① Constraints in Hamiltonian:

$$\psi_k(p_i, q_i, t) = 0$$

$$\tilde{\mathcal{L}} = \mathcal{L} + \lambda_k \psi^k$$

$$\hat{\mathcal{H}} = p_i \dot{q}^i - \tilde{\mathcal{L}}$$

$$S[p_i, q^i, \lambda] = \int \mathcal{L} + \lambda_k \psi^k(p_i, q^i, t)$$

$$\stackrel{!}{=} \int p_i \dot{q}^i - \mathcal{H}(p_i, q^i) - \lambda_k \psi^k(p_i, q^i, t)$$

$$S[p_i + dp_i, q^i + dq^i, \lambda + d\lambda] = \int (p_i + dp_i)(\dot{q}^i + d\dot{q}^i) - \left\{\mathcal{H} + \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial q_i} dq^i\right\}$$
$$- (\lambda_k + d\lambda_k)\left(\psi(p_i, q^i, t) + \frac{\partial \psi}{\partial p_i} dp_i + \frac{\partial \psi}{\partial q_i} dq^i\right)$$

$$\delta S = \int p_i d\dot{q}^i + \dot{q}^i dp_i - \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial q_i} dq^i$$
$$- \lambda_k \left(\frac{\partial \psi}{\partial p_i} dp_i + \frac{\partial \psi}{\partial q_i} dq^i\right) - d\lambda_k \psi(p_i, q^i, t)$$

$$\stackrel{!}{=} \int \cancel{\frac{d}{dt}(p_i dq^i)} + dp_i \left(\dot{q}^i - \frac{\partial \mathcal{H}}{\partial p_i} - \lambda_k \frac{\partial \psi}{\partial p_i}\right)$$
$$+ dq^i \left(-\frac{d}{dt}(p_i) - \frac{\partial \mathcal{H}}{\partial q^i} - \lambda_k \frac{\partial \psi}{\partial q^i}\right)$$
$$- d\lambda_k \cancel{\psi(p_i, q^i, t)}$$

$$\hookrightarrow \boxed{\begin{aligned} \dot{q}^i &= \frac{\partial \mathcal{H}}{\partial p_i} + \lambda_k \frac{\partial \psi^k}{\partial p_i} \\ \frac{d}{dt}(p_i) &= -\left(\frac{\partial \mathcal{H}}{\partial q^i} + \lambda_k \frac{\partial \psi^k}{\partial q^i}\right) \end{aligned}}$$

$$\text{vs.}\quad p_i = \frac{\partial L}{\partial \dot{q}^i} + \lambda_k \frac{\partial \psi^k}{\partial \dot{q}^i}$$

$$\frac{d}{dt}(p_i) = \frac{\partial L}{\partial q^i} + \lambda_k \frac{\partial \psi^k}{\partial q^i}$$

2) $S[q] = \int dt\, L(q, \dot{q})$

Use Lagrange Multiplier called $p(t)$

$$\hat{S}[q(t), v(t), p(t)] \equiv \int dt\, \hat{L}(q, \dot{q}, v, p)$$

for $\hat{L} \equiv L(q, v) - p(v - \dot{q})$

Enforces $v = \dot{q}$

Require $\delta S = 0$ for independent variation of q, v, p

a) Consider Lagrangian $L = \frac{1}{2} m \dot{q}^2 - U(q)$

$$\hat{S}[q, v, p] = \int dt\, L(q, v) - p(v - \dot{q})$$

$$\hat{S}[q+dq, v+dv, p+dp] = \int dt\, L(q+dq, v+dv) - (p+dp)(v+dv-\dot{q}-d\dot{q})$$

$$\hat{S}[q+dq, v+dv, p+dp] = \int dt\, L(q, v) + \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial v} dv$$
$$- p(v + dv - \dot{q} - d\dot{q}) - dp(v - \dot{q})$$

$$\delta S = \int dt\, \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial v} dv - p\,dv + p\,d\dot{q} - dp\,v + dp\,\dot{q}$$

$$\doteq \int dt \left(\frac{\partial L}{\partial v} - p\right) dv + \frac{\partial L}{\partial q}\, dq + \frac{d}{dt}(p\, dq) - \frac{dp}{dt}\, dq - dp\, v + dp\, \dot{q}$$

$$\doteq \int dt \left(\frac{\partial L}{\partial v} - p\right) dv + (\dot{q} - v)\, dp + \left(\frac{\partial L}{\partial q} - \frac{dp}{dt}\right) dq$$

① $\frac{\partial L}{\partial v} - p = 0$ ② $\dot{q} - v = 0$ ③ $\frac{\partial L}{\partial q} - \frac{dp}{dt} = 0$

For $L = \frac{1}{2} m \dot{q}^2 - U(q)$

$$\frac{\partial L}{\partial q} = -\frac{\partial}{\partial q} U(q) = \frac{dp}{dt} \leftarrow p \text{ is momentum.}$$

$$\dot{q} = v$$

$$\frac{\partial L}{\partial v} = p \doteq \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$

b) One way to extremize $\hat{S}$ is to first extremize $\hat{S}$ with respect to p, v with q fixed.

Then $\hat{S}_{red}[q] = S[q]$

$$\hat{S}[q, p+dp, v+dv] = \int dt\ L(q, v+dv) - (p+dp)(v+dv-\dot{q})$$

$$\delta S = \int \left\{ \frac{\partial L}{\partial v} dv - p\, dv - dp\, v + dp\, \dot{q} \right\} dt$$

$$\doteq \int \left\{ \left(\frac{\partial L}{\partial v} - p\right) dv + (\dot{q} - v)\, dp \right\} dt$$

$$=0 \qquad\qquad =0$$

$$\frac{\partial L}{\partial v} = p \qquad \dot{q} = v$$

$$\hat{S}_{red}[q] = \int dt\, L(q,v) - p\cancel{(v-\dot{q})}^{0}$$

$$\hat{S}_{red}[q] \stackrel{!}{=} \int dt\, L(q,\dot{q})$$

c) If $\hat{S}_{red}[q,p]$ then it corresponds to Hamiltonian.

$$\hat{S}[q,p,v+dv] = \int dt\, L(q,v+dv) - p(v+dv-\dot{q})$$

$$\delta S = \int dt\, \frac{\partial L}{\partial v} dv - p\,dv$$

$$\stackrel{!}{=} \int dt\, dv\left(\frac{\partial L}{\partial v} - p\right)$$

$$\frac{\partial L}{\partial v} = p$$

$$\hat{S}_{red}[q,p] = \int dt\, L(q,v) - p(v-\dot{q}) = \int dt\left(p\frac{dq}{dt} - H(p,q)\right)$$

$$\stackrel{!}{=} \int dt\, p\dot{q} - pv + L(q,v)$$

$$\stackrel{!}{=} \int dt\, p\dot{q} - \frac{\partial L}{\partial v} v + L(q,v)$$

$$\stackrel{!}{=} \int dt\, p\dot{q} - \left(\frac{\partial L}{\partial v} v - L(q,v)\right)$$

$$-\int dt \, \Big[\; \underbrace{\Big(\frac{\partial L}{\partial \dot{q}}\dot{q} - L(q,\dot{q})\Big)}_{H(p,q)} \Big]$$

3) Virial Theorem: For periodic motion of a particle

$$\overline{2T} = \overline{\vec{r}\cdot\frac{\partial U(\vec{r})}{\partial \vec{r}}}$$

To single particle: $L = \frac{1}{2}m\dot{r}^2 - U(r)$

then: $$\overline{2T} = \overline{\sum_a \vec{r}_a \cdot \frac{\partial U(r)}{\partial \vec{r}_a}}$$

a) For a closed orbit of $U(r) \propto r^{\beta}$:

For $U(r) = kr^{\beta}$ $$\overline{2T} = \overline{\sum_a \vec{r}\cdot\frac{\partial k r_a^{\beta}}{\partial r}} = \sum_a \vec{r}_a \cdot \beta k r^{\beta-1}$$

$$\overline{2T} = \beta k r^{\beta} = \overline{\beta U(r)}$$

For $\beta = 2$. $\overline{2T} = 2kr^2 = 2U(r)$ or $\overline{T} = \overline{U(r)}$

For $\beta = -1$ $\overline{2T} = -\overline{U}$: Gravitational potential

$$\overline{T} = -\frac{1}{2}\overline{U(r)}$$

b) Consider $H|\psi_n(x)\rangle = E_n|\psi_n\rangle$

show $$\langle 2T \rangle = \langle x \frac{\partial U(x)}{\partial x} \rangle$$

by considering $\langle \psi_n | [xp, H] | \psi_n \rangle$

$$\langle \psi_n | [xp, H] | \psi_n \rangle = \langle \psi_n | xpH - Hxp | \psi_n \rangle$$

$p = -i\hbar \frac{\partial}{\partial x}$

$$\stackrel{!}{=} \langle xp \rangle E_n - E_n \langle xp \rangle = 0$$

since $[ab, c] = a[b,c] + [a,c]b$

$$[xp, H] = x[p,H] + [x,H]p$$

$$\stackrel{!}{=} x[p, \tfrac{p^2}{2m} + V] + [x, \tfrac{p^2}{2m} + V]p$$

$$\stackrel{!}{=} x\left[\underbrace{\cancel{[p, \tfrac{p^2}{2m}]}}_{0} + [p,V]\right] + \left[[x, \tfrac{p^2}{2m}] + \overbrace{\cancel{[x,V]}}^{0}\right]p$$

$$\stackrel{!}{=} x[p,V] + [x, \tfrac{p^2}{2m}]p$$

$$\stackrel{!}{=} x(pV - Vp) + \left(\frac{xp^2}{2m} - \frac{p^2}{2m}x\right)p$$

$$\stackrel{!}{=} xpV - xVp + \frac{xp^3}{2m} - \frac{p^2}{2m}xp$$

$p = -i\hbar \frac{\partial}{\partial x}$

$p^3 = i\hbar^3 \frac{\partial^3}{\partial x^3}$

$$xpV\psi(x) - xVp\psi(x) + \frac{xp^3}{2m}\psi(x) - \frac{p^2}{2m}xp\psi(x)$$

$$\hookrightarrow = -i\hbar x\left(\frac{\partial}{\partial x}(V(x)\psi(x)) - V(x)\frac{\partial}{\partial x}\psi\right) + \frac{i\hbar^3}{2m}\left(x\frac{\partial^3}{\partial x^3}\psi - \frac{\partial^2}{\partial x^2}\left(x\frac{\partial}{\partial x}\psi(x)\right)\right)$$

$$\stackrel{!}{=} -i\hbar x\left(\psi\frac{\partial}{\partial x}V + \cancel{V\frac{\partial}{\partial x}\psi} - \cancel{V\frac{\partial}{\partial x}\psi}\right) + \frac{i\hbar^3}{2m}\left(x\frac{\partial^3}{\partial x^3}\psi - \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\psi + x\frac{\partial^2}{\partial x^2}\psi\right\}\right)$$

$$\stackrel{!}{=} -i\hbar x\psi\frac{\partial}{\partial x}V + \frac{i\hbar^3}{2m}\left(\cancel{x\frac{\partial^3}{\partial x^3}\psi} - \frac{\partial^2}{\partial x^2}\psi - \frac{\partial^2}{\partial x^2}\psi - \cancel{x\frac{\partial^3}{\partial x^3}\psi}\right)$$

$$\stackrel{!}{=} -i\hbar x\psi\frac{\partial}{\partial x}V - \frac{i\hbar^3}{m}\frac{\partial^2}{\partial x^2}\psi$$

$$\langle xp, H\rangle = -i\hbar x \frac{\partial}{\partial x} V - \frac{i\hbar^3}{2m} 2 \frac{\partial^2}{\partial x^2} \qquad p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\frac{\langle xp, H\rangle}{i\hbar} = \langle -x \frac{\partial}{\partial x} V + \frac{p^2}{2m} 2 \rangle$$

$$\hookrightarrow \quad \langle x \frac{\partial}{\partial x} U \rangle = 2T$$

c) Consider a specific variation over one complete period of a periodic classical trajectory Γ?

$$\vec{r} \to (1+\epsilon)\vec{r} = \vec{r} + \epsilon\vec{r}$$

$$S[r_a + \delta r_a] = \int dt \, \mathcal{L}(r_a + \delta r_a, \dot{r}_a + \delta \dot{r}_a)$$

$$\doteq \int dt \, \mathcal{L}(r_a, \dot{r}_a) + \frac{\partial \mathcal{L}}{\partial r_a} \delta r_a + \frac{\partial \mathcal{L}}{\partial r_a} \delta \dot{r}_a$$

$$\delta S = \int dt \, \frac{\partial \mathcal{L}}{\partial r_a} \delta r_a + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_a} \delta r_a\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_a}\right) \delta r_a$$

$$\doteq \frac{\partial \mathcal{L}}{\partial \dot{r}} \delta r_a \Big|_{r_0}^{r_1} + \int dt \, \underbrace{\left\{\frac{\partial \mathcal{L}}{\partial r_a} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_a}\right)\right\}}_{=0 \text{ if on-shell.}} \delta r_a$$

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{r}_a} \epsilon r_a \Big|_{t_0}^{T+t_0} = 0$$

↑ since after full cycle

$$r_a(T+t_0) = r_a(t_0)$$

d) If not on-shell:

$$\delta S = \int dt \left\{\frac{\partial \mathcal{L}}{\partial r_a} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_a}\right)\right\} \epsilon r^a$$

For $L = \frac{1}{2} m \dot{r}^2 - U(r)$

$$\frac{\partial L}{\partial r} = -\frac{\partial}{\partial r} U(r)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) \right\} \epsilon r$$

$$= \int dt \left\{ -\frac{\partial}{\partial r} U(r) - m\ddot{r} \right\} \epsilon r$$

$$= \epsilon \left(\int -\frac{\partial}{\partial r} U(r)\, dt - \int m\, dt \underbrace{\left\{ \frac{d}{dt}(\dot{r} r) - \dot{r}^2 \right\}} \right)$$

Divide by T_0 to get average:

$$\underbrace{\frac{1}{T_0}\int_{t_0}^{T+t_0} -\epsilon \frac{\partial}{\partial r} U(r)\, r\, dt} + \underbrace{\frac{1}{T_0}\int_{t_0}^{T_0} \epsilon m \dot{r}^2\, dt} - \cancel{\dot{r} r \Big|_{t_0}^{T+t_0}}_{0}$$

$$\epsilon \left\{ -\left\langle r \frac{\partial}{\partial r} U(r) \right\rangle + \langle 2T \rangle \right\} = 0$$

$$\left\langle r \frac{\partial}{\partial r} U(r) \right\rangle = \langle 2T \rangle$$

$$\hookrightarrow \frac{1}{T_0}\int_{t_0}^{T+t_0} r \frac{\partial}{\partial r} U(r)\, dt = \frac{1}{T_0}\int_{t_0}^{T+t_0} 2 \frac{1}{2} m \dot{r}^2\, dt$$

4) Foucault Pendulum and Coriolis Effect:

a) $\vec{F}_{eff} \simeq \vec{F} - 2m(\vec{\omega} \times \vec{v}_r)$ ← Derive from Lagrangian.

Stationary: $\mathcal{L} = \frac{1}{2} m \vec{v}_0^{\,2} - U(r)$

$$\vec{r}_0 = \vec{r} + \vec{r}_k$$

$$\vec{v}_0 = \frac{d}{dt}\vec{r}_0 = \frac{d\vec{r}}{dt} + \frac{d\vec{r}_k}{dt}.$$

$$\mathcal{L} = \frac{1}{2} m \left(\frac{d\vec{r}}{dt} + \frac{d\vec{r}_k}{dt}\right)^2 - U(r)$$

$$\stackrel{!}{=} \frac{1}{2} m \left(\frac{d\vec{r}}{dt}^2 + 2\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}_k}{dt} + \frac{d\vec{r}_k}{dt}^2\right) - U(r)$$

Doesn't depend on $\vec{r}$ so ignore

$$= \frac{1}{2} m \left(\frac{d\vec{r}}{dt}^2 + \cancel{2\frac{d}{dt}\left(\vec{r}\cdot\frac{d\vec{r}_k}{dt}\right)} - 2\vec{r}\cdot\frac{d^2\vec{r}_k}{dt}\right) - U(r)$$

$$\stackrel{!}{=} \frac{1}{2} m \left(\frac{d\vec{r}}{dt}^2 - 2\vec{r}\cdot\frac{d^2\vec{r}_k}{dt^2}\right) - U(r)$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} + \vec{\omega}\times\vec{r} = \vec{v}_r + \vec{\omega}\times\vec{r}$$

$$\mathcal{L} \stackrel{!}{=} \frac{1}{2} m \left(\vec{v}_r^{\,2} + 2\vec{v}_r\cdot(\vec{\omega}\times\vec{r}) + (\vec{\omega}\times\vec{r})^2 - 2\vec{r}\cdot\vec{a}_k\right) - U(r)$$

$$P_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\vec{v}_r + m(\vec{\omega}\times\vec{r})$$

$$\frac{\partial\mathcal{L}}{\partial\vec{r}} = \frac{\partial}{\partial\vec{r}}\left\{\frac{1}{2} m\left[2\vec{v}_r\cdot(\vec{\omega}\times\vec{r}) + (\vec{\omega}\times\vec{r})(\vec{\omega}\times\vec{r}) - 2\vec{r}\cdot\vec{a}_k\right] - U(r)\right\}$$

$$\stackrel{!}{=} \frac{1}{2} m\left[2\left(\frac{\partial}{\partial r}\vec{r}\cdot(\vec{v}_r\times\vec{\omega})\right) + 2(\vec{\omega}\times\vec{r})\frac{\partial}{\partial r}(\vec{\omega}\times\vec{r}) - 2\vec{a}_k\right] - \frac{\partial U}{\partial r}$$

$$= \frac{1}{2} m\, 2\,(\vec{v}_r\times\vec{\omega}) + \frac{1}{2} m\, 2\,(\vec{\omega}\times\vec{r})\times\vec{\omega} - \frac{1}{2} m\, 2\,\vec{a}_k - \frac{\partial U}{\partial r}$$

$$\stackrel{!}{=} m(\vec{v}_r\times\vec{\omega}) - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\vec{a}_k - \partial U$$

$= -m(\vec{w}\times\vec{v}_r) - m\vec{w}\times(\vec{w}\times\vec{r}) \quad \cdots \quad \frac{\partial U}{\partial r}$

$$\frac{d}{dt}(P_{\vec{r}}) = \frac{d}{dt}\left\{m\vec{v}_r + m(\vec{w}\times\vec{r})\right\}$$

$$\doteq m\vec{a}_r + m\left\{(\dot{\vec{w}}\times\vec{r}) + (\vec{w}\times\dot{\vec{r}})\right\}$$

Since $L = \frac{1}{2}mv_0^2 - U$ ↖ since not taking

$\frac{\partial L}{\partial r} = -\frac{\partial U}{\partial r} = F$ $\quad \frac{d}{dt}$ in fixed frame, there is no $(\vec{w}\times)$ terms.?

$$ma_r + m(\dot{\vec{w}}\times\vec{r} + \vec{w}\times\vec{v}_r)$$

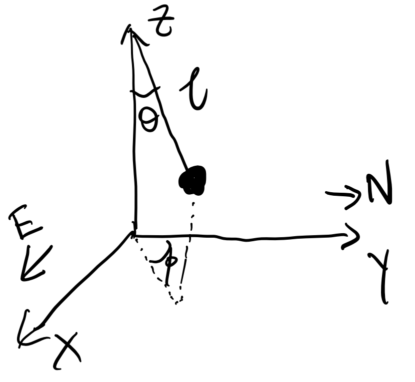
$$= -m(\vec{w}\times\vec{v}_r) - m\vec{w}\times(\vec{w}\times\vec{r}) - m\vec{a}_k - \frac{\partial U}{\partial r}$$

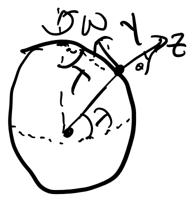
$$ma_r = F - 2m(\vec{w}\times\vec{v}_r) - m(\dot{\vec{w}}\times\vec{r}) - m\vec{w}\times(\vec{w}\times\vec{r}) - m\vec{a}_k$$

b) Consider pendulum of long massless rod of length l attached to mass m. The pendulum is hung in a tower at latitude λ.

$L \doteq \frac{1}{2}m(\vec{v}_r^{\,2} + 2\vec{v}_r\cdot(\vec{w}\times\vec{r}) + (\vec{w}\times\vec{r})^2 - 2\vec{r}\cdot\vec{a}_k) - U(r)$

z, is perpendicular to the Earth's surface





Keep terms of w to first order.

$z = -l\cos\theta$

$x = l\sin\theta\sin\phi$

$y = l\sin\theta\cos\phi$

ignore second order

$$\mathcal{L} = \frac{1}{2}m\left(\vec{v}_r^{\,2} + 2\vec{v}_r\cdot(\vec{\omega}\times\vec{r}) + \cancel{(\vec{\omega}\times\vec{r})^2 - 2\vec{r}\cdot\vec{a}_k}\right) - U(r)$$

No translational acceleration of frame.

$$\approx \frac{1}{2}m\left(v_r^2 + 2\vec{v}_r\cdot(\vec{\omega}\times\vec{r})\right) - U(r)$$

$$\doteq \frac{1}{2}m\left\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})\cdot(\vec{\omega}\times\vec{r})\right\} - mgz$$

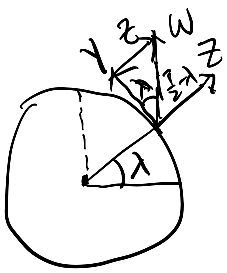
$$\begin{cases} \dot{z} = l\sin\theta\,\dot{\theta} \\ \dot{x} = l(\sin\theta\cos\phi\,\dot{\phi} + \cos\theta\sin\phi\,\dot{\theta}) \\ \dot{y} = l(\sin\theta\sin\phi\,\dot{\phi} + \cos\theta\cos\phi\,\dot{\theta}) \\ \vec{\omega} = \omega(\sin\lambda\,\hat{z} - \cos\lambda\,\hat{y}) \end{cases}$$

$$\frac{1}{2}mv_r^2 = \frac{1}{2}ml^2\left\{\sin^2\theta\,\dot{\theta}^2 + \sin^2\theta\cos^2\phi\,\dot{\phi}^2 + \cos^2\theta\sin^2\phi\,\dot{\theta}^2 + 2\cancel{\sin\theta\cos\phi\cos\theta\sin\phi}\,\dot{\theta}\dot{\phi} + \sin^2\theta\sin^2\phi\,\dot{\phi}^2 + \cos^2\theta\cos^2\phi\,\dot{\theta}^2 - 2\cancel{\sin\theta\cos\theta\sin\phi\cos\phi}\,\dot{\theta}\dot{\phi}\right\}$$

$$\frac{1}{2}mv_r^2 = \frac{1}{2}ml^2\left(\sin^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 + \cos^2\theta\,\dot{\theta}^2\right)$$

$$\doteq \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

$$\frac{1}{2}mv_r^2 \doteq \frac{1}{2}ml^2\left(\dot{\theta}^2 + \theta^2\dot{\phi}^2\right)$$



$$\omega = \cos\lambda\,\hat{y} + \sin\lambda\,\hat{z}$$

$$m\vec{v_r}\cdot(\vec{\omega}\times\vec{r}) = m\vec{v_r}\cdot(\cos\lambda\,\hat{y}+\sin\lambda\,\hat{z})\times(x\,\hat{x}+y\,\hat{y}+z\,\hat{z})$$

$$\hookrightarrow = m\vec{v_r}\cdot\left(-\cos\lambda\, x\,\hat{z}+\cos\lambda\, z\,\hat{x}+x\sin\lambda\,\hat{y}-y\sin\lambda\,\hat{x}\right)$$

$$\doteq m\left(-\cos\lambda\, x\,\dot{z}+(\cos\lambda\, z-\sin\lambda\, y)\dot{x}+x\sin\lambda\,\dot{y}\right)$$

$$\begin{cases} \dot{z} = l\sin\theta\,\dot{\theta} \\ \dot{x} = l(\sin\theta\cos\phi\,\dot{\phi}+\cos\theta\sin\phi\,\dot{\theta}) \\ \dot{y} = l(-\sin\theta\sin\phi\,\dot{\phi}+\cos\theta\cos\phi\,\dot{\theta}) \end{cases}$$

$$z = -l\cos\theta$$
$$x = l\sin\theta\sin\phi$$
$$y = l\sin\theta\cos\phi$$

$$\begin{aligned}\doteq m\Big(&-\cos\lambda\, l\sin\theta\sin\phi\, l\sin\theta\,\dot{\theta} \\ &+\cos\lambda(-l\cos\theta)\, l(\sin\theta\cos\phi\,\dot{\phi}+\cos\theta\sin\phi\,\dot{\theta}) \\ &-\sin\lambda\, l\sin\theta\cos\phi\, l(\sin\theta\cos\phi\,\dot{\phi}+\cos\theta\sin\phi\,\dot{\theta}) \\ &+\sin\lambda\, l\sin\theta\sin\phi\, l(-\sin\theta\sin\phi\,\dot{\phi}+\cos\theta\cos\phi\,\dot{\theta})\Big)\end{aligned}$$

$$\begin{aligned}\doteq m\Big\{&-\cos\lambda\, l^2\sin^2\theta\sin\phi\,\dot{\theta} \\ &-\cos\lambda\, l^2\cos^2\theta\sin\phi\,\dot{\theta} \\ &-\sin\lambda\, l^2\sin\theta\,\cancel{\cos\phi}\cos\theta\sin\phi\,\dot{\theta} \\ &+\sin\lambda\, l^2\,\cancel{\sin\theta}\sin\phi\cos\theta\cos\phi\,\dot{\theta} \\ &-\cos\lambda\, l^2\cos\theta\sin\theta\cos\phi\,\dot{\phi} \\ &-\sin\lambda\, l^2\sin^2\theta\cos^2\phi\,\dot{\phi}\end{aligned}$$

$$-\sin\lambda \ l^2 \sin^2\theta \sin^2\phi \dot{\phi} \}$$

$$\stackrel{!}{=} wm \{ -\cos\lambda \ l^2 \sin\phi \dot{\theta} - \cos\lambda \ l^2 \cos\theta \sin\theta \cos\phi \dot{\phi}$$
$$-\sin\lambda \ l^2 \sin^2\theta \dot{\phi}$$

$$= wm \{ -\cos\lambda \ l^2 (\sin\phi \dot{\theta} + \cos\theta \sin\theta \cos\phi \dot{\phi})$$
$$-\sin\lambda \ l^2 \sin^2\theta \dot{\phi} \}$$

$$\frac{d}{dt}\left(\cos\theta \sin\theta \sin\phi \right) = \cos\phi \dot{\phi} \cos\theta \sin\theta + \sin\phi \left[-\sin^2\theta \dot{\theta} + \cos^2\theta \dot{\theta} \right]$$
$$\stackrel{!}{=} \cos\phi \dot{\phi} \cos\theta \sin\theta + \sin\phi \dot{\theta} \left[\underbrace{-\cos(2\theta)}_{\cong 1} \right]$$
$$\stackrel{!}{=} \cos\phi \dot{\phi} \cos\theta \sin\theta + \sin\phi \dot{\theta}$$

$$\stackrel{!}{=} wm \left\{ \cancel{-\cos\lambda \ l^2 \frac{d}{dt}\left(\cos\theta \sin\theta \sin\phi \right)} - \sin\lambda \ l^2 \sin^2\theta \dot{\phi} \right\}$$

ignore full-time derivative

$$\stackrel{!}{=} -wm \sin\lambda \ l^2 \sin^2\theta \dot{\phi}$$

$$\cong -wm \sin\lambda \ l^2 \theta^2 \dot{\phi}$$

$$L = \frac{1}{2} m \left(v_r^2 + 2\vec{v}_r \cdot (\vec{\omega} \times \vec{r}) \right) - U(r)$$

$$L = \frac{1}{2} m l^2 \left(\sin^2\theta \dot{\phi}^2 + \dot{\theta}^2 \right) - mw \sin\lambda \ l^2 \theta^2 \dot{\phi}$$
$$- mg(-l\cos\theta)$$

$$\cos\theta \cong 1 - \frac{\theta^2}{2}$$

$$\boxed{L \doteq \frac{1}{2}ml^2(\theta^2\dot\phi^2 + \dot\theta^2) - m\omega\sin\lambda\, l^2\theta^2\dot\phi - mgl\frac{\theta^2}{2} + \cancel{mgl}}$$

ignore constant.

c) $x = l\sin\theta\sin\phi = l\theta\sin\phi$

$y = l\sin\theta\cos\phi = l\theta\cos\phi$

$$P_\phi = \frac{\partial L}{\partial\dot\phi} = ml^2\theta^2\dot\phi - m\omega\sin\lambda\, l^2\theta^2$$

Since $L(\theta, \dot\phi, \dot\theta)$, not a function of ϕ, then

$\frac{\partial L}{\partial\phi} = 0$ or $\frac{d}{dt}P_\phi = 0$ or P_ϕ is constant.

Convert back to x, y, since $z = l\cos\theta \cong l$ for θ small.

$l^2 = x^2 + y^2 + z^2$ or $z = \sqrt{l^2 - x^2 - y^2} = l\sqrt{1 - \frac{x^2+y^2}{l^2}}$ since x^2+y^2 is small.

$\doteq l - \frac{1}{2}\frac{x^2+y^2}{l}$

$$\frac{1}{2}mv_r^2 \cong \frac{1}{2}m(\dot x^2 + \dot y^2 + \cancel{\dot z^2}) \cong \frac{1}{2}m(\dot x^2 + \dot y^2)$$

$$m\vec v_r\cdot(\vec\omega\times\vec r) = \omega m\vec v_r\cdot(\cos\lambda\hat y + \sin\lambda\hat z) \times (x\hat x + y\hat y)$$

$$\doteq m\vec v_r\cdot(-\cos\lambda x\hat z + \sin\lambda x\hat y - \sin\lambda y\hat x)$$

$$\dot{=} wm\left(\cancel{-\cos\lambda x}\,\dot{z} + \sin\lambda x\dot{y} - \sin\lambda y\dot{x}\right)$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + wm\sin\lambda(x\dot{y}-y\dot{x}) - mg\frac{x^2+y^2}{2l}$$

$$\frac{\partial L}{\partial \dot{x}} = P_x = m\dot{x} - wm\sin\lambda y$$

$$\frac{\partial L}{\partial \dot{y}} = P_y = m\dot{y} + wm\sin\lambda x$$

$$\frac{d}{dt}(P_x) = \frac{\partial L}{\partial x} = wm\sin\lambda\dot{y} - \frac{mgx}{l}$$

$$\doteq m\ddot{x} - wm\sin\lambda\dot{y}$$

$$\frac{d}{dt}(P_y) = \frac{\partial L}{\partial y} = -wm\sin\lambda\dot{x} - \frac{mg}{l}y$$

$$\doteq m\ddot{y} + wm\sin\lambda\dot{x}$$

$$\ddot{x} = 2w\sin\lambda\dot{y} - \frac{g}{l}x$$

$$\ddot{y} = -2w\sin\lambda\dot{x} - \frac{g}{l}y$$

$$z = x+iy$$

$$\dot{z} = \dot{x}+i\dot{y}$$

$$\ddot{z} = \ddot{x}+i\ddot{y}$$

$$\doteq 2w\sin\lambda\dot{y} - \frac{g}{l}x + i\left(-2w\sin\lambda\dot{x} - \frac{g}{l}y\right)$$

$$\doteq -\frac{g}{l}(x+iy) - i2w\sin\lambda(\dot{x}+i\dot{y})$$

$$\ddot{z} = -\frac{g}{l}z - i2w\sin\lambda\,\dot{z}$$

$\ddot{z}$

$$\ddot{z} + \frac{g}{l}z + i2w\sin\lambda\,\dot{z} = 0$$

let $z = Ae^{-i\Omega t}$

$$\dot{z} = -i\,Ae^{-i\Omega t}$$

$$\ddot{z} = -\Omega^2 Ae^{-i\Omega t} = -\Omega^2 z$$

$$\Big(\underbrace{-\Omega^2 + i2w\sin\lambda(-i\Omega) + g/l}\Big) Ae^{-i\Omega t} = 0.$$

$$-\Omega^2 + 2w\Omega\sin\lambda + g/l = 0$$

$$\Omega = \frac{-2w\sin\lambda \pm \sqrt{(2w\sin\lambda)^2 + 4g/l}}{-2}$$

$$\Omega \stackrel{!}{=} w\sin\lambda \pm \frac{\sqrt{(2w\sin\lambda)^2 + 4g/l}}{-2}$$

$$\Omega \stackrel{!}{=} w\sin\lambda \pm \sqrt{w^2\sin^2\lambda + g/l}$$

dominant

$$z = Ae^{-i\Omega t} = e^{-iw\sin\lambda t}\left(Ae^{-i\sqrt{w^2\sin^2\lambda + g/l}\,t} + Be^{i\sqrt{w^2\sin^2\lambda + g/l}\,t}\right)$$

If $A = B$. we $\frac{e^{ix} + e^{-ix}}{2} = \cos x$.

$$z = Ae^{-iw\sin\lambda t}\cos\left(\sqrt{w^2\sin^2\lambda + g/l}\,t\right)$$

$$\mathrm{Re}(z) = x = A\cos(w\sin\lambda t)\cos\left(\sqrt{w^2\sin^2\lambda + g/l}\,t\right)$$

$$\mathrm{Im}(z) = y = -A\sin(w\sin\lambda t)\cos\left(\sqrt{w^2\sin^2\lambda + g/l}\,t\right)$$

we know $x = \sin\theta\sin\phi$.

$y = \sin\theta\cos\phi$

then by matching,

$$\dot{\phi} = \omega \sin\lambda.$$

if $0 < \lambda < \pi/2$, Northern Hemisphere, $\sin\lambda > 0$
so counterclockwise.

if $-\frac{\pi}{2} < \lambda < 0$, Southern Hemisphere, $\sin\lambda < 0$
so clockwise