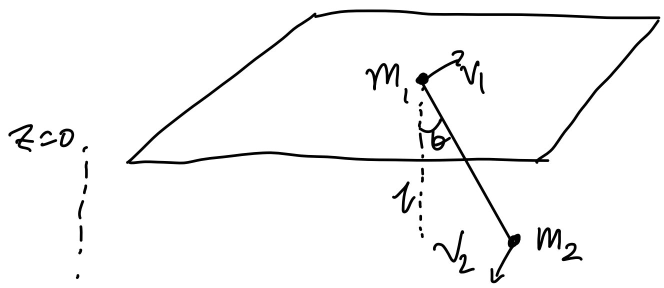
1) A sliding conical pendulum:



First bead constrained to move in x-y plane

Second bead moves in 3D.

a) Determine distance from m_1 to CoM:

$$\vec{r}_1 = \vec{R}_{cm} + \Delta\vec{r}_1$$
$$\vec{r}_2 = \vec{R}_{cm} + \Delta\vec{r}_2$$

$$X_{cm} = \frac{x_1 m_1 + x_2 m_2}{m_1+m_2} = \frac{x_1 m_1 + x_2 m_2}{M}$$

$$Y_{cm} = \frac{y_1 m_1 + y_2 m_2}{m_1+m_2} = \frac{y_1 m_1 + y_2 m_2}{M}$$

$$Z_{cm} = \frac{z_1 m_1 + z_2 m_2}{m_1+m_2} = z_2 \frac{m_2}{M}$$

$$\Delta X_1 = X_1 - X_{cm}$$
$$\stackrel{!}{=} \frac{m_1 x_1 + m_2 x_1}{M} - \frac{x_1 m_1 + x_2 m_2}{M} = -(x_2 - x_1)\frac{m_2}{M} = -x\frac{m_2}{M}$$

$$\Delta Y_1 = y_1 - y_{cm}$$
$$\stackrel{!}{=} \frac{m_1 y_1 + m_2 y_2}{M} - \frac{y_1 m_1 + y_2 m_2}{M} = -(y_2 - y_1)\frac{m_2}{M} = -y\frac{m_2}{M}$$

$$\Delta Z_1 = z_1 - Z_{cm}$$
$$\stackrel{!}{=} -z_2 \frac{m_2}{M} = -z\frac{m_2}{M}$$

$$r_{cm1} = \sqrt{\Delta x_1^2 + \Delta y_1^2 + \Delta z_1^2}$$
$$\stackrel{!}{=} \frac{m_2}{M} l$$

$$\Delta X_2 = X_2 - X_{cm}$$
$$\stackrel{!}{=} m_1 x_2 + m_2 x_2 - m_1 x_1 + m_2 x_2 - (x_2 - x_1)\frac{m_1}{\ } = x\frac{m_1}{\ }$$

$$- \frac{\quad}{M} - \frac{\quad}{M} = \quad M \quad \frac{\quad}{M}$$

$$\stackrel{!}{=} l \sin\theta \cos\phi$$

$$\Delta y_2 = y_2 - y_{cm}$$

$$\stackrel{!}{=} \frac{m_1 y_2 + m_2 y_2}{M} - \frac{m_1 y_1 + m_2 y_2}{M} = (y_2 - y_1)\frac{m_1}{M} = y \frac{m_1}{M}$$

$$\Delta z_2 = z_2 - z_{cm}$$

$$\stackrel{!}{=} \frac{m_1 z_2 + m_2 z_2}{M} - \frac{z_2 m_2}{M} = z_2 \frac{m_1}{M} = z \frac{m_1}{M}$$

$$L = \frac{1}{2} M (\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2) + \frac{1}{2} m_1 \left(\frac{m_2}{M}\right)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$+ \frac{1}{2} m_2 \left(\frac{m_1}{M}\right)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - M g z_{cm}$$

$$L = \frac{1}{2} M (\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2) + \frac{1}{2} \frac{m_1 m_2}{M} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$- M g z_{cm}$$

$$x = l \sin\theta \cos\phi$$

$$\dot{x} = l(\cos\theta \cos\phi\, \dot{\theta} - \sin\theta \sin\phi\, \dot{\phi})$$

$$z_{cm} = \frac{m_2}{M} l \cos\theta$$

$$\dot{z}_{cm} = -\frac{m_2}{M} l \sin\theta\, \dot{\theta}$$

$$y = l \sin\theta \sin\phi$$

$$\dot{y} = l(\cos\theta \sin\phi\, \dot{\theta} + \sin\theta \cos\phi\, \dot{\phi})$$

$$z = -l\cos\theta$$

$$\dot{z} = l \sin\theta\, \dot{\theta}$$

$$L = \frac{1}{2} M \left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \left(\frac{m_2}{M} l\right)^2 \sin^2\theta\, \dot{\theta}^2\right)$$

$$+ \frac{1}{2} \frac{m_1 m_2}{M} (l^2 \dot{\theta}^2 + l^2 \sin^2\theta\, \dot{\phi}^2) + M g l \cos\theta$$

c) $\frac{\partial L}{\quad} = M\dot{v} \quad = P \quad = \text{const}$

$$\frac{\partial L}{\partial \dot{x}_{cm}} = M\dot{x}_{cm} = P_{x_{cm}} = \text{const.}$$

$$\frac{\partial L}{\partial \dot{y}_{cm}} = M\dot{y}_{cm} = P_{y_{cm}} = \text{constant.}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{m_1 m_2}{M} l^2 \sin^2\theta\, \dot{\phi} = P_\phi = \text{constant.} \Rightarrow \dot{\phi} = \frac{P_\phi}{\mu l^2 \sin^2\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = M\left(\frac{m_2}{M} l\right)^2 \sin^2\theta\, \dot{\theta} + \frac{m_1 m_2}{M} l^2 \dot{\theta} = P_\theta$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2} M\left(\frac{m_2}{M} l\right)^2 2\sin\theta\cos\theta\, \dot{\theta}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{M}\right) l^2\, 2\sin\theta\cos\theta\, \dot{\phi}^2 - Mgl\sin\theta$$

$$h = P_x\dot{X} + P_y\dot{Y} + P_\phi\dot{\phi} + P_\theta\dot{\theta} - L$$

$$= \frac{1}{2}M\dot{X}_{cm}^2 + \frac{1}{2}M\dot{Y}_{cm}^2 + \frac{1}{2}\frac{m_1 m_2}{M} l^2 \sin^2\theta\,\dot{\phi}^2 + \left(\frac{1}{2}\frac{m_1 m_2}{M} l^2 + \frac{1}{2}M\left(\frac{m_2}{M} l\right)^2 \sin^2\theta\right)\dot{\theta}^2 - Mgl\cos\theta$$

constant energy

d) since P_x and P_y are constant

$$P_{cmx} = P_{x_1} + P_{x_2} = 0$$
$$P_{cmy} = P_{y_1} + P_{y_2} = m_2 v_0 = M\dot{Y}_{cm} \Rightarrow \dot{Y}_{cm} = \frac{m_2}{M} v_0$$

If center of mass moving along with velocity $m_2 V_0$ then its periodic.

$$P_\phi \Big|_{\theta=\frac{\pi}{2}} = \frac{m_1 m_2}{M} l^2 \dot{\phi} \quad \text{also constant.}$$

since $l\dot{\phi} = V_0$.

$$P_\phi = \frac{m_1 m_2}{M} l V_0$$

e) The pendulum swings down from $\theta = \pi/2$ relative to the vertical to a minimum angle.

$$H = P_x \dot{x} + P_y \dot{y} + P_\phi \dot{\phi} + P_\theta \dot{\theta} - L$$

$$\stackrel{!}{=} \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_\phi^2}{\mu l^2 \sin^2\theta} + \frac{P_\theta^2}{M\left(\frac{m_2}{M} l\right)^2 \sin^2\theta + \mu l^2}$$

$$- \left\{ \left[\frac{1}{2} M \left(\frac{m_2}{M}\right)^2 \sin^2\theta + \frac{1}{2}\mu \right] l^2 \dot{\theta}^2 + \frac{1}{2}\mu \, l^2 \sin^2\theta \frac{P_\phi^2}{(\mu l^2 \sin^2\theta)^2} + Mgl\cos\theta \right\}$$

$$\stackrel{!}{=} \frac{P_x^2}{2M} + \frac{P_y^2}{2M} + \frac{P_\phi^2}{2\mu l^2 \sin^2\theta} - \left(\frac{1}{3} M \left(\frac{m_2}{M}\right)^2 \sin^2\theta + \frac{1}{2}\mu\right) l^2 \frac{P_\theta^2}{\left(M\left(\frac{m_2}{M} l\right)^2 \sin^2\theta + \mu l^2\right)^2}$$

$$= \frac{P_x^2}{2M} + \frac{P_y^2}{2M} + \frac{P_\phi^2}{2\mu l^2 \sin^2\theta} + \frac{P_\theta^2}{2\left(M\left(\frac{m_2}{M}l\right)^2 \sin^2\theta + \mu l^2\right)}$$

$$H_{init} = \frac{1}{2}\mu l^2 \sin^2\theta \dot{\phi}^2 + \frac{1}{2}M \frac{m_1\dot{x}_1 + m_2\dot{x}_2}{M} + \frac{1}{2}M \dot{y}_{cm}^2$$

$$= \frac{1}{2}\mu l^2 \dot{\phi}^2 + \frac{1}{2}M\left(\frac{m_2}{M}v_0\right)$$

$$= \frac{1}{2}\mu v_0^2 + \frac{1}{2}M\left(\frac{m_2}{M}v_0\right)^2$$

$$H_{final} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}\mu l^2 \sin^2\theta \dot{\phi}^2$$

$$+ \left(\frac{1}{2}\frac{m_1 m_2}{M} l^2 + \frac{1}{2}M\left(\frac{m_2}{M}l\right)^2 \sin^2\theta\right)\dot{\theta}^2 - Mgl\cos\theta$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}\mu l^2 \sin^2\theta \dot{\phi}^2 - Mgl\cos\theta$$

Since $\frac{P_x^2}{2M}$ and $\frac{P_y^2}{2M}$ are constant.

$$\frac{1}{2}\mu v_0^2 = \frac{(\mu l v_0)^2}{2\mu l^2 \sin^2\theta} - Mgl\cos\theta$$
$$+ \frac{1}{2}m(\theta)\dot{\theta}^2$$

$$\frac{1}{2}\mu v_0^2\left(1 - \frac{1}{\sin^2\theta}\right) + Mgl\cos\theta = \frac{1}{2}m(\theta)\dot{\theta}^2$$

$$\frac{\sin^2\theta - 1}{\sin^2\theta} = \frac{-\cos^2\theta}{\sin^2\theta}$$

$\hookrightarrow -\frac{1}{2}uv_0^2\cot^2\theta + Mgl\cos\theta = \frac{1}{2}m(\theta)\dot{\theta}^2$

$$\dot{\theta} = \pm\sqrt{\frac{-\frac{1}{2}uv_0^2\cot^2\theta + Mgl\cos\theta}{\frac{1}{2}ul^2 + \frac{1}{2}M\left(\frac{m_2}{M}l\right)^2\sin^2\theta}}$$

Minimum angle when $\dot{\theta} = 0$

$$-\frac{1}{2}uv_0^2\cot^2\theta + Mgl\cos\theta = 0.$$

$$Mgl\cos\theta = \frac{1}{2}uv_0^2\cot^2\theta$$

$$\frac{Mgl}{\frac{1}{2}uv_0^2}\cos\theta = \cot^2\theta$$

$$u\cos\theta = \frac{\cos^2\theta}{1-\cos^2\theta}$$

$u\cos\theta - u\cos^3\theta - \cos^2\theta = 0$ either when $\theta = \frac{\pi}{2}$

or

$u\cos^2\theta + \cos\theta - u = 0$

$$\cos\theta = \frac{-1 \pm \sqrt{1+4u^2}}{2u} \leftarrow$$ take negative sign. to avoid going above $\frac{\pi}{2}$

(ii)
$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\int_{\frac{\pi}{2}}^{\theta_-} \frac{d\theta}{\dot{\theta}} = \int_0^{T/2} dt = T/2$$

$$\int_{\frac{\pi}{2}}^{\theta_-} \pm \sqrt{\frac{\cancel{\frac{1}{2}}\mu \cancel{l^2} + \cancel{\frac{1}{2}} M\left(\frac{m_2}{M}\cancel{l}\right)^2 \sin^2\theta}{-\cancel{\frac{1}{2}}\mu v_0^2 \cot^2\theta + Mgl\cos\theta}}\, d\theta = T/2$$

let $\frac{Mgl}{\frac{1}{2}\mu v_0^2} = u$ $\qquad M\left(\frac{m_2}{M}\right)^2 \frac{M}{m_1 m_2} = \frac{m_2}{m_1}$

$$\int_{\frac{\pi}{2}}^{\theta_-} -\frac{l}{v_0} \sqrt{\frac{1 + \frac{m_2}{m_1}\sin^2\theta}{-\cot^2\theta + u\cos\theta}}\, d\theta = T/2$$

pick negative since $\frac{\pi}{2} \to \theta_-$, θ is decreasing.

or $\int^{\frac{\pi}{2}} l \sqrt{1 + \frac{m_2}{m_1}\sin^2\theta}$

$$\int_{\theta_-} \frac{l}{v_0}\sqrt{\frac{1+\frac{m_2}{m_1}\sin^2\theta}{-\cot^2\theta + u\cos\theta}}\, d\theta = T/2$$

for large v_0, $Mgl \ll \frac{1}{2}uv_0^2 \Rightarrow u \ll 1$

then

$$\frac{T}{2} = \int_{\theta_-}^{\frac{\pi}{2}} \frac{l}{v_0}\sqrt{\frac{1+\frac{m_2}{m_1}\sin^2\theta}{-\cot^2\theta}} \qquad u = \frac{Mgl}{\frac{1}{2}uv_0^2}$$

$$\frac{T}{2} = \int_{\theta_-}^{\frac{\pi}{2}} \frac{l}{v_0}\frac{\sin\theta}{\cos\theta}\sqrt{\frac{(1+\frac{m_2}{m_1})\sin^2\theta + \cos^2\theta}{-1}}$$

Kinetic energy dominates over gravity.

for small v_0, $\frac{1}{2}uv_0^2 \ll Mgl$

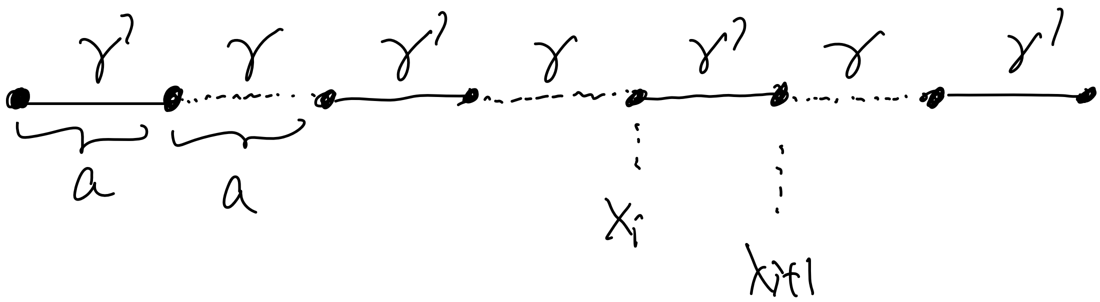
$$\frac{T}{2} = \int_{\theta_-}^{\frac{\pi}{2}} \frac{l}{v_0}\sqrt{\frac{1+\frac{m_2}{m_1}\sin^2\theta}{u\cos\theta}}$$

$$= \int_{\theta_-}^{\frac{\pi}{2}} \frac{l}{v_0}\sqrt{\frac{\frac{1}{2}uv_0^2}{Mgl}}$$

$$\frac{1}{=}\int_{\theta_-}^{\frac{\pi}{2}} \sqrt{\frac{u}{M}}\sqrt{\frac{1}{2}}\sqrt{\frac{l}{g}}\sqrt{\frac{1+\frac{m_2}{m_1}\sin^2\theta}{\cos\theta}}$$

pendulum.

2) Waves from coupled springs



$$T = \sum_{i=\text{even}} \frac{1}{2} m \dot{u}_i^2 + \frac{1}{2} m \dot{u}_{i+1}^2$$

$$V = \sum_{i=\text{even}} \frac{1}{2}\gamma(x_i - x_{i-1} - a)^2 + \frac{1}{2}\gamma'(x_{i+1} - x_i - a)^2$$

$$\overset{!}{=} \sum_{i=\text{even}} \frac{1}{2}\gamma(u_i - u_{i-1})^2 + \frac{1}{2}\gamma'(u_{i+1} - u_i)^2 + \frac{1}{2}\gamma(u_{i+2} - u_{i+1})^2$$

$$L = \sum_{i=\text{even}} \frac{1}{2} m \dot{u}_i^2 + \frac{1}{2} m \dot{u}_{i+1}^2 - \frac{1}{2}\gamma(u_i - u_{i-1})^2$$
$$- \frac{1}{2}\gamma'(u_{i+1} - u_i)^2 - \frac{1}{2}\gamma(u_{i+2} - u_{i+1})^2$$

$$m\ddot{u}_i = -\gamma(u_i - u_{i-1}) + \gamma'(u_{i+1} - u_i)$$
$$\overset{!}{=} -(\gamma + \gamma')u_i + \gamma u_{i-1} + \gamma' u_{i+1}$$

$$m\ddot{u}_{i+1} = -\gamma'(u_{i+1} - u_i) + \gamma(u_{i+2} - u_{i+1})$$
$$\overset{!}{=} -(\gamma' + \gamma)u_{i+1} + \gamma' u_i + \gamma u_{i+2}$$

let $u_i = A\xi_1 e^{-i\omega t + kx_i} = A\xi_1 e^{-i\omega t + kia}$

$$u_{j+1} = A\xi_2 e^{-i\omega t + kx_{j+1}} = A\xi_2 e^{-i\omega t + k(j+1)a}$$

$$-\omega^2 m A\xi_1 e^{-i\omega t + kx_j} = -(\gamma+\gamma')A\xi_1 e^{-i\omega t + kx_j} + \gamma A\xi_2 e^{-i\omega t + kx_{j-1}}$$
$$+\gamma' A\xi_2 e^{-i\omega t + kx_{j+1}}$$

① $$-\omega^2 m\xi_1 = -(\gamma+\gamma')\xi_1 + \gamma\xi_2 e^{-ka} + \gamma'\xi_2 e^{ka}$$
$$\overset{!}{=} -(\gamma+\gamma')\xi_1 + (\gamma e^{-ika} + \gamma' e^{ika})\xi_2$$

$$-\omega^2 m A\xi_2 e^{-i\omega t + kx_{j+1}} = -(\gamma'+\gamma)A\xi_2 e^{-i\omega t + kx_{j+1}} + \gamma' A\xi_1 e^{-i\omega t + kx_j}$$
$$+\gamma A\xi_1 e^{-i\omega t + kx_{j+2}}$$

② $$-\omega^2 m\xi_2 = -(\gamma'+\gamma)\xi_2 + \gamma'\xi_1 e^{-ika} + \gamma\xi_1 e^{ka}$$
$$\overset{!}{=} -(\gamma'+\gamma)\xi_2 + (\gamma' e^{-ika} + \gamma e^{ika})\xi_1$$

$$-\omega^2 \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = \begin{bmatrix} -(\gamma'+\gamma) & (\gamma e^{-ika} + \gamma' e^{ika}) \\ (\gamma' e^{-ika} + \gamma e^{ika}) & -(\gamma'+\gamma) \end{bmatrix} \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}$$

$$\begin{vmatrix} \omega^2-(\gamma'+\gamma)/m & (\gamma e^{-ika}+\gamma' e^{ika})/m \\ (\gamma' e^{-ika}+\gamma e^{ika})/m & \omega^2-(\gamma'+\gamma)/m \end{vmatrix} = 0$$

$$\omega^4-\frac{2(\gamma'+\gamma)}{m}\omega^2+\frac{(\gamma'+\gamma)^2}{m^2}-\frac{1}{m^2}\left(\gamma\gamma' e^{-2ika}+\gamma^2+\gamma'^2+\gamma\gamma' e^{2ika}\right)$$

$$\omega^4-\frac{2(\gamma'+\gamma)}{m}\omega^2+\frac{1}{m^2}\left((\gamma'+\gamma)^2-\gamma^2-\gamma'^2-2\gamma\gamma'\cos 2ka\right)$$

$$\omega^4-\frac{2(\gamma'+\gamma)}{m}\omega^2+\frac{2\gamma\gamma'}{m^2}\left(1-\cos 2ka\right)=0$$

$$\omega^2=\frac{\frac{2(\gamma'+\gamma)}{m}\pm\frac{1}{m}\sqrt{4(\gamma'+\gamma)^2-8\gamma'\gamma(1-\cos 2ka)}}{2}$$

$$=\left(\frac{\gamma'+\gamma}{m}\right)\pm\frac{1}{m}\sqrt{(\gamma'+\gamma)^2-2\gamma'\gamma(1-\cos 2ka)}$$

$$\omega_\pm^2=\frac{\gamma'+\gamma}{m}\pm\frac{1}{m}\sqrt{\gamma'^2+\gamma^2+2\gamma'\gamma\cos 2ka}$$

same.

$$\cos 2ka = 1 - \frac{(2ka)^2}{2}$$

$$\omega_\pm^2 = \frac{1}{m}\left(\gamma'+\gamma \pm \sqrt{(\gamma'+\gamma)^2 - 2\gamma'\gamma\left(1-\left\{1-\frac{(2ka)^2}{2}\right\}\right)}\right)$$

$$\doteq \frac{1}{m}\left(\gamma'+\gamma \pm \sqrt{(\gamma'+\gamma)^2 - 2\gamma'\gamma\,\frac{(2ka)^2}{2}}\right)$$

$$\doteq \frac{1}{m}\left(\gamma'+\gamma \pm (\gamma'+\gamma)\sqrt{1-\frac{4\gamma'\gamma}{(\gamma'+\gamma)^2}(ka)^2}\right)$$

$$\doteq \frac{1}{m}\left(\gamma'+\gamma \pm (\gamma'+\gamma)\left(1-\frac{2\gamma'\gamma}{(\gamma'+\gamma)^2}\,(ka)^2\right)\right)$$

$$\omega_+ = \sqrt{\frac{2}{m}(\gamma'+\gamma) - \frac{2\gamma'\gamma}{\gamma'+\gamma}(ka)^2}$$

$$\doteq \sqrt{\frac{2}{m}(\gamma'+\gamma)}\left(1-\frac{\gamma'\gamma}{2m}(ka)^2\right)$$

$$\omega_- = \sqrt{\frac{2\gamma'\gamma}{\gamma'+\gamma}(ka)^2} = \sqrt{\frac{2\gamma'\gamma}{\gamma'+\gamma}}\;ka$$

c) For long wavelengths $ka \ll 1$

treat oscillators as continuous systems.

$$S[u(t,x)] = \int dt\,dx\,(\partial_t u(t,x))^2 - c_1(\partial_x u(t,x))^2 - c_2(u(t,x))^2$$

$$\mathcal{L}$$

i)

$$S[q+\delta q] = \int dt\,dx\; \mathcal{L}(q+\delta q, \partial_a q + \partial_a \delta q)$$

$$\stackrel{1}{=} \int dt\,dx\; \mathcal{L}(q, \partial_a q) + \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial_a q}\partial_a \delta q$$

$$\Delta S = \int dt\,dx\; \frac{\partial \mathcal{L}}{\partial q}\delta q + \underbrace{\frac{\partial}{\partial_a}\left(\frac{\partial \mathcal{L}}{\partial_a q}\delta q\right)}_{=0} - \frac{\partial}{\partial_a}\left(\frac{\partial \mathcal{L}}{\partial_a q}\right)\delta q$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial}{\partial_a}\frac{\partial \mathcal{L}}{\partial_a q}} \quad *$$

$$-2c_2 u = \frac{\partial}{\partial t}(2\partial_t u) + -c_1 \frac{\partial}{\partial x}(2\partial_x u)$$

$$-2c_2 u = 2\frac{\partial^2}{\partial t^2}u - 2c_1 \frac{\partial^2}{\partial x^2}u$$

$$\frac{\partial^2}{\partial t^2}u - c_1\frac{\partial^2}{\partial x^2}u + c_2 u = 0$$

let $u = Ae^{ikx - i\omega t}$.

$$-\omega^2 u + c_1 k^2 u + c_2 u = 0$$

$$-\omega^2 + c_1 k^2 + c_2 = 0$$

i) $\omega^2 = c_1 k^2 + c_2$

(ii) For $\omega_+^2 = \frac{2}{m}(\gamma'+\gamma) - \frac{2\gamma'\gamma}{\gamma'+\gamma}(ka)^2$

$$C_1 = \frac{-2\gamma'\gamma}{\gamma'+\gamma} a^2$$

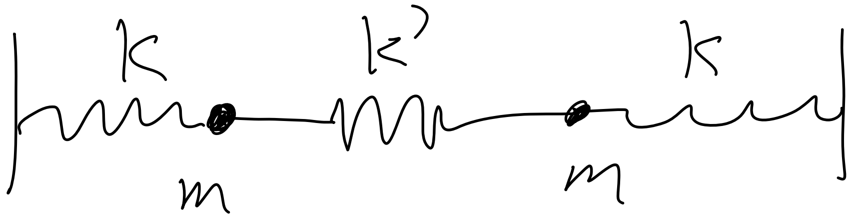
$$C_2 = \frac{2}{m}(\gamma'+\gamma)$$

For $\omega_-^2 = \frac{2\gamma'\gamma}{\gamma'+\gamma}(ka)^2$

$$C_1 = \frac{2\gamma'\gamma}{\gamma'+\gamma} a^2$$

$$C_2 = 0$$

3) Oscillations with similar frequency.



$k' << k$

a) $L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k x_1^2 - \frac{1}{2}k x_2^2$
$\quad - \frac{1}{2}k'(x_1 - x_2)^2$

$$m\ddot{x}_1 = -kx_1 - k'(x_1 - x_2)$$
$$= -(k+k')x_1 + k'x_2$$
$$m\ddot{x}_2 = -kx_2 + k'(x_1 - x_2)$$
$$= k'x_1 - (k+k')x_2$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(\omega_0^2 + \omega'^2) & \omega'^2 \\ \omega'^2 & -(\omega_0^2 + \omega'^2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{vmatrix} \omega^2 - (\omega_0^2 + \omega'^2) & \omega'^2 \\ \omega'^2 & \omega^2 - (\omega_0^2 + \omega'^2) \end{vmatrix} = 0$$

$$\omega^4 - 2\omega^2(\omega_0^2 + \omega'^2) + (\omega_0^2 + \omega'^2)^2 - \omega'^4 = 0$$

$$\omega^2 = \frac{\cancel{2}(\omega_0^2 + \omega'^2) \pm \cancel{2}\sqrt{(\omega_0^2 + \omega'^2)^2 - [\omega_0^4 + 2\omega_0^2\omega'^2]}}{\cancel{2}}$$

$$\omega^2 = (\omega_0^2 + \omega'^2) \pm \sqrt{\omega'^4}$$

$$= \omega_0^2 + \omega'^2 \pm \omega'^2$$

$$\omega_+ = \sqrt{\omega_0^2 + 2\omega'^2}$$

$$\omega_- = \omega_0$$

when $\omega^2 = \omega_0^2 + 2\omega'^2$

$$\begin{vmatrix} \omega^2 - (\omega_0^2 + \omega'^2) & \omega'^2 \\ \omega'^2 & \omega^2 - (\omega_0^2 + \omega'^2) \end{vmatrix}$$

$$\Rightarrow \begin{pmatrix} \omega'^2 & \omega'^2 \\ \omega'^2 & \omega'^2 \end{pmatrix} \begin{pmatrix} E^+_1 \\ E^+_2 \end{pmatrix}$$

$$E^+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

when $\omega^2 = \omega_0^2$

$$\begin{pmatrix} -\omega'^2 & \omega'^2 \\ \omega'^2 & \omega'^2 \end{pmatrix} \begin{pmatrix} E^-_1 \\ E^-_2 \end{pmatrix}$$

$$\vec{E} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{A\cos(\omega_+ t + \phi_+)}_{Q_+} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \underbrace{B\cos(\omega_- t + \phi_-)}_{Q_-} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

At $t=0$, $\quad x_1 = x_0 \qquad x_2 = 0$

$$x_2(t=0) = 0 = A - B = 0$$

$$A = B \qquad \phi_+ = \phi_- = 0$$

$$x_1(t=0) = A + A = x_0$$

$$A = \frac{x_0}{2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_0}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos(\omega_+ t) + \frac{x_0}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t)$$

$$= Q_+ \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} Q_-$$

b) $\omega_+^2 = \omega_0^2 + 2\omega'^2$

$$\omega_+ = \omega_0 \sqrt{1 + 2(\omega')^2}$$

$$\omega_+ = \omega_0 \sqrt{1+2\left(\frac{\omega'}{\omega_0}\right)^2}$$

$$\simeq \omega_0 \left(1+\left(\frac{\omega'}{\omega_0}\right)^2\right)$$

$$\omega_- = \omega_0$$

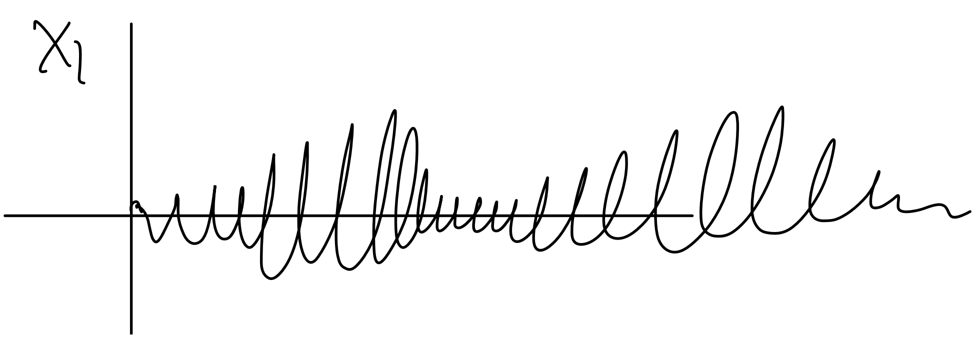
$$x_1 = \frac{x_0}{2}\left(\cos\left(\omega_0 + \frac{\omega'^2}{\omega_0}\, t\right) + \cos \omega_0 t\right).$$

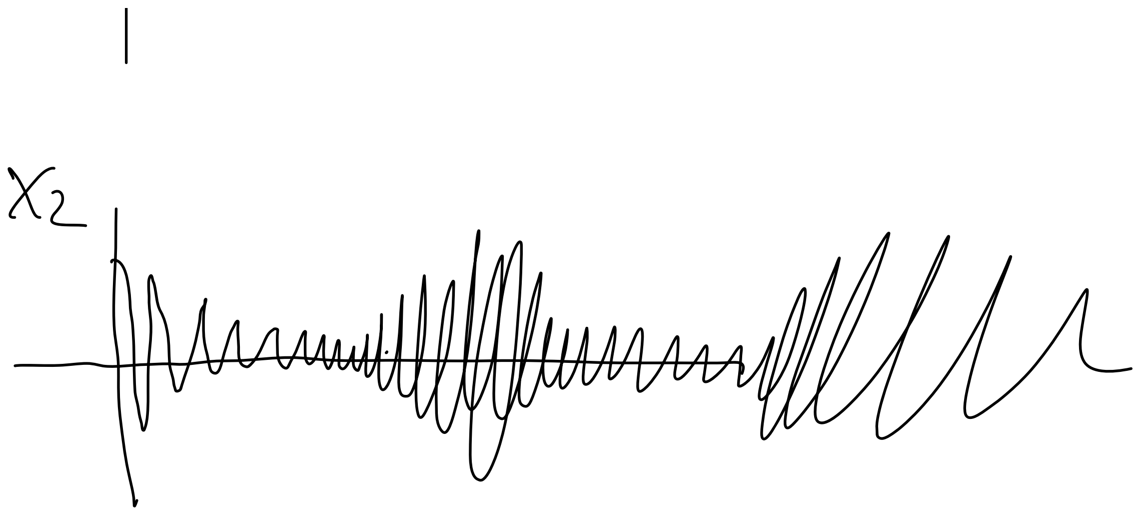
$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$x_1 = x_0 \cos\left(\omega_0\left(2+\left(\frac{\omega'}{\omega_0}\right)^2\right)t\right)\cos\left(\frac{\omega'^2}{\omega_0}\, t\right)$$

$$x_2 = \frac{x_0}{2}\left(\cos\left(\omega_0\left(1+\frac{\omega'^2}{\omega_0}\right)t\right) - \cos\omega_0 t\right)$$

$$\overset{!}{=} -x_0 \sin\left(\omega_0\left(2+\frac{\omega'^2}{\omega_0}\right)t\right)\sin\left(\frac{\omega_0'^2}{\omega_0}\, t\right)$$





c) Now particles also experiences

$$F_{drag} = -m\eta\frac{dx}{dt}$$

After $t>0$, particles experience external force $F(t)$

$$\ddot{x}_1 + (\omega_0^2+\omega'^2)x_1 - \omega'^2 x_2 + \eta\dot{x}_1 + F(t) = 0$$

$$\ddot{x}_2 - \omega'^2 x_1 + (\omega_0^2+\omega'^2)x_2 + \eta\dot{x}_2 - F(t) = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q_+\begin{pmatrix} -1 \\ 1 \end{pmatrix} + Q_-\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \ddot{Q}_+\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \ddot{Q}_-\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

① $-\ddot{Q}_+ + \ddot{Q}_- + (\omega_0^2+\omega'^2)(-Q_+ + Q_-) - \omega'^2(Q_+ + Q_-) + \eta(-\dot{Q}_+ + \dot{Q}_-) + F(t) = 0$

② $\ddot{Q}_+ + \ddot{Q}_- + (\omega_0^2+\omega'^2)(Q_+ + Q_-) - \omega'^2(-Q_+ + Q_-) + \eta(\dot{Q}_+ + \dot{Q}_-) - F(t) = 0$

$$\ddot{Q}_- + (\omega_0^2 + \omega'^2) Q_- - \omega'^2 Q_- + \eta \dot{Q}_- = 0. \qquad ① + ②$$

$$\ddot{Q}_- + \omega_-^2 Q_- + \eta \dot{Q}_- = 0$$

$$\ddot{Q}_+ + \omega_+^2 Q_+ + \eta \dot{Q}_+ = F(t) \qquad ② - ①$$

A damped harmonic oscillator has solution:

$$Q = Ae^{i\omega t}$$

$$-\omega^2 \cancel{Ae^{i\omega t}} + \omega_0^2 \cancel{Ae^{i\omega t}} + \eta i\omega \cancel{Ae^{i\omega t}} = 0.$$

$$\omega^2 - i\eta\omega - \omega_0^2 = 0$$

$$\omega = \frac{i\eta}{2} \pm \frac{1}{2}\sqrt{4\omega_0^2 - \eta^2}$$

$$Q = Ae^{-\frac{\eta}{2}t} e^{i\sqrt{\omega^2 - \left(\frac{\eta}{2}\right)^2}\, t}$$

$$\stackrel{!}{=} Ae^{-\frac{\eta}{2}t} e^{i\sqrt{\omega^2 - \left(\frac{\eta}{2}\right)^2}\, t}$$

So Green's function has a form of:

$$G(t, t_0) = Ae^{-\frac{\eta}{2}t} e^{i\sqrt{\omega^2 - \left(\frac{\eta}{2}\right)^2}\, t} = Ae^{-\frac{\eta}{2}t} \cos\left(\sqrt{\omega^2 - \left(\frac{\eta}{2}\right)^2}\, t + \phi\right)$$

Use Green's function boundary system:

Continuity: $G|_{t=t_{0+}} = 0$ since $G = 0$ for $t < t_0$

$$G(t=t_0<0) = A\cos\phi = 0$$

then

$$G(t,t_0) = Ae^{-\frac{\eta}{2}(t-t_0)}\sin\left(\underbrace{\sqrt{\omega^2-\left(\frac{\eta}{2}\right)^2}}_{\approx\omega \text{ for } \eta\ll\omega}(t-t_0)\right)$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} m\frac{d^2G}{dt^2} + m\eta\frac{d}{dt}G + \omega^2 G\, dt = \int_{t_0-\epsilon}^{t_0+\epsilon}\delta(t-t_0)\,dt$$

$$m\frac{dG}{dt}\Big|_{t_0+\epsilon} = 1$$

then

$$m\frac{dG}{dt} = \cancel{-\frac{\eta}{2}Ae^{-\frac{\eta}{2}(t-t_0)}\sin\omega(t-t_0)} + \omega A e^{-\frac{\eta}{2}(t-t_0)}\cos(\omega(t-t_0))\Big|_{t=t_0}$$

$$\stackrel{!}{=} 1 = \omega A m$$

$$G(t,t_0) \stackrel{!}{=} \frac{\theta(t-t_0)}{\omega m} e^{-\frac{\eta}{2}(t-t_0)}\sin(\omega_1(t-t_0))$$

Then the explicit solution is:

$$Q_+ = \int_{-\infty}^{\infty} dt_0\, F(t_0)\, G(t,t_0)$$

$$\stackrel{!}{=} \int_0^t dt_0\, \frac{F(t_0)}{\omega m} e^{-\frac{\eta}{2}(t-t_0)}\sin(\omega_+(t-t_0))$$

$Q_- = 0$ since $F = 0$ for Q_-.

d) $E = \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\dot{x}^2$ no Q- since its 0.

$E = \frac{1}{2} m \omega_t^2 Q_t^2 + \frac{1}{2} m \dot{Q}_t^2$

$$\frac{d}{dt} Q_t = \int_0^t dt_0 \frac{F(t_0)}{\omega_t m}\left[\left(\frac{-\eta}{2}\right)\sin(\omega_t(t-t_0)) + \omega_t \cos(\omega_t(t-t_0))\right] e^{\frac{-\eta}{2}(t-t_0)}$$

$$E = \frac{1}{2} m\omega_t^2 \int_0^t \int_0^t dt_0 dt' \frac{1}{\omega_t^2 m^2} F(t_0) e^{\frac{-\eta}{2}(t-t_0)} \sin(\omega_t(t-t_0))$$

$$F(t') e^{\frac{-\eta}{2}(t-t')} \sin(\omega_t(t-t'))$$

$$+ \frac{1}{2} m \left(\frac{1}{\omega_t m}\right)^2 \int\int dt_0 dt' F(t_0) F(t') e^{\frac{-\eta}{2}(t-t_0)} e^{\frac{-\eta}{2}(t-t')}$$

$$\left[\cancel{\left(\frac{-\eta}{2}\right)\sin}(\omega_t(t-t_0)) + \omega_t \cos(\omega_t(t-t_0))\right]$$

$$\left[\cancel{\left(\frac{-\eta}{2}\right)\sin}(\omega_t(t-t')) + \omega_t \cos(\omega_t(t-t'))\right]$$

$$E = \int_0^t\int_0^t dt_1 dt_2 \frac{1}{2} \frac{1}{\omega_t^2 m^2} m\omega_t^2 F(t_1) F(t_2) e^{\frac{-\eta}{2}(t-t_1)} e^{\frac{-\eta}{2}(t-t_2)}$$

$$\left[\sin(\omega_t(t-t_1))\sin(\omega_t(t-t_2)) + \cos(\omega_t(t-t_1))\cos(\omega_t(t-t_2))\right]$$

e) If $F(t)$ is time dependent.

$$\langle F(t)\rangle = 0$$

$$\langle F(t)F(t')\rangle = 2Tm\eta\,\delta(t-t')$$

$$E = \int_0^t\int_0^t dt_1 dt_2 \frac{1}{2}\frac{1}{w^2m^2} m w^2 \; 2Tm\eta\,\delta(t_1-t_2) e^{-\frac{\eta}{2}(t-t_1)} e^{-\frac{\eta}{2}(t-t_2)}$$

$$\left[\sin(w_t(t-t_1))\sin(w_t(t-t_2)) + \cos(w_t(t-t_1))\cos(w_t(t-t_2))\right]$$

$$\stackrel{!}{=} \int_0^t dt_1 \frac{1}{2}\frac{1}{w^2m^2} m w^2 \; 2Tm\eta \; e^{-\eta(t-t_1)}$$

$$= \eta T \int_0^t e^{-\eta(t-t_1)} dt_1$$

$$\stackrel{!}{=} \eta T \frac{1}{\eta} \left| e^{-\eta(t-t)} - e^{-\eta t} \right|$$

$$\stackrel{!}{=} T\left(1 - e^{-\eta t}\right)$$