1) The adiabatic oscillator:

$$m\ddot{x} + m\omega^2 x = 0 \qquad \omega(t), \; |\dot{\omega}| \ll \omega^2$$

slow varying.

a) $$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2(t)\,x^2$$

$$x = e^{a(t)+i\theta(t)}$$
$$\dot{x} = (\dot{a}+i\dot{\theta})e^{a(t)+i\theta(t)}$$
$$\ddot{x} = (\ddot{a}+i\ddot{\theta})e^{a(t)+i\theta} + (\dot{a}+i\dot{\theta})^2 e^{a(t)+i\theta}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -m\omega^2 x$$

$$\cancel{m}\ddot{x} + \cancel{m}\omega^2 x = 0$$

$$\ddot{a} + i\ddot{\theta} + (\dot{a}+i\dot{\theta})^2 + \omega^2 = 0$$
$$\ddot{a} + i\ddot{\theta} + \dot{a}^2 + 2i\dot{a}\dot{\theta} - \dot{\theta}^2 + \omega^2 = 0$$

Real: $\ddot{a} + \dot{a}^2 - \dot{\theta}^2 + \omega^2 = 0$

Imaginary: $\ddot{\theta} + 2\dot{a}\dot{\theta} = 0$

b) assume: $\dot{a}^2 \ll \omega^2 \quad \ddot{a} \ll \omega^2$, find real $x(t)$.
with $x(0) = x_0$ and $\dot{x}(0) = v_0$

$$\ddot{a} + \dot{a}^2 - \dot{\theta}^2 + \omega^2 = 0$$

small

So to 0th order

$$\dot{\theta}_0^2 = \omega^2$$

$$\dot{\theta}_0 = \pm\omega$$

$$\theta = \theta_0 \pm \int_0^t \omega(t')\,dt'$$

$$\ddot{\theta} + 2\dot{a}\,\dot{\theta} = 0$$

$$-2\dot{a}\dot{\theta} = \ddot{\theta}$$

$$-2\frac{da}{dt}dt = \frac{d\dot{\theta}}{\dot{\theta}}$$

$$-2(a(t) - a_0) = \ln\frac{\dot{\theta}}{\dot{\theta}_0}$$

$$e^{-2(a-a_0)} = \frac{\dot{\theta}}{\dot{\theta}_0}$$

$$e^{a(t)} = e^{a_0}\sqrt{\frac{\dot{\theta}_0}{\dot{\theta}}}$$

$$\dot{\theta}_0 = \omega(t=0) \qquad \dot{\theta} = \omega(t)$$

$$x = e^{a_0}\sqrt{\frac{\omega(t=0)}{\omega(t)}}\, e^{i\theta}$$

$$\overset{!}{=} e^{a_0}\sqrt{\frac{\omega(0)}{\omega(t)}}\, e^{i\theta_0}\, e^{\pm i\int_0^t \omega(t')dt'}$$

$$\overset{!}{=} \sqrt{\frac{\omega(0)}{\omega(t)}}\left[A\cos\left(\int_0^t \omega(t')dt'\right) + B\sin\left(\int_0^t \omega(t')dt'\right)\right]$$

$$X(t=0) = Ae^{u_0} = X_0$$

$$X(t) = \sqrt{\frac{w(0)}{w(t)}}\left[X_0 \cos\left(\int_0^t w(t')\,dt'\right) + B\sin\left(\int_0^t w(t')\,dt'\right)\right]$$

$$\dot{X}(t) = \sqrt{w(0)}\left(\frac{-1}{2}\right)\frac{\dot{w}}{w(t')^{3/2}}\left[X_0\cos\left(\int_0^t w(t')\,dt'\right) + B\sin\left(\int_0^t w(t')\,dt'\right)\right]$$

$$+ \sqrt{\frac{w(0)}{w(t)}}\left[-X_0\int_0^t \dot{w}(t')\,dt'\,\sin\left(\int_0^t w(t')\,dt'\right)\right.$$

$$\left. + B\int_0^t \dot{w}(t')\,dt'\cos\left(\int_0^t w(t')\,dt'\right)\right]$$

$$\dot{X}(t=0) = V_0 = -\frac{1}{2}\frac{\sqrt{w(0)}}{\sqrt[3/2]{w(t)}}\dot{w}X_0 + \sqrt{\frac{w(0)}{w(t)}}\,B\int_0^t \dot{w}(t')\,dt'$$

$$V_0 + \frac{1}{2}\underbrace{\cancel{\sqrt{\frac{w(0)}{w(t)}}}}_{w(t=0)}\frac{\dot{w}}{w(t)}X_0 = \underbrace{\cancel{\sqrt{\frac{w(0)}{w(t)}}}}_{w(t=0)}\,B\underbrace{\int_0^t \dot{w}(t')\,dt'}_{w(t)|_{t=0}}$$

$$V_0 + \frac{1}{2}\frac{\dot{w}(0)}{w(0)}X_0 = B\,w(0)$$

$$\frac{V_0}{w(0)} + \frac{1}{2}\frac{\dot{w}(0)}{w(0)^2}X_0 = B \qquad A = X_0$$

$$X(t) = \sqrt{\frac{w(0)}{w(t)}}\left[A\cos\left(\int_0^t w(t')\,dt'\right) + B\sin\left(\int_0^t w(t')\,dt'\right)\right]$$

$$\boxed{x(t) = x_0\sqrt{\frac{\omega(0)}{\omega(t)}}\left[\cos\left(\int_0^t \omega(t')dt'\right) + q\sin\left(\int_0^t \omega(t')dt'\right)\right]}$$

$$\boxed{q = \frac{B}{x_0} = \frac{1}{\omega(0)}\left[\frac{v_0}{x_0} + \frac{\dot{\omega}(0)}{2\omega(0)}\right]}$$

c) $E = h = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$

here $\omega(t)$, so

$h(\dot{x}, x, t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2(t)\, x^2$

$\left.\frac{\partial h}{\partial t}\right|_{\dot{x}, x} = \frac{\partial}{\partial t}\left[\omega^2(t)\right]\frac{1}{2}mx^2$

we held $\dot{x}$ and x to be constant but $\omega(t)$

$= 2\omega\dot{\omega}\frac{1}{2}mx^2$

$= m\omega\dot{\omega}x^2$

Not 0, so

energy is not conserved.

d) average over $T = \frac{2\pi}{\omega}$

Show $\overline{\frac{E(t)}{\omega(t)}}$ is constant in time

to first order in $\dot{\omega}$

$$\overline{\frac{E(t)}{\omega(t)}} = \frac{1}{2} m \overline{\left(\frac{\dot{x}^2}{\omega} + \omega x^2 \right)}$$

$$\omega x^2 = \omega(0) x_0^2 \left(\cos^2 + q^2 \sin^2 + 2 \cos \sin \right)$$

$$\overline{\omega x^2} = \frac{1}{T} \int_0^T \omega(0) x_0^2 \left(\cos^2 + q^2 \sin^2 + 2 \cos \sin \right)$$

$$= \frac{\omega(0) x_0^2 (\pi + \pi)}{T}$$

$$\dot{x} = \frac{-1}{2} x_0 \sqrt{\frac{\omega(0)}{\omega(t)}} \frac{\dot{\omega}}{\omega(t)} (\cos + q \sin)$$
$$+ x_0 \sqrt{\frac{\omega(0)}{\omega}} (\omega) \quad (-\sin + q \cos)$$

$$\frac{\dot{X}^2}{W} = \frac{1}{W}\Big\{ \cancel{\frac{1}{4} X_0^2 \frac{W(0)}{W} \frac{\dot{W}^2}{W^2}} (\cos^2 + q^2 \sin^2 + 2q \cos\sin)$$

$$+ \; X_0^2 \frac{W(0)}{W} (W)^2 \; (\sin^2 + q^2\cos - 2q \cos\sin)$$

second order $\left(\frac{\dot{W}}{W^2}\right)^2$

$$- \; X_0^2 \frac{W(0)}{W} \frac{\dot{W}}{W} (W) \; (-\cos\sin + q\cos^2 - q\sin^2 + \cos^2) \Big\}$$

$\dot{W} \ll |W|^2$ so ignore this.

$$\frac{\dot{X}^2}{W} \cong X_0^2 \, W(0) (\sin^2 + q^2 \cos - 2q \sin\cos)$$

$$- X_0^2 W(0) \frac{\dot{W}}{W^2} (-\cos\sin + q\cos^2 - q\sin^2 + q^2 \sin\cos)$$

$$\stackrel{\int}{=} X_0^2 W(0) (\pi + q^2 \pi)$$

$$\cancel{- X_0^2 W(0) \frac{\dot{W}}{W^2} (q\pi - q\pi)}$$

$$= X_0^2 W(0) (1 + q^2) \pi$$

$$\overline{\frac{E}{W}} = \frac{1}{T} \left\{ W(0) X_0^2 \, 2\pi \; + X_0^2 W(0) (1 + q^2) \pi \right\}$$

$$\doteq \frac{1}{T}\left\{ x_0^2 w(0)(3+q^2)\pi \right\}$$

constant.

e) let $l(t) = l_0(1+\beta\cos\Omega t)$

$\Omega = \frac{2\pi}{1\text{day}}$ $\beta \ll 1$ $w^2 = g/l(t)$

Max displacement at 12:00 PM, ϕ_0

what is amplitude and phase at 6:00 PM?

$$x(t) = x_0\sqrt{\frac{w(0)}{w(t)}}\left[\cos\left(\int_0^t w(t')dt'\right) + q\sin\left(\int_0^t w(t')dt'\right)\right]$$

$\frac{2\pi}{24\text{hour}}$ 6hour $= \frac{\pi}{2}$

$$w(t) = \sqrt{\frac{g}{l(t)}} = \sqrt{\frac{g}{l_0}}\left(1 - \frac{1}{2}\beta\cos\Omega t\right)$$

$t=0$ at 12:00 PM:

$$x(t=0) = x_0 = \phi_0$$

for $t = 6$ hour

$$w_0 \int_0^t \left(1 - \tfrac{1}{2}\beta\cos\Omega t'\right) dt'$$

$$= w_0 \left(t - \frac{\Omega}{2}\beta\sin\Omega t\right)$$

$E = mgy = mgl\left(1 - \frac{\theta^2}{2}\right)$

since $\frac{E}{w}$ is constant

$E \sim mgl\phi^2$ and $w \propto \frac{1}{\sqrt{l}}$

$\frac{E}{w} \propto l^{3/2}\phi^2 = \text{const}$

$l^{3/4}\phi = \text{constant}.$

$$\phi(t) = \phi_0 \left(\frac{1+\beta}{1+\beta\cos\Omega t}\right)^{3/4} \left[\cos\left(w_0 t - \frac{w_0\Omega}{2}\beta\sin\Omega t\right)\right.$$

$$\cancel{\left. + q\sin\left(w_0 t - \frac{w_0\Omega}{2}\beta\sin\Omega t\right)\right]}$$

$$q = \frac{1}{w(0)}\left[\cancel{\frac{v_0}{x_0}} + \frac{\dot{w}(0)}{2w(0)}\right]$$

$$= \frac{1}{1-\frac{1}{2}\beta}\left[\frac{\frac{1}{2}\beta\sin\Omega t}{2(1-\frac{1}{2}\beta)}\right]\Bigg|_{t=0} = 0.$$

Amplitude at $t = 6$ hour : $\Omega t = \frac{\pi}{2}$

Amplitude $\boxed{A(\Omega t = \tfrac{\pi}{2}) = \varphi_0 \sqrt{1 - \tfrac{1}{2}B}}$

Phase:

$$\theta = \omega_0 t - \frac{\omega_0 \Omega}{2} B \sin \Omega t \Big|_{\Omega t = \frac{\pi}{2}} = \frac{\omega_0 \pi}{2} - \frac{\omega_0 \Omega}{2} B$$

$$\left(\frac{\omega^2(0)}{\omega^2(t)}\right)^{3/4} = \left(\frac{1+B}{1+B\cos\Omega t}\right)^{3/4}$$

2) Stellar orbits:

$$\Phi(r, z) = \frac{1}{2} v_0^2 \ln\left(r^2 + \frac{z^2}{q^2}\right)$$

gravitational potential.

a) $L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - m\Phi(r, z)$.

$\frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} = P_\phi$ is constant.

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$h = P_\phi \dot{\phi} + P_r \dot{r} + P_z \dot{z} - L$$

$$= \frac{P_\phi^2}{mr^2} + m\dot{r}^2 + m\dot{z}^2 - \left\{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - m\Phi(r,z)\right\}$$

$$E = \frac{P_\phi^2}{2mr^2} + \frac{m\dot{r}^2}{2} + \frac{m\dot{z}^2}{2} + m\Phi(r,z)$$

energy is constant

b) $R = P_\phi \dot{\phi} - L$

$$R = \frac{P_\phi^2}{2mr^2} + m\Phi(r,z) - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}m\dot{z}^2$$

$$L_{eff} = -R$$

$$\frac{\partial L_{eff}}{\partial r} = -\frac{\partial}{\partial r} V_{eff}(r)$$

$$= -\frac{\partial}{\partial r}\left(\frac{P_\phi^2}{2mr^2} + m\Phi(r,z)\right)$$

circular when $\frac{\partial V_{eff}}{\partial r} = 0$

∂r

$$\hookrightarrow \frac{-p_\phi^2}{mr^3} + m\frac{1}{2}v_0^2 \frac{2r}{r^2+\frac{z^2}{q^2}} = 0$$

$$\frac{1}{2}v_0^2 \frac{2rm}{r^2+\frac{z^2}{q^2}} = \frac{p_\phi^2}{mr^3}$$

$$v_0^2 m r^4 = p_\phi^2\left(r^2+\frac{z^2}{q^2}\right)$$

$$v_0^2 m r^4 - p_\phi^2 r^2 - \frac{p_\phi^2 z^2}{q^2} = 0$$

$$r^2 = \frac{p_\phi^2 \pm \sqrt{p_\phi^4 + \frac{4v_0^2 m p_\phi^2 z^2}{q^2}}}{2v_0^2 m^2}$$

$$r_c = \sqrt{\frac{p_\phi^2 + p_\phi^2\sqrt{1+\frac{4v_0^2 m z^2}{p_\phi^2 q^2}}}{2v_0^2 m^2}}$$ $\Leftarrow$ take plus sign to avoid negative r.

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$m\ddot{z} = -\frac{\partial}{\partial z} m\Phi$$

$2z/q^2$

$$= -\frac{1}{2} v_0^2 m \frac{\cdot}{r^2 + \frac{z^2}{q^2}} = 0.$$

let $\underline{z_c = 0}$

$$r_c\Big|_{z=z_c} = \sqrt{\frac{\cancel{2} P_\phi^2}{\cancel{2} V_0^2 m^2}} = \sqrt{\frac{P_\phi^2}{V_0^2 m^2}} = \frac{P_\phi}{V_0 m}$$

period $T = \frac{2\pi}{\dot{\phi}} = \frac{2\pi}{P_\phi} m r_c^2$

$$T = \frac{2\pi}{\cancel{P_\phi}} \cancel{m} \frac{P_\phi^{\cancel{2}}}{V_0^2 m^{\cancel{2}}} = \frac{2\pi P_\phi}{V_0^2 m}$$

c)

$$m\ddot{\delta r} = \cancel{-\frac{\partial}{\partial r} V_{eff}(r,z)\Big|_{r_c, z_c}} - \frac{\partial^2}{\partial r^2} V_{eff}\Big|_{r_c, z_c} \delta r$$

$$m\ddot{\delta r} = -\frac{\partial^2}{\partial r^2} V_{eff}\Big|_{r_c, z_c} \delta r$$

$$= -\frac{\partial}{\partial r}\left(\frac{-P_\phi^2}{m r^3} + \frac{1}{2} V_0^2 \frac{2rm}{r^2 + \frac{z^2}{q^2}} \right) \delta r$$

$$\overset{!}{=} -\left(\frac{3P_\phi^2}{mr^4} + \frac{1}{2}v_0^2 m \frac{2\left(r^2 + \frac{z^2}{q^2}\right) - 4r^2}{\left(r^2 + \frac{z^2}{q^2}\right)^2} \right)\Bigg|_{z_c, r_c} \delta r$$

$$\overset{!}{=} -\left(\frac{3P_\phi^2}{m\left(\frac{P_\phi^4}{v_0^4 m^4}\right)} + \frac{1}{2}v_0^2 m \frac{-2\frac{P_\phi^2}{v_0^2 m^2}}{\frac{P_\phi^4}{v_0^4 m^4}} \right) \delta r$$

$$\overset{!}{=} -\left(\frac{3v_0^4 m^3}{P_\phi^2} - v_0^2 m \frac{v_0^2 m^2}{P_\phi^2} \right) \delta r$$

$$m\ddot{\delta r} \overset{!}{=} -\left(\frac{2v_0^4 m^3}{P_\phi^2} \right) \delta r$$

$$\ddot{\delta r} + \underbrace{\frac{2v_0^4 m^2}{P_\phi^2}}\delta r$$

$$\boxed{\omega_r^2 = \frac{2v_0^4 m^2}{P_\phi^2}} \Rightarrow \boxed{\omega_r = \pm \frac{\sqrt{2}\, v_0^2 m}{P_\phi}}$$

$$m\ddot{\delta z} = -\frac{\partial^2}{\partial z^2} V_{eff}(r,z)\ \delta z$$

$$\overset{!}{=} -\frac{\partial}{\partial z}\left\{ \frac{1}{2}v_0^2 m \frac{2zq^2}{r^2 + \frac{z^2}{q^2}} \right\} \delta z$$

$$\stackrel{!}{=} -\left(\frac{1}{2}\frac{V_0^2}{q^2} m \frac{2\left(r^2+\frac{z^2}{q^2}\right) - 4z^2/q^2}{\left(r^2+\frac{z^2}{q^2}\right)^2}\right)\Bigg|_{r_c,z_c} \delta z$$

$$\stackrel{!}{=} -\frac{1}{2}\frac{V_0^2}{q^2} m \frac{2\cancel{r_c^2}}{r_c^{\cancel{4}\,2}}\, \delta z$$

$$\stackrel{!}{=} -\frac{1}{2}\frac{V_0^2}{q^2} m \frac{2}{\frac{P_\phi^2}{V_0^2 m^2}}\, \delta z$$

$$\stackrel{!}{=} -\frac{V_0^4 m^3}{P_\phi^2 q^2}\, \delta z$$

$$\cancel{m}\,\delta\ddot{z} + \frac{V_0^4 \cancel{m}^3}{P_\phi^2 q^2}\delta z = 0$$

$$\omega_z^2 = \frac{V_0^4 m^2}{P_\phi^2 q^2}$$

$$\boxed{\omega_z = \pm \frac{V_0^2 m}{P_\phi q}}$$

$$\Phi(r,z) = \frac{1}{2}v_0^2 \ln\left((r_{cir} + \delta r)^2 + \frac{\delta z^2}{q^2}\right)$$

$$\doteq \frac{1}{2}v_0^2 \ln\left(r_{circ}\left[\left(1 + 2\frac{\delta r}{r_{circ}}\right) + \frac{\delta z^2}{r_{circ}^2 q^2}\right]\right)$$

$$\doteq \frac{1}{2}v_0^2\left[\ln r_{circ} + \ln\left(1 + 2\frac{\delta r}{r_{circ}} + \frac{\delta z^2}{r_{circ}^2 q^2}\right)\right]$$

small when $\delta r \ll r_{circ}$

$\delta z \ll r_{circ}\, q$

d) $r_c = 2.5 \times 10^{17}$ km $= \frac{P_\phi}{v_0 m}$

$T = 225 \times 10^6$ years $= \frac{2\pi P_\phi}{v_0^2 m}$

$T_z = \frac{2\pi}{\omega_z} = 87 \times 10^6$ years $= \frac{2\pi P_\phi q}{v_0^2 m}$

$T_r = \frac{2\pi}{\omega_r} = 160 \times 10^6$ years $= \frac{2\pi P_\phi}{\sqrt{2} v_0^2 m}$

$\Phi = \frac{G M_{MW}}{r}$

$v_0 m r_c = P_\phi$ ①

$T \cancel{v_0^2} \cancel{m} = 2\pi \cancel{v_0} \cancel{m} r_c$

$\boxed{V_0 = \frac{2\pi r_c}{T}}$ ② ✓

$$T_z = \frac{2\pi\left(v_0 \cancel{m} r_c\right) q}{\left(\frac{2\pi r_c}{T}\right)^2 \cancel{m}}$$

$$= \frac{2\pi\left(\frac{2\pi r_c}{T}\right) r_c\, q}{\left(\frac{2\pi r_c}{T}\right)^2}$$

② to ①

$$\frac{2\pi r_c}{T} m r_c = P_\phi$$

$$\boxed{P_\phi = \frac{2\pi m r_c^2}{T}}$$

$$T_z = qT$$

$$\boxed{\frac{T_z}{T} = q}$$

③ ✓

$$T_r = \frac{2\pi v_0 m r_c}{T_z \left(\frac{2\pi r_c}{T}\right)^2 m} = \frac{2\pi\left(\frac{2\pi r_c}{T}\right)\cancel{m} r_c}{T_z\left(\frac{2\pi r_c}{T}\right)^2 \cancel{m}} = \frac{T}{T_z}$$

$$\nabla^2\Phi = -4\pi G\rho$$

$$\nabla^2\Phi = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Phi$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left[\frac{1}{2}v_0^2 \frac{2r}{r^2 + \frac{z^2}{q^2}}\right]\right)$$

$$+ \frac{1}{2}\frac{v_0^2}{q^2}\,\frac{2\left(r^2+\frac{z^2}{q^2}\right) - 4z^2/q^2}{\left(r^2+\frac{z^2}{q^2}\right)^2}\Bigg)$$

$$= \cancel{\frac{1}{r}}\,\frac{1}{2}v_0^2\,\frac{4\cancel{r}\left[\cancel{r^2}+z^2/q^2\right] - \cancel{4r^2}}{\left(r^2+z^2/q^2\right)^2}$$

$$+ \frac{1}{2}\,\frac{v_0^2}{q^2}\,\frac{2\left(r^2+z^2/q^2\right) - 4z^2/q^2}{\left(r^2+z^2/q^2\right)^2}$$

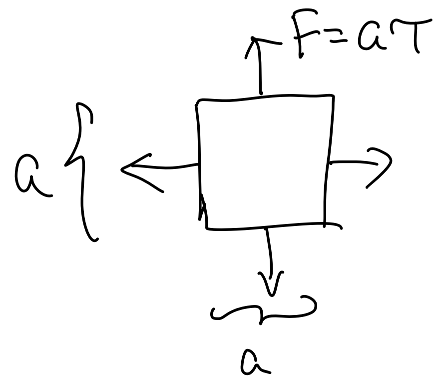
$$= \frac{v_0^2}{q^2}\,\frac{r^2+\left(2-1/q^2\right)z^2}{\left(r^2+z^2/q^2\right)^2}$$

use $v_0 = \left(\frac{2\pi r_c}{T}\right)$ $\qquad$ $q = \frac{T_z}{T} = \frac{T_z}{T_z T_r}$

$$= \frac{(2\pi)(2.5\times10^{17})\text{ km}}{225 \text{ million years}} \qquad = \frac{87 \text{ million years}}{225 \text{ million years}}$$

$$\rho = \frac{\nabla^2\Phi}{4\pi G} = \frac{v_0^2}{4\pi G q^2}\,\frac{r^2+\left(2-1/q^2\right)z^2}{\left(r^2+z^2/q^2\right)^2}$$

3) Waves vs. Oscillation:



τ = tension per unit length

μ = mass per area

a) Derive 2D wave equation describing small transverse displacement.

1D: $\mathcal{L} = \int dx \left[\frac{1}{2} \underbrace{\frac{m}{a}}_{\mu} (\partial_t q)^2 - \frac{1}{2} \Upsilon (\partial_x q)^2 \right]$ $\quad V_0 = \sqrt{\frac{\Upsilon}{\mu}} = \sqrt{\frac{T}{\mu}} = a\omega_0$

$\Upsilon \propto \gamma a \propto$ Tension $\qquad = a\sqrt{\frac{k}{m}}$

$$dT = \frac{1}{2} m \dot{z}^2 = \frac{1}{2} dx\,dy\, \mu (\partial_t z)^2$$

$dV = \sigma(x) \int dS$

$\frac{T}{L} = \tau$

surface tension

$\propto$ tension.

$T \propto ka$

$dS = \frac{dx\,dy}{\hat{n} \cdot \hat{z}}$

$V = \frac{1}{2} k (l - l_0)^2$

projection of

$T = k(l - l_0)$

$\hat{z} - \vec{\nabla} z$

element onto
x-y plane.

$$n - \frac{}{\sqrt{1+(\vec{\nabla} z)^2}}$$

$$k = \frac{I}{\partial t}.$$

$$\hat{n}\cdot\hat{z} = \frac{1}{\sqrt{1+(\vec{\nabla} z)^2}}$$

$$dS = dx\,dy\sqrt{1+(\vec{\nabla} z)^2} = dx\,dy\left(1+\frac{1}{2}(\vec{\nabla} z)^2\right)$$

$$dV = U_0 + \int dx\,dy\,\frac{1}{2}\tau(\vec{\nabla} z)^2$$

$$\mathcal{L} = \int dx\,dy\,\frac{1}{2}u(\partial_t z)^2 - \frac{1}{2}\tau(\vec{\nabla} z)^2$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \mathcal{L}}{\partial(\partial_a q)}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0.$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t q)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x q)} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial_y q)} = 0$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t q)} + \vec{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial(\vec{\nabla} q)} = 0$$

$$u\frac{\partial^2}{\partial t^2}z - \vec{\nabla}\cdot(\tau\vec{\nabla} z) = 0$$

$$\boxed{u\frac{\partial^2}{\partial t^2}z - \tau\nabla_{xy}^2 z = 0}$$

B) Boundary conditions:

$$z = 0 \text{ at } \begin{cases} 0 \le x \le a & \text{for } y = 0, a \\ 0 \le y \le a & \text{for } x = 0, a \end{cases}$$

$\rightarrow$ z on the frame

is 0.

$$u \frac{\partial^2}{\partial t^2} z = \tau \left(\frac{\partial^2}{\partial x^2} z + \frac{\partial^2}{\partial y^2} z \right)$$

let $z = \sum_{n,m} C_{n,m} \; X(x)_n Y(y)_m T(t)_{n,m}$

$$u X Y \frac{\partial^2}{\partial t} T = \left(Y T \frac{\partial^2}{\partial x^2} X + X T \frac{\partial^2}{\partial y^2} Y \right) \tau$$

$$\frac{u}{\tau} \frac{1}{T} \frac{\partial^2}{\partial t^2} T = \frac{1}{X} \frac{\partial^2}{\partial x^2} X + \frac{1}{Y} \frac{\partial^2}{\partial y^2} Y$$

Since left side only has time dependence and right side only has spatial dependence.

Then only way to make them equal is to let them equal to some constant.

$$\frac{1}{T} \frac{\partial^2}{\partial t^2} T = -\omega^2$$

$$\frac{\tau}{u} \left(\frac{1}{X} \frac{\partial^2}{\partial x^2} X + \frac{1}{Y} \frac{\partial^2}{\partial y^2} Y \right) = -\omega^2$$

let $T = A \cos \omega t + B \sin \omega t$

$\hookrightarrow \frac{\partial^2}{\partial t^2} T + \omega^2 T = 0$

∂t^2

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = -\omega^2\frac{\mu}{\tau}$$

let $X = C\cos(k_x x) + D\sin(k_y x)$

$Y = F\cos(k_x x) + G\sin(k_y y)$

$$-k_x^2 - k_y^2 = -\omega^2\frac{\mu}{\tau}$$

$$k_x^2 + k_y^2 = \omega^2\frac{\mu}{\tau}$$

Use boundary condition:

$$z=0 \text{ at } \begin{cases} 0 \le x \le a & \text{for } y=0,a \\ 0 \le y \le a & \text{for } x=0,a \end{cases}$$

$y=0, a.$ $\quad Y(y=0,a) = 0.$

$Y_m(y=0) = F \quad = 0$

$Y_m(y=a) = G\sin(k_y a) = 0$

when $k_y a = m\pi$ for $m = 0, 1, 2 \cdots$

$k_y = \frac{m\pi}{a}$

likewise: $X(x=a) = D\sin(k_x a) = 0$

$$k_x = \frac{n\pi}{a}$$

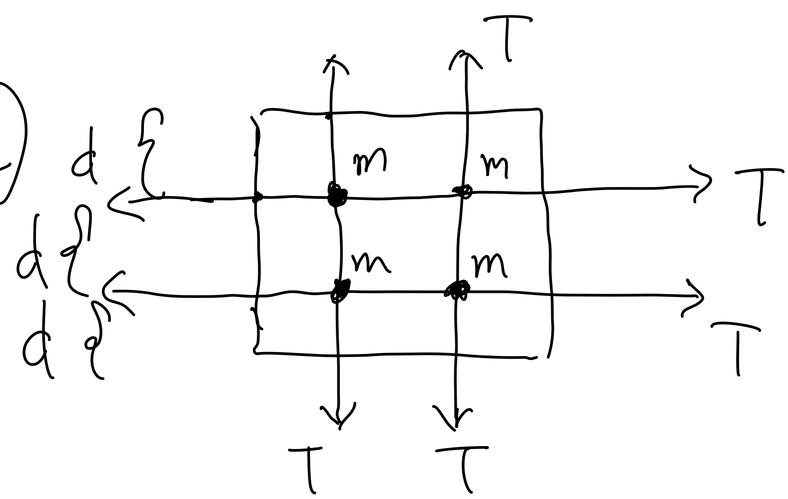
$$k_x^2 + k_y^2 = \omega^2 \frac{\mu}{\tau}$$

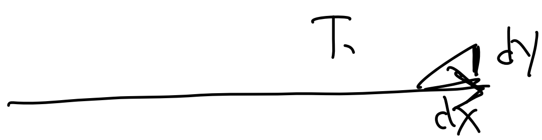
$$(n^2+m^2)\frac{\pi^2}{a^2} = \omega^2 \frac{\mu}{\tau}$$

$$\boxed{\omega^2 = (n^2+m^2)\frac{\tau}{\mu}\frac{\pi^2}{a^2}}$$ ← Frequency of oscillation.

$$z_{n,m} = XYT$$
$$= \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\left[A\cos\omega t + B\sin\omega t\right]$$

c)





$$dV = \int dx\,dy\,\frac{1}{2}T(\vec{\nabla}z)^2$$

$$V = \frac{1}{2}\frac{T}{d}d^2\left[\left(\frac{z_{i,j}-z_{i-1,j}}{d}\right)^2 + \left(\frac{z_{i,j}-z_{i,j-1}}{d}\right)^2 + \left(\frac{z_{i+1,j}-z_{i,j}}{d}\right)^2 + \left(\frac{z_{i,j+1}-z_{i,j}}{d}\right)^2\right]$$

$$\begin{aligned} V = \frac{1}{2}\frac{T}{d}\Big[& z_{i,j}^2 - 2z_{i,j}z_{i-1,j} + z_{i-1,j}^2 \\ & + z_{i,j}^2 - 2z_{i,j}z_{i,j-1} + z_{i,j-1}^2 \\ & + z_{i,j}^2 - 2z_{i,j}z_{i+1,j} + z_{i+1,j}^2 \\ & + z_{i,j}^2 - 2z_{i,j}z_{i,j+1} + z_{i,j+1}^2 \Big] \end{aligned}$$

$$T = \frac{1}{2}m\dot{z}_{i,j}^2$$

$$\begin{aligned} m\ddot{z}_{i,j} = -\frac{1}{2}\frac{T}{d}\Big[& 8z_{i,j} - 2z_{i-1,j} - 2z_{i,j-1} \\ & - 2z_{i+1,j} - 2z_{i,j+1}\Big] \end{aligned}$$

$$\boxed{\begin{aligned} m\ddot{z}_{i,j} = \frac{T}{d}\Big[& z_{i-1,j} + z_{i,j-1} + z_{i+1,j} + z_{i,j+1} \\ & -4z_{i,j}\Big] \end{aligned}}$$

D) Find eigenfrequency and eigenmodes

$$m\underbrace{\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}}_{M}\begin{pmatrix}z_{0,0}\\z_{0,1}\\z_{1,0}\\z_{1,1}\end{pmatrix}, \quad z=\begin{pmatrix}z_{0,0}\\z_{0,1}\\z_{1,0}\\z_{1,1}\end{pmatrix}$$

Symmetry:

$$\vec{Q}_0 = q_0\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

CoM motion

Reflection over x-direction:

$z_{0,0}$	$z_{0,1}$	|	$z_{0,1}$	$z_{0,0}$
$z_{1,0}$	$z_{1,1}$	|	$z_{1,1}$	$z_{1,0}$

$$\vec{z}=\begin{pmatrix}z_{0,0}\\z_{0,1}\\z_{1,0}\\z_{1,1}\end{pmatrix} \qquad \bar{z}=\begin{pmatrix}-z_{0,1}\\-z_{0,0}\\-z_{1,1}\\-z_{1,0}\end{pmatrix}$$

odd

$$-z_{0,0}=-z_{0,1}$$
$$-z_{1,0}=-z_{1,1}$$

odd when $z=-\bar{z}$ even when $z=\bar{z}$

Even:
$$z_{0,0}=-z_{0,1}$$
$$z_{1,0}=-z_{1,1}$$

$$Q_{x,o} = \begin{pmatrix} q_{o1} \\ q_{o1} \\ q_{o2} \\ q_{o2} \end{pmatrix} \qquad Q_{x,e} = \begin{pmatrix} q_{e1} \\ -q_{e1} \\ q_{e2} \\ -q_{e2} \end{pmatrix}$$

Reflection over y-direction:

$$\vec{z} = \begin{pmatrix} z_{0,0} \\ z_{0,1} \\ z_{1,0} \\ z_{1,1} \end{pmatrix} \qquad \bar{z} = \begin{pmatrix} -z_{1,0} \\ -z_{1,1} \\ -z_{0,0} \\ -z_{0,1} \end{pmatrix}$$

Odd: $-z_{0,0} = -z_{1,0}$
$-z_{0,1} = z_{1,1}$

even $z_{0,0} = -z_{1,0}$
$z_{0,1} = -z_{1,1}$

$$Q_{y,o} = \begin{pmatrix} q_{o,1} \\ q_{o,2} \\ q_{o,1} \\ q_{o,2} \end{pmatrix} \qquad Q_{y,e} = \begin{pmatrix} q_{e,1} \\ q_{e,2} \\ -q_{e1} \\ -q_{e2} \end{pmatrix}$$

Do orthogonality:

$$(q_{o1} \; q_{o1} \; q_{o2} \; q_{o2}) \, M \, (1 \; 1 \; 1 \; 1)^T = 0$$

let $q_{o1} = 1 \quad q_{o2} = 1.$

$$2q_{o1} + 2q_{o2} = 0$$

$$q_{01} = -q_{02}$$

$$Q_{0,x} = q_{0x}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$$

$$(q_{0,1} \quad q_{02} \quad q_{0,1} \quad q_{02})\, M\, (1 \quad 1 \quad 1 \quad 1)^T$$

$$\Leftrightarrow = 2q_{0,1} + 2q_{0,2} = 0$$

$$q_{01} = -q_{0,2}$$

$$Q_{0,y} = q_{0y}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

By symmetry?

$$q_0\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\ q_1\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},\ q_2\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\ q_3\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}z_{0,0}\\z_{1,0}\\z_{0,1}\\z_{1,1}\end{pmatrix} = q_0\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + q_1\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} + q_2\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} + q_3\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$$T = \frac{1}{2} m \left(\dot{z}_{0,0}^{2} + \dot{z}_{1,0}^{2} + \dot{z}_{0,1}^{2} + \dot{z}_{1,1}^{2} \right)$$

$$= \frac{1}{2} m \left[(\dot{q}_0 + \dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 + (\dot{q}_0 + \dot{q}_1 - \dot{q}_2 - \dot{q}_3)^2 + (\dot{q}_0 - \dot{q}_1 + \dot{q}_2 - \dot{q}_3)^2 + (\dot{q}_0 - \dot{q}_1 - \dot{q}_2 + \dot{q}_3)^2 \right]$$

$$= \frac{1}{2} m \left[4\dot{q}_0^{2} + 4\dot{q}_1^{2} + 4\dot{q}_2^{2} + 4\dot{q}_3^{2} \right]$$

$$= 2m \left(\dot{q}_0^{2} + \dot{q}_1^{2} + \dot{q}_2^{2} + \dot{q}_3^{2} \right)$$

$$V = \frac{1}{2} \frac{T}{d} d^2 \left[\left(\frac{z_{i,j} - z_{i-1,j}}{d} \right)^2 + \left(\frac{z_{i,j} - z_{i,j-1}}{d} \right)^2 + \left(\frac{z_{i+1,j} - z_{i,j}}{d} \right)^2 + \left(\frac{z_{i,j+1} - z_{i,j}}{d} \right)^2 \right]$$

$$V = \frac{1}{2} \frac{T}{d} \left[2(z_{1,1})^2 + (z_{1,1} - z_{0,1})^2 + (z_{1,1} - z_{1,0})^2 + (z_{1,0} - z_{0,0})^2 + 2(z_{1,0})^2 + (z_{1,1} - z_{1,0})^2 + 2(z_{0,1})^2 + (z_{1,1} - z_{0,1})^2 + (z_{0,1} - z_{0,0})^2 + 2(z_{0,0})^2 + (z_{1,0} - z_{0,0})^2 + (z_{0,1} - z_{0,0})^2 \right]$$

$$= \frac{1}{d}\Big[(z_{1,1})^2 + (z_{0,1})^2 + (z_{1,0})^2 + (z_{0,0})^2$$
$$+ \frac{1}{2}(z_{1,1} - z_{0,1})^2 + \frac{1}{2}(z_{1,0} - z_{1,0})^2 + \frac{1}{2}(z_{0,1} - z_{0,0})^2$$
$$+ (z_{1,0} - z_{0,0})^2 \frac{1}{2}\Big]$$

$$= \frac{T}{d}\Big[(q_0+q_1+q_2+q_3)^2 + (q_0+q_1-q_2-q_3)^2 + (q_0-q_1+q_2-q_3)^2$$
$$+ (q_0-q_1-q_2+q_3)^2$$
$$+ (-2q_2+2q_3)^2 + (-2q_1+2q_3)^2 + (-2q_1-2q_3)^2$$
$$+ (-2q_2-2q_3)^2\Big]$$

$$= \frac{T}{d}\Big[4(q_0^2+q_1^2+q_2^2+q_3^2) + 2q_2^2 + 2q_3^2 + 2q_1^2 + 2q_3^2$$
$$+ 2q_1^2 + 2q_3^2 + 2q_2^2 + 2q_3^2\Big]$$

$$= \frac{T}{d}\Big[4q_0^2 + 8q_1^2 + 8q_2^2 + 12q_3^2\Big]$$

$$4m\ddot{q}_0 = -\frac{T}{d} 8 q_0$$

$$\boxed{\omega_0^2 = 2\frac{T}{md}}$$

$$4m\ddot{q}_1 = -16\frac{T}{d} q_1 \qquad 4m\ddot{q}_2 = 16\frac{T}{d} q_2$$

$$\boxed{\omega_1^2 = 4\frac{T}{md}} \qquad \boxed{\omega_2^2 = 4\frac{T}{md}}$$

$$4m\ddot{q}_3 = -24\frac{T}{md}q_3$$

$$\boxed{\omega_3^2 = 6\frac{T}{md}}$$

E) Membrane

$$\boxed{\omega^2 = (n^2+m^2)\frac{T}{u}\frac{\pi^2}{a^2}}$$

$n=1,\ m=1$

$$\omega^2 = (n^2+m^2)\frac{\frac{T}{d}}{\frac{m}{d^2}}\frac{\pi^2}{(3d)^2}$$

$$\omega_{1,1}^2 = 2\frac{T}{md}\frac{\pi^2}{3^2}$$

$$\omega_{2,1}^2 = \omega_{1,2}^2 = 5\frac{T}{md}\left(\frac{\pi}{3}\right)^2$$

$$\omega_{2,2}^2 = 8\frac{T}{md}\left(\frac{\pi}{3}\right)^2$$