Nonlinear Oscillations:

suppose: $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 - \frac{1}{4}Bmx^4$

then EOM:

$$\ddot{x} + \omega_0^2 x + Bx^3 = 0$$

$$\ddot{x} + \omega_0^2 x = -Bx^3 = \frac{f_{induced}}{m}$$

treat it as some perturbation. if B is "small"

Naive Perturbation Theory: Doesn't work.

Zeroth order solution, solution that get rid of perturbation:

$$x^{(0)} = A\cos(-\omega_0 t + \phi)$$

let $x = A$ to get order of magnitude:

suppose: $BA^3 \ll \omega_0^2 A$

so $\frac{BA^2}{\omega_0^2} \ll 1$

Try $x(t) = x^{(0)} + x^{(1)}$

$$\ddot{x} + \omega_0^2 x = -Bx^3$$

$$\hookrightarrow \underbrace{\ddot{x}^{(0)} + \omega_0^2 x^{(0)}}_{=0} + \underbrace{\ddot{x}^{(1)} + \omega_0^2 x^{(1)}}_{} = -B(x^{(0)} + x^{(1)})^3$$

$$= -B(x^{(0)2} + 2x^{(0)}x^{(1)} + x^{(1)2})(x^{(0)} + x^{(1)})$$

$= 0$ by zeroth order. First order terms. $- B(x^{(0)3} + \ldots)$

$\stackrel{\approx}{=} -Bx^{(0)3}$

Since B is already first order only keep $x^{(0)}$ terms.

$$\hookrightarrow \quad \ddot{x}^{(1)} + w_0^2 x^{(1)} = -Bx^{(0)3}$$

Now $x^{(0)3} = [A\cos(w_0t + \phi)]^3 = A^3\left(\frac{e^{-iw_0t} + e^{iw_0t}}{2}\right)^3$

$$\stackrel{!}{=} \frac{A^3}{8}\left(e^{-3iw_0t} + 3e^{-2iw_0t}e^{iw_0t} + 3e^{2iw_0t}e^{-iw_0t} + e^{3iw_0t}\right)$$

$$\stackrel{!}{=} \frac{A^3}{4}\left(\cos(3w_0t) + 3\cos(-w_0t)\right)$$

$$\hookrightarrow \quad \ddot{x}^{(1)} + w_0^2 x^{(1)} = \frac{-BA^3}{4}\left(\cos(3w_0t) + 3\cos(-w_0t)\right)$$

off resonance   on resonance

Previously: $\ddot{x} + w_0^2 x = \frac{F_0}{m}\cos wt$ had sol $x = \frac{F_0/m}{-w^2 + w_0^2}\cos wt$

so for $\frac{f}{m} = \frac{-BA^3}{4}\cos(3w_0t)$

off-resonance:

$$X(t) = \frac{\frac{-BA^3}{4}}{-(3w_0)^2 + w_0^2}\cos(3w_0t)$$

$$\stackrel{!}{=} \frac{BA^2}{w_0^2}\frac{A}{32}\cos(3w_0t)$$

Note $\frac{BA^2}{\quad} \ll 1$

$\omega_0^2 \cdots \perp$

On - resonance: previously: $X(t) = \frac{f_{ind}/m}{2\omega_0^2} \omega_0 t \sin \omega_0 t$

then: $X^{(1)}(t) = \frac{-\beta A^2}{\omega_0^2} \frac{3A}{8} \omega_0 t \sin \omega_0 t$

grows linearly in time

so it is secular divergence.

Secular Perturbation Theory:

$$\ddot{X} + \omega_0^2 X = -\beta x^3$$

Try a sol: $X(t) = x^{(0)} + x^{(1)}$

let
$$\boxed{x^{(0)} = a(t)\cos(-\omega_0 t + \phi(t)), \quad \phi(t) = -\Delta\omega t}$$

slow, so $a(t)$ and $\phi(t)$ also function of time.

meaning their time derivative is first order.

then

$$\dot{x}^{(0)} = \dot{a}\cos(-\omega_0 t + \phi) + a\frac{d}{dt}\cos(-\omega_0 t + \phi)$$
$$\stackrel{!}{=} \dot{a}\cos(-\omega_0 t + \phi) - a\sin(-\omega_0 t + \phi)(-\omega_0 + \dot{\phi})$$

$$\ddot{x}^{(0)} = \ddot{a}\cancel{\cos(-\omega_0 t + \phi)} - \dot{a}\sin(-\omega_0 t + \phi)(-\omega_0 + \cancel{\dot{\phi}})$$
$$- \dot{a}\sin(-\omega_0 t + \phi)(-\omega_0 + \cancel{\dot{\phi}}) - a\Big(\cos(-\omega_0 t + \phi)(-\omega_0 + \dot{\phi})^2$$
$$+ \cancel{\sin(-\omega_0 t + \phi)}\,\ddot{\phi}\Big)$$

|

$$\stackrel{!}{=} 2\dot{a}\omega_0 \sin(-\omega_0 t + \phi) - a\cos(-\omega_0 t + \phi)[\omega_0^2 - 2\omega_0 \dot{\phi}]$$

$$\ddot{x}^{(0)} \stackrel{!}{=} -\omega_0^2 a\cos(-\omega_0 t+\phi) + 2\dot{a}\omega_0 \sin(-\omega_0 t+\phi) + 2a\omega_0 \dot{\phi}\cos(-\omega_0 t+\phi)$$

Then $\ddot{x}^{(0)} + \omega_0^2 x^{(0)}$

$$\hookrightarrow = \cancel{-\omega_0^2 a\cos(-\omega_0 t+\phi)} + 2\dot{a}\omega_0 \sin(-\omega_0 t+\phi) + 2a\omega_0 \dot{\phi}\cos(-\omega_0 t+\phi)$$
$$+ \cancel{\omega_0^2 a\cos(-\omega_0 t+\phi}$$

$$\stackrel{!}{=} \underbrace{2\dot{a}\omega_0 \sin(-\omega_0 t+\phi) + 2a\omega_0 \dot{\phi}\cos(-\omega_0 t+\phi)}_{\text{First order corrections that get carried over.}}$$

Remember we had :

$$\ddot{x}^{(0)} + \omega_0^2 x^{(0)} + \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -B(x^{(0)} + x^{(1)})^3 = -B x^{(0)3}$$

$$\hookrightarrow \cancel{-\omega_0^2 a\cos(-\omega_0 t+\phi)} + 2\dot{a}\omega_0 \sin(-\omega_0 t+\phi) + 2a\omega_0 \dot{\phi}\cos(-\omega_0 t+\phi)$$
$$+ \cancel{\omega_0^2 a\cos(-\omega_0 t+\phi} + \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = \frac{-Ba^3}{4}\cos(3(-\omega_0 t+\phi)) - \frac{3}{4}Ba^3\cos(-\omega_0 t+\phi)$$

$$\hookrightarrow \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = \frac{-Ba^3}{4}\cos(3(-\omega_0 t+\phi)) - \frac{3}{4}Ba^3\cos(-\omega_0 t+\phi)$$
$$-2\dot{a}\omega_0 \sin(-\omega_0 t+\phi) - 2a\omega_0 \dot{\phi}\cos(-\omega_0 t+\phi)$$

$$\hookrightarrow \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\frac{Ba^3}{4}\cos(3(-\omega_0 t+\phi)) - \left(\frac{3}{4}Ba^3 + 2a\omega_0\dot{\phi}\right)\cos(-\omega_0 t+\phi)$$

$-2\dot{a} w_0 \sin(-w_0 t+\phi)$

We want to get rid of resonance terms, $\cos(-w_0 t+\phi)$ and $\sin(-w_0 t+\phi)$

So adjust $a(t)$ and $\phi(t)$ to satisfy these conditions

① $-2\dot{a}\, w_0 = 0 \Rightarrow$ i.e. $a =$ constant

② $\frac{3}{4}Ba^3 + 2aw_0\dot{\phi} = 0 \Rightarrow \dot{\phi} = -\frac{3}{8}\frac{Ba^2}{w_0}$

or $\phi = \underbrace{\frac{-3}{8}\frac{Ba^2}{w_0}}_{\Delta w} t + \phi_0$

then $x^{(0)} = a\cos\left(-w_0 t + \frac{-3}{8}\frac{Ba^2}{w_0} t + \phi_0\right)$

then $\ddot{x}^{(1)} + w_0^2 x^{(1)} = \frac{-Ba^3}{4}\cos\left(3\left(-w_0 t + \frac{-3}{8}\frac{Ba^2}{w_0} t + \phi_0\right)\right)$

$\stackrel{!}{=} \frac{-Ba^3}{4}\cos(-3(1+\Delta)w_0 t + \phi_0)$

for $\Delta = \frac{3}{8}\frac{Ba^2}{w_0^2} w_0$

$x^{(1)} = A\cos(-3(w_0+\Delta w)t + \phi_0)$

$\dot{x}^{(1)} = A\cos(-3(w_0+\Delta w)t + \phi_0)\,(-3(w_0+\Delta w))$

$\ddot{x}^{(1)} = A\sin(-3(w_0+\Delta w)t + \phi_0)\,(3(w_0+\Delta w))$

$\approx A\cos(-3(w_0+\Delta w)t + \phi_0)(-9w_0^2) + O(\Delta w)$

$$\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = \frac{-Ba^3}{4}\cos\left(-3\left[(\omega_0+\Delta\omega)t + \phi_0\right]\right)$$

$$-9\omega_0^2 A\cos\left(\cancel{-3(\omega_0+\Delta\omega)t+\phi_0}\right) + \omega_0^2 A\cos\left(\cancel{-3[(\omega_0+\Delta\omega)t+\phi_0]}\right)$$

$$= \frac{-Ba^3}{4}\cos\left(\cancel{-3[(\omega_0+\Delta\omega t)+\phi_0]}\right)$$

$$-8\omega_0^2 A = \frac{-Ba^3}{4}$$

$$A = \frac{Ba^3}{32\omega_0^2}$$

Then $\boxed{x^{(1)} = \frac{Ba^3}{32\omega_0^2}\cos\left(-3\left[(\omega_0+\Delta\omega)t + \phi_0\right]\right)}$

Then $x(t) = x^{(0)} + x^{(1)}$

$$\doteq a\cos\left(-(\omega_0+\Delta\omega)t + \phi_0\right)$$

$$+ \frac{Ba^2}{\omega_0^2}\frac{a}{32}\cos\left(-3\left[(\omega_0+\Delta\omega)t + \phi_0\right]\right)$$

for $\Delta\omega = \frac{3}{8}\frac{Ba^2}{\omega_0^2}\omega_0$

Example 2: Damped SHO:

$$\ddot{x} + \eta\dot{x} + \omega_0^2 x = 0$$

For small η, we had solution:

$$x(t) = a_0 e^{-\frac{\eta}{2}t} \cos(-\omega_0' t + \phi_0) \quad \text{for } \omega_0' = \sqrt{\omega_0^2 - \left(\frac{\eta}{2}\right)^2}$$

$$\stackrel{!}{=} a_0 e^{-\frac{\eta}{2}t} \cos(-\omega_0 t + \phi_0) \quad \text{assume } \frac{\eta}{2} \ll \omega_0$$

Now use perturbation Theory to solve.

$$\ddot{x} + \omega_0^2 x = -\eta \dot{x}$$

let $\quad x(t) = x^{(0)} + x^{(1)}$

and $\quad x^{(0)} = a(t)\cos(-\omega_0 t + \phi(t))$

$$\dot{x}^{(0)} = \dot{a}\cos(-\omega_0 t + \phi) + a \frac{d}{dt}\cos(-\omega_0 t + \phi)$$

$$\stackrel{!}{=} \dot{a}\cos(-\omega_0 t + \phi) - a\sin(-\omega_0 t + \phi)(-\omega_0 + \dot{\phi})$$

$$\ddot{x}^{(0)} = -\omega_0^2 a\cos(-\omega_0 t + \phi) + 2\dot{a}\,\omega_0 \sin(-\omega_0 t + \phi) + 2a\omega_0 \dot{\phi}\cos(-\omega_0 t + \phi) + \mathcal{O}$$

$$\hookrightarrow \quad \ddot{x}^{(0)} + \omega_0^2 x^{(0)} + \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\eta a \omega_0 \sin(-\omega_0 t + \phi)$$

$$\hookrightarrow \quad \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\eta a\omega_0 \sin(-\omega_0 t + \phi) - 2\dot{a}\omega_0 \sin(-\omega_0 t + \phi) - 2a\omega_0 \dot{\phi}\cos(-\omega_0 t + \phi)$$

$\hookrightarrow$ Require secular terms, $\cos(-\omega_0 t + \phi)$ and $\sin(-\omega_0 t + \phi)$ go zero.

$$\Rightarrow \quad -a\omega_0 \dot{\phi}\cos(-\omega_0 t + \phi) = 0$$

so $\phi = \text{const} = \phi_0$

$$\Rightarrow \left(-\eta a\omega_0 - 2\dot{a}\omega_0\right)\sin(-\omega_0 t + \phi) = 0$$

$$-\eta a\cancel{\omega_0} = 2\dot{a}\cancel{\omega_0}$$

$$\frac{da}{a} = -\frac{\eta}{2}dt$$

$$\ln a = \frac{-\eta}{2}t + \text{const.}$$

$$a = a_0 e^{-\frac{\eta}{2}t}$$

then $x^{(0)} = a_0 e^{-\frac{\eta}{2}t}\cos(-\omega_0 t + \phi_0)$

$$\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = 0$$