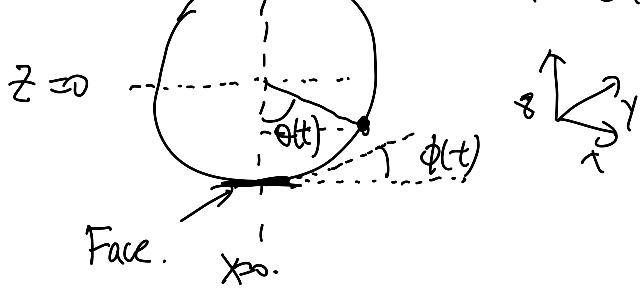
1) A bead on a driven ring:

→ a small bead slides freely on a ring with radius, a, in gravitational field.

→ The face of the ring is at an angle $\phi(t)$ with respect to x-axis, and it is driven harmonically with small amplitude ϕ_0, and high frequency ω,

$$\phi(t) = \phi_0 \cos \omega t$$

ϕ_0 is small, a first order.



a) Determine the lagrangian of the system without approximations.

$$x = a\sin\theta\cos\phi \Rightarrow \dot{x} = a\cos\theta\cos\phi\,\dot\theta - a\sin\theta\sin\phi\,\dot\phi$$
$$y = a\sin\theta\sin\phi \Rightarrow \dot{y} = a\cos\theta\sin\phi\,\dot\theta + a\sin\theta\cos\phi\,\dot\phi$$
$$z = -a\cos\theta \Rightarrow \dot{z} = a\sin\theta\,\dot\theta$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2}ma^2(\cos^2\theta\cos^2\phi\,\dot\theta^2 + \sin^2\theta\sin^2\phi\,\dot\phi^2 + \cos^2\theta\sin^2\phi\,\dot\theta^2 + \sin^2\theta\cos^2\phi\,\dot\phi^2 + \sin^2\theta\,\dot\theta^2)$$
$$= \frac{1}{2}ma^2(\sin^2\theta\,\dot\phi^2 + \dot\theta^2)$$

$$V = mgz = -mga\cos\theta$$

$$\mathcal{L} = \frac{1}{2}ma^2(\sin^2\theta\,\dot\phi^2 + \dot\theta^2) + mga\cos\theta$$

$$\dot\phi = -\omega\phi_0 \sin\omega t$$

$$\mathcal{L} = \frac{1}{2}ma^2\left[(\omega\phi_0\sin\theta\sin\omega t)^2 + \dot\theta^2\right] + mga\cos\theta$$

$$= \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{2}ma^2\omega^2\phi_0^2\sin^2\theta\sin^2\omega t + mga\cos\theta$$

$$= \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{4}ma^2\omega^2\phi_0^2\sin^2\theta(1-\cos 2\omega t) + mga\cos\theta$$

$\sin^2\omega t = \frac{1}{2}(1-\cos 2\omega t)$.

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = P_\theta = ma^2\dot\theta$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{1}{2}ma^2\omega^2\phi_0^2\sin^2\omega t\;2\sin\theta\cos\theta - mga\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \frac{\partial\mathcal{L}}{\partial\theta}$$

$$ma^2\ddot\theta = \left(ma^2\phi_0^2\omega^2\sin^2\omega t\cos\theta - mga\right)\sin\theta$$

$$\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)^2 = \frac{e^{2i\omega t} - 2 + e^{-2i\omega t}}{-4} = \frac{1}{2}(1 + \cos 2\omega t)$$

$$ma^2\ddot\theta = \left(\frac{1}{2}ma^2\phi_0^2\omega^2(1+\cos 2\omega t)\cos\theta - mga\right)\sin\theta$$

$$\boxed{\ddot\theta = \left(\frac{1}{2}\phi_0^2\omega^2(1+\cos 2\omega t)\cos\theta - g/a\right)\sin\theta}$$

b) Determine EOM for $t \gg 1/\omega$.

$$\mathcal{L} = \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{4}ma^2\omega^2\phi_0^2\sin^2\theta(1-\cos 2\omega t) + mga\cos\theta$$

$$\mathcal{L} = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{4}ma^2\omega^2\phi_0^2\sin^2\theta + mga\cos\theta - \frac{1}{4}ma^2\omega^2\phi_0^2\sin^2\theta\cos 2\omega t$$

$$V_0(\theta) = -\frac{1}{4}ma^2\omega^2\phi_0^2\sin^2\theta - mga\cos\theta$$

$$V_1(\theta) = \frac{1}{4}ma^2\omega^2\phi_0^2\sin^2\theta$$

$$\mathcal{L} = \frac{1}{2}ma^2\dot{\theta}^2 - V_0 - V_1\cos 2\omega t$$

$$ma^2\ddot{\theta} = -\frac{\partial V_0}{\partial\theta} - \frac{\partial V_1}{\partial\theta}\cos 2\omega t$$

let $\theta = \varphi + \xi$

slow, large term

fast, small

$$ma^2(\ddot{\varphi}+\ddot{\xi}) = -\frac{\partial V_0}{\partial\theta}\Big|_\varphi - \frac{\partial^2 V_0}{\partial\theta}\Big|_\varphi \xi - \frac{\partial V_1}{\partial\theta}\Big|_\varphi \cos 2\omega t - \frac{\partial^2 V_1}{\partial\theta}\Big|_\varphi \xi\cos 2\omega t$$

$$\cancel{ma^2}\ddot{\xi} = -\frac{\partial V_1}{\partial\theta}\Big|_\varphi = -\frac{1}{2}\cancel{ma^2}\omega^2\phi_0^2\sin\varphi\cos\varphi\cos 2\omega t$$

then $\xi = \frac{1}{8}\phi_0^2\sin\varphi\cos\varphi\cos 2\omega t$

$$ma^2\ddot{\varphi} = -\frac{\partial V_0}{\partial\theta}\Big|_\varphi - \cancel{\overline{\frac{\partial^2 V_0}{\partial\theta^2}\Big|_\varphi \xi}} - \overline{\frac{\partial^2 V_1}{\partial\theta^2}\Big|_\varphi \xi\cos 2\omega t}.$$

since $-\frac{\partial^2 V_0}{\partial\theta^2}\xi \propto \cos 2\omega t$

take average over ξ terms.

and $\frac{1}{T}\int_0^T dt \cos 2\omega t = 0$

and $T \int_0 \cos 2\omega t = 0$.

$$ma^2\ddot{\varphi} = -\frac{\partial V_0}{\partial\theta}\Big|_\varphi - \overline{\frac{\partial^2 V_1}{\partial\theta^2}\Big|_\varphi \xi \cos 2\omega t}$$

$$-\frac{\partial^2 V_1}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(-\frac{1}{2} ma^2\omega^2\phi_0^2 \sin\theta\cos\theta\right)\Big|_{\theta=\varphi}$$

$$\stackrel{!}{=} -\frac{1}{2} ma^2\omega^2\phi_0^2 \left(\cos^2\varphi - \sin^2\varphi\right)$$

$$-\frac{\partial^2 V_1}{\partial\theta^2}\xi\cos 2\omega t = -\frac{1}{2} ma^2\omega^2\phi_0^2 \left(1 - 2\sin^2\varphi\right)\xi\cos 2\omega t$$

$$\stackrel{!}{=} -\frac{1}{2} ma^2\omega^2\phi_0^2 \left(1 - 2\frac{1}{2}(1-\cos 2\varphi)\right)\xi\cos 2\omega t$$

$$\stackrel{!}{=} -\frac{1}{2} ma^2\omega^2\phi_0^2 \cos 2\varphi \left[\frac{1}{8}\phi_0^2 \sin\varphi\cos\varphi\cos^2 2\omega t\right]$$

$$\stackrel{!}{=} -\frac{1}{16} ma^2\omega^2\phi_0^4 \sin\varphi\cos\varphi\cos 2\varphi\cos^2 2\omega t.$$

$$-\frac{\partial^2 V_1}{\partial\theta^2}\xi\cos 2\omega t \propto \phi_0^4 \quad \leftarrow \text{ignore.}$$

$$ma^2\ddot{\varphi} = -\frac{\partial V_0}{\partial\theta}\Big|_\varphi.$$

$$\boxed{ma^2\ddot{\varphi} = \frac{1}{2} ma^2\omega^2\phi_0^2 \sin\varphi\cos\varphi - mga\sin\varphi}$$

c) When bead at bottom, $\theta = 0$

$$V_{eff} = V_0 = -\frac{1}{4} m a^2 w^2 \phi_0^2 \sin^2\theta - mga\cos\theta$$

see unstable if $\frac{\partial^2 V_{eff}}{\partial \theta^2} < 0$

$$\frac{\partial}{\partial \theta} V_{eff} = -\frac{1}{2} m a^2 w^2 \phi_0^2 \sin\theta \cos\theta + mga\sin\theta$$

$$\frac{\partial^2}{\partial \theta^2} V_{eff} = -\frac{1}{2} m a^2 w^2 \phi_0^2 \left(\cos^2\theta - \sin^2\theta\right) + mga\cos\theta$$

$$= -\frac{1}{2} m a^2 w^2 \phi_0^2 \left(1 - 2\sin^2\theta\right) + mga\cos\theta$$

$$= -\frac{1}{2} m a^2 w^2 \phi_0^2 \left(1 - 2\tfrac{1}{2}(1 - \cos 2\theta)\right) + mga\cos\theta$$

$$= -\frac{1}{2} m a^2 w^2 \phi_0^2 \cos 2\theta + mga\cos\theta$$

For $\theta = \delta\theta$ around $\theta = 0$.

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

$$\cos 2\theta = 1 - 2\theta^2$$

$$= -\frac{1}{2} m a^2 w^2 \phi_0^2 \left(1 - 2\theta^2\right) + mga\left(1 - \frac{\theta^2}{2}\right)$$

$$\frac{\partial}{\partial\theta^2} V_{eff} \frac{1}{ma^2} = -\frac{1}{2} w^2 \phi_0^2 \, 1 + w^2 \phi_0^2 \theta^2 + g/a - \frac{1}{2} g/a \, \theta^2$$

$$= \cancel{\left(w^2 \phi_0^2 - \frac{w_0^2}{2} \right) \theta^2} + w_0^2 - \frac{1}{2} w^2 \phi_0^2$$

first order

require $w_0^2 > \frac{1}{2} w^2 \phi_0^2$ to be stable.

or $w^2 < \frac{2 w_0^2}{\phi_0^2} = w_c^2$

$$w_c = \sqrt{\frac{2g}{a\phi_0^2}}$$

$w < w_c$ for stable

$w > w_c$ for unstable.

d) Determine steady state position for $w > w_c$.

Take limit $w \to$ large.

$$ma^2 \ddot{\varphi} = \frac{1}{2} ma^2 w^2 \phi_0^2 \sin\varphi \cos\varphi - mga \sin\varphi$$

Steady state: $\dot{\theta} = 0$ $\ddot{\theta} = 0$

$$0 = \underbrace{\left(\frac{1}{2} w^2 \phi_0^2 \cos\theta - w_0^2 \right)}\sin\theta$$

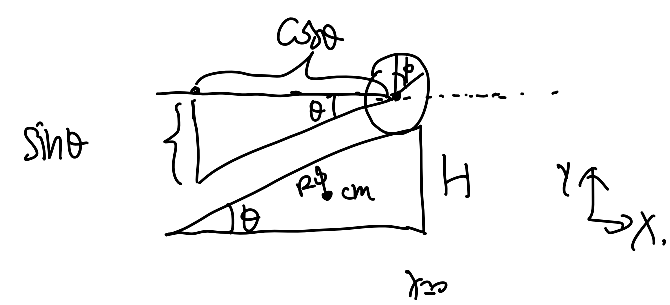
$$= 0$$

$$W_0^2 = \frac{1}{2} W^2 \phi_0^2 \cos\theta$$

$$\frac{2W_0^2}{W^2\phi_0^2} = \cos\theta$$

$$\boxed{\begin{aligned} &\theta = \cos^{-1}\left(\frac{2W_0^2}{W^2\phi_0^2}\right) \\ &\text{if } W^2\phi_0^2 \gg 2W_0^2 \\ &\cos^{-1}(0) = \frac{\pi}{2} = \theta. \end{aligned}}$$

2) A cylinder on a ramp:



a) Solid cylinder of mass m, height, H

$$I_{cylinder} = \int dm\, r^2 = \int \frac{dm}{dA} r^2\, r\,dr\,d\theta = \int \frac{m}{\pi R^2} r^3\,dr\,d\theta$$

$$= \frac{m}{\pi R^2} \frac{R^4}{4} 2\pi$$

$$I_{cylinder} = \frac{mR^2}{2}$$

$$y_{ramp} = 0$$

$$X_{ramp} = X_0 + \text{const}$$
$$X_{cylinder} = R\phi\cos\theta + x_0 + \text{const}$$
$$Y_{cylinder} = R\phi\sin\theta + \text{const}$$

$$\dot{X}_{ramp} = \dot{X}_0$$
$$\dot{X}_{cyl} = R\dot{\phi}\cos\theta - \cancel{R\phi\sin\theta\,\dot{\theta}} + \dot{X}_0$$
$$\dot{Y}_{cyl} = R\dot{\phi}\sin\theta + \cancel{R\phi\cos\theta\,\dot{\theta}}$$
$$L = \tfrac{1}{2}(2m)\dot{X}_0^2 + \tfrac{1}{2}m(R\dot{\phi}\cos\theta + \dot{X}_0)^2 + \tfrac{1}{2}I\dot{\phi}^2$$
$$+ \tfrac{1}{2}m(-R\dot{\phi}\sin\theta)^2 - mgR\phi\sin\theta$$
$$= m\dot{X}_0^2 + \tfrac{1}{2}m\left(R^2\dot{\phi}^2\cos^2\theta + 2R\dot{\phi}\cos\theta\dot{X}_0 + \dot{X}_0^2\right) + \tfrac{1}{2}\tfrac{mR^2}{2}\dot{\phi}^2$$
$$+ \tfrac{1}{2}mR^2\dot{\phi}^2\sin^2\theta - mgR\phi\sin\theta$$
$$\boxed{L = \tfrac{3}{2}m\dot{X}_0^2 + \tfrac{3}{4}mR^2\dot{\phi}^2 + mR\dot{\phi}\cos\theta\dot{X}_0 - mgR\phi\sin\theta}$$

b) $$\frac{\partial L}{\partial \dot{X}_0} = 3m\dot{X}_0 + mR\cos\theta\,\dot{\phi} = P_x = 0 \Rightarrow \dot{X}_0 = \frac{-mR\cos\theta\,\dot{\phi}}{3m}$$
$$\frac{\partial L}{\partial \dot{\phi}} = \tfrac{3}{2}mR^2\dot{\phi} + mR\cos\theta\dot{X}_0 = P_\phi$$
$$\frac{\partial L}{\partial \phi} = -mgR\sin\theta$$

$$\frac{d}{dt}\left(\tfrac{3}{2}mR^2\dot{\phi} + mR\cos\theta\dot{X}_0\right) = -mgR\sin\theta$$
$$\frac{d}{dt}\left(\tfrac{3}{2}mR^2\dot{\phi} + mR\cos\theta\left(-R\cos\theta\dot{\phi}\right)\right) = -mgR\sin\theta$$

$\ldots dt\left(2 \ldots \left(\frac{\ldots}{3}\right)\right) \ldots$

$$\left(\frac{3}{2} - \frac{1}{3}\cos^2\theta\right) mR^2 \ddot{\phi} = -mgR\sin\theta$$

$$h = P_x\dot{x} + P_\phi\dot{\phi} - L$$

$$\stackrel{!}{=} 3m\dot{x}_0^2 + mR\cos\theta\,\dot{\phi}\dot{x}_0 + \frac{3}{2}mR^2\dot{\phi}^2 + mR\cos\theta\,\dot{\phi}\dot{x}_0$$
$$- \left(\frac{3}{2}m\dot{x}_0^2 + \frac{3}{4}mR^2\dot{\phi}^2 + mR\dot{\phi}\cos\theta\,\dot{x}_0 - mgR\phi\sin\theta\right)$$

$$h \stackrel{!}{=} \frac{3}{2}m\dot{x}_0^2 + \frac{3}{4}mR^2\dot{\phi}^2 + mR\cos\theta\,\dot{\phi}\dot{x}_0 + mgR\phi\sin\theta$$

Initially:

$$h = mg\underbrace{R\phi\sin\theta}_{H} = mgH$$

at bottom.

$$mgH = \frac{3}{2}m\dot{x}_0^2 + \frac{3}{4}mR^2\dot{\phi}^2 + mR\cos\theta\,\dot{\phi}\dot{x}_0 + \cancel{mgR\phi\sin\theta}$$

know $\quad \dot{x}_0 = \dfrac{-\cancel{m}R\cos\theta\,\dot{\phi}}{3\cancel{m}}$

$$mgH = \frac{3}{2}m\left(\frac{R^2\cos^2\theta\,\dot{\phi}^2}{9}\right) + \frac{3}{4}mR^2\dot{\phi}^2 + mR\cos\theta\left(\frac{-R\cos\theta\,\dot{\phi}}{3}\right)\dot{\phi}$$
$$\stackrel{!}{=} \frac{1}{6}mR^2\cos^2\theta\,\dot{\phi}^2 + \frac{3}{4}mR^2\dot{\phi}^2 - \frac{1}{3}mR^2\cos^2\theta\,\dot{\phi}^2$$

$$\cancel{m}gH \stackrel{!}{=} \left(\frac{3}{4} - \frac{1}{6}\cos^2\theta\right)\cancel{m}R^2\dot{\phi}^2$$

$$gH = \left(\frac{9}{12} - \frac{2}{12}\cos^2\theta\right) R^2 \dot\phi^2$$

$$\dot\phi = \sqrt{\frac{gH}{R^2}\,\frac{12}{9-2\cos^2\theta}}$$

then $$\dot x_0 = \frac{-\cancel{m}R\cos\theta\,\dot\phi}{3\cancel{m}} = \frac{-R\cos\theta}{3}\sqrt{\frac{gH}{R^2}\,\frac{12}{9-2\cos^2\theta}}$$

c) Determine the normal force.

What is the condition that cylinder remains on the slope.

$$y = R\phi\sin\theta$$

$$L = \frac{1}{2}2m\dot x_0^2 + \frac{1}{2}m(\dot x_0 + R\dot\phi\cos\theta)^2 + \frac{1}{2}m\dot y^2 + \frac{1}{2}\frac{1}{2}mR^2\dot\phi^2$$
$$- mgy + \lambda(y - R\phi\sin\theta)$$

$$\frac{\partial L}{\partial \dot x_0} = 3m\dot x_0 + mR\dot\phi\cos\theta = \text{constant} \Rightarrow 3m\ddot x_0 = -mR\cos\theta\,\ddot\phi$$

$$\frac{\partial L}{\partial \dot\phi} = \frac{1}{2}mR^2\dot\phi + m\dot x_0 R\cos\theta + mR^2\dot\phi\cos^2\theta$$

$$\frac{\partial L}{\partial \phi} = -\lambda R\sin\theta \qquad mR^2\left(\frac{1}{2}+\cos^2\theta\right)\ddot\phi + mR\cos\theta\,\ddot x_0 = -\lambda R\sin\theta \quad ②$$

$$\frac{\partial L}{\partial \dot y} = m\dot y$$

$$\frac{\partial L}{\partial y} = -mg + \lambda$$

$$\Big\} \; m\ddot y = -mg + \lambda \qquad ③$$

$$\frac{\partial L}{\partial \lambda} = y - R\phi\sin\theta = 0 \Rightarrow \ddot{y} = R\ddot{\phi}\sin\theta \quad (4)$$

with (3) and (4)

$$mR\ddot{\phi}\sin\theta + mg = \lambda$$

$$\ddot{\phi} = \frac{\lambda - mg}{mR\sin\theta}$$

Using (1):

$$\ddot{x}_0 = \frac{-mR\cos\theta\,\ddot{\phi}}{3m} = \frac{-R\cos\theta\,\ddot{\phi}}{3} = \frac{-R\cos\theta}{3}\left(\frac{\lambda - mg}{mR\sin\theta}\right)$$

$$= \frac{mg - \lambda}{3m\sin\theta}$$

Using (2):

$$mR^2\left(\frac{1}{2} + \cos^2\theta\right)\ddot{\phi} + mR\cos\theta\,\ddot{x}_0 = -\lambda R\sin\theta \quad (2)$$

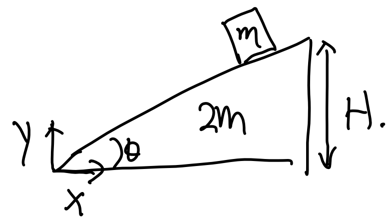
$$\hookrightarrow \cancel{m}R^{\cancel{2}}\left(\frac{1}{2} + \cos^2\theta\right)\left(\frac{\lambda - mg}{\cancel{mR}\sin\theta}\right) + \cancel{m}R\cos^2\theta\left(\frac{mg - \lambda}{3\cancel{m}\sin\theta}\right) = -\lambda R\sin\theta$$

$$\frac{\left(\frac{1}{2} + \cos^2\theta\right)(mg - \lambda) + \cos^2\theta\,(\lambda - mg)\,\frac{1}{3}}{\sin^2\theta} = \lambda$$

$$\lambda = \frac{(mg - \lambda)\left[\frac{1}{2} + \cos^2\theta - \frac{1}{3}\cos^2\theta\right]}{\sin^2\theta}$$

$$\lambda = \frac{mg\left(\frac{1}{2} + \frac{2}{3}\cos^2\theta\right)}{\left(\frac{3}{2} + \frac{2}{3}\cos^2\theta\right)}$$

3) Consider a block of mass on the ramp



$X = X_0 + r\cos\theta$

$Y = r\sin\theta$

$L = \frac{1}{2}(2m)\dot{X}_0^2 + \frac{1}{2}m(\dot{X}_0 + \dot{r}\cos\theta)^2 + \frac{1}{2}m\dot{r}^2\sin^2\theta - mgr\sin\theta$

$L = m\dot{X}_0^2 + \frac{1}{2}m\dot{X}_0^2 + m\dot{X}_0\dot{r}\cos\theta + \frac{1}{2}m\dot{r}^2\sin^2\theta + \frac{1}{2}m\dot{r}^2\cos^2\theta - mgr\sin\theta$

$L = \frac{3}{2}m\dot{X}_0^2 + \frac{1}{2}m\dot{r}^2 + m\dot{X}_0\dot{r}\cos\theta - mgr\sin\theta$

$\frac{\partial L}{\partial \dot{X}_0} = 3m\dot{X}_0 + m\dot{r}\cos\theta = 0$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{X}_0\cos\theta$

$\frac{\partial L}{\partial r} = -mg\sin\theta$

$m\ddot{r} + m\ddot{X}_0\cos\theta = -mg\sin\theta$

$3m\ddot{X}_0 + m\ddot{r}\cos\theta = 0$

$m\ddot{X}_0 = -\frac{m\ddot{r}\cos\theta}{3}$

$m\ddot{r} - \frac{m\ddot{r}\cos^2\theta}{3} = -mg\sin\theta$

$\left(1 - \frac{\cos^2\theta}{3}\right)m\ddot{r} = -mg\sin\theta$

What is $\dot{r}$ at end of ramp?

$$E = h = p_{x_0}\dot{x}_0 + p_r\dot{r} - L$$

$$= 3m\dot{x}_0^2 + m\dot{r}\dot{x}_0\cos\theta + m\dot{r}^2 + m\dot{r}\dot{x}_0\cos\theta$$

$$-\left(\frac{3}{2}m\dot{x}_0^2 + \frac{1}{2}m\dot{r}^2 + m\dot{x}_0\dot{r}\cos\theta - mgr\sin\theta\right)$$

$$E = \frac{3}{2}m\dot{x}_0^2 + \frac{1}{2}m\dot{r}^2 + m\dot{r}\dot{x}_0\cos\theta + mgr\sin\theta$$

at top: $E = mgH$

$$mgH = \frac{3}{2}m\dot{x}_0^2 + \frac{1}{2}m\dot{r}^2 + m\dot{r}\dot{x}_0\cos\theta + mgr\sin\theta.$$

at bottom $mgr\sin\theta = 0$.

$$mgH = \frac{3}{2}m\dot{x}_0^2 + \frac{1}{2}m\dot{r}^2 + m\dot{r}\dot{x}_0\cos\theta$$

we know $3m\dot{x}_0 + m\dot{r}\cos\theta = 0$

$$\dot{x}_0 = \frac{-\dot{r}\cos\theta}{3}$$

$$mgH = \frac{3}{2}m\frac{\dot{r}^2\cos^2\theta}{9} + \frac{1}{2}m\dot{r}^2 + m\dot{r}\left(-\frac{\dot{r}\cos\theta}{3}\right)\cos\theta$$

$$= \frac{1}{6}m\dot{r}^2\cos^2\theta + \frac{1}{2}m\dot{r}^2 - \frac{m\dot{r}^2\cos^2\theta}{3}$$

$$= \frac{1}{2}m\dot{r}^2 - \frac{1}{6}m\dot{r}^2\cos^2\theta$$

$$\cancel{m}gH = \left(\frac{1}{2} - \frac{1}{6}\cos^2\theta\right)\cancel{m}\dot{r}^2$$

$\dot{r} = \sqrt{\left(\frac{10}{3-\cos^2\theta}\right) gH}$ ← at bottom.

To Find normal force:

use constraint $(y - r\sin\theta) = 0$.

$L = \frac{1}{2}2m\dot{x}_0^2 + \frac{1}{2}m(\dot{x}_0 + \dot{r}\cos\theta)^2 + \frac{1}{2}m\dot{y}^2 - mgy + \lambda(y - r\sin\theta)$

$= \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}2m\dot{x}_0^2 + m\cos\theta\,\dot{x}_0\dot{r} + \frac{1}{2}m\dot{r}^2\cos^2\theta + \frac{1}{2}m\dot{y}^2 - mgy + \lambda(y - r\sin\theta)$

$\frac{\partial L}{\partial \dot{x}_0} = 3m\dot{x}_0 + m\cos\theta\,\dot{r} = \text{constant} \Rightarrow 3m\ddot{x}_0 + m\cos\theta\,\ddot{r} = 0$ ①

$\frac{\partial L}{\partial \dot{r}} = m\cos\theta\,\dot{x}_0 + m\dot{r}\cos^2\theta$

$\frac{\partial L}{\partial r} = -\lambda\sin\theta$

$m\cos\theta\,\ddot{x}_0 + m\ddot{r}\cos^2\theta = -\lambda\sin\theta$ ②

$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{\partial L}{\partial y} = -mg + \lambda$

$m\ddot{y} = -mg + \lambda$ ③

$\frac{\partial L}{\partial \lambda} = y - r\sin\theta = 0$ $\quad \ddot{y} = \ddot{r}\sin\theta$ ④

with ③, ④

$m\ddot{r}\sin\theta = -mg + \lambda$

$\ddot{r} = \frac{\lambda - mg}{m\sin\theta}$

$3m\ddot{x}_0 + m\cos\theta\,\ddot{r} = 3\cancel{m}\ddot{x}_0 + \cancel{m}\cos\theta\left(\frac{\lambda - mg}{m\sin\theta}\right) = 0$

$$\ddot{x}_0 = \frac{(mg-\lambda)\cos\theta}{3m\sin\theta}$$

② $m\cos\theta\ddot{x}_0 + m\ddot{r}\cos^2\theta = -\lambda\sin\theta$

$$\frac{(mg-\lambda)}{3\sin\theta}\cos^2\theta + \cancel{m}\cos^2\theta\frac{\lambda-mg}{\cancel{m}\sin\theta} = -\lambda\sin\theta$$

$$\lambda = \frac{\cos^2\theta}{\sin^2\theta}\left(\frac{1}{3}(\lambda-mg) + mg-\lambda\right)$$

$$\lambda = \frac{\cos^2\theta}{\sin^2\theta}\left(-\frac{2}{3}\lambda + \frac{2}{3}mg\right)$$

$$\lambda\left(1+\frac{2}{3}\frac{\cos^2\theta}{\sin^2\theta}\right) = \frac{\cos^2\theta\,\frac{2}{3}mg}{\sin^2\theta}$$

$$\lambda = \frac{\frac{\cos^2\theta}{\sin^2\theta}\frac{2}{3}mg}{1+\frac{2}{3}\frac{\cos^2\theta}{\sin^2\theta}}$$

3) Ion Trapping:

Consider particle with charge q, mass m.

$$\Phi(x,y,z) = \frac{V_1}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right) \qquad \rho = \sqrt{x^2+y^2}$$

$$B = B_0\hat{z}$$

a) Explain why motion of charged particle is unstable at origin without B field.

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\Phi + q\cancel{\frac{\dot{\vec{r}}}{c}\cdot\vec{A}} \quad \text{without B field.}$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\frac{V_0}{L^2}\left(z^2 - \frac{1}{2}(x^2+y^2)\right)$$

$V = q\Phi$

$$\frac{\partial^2 V}{\partial r^2} = q\frac{V_0}{L^2}\left(2\hat{z} - \hat{x} - \hat{y}\right)$$

we see that $\frac{\partial^2 V}{\partial r^2} < 0$ for $\hat{x}$ and $\hat{y}$

so its unstable.

b) $$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\Phi + \underbrace{\frac{q}{c}\frac{B_0}{2}}_{\frac{m\omega_B}{2}}(-\dot{x}y + x\dot{y})$$

$X = \rho\cos\phi$

$\dot{x} = \dot{\rho}\cos\phi - \rho\sin\phi\dot{\phi}$

$\dot{x}^2 = \dot{\rho}^2\cos^2\phi + \rho^2\sin^2\phi\dot{\phi}^2 - 2\cos\phi\sin\phi\rho\dot{\rho}\dot{\phi}$

$y = \rho\sin\phi$

$\dot{y} = \dot{\rho}\sin\phi + \rho\cos\phi\dot{\phi}$

$\dot{y}^2 = \dot{\rho}^2\sin^2\phi + 2\rho\cos\phi\sin\phi\dot{\rho}\dot{\phi} + \rho^2\cos^2\phi\dot{\phi}^2$

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$
$$-q\frac{V_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right) + \frac{m\omega_B}{2}\Big(\rho\cos\phi[\dot{\rho}\sin\phi + \rho\cos\phi\dot{\phi}]$$
$$- \rho\sin\phi[\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi}]\Big)$$

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$
$$-\frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right) + \frac{m\omega_B}{2}\left(\rho^2\dot{\phi}\right)$$

c) $$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial z} = -\frac{2qV_0}{L^2}z$$

$$m\ddot{z} + \frac{2qV_0}{L^2}z = 0$$

$$\ddot{z} + \frac{2qV_0}{mL^2}z = 0$$

$$\ddot{z} + \omega_z^2 z = 0 \qquad \omega_z^2 = \frac{2qV_0}{mL^2}$$

$$z = A\cos\omega_z t + \phi$$

$$\varepsilon_z = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\omega_z^2 z^2$$

$$\frac{\partial E_z}{\partial t} = 0$$

d) $$\frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi + \frac{m\omega_B}{2}\rho^2 = P_\phi = \text{constant.}$$

$$\hookrightarrow \dot\phi = \frac{P_\phi - \frac{m\omega_B}{2}\rho^2}{m\rho^2}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\overset{!}{=} \frac{P_\phi}{m\rho^2} - \frac{\omega_B}{2}$$

$$\frac{\partial L}{\partial \dot\rho} = m\dot\rho$$

$$h = P_z\dot z + P_\phi\dot\phi + P_\rho\dot\rho - L$$

$$\overset{!}{=} m\dot z^2 + \left(m\rho^2\dot\phi + \frac{m\omega_B}{2}\rho^2\right)\dot\phi + m\dot\rho^2$$

$$-\left\{\frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\right)\right.$$

$$\left.-\frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right) + \frac{m\omega_B}{2}\left(\rho^2\dot\phi\right)\right\}$$

$$h \overset{!}{=} \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\phi^2 + \frac{1}{2}m\dot z^2 + \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right)$$

↑ total energy is also conserved.

e) Use Routhian

$$R' = P_\phi \phi - L$$

$$\stackrel{!}{=} P_\phi \left[\frac{P_\phi}{m\rho^2} - \frac{\omega_B}{2}\right]$$

$$- \left\{ \frac{1}{2} m\left(\dot\rho^2 + \rho^2 \dot\phi^2 + \dot z^2\right) - \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right) + \frac{m\omega_B}{2}\left(\rho^2 \dot\phi\right) \right\}$$

$$\stackrel{!}{=} -\frac{1}{2} m\dot\rho^2 - \frac{1}{2} m\dot z^2 + \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right)$$

$$+ \frac{P_\phi^2}{m\rho^2} - \frac{P_\phi \omega_B}{2} - \frac{1}{2} m\rho^2 \left[\frac{P_\phi}{m\rho^2} - \frac{\omega_B}{2}\right]^2$$

$$- \frac{m\omega_B}{2}\rho^2 \left(\frac{P_\phi}{m\rho^2} - \frac{\omega_B}{2}\right)$$

$$\stackrel{!}{=} -\frac{1}{2} m\dot\rho^2 - \frac{1}{2} m\dot z^2 + \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right)$$

$$+ \frac{P_\phi^2}{2m\rho^2} - \frac{1}{2} m\rho^2 \left(\frac{\omega_B}{2}\right)^2 - \frac{P_\phi \omega_B}{2} + \frac{m\omega_B^2 \rho^2}{4}$$

$$\stackrel{!}{=} -\frac{1}{2} m\dot\rho^2 - \frac{1}{2} m\dot z^2 + \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right)$$

$$\frac{p_\phi^2}{2m\rho^2} + \frac{1}{8} m\omega_B^2\rho^2 - \frac{p_\phi \omega_B}{2}$$

$$\boxed{\omega_z^2 = \frac{2qV_0}{mz^2}}$$

$L_{eff} = -R$

$$\frac{\partial L}{\partial \dot\rho} = m\dot\rho$$

$$\frac{\partial L}{\partial \rho} = -\frac{\partial}{\partial \rho}\Bigg(\underbrace{-\frac{1}{4} m\omega_z^2\rho^2 + \frac{p_\phi^2}{2m\rho^2} + \frac{1}{8} m\omega_B^2\rho^2}_{V_{eff}(\rho)}\Bigg)$$

$$V_{eff}(\rho) = \left(-\frac{1}{4} m\omega_z^2 + \frac{1}{8} m\omega_B^2\right)\rho^2 + \frac{p_\phi^2}{2m\rho^2}$$

To have bounded motion

$$\frac{\partial^2 V_{eff}}{\partial \rho^2} = 2\Bigg(\underbrace{-\frac{1}{4} m\omega_z^2 + \frac{1}{8} m\omega_B^2}_{\text{must be} > 0}\Bigg) - \frac{3p_\phi^2}{m\rho^4} > 0$$

$$\frac{1}{8} m\omega_B^2 > \frac{1}{4} m\omega_z^2$$

$$\omega_B^2 > 2\omega_z^2$$

$$\left(\frac{qB}{mc}\right)^2 > 2\left(\frac{2qV_0}{mL^2}\right)$$

$$\boxed{B > \frac{mc}{q}\sqrt{4\frac{qV_0}{mL^2}} = B_{crit}}$$

ii) Find ρ_{min}, ρ_{max}.

$$h = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right)$$

$$h = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\left[\frac{p_\phi}{m\rho^2} - \frac{\omega_B}{2}\right]^2$$
$$+ \frac{1}{2}m\dot{z}^2 + \frac{m\omega_z^2}{2}z^2 - \frac{m\omega_z^2}{4}\rho^2$$

$$E = \frac{1}{2}m\dot{\rho}^2 + \frac{p_\phi^2}{2m\rho^2} - \frac{1}{2}\cancel{m\rho^2}\frac{p_\phi\omega_B}{\cancel{m\rho^2}} + \frac{1}{8}m\omega_B^2\rho^2$$
$$- \frac{m\omega_z^2}{4}\rho^2 + \varepsilon_z$$

$$E - \varepsilon_z + \frac{1}{2}p_\phi\omega_B = \frac{1}{2}m\dot{\rho}^2 + \frac{p_\phi^2}{2m\rho^2} - \frac{m\omega_z^2}{4}\rho^2 + \frac{1}{8}m\omega_B^2\rho^2$$

$$\underbrace{\qquad\qquad\qquad}_{\bar{E}} \qquad = \frac{1}{2}m\dot{\rho}^2 + \frac{P_\phi^2}{2m\rho^2} + \frac{1}{8}m\left(\omega_B^2 - 2\omega_z^2\right)\rho^2$$

At $\rho = \rho_{min,\,max}$ $\dot{\rho} = 0$

$$\bar{E} = \frac{P_\phi^2}{2m\rho_*^2} + \frac{1}{8}m\left(\omega_B^2 - 2\omega_z^2\right)\rho_*^2$$

$$\frac{1}{8}\left(\omega_B^2 - 2\omega_z^2\right)\rho_*^4 - \bar{E}\rho_*^2 + \frac{P_\phi^2}{2m} = 0$$

$$\rho_* = \left[\frac{\bar{E} \pm \sqrt{\bar{E}^2 - 4\frac{1}{8}\left(\omega_B^2 - 2\omega_z^2\right)\frac{P_\phi^2}{2m}}}{\frac{1}{4}\left(\omega_B^2 - 2\omega_z^2\right)}\right]^{1/2}$$

$$\rho_\pm = \left[\frac{\bar{E} \pm \sqrt{\bar{E}^2 - \frac{1}{4}\frac{\left(\omega_B^2 - 2\omega_z\right)P_\phi^2}{2}}}{\frac{1}{4}\left(\omega_B^2 - 2\omega_z^2\right)}\right]^{1/2}$$

Here

$\rho_+ = \rho_{max}$ $\qquad$ $\rho_- = \rho_{min}$

d) For fixed orbit.

$$\frac{\partial V_{eff}}{\partial \rho} = 0$$

$$\frac{\partial}{\partial \rho}\left\{\left(\frac{-1}{4}m w_z^2 + \frac{1}{8}m w_B^2\right)\rho^2 + \frac{P_\phi^2}{2m\rho^2}\right\}$$

$$= 2\left(\frac{-1}{4}m w_z^2 + \frac{1}{8}m w_B^2\right)\rho - \frac{P_\phi^2}{m\rho^3} = 0.$$

$$\left(\frac{-1}{2}\ w_z^2 + \frac{1}{4}\ w_B^2\right)\rho^4 m^2 - P_\phi^2 = 0$$

$$\rho_0 = \left[\frac{P_\phi^2}{m^2}\left[\frac{-1}{2}\ w_z^2 + \frac{1}{4}\ w_B^2\right]^{-1}\right]^{1/4}$$

we know

$$w_0 = \dot{\phi} = \frac{P_\phi - \frac{m w_B}{2}\rho^2}{\quad} = P_\phi \qquad w_K$$

$m\rho^2 \qquad \frac{\quad}{m\rho^2} - \frac{\quad}{2}$

then

$$\dot{\phi}_0 = \frac{\cancel{m\rho_0}}{\cancel{m}\, P_\phi} \sqrt{\left[-\frac{1}{2}W_z^2 + \frac{1}{4}W_B^2\right]} - \frac{W_B}{2}$$

$$W_0 = \dot{\phi}_0 \doteq \sqrt{-\frac{1}{2}W_z^2 + \frac{1}{4}W_B^2} - \frac{W_B}{2}$$

$$\doteq \frac{1}{2}W_B \sqrt{1 - 2\left(\frac{W_z}{W_B}\right)^2} - \frac{W_B}{2}$$

$$\boxed{W_0 \doteq \frac{1}{2}W_B \left(\underbrace{\sqrt{1 - 2\left(\frac{W_z}{W_B}\right)^2}} - 1\right)}$$

since $1 - 2\left(\frac{W_z}{W_B}\right)^2 < 1$

$W_0 = \dot{\phi}_0$ is negative, so <u>clockwise.</u>

For $W_B^2 \gg 2W_z^2$

$$W_0 = \frac{1}{2}W_B \left(1 - \left(\frac{W_z}{W_B}\right)^2 - 1\right)$$

$$\doteq \frac{1}{2}\left(\frac{W_z}{W_B}\right)^2 W_B \quad \text{or} \quad -\frac{1}{2}\frac{W_z^2}{W_B}$$

ii) particle orbits around the center while oscillate harmonically in z direction.

It conserves angular momentum