Look for a general coordinate map:

$$\left.\begin{aligned} q &\to Q(q,p) \\ p &\to P(q,p) \\ H &\to \tilde{H}(Q,P) \end{aligned}\right\}$$

We want a change of variable to have the same form of EOM

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \qquad \dot{P} = -\frac{\partial \tilde{H}}{\partial Q}$$

let:

$$z = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ \vdots \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \leftarrow 2\times n \text{ dimension.}$$

then:

$$\frac{dz^i}{dt} = J^{ij}\frac{\partial H}{\partial z^j} \quad \text{where} \quad \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \\ \hline -1 & & & 0 & & \\ & -1 & & & 0 & \\ & & -1 & & & 0 \end{array}\right)$$

i.e.

$$\begin{pmatrix} \frac{dq}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

Need to do transformation:

$$z^i \rightleftharpoons y^i$$

where $y^i = \begin{pmatrix} Q_1 \\ \vdots \\ P_1 \\ \vdots \end{pmatrix} \Rightarrow \frac{dy^i}{dt} = \underbrace{\frac{dy^i}{dz^j}}_{\text{Jacobian, } M^i{}_j} \frac{dz^j}{dt} = M_{ij} \frac{dz^j}{dt}$

we know $\frac{dz^j}{dt} = J^{jl} \frac{\partial \tilde{H}}{\partial z^l}$

then $\frac{dy^i}{dt} = M^i{}_j J^{jl} \frac{\partial \tilde{H}}{\partial z^l}$

$\stackrel{!}{=} M^i{}_j J^{jl} \frac{\partial \tilde{H}}{\partial y^l} \frac{dy^l}{dz^k}$

$\rightarrow M^l{}_k = (M^T)^k{}_l$

$\stackrel{!}{=} (MJM^T)^{ik} \frac{\partial \tilde{H}}{\partial y^k}$

but for canonical transformation:

$\frac{dy^i}{dt} = J^{ik} \frac{\partial \tilde{H}}{\partial y^k}$

so $\boxed{(MJM^T)^{ik} = J^{ik}}$ For canonical transformation

Therefore, we want $z^i \to y^i$, and y^i to satisfy the same Hamiltonian form:

$$\frac{\partial y^i}{\partial t} = J^{ij} \frac{\partial \tilde{H}}{\partial y^j} = (MJM^T)^{ij} \frac{\partial \tilde{H}}{\partial y^j}$$

So a transform is canonical iff:

Transformed variable

$$\boxed{J = MJM^T \quad \text{where } M = \frac{\partial \zeta}{\partial z} \leftarrow \text{original variable}}$$

<u>Consider Infinitesimal Transform:</u>

for $\lambda \ll 1$

$$q \to Q(q,p) = q + \dot{Q}(q,p)\lambda \qquad \text{for } \dot{Q} = \frac{dQ(q,p)}{d\lambda} = \Delta q$$

$$p \to P(q,p) = p + \dot{P}(q,p)\lambda \qquad \text{for } \dot{P} = \frac{dP}{d\lambda} = \Delta p$$

then the Jacobian: $M^i_{\ j} = \frac{\partial(Q,P)}{\partial(q,p)} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$

$$\hookrightarrow \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\partial \dot{Q}}{\partial q}\lambda & \frac{\partial \dot{Q}}{\partial p}\lambda \\ \frac{\partial \dot{P}}{\partial q}\lambda & 1 + \frac{\partial \dot{P}}{\partial p}\lambda \end{pmatrix}$$

Need $\quad J = MJM^T \quad$ where $M = 1 + M^{(1)}$

$$J = (1 + M^{(1)})\, J\, (1 + M^{(1)T}) \qquad \text{for } M^{(1)} = \begin{pmatrix} \frac{\partial \dot{Q}}{\partial q} & \frac{\partial \dot{Q}}{\partial p} \\ \frac{\partial \dot{P}}{\partial q} & \frac{\partial \dot{P}}{\partial p} \end{pmatrix}$$

To zeroth order

$$J = J$$

To first order:

$$M^{(1)}J + JM^{(1)T} = 0$$

$$\begin{pmatrix} \frac{\partial(\Delta q)}{\partial q} & \frac{\partial(\Delta q)}{\partial p} \\ \frac{\partial(\Delta p)}{\partial q} & \frac{\partial(\Delta p)}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \left[\frac{\partial(\Delta q)}{\partial q}\right]^T & \left(\frac{\partial(\Delta p)}{\partial q}\right)^T \\ \left(\frac{\partial(\Delta q)}{\partial p}\right)^T & \left(\frac{\partial(\Delta p)}{\partial p}\right)^T \end{pmatrix}$$

$$\hookrightarrow \begin{matrix} \cancel{-\frac{\partial(\Delta q)}{\partial p} + \left(\frac{\partial(\Delta q)}{\partial p}\right)^T} & \frac{\partial(\Delta q)}{\partial q} + \left(\frac{\partial(\Delta p)}{\partial p}\right)^T \\ -\frac{\partial(\Delta p)}{\partial p} + \left(\frac{\partial(\Delta q)}{\partial q}\right)^T & \cancel{\frac{\partial(\Delta p)}{\partial q} - \left(\frac{\partial(\Delta p)}{\partial q}\right)^T} \end{matrix}$$

With a constraint:

$$\frac{\partial(\Delta q)}{\partial q} + \frac{\partial(\Delta p)}{\partial p} = 0 \quad \leftarrow \text{curl-free.}$$

$$\text{or} \quad \frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} = 0$$

Suppose we have $G(q,p)$

such that $\boxed{\Delta q = \frac{\partial G}{\partial p} \quad \text{and} \quad \Delta p = -\frac{\partial G}{\partial q}}$

then $\frac{\partial(\Delta q)}{\partial q} + \frac{\partial(\Delta p)}{\partial p}$

$$= \frac{\partial^2 G}{\partial q \partial p} - \frac{\partial^2 G}{\partial q \partial p} = 0$$

Hence $G(q,p)$ is a generator of transformation

Now, all transformation:

$$\boxed{\begin{aligned} q &\to Q(q,p) = q + \underbrace{\frac{\partial G}{\partial p}\lambda}_{\delta q} \\ p &\to P(q,p) = p - \frac{\partial G}{\partial q}\lambda \end{aligned}}$$

Noether Theorem:

$$H(q,p) \to H(q+\delta q, p+\delta p) = H(Q,P)$$

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p = 0$$

$$\overset{!}{=} \frac{\partial H}{\partial q}\frac{\partial G}{\partial p}\lambda - \frac{\partial H}{\partial p}\frac{\partial G}{\partial q}\lambda$$

$$\overset{!}{=} \underbrace{\{H, G\}}_{=0}\lambda = 0$$

If $\{G, H\} = \frac{dG}{dt} = 0$

↳ G is conserved if the canonical transformation from G leaves H invariant.

Note: $\tilde{H}(Q,P) = H(q,p)$ ← always true.

but for invariant:

$$\tilde{H}(Q,P) = H(Q,P) = H(q,p)$$

so $H(Q,P) - H(q,p) = 0$

<u>Example:</u> For $H = \frac{p^2}{2m}$

clearly, H is invariant under:

$$\left.\begin{aligned} q &\to Q = q + \lambda \\ p &\to P = p \end{aligned}\right\}$$ since H doesn't depend on q.

then $H(q,p) = \frac{p^2}{2m} = H(Q,P) = \frac{P^2}{2m} = \frac{p^2}{2m}$

Then $\Delta q = \frac{\partial G}{\partial p} = 1$ and $\Delta p = \frac{-\partial G}{\partial q} = 0$

So $G(q,p) = 1$

since H is invariant

$G = p$ is conserved.

<u>Ex 2:</u> Consider infinitesimal rotation:

$$X = x + y\delta\theta$$
$$Y = y - x\delta\theta$$
$$P_x = p_x + p_y\delta\theta$$
$$P_Y = p_y - p_x\delta\theta$$

$\star$ First find whether transformation is canonical,

check:
$$\frac{\partial(\Delta q^i)}{\partial q^j} + \frac{\partial(\Delta p_j)}{\partial p^i} = 0$$

$$\frac{\partial(y\delta\theta)}{\partial x} + \frac{\partial(P_y\delta\theta)}{\partial P_x} = 0$$

$$\frac{\partial(-x\delta\theta)}{\partial y} + \frac{\partial(-P_x\delta\theta)}{\partial P_y} = 0$$

$$\frac{\partial(y\delta\theta)}{\partial y} + \frac{\partial(-P_x\delta\theta)}{\partial P_x} = \delta\theta - \delta\theta = 0$$

<u>Then find the generator, $G(q,p)$:</u>

$$\frac{\partial G}{\partial P_x} = \Delta X = Y \qquad -\frac{\partial G}{\partial X} = \Delta p = P_y$$

$$\frac{\partial G}{\partial P_y} = \Delta Y = -X \qquad -\frac{\partial G}{\partial Y} = \Delta p = -P_x$$

$$G = -P_y X + P_x Y = -L_z$$

$$\overset{!}{=} -(XP_y - YP_x)$$