Vectors, Cross products:

For every vector, there is an associated anti-symmetric tensor, $\hat{v}$. note $\vec{v} = v^a \vec{e}_a$

$$\boxed{\hat{V}_{ab} = \epsilon_{abc} V^c \iff V^a = \frac{1}{2}\epsilon^{abc}\hat{V}_{bc}}$$

where

$$\hat{V}_{ab} = \begin{pmatrix} 0 & \hat{V}_{xy} & \hat{V}_{xz} \\ -\hat{V}_{xy} & 0 & \hat{V}_{yz} \\ -\hat{V}_{xz} & -\hat{V}_{yz} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V^z & -V^y \\ -V^z & 0 & V^x \\ V^y & -V^x & 0 \end{pmatrix}$$

Note:

$$\epsilon_{ijk}\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m$$

$$\epsilon_{jmn}\epsilon^{imn} = 2\delta_j^i \qquad m=j$$

$$\epsilon_{ijk}\epsilon^{ijk} = 6$$

For instance:

$$\hat{T}_{xy} = T^z$$

$$\hat{L}_{xy} = r_x p_y - r_y p_x = L^z$$

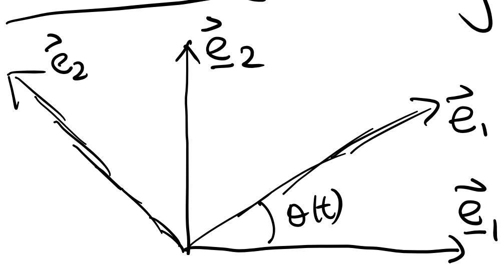
$$V^a = \frac{1}{2}\epsilon^{abc} V_{bc} = \frac{1}{2}\epsilon^{abc}\epsilon_{bca}V^a = \frac{1}{2}\underbrace{\epsilon^{abc}\epsilon_{abc}}_{2\delta^a_a}V^a = \delta^a_a V^a = V^a$$

$\underline{\text{Ex}}$: $(\vec{v}\times\vec{w})_a = \epsilon_{abc}\, v^b w^c = \epsilon_{cab}\, v^b w^c = \hat{v}_{ca}\, w^c = (\vec{w}\cdot\hat{v})_a$

$$(\vec{v}\times\vec{w})_a = (\vec{w}\cdot\hat{v})_a.$$

$$\begin{aligned}
\widehat{(v\times w)}_{bc} &= \epsilon_{bca}\,(\vec{v}\times\vec{w})_a \\
&\overset{!}{=} \epsilon_{bca}\,\epsilon_{a\underline{b}\underline{c}}\, v^{\underline{b}} v^{\underline{c}} \\
&\overset{!}{=} \left(\delta_{b\underline{b}}\,\delta_{c\underline{c}} - \delta_{\underline{b}c}\,\delta_{\underline{c}b}\right) v^{\underline{b}} v^{\underline{c}} \\
&\overset{!}{=} v_b w_c - v_c w_b.
\end{aligned}$$

$\underline{\text{Rotations}}$: (rotating reference frame)



Here $\vec{\underline{e}}_a$ are fixed
$\vec{e}_a$ are rotating.

Therefore:

$$\vec{e}_a\cdot\vec{e}_b = \delta_{ab}$$
$$\vec{\underline{e}}_a\cdot\vec{\underline{e}}_b = \delta_{ab}$$

Define: $\vec{e}_a = R_{ab}\,\vec{\underline{e}}_b$ ← convert from fixed basis $\vec{\underline{e}}_b$ to rotating basis $\vec{e}_a$.

then $\vec{e}_a\cdot\vec{\underline{e}}_b = R_{ab}(t)$ ← Rotation Matrix

$\therefore$

$$\begin{aligned}
R_{11} &= e_1\cdot\underline{e}_1 = \cos\theta \\
R_{12} &= e_1\cdot\underline{e}_2 = \cos\left(\tfrac{\pi}{2}-\theta\right) = \sin\theta \\
R_{21} &= e_2\cdot\underline{e}_1 = -\sin\theta = \cos\left(\tfrac{\pi}{2}+\theta\right) = -\sin\theta \\
R_{22} &= e_2\cdot\underline{e}_2 = \cos\theta
\end{aligned}$$

Properties:

Orthogonality: $\vec{e}_a \cdot \vec{e}_b = \delta_{ab}$

know $\vec{e}_a = R_{ac}\vec{e}_c$ $\hookrightarrow R_{ac}\vec{e}_c \cdot R_{bd}\vec{e}_d = \delta_{ab}$

$\vec{e}_b = R_{bd}\vec{e}_d$ $\hookrightarrow R_{ac}R_{bd}\,\delta_{cd} = \delta_{ab}$

$\hookrightarrow R_{ac}R_{bc} = \delta_{ab}$

$\hookrightarrow R_{ac}(R^T)_{cb} = \delta_{ab}$

$\therefore\ (R)_{ab} = (R^T)_{ba}$

$(RR^T)_{ab} = R_{ac}(R^T)_{cb} = R_{ac}R_{bc} = \delta_{ab}$

Therefore R is an orthogonal matrix

Let $\vec{r} = r^a \vec{e}_a(t)$ — rotating basis, look at $\vec{r}$ as a rotating vector in a fixed frame. here r^a is constant if rotating.

$$\frac{d\vec{r}}{dt} = r^a \frac{d\vec{e}_a}{dt} + \cancel{\frac{dr^a}{dt}}^{\,0}\, \vec{e}_a$$

$$\stackrel{!}{=} r^a \frac{d}{dt}\left(R_{ab}(t)\underline{\vec{e}_b}\right)$$ fixed basis.

know $\vec{e}_a = R_{ab}\underline{\vec{e}_b}$

$$\stackrel{!}{=} r^a \dot{R}_{ab}(t)\underline{\vec{e}_b}$$

$(R^{-1})_{ba}\vec{e}_a = \underline{\vec{e}_b}$

$$\stackrel{!}{=} r^a \underbrace{\dot{R}_{ab}(R^{-1})_{bc}}\vec{e}_c$$

$\hat{\omega}_{ac} = (\dot{R}R^{-1})_{ac}$ $\Rightarrow$ angular momentum matrix

$$\frac{d\vec{r}}{dt} = r^a\hat{\omega}_{ac}\vec{e}_c = \vec{r}\cdot\hat{\omega} = \vec{\omega}\times\vec{r}$$

and $\dfrac{d\vec{e}_a}{dt} = \hat{\omega}_{ac}\vec{e}_c$

proof that $\hat{\omega}_{ac}$ is an anti-symmetric matrix

$$\hat{\omega}_{bc} = (\dot{R} R^{-1})_{bc} = (\dot{R} R^{T})_{bc} =$$

$$\stackrel{!}{=} \dot{R}_{bd} (R^{T})_{dc}$$

$$\stackrel{!}{=} \dot{R}_{bd} R_{cd}$$

Since $(RR^{T})_{ab} = \delta_{ab}$

$$\frac{d}{dt}\left(R_{ac} R_{bc} = \delta_{ab}\right)$$

$$\dot{R}_{ac} R_{bc} + R_{ac} \dot{R}_{bc} = 0$$

$$\hookrightarrow \dot{R}_{ac} (R^{T})_{cb} + \dot{R}_{bc} (R^{T})_{ca} = 0$$

$$\hat{\omega}_{ab} + \hat{\omega}_{ba} = 0$$

$$\hat{\omega}_{ab} = -\hat{\omega}_{ba} \leftarrow \text{anti-symmetric}$$

Then $\vec{e}_a(t+\Delta t) = \vec{e}_a + \hat{\omega}_{ac} \Delta t \, \vec{e}_c$

$$\stackrel{!}{=} \underbrace{(\delta_{ac} + \hat{\omega}_{ba} \Delta t)}_{\text{infinitesimal rotation.}} \vec{e}_c$$

Summary:

① $\vec{e}_a = R_{ab}(t) \vec{e}_b$ or $R_{ab} = \vec{e}_a \cdot \vec{e}_b$

(2) R is orthogonal or

$$R^{-1} = R^T \quad \text{or} \quad (R^{-1})_{ab} = (R^T)_{ab} = R_{ba}$$

(3) $$\frac{d\vec{e}_a}{dt} = \hat{w}_{ab}\vec{e}_b \qquad \frac{dR_{ac}}{dt} = \hat{w}_{ab}R_{bc}$$

$$= \vec{w} \times \vec{e}_a \quad \text{where } \hat{w}_{ab} = \dot{R}_{ac}(R^T)_{cb}$$

so $$w_a = \frac{1}{2}\epsilon_{abc}\hat{w}_{bc}$$
$$w_{ab} = \epsilon_{abc} w^c$$

Notice : $\underbrace{\vec{w} \times \vec{r} = \vec{r} \cdot \hat{w}}_{\text{shown previously}} \Rightarrow (\vec{v} \times \vec{w})_a = (\hat{w} \cdot \vec{v})_a.$

(5) Change of $r(t)$ under rotation:

$$\frac{d}{dt}(r(t)) = \frac{dr^a}{dt}\vec{e}_a + r^a \frac{d}{dt}\vec{e}_a = \frac{dr^a}{dt}\vec{e}_a + r^a \hat{w}_{ab}$$

$$\vec{w} \times \vec{e}_a = \hat{w}_{ab}\vec{e}_b \qquad \vec{r} \cdot \hat{w} = \vec{w} \times \vec{r}$$

$$= \underbrace{\left(\frac{dr^a}{dt}\delta_{ab} + r^a \hat{w}_{ab}\right)}_{(D_t r^b)\vec{e}_b}\vec{e}_b$$

$$\hat{w}_{ab} = (\dot{R}R^{-1})_{ab}$$

$$w^c = \frac{1}{2}\epsilon^{cab}\hat{w}_{ab}$$

for $D_t = \frac{dr^a}{dt}\delta_{ab} + r^a \hat{w}_{ab}.$

Accelerating / Rotating Frame:

$$\frac{d}{dt}\vec{r} = \frac{d}{dt}\left(r^a\vec{e}_a\right) = \frac{d}{dt}r^a\,\vec{e}_a + r^a\frac{d\vec{e}_a}{dt}$$

In fixed Frame.

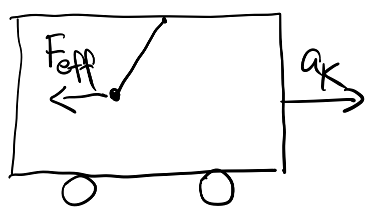
$$\stackrel{!}{=} \frac{d}{dt}r^a\,\vec{e}_a + r^a\omega_{ab}\,\vec{e}_b$$

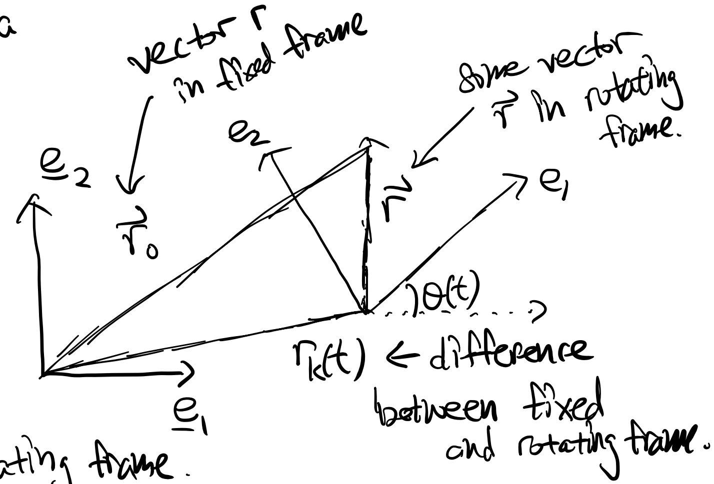
$$\stackrel{!}{=} \frac{d\vec{r}}{dt} + \vec{w}\times\vec{r}$$

← How rotating vector behaves in a fixed frame.

Rotating Frame.

Note $\dfrac{d\vec{e}_a}{dt} = \vec{w}\times\vec{e}_a$





Fixed frame

rotating frame.

$$\vec{r}_0 = \vec{r}(t) + \vec{r}_k(t)$$

$$\frac{d}{dt}\vec{r}_0 = \vec{v}_0 = \left\{\left(\frac{d}{dt}\vec{r}\right)_r + \vec{w}\times\vec{r}\right\} + \frac{d\vec{r}_k}{dt}$$

$$\vec{v}_0 \stackrel{!}{=} \vec{v}_r + \vec{w}\times\vec{r} + \vec{v}_k$$

$$\frac{d}{dt}\vec{v}_0 = a_0 = \left(\left(\frac{d\vec{v}_r}{dt}\right)_r + \vec{w}\times\vec{v}_r\right) + \left[\left(\frac{d\vec{w}}{dt}\right)_r + \overset{0}{\cancel{\vec{w}\times\vec{w}}}\right]\times\vec{r}$$

$$+ \vec{w}\times\left[\left(\frac{d\vec{r}}{dt}\right)_r + \vec{w}\times\vec{r}\right] + \frac{d\vec{v}_k}{dt}$$

in fixed basis

$$\Rightarrow \vec{a}_0 = \vec{a}_r + 2\vec{w}\times\vec{v}_r + \vec{w}\times(\vec{w}\times\vec{r}) + \dot{\vec{w}}\times\vec{r} + \vec{a}_k$$

linear acceleration in fixed frame.

linear acceleration in rotating frame.

Effects due to frame.

work in rotating basis

$\vec{F} = m\vec{a}_0$ ← Acceleration in stationary frame.

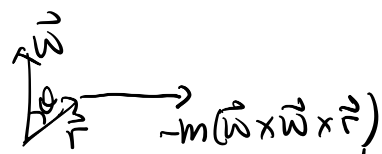
$\vec{F}_{eff} = m\vec{a}_r$ ← Acceleration experienced by object in acceleration/rotating frame.

☆

$$\boxed{\vec{F}_{eff} = \vec{F} - 2m\vec{\omega}\times\vec{v}_r - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\dot{\vec{\omega}}\times\vec{r} - m\vec{a}_k}$$

$-2m\vec{\omega}\times\vec{v}_r$: Coriolis Force

$-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$: Centrifugal Force ⇒



$-m\dot{\vec{\omega}}\times\vec{r}$: angular acceleration of rotating frame with respect to fixed frame.

$-m\vec{a}_k$: Translational term.

Coriolis Force :

← Earth observer.

$$F_{eff} = m\frac{d^2}{dt^2}\vec{r} = -2m\vec{\omega}\times\vec{v}_r$$

$$\stackrel{!}{=} -2m\omega\hat{z}\times v(-\hat{y})$$

$$\stackrel{!}{=} -2m\omega v\hat{x}$$ ← appears to move in $\hat{x}$ as Earth rotates.

Rotating Lagrangian with $\vec{r}_k$ constant. : Rotating Frame

$$\mathcal{L} = \frac{1}{2}mv_0^2 - U(r_0)$$

For $\vec{v}_0 = \vec{v}_r + \vec{\omega}\times\vec{r} + \vec{v}_k$

$$\mathcal{L} = \frac{1}{2}m\left(\vec{v}_r^{\,2} + 2\vec{v}_r\cdot(\vec{\omega}\times\vec{r}) + (\vec{\omega}\times\vec{r})^2\right) - U(\vec{r}_0)$$

$$\overset{!}{=} \frac{1}{2}m\vec{v}_r^{\,2} + \vec{v}_r\cdot(\vec{\omega}\times\vec{r}) + \underbrace{\frac{1}{2}m(\vec{\omega}\times\vec{r})^2 - U(\vec{r}_0)}_{-U_{eff}(r)}$$

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{r}^i} = m\dot{r}_i + m(\vec{\omega}\times\vec{r})$$

If $\vec{\omega}$ is constant, then Energy is conserved.

$$h = \frac{1}{2}m\vec{v}_r^{\,2} - \frac{1}{2}m(\vec{\omega}\times\vec{r})^2 + U$$

conserved quantity.

$$\overset{!}{=} \frac{1}{2}m(\vec{v}_0 - \vec{\omega}\times\vec{r})^2 + U(r) - \frac{1}{2}m(\vec{\omega}\times\vec{r})^2$$

$$\overset{!}{=} \frac{1}{2}mv_0^2 - m\vec{v}_0\cdot(\vec{\omega}\times\vec{r}) + U(r)$$

For $E_0 = \frac{1}{2}mv_0^2 + U(r)$

$$\overset{!}{=} E_0 - m\vec{v}_0\cdot(\vec{\omega}\times\vec{r})$$

$$\overset{!}{=} E_0 - \vec{\omega}\,(\underbrace{m\vec{r}\times\vec{v}_0}_{\text{angular momentum, } L_0})$$

$$h \overset{!}{=} E_0 - \vec{\omega}\cdot\vec{L}_0$$

In rotating frame, instead conserving E_0

$$\boxed{h = E_0 - \vec{\omega}\cdot\vec{L}_0}$$ is conserved.

in rotation Frame.