① A non-relativistic particle of charge q in a electro-magnetic field is described by Lagrangian:

$$\boxed{\mathcal{L} = \frac{1}{2} m \dot{\vec{r}}^2 - q\phi + q\frac{\dot{\vec{r}}}{c} \cdot \vec{A}}$$ ← Remember.

$\phi(t, \vec{r}(t))$ is the scalar potential
$\vec{A}(t, \vec{r}(t))$ is the vector potential

$$\vec{E}(t, \vec{r}) = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

$$\vec{B}(t, \vec{r}) = \vec{\nabla} \times \vec{A}$$

a) Show Euler-Lagrange EOM of the particle is:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$
$$= \dot{r}\hat{r} + r\{\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi}\ \hat{x}$$
$$+ \cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi}\ \hat{y}$$
$$- \sin\theta\,\dot{\theta}\ \hat{z}\}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\,d\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

$$\dot{\vec{r}}^2 = \dot{r}^2 + r^2\{(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})^2$$
$$+ (\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi})^2$$
$$+ (\sin^2\theta\,\dot{\theta})^2\}$$

$$\doteq \dot{r}^2 + r^2\{\cos^2\theta\cos^2\phi\,\dot\theta^2 - \cancel{2\cos\theta\cos\phi\sin\theta\sin\phi\,\dot\theta\dot\phi} + \sin^2\theta\sin^2\phi\,\dot\phi^2$$
$$+ \cos^2\theta\sin^2\phi\,\dot\theta^2 + \cancel{2\cos\theta\sin\phi\sin\theta\cos\phi\,\dot\theta\dot\phi} + \sin^2\theta\cos^2\phi\,\dot\phi^2$$
$$+ \sin^2\theta\,\dot\theta^2\}$$

$$\doteq \dot{r}^2 + r^2(\cos^2\theta\,\dot\theta^2 + \sin^2\theta\,\dot\phi^2 + \sin^2\theta\,\dot\theta^2)$$

$$\doteq \dot{r}^2 + r^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$$

$$\dot{\vec{r}}\cdot\dot{\vec{r}} \doteq \dot{r}^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}^i}\right) = \frac{\partial\mathcal{L}}{\partial r^i}$$

$$\frac{\partial}{\partial\dot{r}^i}\mathcal{L} = \frac{1}{2}m\frac{\partial}{\partial\dot{r}^i}(\dot{r}^{i2}) - q\phi + \frac{q}{c}\dot{r}^i A_i$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{r}^i}\mathcal{L}\right) = m\dot{r}^i + \frac{q}{c}A_i$$

$$\doteq m\ddot{r}^i + \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial r^j}\frac{dr^j}{dt}\right)$$

$$\frac{\partial\mathcal{L}}{\partial r^i} = -q\frac{\partial}{\partial r^i}\phi + \frac{q}{c}\frac{\partial}{\partial r^i}(\dot{r}^j A_j)$$

$$\doteq -q\frac{\partial}{\partial r^i}\phi + \frac{q}{c}\left(\underbrace{\cancel{\frac{\partial}{\partial r^i}\dot{r}^j}}_{=0} A_j + \dot{r}^j\frac{\partial}{\partial r^i}A_j\right)$$

$$m\ddot{r}^i + \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial r^j}\frac{dr^j}{dt}\right) = -q\frac{\partial}{\partial r^i}\phi + \frac{q}{c}\left(\underbrace{\cancel{\frac{\partial}{\partial r^i}\dot{r}^j}}_{=0} A_j + \dot{r}^j\frac{\partial}{\partial r^i}A_j\right)$$

$$m\ddot{r}^i = -q\frac{\partial}{\partial r^i}\phi - \frac{q}{c}\left\{\frac{\partial A_i}{\partial t} - \dot{r}^j\underbrace{\left(\frac{\partial}{\partial r^i}A_j - \frac{\partial}{\partial r^j}A_i\right)}\right\}$$

$$\vec{\nabla}\times\vec{A} = \epsilon_{ijk} B^k$$

$$\epsilon^{ijk}\left(\frac{\partial}{\partial r}\right)_i A_j = B^k$$

$$F = q\left(-\frac{\partial}{\partial r^i}\phi - \frac{1}{c}\frac{\partial}{\partial t}A_i\right) + q\frac{\dot{r}^j}{c}\epsilon_{ijk}B^k$$

$$\stackrel{!}{=} q\underbrace{\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right)}_{\vec{E}} + q\frac{\dot{\vec{r}}}{c}\underbrace{\left(\vec{\nabla}\times\vec{A}\right)}_{\vec{B}}$$

b) Compute canonical momentum $= \vec{p}$.
How is it related to $P_{kin} = m\dot{\vec{r}}$

Determine $\frac{d}{dt}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$

$$P_i = \frac{\partial L}{\partial \dot{r}^i} = \frac{\partial}{\partial \dot{r}^i}\left(\frac{1}{2}m_{jk}\dot{r}^j\dot{r}^k - q\phi + \frac{q}{c}\dot{r}^j A_j\right)$$

$$\stackrel{!}{=} \frac{1}{2}m_{jk}\left(\frac{\partial \dot{r}^j}{\partial \dot{r}^i}\dot{r}^k + \dot{r}^j\frac{\partial \dot{r}^k}{\partial \dot{r}^i}\right) + \frac{q}{c}\left(\frac{\partial \dot{r}^j}{\partial \dot{r}^i}A_j + \dot{r}^j\frac{\partial A_j}{\partial \dot{r}^i}\right)$$

$$\stackrel{!}{=} \frac{1}{2}m_{jk}\left(\delta^j_i\dot{r}^k + \dot{r}^j\delta^k_i\right) + \frac{q}{c}\left(\delta^j_i A_j + \dot{r}^j\frac{\partial A_j}{\partial \dot{r}^i}\right)$$

$$\stackrel{!}{=} \frac{1}{2}\left(m_{ik}\dot{r}^k + m_{ij}\dot{r}^j\right) + \frac{q}{c}\left(A_i + \dot{r}^j\frac{\partial}{\partial \dot{r}^i}A_j\right)$$

$$P_i \stackrel{!}{=} \underbrace{(m\dot{r})_i}_{P_{kin}} + \frac{q}{c}\left(A_i + \underbrace{\cancel{\dot{r}^j\frac{\partial}{\partial \dot{r}^i}A_j}}_{=0}\right)$$

$$\boxed{\vec{P} - \vec{P}_{kin} = \frac{q}{c}A_j}$$

$$\frac{d}{dt}\left(P - \frac{q}{c}\vec{A}\right) = \frac{d}{dt}\left(m_{ik}\dot{r}^k\right)$$

$$\stackrel{!}{=} m_{ik}\ddot{r}^k$$

$$\stackrel{!}{=} (m\ddot{\vec{r}})_i$$

$$\frac{d}{dt}\left(\vec{p}-\frac{q}{c}\vec{A}\right) = \frac{d}{dt}\vec{P}_{kin}$$

c) $h = p_i \dot{q}^i - \mathcal{L}$

$$\stackrel{!}{=} \left(m_{ik}\dot{r}^k + \cancel{\frac{q}{c}A_i}\right)\dot{r}^i - \frac{1}{2}m_{ij}\dot{r}^i\dot{r}^j + q\phi - \cancel{\frac{q}{c}\dot{r}^i A_i}$$

$$\stackrel{!}{=} \frac{1}{2}m_{ik}\dot{r}^k\dot{r}^i + q\phi$$

$\mathcal{H} = p_i \dot{q}^i - \mathcal{L}$ $\qquad\qquad \dot{r}^i = (m^{-1})^{ik}\left(p_k - \frac{q}{c}A_k\right)$

$$\stackrel{!}{=} p_i\left\{(m^{-1})^{ik}\left(p_k-\frac{q}{c}A_k\right)\right\} - \frac{1}{2}m_{ij}\left\{(m^{-1})^{ik}\left(p_k-\frac{q}{c}A_k\right)\right\}\left\{(m^{-1})^{jl}\left(p_l-\frac{q}{c}A_l\right)\right\}$$

$$-\frac{q}{c}A_i\left\{(m^{-1})^{ik}\left(p_k-\frac{q}{c}A_k\right)\right\} + q\phi$$

$$\stackrel{!}{=} (m^{-1})^{ik}\left(p_ip_k - \frac{q}{c}p_iA_k\right) - \frac{1}{2}(m^{-1})^{kl}\left(p_k-\frac{q}{c}A_k\right)\left(p_l-\frac{q}{c}A_l\right)$$

$$-\frac{q}{c}(m^{-1})^{ik}\left(p_k-\frac{q}{c}A_k\right)A_i + q\phi$$

$$\stackrel{!}{=} \frac{1}{2}(m^{-1})^{ik}(p_ip_k) - \frac{q}{c}(m^{-1})^{ik}p_iA_k + \frac{q}{c}(m^{-1})^{kl}p_kA_l$$

$$-\frac{q}{c}(m^{-1})^{ik}p_kA_i + \frac{1}{2}(m^{-1})^{ik}\left(\frac{q}{c}\right)^2 A_kA_i + q\phi$$

$$\stackrel{!}{=} \frac{1}{2}(m^{-1})^{ik}p_ip_k - \frac{q}{c}(m^{-1})^{ik}p_kA_i + \frac{1}{2}(m^{-1})^{ik}\left(\frac{q}{c}\right)^2 A_kA_i + q\phi$$

$$\mathcal{H} \stackrel{!}{=} \frac{1}{2}(m^{-1})^{ik}\left(p_i-\frac{q}{c}A_i\right)\left(p_k-\frac{q}{c}A_k\right) + q\phi$$

d) $$\frac{d}{dt}P_i = \frac{-\partial \mathcal{H}}{\partial r^i} = -q\frac{\partial}{\partial r^i}\phi + \frac{q}{c}(m^{-1})^{jk}P_k\frac{\partial}{\partial r^i}A_j$$
$$-\frac{1}{2}(m^{-1})^{jk}\left(\frac{q}{c}\right)^2\left\{\frac{\partial}{\partial r^i}A_k A_j + \frac{\partial}{\partial r^i}A_j A_k\right\}$$

$$P_k = (m\dot{r})_k + \frac{q}{c}A_k \quad \hookrightarrow = -q\frac{\partial}{\partial r^i}\phi + \frac{q}{c}(m^{-1})^{jk}\left((m\dot{r})_k + \frac{q}{c}A_k\right)\frac{\partial}{\partial r^i}A_j$$
$$-\frac{1}{2}(m^{-1})^{jk}\left(\frac{q}{c}\right)^2 2\left(\frac{\partial}{\partial r^i}A_k A_j\right)$$

$$\frac{dP}{dt} \quad \doteq -q\frac{\partial}{\partial r^i}\phi + \underbrace{\frac{q}{c}\dot{r}^j\frac{\partial}{\partial r^i}A_j}_{\frac{q}{c}\frac{dr^j}{dt}\frac{\partial}{\partial r^i}A_j}$$
$$= \frac{q}{c}\frac{d}{dt}(A_j)$$

$$\hookrightarrow \frac{d}{dt}\left(P - \frac{q}{c}A_j\right) = -q\frac{\partial}{\partial r^j}\phi$$

② Routhian Tutorial:

Consider kepler:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r)$$

Where Hamiltonian, a Legendre Transform respect to r, ϕ.

$$\mathcal{H} = P_r\dot{r} + P_\phi\dot{\phi} - \mathcal{L}(r, \dot{r}, \phi, \dot{\phi})$$

Sometimes, it is convenient to transform part of variables, or the Routhian:

Define: Routhian:

$$R = P_\phi \dot{\phi} - L(r, \dot{r}, \phi, \dot{\phi})$$

Only transform w.r.t. ϕ

useful when some coordinates are cyclic, or $\frac{d}{dt}(P_\phi) = 0$

then P_ϕ is constant in EoM and action. So we have non-constant in Lagrangian

a) Show Routhian EoM: $R(r, \dot{r}, \phi, P_\phi)$ from Lagrange EoM.

know: $\mathcal{L} = P_\phi \dot{\phi} - R(r, \dot{r}, \phi, P_\phi)$

$$\frac{\partial R}{\partial P_\phi} = \dot{\phi} - \cancel{\frac{\partial \mathcal{L}}{\partial P_\phi}}$$ since $\mathcal{L}$ is not function of P_ϕ

$$-\frac{\partial R}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{d}{dt} P_\phi$$

$$\frac{\partial R}{\partial r} = -\frac{\partial \mathcal{L}}{\partial r} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = -\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right)$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) = \frac{d}{dt}\left(-\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)$$

b) Determine Routhian for

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r)$$

$$P_\phi = \frac{\partial}{\partial\dot{\phi}}\mathcal{L} = mr^2\dot{\phi} \quad \Rightarrow \quad \dot{\phi} = \frac{P_\phi}{mr^2}$$

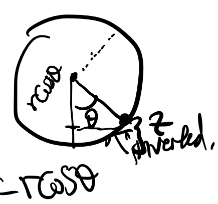
$$R = P_\phi\frac{P_\phi}{mr^2} - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}\frac{P_\phi^2}{mr^2} + U(r)$$

$$= -\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{P_\phi^2}{mr^2} + U(r)$$

$$-R = \frac{1}{2}m\dot{r}^2 - \underbrace{\left(+\frac{1}{2}\frac{P_\phi^2}{mr^2} + U(r)\right)}_{V_{eff}(r, P_\phi)}$$

$$-\frac{d}{dt}\left(\frac{\partial R}{\partial\dot{r}}\right) = \frac{\partial}{\partial r}R$$

c)



$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta\,\dot{\phi})^2\right) + mgr(1-\cos\theta)$$

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$$\dot{\phi} = \frac{P_\phi}{mr^2\sin^2\theta}$$

$$\frac{dP_\phi}{dt} = \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \Rightarrow \phi \text{ is cyclic}$$

$$\frac{\partial\mathcal{L}}{\partial r} \neq 0, \quad \frac{\partial\mathcal{L}}{\partial\theta} \neq 0 \Rightarrow r, \theta \text{ not cyclic.}$$

$$R = P_\phi \dot\phi - L$$

$$\doteq P_\phi \frac{P_\phi}{mr^2\sin^2\theta} - \frac{1}{2}m\left(\cancelto{0}{\dot r^2} + r^2\dot\theta^2 + \left(r\sin\theta \frac{P_\phi}{mr^2\sin^2\theta}\right)^2\right) + mgr(1-\cos\theta)$$

$$R \doteq -\frac{1}{2}mr^2\dot\theta^2 + \frac{P_\phi^2}{2mr^2\sin^2\theta} + mgr(1-\cos\theta)$$

$$\doteq -\frac{1}{2}mr^2\dot\theta^2 - \left[\frac{-P_\phi^2}{2mr^2\sin^2\theta} - mgr(1-\cos\theta)\right]$$

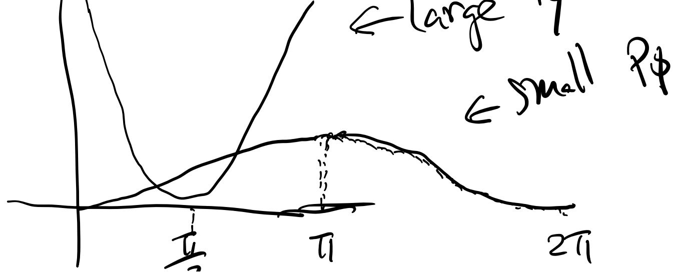
$$-R = \frac{1}{2}mr^2\dot\theta^2 - \Bigg[\underbrace{\frac{P_\phi^2}{2mr^2\sin^2\theta} + mgr(1-\cos\theta)}_{V_{eff}(\theta, P_\phi)}\Bigg]$$

Two potential: $\frac{P_\phi^2}{2mr^2\sin^2\theta}$ ← centrifugal

$mgr(1-\cos\theta)$ ← gravity

P_ϕ is small when $\frac{P_\phi^2}{2mr^2} \ll mgr$

P_ϕ is large when $\frac{P_\phi^2}{2mr^2} \gg mgr$



At large P_ϕ, ball is stationary at $\frac{\pi}{2}$.

If we let $\theta = \frac{\pi}{2}$, then what will be its deviation δ.

$\theta = \frac{\pi}{2} - \delta$

$$V_{eff}\left(\theta = \frac{\pi}{2} + \delta\right) = \frac{P_\phi^2}{2mr^2\sin^2\theta} + mgr(1-\cos\theta)$$

$$\left.\frac{1}{\sin^2\theta}\right|_{\theta=\frac{\pi}{2}+\delta} = \left.\frac{1}{(\sin\theta)^2}\right|_{\theta=\frac{\pi}{2}-\delta} = \left[\sin\frac{\pi}{2}) + \left.\frac{d}{d\theta}\cancel{(\sin)}\right|_{\frac{\pi}{2}}\delta + \frac{1}{2}\left.\frac{d^2}{d\theta^2}(\sin\theta)\right|_{\frac{\pi}{2}}\delta^2\right]$$

$$\doteq \left[1 - \frac{1}{2}\delta^2\right]^{-2}$$

$$\doteq 1 + \delta^2$$

$$\left.\cos\theta\right|_{\theta=\frac{\pi}{2}+\delta} = \cancel{\cos\left(\frac{\pi}{2}\right)} + \left.\frac{d}{d\theta}\cos(\theta)\right|_{\frac{\pi}{2}}\delta$$

$$\doteq -\delta$$

$$V_{eff}\left(\theta = \frac{\pi}{2} + \delta\right) = \frac{P_\phi^2}{2mr^2}\delta^2 + mgr(1+\delta)$$

$$\frac{\partial V_{eff}}{\partial\delta} = \frac{P_\phi^2}{mr^2}\delta + mgr$$

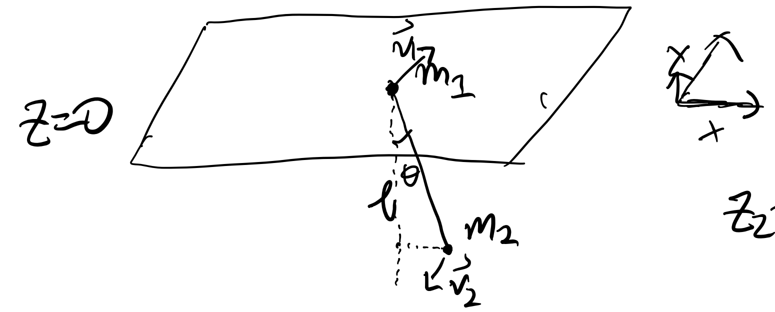
$$\delta = \frac{-m^2gr^3}{P_\phi^2} \leftarrow \text{if } P_\phi^2 \Rightarrow \infty$$

$\delta \Rightarrow 0$

then particle stays at $\frac{\pi}{2}$

else there is a small deviation from $\frac{\pi}{2}$ due to gravity.

③



$z_2 = -l\cos\theta\,\hat{z}$

$(m_1+m_2)X_{cm} = m_1x_1 + m_2x_2$ $\qquad |\vec{r}_1 - \vec{r}_2| = l$

$(m_1+m_2)Y_{cm} = m_1y_1 + m_2y_2$

$(m_1+m_2)Z_{cm} = \cancel{m_1z_1} + m_2z_2$ $\qquad \Rightarrow z = -z_2$

$$\vec{Z}_{cm} = \frac{m_2}{M}z_2 = \alpha z_2 = -\alpha l\cos\theta\,\hat{z}$$

$$\Delta X_1 = X_1 - X_{cm} = X_1 - \frac{m_1x_1+m_2x_2}{M} = \frac{m_2}{M}(x_1-x_2) = \frac{m_2}{M}\vec{x}$$

$$\Delta Y_1 = Y_1 - Y_{cm} = Y_1 - \frac{m_1y_1+m_2y_2}{M} = \frac{m_2}{M}(y_1-y_2) = \frac{m_2}{M}\vec{y}$$

$$\Delta Z_1 = Z_1 - Z_{cm} = Z_1 - \frac{m_1z_1+m_2z_2}{M} = \frac{m_2}{M}(\cancel{z_1}-z_2) = \frac{m_2}{M}\vec{z}$$

$$|\vec{r}_1 - \vec{R}_{cm}| = \frac{m_2}{M}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(-z_2)^2} = \underline{\underline{\alpha l}}$$

$$|\vec{r}_1 - \vec{R}_{cm}| = \alpha l =$$

b) $\vec{r}_2 - \vec{R}_{cm} = x_2 - \frac{m_1x_1+m_2x_2}{M}\hat{x} + y_2 - \frac{m_1y_1+m_2y_2}{M}\hat{y}$

$$+ z_2 - \frac{m_1z_1+m_2z_2}{M}\hat{z}$$

$$\stackrel{!}{=} -\frac{m_1}{M}x\hat{x} + -\frac{m_1}{M}y\hat{y} + -\frac{m_1}{M}z\hat{z}$$

where $(x_1 - x_2) = x \quad (y_1 - y_2) = y \quad (z_1 - z_2) = -z_2 = z$

$$T = \frac{1}{2}\sum_a m_a \dot{\vec{R}}_{cm}^2 + \frac{1}{2}\sum_a m_a \dot{\vec{\Delta r}}_a^2$$

$$= \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2)$$

$$+ \frac{1}{2}m_1\left(\frac{m_2}{M}\right)^2\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}$$

$$+ \frac{1}{2}m_2\left(\frac{m_1}{M}\right)^2\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}$$

$$= \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2) + \frac{1}{2}\frac{m_1 m_2}{M}\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}$$

$$x = l\sin\theta\cos\phi \qquad y = l\sin\theta\sin\theta \qquad z = -l\cos\theta$$

$$z_{cm} = -\frac{m_2}{M}z = \frac{m_2}{M}l\cos\theta$$

$$T = \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \frac{1}{2}M\left(\frac{m_2}{M}l\sin\theta\dot{\theta}\right)^2$$

$$+ \frac{1}{2}\mu\left((l\dot{\theta})^2 + (l\sin\theta\dot{\phi})^2\right)$$

$$U = \cancel{m_1 g z_1} + m_2 g z_2 = m_2 g(-l\cos\theta\hat{z}) = -m_2 g l\cos\theta\hat{z}$$

$$L = \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \frac{1}{2}M(\alpha l\sin\theta)^2\dot{\theta}^2$$

$$+ \frac{1}{2}\mu l^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + M\alpha g l\cos\theta$$

$$\mathcal{L} = \frac{1}{2}M(\dot{X}_{cm}^2 + \dot{Y}_{cm}^2) + \frac{1}{2}\left(M(\alpha\sin\theta)^2 + \mu\right)l^2\dot{\theta}^2$$
$$+\frac{1}{2}\mu l^2\sin^2\theta\,\dot{\phi}^2 + M\alpha g l\cos\theta$$

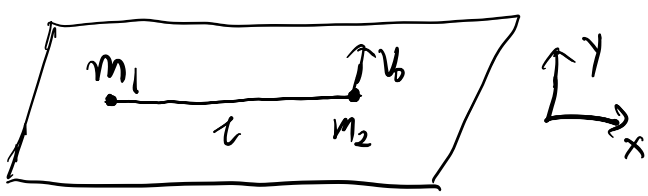
c) $\frac{\partial\mathcal{L}}{\partial\dot{X}_{cm}} = M\dot{X}_{cm} = P_{X_{cm}} = \text{const}$ $\quad \frac{\partial\mathcal{L}}{\partial\dot{Y}_{cm}} = M\dot{Y}_{cm} = P_{Y_{cm}} = \text{const}$

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu l^2\sin^2\theta\,\dot{\phi} = \text{const}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left\{\left(M(\alpha\sin\theta)^2 + \mu\right)l^2\dot{\theta}\right\} = \frac{d}{dt}(P_\theta)$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = M\alpha^2\sin\theta\cos\theta\, l^2\dot{\theta}^2 + \mu l^2\sin\theta\cos\theta\,\dot{\phi}^2$$
$$-M\alpha g l\sin\theta$$

d) Now consider



In center of mass frame, motion is periodic.

Initially: $P_x^1 = P_y^1 = P_x^2 = 0$
$P_y^2 = m_2 v_0$

since $P_x^1 + P_x^2 = 0$
$P_x^1 = -P_x^2 \qquad P_y^1 + P_y^2 = m_2 v_0$

we know $\theta = \frac{\pi}{2}$ and $P_\phi = \mu l^2 \sin^2\theta\, \dot{\phi} \big|_{\theta=\frac{\pi}{2}}$

$$P_\phi = \mu l^2 \dot{\phi} \qquad \text{since } v_0 = l\dot{\phi}$$
$$\stackrel{!}{=} \mu l v_0$$

c) i) pendulum swings down from $\theta = \frac{\pi}{2}$ relative to vertical

since $\mathcal{L}$ is explicitly independent from t,

E is conserved

Know initially: $\theta = \frac{\pi}{2}$, $P_\phi = \mu l^2 \sin^2\theta\, \dot{\phi} \big|_{\theta=\frac{\pi}{2}}$

$$\stackrel{!}{=} \mu l^2 \dot{\phi} \quad \text{and} \quad v_0 = l\dot{\phi}$$
$$P_\phi \stackrel{!}{=} \mu l v_0$$

$$P_x = P_y = P_\theta = 0$$

$$\mathcal{H} = \frac{P_x^2}{2M} + \frac{P_y^2}{2M} + \frac{P_\phi^2}{2\mu l^2 \sin^2\theta} + P_\theta \frac{P_\theta}{m_0 l^2}$$
$$- \frac{1}{2} m_0 l^2 \frac{P_\theta}{(m_0 l^2)^2} - M\alpha g l \cos\theta$$

$$\mathcal{H} \stackrel{!}{=} \frac{P_{x_{cm}}^2}{2M} + \frac{P_{y_{cm}}^2}{2M} + \frac{P_\phi^2}{2\mu l^2 \sin^2\theta} + \frac{P_\theta^2}{2m_0 l^2} - M\alpha g l \cos\theta$$

$$E_{init} = \frac{1}{2} m_2 v_0^2 = \frac{1}{2} \frac{m_2(m_1+m_2)}{M} v_0^2$$

$$\stackrel{!}{=} \frac{1}{2}\frac{m_2^2}{M} v_0^2 + \frac{1}{2}\frac{m_2 m_1}{M} v_0^2$$

$$\stackrel{!}{=} \frac{1}{2}\frac{m_2^2}{M} v_0^2 + \frac{1}{2}\mu v_0^2$$

$$E_{init} = \frac{P_{xcm}^2}{2M} + \frac{P_{ycm}^2}{2M} + \frac{P_\phi^2}{2\mu l^2 \sin^2\theta} + \frac{1}{2}(M\alpha^2\sin^2\theta + \mu) l^2 \dot{\theta}^2 - M\alpha g l \cos\theta$$

$$\stackrel{!}{=} \frac{1}{2}\frac{(m_2 v_0)^2}{M} + \frac{1}{2}\mu v_0^2$$

we see $\frac{1}{2}\frac{(m_2 v_0)^2}{M} = \frac{P_{ycm}^2 + P_{xcm}^2}{2M}$

then

$$\frac{1}{2}\mu v_0^2 = \frac{P_\phi^2}{2\mu l^2 \sin^2\theta} + \frac{1}{2}(M\alpha^2\sin^2\theta + \mu) l^2 \dot{\theta}^2 - M\alpha g l \cos\theta$$

we know $P_\phi = \mu l^2 \sin^2\theta\, \dot{\phi} = \mu l^2 \sin^2\theta \left(\frac{v_0}{l}\right)$

$$\stackrel{!}{=} \mu l v_0 \sin^2\theta$$

we know $P_\phi = \text{const}$, $P_\phi|_{init} = P_\phi = P_\phi|_{\theta=\frac{\pi}{2}} = \mu l v_0$

$$\frac{1}{2}\mu v_0^2 = \frac{(\mu l v_0)^2}{2\mu l^2 \sin^2\theta} + \frac{1}{2}(M\alpha^2\sin^2\theta + \mu) l^2 \dot{\theta}^2 - M\alpha g l \cos\theta$$

$$\frac{1}{2}\mu v_0^2 = \frac{\mu v_0^2}{2\sin^2\theta} + \frac{1}{2}(M\alpha^2\sin^2\theta + \mu) l^2 \dot{\theta}^2 - M\alpha g l \cos\theta$$

$$\sqrt{\frac{\frac{1}{2}\mu v_0^2\left(1 - \frac{1}{\sin^2\theta}\right) + M\alpha g l \cos\theta}{\frac{1}{2}(M\alpha^2\sin^2\theta + \mu) l^2}} = \frac{d\theta}{dt}$$

$$1 - \frac{1}{\sin^2\theta} = -\cot^2\theta$$

$\frac{1}{\sin^2\theta} - \cos\theta$

$$\longrightarrow \pm\sqrt{\frac{-\cot^2\theta + \frac{M\alpha g l}{\frac{1}{2}m v_0^2}\cos\theta}{\frac{l^2}{v_0^2}\left(\frac{M\alpha^3}{m}\sin^2\theta + 1\right)}} = \frac{d\theta}{dt}$$

$$\frac{1}{2}m v_0^2\left(1 - \frac{1}{\sin^2\theta}\right) + M\alpha g l\cos\theta = 0$$

minimum angle when $\frac{d\theta}{dt} = 0$

$$\left(1 - \frac{1}{\sin^2\theta}\right) + u\cos\theta = 0$$

$$-\cot^2\theta + u\cos\theta = 0.$$

for $u = \frac{M\alpha g l}{\frac{1}{2}m v_0^2}$

$$\cot^2\theta = u\cos\theta$$

$$\frac{\cos^2\theta}{1-\cos^2\theta} = u\cos\theta$$

$$\cos^2\theta = u\cos\theta - u\cos^3\theta$$

$$\cos\theta\left(u\cos^2\theta + \cos\theta - u\right) = 0$$

Either let $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$

or:

want (+) since (−) makes unphysical angles

$$\cos\theta = \frac{-1 \pm \sqrt{1+4u^2}}{2u}$$

ii) previously we have:

$$\pm\sqrt{\frac{-\cot^2\theta + \frac{M\alpha g l}{\frac{1}{2}m v_0^2}\cos\theta}{\dots}} = \frac{d\theta}{dt}$$

$$\sqrt{\frac{l}{v_0^2}\left(\frac{M\alpha}{u}\sin^2\theta+1\right)}$$

$$\int_{\theta=\frac{\pi}{2}}^{\theta=\theta_-} \pm\frac{l}{v_0}\sqrt{\frac{\left(\frac{M\alpha^2}{u}\sin^2\theta+1\right)}{-\cot^2\theta+u\cos\theta}}\,d\theta = \int_{t=0}^{t(\theta=\theta_-)} dt = \frac{T}{2}$$

$M\left(\frac{m_2}{M}\right)^2\left(\frac{M}{m_1 m_2}\right)$

$\hookrightarrow = m_2/m_1 = \frac{1}{r}$

$$\hookrightarrow -\frac{l}{v_0}\int_{\theta=\frac{\pi}{2}}^{\theta=\theta_-} \frac{\left(\frac{1}{r}\sin^2\theta+1\right)}{\cot^2\theta+u\cos\theta}\,d\theta = \frac{T}{2}$$

take negative since angle is decreasing

$u = \dfrac{M\alpha g l}{\frac{1}{2}u v_0^2}$, large $v_0 \Rightarrow \frac{1}{2}u v_0^2 \gg M\alpha g l$

$\hookrightarrow v_0^2 \gg \dfrac{2M\alpha g l}{u}$

and small $v_0 \Rightarrow \frac{1}{2}u v_0^2 \ll M\alpha g l$

$v_0^2 \ll \dfrac{2M\alpha g l}{u}$.

If Large v_0, then $u \approx 0$, for $m_1 = m_2$

$$-2\frac{l}{v_0}\int_{\frac{\pi}{2}}^{\theta_-}\sqrt{\frac{\sin^2\theta+1}{-\cot^2\theta}}\,d\theta = T = \frac{2\pi l}{v_0}\sqrt{}$$

If small v_0, u becomes big.

$$-2\frac{l}{v_0}\int_{\frac{\pi}{2}}^{\theta_-}\sqrt{\frac{\sin^2\theta+1}{u\cos\theta}}\,d\theta = T.$$

$\int^{\theta_-}$

$$= -2\,\frac{\ell}{v_0}\sqrt{\frac{\frac{1}{2}u v_0^2}{M\alpha g\ell}}\int_{\frac{\pi}{2}}\int\frac{\cos\theta}{\sin^2\theta+1}\,d\theta = T.$$

$$\stackrel{\downarrow}{\simeq}\ \text{some}\ \#\sqrt{\frac{\ell}{g}} = \frac{2\pi\ell}{v_0}\left(\frac{1.07}{\sqrt{u}}\right)$$

τ pendulum