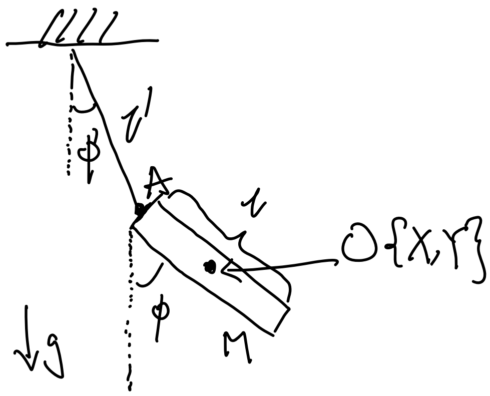
1) <u>A bar on a thread</u>



a) Find eom

$$X = l' \sin\phi' + \frac{l}{2}\sin\phi$$
$$Y = -\left(l'\cos\phi' + \frac{l}{2}\cos\phi\right)$$

$$\dot{X} = l'\cos\phi'\,\dot{\phi}' + \frac{l}{2}\cos\phi\,\dot{\phi}$$
$$\dot{Y} = l'\sin\phi'\,\dot{\phi}' + \frac{l}{2}\sin\phi\,\dot{\phi}$$

$$T = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{1}{2}I\dot{\phi}^2$$

$$V = MgY$$

$$L = \frac{1}{2}M\Big(l'^2\cos^2\phi'\,\dot{\phi}'^2 + \frac{l^2}{4}\cos^2\phi\,\dot{\phi}^2 + l'l\cos\phi'\cos\phi\,\dot{\phi}'\dot{\phi}$$
$$+ l'^2\sin^2\phi'\,\dot{\phi}'^2 + \frac{l^2}{4}\sin^2\phi\,\dot{\phi}^2 + l'l\sin\phi'\sin\phi\,\dot{\phi}'\dot{\phi}\Big)$$
$$+ \frac{1}{2}I\dot{\phi}^2 + Mg\left(l'\cos\phi' + \frac{l}{2}\cos\phi\right)$$

$$= \frac{1}{2}M\left(l'^2\dot{\phi}'^2 + \frac{l^2}{4}\dot{\phi}^2 + l'l\dot{\phi}'\dot{\phi}(\cos\phi'\cos\phi + \sin\phi'\sin\phi)\right)$$

$$+\frac{1}{2}I_A\dot{\phi}^2 + Mg\left(l'\cos\phi' + \frac{l}{2}\cos\phi\right) \quad \cos(\phi-\phi')$$

$$\frac{\partial L}{\partial \dot{\phi}'} = Ml'^2\dot{\phi}' + \frac{l'lM}{2}\cos(\phi-\phi')\dot{\phi}$$

$$\frac{\partial L}{\partial \phi'} = \frac{Ml'}{2}l\dot{\phi}'\dot{\phi}\,\sin(\phi-\phi') - Mgl'\sin\phi'$$

$$\frac{d}{dt}\left(Ml'^2\dot{\phi}' + \frac{Ml'l}{2}\cos(\phi-\phi')\dot{\phi}\right) = \frac{Ml'l}{2}\dot{\phi}'\dot{\phi}\sin(\phi-\phi') - Mgl'\sin\phi'$$

$$Ml'^2\ddot{\phi}' + \frac{Ml'l}{2}\cos(\phi-\phi')\ddot{\phi} - \frac{Ml'l}{2}\sin(\phi-\phi')\dot{\phi}(\dot{\phi}-\dot{\phi}')$$

$$= \frac{Ml'l}{2}\dot{\phi}'\dot{\phi}\sin(\phi-\phi') - Mgl'\sin\phi'$$

$$\boxed{Ml'^2\ddot{\phi}' + \frac{1}{2}Ml l'\cos(\phi-\phi')\ddot{\phi} - \frac{1}{2}Mll'\sin(\phi-\phi')\dot{\phi}^2 + Mgl'\sin\phi' = 0}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{Ml^2}{4}\dot{\phi} + \frac{1}{2}Ml'l\cos(\phi-\phi')\dot{\phi}' + I\dot{\phi}$$

$$= I_A\dot{\phi} + l'l\cos(\phi-\phi')\dot{\phi}'\frac{1}{2}M$$

$$\frac{\partial L}{\partial \phi} = -\frac{1}{2}Ml'\dot{\phi}'\dot{\phi}\sin(\phi-\phi') - \frac{Mgl}{2}\sin\phi$$

$$\frac{d}{dt}\left(I_A\dot{\phi} + \frac{Ml'l}{2}\cos(\phi-\phi')\dot{\phi}'\right) = -\frac{Ml'}{2}l\dot{\phi}'\dot{\phi}\sin(\phi-\phi') - \frac{Mgl}{2}\sin\phi$$

$$\frac{d}{dt}\Big(\cdots \frac{\cdots}{2} \cdots \frac{\cdots}{2} \cdots \frac{\cdots}{2} \cdots \Big)$$

$$I_A\ddot{\phi} + \frac{1}{2}Mll'\cos(\phi-\phi')\ddot{\phi}' - \frac{1}{2}Mll'\sin(\phi-\phi')\dot{\phi}'(\dot{\phi}-\dot{\phi}')$$
$$= -\frac{1}{2}Ml'l\dot{\phi}'\dot{\phi}\sin(\phi-\phi') - \frac{Mgl}{2}\sin\phi$$

$$\boxed{I_A\ddot{\phi} + \frac{1}{2}Mll'\cos(\phi-\phi')\ddot{\phi}' + \frac{1}{2}Mll'\sin(\phi-\phi')\dot{\phi}'^2 + \frac{Mgl}{2}\sin\phi = 0}$$

ii) Linearize around $\phi = \phi' = 0$ to first order.

$$\sin\phi' = \phi$$
$$\cos(\phi-\phi') = 1 - \cancel{\frac{(\phi-\phi')^2}{2}} =$$

$$Ml'\ddot{\phi}' + \frac{1}{2}Mll'\cos(\phi-\phi')\ddot{\phi} - \frac{1}{2}Mll'\sin(\phi-\phi')\dot{\phi}^2 + Mgl'\sin\phi' = 0$$

$$\hookrightarrow \boxed{Ml'^2\ddot{\phi}' + \frac{1}{2}Mll'\ddot{\phi} + Mgl'\phi' = 0}$$

$$I_A\ddot{\phi} + \frac{1}{2}Mll'\cos(\phi-\phi')\ddot{\phi}' + \frac{1}{2}Mll'\sin(\phi-\phi')\dot{\phi}'^2 + \frac{Mgl}{2}\sin\phi = 0$$

$$\hookrightarrow \boxed{I_A\ddot{\phi} + \frac{1}{2}Mll'\ddot{\phi}' + \frac{Mgl}{2}\phi = 0}$$

c) Find eigenfrequencies of small oscillations near equilibrium.

$$I_A \ddot{\phi} + \frac{Mgl}{2}\phi + \frac{1}{2}Mll'\ddot{\phi}' = 0$$

$$Ml'^2\ddot{\phi}' + Mgl'\phi' + \frac{1}{2}Mll'\ddot{\phi} = 0$$

$$\begin{pmatrix} I_A & \frac{1}{2}Mll' \\ \frac{1}{2}Mll' & Ml'^2 \end{pmatrix}\begin{pmatrix} \ddot{\phi} \\ \ddot{\phi}' \end{pmatrix} + \begin{pmatrix} \frac{Mgl}{2} & 0 \\ 0 & Mgl' \end{pmatrix}\begin{pmatrix} \phi \\ \phi' \end{pmatrix} = 0$$

let $\ddot{\Phi} = A e^{-i w^2 t}$

$$\left| -w^2\begin{pmatrix} I_A & \frac{1}{2}Mll' \\ \frac{1}{2}Ml'l & Ml'^2 \end{pmatrix} + \begin{pmatrix} \frac{Mgl}{2} & 0 \\ 0 & Mgl' \end{pmatrix} \right| = 0$$

$$\left(-w^2 I_A + \frac{Mgl}{2}\right)\left(-w^2 Ml'^2 + Mgl'\right) - \frac{1}{4}M^2 l^2 l'^2 w^4 = 0$$

$$w^4 \frac{1}{3} l^2 M^2 l'^2 - w^2 \cancel{I_A} Mgl' - w^2 \frac{Mgl}{2} Ml'^2 + \frac{M^2 g^2 l l'}{2} - \frac{1}{4}M^2 l^2 l'^2 w^4 = 0$$

$$\frac{1}{12}\cancel{M^2} l^2 l'^2 w^4 - w^2\left(\cancel{M^2} g\right)\left(\frac{1}{3}l^2 l' + \frac{1}{2}l l'^2\right) + \frac{\cancel{M^2} g^2 l l'}{2} = 0$$

$$w^4 - w^2\, 12g\left(\frac{1}{3}\frac{1}{l'} + \frac{1}{2}\frac{1}{l}\right) + \frac{12 g^2}{2 l l'} = 0$$

$$w^4 - w^2\left(4\Omega'^2 + 4\Omega^2\right) + 4\Omega^2\Omega'^2 = 0$$

$\frac{3}{2}\frac{g}{l} = \Omega^2$

$\frac{g}{l'} = \Omega'^2$

$$w^2 = \frac{4(\Omega'^2 + \Omega^2) \pm \sqrt{16(\Omega'^2 + \Omega^2)^2 - 16\Omega^2\Omega'^2}}{2}$$

$$\boxed{\omega_\pm^2 = \frac{1}{2}(\Omega'^2+\Omega^2) \pm \frac{1}{2}\sqrt{(\Omega'^2+\Omega^2)^2 - \varrho^2\varrho'^2}}$$

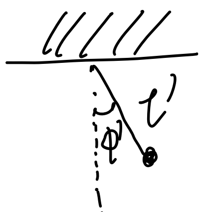
d) i) Discuss motion in limit $l \ll l'$

if $l \ll l'$: then there is no ϕ eq goes away.

$$\cancel{\frac{1}{3}Ml^2\ddot{\phi} + \frac{1}{2}Mll'\ddot{\phi}' + \frac{Mgl}{2}\phi = 0}$$

$$Ml'^2\ddot{\phi}' + \cancel{\frac{1}{2}Mll'\ddot{\phi}} + Mgl'\phi' = 0$$

$$\ddot{\phi}' + \frac{g}{l'}\phi' = 0 \qquad \leftarrow \text{a simple pendulum with } \omega^2 = g/l' = \Omega'^2$$



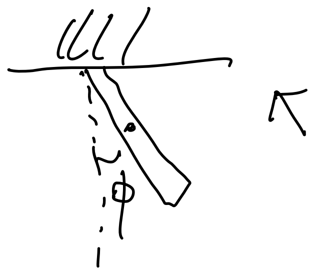
For $l' \ll l$:

we have

$$\frac{1}{3}Ml^2\ddot{\phi} + \cancel{\frac{1}{2}Mll'\ddot{\phi}'} + \frac{Mgl}{2}\phi = 0$$

$$\ddot{\phi} + \frac{3}{2}\frac{g}{l}\phi = 0$$

$\leftarrow \quad \omega'^2 = \frac{3}{2} g/l = \Omega'^2$



a rod hanging

e) Find eigenmodes for $l = l'$

$$\omega_\pm^2 = 2(\Omega'^2 + \Omega^2) \pm 2\sqrt{(\Omega'^2 + \Omega^2)^2 - \Omega^2 \Omega'^2}$$

$$\hookrightarrow \omega_\pm^2 = 2\left(\frac{5}{2}\Omega'^2\right) \pm 2\sqrt{\left(\frac{5}{2}\Omega'^2\right)^2 - \frac{3}{2}\Omega'^4}$$

$$\overset{!}{=} 5\Omega'^2 \pm \Omega'^2 2\sqrt{\frac{25}{4} - \frac{3}{2}}$$

$$\overset{!}{=} 5\Omega'^2 \pm \Omega'^2 2\sqrt{\frac{19}{4}}$$

$$\omega_\pm^2 \overset{!}{=} (5 \pm \sqrt{19})\Omega^2$$

For $\omega^2 = \omega_+^2 \qquad l = l'$

$$\left\{ -(5+\sqrt{19})\Omega^2 \begin{pmatrix} I_A & \frac{1}{2}Ml^2 \\ \frac{1}{2}Ml^2 & Ml^2 \end{pmatrix} + \begin{pmatrix} \frac{Mgl}{2} & 0 \\ 0 & Mgl \end{pmatrix} \right\} \begin{pmatrix} E^+_1 \\ E^+_2 \end{pmatrix} = 0$$

$$\left[-(5+\sqrt{19})\Omega^2\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}+\begin{pmatrix}\Omega^2\frac{1}{2} & 0\\ 0 & \Omega^2\end{pmatrix}\right]\begin{pmatrix}E^{+}_1\\ E^{+}_2\end{pmatrix}=0$$

$$\begin{bmatrix}\left(-\frac{(5+\sqrt{19})}{3}+\frac{1}{2}\right)\Omega^2 & -\frac{(5+\sqrt{19})}{2}\Omega^2\\ -\frac{(5+\sqrt{19})}{2}\Omega^2 & \left(-(5+\sqrt{19})+1\right)\Omega^2\end{bmatrix}\begin{pmatrix}E^{+}_1\\ E^{+}_2\end{pmatrix}=0$$

let $E^{+}_1 = 1$

$$-\left(\frac{5+\sqrt{19}}{3}\right)+\frac{1}{2}-\frac{(5+\sqrt{19})}{2}E^{+}_2=0$$

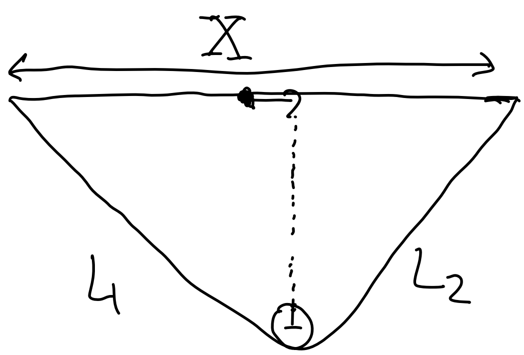
$$-\left(\frac{7+2\sqrt{19}}{6}\right)=\left(\frac{5+\sqrt{19}}{2}\right)E^{+}_2$$

$$E^{+}_2=-\left(\frac{7+2\sqrt{19}}{3(5+\sqrt{19})}\right)$$

$$E_2 = -\left(\frac{\text{[illegible]}}{3(5-\sqrt{19})}\right) \qquad E_1 = 1$$

$$\begin{pmatrix}\phi \\ \phi'\end{pmatrix} = A e^{-i\omega_+ t}\begin{pmatrix}1 \\ -\left(\frac{7+2\sqrt{19}}{3(5+\sqrt{19})}\right)\end{pmatrix} + B e^{-i\omega_- t}\begin{pmatrix}1 \\ -\left(\frac{7-2\sqrt{19}}{3(5-\sqrt{19})}\right)\end{pmatrix}$$

2) Rolling of a spool :



$X \sim L$

$L_1 + L_2 = L$

$R \ll L$

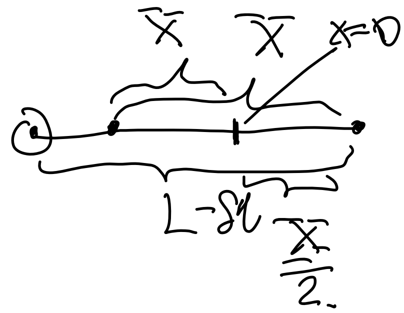
a) know Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Determine a, b in terms of L and X

show: $$L_1 = \frac{L}{2} + \frac{X}{2}x$$

$L_2 = \frac{L}{2} - \frac{\Delta}{L} x$



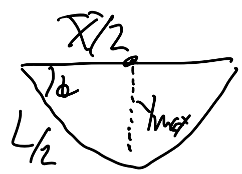
When $Y=0$

$$X_{max} = a = L - \delta l - \frac{\bar{X}}{2}$$

since $\bar{X} \simeq L$

$$\simeq \frac{L}{2}$$

When $X=0$ $\quad Y_{max}^2 = b^2$



$$\left(\frac{\bar{X}}{2}\right)^2 + Y_{max}^2 = \left(L/2\right)^2$$

$$Y_{max} = \sqrt{(L/2)^2 - \left(\frac{\bar{X}}{2}\right)^2}$$

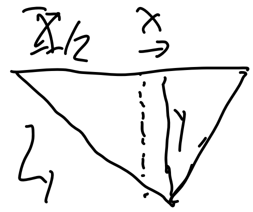
$$b \simeq \frac{1}{2}\sqrt{L^2 - \bar{X}^2}$$

$$\frac{X^2}{\left(\frac{L}{2}\right)^2} + \frac{Y^2}{\frac{1}{4}(L^2 - \bar{X}^2)} = 1$$

$$Y^2 = \left(1 - \frac{4X^2}{L^2}\right)\frac{1}{4}(L^2 - \bar{X}^2)$$

$$\simeq \left(\frac{1}{4} - \frac{X^2}{L^2}\right)(L^2 - \bar{X}^2)$$

$$Y^2 \quad 1 \left(L^2 - X^2\right)\left(L^2 - \bar{X}^2\right)$$



$$L_1^2 = (X/2 + x)^2 + y^2$$

$$\stackrel{!}{=} x^2 + Xx + \frac{X^2}{4} + y^2$$

$$L_2^2 = (X/2 - x)^2 + y^2$$

$$\stackrel{!}{=} x^2 - Xx + \frac{X^2}{4} + y^2$$

$$L_1^2 - L_2^2 = 2Xx$$

$$L_1^2 - (L - L_1)^2 = -L^2 + 2LL_1 = 2Xx$$

$$\boxed{L_1 = \frac{L}{2} + \frac{X}{L}x}$$

$$L - L_1 = L_2$$

$$\boxed{L_2 = \frac{L}{2} - \frac{X}{L}x}$$

b) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\omega^2$

$$I = \int dm\, r^2$$

$$= \int \frac{M}{\pi R^2}\, r^2\, r\, dr\, d\theta$$

$$= \frac{2\pi M}{\pi R^2} \frac{R^4}{4}$$

$$I_{disk} = \frac{MR^2}{2}$$

$$V = Mgy$$

$$y = -\sqrt{\left(\frac{L^2}{4} - x^2\right)\left(\frac{L^2 - x^2}{L^2}\right)}$$

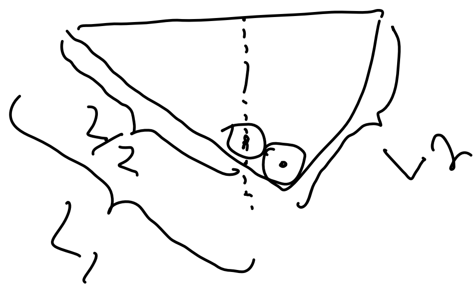
$$\dot{y} = \sqrt{\frac{L^2 - x^2}{L^2}}\, \frac{x\dot{x}}{\sqrt{\frac{L^2}{4} - x^2}}$$

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 + \left(\frac{L^2 - x^2}{L^2}\right)\frac{x^2\dot{x}^2}{\left(\frac{L^2}{4} - x^2\right)}\right)$$

$$+ \frac{1}{2}I\dot{\phi}^2 - Mgy.$$

$$\mathcal{L} = \frac{1}{2}M\left(1 + \left(\frac{L^2 - x^2}{L^2}\right)\frac{x^2}{\left(\frac{L^2}{4} - x^2\right)}\right)\dot{x}^2$$

$$+ \frac{1}{2}I\dot{\phi}^2 - Mg\sqrt{\frac{L^2 - x^2}{L^2}}\sqrt{\frac{L^2}{4} - x^2}$$



$$R\phi = l - l_0$$ ← length at origin

distance traveled

$$= L_1 - L/2$$

$$= \frac{L}{2} + \frac{X}{L}x - L/2$$

$$R\phi = \frac{X}{L}X$$

$$\phi = \frac{X}{RL}X$$

$$\hookrightarrow \dot{\phi} = \frac{X}{RL}\dot{X}$$

$$\mathcal{L} = \frac{1}{3}M\left(1 + \left(\frac{L^2 - X^2}{\ldots}\right)\frac{x^2}{(\ldots)}\right)\dot{X}^2$$

$$\left(L^2 \sqrt{\frac{L^2}{4}-x}\right)$$

$$+\frac{1}{2} \underset{\frac{MR^2}{2}}{I} \left(\frac{x}{RL}\right)^2 \dot{x}^2 + Mg\sqrt{\frac{L^2-x^2}{L^2}}\sqrt{\frac{L^2}{4}-x^2}$$

$$\boxed{L=\frac{1}{2} M\left(1+\frac{x^2}{2L^2}+\left(\frac{L^2-x^2}{L^2}\right)\left(\frac{x^2}{\frac{L^2}{4}-x^2}\right)\right)\dot{x}^2 + Mg\sqrt{\frac{L^2-x^2}{L^2}}\sqrt{\frac{L^2}{4}-x^2}}$$

c) Determine Frequency around equilibrium.

Clearly equilibrium when $x_{eq}=0$

$$x = 0 + \delta x = \delta x$$

$$L \simeq \frac{1}{2} M(x=0)\,\dot{x}^2 + Mg\sqrt{\frac{L^2-x^2}{L^2}}\sqrt{\frac{L^2}{4}-\delta x^2}$$

$$L \simeq \frac{1}{2}M\left(1+\frac{\bar{X}^2}{2L^2}\right)\delta\dot{x}^2 + Mg\sqrt{\frac{L^2-\bar{X}^2}{L^2}}\,\frac{L}{2}\sqrt{1-\frac{4\delta x^2}{L^2}}$$

$$= \frac{1}{2}M\left(1+\frac{\bar{X}^2}{2L^2}\right)\delta\dot{x}^2 + Mg\sqrt{\frac{L^2-\bar{X}^2}{L^2}}\,\frac{L}{2}\left(1-\frac{2\delta x^2}{L^2}\right)$$

$$= \frac{1}{2}M\left(1+\frac{\bar{X}^2}{2L^2}\right)\delta\dot{x}^2 - \frac{Mg}{L}\sqrt{\frac{L^2-\bar{X}^2}{L^2}}\;\delta x^2$$

$$\frac{\partial L}{\partial \delta\dot{x}} = \cancel{M}\left(1+\frac{\bar{X}^2}{2L^2}\right)\delta\dot{x}$$

$$\frac{\partial L}{\partial \delta x} = -\frac{2\cancel{M}g}{L}\sqrt{\frac{L^2-\bar{X}^2}{L^2}}\;\delta x$$

$$\delta\ddot{x} = \underbrace{\frac{-2g}{L}\sqrt{\frac{L^2-\bar{X}^2}{L^2}}\left(1+\frac{\bar{X}^2}{2L^2}\right)^{-1}}_{w^2}\delta x$$

$$w^2 = \frac{2g}{L}\sqrt{\frac{L^2-\bar{X}^2}{L^2}}\left(\frac{2L^2}{2L^2}+\frac{\bar{X}^2}{2L^2}\right)^{-1}$$

$$= \frac{2g}{L^2}\sqrt{L^2-\bar{X}^2}\;\frac{2L^2}{\ldots}$$

$$2L^2+\bar{x}^2$$

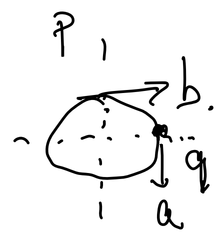
$$\boxed{\omega^2 \stackrel{!}{=} \frac{4g\sqrt{L^2-\bar{x}^2}}{2L^2+\bar{x}^2}}$$

d) Now consider if $\bar{X}$ is slowly increased.

slowly increase → adiabatic Invariant.

Area of phase space.

$$\frac{1}{2\pi}\int dp\,dq = I$$



$$\oint \frac{p\,dq}{2\pi} = I.$$ ← Adiabatic Invariant.

$I = \pi ab$ for ellipse.

If E is constant, then phase space is closed.

$$\ddot{\delta x} + \omega^2 \delta x = 0 \Rightarrow \delta x = A_0 \cos(-\omega t + \phi)$$

$$E = \tfrac{1}{2}m(x)\dot{\delta x}^2 + \tfrac{1}{2}m(x)\omega^2 \delta x^2$$

$$= \frac{1}{2}M\left(1+\frac{\bar{x}^2}{2L^2}\right)\dot{\delta x}^2 + \frac{1}{2}M\left(1+\frac{\bar{x}^2}{2L^2}\right)\omega^2\delta x^2$$

We know $\delta x = A_0\cos(-\omega t + \phi)$

so $\max(\delta x) = A_0 = a$

$$P = \frac{\partial L}{\partial \delta \dot{x}} = m(x)\,\delta \dot{x}$$

$$= M\left(1 + \frac{x^2}{2L^2}\right)\omega A_0 \sin(\omega t + \phi).$$

$$P_{max} = M\left(1 + \frac{x^2}{2L^2}\right)\omega A_0 = b$$

$$I_0 = \pi a b = M A_0^2\left(1 + \frac{x_0^2}{2L^2}\right)\sqrt{\frac{4g\sqrt{L^2 - x_0^2}}{2L^2 + x_0^2}}$$

$$I_f = I_0$$

$$\cancel{M} A_f^2\left(1 + \frac{x_f^2}{2L^2}\right)\sqrt{\frac{4g\sqrt{L^2 - x_f^2}}{2L^2 - x_f^2}} =$$

$$\cancel{M} A_0^2\left(1 + \frac{x_0^2}{2L^2}\right)\sqrt{\frac{4g\sqrt{L^2 - x_0^2}}{2L^2 + x_0^2}}$$

$$A_f = A_0\sqrt{\frac{\left(1 + \frac{x_0^2}{2L^2}\right)\sqrt{\frac{4g\sqrt{L^2 - x_0^2}}{2L^2 + x_0^2}}}{\left(1 + \frac{x_f^2}{2L^2}\right)\sqrt{\frac{4g\sqrt{L^2 - x_f^2}}{2L^2 - x_f^2}}}}$$

3) Dissipation from an external field.

mass moving in 1D with $V(q)$ and

$$F_{friction} = -2m\gamma\dot{q}$$

a) $$\frac{\partial L}{\partial q} = f(t)\frac{\partial}{\partial q}L_0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(f(t)\frac{\partial}{\partial \dot{q}}L_0\right) = \dot{f}\frac{\partial}{\partial \dot{q}}L_0 + f\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}L_0\right)$$

$$\dot{f}\frac{\partial}{\partial \dot{q}}L_0 + f\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}L_0\right) = f(t)\frac{\partial}{\partial q}L_0$$

$$L_0 = \frac{1}{2}m\dot{q}^2 - V(q)$$

$$\dot{f}(t)m\dot{q} + fm\ddot{q} = f\,-\frac{\partial}{\partial q}V(q)$$

$$\frac{\dot{f}}{f}m\dot{q} + m\ddot{q} + \frac{\partial}{\partial q}V(q) = 0$$

$$\frac{\dot{f}}{f}m\dot{q} = 2m\gamma\dot{q}$$

$$\frac{\dot{f}}{f} = 2\gamma$$

$$\ln f = 2\gamma t$$

$$e^{2\gamma t} = f(t)$$

$$\boxed{\mathcal{L}(t) = e^{2\gamma t}\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]}$$

b) $H = p\dot{q} - \mathcal{L}$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = e^{2\gamma t} m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}e^{-2\gamma t}$$

$$H = \frac{p^2}{m}e^{-2\gamma t} - e^{2\gamma t}\left[\frac{1}{2}\frac{p^2}{m}e^{-4\gamma t} - V(q)\right]$$

$$= \frac{p^2}{2m}e^{-2\gamma t} + e^{2\gamma t}V(q)$$

c) let $V(q) = \frac{1}{2}m\omega^2 q^2 = \frac{1}{2}q^2$ by $m = \omega = 1$

Find canonical transform $Q = Q(q, p, t)$

$P = P(q, p, t)$

such that $H(q, p, t) \to K(Q, P)$

$$H = \frac{p^2}{2}e^{-2\gamma t} + e^{2\gamma t}\frac{1}{2}q^2$$

$$\frac{\partial H}{\partial p} = \dot{q} \qquad -\frac{\partial H}{\partial q} = \dot{p}$$

$$\stackrel{!}{=} pe^{-2\gamma t} \qquad \stackrel{!}{=} -qe^{2\gamma t}$$

Find $\tilde{H}(Q,P)$ such that

$$\frac{\partial \tilde{H}}{\partial P} = \dot{Q} \qquad -\frac{\partial \tilde{H}}{\partial Q} = \dot{P}$$

The variation in action should remain unchanged

$$S_1 = \int p\,dq - H\,dt$$

$$S_2 = \int P\,dQ - \tilde{H}\,dt$$

The difference must be a total derivative, else the action is different.

$$\int p\,dq - P\,dQ - (H - \tilde{H})\,dt = \frac{dF}{dt}$$

So:

$$p\dot{q} - H(p,q,t) = P\dot{Q} - \tilde{H}(P,Q,t) + \frac{dF}{dt}$$

if we let $P = pe^{-\gamma t}$ $\quad Q = qe^{\gamma t}$

$\hookrightarrow p = P e^{\gamma t}$ $\quad \hookrightarrow q = Q e^{-\gamma t}$

$\dot{q} = \dot{Q} e^{-\gamma t} + -\gamma Q e^{-\gamma t}$

$$p\dot{q} - \mathcal{H} = P e^{\gamma t}\left(\dot{Q} e^{-\gamma t} - \gamma Q e^{-\gamma t}\right) - \frac{P^2 e^{2\gamma t} e^{-2\gamma t}}{2} - \frac{Q^2 e^{-2\gamma t}}{2}$$

$$\stackrel{!}{=} P\dot{Q} - P\gamma Q - \left(\frac{P^2}{2} + \frac{Q^2}{2}\right)$$

know

$$p\dot{q} - \mathcal{H} = P\dot{Q} - \tilde{\mathcal{H}} + \frac{df}{dt}.$$

$$\cancel{P\dot{Q}} - P\gamma Q - \left(\frac{P^2}{2} + \frac{Q^2}{2}\right) = \cancel{P\dot{Q}} - \tilde{\mathcal{H}} + \frac{df}{dt}$$

$$\tilde{H} = \left(\frac{P^2}{2} + \frac{Q^2}{2}\right) + \gamma P Q \cancel{+ \frac{df}{dt}} \quad \text{let } f = 0$$

Procedure:

let

① $p\dot{q} - \mathcal{H} = P\dot{Q} - \tilde{\mathcal{H}} + \frac{df}{dt}$

② Have some ansatz on P and Q on q and p.

③ revert to $q(Q,P)$, $p(Q,P)$. Plug into ①

Then determine $\tilde{\mathcal{H}}$

d) $\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = -Q - \gamma P$

$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = P + \gamma Q$

$\ddot{Q} = \dot{P} + \gamma \dot{Q}$

$\ddot{Q} = -Q - \gamma P + \gamma \dot{Q}$

$\stackrel{!}{=} -Q - \gamma P + \gamma (P + \gamma Q)$

$\ddot{Q} = -Q + \gamma^2 Q$

$\ddot{Q} \stackrel{!}{=} -(1-\gamma^2) Q$

$Q(t) = A \cos(\sqrt{1-\gamma^2}\, t - \phi)$

for $Q(t) = q e^{\gamma t}$

$q(t) = e^{-\gamma t} A \cos(\sqrt{1-\gamma^2}\, t - \phi)$