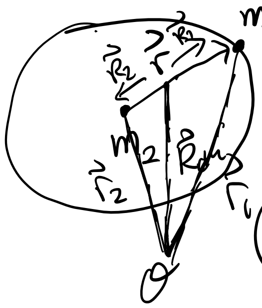


$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec r}_1^{\,2} + \frac{1}{2}m_2\dot{\vec r}_2^{\,2} - U(|\vec r_1 - \vec r_2|)$$

use center of mass coordinate:

$$(m_1+m_2)\vec R_{cm} = M\vec R_{cm} = m_1\vec r_1 + m_2\vec r_2$$

$$\vec r = \vec r_1 - \vec r_2$$

$$\vec R_1 = \vec r_1 - \vec R_{cm}$$

$$\stackrel{!}{=} \vec r_1 - \frac{m_1\vec r_1 + m_2\vec r_2}{m_1+m_2}$$

$$\vec R_1 \stackrel{!}{=} \frac{m_2}{M}(\vec r_1 - \vec r_2) = \frac{m_2}{M}\vec r$$

$$\vec R_2 = \vec r_2 - \vec R_{cm}$$

$$\stackrel{!}{=} \vec r_2 - \frac{m_1\vec r_1 + m_2\vec r_2}{M}$$

$$\vec R_2 \stackrel{!}{=} \frac{m_1}{M}\vec r_2 - \vec r_1 = -\frac{m_1}{M}\vec r$$

$$\mathcal{L} = \frac{1}{2}\sum_a m_a \dot{\vec R}_{cm}^{\,2} + \frac{1}{2}\sum_a m_a \Delta\dot{\vec r}_a^{\,2} - U(|\vec r_1 - \vec r_2|)$$

$$\stackrel{!}{=} \frac{1}{2}M\dot{\vec R}_{cm}^{\,2} + \frac{1}{2}m_1\left(\frac{m_2}{M}\dot{\vec r}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1}{M}\dot{\vec r}\right)^2 - U(|r|)$$

$$\stackrel{!}{=} \frac{1}{2}M\dot{\vec R}_{cm}^{\,2} + \frac{1}{2}\frac{m_1m_2}{M}\dot{\vec r}^{\,2}\left(\cancel{\frac{m_2}{M} + \frac{m_1}{M}}\right) - U(|r|)$$

$$\mu = \frac{m_1m_2}{m_1+m_2}$$

$$\mathcal{L} \stackrel{!}{=} \frac{1}{2}M\dot{\vec R}_{cm}^{\,2} + \frac{1}{2}\mu\dot{\vec r}^{\,2} - U(|r|)$$

Choose orbit motion in x-y plane.

then $\vec{r} = (r\cos\phi, r\sin\phi, 0)$ where it starts from $\vec{r}_2$.

$$\mathcal{L} = \frac{1}{2} M (\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \frac{1}{2} \mu (\dot{r}^2 + (r\dot{\phi})^2) - U(|r|)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = P_\phi = \mu r^2 \dot{\phi} = \text{const} \quad \text{since } \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Rightarrow \dot{\phi} = \frac{P_\phi}{\mu r^2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \mu \ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\mu \ddot{r} \stackrel{!}{=} \frac{P_\phi^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

$$\mu \ddot{r} = -\frac{\partial}{\partial r}\Big[\underbrace{\frac{P_\phi^2}{2\mu r^2} + U}_{V_{eff}(r)}\Big]$$

Since $\mathcal{L}$ is not explicitly function of time, E is conserved.

$$E = T + V_{eff} = \frac{1}{2}\mu \dot{r}^2 + \frac{P_\phi^2}{2\mu r^2} + U$$

<u>Hamiltonian Formalism:</u>

$$\mathcal{H} = P_i \dot{q}^i - \mathcal{L}$$

$$\mathcal{H} = P_{x_{cm}} \dot{X}_{cm} + P_{y_{cm}} \dot{Y}_{cm} + P_\phi \dot{\phi} + P_r \dot{r}$$

$$- \mathcal{L}$$

$$\mathcal{H} = \frac{P_{x_{cm}}^2}{2M} + \frac{P_{y_{cm}}^2}{2M} + P_\phi \frac{P_\phi}{\mu r^2} + P_r \frac{P_r}{\mu}$$

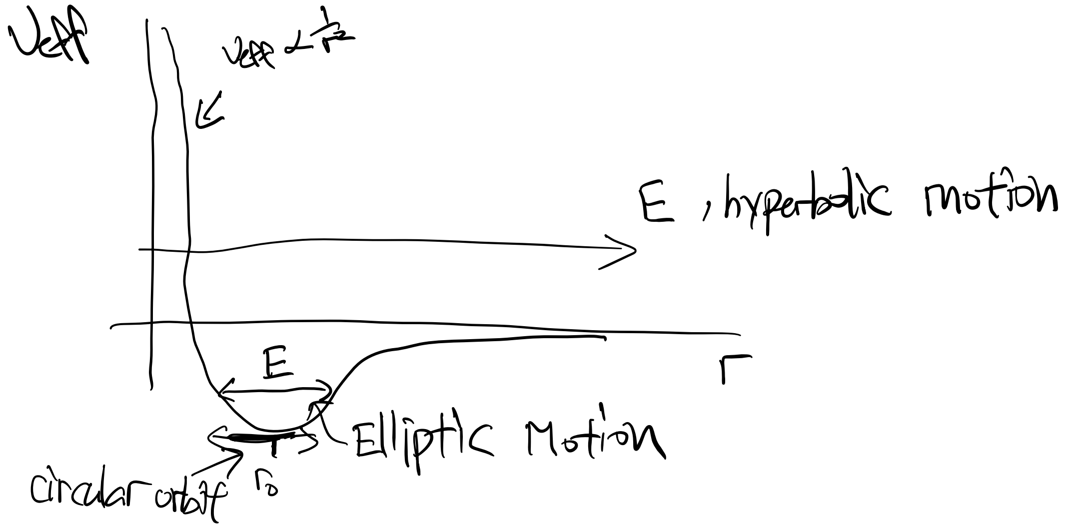
$$- \frac{1}{2}\mu \left(\frac{P_r}{\mu}\right)^2 - \frac{1}{2}\mu r^2 \left(\frac{P_\phi}{\mu r^2}\right)^2 + U$$

$$= \frac{P_{x_{cm}}^2}{2M} + \frac{P_{y_{cm}}^2}{2M} + \frac{P_\phi^2}{2\mu r^2} + \frac{P_r}{2\mu} + U$$

$$\frac{\partial H}{\partial P_\phi} = \dot{\phi} = \frac{P_\phi}{\mu r^2}$$

$$-\frac{\partial H}{\partial \phi} = 0 = \frac{dP_\phi}{dt}$$

$$-\frac{\partial H}{\partial r} = -\frac{\partial}{\partial r}\underbrace{\left\{\frac{P_\phi^2}{2\mu r^2} + U\right\}}_{V_{eff}(r)} = \frac{dP_r}{dt}$$



Since E is conserved.

$$\underbrace{E - \frac{P_x^2}{2M} - \frac{P_y^2}{2M}}_{E'} = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{P_\phi^2}{2\mu r^2} + U}_{V_{eff}}$$

$$\frac{dr}{dt} = \sqrt{\frac{2\left(E - V_{eff}\right)}{\mu}}$$

$$\int_{r_0}^{r} dr \sqrt{\frac{\mu}{2(E-V_{eff})}} = \int_{t_0}^{t} dt = t - t_0$$

$\uparrow$ gives $r(t)$

Get $r(\phi)$

$$\int_{r_0}^{r} dr \sqrt{\frac{\mu}{2(E-V_{eff})}} = \int \frac{dt}{d\phi}\, d\phi$$

since $\dot{\phi} = \frac{P_\phi}{\mu r^2}$

then $d\phi = \frac{P_\phi}{\mu r^2}\, dt$

$$\hookrightarrow \quad \sqrt{\frac{\mu}{2}} \int_{r_0}^{r} \frac{dr}{(E-V_{eff})} = \int_{\phi_0}^{\phi} \frac{\mu r^2}{P_\phi}\, d\phi$$

$$P_\phi \sqrt{\frac{1}{2\mu}} \int_{r_0}^{r} \frac{dr}{r^2\sqrt{E-V_{eff}}} = \phi - \phi_0$$

Gives $\nearrow$ $r(\phi)$

Circular Motion; when $V_{eff}(r)$ is minimum.

$$\frac{\partial V_{eff}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{P_\phi^2}{2\mu r^2} + U(r)\right)$$

$\uparrow = \frac{-k}{r}$

$$= \frac{1\phi}{ur^3} + \frac{k}{r^2}$$

then $\boxed{r_0 = \frac{P_\phi^2}{uk}}$ ← radius of circular motion.

$$U(r=r_0) = \frac{-k}{r_0} \equiv -2\varepsilon_0$$

then $\varepsilon_0 = \frac{k}{2r_0} = \frac{uk^2}{2P_\phi^2}$

$$T(r=r_0) = \frac{P_\phi^2}{2ur_0^2} + \frac{1}{2}\cancel{u\dot{r}^2}$$ $=0$ since circular motion

$$\stackrel{!}{=} \frac{P_\phi^2}{2u} \frac{(uk)^2}{P_\phi^4}$$

$$\stackrel{!}{=} \frac{uk^2}{2P_\phi^2} = \varepsilon_0$$

then

$$T + U = \varepsilon_0 + -2\varepsilon_0 = -\varepsilon_0$$

$$\hookrightarrow -\frac{1}{2}U = T$$ ← virial Theorem

Let $\varepsilon = \frac{E}{\varepsilon_0}$ $\quad \bar{r} = \frac{r}{r_0}$

define: $\frac{V_{eff}}{\varepsilon_0} = \frac{P_\phi^2}{2ur^2} \frac{2r_0^2}{k r_0} - \frac{k}{r} \frac{2r_0}{k}$

$$\stackrel{!}{=} \frac{P_\phi^2}{kur_0} \frac{1}{\bar{r}^2} - \frac{2}{\bar{r}}$$

$r_0 = \frac{P_\phi^2}{uk}$

$$\hookrightarrow \stackrel{!}{=} \frac{1}{\bar{r}^2} - \frac{2}{\bar{r}}$$

Let

let $u = \frac{1}{\bar{r}}$

$$\doteq u^2 - 2u$$

$$\doteq u^2 - 2u + 1 - 1$$

$$\frac{V_{eff}}{\varepsilon_0} \doteq (u-1)^2 - 1$$

$$\varepsilon_0 = \frac{k}{2r_0}$$

$$r_0 = \frac{p_\phi^2}{\mu k}$$

$$\bar{r} = \frac{r}{r_0}$$

$$r_0\, d\bar{r} = dr$$

Then:

$$\phi - \phi_0 = \frac{p_\phi}{\sqrt{2\mu}} \int_{r_0}^{r} \frac{dr/r^2}{\sqrt{E - V_{eff}}}$$

$$\doteq \frac{p_\phi}{\sqrt{2\mu}} \int_{r_0}^{r} \frac{\cancel{r_0}\, d\bar{r}}{r_0^{\cancel{2}} \sqrt{\varepsilon_0}\, \bar{r}^2 \sqrt{\varepsilon - (u-1)^2 + 1}}$$

$$\doteq \cancel{\sqrt{\frac{p_\phi^2}{2\mu}} \sqrt{\frac{2p_\phi^2}{\mu k^2}} \left(\frac{\mu k}{p_\phi^2}\right)} \underbrace{\int_{r_0}^{r} \frac{d\bar{r}/\bar{r}^2}{\sqrt{\varepsilon - (u-1)^2 + 1}}}_{-\int_{r_0}^{r} \frac{du}{\sqrt{(\varepsilon+1) - (u-1)^2}}}$$

$$\phi - \phi_0 \doteq \cos^{-1}\left\{\frac{u-1}{\sqrt{\varepsilon + 1}}\right\}$$

$$\cos(\phi - \phi_0) = \frac{u-1}{\sqrt{\varepsilon+1}}$$

let $\phi_0 = 0$

$$u = \frac{r_0}{r} = 1 + \sqrt{\frac{E}{\varepsilon_0} + 1}\, \cos(\phi - \cancel{\phi_0})$$

$$\frac{1}{r} = \frac{1}{r_0}\left\{1 + \sqrt{\frac{E}{\varepsilon_0} + 1}\, \cos\phi\right\}$$

$$r = \frac{r_0}{1 + \sqrt{\frac{E}{\varepsilon_0} + 1}\, \cos\phi}$$

Where Ellipse equation: $r = \frac{r_0}{1 + e\cos\phi}$

Hence eccentricity: $e = \sqrt{\frac{E}{\varepsilon_0} + 1}$

$$= \sqrt{\frac{E P_\phi^2}{2 u k^2} + 1}$$

when $E = -\varepsilon_0$

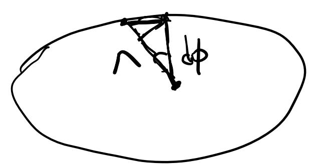
$e = 0 \Rightarrow$ circular motion.

Kepler Laws:

① Equal area per time

due to angular momentum conservation.

$$P_\phi = u r^2 \dot{\phi}$$



$$dA = r\,dr\,d\phi = \frac{1}{2} r^2 d\phi = \underbrace{\frac{P_\phi}{u}}_{\text{const}} dt$$

so dA is same as long as dt is same.

Period:

$$T = \frac{\text{Area}}{\frac{dA}{dt}} \qquad \text{Area} : \pi ab$$

$$\frac{dA}{dt} = \frac{r\,dr\,d\phi}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{1}{2} r^2 \frac{P_\phi}{u r^2} = \frac{P_\phi}{2u}$$

$$\frac{dA}{dt} = \frac{1}{2u} \sqrt{r_0 u k} \qquad r_0 = \frac{P_\phi^2}{u k}$$

$$T = \frac{A}{\frac{dA}{dt}} = \frac{\pi ab}{\frac{1}{2}\sqrt{\frac{r_0 k}{u}}} \qquad \text{since } \pi ab \propto \pi a^2, \quad r_0 \propto a$$

$$T \overset{!}{=} 2\pi a^{3/2} \sqrt{\frac{u}{k}}$$

or $T^2 = a^3$ ← kepler second law.

Is the Ellipse a full-closed orbit:

$$\text{Require} \int_0^{2\pi} d\phi = 2\pi = \underset{\substack{\uparrow \\ \text{back and} \\ \text{forth.}}}{2} \times \frac{P_\phi}{\sqrt{2\mu}} \int_{r_{min}}^{r_{max}} \frac{dr/r^2}{\sqrt{E - V_{eff}}}$$

Which is the case.

Bertrand Theorem!

For $U = Cr^\beta$, only closed orbits are

$\beta = -1$ ← gravity

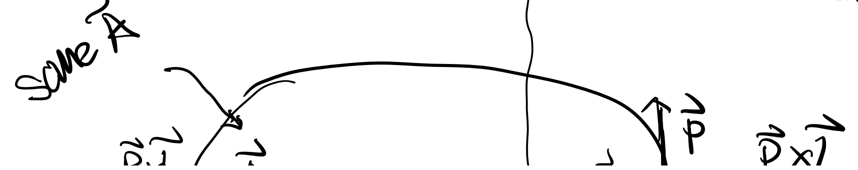
$\beta = 2$ ← SHO

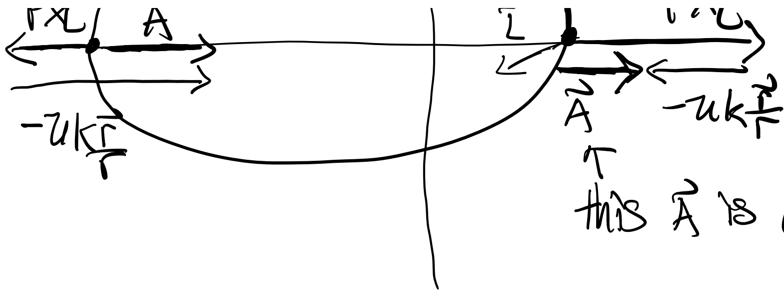
The reason $\beta = -1$ is closed orbit, is due to extra symmetry.

$$\vec{A} = \vec{p} \times \vec{L} - \mu k \frac{\vec{r}}{r}$$ ← Laplace Runge-Lenz Vector.

↳ $\frac{d\vec{A}}{dt} = 0$ ← due to extra symmetry

or a conserved quantity. in time and orients the Ellipse.





this $\vec{A}$ is constant.