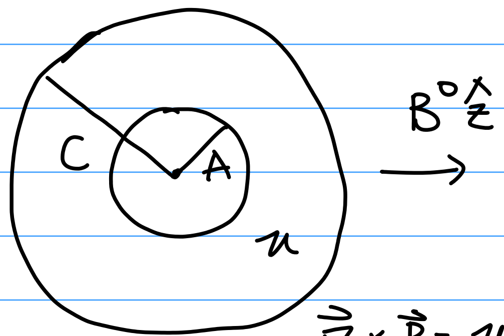
Zhi Chen HW#8

1)



Find $\vec{B}$, $r<A$

With Maxwell: $\vec{\nabla}\cdot\vec{B}=0$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \xrightarrow{\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}=\frac{1}{\mu}\vec{B}} \vec{\nabla}\times\vec{H}=\vec{J}_{cond}=0$$

Because $\vec{\nabla}\times\vec{H}=0$, we can let $\vec{H}=-\vec{\nabla}\chi$

since $\vec{\nabla}\times(-\vec{\nabla}\chi)=0$

If $\vec{H}=-\vec{\nabla}\chi$ then $\vec{B}=-\mu\vec{\nabla}\chi$

by $\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(-\mu\vec{\nabla}\chi)=-\mu\nabla^2\chi=0$

In spherical coord, $\nabla^2\chi=0$ has solution:

$$\chi=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l+B_{lm}r^{-(l+1)}\right)Y_l^m(\theta,\phi)$$

However problem is azimuthal symmetric, so $\boxed{m=0}$

$$\rightarrow \chi=\sum_{l=0}^{\infty}\left(A_l r^l+B_l r^{-(l+1)}\right)P_l(\cos\theta)$$

For $r > C$:

Expect $\vec{B}(r\to\infty) = \vec{B}^{ext} = B_0\hat{z}$

$$-\mu_0\vec{\nabla}\chi(\vec{r})\Big|_{r\to\infty} = B_0\hat{z}$$

or $\chi^{ext} = -\frac{1}{\mu_0}\int^{z} B_0\hat{z}\cdot dz'\hat{z} = -\frac{1}{\mu_0}B_0 z$

$$\boxed{\chi^{ext} = -\frac{1}{\mu_0}B_0 r\cos\theta}$$

as $r\to\infty$, $\chi\to\chi^{ext}$.

$$\chi(r\to\infty) = \sum_l A_l r^l P_l(\cos\theta) = -\frac{1}{\mu_0}B_0 r\cos\theta = A_1 r\cos\theta.$$

By matching, theres only $\boxed{l=1}$

By symmetry, don't expect higher variation terms for $r^{-(l+1)}$ term other than $l=1$ as well.

then $\boxed{\chi^{III}(r) = \left(-\frac{1}{\mu_0}B_0 C\left(\frac{r}{C}\right) + B\left(\frac{C}{r}\right)^2\right)\cos\theta \text{ for } r > C}$

For $A<r<C$: $\chi^{II} = \sum_l \left(a_l r^l + b_l r^{-(l+1)}\right)P_l(\cos\theta)$

Since for $r > C$, we expect only $l=1$ term, then we should expect the same behavior since the only difference between $A<r<C$ and $r > C$ is they have different μ, but μ is homogeneous throughout media. So χ^{II} should have form:

$$\rightarrow \boxed{\chi^{II} = (ar + br^{-2})\cos\theta = \left(a\left(\frac{r}{A}\right) + b\left(\frac{A}{r}\right)^2\right)\cos\theta}$$

For $\underline{r < A}$: By the same argument, it should have form:

$$\chi^{I}(\vec{r}) = (f r + \gamma r^{-2})\cos\theta$$

but in order to avoid divergence at $r \to 0$, set $\boxed{\gamma = 0}$

then $$\boxed{\chi^{I}(\vec{r}) = f r \cos\theta = f \frac{r}{A} \cos\theta}$$

<u>Boundary Cond:</u>

Now since $\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \hat{n} \cdot \vec{B}\big|_{r=c} = \hat{n} \cdot \vec{B}\big|_{r=c}$

$$\hookrightarrow \hat{n} \cdot -\mu^{III} \vec{\nabla}\chi^{III}\big|_{r=c} = \hat{n} \cdot -\mu^{II} \vec{\nabla}\chi^{II}\big|_{r=c}$$

since we have sphere, $\hat{n} = \hat{r}$

$$\hookrightarrow \boxed{-\mu^{III} \partial_r \chi^{III}\big|_{r=c} = -\mu^{II} \partial_r \chi^{II}\big|_{r=c}}$$

and $\vec{\nabla} \times \vec{H} = 0 \rightarrow d\vec{r} \cdot \vec{H}^{III} = d\vec{r} \cdot \vec{H}^{II}$

$$\hookrightarrow -(\vec{\nabla}\chi^{III})_{\parallel}\big|_{r=c} = -(\vec{\nabla}\chi^{II})_{\parallel}\big|_{r=c}$$

Since problem independent of ϕ, choose $d\vec{r}$ in $\hat{\theta}$ direction.

then $$\boxed{-\frac{1}{r} \partial_\theta \chi^{III}\big|_{r=c} = -\frac{1}{r} \partial_\theta \chi^{II}\big|_{r=c}}$$

At $\underline{\underline{r=A}}$ $\Rightarrow$ $\mu^{II}\partial_r \chi^{II}\Big|_{r=A} = \mu^{I}\partial_r\chi^{I}\Big|_{r=A}$

$$\mu\left(\frac{a}{A}-2b\frac{1}{A^3}A^2\right)=\mu_0 f\frac{1}{A}$$

$$\boxed{f=\frac{\mu}{\mu_0}(a-2b) = (1+\varepsilon)(a-2b)}$$

let $\mu/\mu_0 = 1+\varepsilon$

$\Rightarrow$ $-\frac{1}{r}\partial_\theta\chi^{II}\Big|_{r=A} = -\frac{1}{r}\partial_\theta\chi^{I}\Big|_{r=A}$

$$a+b = f$$

$$a+b = (1+\varepsilon)(a-2b)$$

$$b(3+2\varepsilon)=a\varepsilon$$

$$\boxed{b=a\frac{\varepsilon}{3+2\varepsilon}}$$

then $f=(1+\varepsilon)(a-2b)$

$$\stackrel{!}{=}(1+\varepsilon)a\left(1-\frac{2\varepsilon}{3+2\varepsilon}\right)$$

$$f=(1+\varepsilon)a\left(\frac{3}{3+2\varepsilon}\right)$$

then $\chi^{I} = a(1+\varepsilon)\frac{3}{3+2\varepsilon}\left(\frac{r}{A}\right)\cos\theta$

$$\chi^{II} = a\cos\theta\left(\frac{r}{A}+\left(\frac{A}{r}\right)^2\frac{\varepsilon}{3+2\varepsilon}\right)$$

$$\chi^{II} \stackrel{!}{=} a\frac{r}{A}\cos\theta\left[1+\left(\frac{A}{r}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right]$$

At $\underline{\underline{r=c}}$: $\Rightarrow \; -\frac{1}{r}\partial_\theta \chi^{III}\Big|_{r=c} = -\frac{1}{r}\partial_\theta \chi^{II}\Big|_{r=c}$

$$\hookrightarrow \quad -\frac{1}{u_0}B_0 C + B = a\left(\frac{C}{A} + \left(\frac{A}{C}\right)^2 \frac{\varepsilon}{3+2\varepsilon}\right)$$

$$\hookrightarrow \quad B = a\left(\frac{C}{A} + \left(\frac{A}{C}\right)^2 \frac{\varepsilon}{3+2\varepsilon}\right) + \frac{1}{u_0}B_0 C$$

$$\boxed{B \overset{!}{=} a\frac{C}{A}\left(1+\left(\frac{A}{C}\right)^3 \frac{\varepsilon}{3+2\varepsilon}\right) + \frac{1}{u_0}B_0 C}$$

$$\Rightarrow \quad u^{III}\partial_r \chi^{III}\Big|_{r=C} = u^{II}\partial_r \chi^{II}\Big|_{r=C}$$

$$\hookrightarrow \quad u_0\left(-\frac{1}{u_0}B_0 - 2B\frac{\cancel{C^2}}{\cancel{C^3}}\right) = u a\left(\frac{1}{A} - \frac{2A^2}{C^3}\frac{\varepsilon}{3+2\varepsilon}\right)$$

$$-\frac{1}{u_0}B_0 - \frac{2}{C}\left[a\frac{C}{A}\left(1+\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right) + \frac{1}{u_0}B_0 C\right] = \frac{u}{u_0}a\left(\frac{1}{A} - \frac{2A^2}{C^3}\frac{\varepsilon}{3+2\varepsilon}\right)$$

$$-\frac{3}{2u_0}B_0 C - a\frac{C}{A}\left(1+\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right) = (1+\varepsilon)a\frac{C}{A}\left(\frac{1}{2} - \left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)$$

$$-\frac{3}{2u_0}B_0 C = a\frac{C}{A}\left[1 + \cancel{\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}} + \frac{1}{2} - \cancel{\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}} + \frac{1}{2}\varepsilon - \left(\frac{A}{C}\right)^3\frac{\varepsilon^2}{3+2\varepsilon}\right]$$

$$-3B_0 C \overset{!}{=} a u_0 \frac{C}{A}\left[3 + \left(1 - 2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon\right]$$

So $$\boxed{a = \frac{-3B_0 C}{u_0\frac{C}{A}\left[3 + \left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon\right]}}$$

then $$B = a\frac{C}{A}\left(1+\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)+\frac{1}{u_0}B_0C$$

$$= \frac{B_0C}{u_0}\left[\frac{-3-3\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}}{3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon}+1\right]$$

$$\boxed{B = \frac{-B_0C}{u_0}\left[\frac{(3+2\varepsilon)\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}-\varepsilon}{3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon}\right]}$$

then

$$\chi^{I} = \frac{-3B_0C}{\underbrace{u_0\frac{C}{A}\left[3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon\right]}_{=a}}(1+\varepsilon)\frac{3}{3+2\varepsilon}\left(\frac{r}{A}\right)\cos\theta,\quad r<A$$

$$\chi^{II} = \frac{-3B_0C}{\underbrace{u_0\frac{C}{A}\left[3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon\right]}_{=a}}\left[\left(\frac{r}{A}\right)+\left(\frac{A}{r}\right)^2\frac{\varepsilon}{3+2\varepsilon}\right]\cos\theta,\quad A<r<C$$

$$\chi^{III} = -\frac{1}{u_0}B_0C\left[\frac{r}{C}+\frac{(3+2\varepsilon)\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}-\varepsilon}{3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon}\left(\frac{C}{r}\right)^2\right]\cos\theta,\quad r>C$$

$\vec{B} = -u\vec{\nabla}\chi$,

So for $r<A$

$$\vec{B} = -u_0\left(\partial_r\chi\,\hat{r} + \frac{1}{r}\partial_\theta\chi\,\hat{\theta}\right)$$

$$\overset{!}{=} -u_0 a(1+\varepsilon)\frac{3}{3+2\varepsilon}\frac{1}{A}\underbrace{\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)}_{\hat{z}}$$

$$\overset{!}{=} -u_0 a(1+\varepsilon)\frac{3}{3+2\varepsilon}\frac{1}{A}\hat{z}$$

$$\overset{!}{=} \frac{\cancel{(-u_0)}\ \cancel{-}3B_0\cancel{C}}{\cancel{u_0}\frac{\cancel{C}}{\cancel{A}}\left[3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon\right]}(1+\varepsilon)\frac{3}{3+2\varepsilon}\frac{\cancel{1}}{\cancel{A}}\hat{z}$$

$$\vec{B} \overset{!}{=} \frac{3}{3+\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon}(1+\varepsilon)\frac{3}{3+2\varepsilon}B_0\hat{z} \qquad r<A$$

$$\overset{!}{=} \frac{1}{1+\frac{1}{3}\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon}(1+\varepsilon)\frac{1}{1+\frac{2}{3}\varepsilon}B_0\hat{z} \qquad r<A$$

If $u<u_0$ then $\frac{u}{u_0} = 1+\varepsilon < 1$, or $\varepsilon < 0$

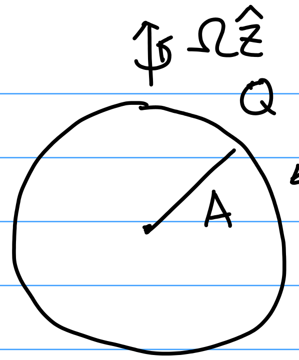
→ then we recognize $1+\varepsilon < 1+\frac{2}{3}\varepsilon$, so $\frac{1+\varepsilon}{1+\frac{2}{3}\varepsilon} > 1$

→ since $3+2\varepsilon > 0$, and $\varepsilon<0$, then $-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon} > 0$

→ so $\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon < 0$, so $\frac{1}{1+\frac{1}{3}\left(1-2\left(\frac{A}{C}\right)^3\frac{\varepsilon}{3+2\varepsilon}\right)\varepsilon} > 1$

→ Now since all constants are >1, we see B-field for $r<A$ is greater than B_0 if $u<u_0$, so the shell helped strengthen the field inside.

2) a)



shell. Find. B-field.

$\vec{\nabla}\cdot\vec{B}$, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

only when $r=A$ for this problem

For region inside sphere, $r < A$

$\vec{\nabla}\times\vec{B} = 0$, so lets introduce scalar potential χ where $\vec{B} = -\vec{\nabla}\chi$

so $\vec{\nabla}\times(-\vec{\nabla}\chi) = 0$

Then with $\vec{\nabla}\cdot\vec{B} = -\nabla^2\chi = 0$

Since problem has no ϕ-dependence, χ follows:

$$\chi = \sum_{l=0}^{\infty}\left(\alpha_l r^l + \beta_l r^{-(l+1)}\right)P_l(\cos\theta)$$

Set $\boxed{\beta_l = 0}$ to avoid singularity at $r\to 0$.

$\rightarrow \chi \overset{I}{=} \sum \alpha_l r^l P_l(\cos\theta)$ for $r < A$.

For $r > A$, we still have $\vec{\nabla}\times\vec{B} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$.

But since we want $\chi \to 0$ as $r\to\infty$, let $\boxed{\alpha_l = 0}$

$\rightarrow \chi \overset{II}{=} \sum \beta_l r^{-(l+1)} P_l(\cos\theta)$ for $r > A$

At boundary: $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \hat{n}\cdot\vec{B}^{II} = \hat{n}\cdot\vec{B}^{I}$

$$\boxed{\text{I} \hookrightarrow \partial_r \chi^{II}\big|_{r=A} = \partial_r \chi^{I}\big|_{r=A}}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \rightarrow \oint d\vec{r}\cdot\vec{B} = \mu_0\int d^2\vec{S}\cdot\vec{J}$$

$$\text{II} \hookrightarrow \int\left(B_{\parallel}^{II} - B_{\parallel}^{I}\right)A\,d\theta = \mu_0\int d^2\vec{S}\cdot\vec{J} = \mu_0 I$$

Find I: $\vec{I} = \frac{dQ}{dt}\hat{\theta} = \underbrace{\frac{dQ}{d^2S}}_{=\sigma}\frac{d^2S}{dt}\hat{\theta}$

since uniform distribution: $\sigma = \frac{Q}{4\pi A^2}$

$d^2S = dl_\theta \times dl_\phi = (A\,d\theta)(A\sin\theta\,d\phi) = A^2\sin\theta\,d\phi\,d\theta$

then $\vec{I} = \frac{Q}{4\pi A^2}A^2\sin\theta\frac{d\phi\,d\theta}{dt}\hat{\theta}$

but we know $\Omega = d\phi/dt$

then $\vec{I} = \frac{Q}{4\pi}\sin\theta\,\Omega\,d\theta\,\hat{\theta}$

then $\left(B_{\parallel}^{II} - B_{\parallel}^{I}\right)A\,\cancel{d\theta} = \mu_0\frac{Q}{4\pi}\sin\theta\,\Omega\,\cancel{d\theta}$

$$\boxed{\hookrightarrow \text{II}\ \left(-\frac{1}{r}\partial_\theta\chi^{II} + \frac{1}{r}\partial_\theta\chi^{I}\right) = \mu_0\frac{Q}{4\pi A}\Omega\sin\theta}$$

$\rightarrow$ Since we see $\partial_\theta \chi \propto \sin\theta$, this suggests

$$\chi \propto \cos\theta \propto P_1(\cos\theta), \quad \text{i.e. } \boxed{l=1}$$

$\rightarrow$ So

$$\chi^{I} = \alpha\left(\frac{r}{A}\right)\cos\theta \qquad r<A$$

$$\chi^{II} = \beta\left(\frac{A}{r}\right)^2\cos\theta \qquad r>A$$

use B.C. $\partial_r \chi^{I}\big|_{r=A} = \partial_r \chi^{II}\big|_{r=A}$

$$\hookrightarrow \quad \alpha\frac{1}{A} = -\beta 2\frac{A^2}{A^3} = -2\beta\frac{1}{A}$$

or $\boxed{\alpha = -2\beta}$

With 2^{nd} B.C.: $\left(-\frac{1}{r}\partial_\theta \chi^{II}\Big|_{r=A} + \frac{1}{r}\partial_\theta \chi^{I}\Big|_{r=A}\right) = \frac{Q}{4\pi A}\Omega \sin\theta\, u_0$

$$-\frac{1}{\cancel{A}}(-\sin\theta)(\beta) + \frac{1}{\cancel{A}}(-\sin\theta)(-2\beta) = \frac{Q}{4\pi \cancel{A}}\Omega \cancel{\sin\theta}\, u_0$$

$$\rightarrow \boxed{\beta = \frac{Q}{12\pi}\Omega u_0}$$

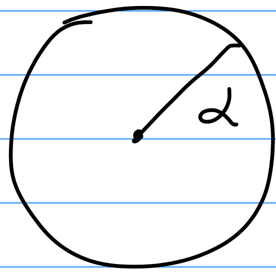
For $r<A$: $\chi^{I} = -\frac{u_0 Q}{6\pi}\Omega\left(\frac{r}{A}\right)\cos\theta$

$$\vec{B}^{I} = -\vec{\nabla}\chi = \frac{u_0}{6\pi}Q\frac{1}{A}\Omega\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) = \frac{u_0}{6\pi}\frac{Q}{A}\Omega\,\hat{z}$$

For $r>A$: $\chi^{II} = \frac{u_0 Q}{12\pi}\Omega\left(\frac{A}{r}\right)^2\cos\theta$

$$\vec{B}^{II} = \frac{u_0}{12\pi}Q\Omega A^2\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

c) For $\vec{M} = M_0 \hat{z} = M_0(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$



$$K_b = \vec{M}\times\hat{n}$$

$$\stackrel{!}{=} M_0(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\times\hat{r}$$

$$\vec{K}_b \stackrel{!}{=} M_0\sin\theta\,\hat{\phi}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

$$\stackrel{!}{=} M_0\,\vec{\nabla}\times(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$\stackrel{!}{=} \frac{1}{r}\Big(\partial_r(-r\sin\theta) - \partial_\theta(\cos\theta)\Big) M_0\,\hat{\phi}$$

$$\stackrel{!}{=} \frac{M_0}{r}(-\sin\theta + \sin\theta)\,\hat{\phi}$$

$$\boxed{\vec{J}_b \stackrel{!}{=} 0}$$

we have $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = -\vec{\nabla}\cdot\vec{M}$

$$\rightarrow \boxed{\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\Big(\cancelto{0}{\vec{J}_{cond}} + \vec{\nabla}\times\vec{M}\Big) = \mu_0(\vec{\nabla}\times\vec{M}) = 0$$

$$\vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \boxed{\vec{\nabla}\times\vec{H} = \vec{J}_{cond} = 0}$$

Since $\vec{\nabla}\times\vec{H}=0$, introduce $\vec{H}=-\vec{\nabla}\chi$

then $\vec{\nabla}\cdot\vec{H}=-\nabla^2\chi=-\vec{\nabla}\cdot\vec{M}$

or $\nabla^2\chi=\vec{\nabla}\cdot\vec{M}$

but $\vec{\nabla}\cdot\vec{M}=\partial_z M_0=0$

then $\nabla^2\chi=0$

By the same argument as we did in part a), we require azimuthal symmetry, non-divergent as $r\to 0$ for $r<a$ and convergence as $r\to\infty$, then we have

$$\chi^{I}=\sum_l A_l r^l P_l(\cos\theta) \quad \text{for} \quad r<a$$

$$\chi^{II}=\sum_l B_l r^{-(l+1)} P_l(\cos\theta) \quad \text{for} \quad r>a$$

But Boundary Condition is slightly different:

I $\quad \vec{\nabla}\times\vec{H}=0 \rightarrow -\frac{1}{r}\partial_\theta\chi^{II}\Big|_{r=a}+\frac{1}{r}\partial_\theta\chi^{I}\Big|_{r=a}=0$

II $\quad \vec{\nabla}\cdot\vec{H}=-\vec{\nabla}\cdot\vec{M} \rightarrow -\partial_r\chi^{II}\Big|_{r=a}+\partial_r\chi^{I}_{r=a}=-\hat{r}\cdot\vec{M}=-M_0\cos\theta$

By B.C. II, we know $\partial_r\chi\sim\cos\theta$, which is when $\boxed{l=1}$

so $\quad \chi^{I}=A\left(\frac{r}{a}\right)\cos\theta \qquad r<a$

$\quad \chi^{II}=B\left(\frac{a}{r}\right)^2\cos\theta \qquad r>a$

using B.C. (I): $\frac{1}{r}\partial_\theta \chi^{II}\Big|_{r=\alpha} = \frac{1}{r}\partial_\theta \chi^{I}\Big|_{r=\alpha}$

we observe $\boxed{A = B}$

Then with B.C. (II):

$$\left(-A\frac{1}{\alpha} - 2A\frac{1}{\alpha}\right)\cancel{\cos\theta} = -M_0 \cancel{\cos\theta}$$

then $A = \frac{1}{3}M_0\,\alpha$

then
$$\boxed{\begin{aligned} \chi^{I} &= \frac{1}{3}M_0\,\alpha\left(\frac{r}{\alpha}\right)\cos\theta && r<\alpha \\ \chi^{II} &= \frac{1}{3}M_0\,\alpha\left(\frac{\alpha}{r}\right)^2\cos\theta && r>\alpha \end{aligned}}$$

then

$$H = \frac{1}{\mu_0}\vec{B} - \vec{M} \quad \text{or}$$

$$\vec{B}^{I} = \mu_0\left(-\vec{\nabla}\chi + \vec{M}\right)$$

$$= \mu_0\left(-\frac{1}{3}M_0\underbrace{\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)}_{\hat{z}} + M_0\hat{z}\right)$$

$$\boxed{\vec{B}^{I} = \frac{2}{3}\mu_0 M_0\,\hat{z} \quad \text{for} \quad r<\alpha}$$

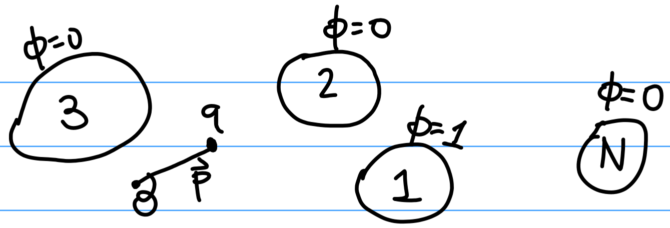
Similarly:

$$\vec{B}^{II} = \mu_0\left(-\vec{\nabla}\chi + \cancel{\vec{M}}\right)$$, but $\vec{M}=0$ outside sphere.

$$= \mu_0\left[-\frac{1}{3}M_0\,\alpha^3\left(-2\frac{1}{r^3}\cos\theta\,\hat{r} - \frac{1}{r^3}\sin\theta\,\hat{\theta}\right)\right]$$

$$\boxed{\vec{B}^{II} = \frac{\mu_0}{3}M_0\left(\frac{\alpha}{r}\right)^3\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \quad \text{for} \quad r>\alpha}$$

6) <u>Shockley - Ramo Theorem:</u>



Introduce G for the <u>vacuum region</u> with charge q.

$$-\nabla^2 G(\vec{r}, \vec{r}'=\vec{p}) = \frac{1}{\varepsilon_0}\delta(\vec{r}, \vec{r}'=\vec{p}) \quad \text{with } G(\vec{r},\vec{p}) = 0 \text{ for all conductors}$$

Now introduce $\Phi(\vec{r})$, $-\nabla^2\Phi(\vec{r}) = \frac{\rho}{\varepsilon_0} = 0$ ← here assume no charge.
with Dirichlet B.C. that $\Phi = 0$ except for one of conductor
$\Phi(r=a) = 1$.

a) With Green's Theorem:

$$\int_{\text{outside cond}} d^3r \left[G(\vec{r},\vec{p}) \underbrace{\nabla_r^2 \Phi(\vec{r})}_{=0} - \Phi(\vec{r}) \underbrace{\nabla_r^2 G(\vec{r},\vec{p})}_{-\frac{1}{\varepsilon_0}\delta(\vec{r},\vec{p})} \right] = \int_{\text{cond}} d^2\vec{S} \left[\underbrace{G(\vec{r},\vec{p})}_{=0 \text{ by homogeneous Dirichlet.}} \vec{\nabla}_r \Phi(\vec{r}) - \Phi(\vec{r}) \vec{\nabla}_r G(\vec{r},\vec{p}) \right]$$

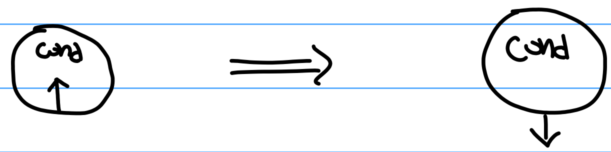
$$\hookrightarrow \frac{1}{\varepsilon_0}\Phi(\vec{p}) = -\int_{\text{cond}} d^2\vec{S}\, \Phi(\vec{r}) \vec{\nabla}_r G(\vec{r},\vec{p})$$

but $\Phi(\vec{r})$ is constant on the boundary conductor, so we pull this out.
Further more $\Phi(\vec{r})$ is only nonzero for 1 conductor, call it conductor a.

$$\hookrightarrow \frac{1}{\varepsilon_0}\Phi(\vec{p}) = -\Phi(\vec{r}=\partial S_a) \int_a d^2\vec{S} \cdot \vec{\nabla}_r G(\vec{r},\vec{p})$$

Here the outward normal points into the conductor, but lets change the definition so that the normal points outside, so we have a <u>sign flip</u>.

i.e.



then

$$\Phi(\vec{p}) = \varepsilon_0 \Phi(\vec{r}=\partial S_a) \int_a d^2\vec{S} \cdot \vec{\nabla}_r G(\vec{r},\vec{p})$$

Now multiple q on both sides, and set $\Phi(\vec{r}=\partial S_a)=1$.

$$q\,\Phi(\vec{p}) = \Phi(\vec{r}=\partial S_a)\,\underbrace{q\varepsilon_0 \int_{\text{cond},a} d^2\vec{S}\cdot\vec{\nabla}_r G(\vec{r},\vec{p})}_{\text{Analogous to } \varepsilon_0\int d^2\vec{S}\cdot\vec{\nabla}_r\Phi = \int d^3r(-\rho) \overset{!}{=} -Q}$$

$$q\,\Phi(\vec{p}) = \underbrace{\Phi(\vec{r}=\partial S_a)}_{=1}(-Q)$$

$$\boxed{q\,\Phi(\vec{p}) = -Q}$$

b) $I = \dfrac{dQ}{dt} = \dfrac{d}{dt}\left(-q\,\Phi(\vec{p})\right)$

$$= -q\,\frac{d}{d\vec{p}}\Phi\,\frac{d\vec{p}}{dt}$$

$$\overset{!}{=} -q\,\dot{\vec{p}}\cdot\vec{\nabla}_p\,\Phi(\vec{p})$$

$\tilde{E}(\vec{p}) = -\vec{\nabla}_p\Phi(\vec{p})$

$$\boxed{I \overset{!}{=} q\,\dot{\vec{p}}\cdot\tilde{E}(\vec{p})}$$