We have:

$$m \frac{d^2q}{dt^2} + m\eta \frac{dq}{dt} + m\omega_0^2 q = F = f_0 \cos\omega t$$

When near resonance, nonlinearity becomes important.
$\omega \sim \omega_0$

We have eq:
$$\ddot{x} + \eta\dot{x} + \omega_0^2 x + \beta x^3 = \frac{f(t)}{m}$$

First assume $\beta = 0$, work with linear case

then $$\ddot{x} + \eta\dot{x} + \omega_0^2 x = \frac{f(t)}{m} = \frac{f_0}{m}\cos\omega t$$

let $x^{(0)} = a(t)\cos(-\omega t + \phi(t))$

$$\dot{x}^{(0)} = \cancel{\dot{a}\cos(-\omega t+\phi)} - a\sin(-\omega t+\phi)\{-\omega + \cancel{\dot{\phi}}\}$$

$$\ddot{x}^{(0)} = \ddot{a}\cos(-\omega t+\phi) - \dot{a}\sin(-\omega t+\phi)(-\omega+\dot{\phi})$$
$$-\dot{a}\sin(-\omega t+\phi)\{-\omega+\dot{\phi}\} - a\{\cos(-\omega t+\phi)(-\omega+\dot{\phi})^2$$
$$+\ddot{\phi}\sin(-\omega t+\phi)\}$$

$$\downarrow$$

$$\simeq \cancel{-2\dot{a}\sin(-\omega t+\phi)} - \omega^2 a\cos(-\omega t+\phi)$$
$$\cancel{+2a\omega\dot{\phi}\cos(-\omega t+\phi)} + \mathcal{O}$$

Assume $\dot{a} = \dot{\phi} = 0$

$$\ddot{x} + \eta \dot{x} + \omega_0^2 x = \frac{f_0}{m}\cos(\omega t)$$

$$(-\omega^2 + \omega_0^2) a\cos(-\omega t + \phi) + \eta \omega a \sin(-\omega t + \phi) = \frac{f_0}{m}\cos(\omega t)$$

let $\omega = \omega_0 + (\omega - \omega_0)$

$\hookrightarrow \omega^2 = \omega_0^2 + (\omega - \omega_0)^2 + 2\omega_0(\omega - \omega_0)$

$$\Big[-\cancel{(\omega-\omega_0)^2} - 2\omega_0(\omega - \omega_0)\Big] a\cos(-\omega t + \phi) + \eta \omega a\sin(-\omega t + \phi) = \frac{f_0}{m}\cos(\omega t)$$

let $\omega \simeq \omega_0$

since $\omega - \omega_0 = \mathcal{O}(\epsilon)$

ignore $\mathcal{O}(\epsilon^2)$

$\hookrightarrow -2\omega_0(\omega - \omega_0) a\cos(-\omega t + \phi) + \eta \omega_0 a \sin(-\omega t + \phi) = \frac{f_0}{m}\cos(\omega t)$

$\hookrightarrow -(\omega - \omega_0) a\cos(-\omega t + \phi) + \frac{\eta}{2} a\sin(-\omega t + \phi) = \frac{f_0}{2m\omega_0}\cos(\omega t)$

let $\gamma = \sqrt{(\omega - \omega_0)^2 + (\eta/2)^2}$

$-(\omega - \omega_0) = \gamma\cos\phi_0$

$\eta/2 = \gamma\sin\phi_0$

$$\gamma\cos\phi_0\cos(-\omega t + \phi_0) + \gamma\sin\phi_0\sin(-\omega t + \phi_0) = \frac{f_0}{2m\omega_0}\cos\omega t$$

get $a = \frac{f_0}{2m\omega_0}\frac{1}{\left[(\omega - \omega_0)^2 + \left(\frac{\eta}{2}\right)^2\right]^{1/2}} \Rightarrow a_{max} = \frac{f_0}{m\omega_0\eta}$

$\tan\phi_0 = \frac{\eta/2}{\quad}$

$\omega_0 - \omega$

For nonlinear oscillator: $\ddot{x} + \eta\dot{x} + \omega_0^2 x + \beta x^3 = \frac{f(t)}{m}$

$$\Delta\omega = \frac{3}{8}\frac{\beta a^2}{\omega_0} = K a^2$$

frequency shift depends on amplitude.

Therefore, nonlinearity becomes important when

$$\Delta\omega \sim \eta/2$$

$$\hookrightarrow K a_{max}^2 \sim \eta/2$$

$$\left(\frac{K}{\eta/2}\right)\left(\frac{f_0}{m\omega\eta}\right)^2 \sim 1$$

Now for $\ddot{x} + \eta\dot{x} + \omega_0^2 x + \beta x^3 = \frac{f(t)}{m}$

use $x = a\cos(-\omega t + \phi)$

To zeroth order, find

$$\beta x^3 = 2\omega_0 \Delta\omega\, a\cos(-\omega t + \phi)$$

$$\stackrel{\downarrow}{=} 2\omega_0 K a^2\, a\cos(-\omega t + \phi)$$

Then

$$-2\omega_0(\omega - \omega_0) a\cos(-\omega t + \phi) + \eta\omega_0 a\sin(-\omega t + \phi) + 2\omega_0 \Delta\omega\, a\cos(-\omega t + \phi)$$

$$= \frac{f_0}{m}\cos(-\omega t + \phi)$$

$$\hookrightarrow = \left[-(\omega-\omega_0-\kappa a^2)\, a\cos(-\omega t+\phi) + \frac{\eta}{2}\sin(-\omega t+\phi)\right] = \frac{f_0}{2m\omega_0}\cos(-\omega t+\phi)$$

$$\hookrightarrow \quad a\gamma\big[\underbrace{\cos\phi_0\cos(-\omega t+\phi) + \sin\phi_0\sin(-\omega t+\phi)}_{\cos(\omega t)}\big] = \frac{f_0\cos\omega t}{2m\omega_0}$$

$$\gamma = \sqrt{(\omega-\omega_0-\kappa a^2)^2 + (\eta/2)^2}$$

then $\tan\phi_0 = \dfrac{\eta/2}{\omega_0+\kappa a^2-\omega}$

$$a^2\gamma^2 = \left(\frac{f_0}{2m\omega_0}\right)^2$$

$$a^2(\omega-\omega_0-\kappa a^2)^2 + a^2(\eta/2)^2 = \left(\frac{f_0}{2m\omega_0}\right)^2$$

Solve to get amplitude.

define: $\bar{f}^2 = \left(\dfrac{\kappa}{\eta/2}\right)\left(\dfrac{f_0}{m\omega_0\eta}\right)^2$

$$\bar{a}^2 = \frac{\kappa}{\eta/2}a^2$$

$$\bar{\xi} = \frac{\omega-\omega_0}{\eta/2}$$

$$\bar{a}^2(\xi-\bar{a}^2)^2 + \bar{a}^2 = \bar{f}^2$$

So $\bar{f} \ll 1$, $\bar{a} \ll 1$

then $\bar{a}^2\xi^2 + \bar{a}^2 = \bar{f}^2$

$$\bar{a} = \frac{\bar{f}}{\bar{\zeta}^2+1}$$ ← simple harmonic oscillator result.

For small $\bar{f}$ ⇒ one real root.

For large $\bar{f}$ ⇒ three real roots