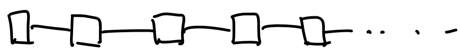
Waves : ∞ oscillators



Find normal modes + freq:

$$\mathcal{L} = \sum_j \frac{1}{2} m \dot{q}_j^2 - \frac{1}{2}\gamma(q_{j+1} - q_j)^2$$

EOM:

$$m\ddot{q}_j = \gamma(q_{j+1} - q_j) - \gamma(q_j - q_{j-1})$$

$$m\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} -2\gamma & \gamma & 0 & 0 & 0 \\ \gamma & -2\gamma & \gamma & 0 & 0 \\ 0 & \gamma & -2\gamma & \gamma & \\ 0 & 0 & \gamma & -2\gamma & \gamma \\ & & & & \ddots \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \end{pmatrix}$$

let $q_j = AE_j e^{-i\omega t}$

with eigenvector $E_j = e^{ikx_j}$

$$\boxed{q_j = Ae^{ikx_j - i\omega t}}$$

$$m\ddot{q}_j = \gamma(q_{j+1} - q_j) - \gamma(q_j - q_{j-1})$$

$$-\omega^2 m Ae^{ikx_j - i\omega t} = \gamma Ae^{-i\omega t}\left\{e^{ikx_{j+1}} - e^{ikx_j} + e^{ikx_{j-1}} - e^{ikx_j}\right\}$$

$$-\omega^2 m \cancel{Ae^{ikx_j - i\omega t}} = \gamma \cancel{Ae^{-i\omega t + ikx_j}} \left\{ e^{ika} + e^{-ika} - 2 \right\}$$

$$-\omega^2 m = 2\gamma\left(\cos ka - 1\right)$$

$$\omega^2 = \frac{\gamma}{m}\left(2 - 2\cos ka\right)$$

$$\omega^2(k) = \omega_0^2 \, 4\sin^2\left(\frac{ka}{2}\right)$$ $\leftarrow$ k is the kth eigenmode.

$$\omega_\pm = \pm\omega_0 \, 2\sin\left(\frac{ka}{2}\right)$$

In general:

$$q(t) = \sum_k C_k E_j^k e^{-i\omega(k)t} + \sum_k D_k E_j^k e^{i\omega(k)t}$$

Ex:

$$q_j = Ae^{ikx - i\omega t} + Ae^{-ikx + i\omega t} = A\cos(kx - \omega t)$$