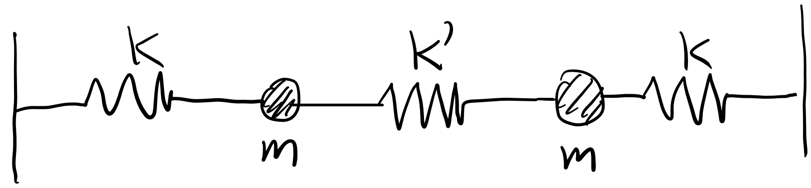
1) Oscillations with similar frequencies:



a) If at time $t=0$, left particle is displaced by an initial position x_0, and the right particle at rest, determine the subsequent oscillations.

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \left\{\frac{1}{2}kx_1^2 + \frac{1}{2}k'(x_1 - x_2)^2 + \frac{1}{2}kx_2^2\right\}$$

$$\stackrel{!}{=} \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \left\{\frac{1}{2}kx_1^2 + \frac{1}{2}k'(x_1^2 - 2x_1x_2 + x_2^2) + \frac{1}{2}kx_2^2\right\}$$

$$m\ddot{x}_1 = -kx_1 - k'x_1 - k'x_2$$

$$\stackrel{!}{=} -(k+k')x_1 - k'x_2$$

$$\ddot{x}_1 + (\omega_0^2 + \omega'^2)x_1 + \omega'^2 x_2 = 0$$

$$m\ddot{x}_2 = -k'x_1 - (k+k')x_2$$

$$\ddot{x}_2 + (\omega_0^2 + \omega'^2)x_2 + \omega'^2 x_1 = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \omega_0^2 + \omega'^2 & \omega'^2 \\ \omega'^2 & \omega_0^2 + \omega'^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left| \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} + \begin{pmatrix} \omega_0^2+\omega'^2 & \omega'^2 \\ \omega'^2 & \omega_0^2+\omega'^2 \end{pmatrix} \right| = 0$$

$$(\omega_0^2+\omega'^2-\omega^2)^2 - \omega'^4 = 0$$

$$\omega_0^4 + 2\omega_0^2\omega'^2 - 2\omega_0^2\omega^2 + \cancel{\omega'^4} - 2\omega'^2\omega^2 + \omega^4 - \cancel{\omega'^4} = 0$$

$$\omega^4 - \omega^2(2\omega_0^2+2\omega'^2) + \omega_0^4 + 2\omega_0^2\omega'^2 = 0$$

$$\omega^2 = \frac{\cancel{2}(\omega_0^2+\omega'^2) \pm \sqrt{(\cancel{2}\omega_0^2+\cancel{2}\omega'^2)^2 - \cancel{4}(\omega_0^4-2\omega_0^2\omega'^2)}}{\cancel{2}}$$

$$\pm\, \omega_0^2+\omega'^2 \pm \omega'^2$$

$$\boxed{\begin{aligned} \omega_+^2 &= \omega_0^2+2\omega'^2 \\ \omega_-^2 &= \omega_0^2 \end{aligned}}$$

For $\omega^2 = \omega_+^2 = \omega_0^2+2\omega'^2$

$$\left[\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} + \begin{pmatrix} \omega_0^2+\omega'^2 & \omega'^2 \\ \omega'^2 & \omega_0^2+\omega'^2 \end{pmatrix} \right] \begin{pmatrix} E_+^1 \\ E_+^2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -\omega'^2 & \omega'^2 \\ \omega'^2 & -\omega'^2 \end{pmatrix} \begin{pmatrix} E_+^1 \\ E_+^2 \end{pmatrix} = 0$$

$$E_+ = (1, 1)$$

For $w^2 = w_-^2 = w_0^2$

$$\begin{pmatrix} w'^2 & w'^2 \\ w'^2 & w'^2 \end{pmatrix} \begin{pmatrix} E_-^1 \\ E_-^2 \end{pmatrix} = 0$$

$$E_- = (1, -1)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q^+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Q^- \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 &= Q^+ + Q^- \\ x_2 &= Q^+ - Q^- \end{aligned}\right\} \qquad \begin{aligned} Q^+ &= A_+ \cos(w_+ t + \phi_+) \\ Q^- &= A_- \cos(w_- t + \phi_-) \end{aligned}$$

Then:
$$\ddot{x}_1 = \ddot{Q}^+ + \ddot{Q}^-$$
$$= -w_+^2 Q^+ - w_-^2 Q^-$$
$$\ddot{x}_2 = -w_+^2 Q^+ + w_-^2 Q^-$$

$$x_1 = A_+ \cos(w_+ t + \phi_+) + A_- \cos(w_- t + \phi_-)$$
$$x_2 = A_+ \cos(w_+ t + \phi_+) - A_- \cos(w_- t + \phi_-)$$

Initially $(t=0)$: $\quad \dot{x}_2(t=0) = 0 \qquad \dot{x}_1(t=0) = 0.$

So
$$x_1 = A_+ \cos(w_+ t) + A_- \cos(w_- t)$$
$$x_2 = A_+ \cos(w_+ t) - A_- \cos(w_- t)$$

Know: $x_2(t=0) = 0, \qquad x_1(t=0) = x_0$

$$x_1 = A_+ + A_- = x_0$$

$$x_2 = A_+ - A_- = 0. \Rightarrow A_+ = A_-$$

then $x_1 = 2A_+ = x_0$

$$A_+ = \frac{x_0}{2}$$

$$x_1 = \frac{x_0}{2}\left(\cos(w_+ t) + \cos(w_- t)\right) \qquad w_+ = \sqrt{w_0^2 + 2w'^2}$$

$$x_2 = \frac{x_0}{2}\left(\cos(w_+ t) - \cos(w_- t)\right) \qquad w_- = w_0$$

b) Plot $x_1(t)$ and $x_2(t)$ for $k' \ll k$.

Given a signal which is $A\cos(w_1 t) + B\sin(w_2 t + \phi)$,

what is required to have prominent beats.

If $k' \ll k$, $\quad w_+ = \sqrt{w_0^2 + 2w'^2}$

$$\stackrel{!}{=} w_0\sqrt{1 + 2\left(\frac{w'}{w_0}\right)^2}$$

$$\stackrel{!}{=} w_0\left(1 + \left(\frac{w'}{w_0}\right)^2\right)$$

$$x_1 = \frac{x_0}{2}\left(\cos(w_+ t) + \cos(w_- t)\right)$$

$$x_2 = \frac{x_0}{2}\left(\cos(w_+ t) - \cos(w_- t)\right)$$

$\hookrightarrow \quad x_1 = \frac{x_0}{2}\left(\cos\left[w_0\left(1 + \left(\frac{w'}{w_0}\right)^2\right) t\right] + \cos w_0 t\right)$

$$X_2 = \frac{X_0}{2}\left(\cos\left[w_0\left(1+\left(\frac{w'}{w_0}\right)^2\right)t\right] - \cos w_0 t\right)$$

use:

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right).$$

$$X_1 = X_0\left\{\cos\left[w_0\left(2+\frac{w'}{w_0}^2\right)t\right]\cos\left(\frac{w'}{w_0}^2 t\right)\right\}$$

$$X_2 = -X_0\left\{\sin\left[w_0\left(2+\frac{w'}{w_0}^2\right)t\right]\sin\left(\frac{w'}{w_0}^2 t\right)\right\}$$

If $x(t) = A\cos w_1 t + B\cos w_2 t + \phi$

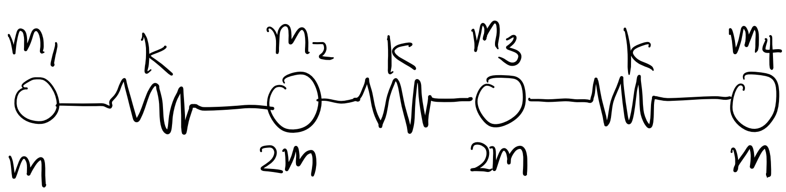
$$x(t) = (A-B)\cos w_1 t + B\cos w_1 t + B\cos w_2 t + \phi$$

$$\stackrel{!}{=} (A-B)\cos w_1 t + 2B\left[\cos\left(\frac{(w_1+w_2)t+\phi}{2}\right)\cos\left(\frac{(w_1-w_2)t-\phi}{2}\right)\right]$$

To have prominent beat, $A = B$ and $w_1 \cong w_2$

so $w_1 - w_2$ is small.

2) <u>Four masses with a kick:</u>



a) Write down a set of coordinates which parameterize deformations that are even and odd.

First we start with zero mode:

$$\vec{Q}_0^i = Q_0^i (1, 1, 1, 1)$$

$$M = m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X = (X_1, X_2, X_3, X_4) \qquad \underline{X} = (-X_4, -X_3, -X_2, -X_1).$$

odd mode if: $\underline{X} = -X$

$$\left.\begin{array}{l} -X_4 = -X_1 \\ -X_3 = -X_2 \\ -X_2 = -X_3 \\ -X_1 = -X_4 \end{array}\right\} \vec{Q}_{odd} = (X_1, X_2, X_2, X_1)$$

even mode if: $\underline{X} = X$

$$\left.\begin{array}{l} -X_4 = X_1 \\ -X_3 = X_2 \\ -X_2 = X_3 \\ -X_1 = X_4 \end{array}\right\} Q_{even} = (X_1, X_2, -X_2, -X_1)$$

$$\vec{Q}_{odd} M \vec{Q}_0 = X_1 + 2X_2 + 2X_2 + X_1 = 0.$$

$$-4x_2 = 2x_1$$
$$-2x_2 = x_1$$

let $x_1 = 2$, $x_2 = -1$

$$Q_{odd} = (2, -1, -1, 2)$$

$$\vec{Q}_{even} M \vec{Q}_0 = x_1 + 2x_2 - 2x_2 - x_1 = 0$$

let $x_1 = 1$
$x_2 = 1$.

$$\vec{Q}_{even} = (1, 0, 0, -1) \quad \text{and} \quad (0, 1, -1, 0)$$

$$\left.\begin{array}{l} \vec{Q}_0 = (1, 1, 1, 1) \\ \vec{Q}_{odd} = (2, -1, -1, 2) \\ Q_{even,1} = (1, 0, 0, -1) \\ Q_{even,2} = (0, 1, -1, 0) \end{array}\right\} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = Q_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + Q_{odd} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + Q_{e_1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + Q_{e2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

b) write down the lagrangian using eigenbasis

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} 2m \dot{x}_2^2 + \frac{1}{2} 2m \dot{x}_3^2 + \frac{1}{2} m \dot{x}_4^2$$

$$= \frac{1}{2} m (\dot{Q}_0 + 2\dot{Q}_{odd} + \dot{Q}_{e1})^2 + \frac{1}{2} 2m (\dot{Q}_0 - \dot{Q}_{odd} + \dot{Q}_{e2})^2$$

$$+ \frac{1}{2} 2m (\dot{Q}_0 - \dot{Q}_{odd} - \dot{Q}_{e2})^2 + \frac{1}{2} m (\dot{Q}_0 + 2\dot{Q}_{odd} - \dot{Q}_{e1})^2$$

$$\doteq \frac{1}{2}m(6)\dot{Q}_0^2 + \frac{1}{2}m(12)\dot{Q}_{odd}^2 + \frac{1}{2}m(4)\dot{Q}_{e2}^2$$
$$+ \frac{1}{2}m(2)\dot{Q}_{e1}^2$$

$$V = \frac{1}{2}k\left((x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_4)^2\right)$$
$$\doteq \frac{1}{2}k\Big([\cancel{Q_0} + 2Q_{odd} + Q_{e1} - \cancel{Q_0} + Q_{odd} - Q_{e2}]^2$$
$$+ [\cancel{Q_0} - \cancel{Q_{odd}} + Q_{e2} - \cancel{Q_0} + \cancel{Q_{odd}} + Q_{e2}]^2$$
$$+ [\cancel{Q_0} - Q_{odd} - Q_{e2} - \cancel{Q_0} - 2Q_{odd} + Q_{e1}]^2\Big)$$
$$\doteq \frac{1}{2}k\left[(3Q_{odd} + Q_{e1} - Q_{e2})^2 + 4Q_{e2}^2 + (-3Q_{odd} + Q_{e1} - Q_{e2})^2\right]$$
$$\doteq \frac{1}{2}k\left[18Q_{odd}^2 + 6Q_{e2}^2 + 2Q_{e1}^2 - 4Q_{e1}Q_{e}\right]$$

$$L = \frac{1}{2}m(6)\dot{Q}_0^2 + \frac{1}{2}m(12)\dot{Q}_{odd}^2 + \frac{1}{2}m(4)\dot{Q}_{e2}^2$$
$$+ \frac{1}{2}m(2)\dot{Q}_{e1}^2$$
$$- \frac{1}{2}k\left[18Q_{odd}^2 + 6Q_{e2}^2 + 2Q_{e1}^2 - 4Q_{e1}Q_{e2}\right]$$

$$6m\ddot{Q}_0 = 0 \quad \Rightarrow \quad \omega_0^2 = 0$$
$$Q_0 = A + Bt.$$

$$12m\ddot{Q}_{odd} = -18kQ_{odd}.$$
$$\ddot{Q}_{odd} = -\frac{18}{12}\frac{k}{m}Q_{odd} \qquad \omega_{odd}^2 = \frac{3}{2}\frac{k}{m}$$

$$2m\ddot{Q}_{e1} = -2kQ_{e1} + 2kQ_{e2}$$
$$4m\ddot{Q}_{e2} = -6kQ_{e2} + 2kQ_{e1}$$

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} (-w^2) + \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \frac{k}{m} \right| = 0$$

$$= \begin{pmatrix} -w^2 + w_0^2 & -w_0^2 \\ -w_0^2 & -2w^2 + 3w_0^2 \end{pmatrix}$$

$$\stackrel{!}{=} (-w^2 + w_0^2)(-2w^2 + 3w_0^2) - w_0^4 = 0.$$

$$2w^4 - 3w^2 w_0^2 - 2w^2 w_0^2 + 3w_0^4 - w_0^4 = 0.$$

$$2w^4 - 5w^2 w_0^2 + 2w_0^4 = 0.$$

$$w_\pm^2 = \frac{5w_0^2 \pm \sqrt{25w_0^4 - 4(2)(2w_0^4)}}{4}.$$

$$\stackrel{!}{=} \frac{5w_0^2 \pm 3w_0^2}{4}$$

$$w_+^2 = 2w_0^2$$
$$w_-^2 = \frac{1}{2} w_0^2$$

$$w^2 = w^2 = 2w_0^2$$

For $w^2 = w_+^2$

$$\begin{pmatrix} -2w_0^2 + w_0^2 & -w_0^2 \\ -w_0^2 & -4w_0^2 + 3w_0^2 \end{pmatrix} \begin{pmatrix} E_+^0 \\ E_-^0 \end{pmatrix}$$

$$\begin{pmatrix} -w_0^2 & -w_0^2 \\ -w_0^2 & -w_0^2 \end{pmatrix} \Rightarrow$$

$$E_+ = (1, -1)$$

For $w^2 = w_-^2 = \frac{1}{2} w_0^2$

$$\begin{pmatrix} -\frac{1}{2} w_0^2 + w_0^2 & -w_0^2 \\ -w_0^2 & -2\left(\frac{1}{2} w_0^2\right) + 3w_0^2 \end{pmatrix} \begin{pmatrix} E_-^0 \\ E_-^1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} w_0^2 & -w_0^2 \\ w_0^2 & 2w_0^2 \end{pmatrix} \begin{pmatrix} E_-^0 \\ E_-^1 \end{pmatrix}$$

$$E_- = (2, 1)$$

$$\begin{pmatrix} Q_{even\,1} \\ Q_{even\,2} \end{pmatrix} = Q^+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + Q^- \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

x_1

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = Q_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + Q_{odd} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + Q_{e_1} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + Q_{e2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\stackrel{!}{=} Q_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + Q_{odd} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + (Q^+ + 2Q^-) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + (Q^- - Q^+) \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \stackrel{!}{=} \underset{\downarrow \atop w_0 = 0}{Q_0} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \underset{\downarrow \atop w_{odd}^2 = \frac{3}{2} w_0^2}{Q_{odd}} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + \underset{\downarrow \atop 2w_0^2}{Q^+} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \underset{\downarrow \atop \frac{w_0^2}{2}}{Q^-} \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix}$$

c) If the most left mass is given an impulsive kick with force $F(t) = P_0 \delta(t)$ at $t=0$, determine position at subsequent times.

$$L_{ext} = F(t)\, x_1 = F(t)\left(Q_0 + 2Q_{odd} + Q^+ + 2Q^- \right)$$

$$L = \frac{1}{2}\left(12m\dot{Q}^{+2} + 6m\dot{Q}^{-2} + 12m\dot{Q}_{odd}^2 + 6m\dot{Q}_0^2\right) - \frac{1}{2}\left[12kQ^{-2} + 6kQ^{+2} + 18kQ_{odd}^2\right] + L_{ext}.$$

$$\ddot{Q}_0 = F(t)/6m$$

$$\ddot{Q}_{odd} = -w_0^2 Q_{odd} + 2F(t)/12m$$

$$\ddot{Q}^+ = -w_+^2 Q^+ + F(t)/6m$$

$$\ddot{Q}^- = -w_-^2 Q^- + 2F(t)/12m$$

$$G_{R,0} = A + Bt.$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{d^2}{dt^2} G_{R,0}\, dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0)\, dt_0 = 1.$$

$$\left.\frac{d}{dt} G_{R,0}\right|_{t_0+\epsilon} = 1 \qquad \leftarrow \text{First condition.}$$

$$\frac{d}{dt}(A+Bt) = 1 \Rightarrow B = 1.$$

Since. $\left. G_{R,0}\right|_{t_0+\epsilon} = 0.$

$$A - Bt_0 = 0.$$
$$A = Bt_0$$

$$G_{R,0} = (t-t_0)\,\theta(t-t_0)$$

$$Q_0 = \int_{-\infty}^{\infty} dt_0\, G_{R,0}\, F(t_0) = \int_{-\infty}^{\infty} dt_0\, (t-t_0) \frac{P_0}{6m} \delta(t_0)\, \theta(t-t_0)$$
$$\downarrow$$
$$= \frac{P_0 t}{6m}\, \theta(t)$$

For $G_R = A\cos(\omega t + \phi)\, \theta(t-t_0)$

same condition of $\left.\frac{dG_R}{dt}\right|_{t_0+\epsilon} = 1$ and $\left.G_R\right|_{t_0+\epsilon} = 0.$

$$G_R(t=t_0) = A\cos(\omega t_0 + \phi) = 0.$$

let $G_R(t) = A\sin(\omega(t-t_0)) = 0$

For $\frac{dG_R}{dt}(t=t_0+\epsilon) = wA\cos(w(t-t_0)) = 1$

$$A = \frac{1}{w}$$

$$Q = \int_{-\infty}^{\infty} dt_0 \, \frac{1}{w}\sin(w(t-t_0)) \, P_0 \, \delta(t_0) \, \theta(t-t_0)$$

$$\stackrel{!}{=} \frac{1}{w}\sin(wt) \quad P_0 \, \theta(t)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \stackrel{!}{=} \underbrace{Q_0}_{\downarrow \, w_1=0} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \underbrace{Q_{odd}}_{\downarrow \, w_{odd}^2 = \frac{3}{2}w_0^2} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + \underbrace{Q^{+}}_{\downarrow \, 2w_0^2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \underbrace{Q^{-}}_{\downarrow \, \frac{w_0^2}{2}} \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix}$$

For $Q_0 = P_0 t \, \theta(t-t_0)/6m$

$Q_{odd} = \frac{1}{w_{odd}}\sin(w_{odd}t)\frac{2P_0}{12m}\theta(t)$, $w_{odd} = \sqrt{\frac{3}{2}w_0^2}$

$Q_+ = \frac{1}{w_+}\sin(w_+ t)\frac{P_0}{6m}\theta(t)$ $\quad w_+ = \sqrt{2w_0^2}$

$Q_- = \frac{1}{w_-}\sin(w_- t)\frac{2P_0}{12m}\theta(t)$ $\quad w_- = \sqrt{\frac{w_0^2}{2}}$

If we follow the frame of Q_0, which is the center of mass frame, then $Q_0 = 0$.

So we are periodic

In origional frame.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{P_0}{6m} + 2\left(\frac{2P_0}{12m}\right) + \frac{P_0}{6m} + 2\left(\frac{2}{12m}P_0\right) = 6\frac{P_0}{m} \\ \frac{P_0}{6m} - \frac{P_0}{6m} - \frac{P_0}{6m} + \frac{P_0}{6m} = 0 \\ \text{likwise: } 0 \\ 0 \end{pmatrix}$$

In moving frame of V_{cm}.

$$\begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \\ v_4^* \end{pmatrix} = \begin{pmatrix} 5V_{cm} \\ -V_{cm} \\ -V_{cm} \\ -V_{cm} \end{pmatrix}$$

$$Q_0 = P_0 t\,\theta(t) \,/6m = V_{cm} t\,\theta(t)$$

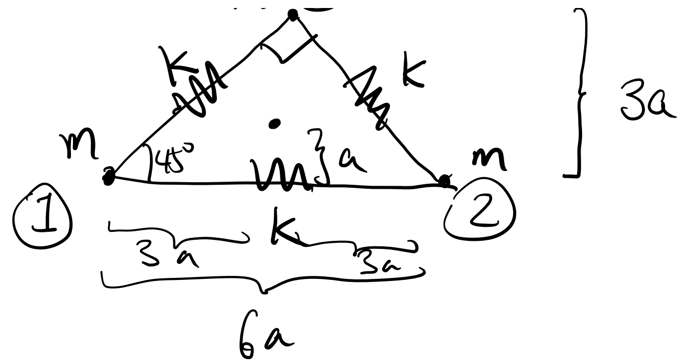
$$Q_{odd} = \frac{1}{W_{odd}} \sin(W_{odd} t) \frac{2P_0}{12m} \theta(t) = V_{cm} \frac{1}{W_{odd}} \sin(W_{odd} t)\theta(t)$$

$$Q_+ = \frac{1}{W_+} \sin(W_+ t) \frac{P_0}{6m} \theta(t) = V_{cm} \frac{1}{W_+} \sin(W_+ t)\, \theta(t)$$

$$Q_- = \frac{1}{W_-} \sin(W_- t) \frac{2P_0}{12m} \theta(t) = V_{cm} \frac{1}{W_-} \sin(W_- t)\, \theta(t)$$

3) A molecule with a right triangle.

m ③



a) $\vec{Q} = (x_1, y_1, x_2, y_2, x_3, y_3)$

Show $\vec{Q}_{rot-z} = a\delta\theta(1, -3, 1, 3, -2, 0)$

First find center of mass:

$$\vec{r} = \frac{m_i r_i}{m_i}$$

$$X_{cm} = \frac{0 + 6am + 3am}{3m} = 3a$$

$$Y_{cm} = \frac{0 + 0 + 3am}{3m} = a.$$

$$\delta\vec{r}_1 = (-3a, -a)$$

$$\delta\vec{r}_2 = (3a, -a)$$

$$\delta\vec{r}_3 = (0, 2a)$$

$$Q_{rotz} = (\delta\theta \times r_{eq1},\ \delta\theta \times r_{eq2},\ \delta\theta \times r_{eq3})$$

$$Q_{rotz} \overset{!}{=} \delta\theta\hat{z} \times -3a\hat{x},\ \delta\theta\hat{z} \times -a\hat{y}$$
$$\delta\theta\hat{z} \times 3a\hat{x},\ \delta\theta\hat{z} \times -a\hat{y}$$
$$\delta\theta\hat{z} \times 0\hat{x},\ \delta\theta\hat{z} \times 2a\hat{y}$$

$$= \delta\theta\left([a\hat{x}, -3a\hat{y}], [a\hat{x}, 3a\hat{y}], [-2a\hat{x}, 0\hat{y}]\right)$$

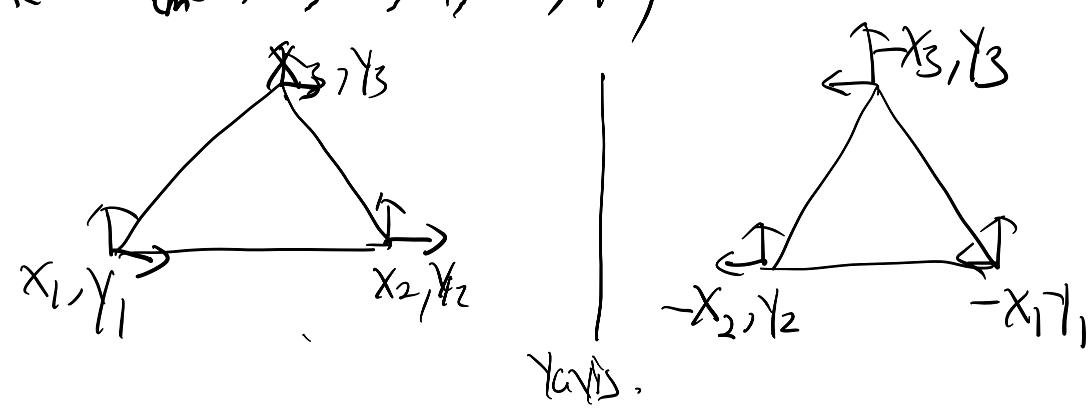
$$Q_{rotz} \overset{!}{=} (1, -3, 1, 3, -2, 0)\, a\delta\theta$$

b)

$$X_{cm} = X_{cm}(1, 0, 1, 0, 1, 0) \Rightarrow \text{odd.}$$

$$Y_{cm} = Y_{cm}(0, 1, 0, 1, 0, 1) \Rightarrow \text{Even}$$

c)



Over Reflection of Y axis

$$\Delta = (-X_2, Y_2, -X_1, Y_1, -X_3, Y_3)$$

if $\underline{X} = -X$ odd

$$\hookrightarrow \left.\begin{aligned} -X_2 &= -X_1 \\ Y_2 &= -Y_1 \\ -X_1 &= -X_2 \\ Y_1 &= -Y_2 \\ -X_3 &= -X_3 \\ Y_3 &= -Y_3 \end{aligned}\right\} \begin{pmatrix} X_1 \\ Y_1 \\ X_1 \\ -Y_1 \\ X_3 \\ 0 \end{pmatrix} Q_{odd} = Q_{odd} E_{odd}$$

if $\underline{X} = X$ Even:

$$\left.\begin{aligned} -X_2 &= X_1 \\ Y_2 &= Y_1 \\ -X_1 &= X_2 \\ Y_1 &= Y_2 \\ -X_3 &= X_3 \\ Y_3 &= Y_3 \end{aligned}\right\} \begin{pmatrix} X_1 \\ Y_1 \\ -X_1 \\ Y_1 \\ 0 \\ Y_3 \end{pmatrix} Q_{even} = Q_{even} E_{even}$$

We see

$$Q_{rot\text{-}z} = (1, -3, 1, 3, -2, 0)$$

$$\underline{Q}_{rot\text{-}z} = (-1, 3, -1, -3, 2, 0)$$

$$\overset{!}{=} -Q_{rot\text{-}z}$$

so Q_{rot-z} is odd.

d) $$T = \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} m(\dot{x}_3^2 + \dot{y}_3^2)$$

$$V = \frac{1}{2} k(\delta\vec{r}_1 - \delta\vec{r}_2)^2 + \frac{1}{2} k(\delta\vec{r}_2 - \delta\vec{r}_3)^2 + \frac{1}{2} k(\delta\vec{r}_3 - \delta\vec{r}_1)^2$$

$$= \frac{1}{2} k \left\{ (x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 + (x_3 - x_1)^2 + (y_3 - y_1)^2 \right\}$$

$$M = m \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$E_{odd} \, M \, E_{xcm} = 0$$

$$= (x_1, y_1, x_1, -y_1, x_3, 0)(1, 0, 1, 0, 1, 0)^T = 0$$

$$2x_1 + x_3 = 0$$

$$x_3 = -2x_1$$

Then $E_{odd} = (x_1, y_1, x_1, -y_1, -2x_1, 0)$

Use E_{rot-z}:

$E_{odd} M E_{rot-z} = 0$

$\hookrightarrow (x_1, y_1, x_1, -y_1, -2x_1, 0)(1, -3, 1, 3, -2, 0)^T = 0$

$x_1 - 3y_1 + x_1 - 3y_1 + 4x_1 = 0$

$6x_1 - 6y_1 = 0$

$x_1 = 1, \quad y_1 = 1$

Then

$\vec{q}_0 = q_0(1, 1, 1, -1, -2, 0)$

Find Even modes using T_{cm}.

$E_{even} M E_{Tcm} = 0$

$\hookrightarrow (x_1, y_1, -x_1, y_1, 0, y_3)(0, 1, 0, 1, 0, 1)^T = 0$

$y_1 + y_1 + y_3 = 0$

$2y_1 + y_3 = 0$

$\because$

$$y_3 = -2 \qquad y_1 = 1 \qquad , \text{let } x_1 = 0$$

$$E_{even_1} = (0, 1, 0, 1, 0, -2)$$

or let $x_1 = 1$, $y_3 = y_1 = 0$

$$E_{even\,2} = (1, 0, -1, 0, 0, 0)$$

Summary:

$$\vec{Q}_{rot\,z} = Q_{rot\text{-}z}\,(1, -3, 1, 3, -2, 0)$$

$$\vec{Y}_{cm} = Y_{cm}\,(0, 1, 0, 1, 0, 1)$$

$$\vec{X}_{cm} = X_{cm}\,(1, 0, 1, 0, 1, 0)$$

$$\vec{Q}_0 = Q_0\,(1, 1, 1, -1, -2, 0)$$

$$\vec{Q}_1 = Q_1\,(0, 1, 0, 1, 0, -2)$$

$$Q_2 = Q_2\,(1, 0, -1, 0, 0, 0)$$

e) Write down Lagrangian:

$$T = \frac{1}{2}m\left(\dot{x}_1^{\,2} + \dot{y}_1^{\,2}\right) + \frac{1}{2}m\left(\dot{x}_2^{\,2} + \dot{y}_2^{\,2}\right) + \frac{1}{2}m\left(\dot{x}_3^{\,2} + \dot{y}_3^{\,2}\right)$$

$$= \frac{1}{2}m\Big(\left[\dot{Q}_z + \dot{X}_{cm} + \dot{Q}_0 + \dot{Q}_2\right]^2 + \left[-3\dot{Q}_z + \dot{Y}_{cm} + \dot{Q}_0 + \dot{Q}_1\right]^2$$

$$+ \left[\dot{Q}_z + \dot{X}_{cm} + \dot{Q}_0 - \dot{Q}_2\right]^2 + \left[3\dot{Q}_z + \dot{Y}_{cm} - \dot{Q}_0 + \dot{Q}_1\right]^2$$

$$+\left[-2\dot{Q}_z+\dot{X}_{cm}-2\dot{Q}_0\right]^2+\left[\dot{Y}_{cm}-2\dot{Q}_1\right]^2\Big)$$

$$T\overset{!}{=}\frac{1}{2}m\Big(24\dot{Q}_z^{\,2}+3\dot{X}_{cm}^{\,2}+8\dot{Q}_0^{\,2}+2\dot{Q}_2^{\,2}+3\dot{Y}_{cm}^{\,2}+6\dot{Q}_1^{\,2}\Big)$$

$$V=\frac{1}{2}k\left(\Delta l_{12}\right)^2+\frac{1}{2}k\left(\Delta l_{23}\right)^2+\frac{1}{2}k\left(\Delta l_{31}\right)^2$$

$$\Delta l_{12}=\sqrt{(\vec{r}_1-\vec{r}_2)^2}-\sqrt{(r_{eq_1}-r_{eq_2})^2}$$

$$\overset{!}{=}\Big[\left[(x_1-3a)-(x_2+3a)\right]^2+\left[(y_1-a)-(y_2-a)\right]^2\Big]^{1/2}-\sqrt{(-3a-3a)^2+(-a--a)^2}$$

$$\overset{!}{=}\sqrt{(x_1-x_2-6a)^2+\cancel{(y_1-y_2)^2}}-6a$$

$$\overset{!}{=}\sqrt{(\delta x-6a)^2}-6a$$

$$\overset{!}{=}\sqrt{36a^2-12a\,\delta x}-6a$$

$$\overset{!}{=}6a\sqrt{1-\frac{1}{3a}\delta x}-6a$$

$$\overset{!}{=}6a\left(1-\frac{1}{6a}\delta x\right)-6a$$

$$\doteq -\delta x$$

$$\doteq x_2 - x_1$$

$$\Delta l_{23} = \sqrt{(\vec{r}_2 - \vec{r}_3)^2} - \sqrt{(r_{eq1} - r_{eq2})^2}$$

$$\doteq \sqrt{(x_2 + 3a - x_3)^2 + (y_2 - a - (y_3 + 2a))^2} - \sqrt{(3a-0)^2 + (-a-2a)^2}$$

$$\doteq \sqrt{(\delta x_{23} + 3a)^2 + (\delta y_{23} - 3a)^2} - \sqrt{18a^2}$$

$$\doteq \sqrt{9a^2 + 6a\,\delta x_{23} + 9a^2 - 6a\,\delta y_{23}} - \sqrt{18a^2}$$

$$\doteq \sqrt{18a^2}\left(\sqrt{1 + \frac{1}{3a}(\delta x_{23} - \delta y_{23})}\right) - \sqrt{18a^2}$$

$$\doteq \sqrt{18a^2}\left(1 + \frac{1}{6a}(\delta x_{23} - \delta y_{23})\right) - \sqrt{18a^2}$$

$$\doteq \frac{1}{\sqrt{2}}(\delta x_{23} - \delta y_{23})$$

$$\doteq \frac{1}{\sqrt{2}}(x_2 - x_3 - (y_2 - y_3))$$

$$\Delta l_{31} = \frac{1}{\sqrt{2}}(x_3 - x_1 + (y_3 - y_1))$$ by symmetry to Δl_{23}.

$$V = \frac{1}{2}k\left(\frac{1}{2}(x_2 - x_3 - y_2 + y_3)^2 + \frac{1}{2}(x_3 - x_1 - y_3 + y_1)^2 + (x_2 - x_1)^2\right)$$

$$
\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} = Q_z \begin{pmatrix} 1 \\ -3 \\ 1 \\ 3 \\ -2 \\ 0 \end{pmatrix} + X_{cm} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + Y_{cm} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + Q_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} + Q_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + Q_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
x_3 - x_2 - y_3 + y_2 &= \cancel{-2Q_z} - \cancel{X_{cm}} + \cancel{2Q_0} + \cancel{Q_z} + \cancel{X_{cm}} + \cancel{Q_0} - Q_2 \\
&\quad + \cancel{Y_{cm}} - 2Q_1 - \cancel{3Q_z} - \cancel{Y_{cm}} + \cancel{Q_0} - Q_1 \\
&\overset{!}{=} -3Q_1 + 4Q_0 - Q_2
\end{aligned}
$$

$$
\begin{aligned}
x_3 - x_1 + y_3 - y_1 &= \cancel{-2Q_z} + \cancel{X_{cm}} - 2Q_0 - \cancel{Q_z} - \cancel{X_{cm}} - Q_0 - Q_2 \\
&\quad + \cancel{Y_{cm}} - 2Q_1 + \cancel{3Q_z} - \cancel{Y_{cm}} - Q_0 - Q_1 \\
&\overset{!}{=} -4Q_0 - Q_2 - 3Q_1
\end{aligned}
$$

$$
\begin{aligned}
x_2 - x_1 &= Q_z + x_{cm} + Q_0 - Q_2 - Q_z - X_{cm} - Q_0 - Q_2 \\
&\overset{!}{=} -2Q_2
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{1}{2}k\left(\frac{1}{2}(-3Q_1 + 4Q_0 - Q_2)^2 + \frac{1}{2}(-4Q_0 - Q_2 - 3Q_1)^2 \right. \\
&\quad \left. + 4Q_2^2\right) \\
&\overset{!}{=} \frac{1}{2}k\left(\frac{1}{2}(\cancel{9Q_1^2} - \cancel{24Q_1Q_0} + 6Q_1Q_2 + \cancel{16Q_0^2} + \cancel{Q_2^2} - \cancel{8Q_0Q_2}\right.
\end{aligned}
$$

$(16Q_0^2 + \cancel{8Q_0}Q_2 + \cancel{24Q_0}Q_1 + Q_2 + \cancel{7Q_1} + 6Q_2Q_1$
$+4Q_2^2)$

$$= \frac{1}{2}k\left(16Q_0^2 + 9Q_1^2 + 5Q_2^2 + 6Q_1Q_2\right).$$

with

$$T = \frac{1}{2}m\left(24\dot{Q}_z^2 + 3\dot{X}_{cm}^2 + 8\dot{Q}_0^2 + 2\dot{Q}_2^2 + 3\dot{Y}_{cm}^2 + 6\dot{Q}_1^2\right)$$

$$\left.\begin{aligned} 24m\ddot{Q}_z &= 0 \\ 3\ddot{X}_{cm} &= 0 \\ 3\ddot{Y}_{cm} &= 0. \end{aligned}\right\} \quad \omega^2 = 0$$

$$8m\ddot{Q}_0 = -16kQ_0$$

so $\boxed{\omega_0^2 = \frac{2k}{m} = 2\omega_0^2}$

$$2m\ddot{Q}_2 = -5kQ_2 - 3kQ_1$$
$$6m\ddot{Q}_1 = -9kQ_1 - 3kQ_2$$

$\left| \begin{pmatrix} 6 & 0 \end{pmatrix} \omega^2 - \begin{pmatrix} 9 & 3 \end{pmatrix} \omega^2 \right| = \ldots$

$$\left| \begin{pmatrix} & \\ 0 & 2 \end{pmatrix} -\omega + \begin{pmatrix} \dot{3} & 5 \end{pmatrix} \omega_0 \right| = 0$$

$$\begin{vmatrix} -6\omega^2 + 9\omega_0^2 & 3\omega_0^2 \\ 3\omega_0^2 & -2\omega^2 + 15\omega_0^2 \end{vmatrix} = 0$$

$$\Leftrightarrow = (-6\omega^2 + 9\omega_0^2)(-2\omega^2 + 15\omega_0^2) - 9\omega_0^4$$

$$\stackrel{!}{=} 12\omega^4 - 30\omega^2\omega_0^2 - 18\omega^2\omega_0^2 + 45\omega_0^4 - 9\omega_0^4$$

$$\stackrel{!}{=} 12\omega^4 - 48\omega^2\omega_0^2 + 36\omega_0^4$$

$$\stackrel{!}{=} \omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4.$$

$$(\omega^2 - \omega_0^2)(\omega^2 - 3\omega_0^2)$$

$$\omega^2 = \omega_0^2 \,,\; 3\omega_0^2$$

For $\omega_-^2 = \omega_0^2$

$$\begin{pmatrix} -6\omega_0^2 + 9\omega_0^2 & 3\omega_0^2 \\ 3\omega_0^2 & -2\omega_0^2 + 15\omega_0^2 \end{pmatrix} \begin{pmatrix} E_-^1 \\ E_-^2 \end{pmatrix}$$

$$= \begin{pmatrix} 3\omega_0^2 & 3\omega_0^2 \\ 3\omega_0^2 & 3\omega_0^2 \end{pmatrix} \begin{pmatrix} E_-^1 \\ E_-^2 \end{pmatrix}$$

$$\underline{E}_- = (1, -1)$$

For $w_+^2 = 3w_0^2$

$$\begin{pmatrix} -18w_0^2+9w_0^2 & 3w_0^2 \\ 3w_0^2 & -6w_0^2+5w_0^2 \end{pmatrix}\begin{pmatrix} E_+ \end{pmatrix}$$

$$-\overset{3}{\cancel{9}}w_0^2 E_+^1 + \cancel{3}w_0^2 E_+^2 = 0.$$

$$E_+^1 = 1 \qquad E_+^2 = 3$$

then

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = Q_+\begin{pmatrix} 1 \\ 3 \end{pmatrix} + Q_-\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Q_1 = Q_+ + Q_-$$

$$Q_2 = 3Q_+ - Q_-$$

$$(Q_+ + Q_-)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + (3Q_+ - Q_-)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = Q_+\begin{pmatrix} 3 \\ 1 \\ -3 \\ 1 \\ 0 \\ -2 \end{pmatrix} + Q_-\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

$(X_1$

$$\begin{pmatrix} y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} = Q_z \begin{pmatrix} 1 \\ -3 \\ 1 \\ 3 \\ -2 \\ 0 \end{pmatrix} + X_{cm} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + Y_{cm} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + Q_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} Q_+ \begin{pmatrix} 3 \\ 1 \\ -3 \\ 1 \\ 0 \\ -2 \end{pmatrix} + Q_- \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$