Euler Angles :

ZXZ rotation

$\left\{ \begin{array}{l} ① \text{ Rotate by } \phi \text{ around } z, \ R_1(\phi) \\ ② \text{ Rotate by } \theta \text{ around the new x-axis}, \ R_2(\theta) \\ ③ \text{ Rotate by } \psi \text{ around new z-axis}, \ R_3(\psi) \end{array} \right.$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Then :

$$\hat{\omega}_{ab} = (\dot{R} R^{-1})_{ab} = \begin{pmatrix} -\sin\theta\,\dot{\theta} & \cos\theta\,\dot{\theta} \\ -\cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= (\dot{R} R^{T})_{ab} = \begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix}$$

For 3D :

General Rotation :

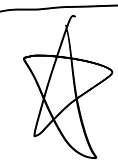
$$R(\phi, \theta, \psi) = R_3^{z}(\psi)\, R_2^{x}(\theta)\, R_1^{z}(\phi)$$

$$R^{-1}(\phi, \theta, \psi) = R_1^{z(-1)}(\phi)\, R_2^{x(-1)}(\theta)\, R_3^{z(-1)}(\psi)$$

$$R = R_3^z(\psi)\, R_2^x(\theta)\, R_1^z(\phi) + R_3^z(\psi)\, R_2^x(\theta)\, R_1(\phi) + R_3^z(\psi)\, R_2(\theta)\, R_1(\phi)$$

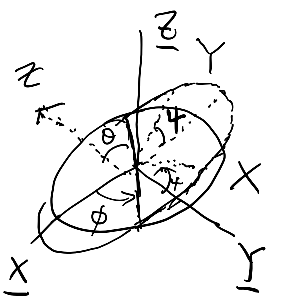
Then $\omega = \left(\dot{R} R^{-1}\right)$ In Fixed Frame:

$$\omega^X = \dot\psi \sin\theta \sin\phi + \dot\theta \cos\phi$$
$$\omega^Y = -\dot\psi \sin\theta \cos\phi + \dot\theta \sin\phi$$
$$\omega^Z = \dot\psi \cos\theta + \dot\phi$$



Rotated Frame:

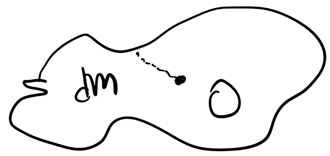
$$\omega^x = \omega_{yz} = \dot\phi \sin\theta \sin\phi + \dot\theta \cos\psi$$
$$\omega^y = \omega_{zx} = \dot\phi \sin\theta \cos\phi - \dot\theta \sin\psi$$
$$\omega^z = \omega_{xy} = \dot\phi \cos\theta + \dot\psi$$



θ: Final tilt angle from Z to z
$\dot\psi$: Final spinning rate of z
$\dot\phi$: precession rate, spinning rate of Z

Kinetic Energy in 3D:

O is fixed



$$T = \sum_i \frac{1}{2} dm_i \left(\frac{d\vec{r}_i}{dt}\right)^2 = \int \frac{1}{2} dm\, \dot{\vec{r}}^2$$

then $\dot{\vec{r}}_i = \dfrac{d\vec{r}_i}{dt} = \vec{\omega} \times \vec{r}_i + \cancel{\left(\dfrac{d\vec{r}}{dt}\right)_r} \; 0$ since dealing with rigid body its not deforming

$$T = \frac{1}{2}\int dm \left(\vec{\omega} \times \vec{r}\right)\cdot\left(\vec{\omega} \times \vec{r}\right)$$
$$= \frac{1}{2}\int dm \left(\omega r \sin\theta\right)^2$$
$$= \frac{1}{2}\int dm\, \omega^2 r^2 \left(1 - \cos^2\theta\right)$$
$$= \frac{1}{2}\int dm \left\{\omega^2 r^2 - \left(\vec{\omega}\cdot\vec{r}\right)^2\right\}$$
$$= \frac{1}{2}\int dm\, \omega_a \omega_b r^2 \delta_{ab} - \omega_a r_a \omega_b r_b$$

$r^2 = r_a r_b \delta_{ab}$
$\vec{\omega}\cdot\vec{r} = \omega_a r_a$
$\omega^2 = \omega_a \omega_b \delta_{ab}$

$$= \frac{1}{2}\int dm\, \omega_a \left(r^2\delta_{ab} - r_a r_b\right)\omega_b$$

$$☆ \quad \boxed{T = \frac{1}{2}\,\omega_a\omega_b \int dm \left(r^2\delta_{ab} - r_a r_b\right)}$$

we can pull these terms out since orientation is independent of where you are in the body. Or, ω doesn't depend on $\vec{r}$.

I_{ab}, moment of inertia tensor. for fixed frame.

$$\boxed{\begin{aligned} I_{ab} &= \int d^3r\, \rho(r)\left(r^2\delta_{ab} - r_a r_b\right) \\ &= \int dm \left(r^2\delta_{ab} - r_a r_b\right) \end{aligned}}$$

← Moment of inertia tensor for fixed origin.

Ex :

$$I = \int d^3r\, \rho(r) \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} x^2+y^2+z^2-x^2 & -xy & -xz \\ -yx & x^2+y^2+z^2-y^2 & -yz \\ -zx & -zy & x^2+y^2+z^2-z^2 \end{pmatrix} \end{array} \begin{array}{c} x \\ y \\ z \end{array}$$

$$= \int d^3r\, \rho(r) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

* Here I_{xx}, I_{yy}, and I_{zz}, are moment of inertia of x-, y-, and z-axis

For non-fixed origin, use center of mass: $\vec{R} = \frac{\sum m_a r_a}{\sum m_a}$

$\Delta\vec{r}_i = \vec{r}_i - \vec{R}$

$$\vec{r}_i = \vec{R}(t) + \Delta\vec{r}_i$$

$O \quad \vec{R}(t)$

$$\frac{d\vec{r}_i}{dt} = \frac{d\vec{R}(t)}{dt} + \frac{d\Delta\vec{r}_i}{dt}$$
$$= \vec{V}_{cm} + \vec{\omega} \times \Delta\vec{r}_i$$

$$KE = \frac{1}{2}\sum m_i \dot{\vec{r}}_i^{\,2}$$
$$= \frac{1}{2}\sum m_i \left\{ v_{cm}^2 + \underbrace{\cancel{2\,\vec{v}_{cm}\cdot(\vec{\omega}\times\Delta\vec{r}_i)}}_{\text{due to center of mass}}^{\,0} + (\vec{\omega}\times\Delta\vec{r}_i)^2 \right\}$$

$$\sum m_i \Delta\vec{r}_i = 0$$

$$= \frac{1}{2}\sum m_i v_{cm}^2 + \frac{1}{2}\sum m_i (\vec{\omega}\times\Delta\vec{r}_i)^2$$
$$= \frac{1}{2}\sum m_i V_{cm}^2 + \frac{1}{2}\sum m_i \omega^2 \Delta r_i^2 \sin^2\theta$$
$$= \frac{1}{2} M_{tot} V_{cm}^2 + \frac{1}{2}\sum m_i \underbrace{\omega^2 \Delta r_i^2 (1-\cos^2\theta)}$$

$$\omega_a \omega_b \delta_{ab} \Delta r_i^2 - \omega_a r_a \omega_b r_b$$
$$= \omega_a (\Delta r_i^2 \delta_{ab} - r_a r_b)\,\omega_b =$$

☆

$$\boxed{KE = \frac{1}{2} M_{tot} V_{cm}^2 + \frac{1}{2}\,\omega_a I_{ab}^{cm}\,\omega_b}$$

← For moving object around center of mass

Translational KE — Rotational KE

Angular Momentum:

$$L = \sum m_i \vec{r}_i \times \dot{\vec{r}}_i$$
$$= \sum m_i \vec{r}_i \times (\vec{\omega}\times\vec{r}_i)$$

use $A\times(B\times C) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$L = \sum m_i \left\{ \vec{w}(\vec{r_i}\cdot\vec{r_i}) - \vec{r_i}(\vec{r_i}\cdot\vec{w}) \right\}$$

$$\stackrel{!}{=} \sum m_i \left\{ w_b\, r_i^2 \delta_{ab} - r_a r_b w_b \right\}$$

$$\stackrel{!}{=} \underbrace{\sum m_i \left(r_i^2 \delta_{ab} - r_a r_b \right)}_{I_{ab}} w_b$$

$$\boxed{L_a \stackrel{!}{=} I_{ab} w_b}$$

$\leftarrow$ Angular momentum around some fixed coordinate.

<u>For CoM: or moving coordinate.</u>

$$\vec{r_i} = \vec{R}(t) + \vec{\Delta r_i}$$

$$\dot{\vec{r_i}} = \dot{\vec{R}}(t) + \vec{w}\times\vec{\Delta r_i}$$

$$L = \sum m_i \vec{r_i}\times\dot{\vec{r_i}}$$

$$\stackrel{!}{=} \sum m_i (\vec{R} + \vec{\Delta r_i}) \times (\dot{\vec{R}} + \vec{w}\times\vec{\Delta r_i})$$

$$\stackrel{!}{=} \sum m_i \left(\vec{R}\times\dot{\vec{R}} + \underbrace{\vec{R}\times(\vec{w}\times\vec{\Delta r_i})}_{=0} + \underbrace{\vec{\Delta r_i}\times\dot{\vec{R}}}_{=0} + \vec{\Delta r_i}\times(\vec{w}\times\vec{\Delta r_i}) \right)$$

since $\sum m_i \vec{\Delta r_i} = 0$

$$\boxed{L_a \stackrel{!}{=} L_{cm,a} + I^{cm}_{ab} w_b}$$

$\uparrow$ spin $\quad \uparrow$ orbital

L of cm. $\quad L$ around cm

Summary:

For fixed origin (fixed object):

$$I_{ab} = \int d^3r \, \rho(\vec{r}) \left(r^2 \delta_{ab} - r_a r_b\right)$$

$$T = \frac{1}{2} \omega_a I_{ab} \omega_b$$

$$L_a = I_{ab} \omega_b$$

For moving origin (use center of mass):

$$T = \frac{1}{2} M_{tot} V_{cm}^2 + \frac{1}{2} \omega_a I_{ab}^{cm} \omega_b$$

$$L_a = \underbrace{M_{tot} \vec{R} \times \vec{V}_{cm}}_{L_{cm}} + I_{ab} \omega_b$$

Since I_{ab} is a matrix, find some axis orientation or find principal axes such that I_{ab} is diagonalized, or non-diagonal elements go zero.

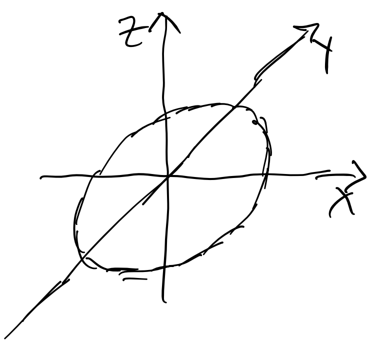
$$I' = R I R^T$$

then

$$I' = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

In principal axis

Ex: Here we see that off-diagonal are all zero.



$$I_{xy} = M\langle xy\rangle = M \frac{\int d^3r\, \rho(r)\, xy}{M} = 0$$

$$I_{xz} = 0$$

$$I_{xx} = \int d^3r\, \rho(\vec{r})\left(y^2 + \cancel{z^2}\right)$$

$$\stackrel{!}{=} M\langle y^2\rangle$$

$$I_{yy} = M\langle x^2\rangle$$

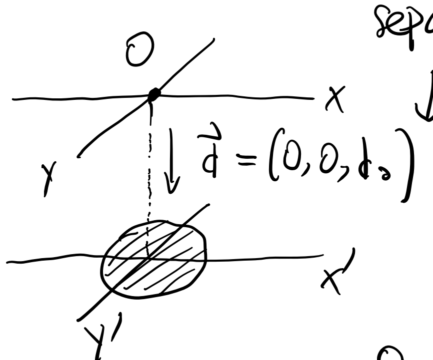
$$I_{zz} = M\langle x^2+y^2\rangle = \int \rho(\vec{r})\, d^3r \left(x^2+y^2\right)$$

$$\stackrel{!}{=} \int r\, d\theta\, dr \frac{M}{\pi r^2} r^2$$

$$\stackrel{!}{=} \frac{2\pi M}{\pi} \frac{R^2}{2}$$

Parallel-Axis Theorem:

ex:



separated by vector $\vec{d}$, from O

distance away from cm and O

$$I^{O}_{ab} = I^{cm}_{ab} + M\left(d^2\delta_{ab} - d_a d_b\right)$$

$$I^{O}_{ab} = \begin{pmatrix} \frac{1}{4}MR^2 & & \\ & \frac{1}{4}MR^2 & \\ & & \ddots \end{pmatrix} + M\begin{pmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \end{pmatrix}$$

$$\frac{1}{2}MR^2 \Big) \quad \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{Parallel Axis Theorem:} \\ \qquad I^{O}_{ab} = I^{cm}_{ab} + M\left(d^2\delta_{ab} - d_a d_b\right) \\ \text{where } \vec{d} \text{ is separation vector between O and cm} \end{array}}$$

Then suppose $\vec{\omega} = \omega(0, 1, 0)$

$$T = \frac{1}{2}\omega_y I^{O}_{yy}\omega_y$$

$$= \frac{1}{2}\omega^2\left(I^{cm}_{ab} + M\left(d^2\delta_{ab} - d_a d_b\right)\right)$$

$$= \frac{1}{2}\omega^2\left(\frac{1}{4}MR^2 + Md_0^2\right)$$

$$= \underbrace{\frac{1}{8}M\omega^2R^2}_{\text{KE about cm}} + \underbrace{\frac{1}{2}M\omega^2 d_0^2}_{\frac{1}{2}MV_{cm}^2} \quad , V_{cm} = \omega d_0$$

$$(L_O)_a = \vec{d} \times (m\vec{v}_{cm}) + I^{cm}_{ab}\omega_b$$

$$= \vec{d} \times (\vec{\omega} \times \vec{d})\, m + I^{cm}_{ab}\omega_b$$

$$= \left\{\vec{\omega}(\vec{d}\cdot\vec{d}) - \cancel{\vec{d}(\vec{d}\cdot\vec{\omega})}\right\} m + I^{cm}_{ab}\omega_b$$

$$= \left(\omega_b d^2\delta_{ab} - d_a d_b\omega_b\right)m + I^{cm}_{ab}\omega_b$$

$$= \omega_b\left(d^2\delta_{ab} - d_a d_b\right)m + I^{cm}_{ab}\omega_b$$

$$= I^{O}$$

$$\doteq I_{ab}\,\omega_b$$

Euler Eqns:

$$\underset{\text{In fixed frame}}{\frac{d\vec{L}}{dt}} = \underbrace{\left(\frac{d\vec{L}}{dt}\right)_r}_{\text{rotating frame.}} + \vec{\omega}\times\vec{L}$$

For $\vec{L} = L_a \vec{e}_a(t)$

$$I_{ab}\dot{\omega}_b = \left(\frac{dL}{dt}\right)_a = \left(\frac{dL_a}{dt}\right)_r \vec{e}_a + \epsilon_{abc}\,\omega^b L^c \vec{e}_a = \underbrace{\cancel{(\vec{\tau}_{ext})_a}}_{\text{0 for free motion}}$$

$$I_a\dot{\omega}_a = -\epsilon_{abc}\,\omega^b L^c \vec{e}_a$$

$$\boxed{\left.\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) \end{aligned}\right\} \text{Euler Eqns.}}$$

For plate: $I_x = I_y = \frac{1}{4}MR^2 \quad I_z = \frac{1}{2}MR^2$

then

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) = 0 \text{ for } I_1 = I_2$$

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$$

$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) = \omega_3\omega_1(I_3 - I_2)$$

then

$$\dot{\omega}_2 = \frac{\omega_3 (I_3 - I_2)}{I_2}\, \omega_1 = \Omega \omega_1$$

$$\dot{\omega}_1 = \frac{\omega_3 (I_2 - I_3)}{I_2}\, \omega_2 = -\Omega \omega_2$$

Solve, get: $(\omega_1, \omega_2) = \omega_T(-\sin\Omega t, \cos\Omega t)$

$\omega_3 = \text{constant}$

$$\vec{L} = I_1 \omega_1 \vec{e}_1 + I_2 \omega_2 \vec{e}_2 + I_3 \omega_3 \vec{e}_3$$

$$\stackrel{!}{=} I_1 (\omega_1 \vec{e}_1 + \omega_2 \vec{e}_2) + I_3 \omega_3 \vec{e}_3$$

$$\stackrel{!}{=} I_1 \vec{\omega}_T + I_3 \vec{\omega}_3$$

$I_1 = I_2$

$I_3 = 2 I_1$

$$\frac{\vec{L}}{I_3} = \frac{1}{2} \vec{\omega}_T + \vec{\omega}_3$$

know $\vec{\omega}_3$ is constant, $\vec{L}$ is constant

$\vec{L} \cdot \vec{\omega}_3 = L \omega_3 \cos\theta = \text{constant}$

then θ is constant.

$$\omega_x = \dot{\phi} \sin\theta \sin\psi + \cancel{\dot{\theta} \cos\psi} \quad = -\omega_T \sin\Omega t$$

$$w_y = \dot\phi \sin\theta \cos\psi - \cancel{\dot\theta} \sin\psi = w_T \cos\Omega t$$

$$w_z = \dot\phi \cos\theta + \dot\psi = \text{constant}$$

By matching, $\psi = -\Omega t \Rightarrow \dot\psi = -\Omega$

precession rate in body frame

$w_z =$ constant

$$\dot\phi \cos\theta + \dot\psi = \dot\phi\cos\theta - \Omega = w_3$$

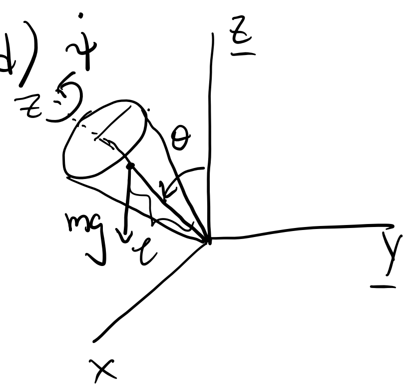
$$\boxed{\dot\phi = \frac{w_3 + \Omega}{\cos\theta}}$$

← precession rate in lab frame.

For $\theta \ll 1$

$$\dot\phi \cong w_3 + \Omega$$

Symmetric Top Problem: (base is fixed)



Meaning:

$$I^0 = \begin{pmatrix} I_{11}^0 & 0 & 0 \\ 0 & I_{22}^0 & 0 \\ 0 & 0 & I_{33}^0 \end{pmatrix}$$

Moment of inertia w.r.t. base point, O.

Lagrangian:

$$L = \frac{1}{2} w_a I_{ab} w_b$$

$$\stackrel{!}{=} \frac{1}{2} w_1^2 I_1 + \frac{1}{2} w_2^2 I_2 + \frac{1}{2} w_3^2 I_3$$

symmetry in $\vec{e}_1$ and $\vec{e}_2$, so $I_1 = I_2$

$$\stackrel{!}{=} \frac{1}{2} I_1 \left(w_1^2 + w_2^2\right) + \frac{1}{2} w_3^2 I_3$$

Rotating Frame.

$$w^x = w_{yz} = \dot\phi \sin\theta \sin\phi + \dot\theta \cos\psi$$

$$w^y = w_{zx} = \dot\phi \sin\theta \cos\phi - \dot\theta \sin\psi$$

$$w^z = w_{xy} = \dot\phi \cos\theta + \dot\psi$$

$$\mathcal{L} \stackrel{!}{=} \frac{1}{2} I_1 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot\phi \cos\theta + \dot\psi\right)^2 - mgl\cos\theta$$

↑ l : distance to center of mass

$$P_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = I_3\left(\dot\phi\cos\theta + \dot\psi\right) = \text{constant}$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = I_1 \dot\phi \sin^2\theta + \underbrace{I_3\left(\dot\phi\cos\theta + \dot\psi\right)}_{P_\psi} \cos\theta = \text{constant}$$

↳ $\dot\phi = \dfrac{P_\phi - P_\psi \cos\theta}{I_1 \sin^2\theta}$ ⇒ precession rate around $\underline{z}$

↳ $\dot\psi = \dfrac{P_\psi}{I_3} - \dot\phi\cos\theta$ ⇒ spinning rate around z

$$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = I_1 \dot\theta$$

Inorder to get $\dot\phi$, need θ.

Use $E = h = P(\dot q)\dot q - \mathcal{L}$ since $\mathcal{L}$ independent of t, energy conservation

$$\stackrel{!}{=} I_1 \dot\theta^2 + I_3 \dot\psi\left(\dot\phi\cos\theta + \dot\psi\right) + I_1 \dot\phi^2 \sin^2\theta + I_3 \dot\phi\left(\dot\phi\cos\theta + \dot\psi\right)\cos\theta$$

$$- \left\{ \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 - mgl\cos\theta \right\}$$

$$E = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 + mgl\cos\theta$$

Express E in integral of motion (constants):

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \sin^2\theta \left(\frac{P_\phi - P_\psi \cos\theta}{I_1 \sin^2\theta} \right)^2 + \frac{1}{2} \frac{P_\psi^2}{I_3} + mgl\cos\theta$$

$$= \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \frac{(P_\phi - P_\psi \cos\theta)^2}{I_1 \sin^2\theta} + \underbrace{\frac{1}{2} \frac{P_\psi^2}{I_3}}_{\text{constant term}} + mgl\cos\theta$$

$$\doteq \frac{1}{2} I_1 \dot{\theta}^2 + \underbrace{\frac{(P_\phi - P_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + mgl\cos\theta}_{U_{eff}(\theta)}$$

Routhian: $L_{eff} = \frac{1}{2} I_1 \dot{\theta}^2 - U_{eff}(\theta) = -R$

$$\frac{d}{dt} \frac{\partial R}{\partial q} = \frac{\partial R}{\partial q} \qquad \frac{d}{dt} \frac{\partial L}{\partial q} = \frac{\partial L}{\partial q}$$

$$\hookrightarrow \quad I_1 \ddot{\theta} = -\frac{\partial}{\partial \theta} U_{eff}(\theta)$$

At fixed angle θ, $\frac{\partial}{\partial\theta} U_{eff}(\theta) = 0$, so U_{eff} is at minimum.
(if initial condition is at θ_{min})

$$\frac{\partial}{\partial\theta} U_{eff} = \frac{\partial}{\partial\theta} \left\{ \frac{(P_\phi - P_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + mgl\cos\theta \right\} = 0$$

define $\bar{g} = \dfrac{mgl}{P_\psi^2/I_1}$ and $B = \dfrac{P_\phi}{P_\psi}$

$$\frac{U_{eff}}{P_\psi^2/I_1} = \bar{g}\cos\theta + \frac{(B-\cos\theta)^2}{2\sin^2\theta}$$

For $\bar{g} \ll 1$, zeroth approximation:

$$\frac{U_{eff}}{P_\psi^2/I_1} = \frac{(B-\cos\theta)^2}{2\sin^2\theta}$$

$$\frac{\partial}{\partial\theta}\left(\frac{U_{eff}}{P_\psi^2/I_1}\right) = \frac{4(B-\cos\theta)\sin^2\theta - (B-\cos\theta)^2\, 2\cos\theta}{4\sin^2\theta}$$

$$\stackrel{!}{=} \frac{(B-\cos\theta)\{4\sin^2\theta - 2B\cos\theta + 2\cos^2\theta\}}{4\sin^4\theta}$$

at min : $B - \cos\theta = 0$ or $B = \cos\theta_{min}$

$$B = \frac{P_\phi}{P_\psi} = \cos\theta_{min}$$

$$P_\phi = P_\psi \cos\theta_{min}$$

and

$$\dot{\phi} = \frac{P_\phi - P_\psi\cos\theta}{I_1\sin^2\theta}$$

$$\stackrel{!}{=} \frac{P_\psi\cos\theta - P_\psi\cos\theta}{I_1\sin^2\theta} = 0.$$

Therefore need first order:

$$\underset{\text{0th order ans}}{\cos\theta_{min} = \underset{\uparrow}{B}} + \underset{\text{first order answer}}{\underset{\nwarrow}{\delta}}$$

Find δ by plug into $U_{eff}(\theta)$:

$$\frac{U_{eff}}{P_\psi^2/I_1} = \bar{g}(B+\delta) + \frac{\{B-(B+\delta)\}^2}{2(1-(B+\delta)^2)}$$

$$\stackrel{!}{=} \bar{g}(B+\delta) + \frac{\delta^2}{2(1-B^2)}$$

$$\frac{\partial U_{eff}}{\partial \delta} = \bar{g} + \frac{\delta}{1-B^2} = 0$$

$$\Rightarrow \delta = (1-B^2)(-\bar{g})$$

then plug into $\dot\phi$

$$\dot\phi = \frac{P_\phi - P_\psi \cos\theta_{min}}{I_1 \sin^2\theta} = \frac{P_\psi}{I_1}\left(\frac{B-\cos\theta}{\sin^2\theta}\right)$$

$$\stackrel{!}{=} \frac{P_\psi}{I_1}\left(\frac{B-(B+\delta)}{1-(B+\delta)^2}\right)$$

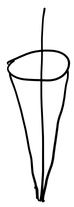
$$\stackrel{!}{=} \frac{P_\psi}{I_1}\frac{-\delta}{1-B^2}$$

$$\stackrel{!}{=} \frac{P_\psi}{I_1}\frac{(1-B^2)\bar{g}}{1-B^2}$$

$$\dot\phi \stackrel{!}{=} \frac{P_\psi \bar{g}}{I_1} = \frac{P_\psi mgl}{I_1 P_\psi^2/I_1}$$

$$\dot\phi = \frac{mgl}{P_\psi}$$

Sleeping Top:



$$U_{eff}(\theta) = mgl\cos\theta + \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta}$$

If $\theta = 0$, then the only way to have minimum potential is to have

$$P_\phi = P_\psi$$

then $$U_{eff}(\theta) = mgl\cos\theta + \frac{P_\phi}{2I_1}\frac{(1-\cos\theta)^2}{\sin^2\theta}$$

then we expand around $\theta = 0$

$$U_{eff}(\theta) = mgl\left(1 - \frac{\theta^2}{2}\right) + \frac{P_\phi^2}{2I_1}\frac{\left\{1 - \left(1 - \frac{\theta^2}{2}\right)\right\}^2}{\theta^2}$$

$$\doteq \underbrace{mgl}_{\text{const}} - mgl\frac{\theta^2}{2} + \frac{P_\phi^2}{2I_1}\frac{\theta^2}{4}$$

$$\doteq \text{const} + \left(\frac{P_\phi^2}{8I_1} - \frac{mgl}{2}\right)\theta^2$$

In order for stability, $U_{eff}(\theta)$ must be concave up near $\theta = 0$

or $$\left(\frac{P_\phi^2}{8I_1} - \frac{mgl}{2}\right) > 0$$

so need $P_\phi > \sqrt{16 mgl I_1}$