1) b) Given a tensor $\vec{I} = I_{ab}\, \vec{e}_a \otimes \vec{e}_b$ in the rotating frame.
and $\underline{I} = \underline{I}_{ab}\, \underline{\vec{e}}_a \otimes \underline{\vec{e}}_b$ , for $\vec{e}_a = R_{ab}\, \underline{\vec{e}}_b$

Show: $I_{ab} = R_{ac} R_{bd}\, \underline{I}_{cd}$

$$I_{ab}\, \vec{e}_a\, \vec{e}_b = \underline{I}_{cd}\, \underline{\vec{e}}_c\, \underline{\vec{e}}_d$$

$$I_{ab}\, R_{ac}\, \underline{\vec{e}}_c\, R_{bd}\, \underline{\vec{e}}_d = \underline{I}_{cd}\, \underline{\vec{e}}_c\, \underline{\vec{e}}_d$$

$$I_{ab}\, R_{ac}\, R_{bd} = \underline{I}_{cd}$$

note $R^{-1} = R^{T}$

$$I_{ab}\, R_{ac} R_{bd} (R^{-1})_{db} (R^{-1})_{ca} = \underline{I}_{cd} (R^{-1})_{db} (R)^{-1}_{ca}$$

$$I_{ab} = R_{ac} R_{bd}\, \underline{I}_{cd}$$

c) Show: $\vec{\omega} \times \vec{v} = \hat{v} \cdot \vec{\omega} = \vec{v} \cdot \hat{\omega}$

$$\hat{v} = \hat{v}_{ab}\, \vec{e}_a \vec{e}_b = \epsilon_{abc}\, v^c\, \vec{e}_a \vec{e}_b$$

$$\vec{w} \times \vec{v} = \epsilon_{abc}\, w^b v^c\, \vec{e}_a = \underbrace{\hat{v}_{ab}\, \vec{e}_a \vec{e}_b}_{\hat{v}}\ \underbrace{w^b \vec{e}_b}_{\vec{w}}$$

$$\overset{!}{=} \epsilon_{cab}\, w^b v^c\, \vec{e}_a = \hat{w}_{ca}\, v^c\, \vec{e}_a\, \vec{e}_c\, \vec{e}_c$$

$$\overset{!}{=} \underbrace{\hat{w}_{ca}\, \vec{e}_a \vec{e}_c}_{\hat{w}} \cdot \underbrace{v^c \vec{e}_c}_{\vec{v}}$$

d) Show $\underline{\vec{w}}_{ac} = (R^{-1} \dot{R})_{ac}$

$$\underline{d\vec{r}} - \cancel{(\underline{dr_a})}\ \vec{\ } \quad + \quad r\ \underline{d\vec{e}_a}$$

$$dt \quad -\left(\cancel{d}t\right)\Gamma \quad e_a \qquad dt$$

$$\stackrel{!}{=} \Gamma^a \frac{d}{dt}\left\{R_{ab}\vec{e}_b\right\} \qquad \vec{e}_a = R_{ab}\vec{e}_b$$

$$\stackrel{!}{=} \Gamma^a \dot{R}_{ab}\vec{e}_b \qquad \vec{e}_c = R_{cb}\vec{e}_b$$

$$\stackrel{!}{=} \Gamma^a \underbrace{\dot{R}_{ab}(R^{-1})_{bc}}_{\omega_{ac}}\vec{e}_c \qquad \vec{e}_b = (R^{-1})_{bc}\vec{e}_c$$

From part b:

$$I_{ab}R_{ac}R_{bd} = I_{cd}$$

Know: $\omega_{bd} = (\dot{R}R^{-1})_{bd}$

$$\underline{\omega}_{ac} = \omega_{bd}R_{ba}R_{dc} = (R^T)_{ab}\,\omega_{bd}\,R_{dc}$$

$$\stackrel{!}{=} (R^T)_{ab}\dot{R}_{bc}R^{-1}_{cd}R_{dc}$$

$$\stackrel{!}{=} (R^T_{ab})\dot{R}_{bc}\delta_{cc}$$

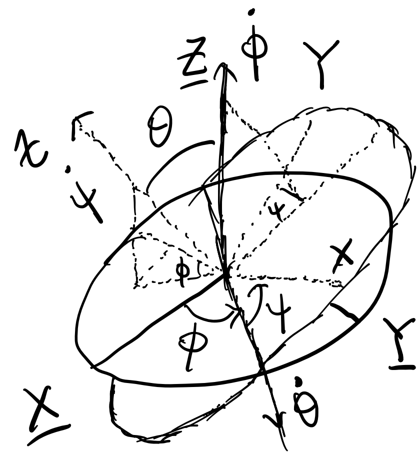
$$\stackrel{!}{=} (R^{-1})_{ab}\dot{R}_{bc}$$

$$\stackrel{!}{=} (R^{-1}\dot{R})_{ac}$$

e) Determine the projection of $\vec{\omega}$ on lab frame axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

$\omega_z = \dot\phi\cos\theta + \dot\psi$



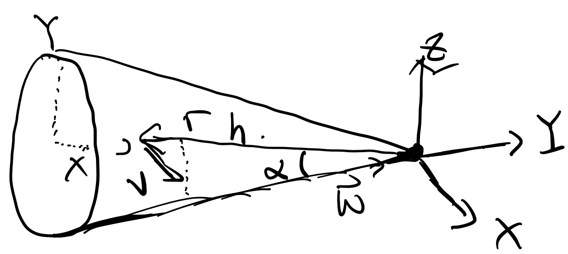

Know in rotated frame:

$$\omega_x = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$

$$\omega_y = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$

$$\omega_z = \dot\phi\cos\theta + \dot\psi$$

## Problem 2: A rolling Cone:




a uniform right circular cone of height, $h$, half angle, $\alpha$, and density, $\rho$ rolls on its side without slipping. It returns to original position in time, $\tau$.

a) Find the moment of inertia tensor for the body or principal axes centered at tip.

$$I^{0}_{ab} = \int dm \left( r^2 \delta_{ab} - r_a r_b \right)$$

$$= \int d^3r\, \rho(\vec{r}) \left( r^2 \delta_{ab} - r_a r_b \right)$$

$$
= \int d^3r\, \rho \begin{pmatrix} y^2+z^2 & -xy & \cdots \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}
$$

$$
V = \pi R^2 h = \pi (\tan\alpha\, h)^2 h
$$

$$
\stackrel{!}{=} \frac{\pi r^2 h}{3}
$$

$$
\stackrel{!}{=} \int d^3r \frac{3M}{\pi r^2 h} \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}
$$

$$
\stackrel{!}{=} \int d^3r \frac{3M}{\pi \tan^2\alpha\, h^3} \begin{pmatrix} r^2\sin^2\theta + z^2 & & \\ & r^2\cos^2\theta + z^2 & \\ & & r^2 \end{pmatrix}
$$

$$
\stackrel{!}{=} \int_0^h \int_0^{\tan\alpha z} \int_0^{2\pi} \frac{3M}{\pi \tan^2\alpha\, h^3}\, r\,dr\,d\theta\,dz \begin{pmatrix} r^2\sin^2\theta + z^2 & & \\ & r^2\cos^2\theta + z^2 & \\ & & r^2 \end{pmatrix}
$$

$$
\stackrel{!}{=} \int_0^h \int_0^{\tan\alpha z} \frac{3M}{\pi \tan^2\alpha\, h^3}\, r\,dr\,dz \begin{pmatrix} \pi r^2 + 2\pi z^2 & & \\ & \pi r^2 + 2\pi z^2 & \\ & & 2\pi r^2 \end{pmatrix}
$$

$$
\stackrel{!}{=} \int_0^h \frac{3M}{\pi \tan^2\alpha\, h^3} \begin{pmatrix} \frac{\pi}{4}\tan^4\alpha\, z^4 + \frac{2\pi z^2}{2}\tan^2\alpha z^2 & & \\ & \frac{\pi}{4}\tan^4\alpha\, z^4 + \frac{2\pi z^2}{2}\tan^2\alpha z^2 & \\ & & \frac{(\tan\alpha z)^4}{24} \end{pmatrix}
$$

$$
\stackrel{!}{=} \frac{3M}{\pi \tan^2\alpha\, h^3} \begin{pmatrix} \frac{\pi}{20}\tan^4\alpha\, h^5 + \frac{2\pi \tan^2\alpha\, h^5}{10} & & \\ & \frac{\pi}{20}\tan^4\alpha\, h^5 + \frac{2\pi \tan^2\alpha\, h^5}{10} & \\ & & \end{pmatrix}
$$

$\frac{2\pi \tan^4\alpha \, h^5}{20}$ /

$$\boxed{I^0_{ab} \doteq \frac{3}{5}Mh^2 \begin{pmatrix} \frac{1}{4}\tan^2\alpha + 1 & & \\ & \frac{1}{4}\tan^2\alpha + 1 & \\ & & \frac{1}{2}\tan^2\alpha \end{pmatrix}}$$

b) Cone is turning around $\underline{z}$-axis in counterclockwise.

Consider displacement:

$$\vec{r} \to \vec{r} + \vec{\delta\theta} \times \vec{r}$$

$\vec{\delta\theta}$ points in $\underline{Y}$ axis

$$\delta\theta \, \hat{\underline{y}} \times (y\hat{\underline{y}} + z\hat{\underline{z}}) = \delta\theta \, z \, \hat{\underline{x}}$$ points in $\hat{\underline{x}}$ direction.

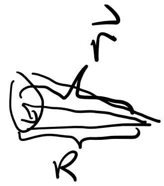
$$\vec{\underline{w}} = \underline{w}_0 \left(-\sin\left(\frac{2\pi}{T}t\right)\hat{\underline{x}} + \cos\left(\frac{2\pi}{T}t\right)\hat{\underline{y}} + 0\,\hat{\underline{z}}\right)$$

$\underline{w_z} = 0$ otherwise:

$$\delta\theta \, \hat{\underline{z}} \times (y\hat{\underline{y}} + z\hat{\underline{z}}) = -\delta\theta \, y \, \hat{x}$$ ← a component in clockwise rotation.

Find $w_0$:

know $T = \tau$ = take it takes for rotation.



actual length sweep over.

since $\vec{w} \times \vec{r} = \vec{v} = w_0 h \sin\alpha$

$R_1 = h\cos\alpha$

$v = \frac{2\pi R_1}{T} = w_0 h \sin\alpha$

$\doteq 2\pi h \cos\alpha \qquad w_0 h \sin\alpha = 2\pi h \cos\alpha$

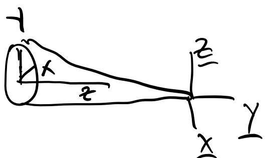
$$\frac{\quad}{T} = \omega_0 \sin\alpha - \frac{\quad}{T}\cos\alpha$$

$$\hookrightarrow \omega_0 = \frac{2\pi}{T}\cot\alpha$$

$$\boxed{\underline{\underline{\omega}} = \frac{2\pi}{T}\cot\alpha \left\{-\sin\left(\frac{2\pi}{T}t\right)\underline{\hat{x}} + \cos\left(\frac{2\pi}{T}t\right)\underline{\hat{y}} + 0\,\underline{\hat{z}}\right)}$$

c) Euler Angle in rotating frame:

$$\omega_x = \dot\phi \sin\theta \sin\psi + \dot\theta\cos\psi$$




$$\omega_y = \dot\phi \sin\theta\cos\psi - \dot\theta \sin\psi$$

$$\omega_z = \dot\phi\cos\theta + \dot\psi$$

$$\omega_{\underline{x}} = \dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi = \frac{2\pi}{T}\cot\alpha\left(-\sin\left(\frac{2\pi}{T}t\right)\right)$$
$$\omega_{\underline{y}} = \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi = \frac{2\pi}{T}\cot\alpha\cos\left(\frac{2\pi}{T}t\right)$$
$$\omega_{\underline{z}} = \dot\psi\cos\theta + \dot\phi \quad = 0$$

know $\dot\theta = 0$

$$\dot\psi\sin\theta\sin\phi = -\frac{2\pi}{T}\cot\alpha\sin\left(\frac{2\pi}{T}t\right)$$

$$-\dot\psi\sin\theta\cos\phi = \frac{2\pi}{T}\cot\alpha\cos\left(\frac{2\pi}{T}t\right)$$

then $\dot\psi\sin\theta = -\frac{2\pi}{T}\cot\alpha$

$\theta = \frac{\pi}{2} - \alpha$

then $\dot\psi = \frac{-2\pi}{T}\frac{\cos\alpha}{\sin\alpha}\frac{1}{\cos\alpha} = \frac{-2\pi}{T}\csc\alpha = -\dot\phi\frac{1}{\sin\alpha}$

$\dot\phi = \ldots$

$$\dot\theta = 0$$

$$\dot\psi \cos\theta + \dot\phi = 0$$

$$-\frac{2\pi}{\tau}\frac{1}{\sin\alpha}\sin\alpha = -\dot\phi$$

$$\dot\phi = \frac{2\pi}{\tau}$$

$$\boxed{\begin{array}{lll} \dot\psi = -\frac{2\pi}{\tau}\frac{1}{\sin\alpha} & \Rightarrow \psi = -\frac{2\pi}{\tau}\frac{1}{\sin\alpha} t & \\ \dot\theta = 0 & \theta = \frac{\pi}{2} - \alpha & \text{Euler Angles} \\ \dot\phi = \frac{2\pi}{\tau} & \phi = \frac{2\pi}{\tau} t & \psi, \theta, \phi \end{array}}$$

d) Find Kinetic Energy?

$$T = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \omega_a I_{ab}^{cm} \omega_b$$

$$\stackrel{!}{=} \frac{1}{2} \omega_a I_{ab}^{0} \omega_b$$

we know moment of inertia at rotating frame:

$$I_{ab}^{0} \stackrel{!}{=} \frac{3}{5} M h^2 \begin{pmatrix} \frac{1}{4}\tan^2\alpha + 1 & & \\ & \frac{1}{4}\tan^2\alpha + 1 & \\ & & \frac{1}{2}\tan^2\alpha \end{pmatrix}$$

Need $\vec\omega$ at rotating frame:

$$\omega_x = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi = \frac{2\pi}{\tau}\cos\alpha \sin\left(\frac{2\pi}{\tau} t\right)$$

$$\omega_y = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi = \frac{2\pi}{\tau}\cos\alpha \cos\left(\frac{2\pi}{\tau} t\right)$$

$$\omega_z = \dot\phi\cos\theta + \dot\psi \qquad = \frac{2\pi}{T}\sin\alpha + \frac{2\pi}{T}\frac{1}{\sin\alpha}$$

at $t=0$

$$\omega = \left(0\,\hat{x} + \frac{2\pi}{T}\cos\alpha\,\hat{y} + \frac{2\pi}{T}\left(\sin\alpha - \frac{1}{\sin\alpha}\right)\hat{z}\right)$$

$$\stackrel{!}{=} \frac{2\pi}{T}\cot\alpha\sin\alpha\,\hat{y} + \frac{2\pi}{T}\cot\alpha\left(\frac{\sin^2\alpha}{\cos\alpha} - \frac{1}{\cos\alpha}\right)\hat{z}$$

$$= \frac{2\pi}{T}\cot\alpha\sin\alpha\,\hat{y} - \frac{2\pi}{T}\cot\alpha\cos\alpha\,\hat{z}\,.$$

$$T = \frac{1}{2}\omega_y I_{yy}\omega_y + \frac{1}{2}\omega_z I_{zz}\omega_z$$

$$\stackrel{!}{=} \frac{1}{2}\left(\frac{2\pi}{T}\cot\alpha\sin\alpha\right)^2\frac{3}{5}Mh^2\left(\frac{1}{4}\tan^2\alpha + 1\right)$$

$$+\frac{1}{2}\left(\frac{2\pi}{T}\right)^2\left(\cot\alpha\cos\alpha\right)^2\frac{3}{5}Mh^2\frac{1}{2}\tan^2\alpha$$

$$\stackrel{!}{=} Mh^2\left(\frac{2\pi}{T}\right)^2\frac{3}{10}\left\{\frac{\sin^2\alpha}{4} + \cos^2\alpha + \frac{\cos^2\alpha}{2}\right\}$$

$$\stackrel{!}{=} Mh^2\left(\frac{2\pi}{T}\right)^2\frac{3}{40}\left\{\sin^2\alpha + 4\cos^2\alpha + 2\cos^2\alpha\right\}$$

$$T = Mh^2\left(\frac{2\pi}{T}\right)^2\frac{3}{40}\left\{\underbrace{\sin^2\alpha + \cos^2\alpha}_{=1} + 5\cos^2\alpha\right\}$$

e) $L(t)$ in Fixed frame:

$L_a = I_{ab}\,\omega_b$ < In Rotated Frame. then apply $R$

$$L_x = I_{xx}\,\omega_x$$

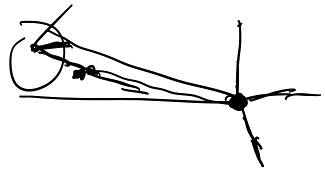
$$\stackrel{!}{=} \frac{3}{5}Mh^2\left(\frac{1}{4}\tan^2\alpha + 1\right)\frac{2\pi}{T}\cos\alpha\sin\left(\frac{2\pi}{T}t\right)$$

$$L_y = I_{yy}\omega_y = \frac{3}{5}Mh^2\left(\frac{1}{4}\tan^2\alpha+1\right)\frac{2\pi}{T}\cos\alpha\cos\left(\frac{2\pi}{T}t\right)$$

$$L_z = I_{zz}\omega_z = -\frac{3}{5}Mh^2\frac{1}{2}\tan^2\alpha\ \frac{2\pi}{T}\cot\alpha\cos\alpha$$

f) Find $I_{cm}$:

Use parallel axis Theorem:



$$I^{0} = I^{cm} + M(d^2\delta_{ab} - d_a d_b)$$

or $$I^{cm} = I^{0} - M(d^2\delta_{ab} - d_a d_b)$$

$$\vec{R}_{cm} = \frac{\int dm\,\vec{r}}{\int dm} = \int \frac{3M}{\pi\tan^2\alpha\, h^3}\left(r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y} + z\,\hat{z}\right) r\,dr\,d\theta\,dz$$

$$\doteq \int_0^h \frac{\frac{3\cancel{M}}{\cancel{\pi}\cancel{\tan^2\alpha}h^3}}{\cancel{M}}\frac{(\cancel{\tan\alpha}\,z)^2}{\cancel{2}}\cancel{2\pi}\,z\,dz$$

$$\doteq \int_0^h \frac{3}{h^3}z^3\,dz$$

$$\vec{R}_{cm} \doteq \frac{3}{4}h\,\hat{z} + 0\,\hat{x} + 0\,\hat{y}$$

$$I_{cm} = I^{0} - M\begin{pmatrix}\left(\frac{3}{4}h\right)^2 - \cancel{d_x^2} & & \\ & \left(\frac{3}{4}h\right)^2 - \cancel{(d_y)^2} & \\ & & \cancel{\left(\frac{3}{4}h\right)^2 - (d_z)^2} = 0\end{pmatrix}$$

$$I_{cm} \doteq \frac{3}{5}Mh^2\begin{pmatrix}\frac{1}{4}\tan^2\alpha + \frac{1}{16} & & \\ & \frac{1}{16} + \frac{1}{4}\tan^2\alpha & \\ & & \frac{1}{2}\tan^2\alpha\end{pmatrix}$$

$$T = \frac{1}{2} M v_{cm}^2 + \frac{1}{2}\omega_a I^{cm}_{ab}\,\omega_b$$

$$\doteq \frac{1}{2} M\left(\cancel{\dot r^2} + (r\dot\theta)^2 + \cancel{\dot z^2}\right) + \frac{1}{2}\omega_y^2 I^{cm}_{yy} + \frac{1}{2}\omega_z^2 I^{cm}_{zz}$$

$$= \frac{1}{2} M\left(\dot x_{cm}^2 + \dot y_{cm}^2 + \dot z_{cm}^2\right)$$

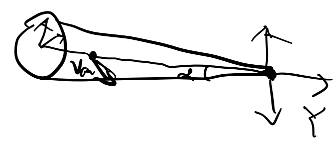
$$+ \frac{1}{2}\left(\frac{2\pi}{T}\cot\alpha\sin\alpha\right)^2 \frac{3}{5} M h^2 \left(\frac{1}{4}\tan^2\alpha + \frac{1}{16}\right)$$

$$+ \frac{1}{2}\left(\frac{2\pi}{T}\right)^2 \left(\cot\alpha\cos\alpha\right)^2 \frac{3}{5} M h^2 \frac{1}{2}\tan^2\alpha$$

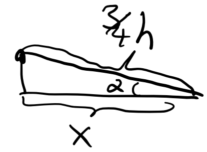
$$\doteq \frac{1}{2} M\left(\dot x_{cm}^2 + \cancel{\dot y_{cm}^2} + \cancel{\dot z_{cm}^2}\right)$$

$$+ M h^2 \left(\frac{2\pi}{T}\right)^2 \left\{\frac{3}{20}\cos^2\alpha + \left(\frac{3}{40}\sin^2\alpha + \frac{3}{160}\cos^2\alpha\right)\right\}$$

$$= \frac{1}{2} M\left(\dot r_{cm}^2 + r_c^2\dot\varphi^2 + \dot z_{cm}^2\right)$$



$$+ M h^2 \left(\frac{2\pi}{T}\right)^2 \frac{3}{40}\Big(\underbrace{\sin^2\alpha + \overbrace{\frac{1}{4}\cos^2\alpha + 2\cos^2\alpha}^{4\cos^2\alpha}}_{1+\frac{5}{4}\cos^2\alpha}\Big)$$




$$\doteq \frac{1}{2} M\left(\frac{3}{4} h\cos\alpha\,\frac{2\pi}{T}\right)^2 \qquad \frac{1}{2}\left(\frac{3}{4}h\right)^2 M\left(\frac{2\pi}{T}\cos\alpha\right)^2$$

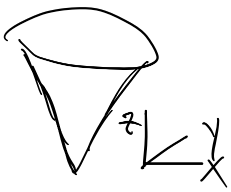
$$+ M h^2 \left(\frac{2\pi}{T}\right)^2 \frac{3}{40}\left(1 + \frac{5}{4}\cos^2\alpha\right)$$

$x_{cm} = \frac{3}{4} h\cos\alpha\sin\left(\frac{2\pi}{T}t\right)$ in fixed frame.

$$\doteq M h^2 \left(\frac{2\pi}{T}\right)^2 \frac{3}{40}\left(1 + 5\cos^2\alpha\right)$$

$\dot x_{cm} = \frac{3}{4}\cos\alpha\,\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}t\right)$

<u>Problem 3: Nutation of a Heavy Symmetric Top:</u>




$$\mathcal{L} \stackrel{!}{=} \frac{1}{2} I_1 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \underbrace{\frac{1}{2} I_3 \left( \dot{\phi} \cos\theta + \dot{\psi} \right)^2}_{\frac{P_\psi^2}{2I_3}} - mgl\cos\theta$$

$l$ : distance to center of mass

$$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_1 \dot{\theta}$$

$$P_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3 \left( \dot{\phi}\cos\theta + \dot{\psi} \right) = \text{const} \Rightarrow \boxed{\dot{\psi} = \frac{P_\psi}{I_3} - \dot{\phi}\cos\theta}$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2\theta + I_3 \left( \dot{\phi}\cos\theta + \dot{\psi} \right) \cos\theta = \text{const.}$$

$$\stackrel{!}{=} I_1 \dot{\phi} \sin^2\theta + P_\psi \cos\theta \Rightarrow \boxed{\dot{\phi} = \frac{P_\phi - P_\psi \cos\theta}{I_1 \sin^2\theta}}$$

$$R = P_\phi \dot{\phi} + P_\psi \dot{\psi} - \mathcal{L}$$

$$\stackrel{!}{=} P_\phi \left( \frac{P_\phi - P_\psi\cos\theta}{I_1 \sin^2\theta} \right) + P_\psi \left\{ \frac{P_\psi}{I_3} - \cancel{\left( \frac{P_\phi - P_\psi\cos\theta}{I_1 \sin^2\theta} \right) \cos\theta} \right\}$$

$$- \left( \frac{1}{2} I_1 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \underbrace{\cancel{\frac{1}{2} I_3 \left( \dot{\phi}\cos\theta + \dot{\psi} \right)^2}}_{\frac{P_\psi^2}{2I_3}} - mgl\cos\theta \right)$$

$$\stackrel{!}{=} \frac{(P_\phi - P_\psi\cos\theta)^2}{\ldots} + \frac{P_\psi^2}{\ldots} - \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} I \sin^2\theta \left( \frac{P_\phi - P_\psi\cos\theta}{\ldots} \right)^2$$

$$\frac{\cdots}{I_1\sin^2\theta} + \frac{\cdots}{2I_3} - \frac{\cdots}{2}I_1\theta - 2I_1\sin^2\theta\left(\frac{\cdots}{I_1\sin^2\theta}\right)$$

$$+mgl\cos\theta$$

$$R = -\frac{1}{2}I_1\dot{\theta}^2 + \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{P_\psi^2}{2I_3} + mgl\cos\theta$$

$$L_{eff} = -R = \frac{1}{2}I_1\dot{\theta}^2 - \left\{\underbrace{\frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \overset{\text{const.}}{\cancel{\frac{P_\psi^2}{2I_3}}} + mgl\cos\theta}_{U_{eff}(\theta)}\right\}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta}$$

$$I_1\ddot{\theta} = -\frac{\partial}{\partial\theta}U_{eff}(\theta)$$

$$\boxed{U_{eff}(\theta) = \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta}$$

b) Determine the minimum of the effective potential, or when $\dot{\theta}=0$ or $\theta$ is constant.

$$\boxed{U_{eff}(\theta) = \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta}$$

We have: $I_1\ddot{\theta} = -\frac{\partial}{\partial\theta}U_{eff}(\theta)$

① Do change of variables:

let $r = P_\phi / P_\psi$ $\quad \bar{g} = mgl \frac{I_1}{P_\psi^2}$

$$\overline{U_{eff}}(\theta) = \frac{U_{eff}}{\left(P_\psi^2/I_1\right)} = \frac{(r-\cos\theta)^2}{2\sin^2\theta} + \bar{g}\cos\theta$$

$\Rightarrow$ Here we note $\bar{g} \ll 1$, so $\bar{g}\cos\theta$ is small to first order.

Now find $\frac{\partial}{\partial\theta}\overline{U_{eff}} = 0$

$$\frac{\partial}{\partial\theta}\overline{U_{eff}} = \frac{2(r-\cos\theta)\sin\theta \, 2\sin^2\theta - 4\sin\theta\cos\theta(r-\cos\theta)^2}{4\sin^4\theta} - \bar{g}\sin\theta$$

$$0 \overset{!}{=} \frac{(r-\cos\theta)\sin^2\theta - \cos\theta(r-\cos\theta)^2}{\sin^3\theta} - \bar{g}\sin\theta.$$

Since $\bar{g} \ll 1$, $\bar{g}\sin\theta$ is first order.

First consider zeroth order, so neglect $\bar{g}\sin\theta$ since its first order.

$$0 = \frac{(r-\cos\theta)\sin^2\theta - \cos\theta(r-\cos\theta)^2}{\sin^3\theta}$$

i.e. $$r-\cos\theta = \frac{-\sin^2\theta \pm \sqrt{\sin^4\theta}}{-2\cos\theta}$$

$$\;\;\;\;\;\;\;\;= 0 \quad \text{or} \quad \frac{\sin^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta}$$

$$\underline{\underline{r = \cos\theta}}$$

or $$r-\cos\theta = \frac{1}{\cos\theta} - \cos\theta$$

$$r = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{r}$$

Use zero order solution $\cos\theta = r = \frac{P_\phi}{P_\psi}$

to calculate $\dot{\phi}$ : precession rate.

$$\dot{\phi} = \frac{P_\phi - P_\psi\cos\theta}{I_1\sin^2\theta} \Rightarrow \frac{P_\phi - P_\psi\left(\frac{P_\phi}{P_\psi}\right)}{I_1\sin^2\theta} = 0$$

Since we don't want $\dot{\phi} = 0$, expand to first order.

Now we want to include first order terms, we include gravity.

$$\overline{U}_{eff} = \frac{(r-\cos\theta)^2}{2\sin^2\theta} + \bar{g}\cos\theta$$

$2\sin\theta$

Now we suppose $\cos\theta = \underset{\substack{\uparrow \\ \text{zeroth order} \\ \text{solution}}}{r} + \underset{\substack{\uparrow \\ \text{first order} \\ \text{solution.}}}{\delta}$

$$U_{eff} = \frac{\{r-(r+\delta)\}^2}{2(1-(r+\delta)^2)} + \bar{g}(r+\delta)$$

$$\doteq \frac{\delta^2}{2(1-r^2)} + \bar{g}(r+\delta)$$

then

$$\frac{\partial U_{eff}}{\partial\theta} \Rightarrow \frac{\partial U_{eff}}{\partial\delta} = \frac{\delta}{1-r^2} + \bar{g} = 0$$

$$\delta = -\bar{g}\,(1-r^2)$$

$$\frac{\partial}{\partial\delta}\underbrace{\frac{d\delta}{d\cos\theta}}_{=1}\underbrace{\frac{d\cos\theta}{d\theta}}_{-\sin\theta}$$

$$-\sqrt{1-\cos^2\theta}$$

$$-\sqrt{1-r^2}$$

then $\cos\theta = r - \bar{g}(1-r^2)$

plug into $\dot{\phi}$

$$\dot{\phi} = \frac{P_\phi - P_\psi\left(r-\bar{g}(1-r^2)\right)}{I_1(1-r^2)} =$$

$1 \quad \bar{g} \quad mgl$

$$\dot{\varphi} \doteq J/I_1 = \frac{J}{P_4}$$

c) Now compute the period of $\theta$ oscillation, for a given E with E just larger than min of $U_{eff}$. Determine $\dot{\phi}(t)$.

Now we still consider $\cos\theta = \underset{\substack{\uparrow \\ \text{zeroth order} \\ \text{solution}}}{r} + \underset{\substack{\uparrow \\ \text{first order} \\ \text{solution.}}}{\delta}$

Now since $E > U_{eff\,min}$, $\frac{\partial}{\partial\theta} U_{eff} \neq 0$

$$I_1 \ddot{\theta} = -\frac{\partial}{\partial\theta} U_{eff}(\theta)$$

still consider $u = \cos\theta = r + \delta$

change $L$ in $\delta$:

$$L_{eff} = -R = \frac{1}{2} I_1 \dot{\theta}^2 - \Big\{ \underbrace{\frac{(P_\phi - P_4\cos\theta)^2}{2I_1 \sin^2\theta} + \cancel{\frac{P_4^2}{2I_3}}^{\text{const.}} + mgl\cos\theta}_{U_{eff}(\theta)} \Big\}$$

$$\dot{u} = -\sin\theta\,\dot{\theta} = -\sqrt{1-u^2}\,\dot{\theta}$$

then $\dot{\theta} = \dfrac{-\dot{u}}{\sqrt{1-u^2}} \simeq \dfrac{-\dot{u}}{\sqrt{1-r^2}}$

$$L_{eff} = \frac{1}{2} I_1 \frac{\dot{u}^2}{1-r^2} - \left\{ \frac{(P_\phi - P_4 u)^2}{2I_1(1-r^2)} + mgl\,u \right\}$$

$$\frac{d}{dt}\frac{\partial L_{eff}}{\partial \dot{u}} = \frac{d}{dt}\left( \frac{I_1}{1-r^2}\dot{u} \right)$$

$$\frac{\partial L_{eff}}{\partial u} = -\left( mgl + \frac{2(P_\phi - P_4 u)(-P_4)}{2I_1(1-r^2)} \right)$$

$$\frac{I_1}{1-r^2}\ddot{u} = -mgl + \frac{(P_\phi - P_4 u)P_4}{I_1(1-r^2)}$$

$$\frac{I_1}{1-r^2}\ddot{\delta} = -mgl + \frac{P_\phi P_4}{I_1(1-r^2)} - \frac{P_4^2(r+\delta)}{I_1(1-r^2)}$$

$$\stackrel{!}{=} -mgl + \frac{P_\phi P_4}{I_1(1-r^2)} - \frac{P_4^2 \frac{P_\phi}{P_4}}{I_1(1-r^2)} - \frac{P_4^2\delta}{I_1(1-r^2)}$$

$$\frac{I_1}{1-r^2}\ddot{\delta} = -mgl - \frac{P_4^2\delta}{I_1(1-r^2)}$$

$$\ddot{\delta} = -\frac{mgl(1-r^2)}{I_1} - \frac{P_4^2}{I_1^2}\delta$$

$$\frac{I_1^2}{P_4^2}\ddot{\delta} = -\frac{mgl}{P_4^2/I_1}(1-r^2) - \delta$$

$$\ddot{\delta} \overset{!}{=} -\left(\frac{P_\psi}{I_1}\right)^2\left(\bar{g}(1-r^2)+\delta\right)$$

let $\delta = Ae^{\pm i\omega t}$ + constant.

$$\dot{\delta} = \pm i\omega Ae^{\pm i\omega t}$$

$$\ddot{\delta} = -\omega^2 Ae^{\pm i\omega t} = -\omega^2(\delta - \text{constant}).$$

$$\ddot{\delta} = -\omega_0^2\left(\delta+\bar{g}(1-r^2)\right)$$

$$\delta - \left(-\bar{g}(1-r^2)\right)$$

so $\delta = A\cos\omega_0 t - \bar{g}(1-r^2)$

or $u = \underbrace{r-\bar{g}(1-r^2)}_{u_{min}} + \underbrace{A\cos\omega_0 t}_{\text{oscillation term}}$

then $$\dot{\phi} = \frac{P_\phi - P_\psi u}{I_1\sin^2\theta}$$

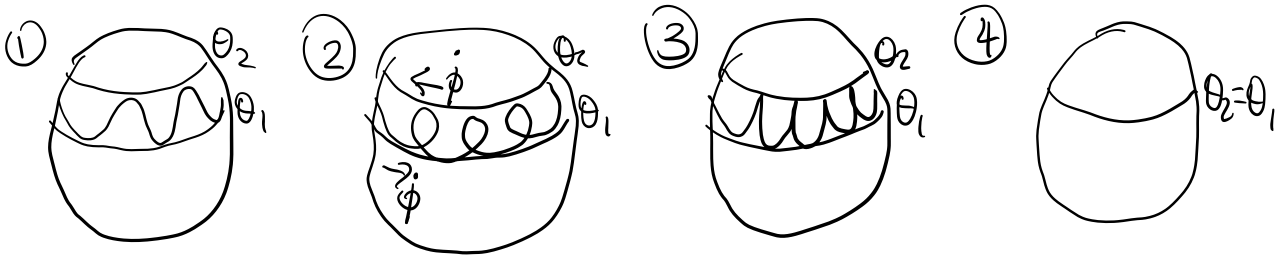
$$\overset{!}{=} \frac{\cancel{P_\phi} - P_\psi\left(\cancel{r} - \bar{g}(1-r^2) + A\cos\omega_0 t\right)}{I_1\sin^2\theta}$$

$$\overset{!}{=} \frac{P_\psi \bar{g}(1-r^2)}{I_1(1-r^2)} + \frac{-P_\psi A\cos\omega_0 t}{I_1(1-r^2)}$$

$$\overset{!}{=} \frac{P_\psi \frac{mgl}{P_\psi^2/I_1}(1-r^2)}{I_1(1-r^2)} + \frac{-P_\psi A\cos\omega_0 t}{I_1(1-r^2)}$$

$$\doteq \frac{mgl}{P_\psi} - \frac{P_\psi}{I_1}\frac{1}{1-r^2} A\cos\omega_0 t$$

e) Describe and determine Energy range for different nutation.




For ②, it experience change of signs for precession, $\dot\phi$

we know:

$$\dot\phi = \frac{P_\phi - P_\psi\cos\theta}{I_1\sin^2\theta} = \frac{P_\psi}{I_1\sin^2\theta}\left(r-\cos\theta\right)$$

so $\dot\phi$ is positive when $r > \cos\theta$

and negative when $r < \cos\theta$

we know:

$$\overline{U_{eff}}(\theta) = \frac{U_{eff}}{\left(P_\psi^2/I_1\right)} = \frac{(r-\cos\theta)^2}{2\sin^2\theta} + \bar{j}\cos\theta$$

From part c:

Since Energy is high, we can neglect first order terms

then $\underline{\cos\theta = r}$

and plug int $U_{eff}$:

Energy when $r = \cos\theta$ or $\dot\phi = 0$

$$\frac{U_{eff}}{\frac{P_\psi^2}{I_1}} = \left\{ \frac{\cancel{(r-r)^2}^{\,0}}{\cancel{2\sin^2\theta}} + \bar{g} r \right\} = E_{crit}$$

so If $E = \bar{g}\cos\theta > E_{crit} = \bar{g} r$

$\cos\theta > r = \frac{P_\phi}{P_\psi}$

we have negative precession.

If $E = \bar{g}\cos\theta < E_{crit} = \bar{g} r$

$\cos\theta < r$

we have positive precession

$\Rightarrow$ Therefore, in order to get negative precession,

we need $E > \bar{g} r = E_{crit}$. for figure ②

$\Rightarrow$ For Figure 3, we need $E = \bar{g} r = E_{crit}$ to have $\dot\phi = 0$

at $\cos(\theta_2)$ which is $U_{eff}(\theta_2)$ when everything in potential

$\Rightarrow$ For Figure 1, we need $E < \bar{g} r$, but still need a lower bound, which is closer to $U_{min}$, so we need to include gravity, which we need to include first order terms

$$\cos\theta_{min} = r - \bar{g}(1-r^2)$$

$$\frac{U_{eff}}{\frac{P_\psi^2}{I_1}} = \left\{ \frac{\left(r - r + \bar{g}(1-r^2)\right)^2}{2\sin^2\theta} + \bar{g}\left(r - \bar{g}(1-r^2)\right) \right\}$$

$$\doteq \frac{\left[\bar{g}(1-r^2)\right]^2}{2(1-r^2)} + \bar{g}r - \bar{g}^2(1-r^2)$$

$$\doteq \frac{\bar{g}(1-r^2)}{2} + \bar{g}r - \bar{g}^2(1-r^2)$$

$$\left(\frac{U_{eff}}{\frac{P_\psi^2}{I_1}}\right)_{min} \doteq \bar{g}r - \frac{1}{2}\bar{g}^2(1-r^2)$$

To get figure (1), we need

$$\bar{g}r < E < \left(\frac{U_{eff}}{\frac{P_\psi^2}{I_1}}\right)_{min}$$

$$\boxed{\tilde{g} r < E < g r - \frac{1}{2}\tilde{g}^2(1-r^2)}$$