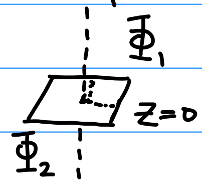
Zhi Chen HW#2

1) Laplace Equation with a flat and then rippled boundary.



a) Boundary Conditions:

$z=0$: $\Gamma(x,y) = \Phi(x,y,0) = g\cos(2\pi x/\lambda)$

$z\to\infty$: $\Phi(x,y,z\to\infty) = 0$

Fourier Transform:

$$\Phi(x,y,z) = \int d^2q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, \hat{\Phi}(\vec{q}_{\parallel}, z)$$

Apply Laplace:

$$\nabla^2\Phi = \int d^2q_{\parallel}\, \nabla^2\left(e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, \hat{\Phi}(\vec{q}_{\parallel}, z)\right) = 0$$

$$0 \stackrel{!}{=} \int d^2q_{\parallel} \underbrace{\left[(-q_x^2 - q_y^2)\hat{\Phi}(\vec{q}_{\parallel}, z) + \partial_z^2 \hat{\Phi}(\vec{q}_{\parallel}, z)\right]}_{\text{then } = 0} e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}$$

$$\hookrightarrow \partial_z^2 \hat{\Phi}(\vec{q}_{\parallel}, z) = (q_x^2 + q_y^2)\,\hat{\Phi}(\vec{q}_{\parallel}, z) = |q_{\parallel}|^2\,\hat{\Phi}(\vec{q}_{\parallel}, z)$$

$$\hookrightarrow \hat{\Phi}(\vec{q}_{\parallel}, z) = A(\vec{q}_{\parallel})e^{|q_{\parallel}|z} + B(\vec{q}_{\parallel})e^{-|q_{\parallel}|z}$$

With Boundary Condition: $\Phi(x,y,z\to\infty) = 0$

$$\lim_{z\to\infty} \hat{\Phi}(\vec{q}_{\parallel}, z) = \underbrace{A(\vec{q}_{\parallel})e^{|q_{\parallel}|z}}_{\infty} + \underbrace{B(\vec{q}_{\parallel})e^{-|q_{\parallel}|z}}_{0} = 0$$

then require $A(\vec{q}_{\parallel}) = 0$

then we have $\boxed{\hat{\Phi}(\vec{q}_{\parallel}, z) = B(\vec{q}_{\parallel})e^{-|q_{\parallel}|z}}$

Now apply 2^{nd} boundary condition

If $\quad \Phi(x,y,z=0) = T(x,y) = g\cos(\frac{2\pi}{\lambda}x)$

then $\quad \hat{\Phi}(\vec{q}_{\parallel}, z)\big|_{z=0} = B(\vec{q}_{\parallel}) = \hat{\Gamma}(\vec{q}_{\parallel})$

so $\quad \hat{\Phi}(\vec{q}_{\parallel}, z) = \hat{\Gamma}(\vec{q}_{\parallel}) e^{-|q_{\parallel}|z}$

$$\Phi(x,y,z) = \int d^2q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, \hat{\Gamma}(\vec{q}_{\parallel})\, e^{-|q_{\parallel}|z}$$

Find $\quad \hat{\Gamma}(\vec{q}_{\parallel}) = \int d^2r_{\parallel}\, e^{-i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, g\cos(\frac{2\pi}{\lambda}x)$

$$\overset{!}{=} \int d^2r_{\parallel}\, e^{-i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, g\, \frac{e^{i\frac{2\pi}{\lambda}x} + e^{-i\frac{2\pi}{\lambda}x}}{2}$$

$$\overset{!}{=} \frac{g}{2}\int dx \left(e^{-i(q_x - \frac{2\pi}{\lambda})x} + e^{-i(\vec{q}_{\parallel} + \frac{2\pi}{\lambda})x}\right)\int dy\, e^{-iq_y y}$$

$$\boxed{\hat{\Gamma}(\vec{q}_{\parallel}) \overset{!}{=} \frac{g}{2}\left[\delta(q_x - \frac{2\pi}{\lambda}) + \delta(q_x + \frac{2\pi}{\lambda})\right]\delta(q_y)}$$

then $\quad \Phi(x,y,z) = \int d^2q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, \frac{g}{2}\left[\delta(q_x - \frac{2\pi}{\lambda}) + \delta(q_x + \frac{2\pi}{\lambda})\right]\delta(q_y)\, e^{-|q_{\parallel}|z}$

$$\overset{!}{=} \frac{g}{2}\int dq_x\, e^{iq_x x}\left[\delta(q_x - \frac{2\pi}{\lambda}) + \delta(q_x + \frac{2\pi}{\lambda})\right]\int dq_y\, \delta(q_y)\, e^{-\sqrt{q_x^2 + q_y^2}\,z}$$

$$\overset{!}{=} \frac{g}{2}\int dq_x\, e^{iq_x x}\left[\delta(q_x - \frac{2\pi}{\lambda}) + \delta(q_x + \frac{2\pi}{\lambda})\right] e^{-\sqrt{|q_x|^2}\,z}$$

$$\overset{!}{=} \frac{g}{2}\left[e^{i\frac{2\pi}{\lambda}x} + e^{-i\frac{2\pi}{\lambda}x}\right] e^{-\frac{2\pi}{\lambda}z}$$

$$\boxed{\Phi(x,y,z) \overset{!}{=} g\cos(\frac{2\pi}{\lambda}x)\, e^{-\frac{2\pi}{\lambda}z}}$$

Suppose Σ is a small distortion of $z=0$ plane, specified by $z = H(x,y)$. Now $\Gamma(x,y)$ lives on $\Sigma = (x, y, z = H(x,y))$

b) Assume $H(x,y)$ is small, show Φ satisfies

$$\Phi(x,y,z)\Big|_{z=0} + H(x,y)\frac{\partial}{\partial z}\Phi(x,y,z)\Big|_{z=0} = \Gamma(x,y)$$

$\Rightarrow$ Since $\Gamma(x,y)$ is the potential at boundary, and boundary is satisfied by the surface $\Sigma = (x, y, z = H(x,y))$ then:

$$\Phi(\Sigma) = \Phi(x,y, z = H(x,y)) = \Gamma(x,y)$$

if $H(x,y)$ is small, i.e. $H(x,y) \ll 1$, then we can taylor expand around $z_0 = 0$

$$\underbrace{\Phi(x,y, z = 0 + H(x,y))}_{=\Gamma(x,y)} = \Phi(x,y,z)\Big|_{z=0} + H(x,y)\frac{\partial}{\partial z}\Phi(x,y,z)\Big|_{z=0} + \mathcal{O}(H^2)$$

ignore higher order terms in H if H is small.

c) Suppose the 2^{nd} boundary condition is changed:

$$\Phi(x,y,z)\big|_{z=0} + H(x,y)\,\partial_z \Phi(x,y,z)\big|_{z=0} = \Gamma(x,y)$$

we still have $\hat{\Phi}(\vec{q}_{\parallel}, z) = B(\vec{q}_{\parallel})\, e^{-|q_{\parallel}|z}$ from part a)

so $$\Phi(x,y,z)\big|_{z=0} = \int d^2 q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, B(q_{\parallel})$$

then

$$\underbrace{\int d^2 q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, B(q_{\parallel})}_{B(\vec{r}_{\parallel})} - H(x,y)\int d^2 q_{\parallel}\, e^{i\vec{q}\cdot\vec{r}_{\parallel}}\, |\vec{q}_{\parallel}|\, B(\vec{q}_{\parallel}) = \Gamma(x,y)$$

Multiply both sides by $\int d^2\vec{k}_{\parallel}\, e^{-i\vec{k}_{\parallel}\cdot\vec{r}_{\parallel}}$ to inverse transform.

$$\underbrace{\int d^2 r_{\parallel}\, e^{-i\vec{k}_{\parallel}\cdot\vec{r}_{\parallel}}\, B(\vec{r}_{\parallel})}_{B(\vec{k}_{\parallel})} = \underbrace{\int d^2\vec{r}_{\parallel}\, e^{-i\vec{k}_{\parallel}\cdot\vec{r}_{\parallel}}\,\Gamma(\vec{r}_{\parallel})}_{\Gamma(\vec{k}_{\parallel})} + \int d^2\vec{r}_{\parallel}\, e^{-i\vec{k}_{\parallel}\cdot\vec{r}_{\parallel}}\, H(\vec{r}_{\parallel})\int d^2 q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, |q_{\parallel}|\, B(\vec{q}_{\parallel})$$

$$\hookrightarrow B(\vec{k}_{\parallel}) = \Gamma(\vec{k}_{\parallel}) + \int d^2 q_{\parallel}\, |q_{\parallel}|\, B(\vec{q}_{\parallel}) \underbrace{\int d^2 r_{\parallel}\, e^{-i(\vec{k}_{\parallel}-\vec{q}_{\parallel})\cdot\vec{r}_{\parallel}}\, H(\vec{r}_{\parallel})}_{\hat{H}(\vec{k}_{\parallel}-\vec{q}_{\parallel})}$$

$$B(\vec{k}_{\parallel}) = \Gamma(\vec{k}_{\parallel}) + \int d^2 q_{\parallel}\, |q_{\parallel}|\, B(\vec{q}_{\parallel})\, \hat{H}(\vec{k}_{\parallel}-\vec{q}_{\parallel})$$

Now we do iteration procedure, since we know B in terms of a function in itself:

$$B(\vec{k}_{\parallel}) = \Gamma(\vec{k}_{\parallel}) + \int d^2 q_{\parallel}\, |q_{\parallel}|\, \hat{H}(\vec{k}_{\parallel} - \vec{q}_{\parallel}) \underbrace{\left[\Gamma(\vec{q}_{\parallel}) + \int d^2 q_{\parallel}'\, |q_{\parallel}'|\, B(\vec{q}_{\parallel}')\, \hat{H}(\vec{q}_{\parallel} - \vec{q}_{\parallel}')\right]}_{B(\vec{q}_{\parallel})}$$

But since $\hat{H}$ is small (first order), the we neglect second order term $\mathcal{O}(\hat{H}^2)$

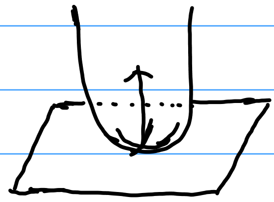
$$\boxed{B(\vec{k}_{\parallel}) = \Gamma(\vec{k}_{\parallel}) + \int d^2 q_{\parallel}\, |q_{\parallel}|\, \hat{H}(\vec{k}_{\parallel} - \vec{q}_{\parallel})\, \Gamma(\vec{q}_{\parallel}) + \mathcal{O}(\hat{H}^2)}$$

Then

$$\Phi(x,y,z) = \int d^2 q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, B(\vec{q}_{\parallel})\, e^{-|q_{\parallel}|z}$$

$$\boxed{\Phi(x,y,z) \overset{!}{=} \int d^2 q_{\parallel}\, e^{i\vec{q}_{\parallel}\cdot\vec{r}_{\parallel}}\, e^{-|q_{\parallel}|z} \Big\{ \underbrace{\Gamma(\vec{q}_{\parallel}) + \int d^2 q_{\parallel}'\, |q_{\parallel}'|\, \hat{H}(\vec{q}_{\parallel} - \vec{q}_{\parallel}')\, \Gamma(\vec{q}_{\parallel}') + \mathcal{O}(\hat{H}^2)}_{B(\vec{q}_{\parallel})} \Big\}}$$

2) Show $\frac{-\hat{n}\cdot\vec{\nabla}E}{2E} = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$



Note: surface of conductor is equipotential
outside the conductor, $\nabla^2\phi = 0$ or $\vec{\nabla}\cdot\vec{E} = 0$

We know given principal radii, $z = \frac{1}{2}\left(\frac{x^2}{R_1}+\frac{y^2}{R_2}\right)$

Consider electric potential near the origin, and we expand around the origin

$$\phi(x,y,z) = \phi(0,0,0) + x\partial_x\phi + y\partial_x\phi + z\partial_z\phi + \frac{1}{2}x^2\partial_x^2\phi + \frac{1}{2}y^2\partial_y^2\phi + \frac{1}{2}z^2\partial_z^2\phi + xy\,\partial_{xy}\phi + xz\,\partial_{xz}\phi + yz\,\partial_{yz}\phi$$

$\Rightarrow$ Since the surface of conductor is equipotential, then there is no change in ϕ in on the surface, which is the $\hat{x}$ and $\hat{y}$ direction of the plane.
Therefore, $\partial_x\phi = \partial_y\phi = 0 = \partial_{xy}\phi = 0$, i.e. parallel component of E is zero.

$\rightarrow$ Since $z = \frac{1}{2}\left(\frac{x^2}{R_1}+\frac{y^2}{R_1}\right)$, and x, y are small,
so z is already 2^{nd} order.

$\rightarrow$ So discard terms higher than 2^{nd} order, which are:
$$\frac{1}{2}z^2\partial_z^2\phi,\ xz\,\partial_{xz}\phi,\ yz\,\partial_{yz}\phi \Rightarrow 0$$

then:

$$\phi(0,0,0) = \phi(0,0,0) + \underbrace{z\,\partial_z\phi + \tfrac{1}{2}x^2\partial_x^2\phi + \tfrac{1}{2}y^2\partial_y^2\phi}_{=0}$$

$$\hookrightarrow \tfrac{1}{2}\left(\frac{x^2}{R_1}+\frac{y^2}{R_2}\right)\partial_z\phi + \tfrac{1}{2}x^2\partial_x^2\phi + \tfrac{1}{2}y^2\partial_y^2\phi = 0$$

$$\hookrightarrow x^2\underbrace{\left(\tfrac{1}{2}\tfrac{1}{R_1}\partial_z\phi + \tfrac{1}{2}\partial_x^2\phi\right)}_{=0} + y^2\underbrace{\left(\tfrac{1}{2}\tfrac{1}{R_2}\partial_z\phi + \tfrac{1}{2}\partial_y^2\phi\right)}_{=0} = 0$$

From equation above, we see that

$$\frac{1}{2}\frac{1}{R_1}\partial_z\phi = -\frac{1}{2}\partial_x^2\phi$$

and

$$\frac{1}{2}\frac{1}{R_2}\partial_z\phi = -\frac{1}{2}\partial_y^2\phi$$

Now we add these equations

$$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\partial_z\phi = -\frac{1}{2}\left(\partial_x^2\phi + \partial_y^2\phi\right)$$

add $-\frac{1}{2}\partial_z^2\phi$ on both sides

$$\hookrightarrow -\frac{1}{2}\partial_z^2\phi + \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\partial_z\phi = -\frac{1}{2}\underbrace{\left(\partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi\right)}_{\nabla^2\phi = 0}$$

So $$\frac{1}{2}\partial_z^2\phi = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\partial_z\phi$$

By noting $\vec{E}$ only has perpendicular component (i.e. $\hat{z}$), then:

$$\hat{n}\cdot\vec{\nabla}E = (-\hat{z}\cdot\partial_z\hat{z})E = -\partial_z E_{\parallel} = \partial_z^2\phi$$

and

$$E = E_{\parallel} = -\partial_z\phi$$

$$\boxed{\therefore\ -\frac{\hat{n}\cdot\vec{\nabla}E}{2E} = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}$$

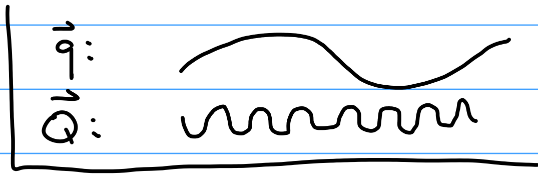
3) Lorentz coarse-graining:

Consider electric charge distribution in 3D,

$$R(\vec{r}) = \cos(\vec{q}\cdot\vec{r}) + \cos(\vec{Q}\cdot\vec{r})$$

Suppose $|\vec{q}| \ll |\vec{Q}|$, so $\vec{q}$: smaller wave vector or higher wavelength

Introduce Gaussian averaging function:



$$F(\vec{r}) = (2\pi L^2)^{-3/2} \exp\left(-|r|^2/2L^2\right)$$

let $\bar{R}(\vec{r}) \equiv \int d^3\vec{s}\; F(\vec{s})\, R(\vec{r}-\vec{s})$

a) $$\bar{R}(\vec{r}) \equiv \int d^3\vec{s}\; (2\pi L^2)^{-3/2} e^{-|s|^2/2L^2} \Big[\underbrace{\cos(\vec{q}\cdot(\vec{r}-\vec{s}))}_{\mathrm{Re}[e^{i\vec{q}\cdot(\vec{r}-\vec{s})}]} + \underbrace{\cos(\vec{Q}\cdot(\vec{r}-\vec{s}))}_{\mathrm{Re}[e^{i\vec{Q}\cdot(\vec{r}-\vec{s})}]}\Big]$$

$$\stackrel{!}{=} (2\pi L^2)^{-3/2}\, \mathrm{Re}\left[\int d^3\vec{s}\; e^{-|s|^2/2L^2} \left(e^{i\vec{q}\cdot\vec{r}} e^{-i\vec{q}\cdot\vec{s}} + e^{i\vec{Q}\cdot\vec{r}} e^{-i\vec{Q}\cdot\vec{s}}\right)\right]$$

$$\stackrel{!}{=} (2\pi L^2)^{-3/2}\, \mathrm{Re}\left[e^{i\vec{q}\cdot\vec{r}} \int d^3\vec{s}\; e^{-|s|^2/2L^2} e^{-i\vec{q}\cdot\vec{s}} + e^{i\vec{Q}\cdot\vec{r}} \int d^3\vec{s}\; e^{-|s|^2/2L^2} e^{-i\vec{Q}\cdot\vec{s}}\right.$$

lets calculate the fourier transform:

$$\int d^3\vec{s}\; \exp\left(\tfrac{-1}{2L^2}(s_x^2+s_y^2+s_z^2)\right) \exp\left(-i(q_x s_x + q_y s_y + q_z s_z)\right)$$

$$\hookrightarrow = \underbrace{\int_{-\infty}^{\infty} ds_x \exp\left(\tfrac{-1}{2L^2}s_x^2 - i q_x s_x\right)}_{=\sqrt{2\pi L^2}\, e^{\frac{-2L^2 q_x^2}{4}}} \underbrace{\int_{-\infty}^{\infty} ds_y \exp\left(\tfrac{-1}{2L^2}s_y^2 - i q_y s_y\right)}_{=\sqrt{2\pi L^2}\, e^{\frac{-2L^2 q_y^2}{4}}} \underbrace{\int_{-\infty}^{\infty} ds_z \exp\left(\tfrac{-1}{2L^2}s_z^2 - i q_z s_z\right)}_{=\sqrt{2\pi L^2}\, e^{\frac{-2L^2 q_z^2}{4}}}$$

$$\stackrel{!}{=} (2\pi L^2)^{3/2}\, e^{-\frac{L^2}{2}|q|^2}$$

Similarly: $\int d^3s\, e^{-|s|^2/2L^2} e^{-i\vec{Q}\cdot\vec{s}} = (2\pi L^2)^{3/2} e^{-\frac{L^2}{2}|Q|^2}$

All together:

$$\bar{R}(\vec{r}) = \cancel{(2\pi L^2)^{-3/2}} \,\mathrm{Re}\left[e^{i\vec{q}\cdot\vec{r}} \cancel{(2\pi L^2)^{3/2}} e^{-\frac{L^2}{2}|q|^2} + e^{i\vec{Q}\cdot\vec{r}} \cancel{(2\pi L^2)^{3/2}} e^{-\frac{L^2}{2}|Q|^2}\right]$$

$$\boxed{\bar{R}(\vec{r}) = \cos(\vec{q}\cdot\vec{r}) e^{-\frac{L^2}{2}|q|^2} + \cos(\vec{Q}\cdot\vec{r})\, e^{-\frac{L^2}{2}|Q|^2}}$$

If $\frac{1}{Q} \ll L \ll \frac{1}{q}$, then $LQ \gg 1$

and $Lq \ll 1$

Since the exponential is negative

$\Rightarrow$ $\exp\left(-\frac{1}{2}(L|q|)^2\right)\Big|_{Lq \ll 1} \to$ large

$\Rightarrow$ $\exp\left(-\frac{1}{2}(L|Q|)^2\right)\Big|_{LQ \gg 1} \to$ small.

or $$\bar{R}(\vec{r}) = \cos(\vec{q}\cdot\vec{r}) \underbrace{e^{-\frac{1}{2}L^2|q|^2}}_{\text{large}} + \underbrace{\cos(\vec{Q}\cdot\vec{r}) \underbrace{e^{-\frac{1}{2}L^2|Q|^2}}_{\text{small}}}_{\text{this term is surpressed due to exponential}}$$

$$\approx \cos(\vec{q}\cdot\vec{r}) \exp\left(-\frac{1}{2}L^2|q|^2\right)$$

And indeed we're left with coarse component, i.e. high wavelength.

b) $\bar{R}(\vec{r}) = \int d^3s \; F(s) \, R(r-s)$

$$= \int d^3s \; F(s) \int d^3q \; e^{-i\vec{q}\cdot(\vec{r}-\vec{s})} \hat{R}(\vec{q})$$

$$= \int d^3q \; e^{-i\vec{q}\cdot\vec{r}} \hat{R}(\vec{q}) \underbrace{\int d^3s \; F(s) e^{i\vec{q}\cdot\vec{s}}}_{\hat{F}(\vec{q})}$$

Now given $F(s) = (2\pi L^2)^{-3/2} \exp(-|s|^2/2L^2)$

then:

$$\bar{R}(\vec{r}) = \int d^3q \; e^{-i\vec{q}\cdot\vec{r}} \hat{R}(\vec{q}) \; \cancel{(2\pi L^2)^{-3/2}} \underbrace{\int d^3s \exp(-|s|^2/2L^2) e^{i\vec{q}\cdot\vec{s}}}_{\text{we found in part a)}}$$

$$= \cancel{(2\pi L^2)^{3/2}} e^{\frac{-L^2|q|^2}{2}}$$

$$\boxed{\bar{R}(\vec{r}) = \int d^3q \; e^{i\vec{q}\cdot\vec{r}} e^{\frac{-L^2|q|^2}{2}} \hat{R}(\vec{q})}$$

We see that there is a $e^{-\frac{L^2|q|^2}{2}}$ factor in the integral, So if $\frac{1}{q} << L$ or $L|q| >> 1$, then $e^{-\frac{L^2|q|^2}{2}}$ becomes small, surpressing the whole term making it vary weakly.

c) If smooth function is given $e^{-|q|L}$, find $\bar{R}(\vec{r})$

From part b), we saw:

$$\bar{R}(\vec{r}) = \int d^3q \, e^{-i\vec{q}\cdot\vec{r}} \hat{R}(q) \hat{F}(q) = \int d^3q \, e^{-i\vec{q}\cdot\vec{r}} e^{-|q|L} \hat{R}(q)$$

or

$$\overset{!}{=} \int d^3s \, F(s) \underbrace{\int d^3q \, e^{-i\vec{q}\cdot(\vec{r}-\vec{s})} \hat{R}(q)}_{R(\vec{r}-\vec{s})}$$

and

$$F(s) = \int d^3q \, e^{-i\vec{q}\cdot\vec{s}} F(\vec{q})$$

$$\overset{!}{=} \int d^3q \, e^{-i\vec{q}\cdot\vec{s}} e^{-|q|L}$$

$$\overset{!}{=} \int |q|^2 \sin\theta \, d\theta \, d\phi \, dq \; e^{-i|q||s|\cos\theta} e^{-|q|L}$$

$$\overset{!}{=} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\infty} dq \, q^2 \int_0^{\pi} \sin\theta \, d\theta \, e^{-iqs\cos\theta} e^{-qL}$$

let $\cos\theta = u \;\rightarrow\; du = -\sin\theta \, d\theta$

$$\overset{!}{=} 2\pi \int_0^{\infty} dq \, q^2 \int_1^{-1} -du \, e^{-iqsu} e^{-|q|L}$$

let $v = -iqsu \;\rightarrow\; du = -dv/(iqs)$

$$\overset{!}{=} 2\pi \int_0^{\infty} dq \, q^{\cancel{2}} e^{-qL} \int_{-iqs}^{iqs} \frac{dv}{i\cancel{q}s} e^{v}$$

$$\overset{!}{=} \frac{2\pi}{is} \int_0^{\infty} dq \, q \, e^{-qL} \left(e^{v} \Big|_{-iqs}^{iqs} \right)$$

$$\overset{!}{=} \frac{2\pi}{is}\int_0^\infty dq\; q\; e^{-qL}\left(e^{iqs} - e^{-iqs}\right)$$

$$\overset{!}{=} \frac{2\pi}{is}\left[\underbrace{\int_0^\infty dq\;\; q\;\; e^{-q(L-is)}} - \underbrace{\int_0^\infty dq\; q\; e^{-q(L+is)}}\right]$$

$$\overset{!}{=} \frac{2\pi}{is}\left(\frac{1}{(L-is)^2} - \frac{1}{(L+is)^2}\right)$$

$$\overset{!}{=} \frac{2\pi}{is}\left(\frac{1}{L^2-2isL-s^2} - \frac{1}{L^2+2isL-s^2}\right)$$

$$\overset{!}{=} \frac{2\pi}{is}\left(\frac{L^2-s^2+2isL}{(L^2-s^2)^2+4s^2L^2} - \frac{L^2-s^2-2isL}{(L^2-s^2)^2+4s^2L^2}\right)$$

$$\overset{!}{=} \frac{2\pi}{\cancel{is}}\;\frac{4\cancel{is}L}{(L^2-s^2)^2+4s^2}$$

$$\overset{!}{=} 8\pi\;\frac{L}{(L^2-s^2)^2+4s^2L^2}$$

$$F(s) \overset{!}{=} 8\pi\;\frac{L}{(L^2+s^2)^2}$$

Then

$$\boxed{\begin{aligned} \bar{R}(\vec{r}) &= \int d^3s\; F(s)\; R(\vec{r}-\vec{s}) \\ \bar{R}(\vec{r}) &\overset{!}{=} \int d^3s\; 8\pi\frac{L}{(L^2+s^2)^2}\; R(\vec{r}-\vec{s}) \end{aligned}}$$