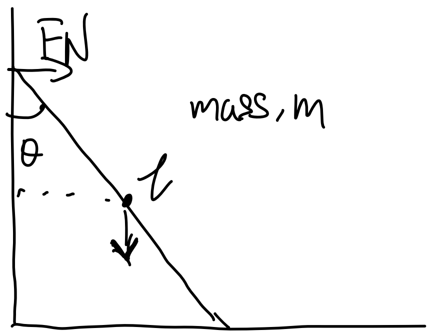
1) Thin rod sliding:



a) gravity makes it accelerate downward, there is a normal force from the wall pushing rod to the right

b) Constraint:

$x_{cm} - \frac{l}{2}\sin\theta = 0$

$y_{cm} = \frac{l}{2}\cos\theta$

$\dot{y}_{cm} = \frac{l}{2}\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2}m\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right) + \frac{1}{2}I\dot{\theta}^2 + \lambda\left(x_{cm} - \frac{l}{2}\sin\theta\right)$$

$$I = \int dm\, r^2 = \int_{-l/2}^{l/2} \frac{m}{l} r^2 dr = \frac{m}{l}\,\frac{\left(\frac{l}{2}\right)^3}{3} - \frac{\left(-\frac{l}{2}\right)^3}{3} = ml^2\frac{1}{12}$$

$$T = \frac{1}{2}m\dot{x}_{cm}^2 + \frac{1}{2}m\left(\frac{l}{2}\right)^2\sin^2\theta\,\dot{\theta}^2 + \frac{1}{2}ml^2\frac{1}{12}\dot{\theta}^2$$

$$V = mgy = +mg\frac{l}{2}\cos\theta$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}_{cm}^2 + \left(\frac{1}{8}ml^2\sin^2\theta + \frac{1}{24}ml^2\right)\dot{\theta}^2 - \frac{mgl}{2}\cos\theta$$
$$+\lambda\left(x_{cm} - \frac{l}{2}\sin\theta\right) = 0$$

$$x_{cm} = \frac{l}{2}\sin\theta$$

$$m\ddot{x}_{cm} = \lambda$$

$$\frac{\partial L}{\partial \dot{\theta}} = \left(\frac{1}{4}\sin^2\theta + \frac{1}{12}\right) ml^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\frac{1}{4}\sin^2\theta + \frac{1}{12}\right) ml^2 \ddot{\theta} + \frac{1}{2}\sin\theta\cos\theta\, ml^2 \dot{\theta}^2$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{4} ml^2 \sin\theta\cos\theta\,\dot{\theta}^2 + \frac{mgl}{2}\sin\theta - \lambda\frac{l}{2}\cos\theta$$

$$\left(\frac{1}{4}\sin^2\theta + \frac{1}{12}\right) ml^2 \ddot{\theta} + \frac{1}{2}\sin\theta\cos\theta\, ml^2 \dot{\theta}^2 = \frac{1}{4} ml^2 \sin\theta\cos\theta\,\dot{\theta}^2 + \frac{mgl}{2}\sin\theta - \lambda\frac{l}{2}\cos\theta$$

$$\left(\frac{1}{4}\sin^2\theta + \frac{1}{12}\right) ml^2 \ddot{\theta} = -\frac{1}{4} ml^2 \sin\theta\cos\theta\,\dot{\theta}^2 + \frac{mgl}{2}\sin\theta - \lambda\frac{l}{2}\cos\theta \quad ①$$

$$m\ddot{x}_{cm} = \lambda \quad ②$$

$$x_{cm} = \frac{l}{2}\sin\theta \quad ③$$

$$\dot{x}_{cm} = \frac{l}{2}\cos\theta\,\dot{\theta}$$

$$m\ddot{x}_{cm} = m\frac{l}{2}\left[\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right] = \lambda$$

$\lambda = 0$ when fall off.

$$\ddot{\theta} = \frac{\sin\theta}{\cos\theta}\dot{\theta}^2$$

$$\left(\frac{1}{4}\sin^2\theta + \frac{1}{12}\right) m l^2 \ddot{\theta} = -\frac{1}{4} m l^2 \sin\theta\cos\theta\,\dot{\theta}^2 + \frac{mgl}{2}\sin\theta - \frac{ml^2}{4}\left(\cos^2\theta\,\ddot{\theta} - \cos\theta\sin\theta\right)$$

$$\left(\frac{1}{4} + \frac{1}{12}\right) m l^2 \ddot{\theta} = \frac{mgl}{2}\sin\theta$$

$$\left(\frac{4}{12}\right) m l^2 \ddot{\theta} = \frac{mgl}{2}\sin\theta$$

$$\boxed{\ddot{\theta} = \frac{3}{2}\frac{g}{l}\sin\theta}$$

$$\frac{ml}{2}\left(\frac{3}{2}\frac{g}{l}\sin\theta\cos\theta - \sin\theta\,\dot{\theta}^2\right) = \lambda$$

$$\frac{ml}{2}\sin\theta\left(\frac{3}{2}\frac{g}{l}\cos\theta - \dot{\theta}^2\right) = \lambda$$

Initially, $\theta = 0$, held upright.

$$E = \frac{mgl}{2}\cos(\theta = 0) = \frac{mgl}{2}$$

$$E = \left(\frac{1}{8} + \frac{1}{24}\right) m l^2 \dot{\theta}^2 + \frac{mgl}{2}\cos\theta = \frac{mgl}{2}$$

$$\frac{mgl}{2}(1 - \cos\theta) = \frac{1}{6} m l^2 \dot{\theta}^2$$

$$\dot{\theta}^2 = 3g(1 - \cos)$$

$\theta \qquad \frac{\quad}{l}(1-\cos\theta)$

$$\frac{ml}{2}\sin\theta\left(\frac{3}{2}\frac{g}{l}\cos\theta-\frac{3g}{l}(1-\cos\theta)\right)=\lambda$$

$$\frac{ml}{2}\sin\theta\,\frac{3g}{l}\underbrace{\left(\frac{3}{2}\cos\theta-1\right)}=\lambda$$

$$1=\frac{3}{2}\cos\theta$$

$$\boxed{\theta=\cos^{-1}\left(\frac{2}{3}\right)\cong 48.2^\circ}$$

V_x is max because bar leaves wall when normal force, $\lambda=0$.

Which means $m\ddot{x}_{cm}=0$

So velocity, $\dot{x}_{cm}$, is at max

c) $\dot{x}_{cm}=\frac{l}{2}\cos\theta\,\dot{\theta}$

$$\stackrel{!}{=}\frac{l}{2}\cos\theta\sqrt{\frac{3g}{l}(1-\cos\theta)}$$

For $\theta=\theta_c=\cos^{-1}(2/3)$

$$\dot{x}_{cm}=\frac{l}{2}\left(\frac{2}{3}\right)\sqrt{\frac{3g}{l}(1-2/3)}$$

$$\dot{x}_{cm} = \frac{l}{3}\sqrt{\frac{g}{l}}$$

d) What is the rod's com velocity when it hits the floor.

Find θ and $\dot{\theta}$ when it hits floor. $\theta_{final} = \frac{\pi}{2}$

$$\mathcal{L} = \frac{1}{2}m\dot{x}_{cm}^2 + \left(\frac{1}{8}ml^2\sin^2\theta + \frac{1}{24}ml^2\right)\dot{\theta}^2 - \frac{mgl}{2}\cos\theta$$
$$+ \lambda\left(x_{cm} - \frac{l}{2}\sin\theta\right) = 0$$

$\lambda = 0$ for rod off the wall.
$\dot{\theta} = 0$ when it hits the floor.

Need initial energy after rod off the wall. or is it $\frac{mgl}{2}$

$$\dot{x}_{cm,init} = \frac{l}{3}\sqrt{g/l} = \text{constant} = V_{x,end}.$$

$$\dot{\theta} = \sqrt{\frac{3g}{l}(1-\cos\theta)}\,\Bigg|_{\theta = \cos^{-1}(2/3)}$$

$$\dot{\theta}_{init} = \sqrt{g/l}$$

$$\dot{y}_{cm} = \frac{l}{2}\sin\theta\,\dot{\theta}$$

$$= \frac{l}{2}\sqrt{g/l}\ \sin\left(\cos^{-1}(2/3)\right)$$

$$= \frac{1}{2}\sqrt{g/l}\sqrt{\left(1-\left(2/3\right)^2\right)}$$

$$\overset{!}{=} \frac{1}{2}\sqrt{g/l}\sqrt{5/9}$$

$$E = \frac{1}{2}m\dot{x}_{cm}^2 + \left(\frac{1}{8}ml^2\sin^2\theta + \frac{1}{24}ml^2\right)\dot{\theta}^2 + \frac{mgl}{2}\cos\theta$$

$$\overset{!}{=} \frac{1}{2}m\left(\frac{l^2}{9}g/l\right) + \left(\frac{1}{8}ml^2(1-\cos^2\theta) + \frac{1}{24}ml^2\right)g/l$$

$$+ \frac{mgl}{2}\frac{2}{3}$$

$$\overset{!}{=} \frac{1}{18}mgl + \left(\frac{1}{8}ml^2\left(1-\frac{4}{9}\right) + \frac{1}{24}ml^2\right)g/l + \frac{mgl}{3}$$

$$\overset{!}{=} \left(\frac{1}{18} + \frac{5}{72} + \frac{1}{24} + \frac{1}{3}\right)mgl$$

$$\overset{!}{=} \left(\frac{1}{18} + \frac{5}{72} + \frac{3}{72} + \frac{24}{72}\right)mgl$$

$$\frac{mgl}{2} \overset{!}{=} \frac{1}{18}mgl + \underbrace{\frac{32}{72}mgl}_{\text{energy of } \frac{1}{2}m\dot{y}_{cm}^2 + \frac{1}{2}I\dot{\theta}^2}$$

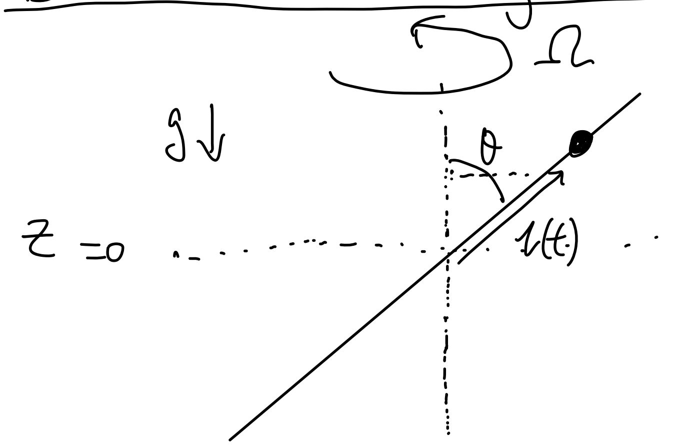
when rod reaches floor, $\dot{\theta} = 0$, $\theta = \frac{\pi}{2}$

$$\frac{1}{2} m \dot{y}_{cm}^2 = \frac{4}{9} mgl$$

$$\dot{y}_{cm} = \sqrt{\frac{8}{9} gl}$$

$$\boxed{(v_x, v_y) = \left(\frac{1}{3}\sqrt{gl}, \frac{2\sqrt{2}}{3}\sqrt{gl}\right)}$$

2) Bead on a rotating wire:



rod forced to rotate with constant Ω

a) Find Lagrangian and EoM:

$$x = l(t)\sin\theta\cos\phi \qquad \dot{x} = \dot{l}\sin\theta\cos\phi - l\sin\theta\sin\phi\,\dot{\phi}$$
$$y = l(t)\sin\theta\sin\phi \qquad \dot{y} = \dot{l}\sin\theta\sin\phi + l\sin\theta\cos\phi\,\dot{\phi}$$
$$z = l(t)\cos\theta \qquad \dot{z} = \dot{l}\cos\theta$$

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$$= \frac{1}{2} m \Big([\dot{l}\sin\theta\cos\phi - l\sin\theta\sin\phi\,\dot{\phi}] + [\dot{l}\sin\theta\sin\phi + l\sin\theta\cos\phi\,\dot{\phi}]$$

$$+ \dot{l}^2\cos^2\theta \Big) - mgl\cos\theta$$

$$= \frac{1}{2} m \left(\dot{l}^2 + l^2\sin^2\theta\,\dot{\phi}^2 \right) - mg\cos\theta\, l(t)$$

$$\frac{\partial L}{\partial \dot{\phi}} = P_\phi = m l^2 \sin^2\theta\,\dot{\phi} = \text{constant}$$

$$\dot{\phi} = \Omega$$

$$R = P_\phi \dot{\phi} - L$$

$$= \frac{P_\phi^2}{m l^2 \sin^2\theta} - \frac{1}{2} m \dot{l}^2 - \frac{1}{2} m l^2 \sin^2\theta \left[\frac{P_\phi}{m l^2 \sin^2\theta} \right]^2$$

$$+ mgl\cos\theta$$

$$= -\frac{1}{2} m \dot{l}^2 + \frac{P_\phi^2}{2 m l^2 \sin^2\theta} + mgl\cos\theta$$

$$L_{eff} = -R$$

$$= \frac{1}{2} m \dot{l}^2 - V_{eff}(l)$$

$$= \frac{1}{2} m \dot{l}^2 - \left(\frac{P_\phi^2}{2 m l^2 \sin^2\theta} + mgl\cos\theta \right)$$

$$\frac{\partial L}{\partial \dot{l}} = m\dot{l}$$

$$\frac{\partial L}{\partial l} = -\frac{\partial}{\partial l}\left(\frac{P_\phi^2}{2ml^2\sin^2\theta} + mgl\cos\theta\right)$$

$$= -\left(\frac{-P_\phi^2}{ml^3\sin^2\theta} + mg\cos\theta\right)$$

$$\boxed{m\ddot{l} = \frac{P_\phi^2}{ml^3\sin^2\theta} - mg\cos\theta}$$

b) For equilibrium: $m\ddot{l} = 0$

$$\frac{P_\phi^2}{ml^3\sin^2\theta} = mg\cos\theta$$

$$P_\phi^2 = (ml^2\sin^2\theta\,\Omega)^2$$

$$\frac{(ml^2\sin^2\theta\,\Omega)^2}{ml^3\sin^2\theta} = mg\cos\theta$$

$$\frac{\cancel{m^2}l^4\sin^{\cancel{4}2}\theta\,\Omega^2}{\cancel{m}l^3\cancel{\sin^2\theta}} = \cancel{m}g\cos\theta$$

$$\boxed{l_0 = \frac{g\cos\theta}{\sin^2\theta\,\Omega^2}}$$

l_0 is where $F = m\vec{a} = 0$, or no acceleration in $\hat{l}$

c) assume $l(t) = l_0 + \eta(t)$

$$V_{eff}(l) = \frac{P_\phi^2}{2ml^2\sin^2\theta} + mgl\cos\theta$$

$$V_{eff}(l_0+\eta) = V_{eff}(l_0) + \frac{\partial V_{eff}}{\partial l}\eta$$

$$m\ddot{l} = -\frac{\partial}{\partial l}\left(V_{eff}(l=l_0) + \frac{\partial}{\partial l}V_{eff}\Big|_{l=l_0}\eta\right)$$

$$\ddot{l} = \ddot{\eta}$$

$$m\ddot{\eta} = \cancel{-\frac{\partial}{\partial l}V_{eff}\Big|_{l=l_0}} - \frac{\partial^2}{\partial l^2}V_{eff}\Big|_{l=l_0}\eta$$

$$\frac{\partial}{\partial l}V_{eff} = \frac{-P_\phi^2}{ml^3\sin^2\theta} + mg\cos\theta$$

$$\frac{\partial^2}{\partial l^2}V_{eff} = \frac{P_\phi^2}{3ml^4\sin^2\theta} = \frac{P_\phi^2}{3m\sin^2\theta}\frac{1}{l^4} > 0$$

Here we see $\frac{\partial^2}{\partial l^2}V_{eff} > 0$

is always positive so
it is stable around equilibrium

$$m\ddot{\eta} = \frac{-P_\phi^2}{3m\sin^2\theta}\frac{1}{l_0^4}\eta$$

$$\ddot{\eta} + \underbrace{\frac{P_\phi^2}{3m^2\sin^2\theta}\frac{1}{l_0^4}}_{\omega^2}\eta = 0$$

$$\eta = A\cos(\omega t + \phi)$$

If $\dot{l} = \dot{\eta}(t=0) = 0 = -A\omega\sin(\phi)$
then let $\phi = 0$.

$$\eta = A\cos\omega t$$

With $\eta(t=0) = \eta_0 = A$

$$\eta(t) = \eta_0\cos\omega t$$

for $\omega = \frac{1}{\sqrt{3}} \frac{P_\phi}{m \sin\theta\, l_0^2}$

d) $\tau = \vec{r} \times \vec{F} = \frac{d}{dt} L$

W done by torque

$W = \tau \theta$

Consider a general solution $l(t)$,

i) Find external torque so $\dot{\phi} = \Omega$

$$L = \frac{1}{2} m \left(\dot{l}^2 + l^2 \sin^2\theta\, \dot{\phi}^2 \right) - mg\cos\theta\, l(t)$$
$$+ \lambda(\phi - \Omega t)$$

$$\frac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2\theta\, \dot{\phi} \qquad \frac{\partial L}{\partial \phi} = \lambda$$

$$m \sin^2\theta \left[l^2 \ddot{\phi} + 2 l \dot{l} \dot{\phi} \right] = \lambda \quad (1)$$

$$\frac{\partial L}{\partial \dot{l}} = m\dot{l}$$

$$\frac{\partial L}{\partial l} = ml\sin^2\theta\,\dot{\phi}^2 - mg\cos\theta$$

$$m\ddot{l} = ml\sin^2\theta\,\dot{\phi}^2 - mg\cos\theta \quad (2)$$

$$\phi = \Omega t \quad (3)$$

so $\dot{\phi} = \Omega$

$$\ddot{\phi} = 0$$

$$m\sin^2\theta\left[2l\dot{l}\Omega\right] = \lambda = T$$

$$m\ddot{l} = ml\sin^2\theta\,\Omega^2 - mg\cos\theta$$

$$\sqrt{\frac{\ddot{l} + g\cos\theta}{ml\sin^2\theta}} = \Omega$$

$$\boxed{T = \lambda = 2m\sin^2\theta\, l\dot{l}\sqrt{\frac{\ddot{l} + g\cos\theta}{ml\sin^2\theta}}}$$

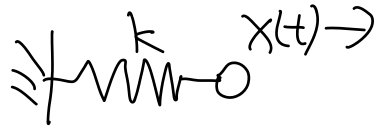
(i) How torque is determined by inertial force (centripetal force)

$$F_{cent} = m\frac{v^2}{r} = m\frac{v^2}{\ell\sin\theta}$$

$$\vec{v} = \dot{\ell}(t)\sin\theta$$

$$F_{cen} = m\frac{\dot{\ell}(t)^2\sin^{\cancel{2}}\theta}{\ell\,\cancel{\sin\theta}} = \frac{m\dot{\ell}(t)^2\sin\theta}{\ell} \quad ?$$

3) A driven set of oscillators:

a) 

at $t=0$, $F(t) = F_0 e^{-\alpha t}$ is applied.

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + F(t)x$$

$m\ddot{x} = -kx + F(t)$

$m\ddot{x} + kx = F_0 e^{-\alpha t}$

know Green Function is solution when

$m\ddot{x} + kx = \delta(t - t_0)$

at $t > t_0 = 0$

$m\ddot{x} + kx = 0$

$G = A\cos w_0 t + \phi$ for $w_0 = \sqrt{\frac{k}{m}}$

Continuity condition requires

$G(t = t_0) = 0$

So $G(t, t_0) = A\sin(w_0(t - t_0))$

Also have

$$\int_{t_0-\epsilon}^{t_0+\epsilon} m\ddot{x} + kx \, dt = 1$$

$$m\frac{dG}{dt}\bigg|_{t_0} = 1$$

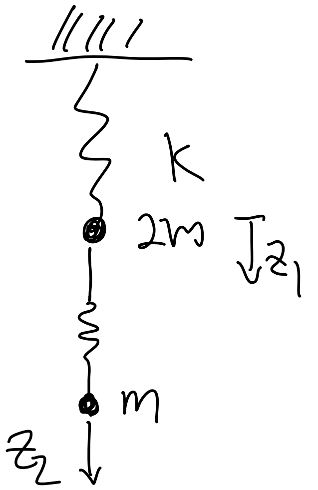
So $m w_0 A\cos(w_0(t_0 - t_0)) = 1$

$$A = \frac{1}{m\omega_0}$$

$$G(t,t_0) = \frac{1}{m\omega_0}\sin(\omega_0(t-t_0))$$

$$x(t) = \int_{-\infty}^{\infty} dt_0\, F_0 e^{-\alpha t_0} \frac{1}{m\omega_0}\sin(\omega_0(t-t_0))$$

$$\stackrel{!}{=} \frac{F_0}{m\omega_0}\int_0^{\infty} dt_0\, e^{-\alpha t_0}\sin(\omega_0(t-t_0))$$



$$L = \frac{1}{2}2m\dot{z}_1^2 + \frac{1}{2}m\dot{z}_2^2 - \frac{1}{2}kz_1^2 - \frac{1}{2}k(z_2-z_1)^2$$

$$L = m\dot{z}_1^2 + \frac{1}{2}m\dot{z}_2^2 - kz_1^2 - \frac{1}{2}kz_2^2 + kz_2z_1$$

$$\frac{\partial L}{\partial \dot{z}_1} = 2m\dot{z}_1$$

$$\frac{\partial L}{\partial z_1} = -2kz_1 + kz_2$$

$$\frac{\partial L}{\partial \dot{z}_2} = m\dot{z}_2$$

$$\frac{\partial L}{\partial z_2} = -kz_2 + kz_1$$

$$\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = \begin{pmatrix} -2w_0^2 & w_0^2 \\ w_0^2 & -w_0^2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\left|\begin{pmatrix} -2w^2+2w_0^2 & -w_0^2 \\ -w_0^2 & -w^2+w_0^2 \end{pmatrix}\right| = 0$$

$$(-2w^2+2w_0^2)(-w^2+w_0^2) - w_0^4 = 0$$

$$2w^4 - 4w^2w_0^2 + 2w_0^4 - w_0^4 = 0$$

$$2w^4 - 4w^2w_0^2 + w_0^4 = 0$$

$$w_f^2 = \frac{4w_0^2 \pm \sqrt{16w_0^4 - 8w_0^4}}{4}$$

$$\frac{1}{[illegible]} w_0^2 + \frac{1}{[illegible]} w_0^2$$

$$\omega_+^2 = \omega_0^2\left(1+\frac{1}{\sqrt{2}}\right)$$

$$\begin{pmatrix} -2\omega^2+2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2+\omega_0^2 \end{pmatrix}\begin{pmatrix} E_1^+ \\ E_2^+ \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2}{\sqrt{2}}\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\frac{1}{\sqrt{2}}\omega_0^2 \end{pmatrix}\begin{pmatrix} E_1^+ \\ E_2^+ \end{pmatrix} = 0$$

$$\frac{-2}{\sqrt{2}}E_1^+ - E_2^+ = 0$$

let $E_1^+ = 1$, $E_2^+ = \frac{-2}{\sqrt{2}}$

$$\vec{Q}^+ = Q^+\begin{pmatrix} 1 \\ \frac{-2}{\sqrt{2}} \end{pmatrix}\cos(\omega_+ t + \phi_+)$$

$$\omega^2 = \omega_-^2 = \omega_0^2\left(1-\frac{1}{\sqrt{2}}\right)$$

$$\begin{pmatrix} -2\omega^2+2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2+\omega_0^2 \end{pmatrix}\begin{pmatrix} E_1^- \\ E_2^- \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & -1 \\ -1 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} E_1^- \\ E_2^- \end{pmatrix}$$

let $E_1^- = 1$ $\quad E_2^- = \frac{2}{\sqrt{2}}$

$$\vec{Q^-} = Q^- \begin{pmatrix} 1 \\ \frac{2}{\sqrt{2}} \end{pmatrix} \cos \omega_- t + \phi_-$$

c) $\mathcal{L} = \frac{1}{2} 2m\dot{z}_1^2 + \frac{1}{2} m\dot{z}_2^2 - \frac{1}{2} k z_1^2$
$\quad - \frac{1}{2} k (z_2 - z_1)^2 + F(t) z_2$

$z_1 = Q^+ + Q^-$

$z_2 = \frac{2}{\sqrt{2}} (Q^- - Q^+)$

$$\mathcal{L} = m(\dot{Q}^+ + \dot{Q}^-)^2 + \frac{1}{2}\frac{4}{2}m(\dot{Q}^- - \dot{Q}^+)^2 - \frac{1}{2}k(Q^+ + Q^-)^2$$

$$-\frac{1}{2}k\left(\frac{2}{\sqrt{2}}(Q^- - Q^+) - (Q^+ + Q^-)\right)^2$$

$$\frac{1}{2}m\dot{Q}^{+2} + m\dot{Q}^{-2}$$

$$\ddot{Q}^+ + \omega_+^2 Q^+ = -\frac{2}{\sqrt{2}}F(t)Q^+$$

$$\ddot{Q}^- + \omega_-^2 Q^- = \frac{2}{\sqrt{2}}F(t)Q^-$$