Newton Laws

$\vec{F}_{ab}$ = force act on a by b.

$$\vec{F}_a = \vec{F}_a^{ext} + \underbrace{\sum_b \vec{F}_{ab}}_{\text{internal}} = \frac{d\vec{p}_a}{dt} \leftarrow \text{Force act on a}$$

$$\frac{d\vec{P}_{tot}}{dt} = \sum_a \frac{d\vec{p}_a}{dt} = \sum_a \vec{F}_a^{ext} + \cancel{\sum_{ab} F_{ab}} \xrightarrow{} 0 \quad \text{since } \vec{F}_{ab} = -\vec{F}_{ba}$$

total force of system

$$\boxed{\frac{d\vec{P}_{tot}}{dt} = \vec{F}_{tot}^{ext}} \leftarrow \text{conservation of momentum}$$

Center of Mass: weight by mass.

$$\boxed{\begin{aligned} \vec{R}_{cm} &= \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a} \\ \vec{v}_{cm} &= \frac{\sum_a m_a \vec{v}_a}{M_{tot}} = \frac{\vec{P}_{tot}}{M_{tot}} \end{aligned}}$$

Angular Momentum

$$\vec{\tau} = \vec{r}_a \times \frac{d\vec{p}_a}{dt} = \vec{r}_a \times \vec{F}_a = \text{Torque.}$$

$$\vec{\tau} = \frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \cancel{\vec{v} \times m\vec{v}} + \vec{r} \times \vec{F}$$

$$\hookrightarrow \boxed{\vec{\tau}_a = \frac{d\vec{l}_a}{dt} = \vec{r}_a \times \vec{F}_a}$$: Torque

$$\boxed{\vec{l}_a = \vec{r}_a \times \vec{p}_a}$$: Angular Momentum.

$$\frac{dL_{tot}}{dt} = \sum_a \frac{d\vec{l}_a}{dt} = \sum_a \vec{r}_a \times \vec{F}_a$$

$$\hookrightarrow = \sum_a \vec{r}_a \times \left\{ \vec{F}_a^{ext} + \sum_b \vec{F}_{ab} \right\}$$

$$\hookrightarrow = \sum_a \vec{r}_a \times \vec{F}_a^{ext} + \underbrace{\sum_{ab} \vec{r}_a \times \vec{F}_{ab}}$$

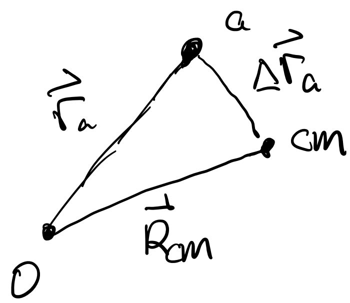
$$= \frac{1}{2}\left(\sum_{ab} \vec{r}_a \times \vec{F}_{ab} + \vec{r}_b \times \underset{\substack{\parallel \\ -\vec{F}_{ab}}}{\vec{F}_{ba}}\right)$$

$$\hookrightarrow = \vec{\tau}_{tot}^{ext} + \frac{1}{2}\sum_{ab}(\vec{r}_a - \vec{r}_b) \times \vec{F}_{ab}$$

but $\vec{F}_{ab} \propto \vec{r}_a - \vec{r}_b$

so $(\vec{r}_a - \vec{r}_b) \times \vec{F}_{ab} = 0$

$$\boxed{\frac{d\vec{L}_{tot}}{dt} = \vec{\tau}^{\,ext}_{tot}}$$ conservation of angular momentum.



$$\vec{r}_a = \vec{R}_{cm} + \Delta\vec{r}_a$$

$$\vec{L}_{O,tot} = \sum_a \vec{r}_a \times \vec{p}_a$$

$$\overset{!}{=} \sum_a (\vec{R}_{cm} + \Delta\vec{r}_a) \times \vec{p}_a$$

$$\overset{!}{=} \sum_a \left(\vec{R}_{cm} \times \vec{p}_a + \Delta\vec{r}_a \times \vec{p}_a\right)$$

$$L_{O,tot} \overset{!}{=} \underbrace{\vec{R}_{cm} \times \vec{P}_{tot}}_{\text{Rotating Angular Momentum (spin)}} + \vec{L}_{cm}$$

$\vec{L}_{cm}$ → Angular momentum relative to cm (orbital)

$\vec{L}_{O,tot}$ = spin + orbital angular momentum.

$$\vec{\tau}_{O,tot} = \frac{d\vec{L}_{O,tot}}{dt} = \frac{d}{dt}\left(\vec{R}_{cm} \times \vec{P}_{tot} + \vec{L}_{cm}\right)$$

$$\overset{!}{=} \frac{d}{dt}\left(R_{cm} \times P_{tot}\right) + \frac{dL_{cm}}{dt}$$

$$\hookrightarrow = \overset{0}{\cancel{\frac{d\vec{R}_{cm}}{dt} \times \vec{P}_{tot}}} + \vec{R}_{cm} \times \frac{d\vec{P}_{tot}}{dt} + \frac{d\vec{L}_{cm}}{dt}$$

$$\overset{!}{=} \vec{R}_{cm} \times \frac{d\vec{P}_{tot}}{dt} + \frac{d\vec{L}_{cm}}{dt}$$

$$\overset{!}{=} \vec{R}_{cm} \times F^{ext}_{tot} + \frac{d\vec{L}_{cm}}{dt}$$

$$\overset{!}{=} \vec{R}_{cm} \times F^{ext}_{tot} + \sum_a \frac{d}{dt}\left(\Delta\vec{r}_a \times \vec{p}_a\right)$$

$$\overset{!}{=} \vec{R}_{cm} \times \vec{F}^{ext}_{tot} + \sum_a \left(\cancel{\frac{d}{dt}\Delta\vec{r}_a \times \vec{p}_a} + \underbrace{\Delta\vec{r}_a \times \overset{F^{ext}_{tot}}{\frac{d}{dt}\vec{p}_a}}_{\tau^{ext}_{cm,tot}} \right)$$

$$\boxed{\vec{\tau}_{o,tot} = \vec{R}_{cm} \times \vec{F}^{ext}_{tot} + \tau^{ext}_{cm,tot}.}$$

Energy:

Single particle:

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = \vec{v} \cdot \vec{F}$$

$$\hookrightarrow \int \vec{v} \cdot \frac{d}{dt} m\vec{v} \, dt = \int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt$$

$$\hookrightarrow \frac{1}{2} m \left(v_f^2 - v_i^2\right) = \int \vec{v} \cdot \vec{F} \, dt$$

$$\overset{!}{=} \int \frac{d\vec{r}}{\cancel{dt}} \cdot \vec{F} \, \cancel{dt}$$

$$\hookrightarrow \quad \frac{1}{2}m\left(v_f^2 - v_i^2\right) = \boxed{\int_{r_i}^{r_f} d\vec{r}\cdot\vec{F} = W}$$

This is always true

If work is independent of path taken, and only dependent on the start and end point, then the force is conservative.

If force is conservative, then force can be describe using a potential field:

$$\boxed{-\nabla_{\vec{r}}\, U = F} \Leftarrow \text{if force is conservative}$$

$$W = \int d\vec{r}\cdot\vec{F} = -\int d\vec{r}\cdot\nabla_{\vec{r}}\, U$$

then $W_{if} = -\int_i^f d\vec{r}\cdot\nabla_{\vec{r}}\, U = -\left(U(r_f) - U(r_i)\right)$

For systems of particles:

center of mass frame

$$E = \sum_a \frac{1}{2} m v_a^2 + U^{ext}(\vec{r}_a) + \frac{1}{2}\sum_{\substack{ab \\ a\neq b}} U^{int}\left(|\vec{r}_a - \vec{r}_b|\right)$$

let $T = \sum_a \frac{1}{2} m_a v_a^2$ (kinetic energy of total system)

$$\vec{v}_a = \vec{v}_{cm} + \Delta\vec{v}_a$$

$$T = \sum_a \frac{1}{2} m_a \left(\vec{v}_{cm} + \Delta\vec{v}_a\right)\cdot\left(\vec{v}_{cm} + \Delta\vec{v}_a\right)$$

$$= \sum_a \frac{1}{2} m_a \left\{ v_{cm}^2 + \Delta v_a^2 + 2\,\vec{v}_{cm} \cdot \Delta\vec{v}_a \right\}$$

$$= \sum_a \frac{1}{2} m_a v_{cm}^2 + \frac{1}{2} m_a \Delta v_a^2 + \underbrace{m_a \vec{v}_{cm} \cdot \frac{d}{dt} \Delta\vec{r}_a}$$

$$\sum_a m_a \Delta\vec{r}_a = 0$$

since we're in CoM, $M_{tot} \vec{R}_{cm} = \sum_a m_a \vec{r}_a = \sum_a m_a (\vec{R}_{cm} + \Delta\vec{r}_a)$

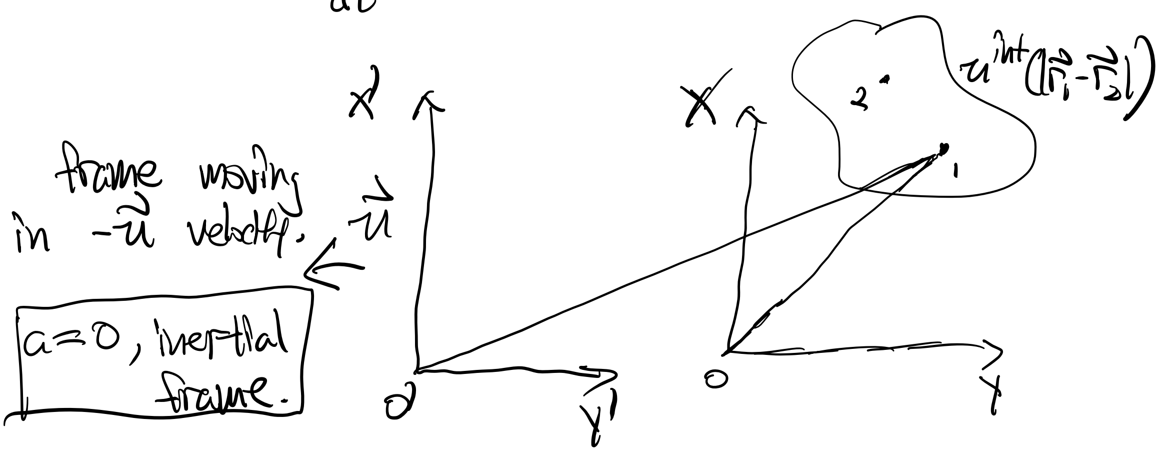
$$M_{tot} R_{cm} = M_{tot} R_{cm} + \sum_a m_a \Delta\vec{r}_a$$

$$0 = \sum_a m_a \Delta\vec{r}_a$$

$$\boxed{T = \sum_a \frac{1}{2} m_a v_{cm}^2 + \frac{1}{2} m_a \Delta v_a^2}$$

Galilean Invariance:

Newton: $\frac{d\vec{p}}{dt} = -\vec{\nabla} U$



$\vec{r}_1' = \vec{r}_1 + \vec{u}t$, $\vec{u}$ is constant and relative to original frame.

Since: $|r_1' - r_2'| = |r_1 - r_2|$

then $u(|r_1' - r_2'|) = u(|r_1 - r_2|)$

we have: $\frac{d\vec{p}}{dt} = -\frac{\partial}{\partial r_1} u(|r_1 - r_2|)$

$$\frac{d\vec{p}'}{dt} = -\frac{\partial}{\partial r_1'} u(|r_1' - r_2'|)$$

$$\vec{p}' = m\frac{d}{dt} r_1' = m\left(\frac{d}{dt} r_1 + \frac{d}{dt}\vec{u}t\right)$$

$$\overset{!}{=} \vec{p} + m\frac{d}{dt}(\vec{u}t)$$

$$\vec{p}' \overset{!}{=} \vec{p} + m\vec{u}$$

$$\frac{d\vec{p}'}{dt} = \frac{d\vec{p}}{dt} + \cancel{\frac{d}{dt}(m\vec{u})} = \frac{d\vec{p}}{dt}.$$

$\hookrightarrow$ $\boxed{\text{invariant under inertial frame}}$