Legendre Transform:

Consider $\mathcal{L}(q, \dot{q}, t)$

$$\hookrightarrow d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} + \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$\overset{!}{=} p\,d\dot{q} + \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial t} dt$$

Define: $\mathcal{H} = p\dot{q} - \mathcal{L} = p\dot{q}(p) - \mathcal{L}(q, \dot{q}(p), t)$

Legendre Transform of Lagrangian.

$$d\mathcal{H} = p\,d\dot{q} + dp\,\dot{q} - d\mathcal{L}$$

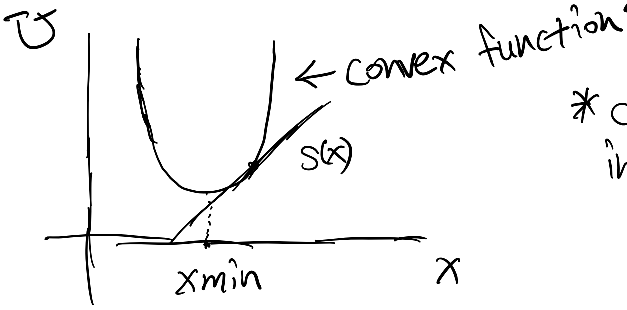
$$\overset{!}{=} \dot{q}\,dp - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial t} dt$$

So: $\frac{\partial \mathcal{H}}{\partial p} = \dot{q}$ $\quad \frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{dp}{dt}$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

from: $\frac{\partial \mathcal{L}}{\partial \dot{q}} = p$ $\quad \frac{\partial \mathcal{L}}{\partial q} = \frac{dp}{dt}$

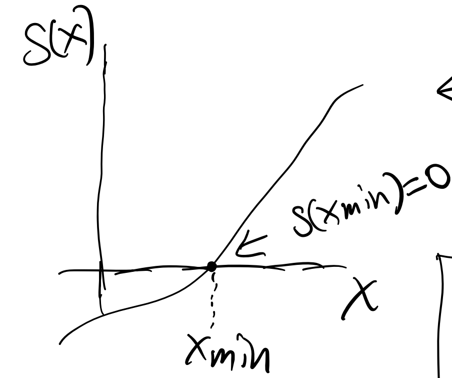
Legendre Transform:



← only applicable to convex function.

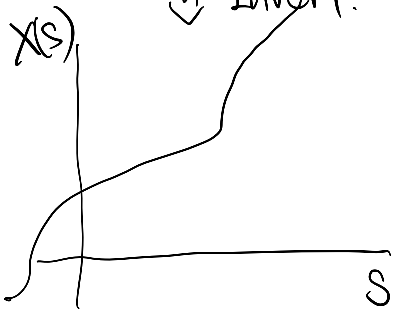
* characterize by slopes, $S(x)$ instead of x.

$$S = \frac{\partial U}{\partial x}$$



← when $S(x) = 0$, $x = x_{min}$ since it's when $\frac{\partial U}{\partial x} = 0$.

$\Downarrow$ Invert.



$$\boxed{\begin{array}{l}\underline{\text{Legendre Transform}} \\ V(s) = s\,x(s) - U(x(s)) \\ \text{where } s = \frac{\partial U}{\partial x}\end{array}}$$

$$dV = ds\,x + s\,dx - dU$$
$$= ds\,x + s\,dx - \frac{\partial U}{\partial x}dx$$
$$= ds\,x + s\,dx - s\,dx$$
$$dV = ds\,x$$

or $\boxed{\frac{dV}{ds} = X(s)}$ and $S(x) = \frac{\partial U}{\partial x}$

<u>Alternatively:</u>

$$V(s_0) = \underset{x}{\text{extrem}}\left(s\,x - U(x)\right)$$

$$0 = \frac{d}{dx}V = \frac{d}{dx}\left(s\,x - U(x)\right)$$

$$0 = s - \frac{dU}{dx}$$

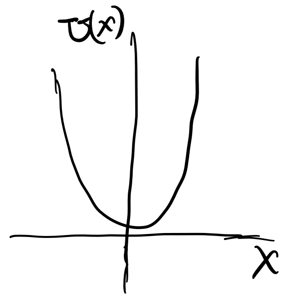
$$S(x) = \frac{dU}{dx}$$

know $S(x)$, now we can invert and find $\underline{\underline{x(s)}}$.

- write $V(s) = s\,x(s) - U(x(s))$

$\therefore$ write $V(s) = s\,x(s) - U(x(s))$

If given $U(x)$:



After apply external force, f_0.

$$U(x, f_0) = U(x) - f_0 x \quad \Rightarrow$$

$$V(s) = \underset{x}{\text{extrem}} \{ sx - U(x, f_0) \}$$

$$0 = s - \frac{\partial U}{\partial x} + f_0$$

$$s = \frac{\partial U}{\partial x} - f_0$$

A map between $x, U(x) \Longleftrightarrow s, V(s)$

Reverse relationship:

$$U(x) = \underset{s}{\text{extrem}} \left(sx - V(s) \right)$$

Legendre Transform in depth:

$$V(s) = \underset{x}{\text{extrem}} \left[sx - U(x) \right]$$

a) Determine the Legendre Transform of $\frac{1}{2}k(x - x_0)^2$

$$V(s) = \underset{x}{\text{extrem}} \left(sx - U(x) \right)$$

$$0 = s - \frac{\partial U}{\partial x}$$

$$s = \frac{\partial U}{\partial x}$$

invert

$$S(x) = k(x - x_0)$$

$$X(S) = \frac{S}{k} + x_0$$

$$U(X(S)) = \frac{1}{2}k\left(\frac{S}{k} + x_0 - x_0\right)^2$$

$$V(S) \overset{!}{=} Sx - U$$

$$\overset{!}{=} S\left(\frac{S}{k} + x_0\right) - \frac{1}{2}\frac{S^2}{k}$$

$$\overset{!}{=} \frac{S^2}{k} + x_0 S - \frac{1}{2}\frac{S^2}{k}$$

$$V(S) \overset{!}{=} \frac{1}{2}\frac{S^2}{k} + x_0 S$$

b) $U(x) = \log(1 - e^{-x})$ with $x > 0$

$$V(S) = \underset{x}{\text{extrem}}\,(Sx - U)$$

$$0 \overset{!}{=} S - \frac{\partial U}{\partial x}$$

$$S(x) = \frac{e^{-x}}{1 - e^{-x}}$$

$$S - Se^{-x} = e^{-x}$$

$$S = e^{-x}(1 + S)$$

$$-\log\left(\frac{S}{1+S}\right) = X(S) \quad \text{for } x > 0$$

$$X(S) = -\log S + \log(1+S)$$

$$U(X(S)) = \log\left(1 - e^{-[-\log S + \log(1+S)]}\right)$$

$$\overset{!}{=} \log\left(1 - e^{\log S} e^{-\log(1+S)}\right)$$

$$\overset{!}{=} \log\left(1 - S(1+S)^{-1}\right)$$

$\stackrel{!}{=} \log\left(1 - \frac{s}{1+s}\right)$

$\stackrel{!}{=} \log\left(\frac{1+s-s}{1+s}\right)$

$\stackrel{!}{=} -\log(1+s)$

$v(s) = s(-\log s + \log(1+s)) + \log(1+s)$

$v(s) \stackrel{!}{=} -s\log s + (1+s)\log(1+s)$

$x(s) = -\log s + \log(1+s) > 0$

$s(x) = \frac{e^{-x}}{1-e^{-x}} \quad x > 0.$

c) consider $U(x)$; concave up

Now external force, f_0, then $U(x) \Rightarrow U(x, f_0)$

Relate minimum value of $U(x, f_0)$ to Legendre Transform of $U(x) \Rightarrow V(s)$

$V(s) = \underset{x}{\text{extrem}}\,[sx - U(x)]$ select x such that we get max or min

$0 = s - \frac{\partial U}{\partial x}$

Now invert write, $x(s)$ $\Leftarrow$ $s = \frac{\partial U(x)}{\partial x}$ of $(sx - U(x))$

After Apply external force, f_0

$$U(x) \Rightarrow U(x, f_0) = U(x) - f_0 x$$

To find minimum of $U(x, f_0)$:

require: $\frac{\partial U(x, f_0)}{\partial x} = 0$

$\hookrightarrow \frac{\partial U(x)}{\partial x} - f_0 = 0$

previously $\Rightarrow S(x) = \frac{\partial U(x)}{\partial x} = f_0$

we know it is $S(x)$ of $U(x)$, and $U(x, f_0)$ is min when $\frac{\partial U(x)}{\partial x} = S(x) = f_0$

Know $V(S) = \underset{x}{\text{extrem}}[Sx - U(x)]$

when $S(x) = f_0$, we have minimum of $U(x, f_0)$

$V(S = f_0) = \min_x [f_0 x - U]$

or $-V(S = f_0) = \min [U - f_0 x]$

$$\boxed{-V(S = f_0) = \min[U(x, f_0)]}$$

then get minimum of $U(x)$

$V(S) = \underset{x}{\text{extrem}}[Sx - U(x)]$

let $S = 0$

$V(S = 0) = \underset{x}{\text{extrem}}[-U(x)]$

$$-V(s=0) = \underset{x}{\text{extrem}}\left[U(x)\right]$$

check with part a):

$$-V(s=0) = \min\left[U(x)\right]$$

$$V(s=0) = 0 \quad \text{which is min of } U(x) = \tfrac{1}{2}k(x-x_0)^2$$

d) Show that we can find x_{min} of $U(x,f_0)$ or $U(x)$ from $V(s)$

$$V(s) = \underset{x}{\text{extrem}}\left[sx - U(x)\right]$$

$$s = \frac{\partial U}{\partial x} = 0 \quad \Leftarrow \text{got } x_{min} \text{ such } s=0.$$

$$\frac{dV}{ds} = x + s\frac{dx}{ds} - \frac{\partial U}{\partial x}\frac{dx}{ds}$$

know $s = \frac{\partial U}{\partial x}$

$$= x + \frac{\partial U}{\partial x}\frac{dx}{ds} - \frac{\partial U}{\partial x}\frac{dx}{ds}$$

$$\boxed{\frac{dV}{ds} = x}$$

From part c, we know $s=0$ for $U(x)_{min}$
know $s=f_0$ for $U(x,f_0)_{min}$.

then

$$\frac{dV}{ds}(s=0) = x_{min} \quad \text{for } U(x)$$

$$\frac{dV}{ds}(s=f_0) = x_{min} \quad \text{for } U(x,f_0)$$

$$\frac{\partial}{\partial s}(s-s_0) = x_{min} \quad \ldots$$

check:

$$v(s) = \frac{1}{2}\frac{s^2}{k} + x_0 s$$

$$\frac{\partial v}{\partial s} = \frac{s}{k} + x_0$$

$$\frac{\partial v}{\partial s}(s=0) = x_0$$

e) Show $\frac{\partial^2 V(s)}{\partial s^2}\frac{\partial^2 U}{\partial x^2} = 1$

$$V(s) = sx - U(x(s))$$
$$U = sx - V(s(x))$$

$$\frac{\partial V}{\partial s} = x + s\frac{dx}{ds} - \frac{\partial U}{\partial x}\frac{dx}{ds} = x$$

$$\frac{\partial^2 V}{\partial s^2} = \frac{\partial x}{\partial s}$$

$$\frac{\partial U}{\partial x} = s + x\frac{ds}{dx} - \frac{\partial V}{\partial s}\frac{ds}{dx} = s$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial s}{\partial x}$$

$$\frac{\partial^2 V}{\partial s^2}\frac{\partial^2 U}{\partial x^2} = \frac{\partial x}{\partial s}\frac{\partial s}{\partial x} = 1$$

$$V(s) = s_i x^i - U(x^i(s^i))$$

$$\frac{\partial V}{\partial s_i} = x^i + s_i\frac{\partial x^i}{\partial s_i} - \frac{\partial U}{\partial x^i}\frac{\partial x^i}{\partial s_i} = x^i$$

$$\frac{\partial^2 V}{\partial S_i \partial S_j} = \frac{\partial x^i}{\partial S_j}$$

$$U(x) = S_i x^i - V(S_i(x^i))$$

$$\frac{\partial U}{\partial x^i} = S_i + x^i \frac{\partial S_i}{\partial x^i} - \frac{\partial V}{\partial S_i}\frac{\partial S_i}{\partial x^i} = S_i$$

$$\frac{\partial^2 U}{\partial x^i \partial x^j} = \frac{\partial S_i}{\partial x^j}$$

$$\frac{\partial^2 V}{\partial S_i \partial S_j}\frac{\partial^2 U}{\partial x^j \partial x^l} = \frac{\partial x^i}{\partial S_j}\frac{\partial S_j}{\partial x^l} = \frac{\partial x^i}{\partial x^l} = \delta^i_l$$

$$L = \frac{1}{2} a_{ij}\dot q^i \dot q^j + b_i \dot q^i - U(q)$$

$$p_i = \frac{\partial L}{\partial \dot q^i} = \frac{1}{2} a_{ij}\left(\dot q^j + \dot q^i \frac{\partial \dot q^j}{\partial \dot q^i}\right) + b_i$$

$$p_i = a_{ij}\dot q^j + b_i$$

$$\dot q^j = (a^{-1})^{ij}(p_i - b_i)$$

$$\left.\begin{aligned} V(S) &= S_i x^i - U(x(S)) \\ H &= p_i \dot q^i - L \end{aligned}\right\} S_i \Leftrightarrow p_i, \quad x^i \Leftrightarrow \dot q^i, \quad U \Leftrightarrow L$$

$$S_i = \frac{\partial U}{\partial x} \Leftrightarrow p_i = \frac{\partial L}{\partial \dot q}$$

$$H = p_i \dot q^i - L$$

$$\overset{!}{=} p_i (a^{-1})^{ij}(p_j - b_j) - \frac{1}{2} a_{ij}\left((a^{-1})^{ij}(p_j - b_j)\right)\left((a^{-1})^{ij}(p_i - b_i)\right)$$

$$- b_i \left((a^{-1})^{ij}(p_j - b_j)\right) + U(q)$$

$$\frac{\partial L}{\partial \dot q^i} = a_{ij}\,\dot q^j + b_i$$

$$\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} = a_{ij}\,\frac{\partial \dot q^j}{\partial \dot q^j} = a_{ij}$$

$$\frac{\partial H}{\partial p_j} = (a^{-1})^{lj}\left\{\frac{\partial p_l}{\partial p_j}(p_j - b_j) + p_l\right\} - \frac{1}{2}(a^{-1})^{lj}\left\{(p_l - b_l) + (p_j - b_j)\frac{\partial p_l}{\partial p_j}\right\}$$

$$- (a^{-1})^{lj}\,b_l$$

$$\frac{\partial^2 H}{\partial p_j \partial p_l} = (a^{-1})^{jl}\{1+1\} - \frac{1}{2}(a^{-1})^{jl}\{1+1\}$$

$$\stackrel{!}{=} (a^{-1})^{jl}$$

$$\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\,\frac{\partial^2 H}{\partial p_j \partial p_j} = a_{ij}\,(a^{-1})^{jl} = \delta_i^l$$