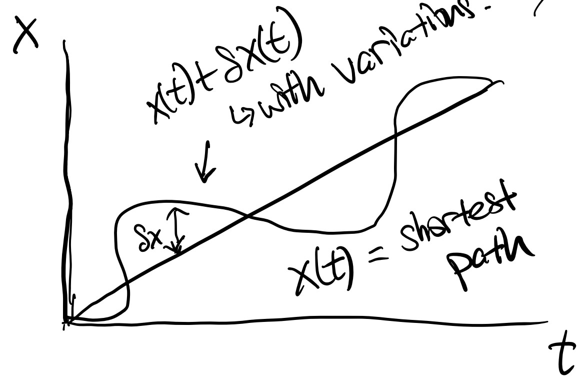
Action:

Suppose 1D, free fall particle:

Equation of motion: $\frac{d}{dt}\left(m\frac{dx}{dt}\right) = -\cancel{\frac{\partial U}{\partial x}}$ 0 for free particle.



← particle ~~follows~~ follows straight line.

If we vary path: $x(t) \to x(t) + \delta x(t)$
but require end points to be fixed.

If we require $\delta x(t) = 0$, then $x(t) \to x(t)$.

Define action:

$$S[x(t)] = \int_{t_1}^{t_2} dt \underbrace{\mathcal{L}(x, \dot{x}, t)}_{\text{Lagrangian, action density.}}$$

action is a functional that takes a path described by $\mathcal{L}(x, \dot{x}, t)$ and returns a # that is used as a measure of the path distance.

__Action principal:__ the classical path, or on-shell path are path that satisfy equation of motion. by finding the extrema.

or: $$S[x(t)+\delta x(t)] = S[x(t)] + \mathcal{O}(\delta x^2)$$

i.e. to first order of δx, $\frac{dS[x]}{dx} = 0$

__Derivation:__

$$S[x+\delta x] = \int dt\, \mathcal{L}(x+\delta x, \dot{x}+\delta\dot{x}, t)$$

$$\overset{!}{=} \int dt \left\{ \mathcal{L}(x,\dot{x},t) + \delta x \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) + \frac{d}{dt}\delta x \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) + \mathcal{O}(\delta x^2, \delta\dot{x}^2) \right\}$$

$$\cancel{S[x]} + \delta x \frac{\partial S}{\partial x}[x] = \int dt\, \cancel{\mathcal{L}}(x,\dot{x},t)$$

set this to zero. $$+\int dt \left\{ \delta x \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) + \frac{d}{dt}\delta x \frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x},t) \right\}$$

$$\underbrace{\delta x \frac{\partial S}{\partial x}[x]}_{\delta S[x,\delta x]} = \int dt\, \delta x \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) + \int dt \left[\frac{d}{dt}\left\{ \delta x \frac{\partial \mathcal{L}}{\partial \dot{x}} \right\} - \delta x \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \right]$$

$$\delta S[x,\delta x] \stackrel{!}{=} \int dt\, \delta x \left\{ \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x},t)\right)\right\}$$

$$+ \cancel{\delta x \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{t_1}^{t_2}}$$

$\delta x(t_2) = \delta x(t_1) = 0$
since we fix end points.

$$\delta S[x,\delta x] \stackrel{!}{=} \int dt\, \delta x \left[\underbrace{\frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) - \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x},t)\right\}} \right]$$

$$\boxed{\delta S[x,\delta x] = 0 \text{ if } \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x},t) - \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x},t)\right\} = 0}$$

For a generalized coordinate: q_i

$$\boxed{\frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q_i,\dot{q}_i,t)\right\} - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

<u>Euler - Lagrange</u>: let $\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - u(x)$

then: $\frac{d}{dt}(m\dot{x}) = -\frac{\partial u}{\partial x}$

Generalized Momentum:

$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \Rightarrow$ generalized momentum.

$- \,ll$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{d}{dt} p_i = F_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

☆

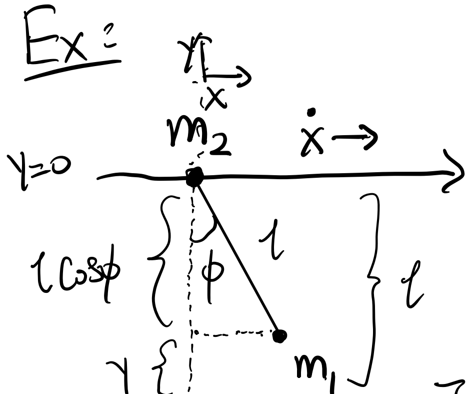
If $\mathcal{L}$ doesn't explicitly depend on q_i, then q_i is a cyclic coordinate.

i.e. $\mathcal{L} = \mathcal{L}(\dot{q}_i, t)$ instead of $\mathcal{L}(q_i, \dot{q}_i, t$

$$\therefore \quad \frac{\partial \mathcal{L}}{\partial q}(\dot{q}_i, t) = 0 = \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{q}}(\dot{q}_i, t)\right\}$$

$$\text{or } \frac{\partial \mathcal{L}}{\partial \dot{q}}(\dot{q}_i, t) = p_i = \text{const}$$

Ex:



$x_1 = l\sin\phi + x_2 \qquad y_2 = l\cos\phi$

$$T = \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \left\{ \frac{d}{dt}(l\sin\phi + x_2)^2 + \frac{d}{dt}(l\cos\phi)^2 \right\}$$

$$T = \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \left\{ (l\cos\phi\,\dot{\phi} + \dot{x}_2)^2 + l^2\sin^2\phi\,\dot{\phi} \right\}$$

$$T = \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \left\{ l^2\cos^2\phi\,\dot{\phi}^2 + \dot{x}_2^2 + 2l\cos\phi\,\dot{\phi}\,\dot{x}_2 + l^2\sin^2\phi\,\dot{\phi} \right\}$$

$$= \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \left\{ l^2\dot{\phi}^2 + \dot{x}_2^2 + 2l\cos\phi\,\dot{\phi}\,\dot{x}_2 \right\}$$

$V = -mg\,l\cos\phi$

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \left\{ l^2 \dot{\phi}^2 + \dot{x}_2^2 + 2l\cos\phi\, \dot{\phi} \dot{x}_2 \right\} + mgl\cos\phi$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_2^2 + \frac{1}{2} m_1 l^2 \dot{\phi}^2 + m_1 l \cos\phi\, \dot{\phi} \dot{x}_2 + mgl\cos\phi$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = (m_1 + m_2)\dot{x}_2 + m_1 l \cos\phi\, \dot{\phi} = P_{x_2} = \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m_1 l^2 \dot{\phi} + m_1 l \cos\phi\, \dot{x}_2 = P_\phi$$

x-momentum conservation

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m_1 l \sin\phi\, \dot{\phi}\, \dot{x}_2 - m_1 g l \sin\phi$$

Here $\mathcal{L}(\dot{x}_2, \phi, \dot{\phi})$ so x_2 is cyclic coordinate.

$$\frac{d}{dt}\left\{ (m_1 + m_2)\dot{x}_2 + m_1 l \cos\phi\, \dot{\phi} \right\} = 0$$

$$\frac{d}{dt}\left\{ m_1 l^2 \dot{\phi} + m_1 l \cos\phi\, \dot{x}_2 \right\} = -m_1 l \sin\phi\, \dot{\phi} \dot{x}_2 - m_1 g l \sin\phi$$

LHS: $\frac{d}{dt}(m_1 l^2 \dot{\phi}) + m_1 l \left\{ \ddot{x}_2 \cos\phi - \cancel{\sin\phi\, \dot{\phi} \dot{x}_2} \right\}$

RHS: $-\cancel{m_1 l \sin\phi\, \dot{\phi} \dot{x}_2} - m_1 g l \sin\phi$

$$\hookrightarrow \frac{d}{dt}(m_1 l^2 \dot{\phi}) = -m_1 l \cos\phi\, \ddot{x}_2 - \underbrace{m_1 g l \sin\phi}_{\text{Torque by gravity}}$$

if $\dot{x}$ is constant, then particle 2 is another inertial frame, and EOM should be same in all frames.

Meaning simply an EOM for a pendulum.

$$\frac{d}{dt}\left(m_1 l^2 \dot{\phi}\right) = -m_1 g l \sin\phi$$

Or solve using center of mass frame: X_{cm}.

$$X = X_{cm} + \Delta X$$

$$X_{cm} = \frac{\sum m_a X_a}{\sum m_a} = \frac{m_1(x + l\sin\phi) + m_2 x}{m_1 + m_2}$$

$$X_{cm} = \frac{x(m_1+m_2) + m_1 l \sin\phi}{m_1+m_2}$$

$$X_{cm} = X + \frac{m_1 l \sin\phi}{m_1+m_2}$$

$$X = X_{cm} - \frac{m_1 l \sin\phi}{m_1+m_2}$$

$$L = T - V$$

$$= \frac{1}{2}m_2 \dot{x}_2^2 + \frac{1}{2}m_1\left\{l^2\dot{\phi}^2 + \dot{x}_2^2 + 2l\cos\phi\,\dot{\phi}\dot{x}_2\right\} + mgl\cos\phi$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}_2^2 + \frac{1}{2}m_1 l^2\dot{\phi}^2 + m_1 l\cos\phi\,\dot{\phi}\dot{x}_2 + mgl\cos\phi$$

From previous calculation.

$$L = \frac{1}{2}(m_1+m_2)\left(\dot{X}_{cm} - \frac{m_1 l}{m_1+m_2}\cos\phi\,\dot{\phi}\right)^2 + \frac{1}{2}m_1 l^2\dot{\phi}^2$$

$$+m_1 l\cos\phi\,\dot\phi\left(\dot x_{cm}-\frac{m_1 l}{m_1+m_2}\cos\phi\,\dot\phi\right)+mgl\cos\phi$$

$$L=\frac12(m_1+m_2)\left(\dot x_{cm}^2-\cancel{\frac{2m_1 l\cos\phi}{m_1+m_2}}\dot x_{cm}\dot\phi+\left(\cancel{\frac{m_1 l}{m_1+m_2}\cos\phi\,\dot\phi}\right)^2\right)$$

$$+\frac12 m_1 l^2\dot\phi^2+m_1 l\cos\phi\,\dot\phi\left(\cancel{\dot x_{cm}}-\frac{m_1 l}{m_1+m_2}\cos\phi\,\dot\phi\right)+mgl\cos\phi$$

$$=\frac12(m_1+m_2)\dot x_{cm}^2+\frac12 m_1 l^2\dot\phi^2-\frac12\frac{(m_1 l\cos\phi\,\dot\phi)^2}{m_1+m_2}+mgl\cos\phi$$

$$=\frac12(m_1+m_2)\dot x_{cm}^2+\frac12 m_1 l^2\left(1-\frac{m_1}{m_1+m_2}\cos^2\phi\right)\dot\phi^2$$

$$+mgl\cos\phi$$

$$=\frac12(m_1+m_2)\dot x_{cm}^2+\frac12 m_1 l^2\left(\frac{m_1+m_2}{m_1+m_2}-\frac{m_1}{m_1+m_2}(1-\sin^2\phi)\right)\dot\phi^2$$

$$+mgl\cos\phi$$

$$=\frac12(m_1+m_2)\dot x_{cm}^2+\frac12 m_1 l^2\left(\frac{m_2}{m_1+m_2}+\frac{m_1}{m_1+m_2}\sin^2\phi\right)\dot\phi^2$$

$$+mgl\cos\phi$$