Noether Theorem: (A relation between continuous symmetry shifts)

Transformation is a symmetric if $\delta S = 0$

$$\delta S[r, \delta_s \vec{r}] = S[r + \delta_s r] - S[r] = 0$$

for transformation : $\vec{r} \to \vec{r} + \delta_s \vec{r}$

Lagrangian is invariant or $L \to L$ for any $r(t)$, meaning it doesn't need to satisfy EOM, or on-shell path.

Derivation:

$$S[\vec{r}_a + \delta_s \vec{r}_a] = S[\vec{r}] + \int dt \; L\left(\dot{\vec{r}}_a + \frac{d}{dt}\delta_s \vec{r}_a \, , \vec{r}_a + \delta_s \vec{r}_a\right)$$

$$\delta S = \int dt \; \frac{\partial L}{\partial \vec{r}_a}\delta_s \vec{r}_a + \frac{\partial L}{\partial \dot{\vec{r}}_a}\frac{d}{dt}(\delta_s \vec{r}_a)$$

$$\stackrel{!}{=} \int dt \; \frac{\partial L}{\partial \vec{r}_a}\delta_s \vec{r}_a + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\delta_s \vec{r}_a\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\right)\delta_s \vec{r}_a$$

$$\stackrel{!}{=} \frac{\partial L}{\partial \dot{\vec{r}}_a}\delta_s \vec{r}_a \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \vec{r}_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\right)\right)\delta_s \vec{r}_a \, dt$$

Here $\delta S = 0$ for any arbitrary $\vec{r}(t)$ if transformation is symmetric.

If consider on-shell path: $\frac{\partial L}{\partial \vec{r}_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\right) = 0$

then

$$\delta S = \frac{\partial L}{\partial \dot{\vec{r}}_a} \delta_s \vec{r}_a \Big|_{t_1}^{t_2} = 0$$

for transformation to be symmetric and satisfy EOM.

Therefore find: $Q = \sum_a \frac{\partial L}{\partial \dot{\vec{r}}_a} \delta_s \vec{r}^a = \sum_a P_a \delta_s \vec{r}^a$ is constant

☆ or $Q(t_2) - Q(t_1) = 0.$

then Q is conserved.

Ex: Momentum Conservation (Translational Symmetry)

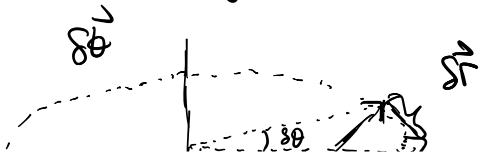
$$\vec{r}_a \to \vec{r}_a + \underbrace{\epsilon \vec{n}}_{\delta_s \vec{r}}$$ ← here $\vec{n} = \hat{x}, \hat{y}$ or $\hat{z}$ a shift in n direction.

then $Q = \sum_a P_a \cdot \epsilon \vec{n}$

$$\sum_a \vec{P}_a \cdot \epsilon \vec{n} \Big|_{t_{final}} - \sum_a \vec{P}_a \cdot \epsilon \vec{n} \Big|_{t_{init}} = 0$$

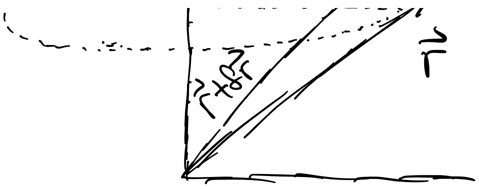
So $\sum_a \vec{P}_a \cdot \epsilon \vec{n}$ is conserved or conservation in the $\hat{n}$ direction.

Ex: Angular Momentum (Rotation Symmetry)



$\delta\theta$ in plane around z.

$$\delta_s \vec{r} = \delta\vec{\theta} \times \vec{r} = r\delta\theta \sin\theta$$



then $\vec{r} \to \vec{r} + \delta_s \vec{r}$

$$Q = \sum_a \vec{P}_a \cdot (\delta\vec{\theta} \times \vec{r})$$

with
triple product:
$a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$

$$\perp \delta\vec{\theta} \cdot \vec{\sum_a} \underbrace{(\vec{r} \times \vec{P}_a)}_{\text{angular momentum}} = 0$$

Symmetry Refinement:

shift of coordinate: $\vec{r}_a \to \vec{r}_a + \delta_s \vec{r}_a$

If $\mathcal{L} \to \mathcal{L}$, invariant after transformation,
then there is a symmetry.

We know: Lagrangian is unchanged under total time derivative

$$\mathcal{L} \to \mathcal{L} + \frac{dk}{dt} = \mathcal{L}'$$

$$S[\vec{r}] = \int \mathcal{L}\, dt$$

$$S[\vec{r} + \delta\vec{r}] = \int \mathcal{L}\, dt + \int_{t_1}^{t_2} dt\, \frac{dk}{dt}$$

$$\delta S[\vec{r} + \delta_s \vec{r}] = k_2 - k_1$$

But previously with EOM, we find

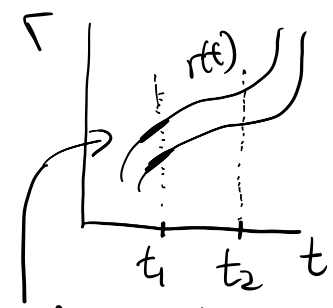
$$\delta S[\vec{r} + \delta_s \vec{r}] = \sum_a P_a\, \delta\vec{r}^{\,a}$$ ← end-point contribution for arbitrary variation.

Then $\delta S[\vec{r}+\delta_s\vec{r}] = K = \sum_a p_a \delta \vec{r}_a$

Thus $\boxed{Q = \sum_a p_a \delta_s \vec{r}^a - K}$ is conserved.

since $Q(t_2) - Q(t_1) = 0$ ← general form

<u>Ex</u>: Energy Conservation (time symmetry)



Action is changed is slightly at end points which is described by K

$$r(t-\Delta t) = r'(t)$$

$$\mathcal{L}'(t) = \mathcal{L}(t-\Delta t)$$ constant

$$\overset{!}{=} \mathcal{L}(t) - \underbrace{\Delta t \frac{d\mathcal{L}}{dt}}_{\frac{dK}{dt}} \Rightarrow K = -\Delta t \mathcal{L}$$

then $\vec{r}_a(t-\Delta t) = \vec{r}_a(t) \underbrace{- \Delta t \frac{d}{dt} \vec{r}_a}_{\delta_s \vec{r}_a}$

$$Q = \sum_a p_a \vec{r}^a - K = \sum_a \vec{p}_a \cdot (-\Delta t\, \dot{\vec{r}}^a) - (-\Delta t \mathcal{L})$$

$$\overset{!}{=} -\Delta t(\underbrace{\vec{p}_a \cdot \dot{\vec{r}}^a - \mathcal{L}}_{h = \text{total energy of system}}) = 0$$

↑ energy conservation. due to time symmetry