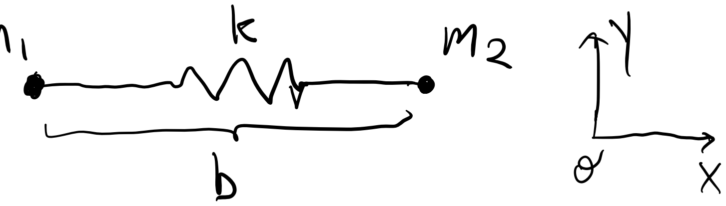
(1)



a)

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2 + \dot{y}_2^2)$$

$$q^i = \{x_1, y_1, x_2, y_2\}$$

$$V = \frac{1}{2} k \Delta r^2 = \frac{1}{2} k (\Delta r)^2$$

$$\stackrel{!}{=} \frac{1}{2} k \left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} - b \right)^2$$

$$\mathcal{L} = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$- \frac{1}{2} k \left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} - b \right)^2$$

b) Write in center of mass frame:

$\hookrightarrow$ written in $\underline{\underline{\vec{r}_{cm}}}$ and $\underline{\underline{\Delta \vec{r}_i}}$

$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{R}_1 = \vec{r}_1 - \vec{R}_{cm} = \Delta \vec{r}_1$$

$$\stackrel{!}{=} \vec{r}_1 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\stackrel{!}{=} \frac{m_1 + m_2}{m_1 + m_2} \vec{r}_1 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

① Write $\vec{R}_{cm}$
② Find $\Delta \vec{r}_i$
③ express in same coordinate $\vec{r}$
④ Find T.
$T = \frac{1}{2} \left(\sum_a m_a \dot{\vec{R}}_{cm} + \sum m_a \Delta \dot{\vec{r}}_i^2 \right)$

$$\vec{R}_1 \stackrel{!}{=} \frac{m_2}{m_1 + m_2} (\vec{r}_1 - \vec{r}_2) = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$R_2 = \vec{r}_2 - \vec{R}_{cm} = \Delta\vec{r}_2$$

$$\stackrel{!}{=} \vec{r}_2 - \left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}\right)$$

$$\stackrel{!}{=} \left(\frac{m_1}{m_1+m_2}\right)\left(\vec{r}_2 - \vec{r}_1\right) = -\frac{m_1}{m_1+m_2}\vec{r}$$

In CoM frame:

$$\boxed{T = \sum_a \frac{1}{2} m_a \dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2} m_a \Delta\dot{\vec{r}}_a^{\,2}}$$

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2}m_1\dot{\vec{R}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{R}}_2^{\,2}$$

Know?

$$\vec{R}_{cm} = X_{cm}\hat{x} + Y_{cm}\hat{y}$$

$$\vec{R}_1 = \frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{R}_2 = \frac{-m_1}{m_1+m_2}\vec{r}$$

$$\vec{r} = r\hat{r} = r\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)$$

$$\frac{d\vec{r}}{dt} = \dot{r}\cos\theta\,\hat{x} - r\sin\theta\,\dot{\theta}\,\hat{x} + \dot{r}\sin\theta\,\hat{y} + r\cos\theta\,\dot{\theta}\,\hat{y}$$

$$\dot{r}^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\sin^2\theta\,\dot{\theta}^2$$

$$+ \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\cos^2\theta\,\dot{\theta}^2$$

$$\stackrel{!}{=} \dot{r}^2 + r^2\dot{\theta}^2$$

$$= \frac{1}{2}(m_1+m_2)\left[\dot{X}_{cm}^2 + \dot{Y}_{cm}^2\right] + \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$+ \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$\stackrel{!}{=} \frac{1}{2}(m_1+m_2)\left[\dot{x}_{cm}^2+\dot{y}_{cm}^2\right]+\frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}}\left(\dot{r}^2+r^2\dot{\theta}^2\right)\cancel{\left[m_1+m_2\right]}$$

$$T \stackrel{!}{=} \frac{1}{2}(m_1+m_2)\left[\dot{x}_{cm}^2+\dot{y}_{cm}^2\right]+\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)$$

$$V = \frac{1}{2}k\Delta r^2$$

$$\stackrel{!}{=} \frac{1}{2}k\left(|\vec{r}_1-\vec{r}_2|-b\right)^2$$

$$\stackrel{!}{=} \frac{1}{2}k(r-b)^2$$

$$\boxed{\mathcal{L} = \frac{1}{2}(m_1+m_2)\left[\dot{x}_{cm}^2+\dot{y}_{cm}^2\right]+\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\dot{r}^2+r^2\dot{\theta}^2\right) - \frac{1}{2}k(r-b)^2 \quad \text{where } q^i=\{x_{cm}, y_{cm}, r, \theta\}}$$

c) Calculate Momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_{cm}} = (m_1+m_2)\dot{x}_{cm} = P_{x,cm} = \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}_{cm}} = (m_1+m_2)\dot{y}_{cm} = P_{y,cm} = \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{m_1 m_2}{m_1+m_2} r^2\dot{\theta} = P_\theta = \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{m_1 m_2}{m_1+m_2}\dot{r} = P_r$$

$$\frac{d}{dt} P_r = \frac{\partial \mathcal{L}}{\partial r} = \frac{m_1 m_2}{m_1+m_2} r\dot{\theta}^2 - k(r-b)$$

$$\underbrace{\frac{m_1 m_2}{m_1+m_2}}_{m_{eff}} \ddot{r} = \frac{m_1 m_2}{m_1+m_2} r\dot{\theta}^2 - k(r-b) = \underbrace{\frac{P_\theta^2}{\frac{m_1 m_2}{m_1+m_2} r^3} - k(r-b)}_{-\frac{\partial}{\partial r} V_{eff}(r)}$$

$$V_{eff}(r) = -\int dr \frac{P_\theta^2}{\frac{m_1 m_2}{m_1+m_2} r^3} - k(r-b)$$

$\leftarrow$ Convert to P when do integration.

$$\stackrel{!}{=} \frac{P_\theta^2}{2\frac{m_1 m_2}{m_1+m_2} r^2} + \frac{1}{2} k r^2 - kbr + C$$

d) write down $h(q, \dot{q}, t)$. Show $\dot{r}$ can be determined from energy E and $V_{eff}(r)$. i.e. show $\frac{1}{2} m_{eff} \dot{r}^2 + V_{eff}(r) = E$

$$h(q, \dot{q}, t) = P_i \dot{q}^i - \mathcal{L}$$

$$\stackrel{!}{=} P_x \dot{x} + P_y \dot{y} + P_\theta \dot{\theta} + P_r \dot{r} - \mathcal{L}$$

$$\stackrel{!}{=} (m_1+m_2)\dot{x}^2 + (m_1+m_2)\dot{y}^2 + \left(\frac{m_1 m_2}{m_1+m_2}\right) r^2 \dot{\theta}^2 + \frac{m_1 m_2}{m_1+m_2} \dot{r}^2$$

$$- \left\{ \frac{1}{2}(m_1+m_2)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} k(r-b)^2 \right\}$$

$$E = h(q, \dot{q}) \stackrel{!}{=} \frac{1}{2}(m_1+m_2)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} k(r-b)^2$$

Here h is not explicitly function of time, so energy is const, or conserved.

since $P_x = (m_1+m_2)\dot{x}$, $P_y = (m_1+m_2)\dot{y}$ } are conserved, they're constants.

$$E - \frac{P_x^2}{2m} - \frac{P_y^2}{2m} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}k(r-b)^2$$

$$P_\theta = \frac{m_1 m_2}{m_1+m_2} r^2\dot{\theta} = \text{const}$$

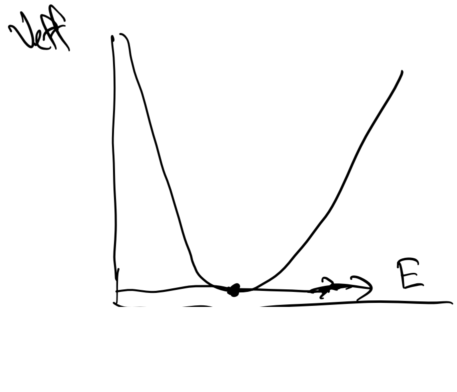
then $\frac{1}{2}\frac{m_1 m_2}{m_1+m_2} r^2\dot{\theta}^2 = \frac{P_\theta^2}{2\frac{m_1 m_2}{m_1+m_2} r^2}$ ← Function of r

$$\hookrightarrow \frac{1}{2}\underbrace{\frac{m_1 m_2}{m_1+m_2}}_{m_{eff}}\dot{r}^2 + \underbrace{\frac{P_\theta^2}{2\frac{m_1 m_2}{m_1+m_2} r^2} + \frac{1}{2}k(r-b)^2}_{V_{eff}(r)} = E - \frac{P_x^2}{2m} - \frac{P_y^2}{2m}$$

d) $V_{eff}(r) = \frac{1}{2}\mu r^2\dot{\theta}^2 + \frac{1}{2}k(r-b)^2$

let $\mu = \frac{m_1 m_2}{m_1+m_2}$ $\quad \doteq \frac{P_\theta^2}{2\mu r^2} + \frac{1}{2}kr^2 - krb + C$

$$\frac{V_{eff}}{kb^2} = \frac{P_\theta^2}{2\mu k(rb)^2} + \frac{1}{2}\left(\frac{r}{b}\right)^2 - \frac{r}{b}$$



if $E = \min(V_{eff})$, then there is is no oscillation.

If $\dot{\theta}\uparrow$, since $P_\theta = $ const

r/b $\quad p_\theta \propto r^2\dot{\theta}$, then r must increase, or spring stretches longer.

2) Show $\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \frac{\partial L}{\partial q^i} = 0$

with $L(q^i, \dot{q}^i, \ddot{q}^i, t)$

$$S[q+\delta q, \dot{q}+\delta\dot{q}, \ddot{q}+\delta\ddot{q}] = \int dt\, L(q+\delta q, \dot{q}+\delta\dot{q}, \ddot{q}+\delta\ddot{q}, t)$$

$$S[q,\dot{q},\ddot{q}] + \delta S = \int dt \Big\{ L(q,\dot{q},\ddot{q},t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \frac{\partial L}{\partial \ddot{q}}\frac{d^3}{dt}\delta q + \mathcal{O}(\delta q^2) \Big\}$$

$$\delta S = \int dt \Big\{ \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \frac{\partial L}{\partial \ddot{q}}\frac{d^2}{dt^2}\delta q \Big\}$$

$$\overset{!}{=} \int dt \Big\{ \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \ddot{q}}\frac{d}{dt}\delta q\right] - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\frac{d}{dt}\delta q \Big\}$$

$$\overset{!}{=} \int dt \Big\{ \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \ddot{q}}\frac{d}{dt}\delta q\right] - \left(\frac{d}{dt}\left[\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\delta q\right] - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta q\right) \Big\}$$

$$0 \overset{!}{=} \cancel{\frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_0}^{t_1}} + \cancel{\frac{\partial L}{\partial \ddot{q}} \frac{d}{dt} \delta q \Big|_{t_0}^{t_1}}$$

$$+ \int dt \ \underbrace{\left\{ \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \right\}}_{=0} \delta q$$

3) a) If $L(q, \dot{q}, t)$ is Lagrangian satisfy Euler-Lagrange equations, then:

$$L' = L + \frac{dF(q,t)}{dt}$$

Yields the same E-L equations as L where F is an arbitrary differentiable function.

with L, E-L equations are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

$$S = \int dt \ L' = \int dt \ L + \frac{dF(q,t)}{dt}$$

$$S[q+\delta q, \dot{q}+\delta\dot{q}] = \int dt \Big\{ L(q,\dot{q},t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}\delta q$$

$$\frac{d}{dt}\left(F(q,t) + \frac{\partial F}{\partial q}\delta q\right)\}$$

$$\delta S = \int dt \left\{ \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right] - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q \right.$$

$$\left. + \frac{d}{dt}\left[\frac{\partial F}{\partial q}\delta q\right]\right.$$

$$\stackrel{!}{=} \underbrace{\cancel{\frac{\partial F}{\partial q}\delta q\Big|_{t_0}^{t_1}}}_{0} + \underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\Big|_{t_0}^{t_1}}}_{0}$$

$$+ \int dt\, \delta q \left\{ \underbrace{\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)}_{=0} \right\}$$

same as $\mathcal{L}$.

b) $$\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial \mathcal{L}}{\partial q} = 0$$

$$\mathcal{L} = -\frac{1}{2}m\ddot{q}q - \frac{1}{2}\omega_0^2 q^2$$

$$\frac{d^2}{dt^2}\left(-\frac{1}{2}mq\right) - \omega_0^2 q - \frac{1}{2}m\ddot{q} = 0$$

$$-\frac{1}{2}m\ddot{q} - \omega_0^2 q - \frac{1}{2}m\ddot{q} = 0$$

$$-m\ddot{q} - \omega_0^2 q = 0. \quad \Rightarrow \quad m\ddot{q} = -\omega_0^2 q$$

Harmonic Oscillator.

$$\ddot{q}q = \frac{d}{dt}(\dot{q}q) - \dot{q}^2$$

$$-\ddot{q}q + \frac{d}{dt}(\dot{q}q) = \dot{q}^2$$

usually $\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}\omega_0^2 q^2$

but since $\dot{q}^2 = -\ddot{q}q + \frac{d}{dt}(\dot{q}q)$

$$\mathcal{L} = -\frac{1}{2}m\ddot{q}q + \underbrace{\frac{d}{dt}\left(\frac{1}{2}m\dot{q}q\right)}_{} - \frac{1}{2}\omega_0^2 q^2$$

due to part a, this term can be ignored.

$$\mathcal{L} = -\frac{1}{2}m\ddot{q}q - \frac{1}{2}\omega_0^2 q^2$$

c) Consider the action of a free particle.

$$S[\vec{r}(t)] = \int dt\, C\vec{v}^2$$

Show action is unchanged under Galilean Transformation:

$$\vec{r}' = \vec{r} + \vec{u}t \qquad \vec{v}' = \dot{\vec{r}}' = \vec{v} + \vec{u}$$

$$S[\vec{r}'(t)] = \int dt\, C\, \vec{v}'^2$$
$$\overset{!}{=} \int dt\, C\{\vec{v} + \vec{u}\}^2$$
$$\overset{!}{=} \int dt\, C\{v^2 + 2\vec{v}\cdot\vec{u} + u^2\}$$
$$\overset{!}{=} \int dt\, Cv^2 + \int dt\,\{2\frac{d}{dt}\vec{r}\cdot\vec{u} + u^2\}$$

Show $2\frac{d}{dt}\vec{r}\cdot\vec{u} + u^2 = \frac{dF(r,t)}{dt}$

$$2\left(\frac{d}{dt}[\vec{r}\cdot\vec{u}] - \cancel{\vec{r}\cdot\frac{d}{dt}\vec{u}}\right) + u^2 = \frac{dF(r,t)}{dt}$$
0

$$F = 2\vec{r}\cdot\vec{u} + u^2 t$$

since $\mathcal{L}' = \mathcal{L} + \frac{dF(\vec{r},t)}{dt}$

then if $F(\vec{r},t) = 2\vec{r}\cdot\vec{u} + u^2 t$

then $\boxed{\mathcal{L}' = \mathcal{L} + \frac{dF(\vec{r},t)}{dt}}$

If $\mathcal{L} = Cv^4$

$$S[\vec{r}'] = \int dt \; Cv'^4$$
$$\stackrel{!}{=} \int dt \; C[\vec{v} + \vec{u}]^4$$
$$\stackrel{!}{=} \int dt \; C[v^2 + 2\vec{v}\cdot\vec{u} + u^2]^2$$
$$\stackrel{!}{=} \int dt \; C\left[v^2 + 2\frac{d}{dt}[\vec{r}\cdot\vec{u}] + u^2\right]^2$$

can't make extra parts total derivative of t.
hence, not the same.

d) consider frictionless block of mass m, it sits on train with acceleration, a_0. The block experiences no force, so it's in free-fall.

$$S = \int dt \, \frac{1}{2} m v_g^2$$

where $v_g(t)$ is velocity relative to the ground.

Let $v(t)$ denote the velocity of the block relative to the back of the train.

$$v(t) = v_g(t) - a_0 t \qquad \text{or} \qquad v_g(t) = v(t) + a_0 t$$

a)
$$S[v(t)] = \int dt \, \frac{1}{2} m \left(\vec{v}(t) + \vec{a}_0 t \right)^2$$

$$= \int dt \, \frac{1}{2} m \left\{ v^2 + 2 \vec{a}_0 \cdot \vec{v} t + a_0^2 t^2 \right\}$$

$$= \int dt \, \frac{1}{2} m \left\{ v^2 + 2 \vec{a}_0 \cdot \left(\frac{d}{dt}[\vec{r} t] - \vec{r} \right) + \frac{d}{dt}\left[\frac{a_0^2 t^3}{3} \right] \right\}$$

$$= \int dt \left\{ \frac{1}{2} m v^2 - m \vec{a}_0 \cdot \vec{r} + \underbrace{\frac{d}{dt}\left[2 \vec{a}_0 \cdot \vec{r} t + \frac{a_0^2 t^3}{3} \right]}_{\text{ignorable total time derivative.}} \right\}$$

$$= \int dt \underbrace{\left\{ \frac{1}{2} m v^2 - m \vec{a}_0 \cdot \vec{r} \right\}}$$

$$L = \frac{1}{2} m v^2 - m \vec{a}_0 \cdot \vec{r} = T - U$$

$$\underline{U(\vec{r}) = m \vec{a}_0 \cdot \vec{r}}$$

acceleration acts as a fictious potential that pulls the block to the back of the train.

$$F = \frac{-\partial U}{\partial r} = -ma_0$$

4) A cylinder on a train:

cylinder: $\dot{X} = R\dot{\phi} + a_0 t$

$Y = 0$

Bead: $X_{Bead} = R\phi + \frac{1}{2}a_0 t + R\sin\phi \Rightarrow \dot{X}_B = R\dot{\phi} + a_0 t + R\cos\phi\,\dot{\phi}$

$Y_{Bead} = R\cos\phi \Rightarrow \dot{Y}_B = -R\sin\phi\,\dot{\phi}$

$$T_{cylinder} = KE_{translational} + KE_{Rotational}$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m(R\dot{\phi} + a_0 t)^2 + \frac{1}{2}\underbrace{mR^2}_{I}\dot{\phi}^2$$

$$= \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\phi}^2 + mR\dot{\phi}a_0 t + \frac{1}{2}ma_0^2t^2$$

$$T_{cylinder} = mR^2\dot{\phi}^2 + \frac{d}{dt}\left[\cancel{mRa_0\phi t + \frac{1}{2}ma_0^2t^3}\right] - mRa_0\phi$$

$$T_{bead} = \frac{1}{2} m_0 \left(\dot{x}_b^2 + \dot{y}^2 \right)$$

$$\doteq \frac{1}{2} m_0 \left\{ \left(R\dot{\phi} + a_0 t + R\cos\phi\, \dot{\phi} \right)^2 + \left(-R\sin\phi\, \dot{\phi} \right)^2 \right\}$$

$$\doteq \frac{1}{2} m_0 \left\{ R^2\dot{\phi}^2 + 2Ra_0\dot{\phi}t + 2R^2\cos\phi\, \dot{\phi}^2 + a_0^2 t^2 + 2Ra_0\cos\phi\, \dot{\phi} t + R^2\cos^2\phi\, \dot{\phi}^2 + R^2\sin^2\phi\, \dot{\phi}^2 \right\}$$

$$\doteq m_0 R^2 \dot{\phi}^2 + \frac{1}{2} m_0 \left\{ \cancel{\frac{d}{dt}\left[2Ra_0\phi t \right]} - 2Ra_0\phi + 2R^2\cos\phi\, \dot{\phi}^2 + \cancel{\frac{d}{dt}\left[\frac{1}{3} a_0^2 t^3 \right]} + \cancel{\frac{d}{dt}\left[2Ra_0\sin\phi\, t \right]} - 2Ra_0\sin\phi \right\}$$

$$\doteq m_0 R^2 \left[1 + \cos\phi \right] \dot{\phi}^2 - m_0 R a_0 \phi - m_0 R a_0 \sin\phi$$

$$T_{tot} = T_{cylinder} + T_{bead}$$

$$\doteq mR^2\dot{\phi}^2 - mRa_0\phi + m_0R^2\left[1+\cos\phi \right]\dot{\phi}^2 - m_0Ra_0\phi - m_0Ra_0\sin\phi$$

$$\doteq \left(m + m_0\left[1+\cos\phi \right] \right) R^2\dot{\phi}^2 - (m+m_0)Ra_0\phi - m_0Ra_0\sin\phi$$

$$V = m_0 g y = m_0 g R\cos\phi$$

$$L = T_{tot} - V$$

$$\stackrel{!}{=} \frac{1}{2}\underbrace{\left(2m+2m_0[1+\cos\phi]\right)}_{m_{eff}}R^2\dot{\phi}^2 - (m+m_0)Ra_0\phi - m_0Ra_0\sin\phi$$
$$- m_0gR\cos\phi$$

Not explicitly function of time.

$\hookrightarrow$ Energy Conservation.
or constant Energy.

$R\phi = x$

$$V_{eff} = (m+m_0)Ra_0\phi + m_0Ra_0\sin\phi + m_0gR\cos\phi$$

b) Speed of cylinder after 2-cycles.

Initially: at rest. m_0 is at $\phi=0$.

$$E = h = P_\phi\dot{\phi} - \mathcal{L}$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial\dot{\phi}} = m_{eff}(\phi)R^2\dot{\phi}$$

$$E = h = \frac{1}{2}m_{eff}(\phi)R^2\dot{\phi}^2 + V_{eff}(\phi)$$

$$h(t=0,\phi=0,\dot{\phi}=0) = V_{eff}(\phi=0)$$

$$E \stackrel{!}{=} m_0gR$$

$$E = m_0gR = \frac{1}{2}m_{eff}(\phi)R^2\dot{\phi}^2 + V_{eff}(\phi)$$

after 2 full cycles, $\phi = 4\pi$

$$m_0 g R - V_{eff}(\phi = 4\pi) = \frac{1}{2} m_{eff}(\phi = 4\pi) R^2 \dot{\phi}^2$$

$$\cancel{m_0 g R} + (m+m_0) R a_0 4\pi - \cancel{m_0 g R \cos(4\pi)} - \cancel{m_0 R a_0 \sin(4\pi)} = \frac{1}{2}\left(2m + 2m_0[1+\cos(4\pi)]\right) R^2 \dot{\phi}^2$$

$$+(m+m_0) R a_0 4\pi = (m+2m_0) R^2 \dot{\phi}^2$$

$$\frac{(m+m_0) R a_0 4\pi}{m+2m_0} = R^2 \dot{\phi}^2$$