; P(x,p)\, H}{\int dx\,dp\; P(x,p)} = T$$

If at rest at $t=0$:

$$x(t) = \int_0^{\infty} G_R(t,t_0)\, F(t_0)\, dt_0$$

$$x(t) = \int_0^{t} \frac{F(t_0)}{m\omega_0} e^{-\frac{\eta}{2}(t-t_0)} \sin \omega_0 (t-t_0)\, dt_0$$

Then the energy:

$$E = \frac{1}{2} m w_0^2 x^2 + \frac{1}{2} m \dot{x}^2$$

$$\doteq \int_0^t dt_0 \int_0^t dt_0' \frac{F(t_0)F(t_0')}{m^2 w_0^2} e^{\frac{-\eta}{2}(t-t_0)} e^{\frac{-\eta}{2}(t-t_0')}$$

$$\times \left(\frac{1}{2} m w_0^2 \left[\sin w_0(t-t_0) \sin w_0(t-t_0') + \cos w_0(t-t_0) \cos w_0(t-t_0') \right] \right)$$

Since $\langle F(t) F(t') \rangle = A\delta(t-t')$

then

$$\bar{E} = \int_0^t dt_0 \int_0^t dt_0' \frac{\langle F(t_0)F(t_0') \rangle}{m^2 w_0^2} e^{\frac{-\eta}{2}(t-t_0)} e^{\frac{-\eta}{2}(t-t_0')}$$

$$\times \left(\frac{1}{2} m w_0^2 \left[\sin w_0(t-t_0) \sin w_0(t-t_0') + \cos w_0(t-t_0) \cos w_0(t-t_0') \right] \right)$$

$$\doteq A \int_0^t dt_0 \frac{\delta(t_0 - t_0')}{m^2 w_0^2} e^{\frac{-\eta}{2}(t-t_0)} e^{\frac{-\eta}{2}(t-t_0')}$$

$\times \frac{1}{2} m w_0^2 ($ ... [illegible]

$$\cdots \frac{1}{2m\omega_0}\left(\sin\omega_0(t-t_0)\sin\omega_0(t-t')\right.$$

$$\left.+\cos\omega_0(t-t_0)\cos\omega_0(t-t')\right)\Big)$$

$$\stackrel{!}{=} \int_0^t dt_0\, A e^{-\eta(t-t_0)}\, \frac{1}{2} m\omega_0^2 \frac{1}{m^2\omega_0^2}\left(\sin^2\omega_0(t-t_0)+\cos^2\omega_0(t-t_0)\right)$$

$$\stackrel{!}{=} \int_0^t dt_0\, \frac{A}{2m}\, e^{-\eta(t-t_0)}$$

$$\stackrel{!}{=} \frac{A}{2m\eta}\, e^{-\eta(t-t_0)}\Big|_0^t$$

$$\stackrel{!}{=} \frac{A}{2m\eta}\left(1-e^{-\eta t}\right)$$

at $t > \infty$, $E = T$

so $\boxed{A = 2m\eta T}$ $\langle F(t)F(t')\rangle = 2m\eta T\,\delta(t-t')$

Example 3: HW#7 problem 2:

a) Determine the retarded green function for the following:

i) $\frac{da}{dt} - i\omega_0 a = 0$

Determine green function by integrating dt around $\int_{t_0-\epsilon}^{t_0+\epsilon}$

$$\int^{t_0+\epsilon}\left(\frac{dG_R}{} - i\omega_0 G(t,t_0)\right)dt = \int^{t+\epsilon}\delta(t-t_0)\,dt = 1$$

$$\int_{t_0-\epsilon} \left(\frac{d}{dt} \quad i\omega_0 G_R(t,t_0)\right)dt = \int_{t_0-\epsilon} \delta(t-t_0)\,dt = 1$$

$$G_R\Big|_{t_0-\epsilon}^{t_0+\epsilon} - \underbrace{\int_{t_0-\epsilon}^{t_0+\epsilon} i\omega_0 G_R(t,t_0)\,dt}_{=0} = 1$$

$G_R(t,t_0)=0$ for $t<t_0$
$\frac{d}{dt} G_R(t,t_0)=0$ for $t>t_0$
Homogeneous condition.

$$G_R\Big|_{t_0+\epsilon} - \underbrace{G_R\Big|_{t_0-\epsilon}}_{G_R=0 \text{ for } t<t_0} = 1$$

$$G_R\Big|_{t_0+\epsilon} = 1$$

For $\frac{da}{dt} - i\omega_0 a = 0$

$$G_R(t,t_0) = Ae^{-i\omega_0(t-t_0)+\phi}$$

$$G_R(t_0+\epsilon, t_0) = Ae^{i\phi} = 1$$

$$A = e^{-i\phi}$$

$$\boxed{\begin{array}{ll} G_R(t,t_0) = e^{-i\omega_0(t-t_0)} & \text{for } t>t_0 \\ G_R(t,t_0) = 0 & \text{for } t<t_0 \end{array}}$$

ii) $\ddot{x} + \eta\dot{x} = 0$

Get green function, replace equation with

$$\frac{d^2}{dt^2} G_R(t,t_0) + \eta \frac{d}{dt} G_R(t,t_0) = \delta(t-t_0)$$

$$\frac{d^2}{dt^2} G_R(t,t_0) + \eta \frac{d}{dt} G_R(t,t_0) = \delta(t-t_0)$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{d^2}{dt^2} G_R(t,t_0) + \eta \frac{d}{dt} G_R(t,t_0) \quad dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0) = 1$$

$$\frac{d}{dt} G_R(t,t_0) \Big|_{t_0-\epsilon}^{t_0+\epsilon} + \eta \, G_R \Big|_{t_0-\epsilon}^{t_0+\epsilon} = 1$$

$$\frac{d}{dt} G_R(t,t_0) \Big|_{t_0+\epsilon} - \underbrace{\frac{d}{dt} G_R(t,t_0) \Big|_{t_0-\epsilon}}_{\substack{=0 \\ \text{since } G_R|_{t_0-\epsilon}=0}} + \eta \Big(\underbrace{G_R \Big|_{t_0+\epsilon}}_{\substack{=0 \\ \text{due to} \\ \text{continuity}}} - \underbrace{G_R \Big|_{t_0-\epsilon}}_{\substack{=0 \\ \text{since } G_R = 0 \text{ for } t<t_0}} \Big) = 1$$

$$\frac{d}{dt} G_R(t,t_0) \Big|_{t_0+\epsilon} = 1$$

For $t > t_0$, $\ddot{G}_R + \eta \dot{G}_R = 0$

$$\frac{d}{dt} G_R(t,t_0) = A e^{-\eta(t-t_0)+\phi}$$

$$G_R(t,t_0) = \frac{A}{\eta} e^{-\eta(t-t_0)+\phi} = \frac{A}{\eta} e^{-\eta(t-t_0)} e^{\phi} + \text{const}$$

$$\stackrel{!}{=} A' e^{-\eta(t-t_0)} + B$$

Know $G_R(t,t_0)\Big|_{t=t_0} = 0$

$$G_R(t_0,t_0) = A' + B = 0 \quad \text{or} \quad A' = -B$$

$$-\eta(t-t_0)$$

So $\quad G_R(t,t_0) = B(-e^{\eta(t-t_0)} + 1)$

Know $\left.\dfrac{dG_R}{dt}\right|_{t_0+\epsilon} = 1$

$\hookrightarrow \quad B\eta e^{\eta(t-t_0)}\Big|_{t_0} = 1$

$\hookrightarrow \quad B\eta = 1$

$B = \dfrac{1}{\eta}$

then
$$\boxed{\begin{array}{ll} G_R(t,t_0) = \frac{1}{\eta}\left(1 - e^{-\eta(t-t_0)}\right) & \text{for } t > t_0 \\ G_R(t,t_0) = 0 & \text{for } t < t_0 \end{array}}$$

b) Consider : $\quad \ddot{x} + \omega_0^2 x = \dfrac{f(t)}{m}$

write as an equation for $a = \dot{x} + i\omega x$ and use previous green function to find the specific sol, $a(t)$.

$$a = \dot{x} + i\omega_0 x$$

$$\frac{da}{dt} = \ddot{x} + i\omega_0 \dot{x}$$

$$\frac{da}{dt} - i\omega_0 a = \ddot{x} + i\omega_0 \dot{x} - i\omega_0 \dot{x} + \omega_0^2 x$$

$$\overset{!}{=} \ddot{x} + \omega_0^2 x$$

$$\hookrightarrow \frac{da}{dt} - i\omega_0 a = \frac{f(t)}{m}$$

Find specific solution by integrating the green function:

$$a(t) = \int dt_0\, G_R(t,t_0) F(t_0)$$

$$= \int_{-\infty}^{\infty} dt_0\, \theta(t-t_0)\, e^{-i\omega_0(t-t_0)} \frac{f(t_0)}{m}$$

Since $G_R(t,t_0) = 0$ for $t < t_0$ or $t_0 > t$

$$= \int_{-\infty}^{t} dt_0\, \theta(t-t_0) e^{-i\omega_0(t-t_0)} \frac{f(t)}{m}$$

$$= e^{i\omega_0 t} \int_{-\infty}^{t} dt_0\, e^{i\omega_0 t_0} \frac{f(t)}{m}$$

c) Suppose the force approaches zero at $\pm\infty$.

If the oscillator was initially at rest
determine the total work done by external force.

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$$

$$= \frac{1}{2} m (\dot{x}^2 + \omega_0^2 x^2)$$

$$= \frac{1}{2} m (a a^*)$$

$$= \frac{1}{2} m |a|^2$$

$$= \frac{1}{2} m \left| \frac{e^{i\omega_0 t}}{m} \int^{t} dt_0\, e^{i\omega_0 t_0} f(t) \right|^2$$

$$-\infty \qquad \underbrace{\qquad\qquad}_{f(\hat{\omega})}$$

$$E \overset{!}{=} \frac{1}{2} m \frac{1}{m^2} \left|f(\hat{\omega})\right|^2 \cancel{e^{i\omega_0 t} e^{-i\omega_0 t}}$$

$$E \overset{!}{=} \frac{1}{2m} \left|f(\hat{\omega})\right|^2$$

d)

consider $f(t) = \begin{cases} F_0 & 0 < t < \tau \\ 0 & \text{otherwise.} \end{cases}$

$$E = \frac{1}{2m} \left| \int_0^\tau dt_0 \, e^{i\omega_0 t_0} F_0 \right|^2$$

$$\overset{!}{=} \frac{1}{2m} F_0^2 \left| \int_0^\tau dt_0 \, e^{i\omega_0 t_0} \right|^2$$

$$\overset{!}{=} \frac{1}{2m} F_0^2 \left| \frac{-i}{\omega_0} \left(e^{i\omega_0 \tau} - 1 \right) \right|^2$$

$$\overset{!}{=} \frac{1}{2m} F_0^2 \left(\frac{-i}{\omega_0} \left(e^{i\omega_0 \tau} - 1 \right) \right) \left(\frac{i}{\omega_0} \left(e^{-i\omega_0 \tau} - 1 \right) \right)$$

$$\overset{!}{=} \frac{1}{2m\omega_0^2} F_0^2 \left(2 - \left(e^{i\omega_0 \tau} + e^{-i\omega_0 \tau} \right) \right)$$

$$\overset{!}{=} \frac{1}{m\omega_0^2} F_0^2 \left(\underbrace{1 - \cos \omega_0 \tau}_{2\sin^2\left(\frac{\omega_0 \tau}{2}\right)} \right)$$

$$\overset{!}{=} \frac{2F_0^2}{m\omega_0^2} \sin^2\left(\frac{\omega_0 \tau}{2}\right)$$