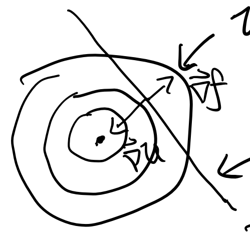
Constraints: (used if interested to find constraint force, like Tension)

$u = \frac{1}{2}k(x^2+y^2)$



← particle constrained to move along this path. $f(x,y) = y + x - 1 = 0$

Interpretation:

$$d\hat{u} = 0 = du(x,y) + d(\lambda f)$$

here the constraint acts as a force that balances original potential holding it in place.

Normally: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

↖ require to be zero so we're at minimum.

but now y and x are not independent coordinates due to $f(x,y)$

Goal: Need to find (x_0, y_0) such that they satisfy both $du = 0$ and $df = 0$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Since we have 2 unknowns, two equations, we multiply original by a constant λ → Lagrange Multiplier

$$du + \lambda df = \left(\frac{\partial u}{\partial x} + \lambda\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial f}{\partial y}\right)dy = 0$$

↖ force of constraint.

Now we have 3 equations, 3 unknowns:

① $\frac{\partial u}{\partial x} + \lambda\frac{\partial f}{\partial x}$ ① x

② $\frac{\partial u}{\partial y} + \lambda\frac{\partial f}{\partial y}$ ② y

③ $f(x,y) = 0$ ③ λ

Ex: $U = \frac{1}{2}k(x^2+y^2)$ $\quad f = x+y-1$

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x} = kx + \lambda = 0$$

$$\frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y} = ky + \lambda = 0$$

$$f(x,y) = x+y-1 = 0$$

$$\left.\begin{array}{l} \lambda = -kx \\ ky - kx = 0 \\ y = x \end{array}\right\} \quad \begin{array}{l} y+y-1=0 \\ \boxed{\begin{array}{l} x = y = 1/2 \\ \lambda = \frac{-k}{2} \end{array}} \end{array}$$

This is equivalent of extremizing: $u(x,y) + \lambda f(x,y) = \hat{u}(x,y)$

$$d\hat{u}(x,y,\lambda) = \frac{\partial \hat{u}}{\partial x}dx + \frac{\partial \hat{u}}{\partial y}dy + \frac{\partial \hat{u}}{\partial \lambda}d\lambda$$

$$= \left(\frac{\partial u}{\partial x} + \lambda\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial f}{\partial y}\right)dy + \frac{\partial \hat{u}}{\partial \lambda}d\lambda$$

Generalization:

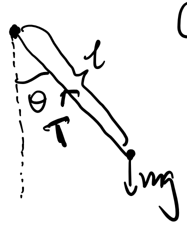
For: x^A : $A = 1 \cdots N$

$\quad \psi^\alpha(x^A)$ $\alpha = 1 \cdots m$ for $m < N$

$$\hat{u}(x^A, \lambda) = u(x^A) + \lambda_\alpha \psi^\alpha(x^A)$$

$$\frac{\partial u}{\partial x^A} = \frac{\partial}{\partial x^A} U(x^y) + \sum_\alpha \lambda_\alpha \frac{\partial}{\partial x^A} \psi(x^y)$$

Ex:



Constraint : $f(x,y) = x^2 + y^2 - l^2 = 0$

↳ forced to move in a circle.

$$ma^x = T_x$$
$$ma^y = T_y - mg$$

where $\vec{T} = -T_0(\sin\theta\,\hat{x}, -\cos\theta\,\hat{y})$

Since force of constraint do no work,

or $\vec{\nabla} f = 0$, where f is like the potential of constraint.

$$\vec{T} \propto \vec{\nabla} f \Rightarrow \vec{T}\cdot d\vec{r} = T_x\,dx + T_y\,dy = 0$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

then $\vec{T} = \lambda \vec{\nabla} f$ ← Tension is the force that creates the constraint:

$$f(x,y) = x^2 + y^2 - l^2 = 0$$

$$\vec{T} = \lambda\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}\right)$$
$$\stackrel{!}{=} \lambda(2x\,\hat{x} + 2y\,\hat{y})$$
$$\stackrel{!}{=} \underbrace{2\lambda x}_{-T_0\sin\theta}\,\hat{x} + \underbrace{2\lambda y}_{T_0\cos\theta}\,\hat{y}$$
$$\stackrel{!}{=} 2\lambda l\sin\theta\,\hat{x} - 2\lambda l\cos\theta\,\hat{y}$$

$$-2\lambda l\sin\theta\,\hat{x} \quad 2\lambda l\cos\theta\,\hat{y}$$

$$2\lambda l = -T_0$$

$$ma^x = 2\lambda l\sin\theta$$
$$ma^y = -2\lambda l\cos\theta - mg$$

<u>Constraints with Lagrangian:</u>

With constraints, Lagrangian is modified as?

$$\hat{L}(q^i, \dot{q}^i, t) = L(q^i, \dot{q}^i, t) + \lambda(t)\,\psi(q^i, t)$$

$$S[q^i, \lambda(t)] = \int L(q^i, \dot{q}^i, t) + \lambda(t)\psi(q^i, t)\, dt$$

$$S[q^i, \delta q^i, \lambda(t), \delta\lambda] = \int L + \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i + (\lambda + \delta\lambda)\left(\psi + \frac{\partial \psi}{\partial q^i} dq^i\right) dt$$

$$\frac{\partial \hat{L}}{\partial \lambda} = \psi$$

$$\delta S = \int \left\{ \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i + \lambda \frac{\partial \psi}{\partial q^i} dq^i + \delta\lambda\,\psi \right\} dt$$

$$\stackrel{!}{=} \int \left\{ \frac{\partial L}{\partial q^i} dq^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i} dq^i\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\right] dq^i + \lambda \frac{\partial \psi}{\partial q^i} dq^i + \delta\lambda\,\psi \right\} dt$$

$$\stackrel{!}{=} \int \left\{ \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\right] + \lambda \frac{\partial \psi}{\partial q^i} \right\} dq^i + \cancel{\frac{\partial L}{\partial \dot{q}^i} dq^i \Big|_{q_0}^{q_1}} + \cancel{\int \delta\lambda\,\psi\, dt}$$

$$dq^i \Big|_{q_0}^{q_1} = 0 \qquad \psi = 0$$

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New EoM equations with constraints:

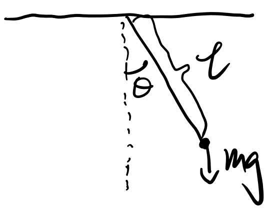
$$\boxed{\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\right] + \lambda \frac{\partial \psi}{\partial q^i} = 0 \quad \text{with} \quad \psi(q^i, t) = 0 = \frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right)}$$

*Use constraint only after solving for $\frac{\partial L}{\partial q^i}$ and $\frac{\partial L}{\partial \dot{q}^i}$

If $L = \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r})$ with $\psi(\vec{r}, t) = 0$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\right] = \frac{\partial L}{\partial q^i} + \lambda \frac{\partial \psi}{\partial q^i}$$

Example problem:



$x = \sin\theta\, l \qquad y = -\cos\theta\, l \qquad r = l$

$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$

$U = mgy = -mgl\cos\theta$

constraint: $r - l = 0$

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + mgr\cos\theta + \lambda(r - l)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{d}{dt}(m\dot{r}) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \psi}{\partial r} = m r \dot{\theta}^2 + mg\cos\theta + \lambda$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m r^2 \dot{\theta}\right) = \frac{\partial L}{\partial \theta} + \lambda \frac{\partial \psi}{\partial \theta} = -mgr\sin\theta$$

$$r - l = 0 \quad \text{or} \quad r = l \Rightarrow \dot{r} = \ddot{r} = 0$$

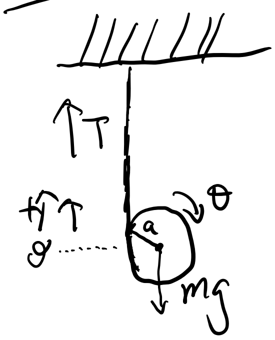
$$\frac{d}{dt}(m\dot{r}) = 0 = m r \dot{\theta}^2 + mg\cos\theta + \lambda$$

$$\frac{d}{dt}(m\dot{r}) = 0 = mr\dot{\theta}^2 + mg\cos\theta + \lambda$$

$$-T = \lambda = -mr\dot{\theta}^2 - mg\cos\theta$$

or $\boxed{T = mr\dot{\theta}^2 + mg\cos\theta}$ ⇐ λ is tension in r-direction.

Ex 2:



$y = a\theta$

$\psi = y - a\theta = 0$

T = Translational + rotational

$= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\omega^2$, $\omega^2 = \dot{\theta}^2$

$= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\int r^2 dm$

$= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\left(\frac{1}{2}ma^2\right)\dot{\theta}^2$

$$I = \int r^2 dm = \int_0^{2\pi} r^2 \frac{dm}{dA} dr\, r d\theta = \int_0^{2\pi}\int_0^a r^3 \frac{m}{\pi a^2} dr\, d\theta = \frac{a^4 m}{4\pi a^2} 2\pi = \frac{1}{2}ma^2$$

$U = mgy$

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}ma^2\dot{\theta}^2 - mgy + \lambda(y - a\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{y}) = \frac{\partial L}{\partial y} = -mg + \lambda \qquad ①$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(\frac{1}{2}ma^2\dot{\theta}\right) = \frac{\partial L}{\partial \theta} = -\lambda a \qquad ②$$

$$y = a\theta \Rightarrow \dot{y} = a\dot{\theta} \qquad ③$$

$$\left\{\frac{d}{dt}(m\dot{y}) = \frac{d}{dt}(ma\dot{\theta}) = -mg + \lambda\right\} a$$

$$\frac{d}{dt}\left(\frac{1}{2}ma^2\dot{\theta}\right) = -\lambda a$$

$$ma^2\ddot\theta = -mag + \lambda a$$
$$\frac{1}{2}ma^2\ddot\theta = -\lambda a$$
$$\frac{3}{2}ma^2\ddot\theta = -mag$$
$$\ddot\theta = -\frac{2}{3}\frac{g}{a}$$

$$\frac{1}{2}ma^2\ddot\theta = -\lambda a$$
$$\frac{1}{2}ma^2\left(-\frac{2}{3}\frac{g}{a}\right) = \frac{-1}{3}mg\cancel{a} = -\lambda\cancel{a}$$
$$-T = \lambda = \tfrac{1}{3}mg$$

Hamiltonian with constraints:

$$\mathcal{H} = p_i\dot q^i(p_i) - L$$
$$\stackrel{!}{=} p_i\dot q^i(p_i) - \hat L(q^i, \dot q^i, t, \lambda(t))$$
$$\stackrel{!}{=} p_i\dot q^i(p_i) - L(q^i, \dot q^i, t) - \lambda(t)\psi(q^i; t)$$
$$S[p+dp, q+dq, \lambda+d\lambda] = \int dt\,(p+dp)(\dot q + d\dot q) - \hat{\mathcal{H}}(p+dp, q+dq, \lambda+d\lambda)$$
$$\stackrel{!}{=} \int (p+dp)(\dot q + d\dot q) - \left\{\mathcal{H} + \frac{\partial\mathcal{H}}{\partial p}dp + \frac{\partial\mathcal{H}}{\partial q}dq + \frac{\partial\mathcal{H}}{\partial\lambda}d\lambda\right\}$$
$$\delta S \stackrel{!}{=} \int p\,d\dot q + \dot q\,dp - \left\{\frac{\partial\mathcal{H}}{\partial p}dp + \frac{\partial\mathcal{H}}{\partial q}dq + \frac{\partial\mathcal{H}}{\partial\lambda}d\lambda\right\} dt$$
$$\delta S = \int \left(\dot q - \frac{\partial\mathcal{H}}{\partial p}\right)dp + \frac{d}{dt}(p\,dq) - \dot p\,dq - \frac{\partial\mathcal{H}}{\partial q}dq - \frac{\partial\mathcal{H}}{\partial\lambda}d\lambda\,dt$$
$$\stackrel{!}{=} \int \left(\dot q - \frac{\partial\mathcal{H}}{\partial p}\right)dp + \left(-\dot p - \frac{\partial\mathcal{H}}{\partial q}\right)dq - \frac{\partial\mathcal{H}}{\partial\lambda}d\lambda\,dt$$

$$\frac{\partial H}{\partial p} = \dot{q} \qquad -\frac{\partial H}{\partial q} = \dot{p} \qquad \frac{\partial H}{\partial \lambda} = 0 = \psi(q^i, t)$$

$$\frac{\partial \hat{L}}{\partial \dot{q}} = p \qquad \dot{p} = \frac{\partial \hat{L}}{\partial q} \qquad \frac{\partial L}{\partial \lambda} = 0 = \psi(q^i, t)$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$