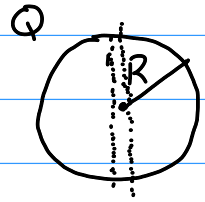
Zhi Chen HW#2 EM

1) Electrostatic Stress Tensor:



$$T_{ij} = \varepsilon_0 E_i E_j - \frac{1}{2}\varepsilon_0 \delta_{ij} E_i E_j$$

Find electric field first.

Since uniform sphere, $\rho = \frac{Q}{\int d^3r} = \frac{Q}{\frac{4}{3}\pi R^3}$

$$\int d^3r \, \vec{\nabla}\cdot\vec{E} = \int \frac{\rho}{\varepsilon_0} d^3r$$

For $r < R$,

$$\int_r d^2\vec{S}\cdot\vec{E} = \int_0^r r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \frac{Q}{\frac{4}{3}\pi R^3 \varepsilon_0}$$

$$\Downarrow$$

$$= \frac{r^3}{3} \frac{Q}{\frac{4}{3}\pi R^3 \varepsilon_0} 4\pi$$

$$E_r r^2 \underbrace{\int \sin\theta d\theta d\phi}_{4\pi} \;\Big|\; = \frac{Q}{\varepsilon_0}\left(\frac{r}{R}\right)^3$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r}{R}\right)^3 \frac{1}{r^2}\hat{r} \quad \text{for } r < R}$$

For $r > R$:

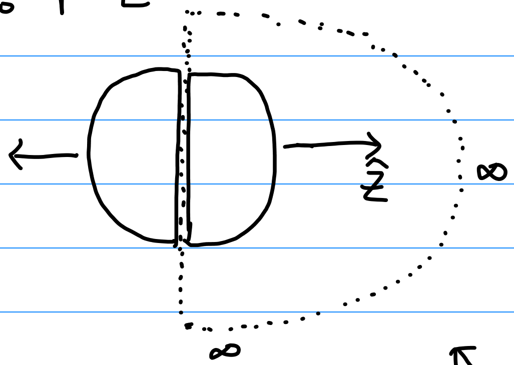
$$\int d^2\vec{S}\cdot\vec{E} = \int_0^\pi \int_0^{2\pi} \sin\theta d\theta d\phi \left[\int_0^R r^2 dr \frac{\rho}{\varepsilon_0} + \int_R^\infty 0\right]$$

$$E r^2 \int \sin\theta d\theta d\phi \;\Big|\; = \frac{Q}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r} \quad \text{for } r > R}$$

Construct T_{jk} using $T_{jk} = \varepsilon_0 \left(E_j E_k - \frac{1}{2} |E|^2 \delta_{jk} \right)$

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\varepsilon_0} \left(\frac{r}{R}\right)^3 \frac{1}{r^2} \hat{r} = \frac{Q}{4\pi\varepsilon_0} \left(\frac{r}{R}\right)^3 \frac{1}{r^2} \left[\sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z} \right] & \text{for } r < R \\ \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2} \left[\sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z} \right] & \text{for } r > R \end{cases}$$



$$F_k = \int d^2S_j\, T_{jk}$$

lets consider the surface wrapping from middle axis to ∞,

$$F_k = \int_{\text{axis}} d^2S_j\, T_{jk} + \underbrace{\cancel{\int_{\infty} d^2S_j\, T_{jk}}}$$

since $T \propto |E|^2 \propto \frac{1}{r^4}$, and $d^2S \propto r^2$

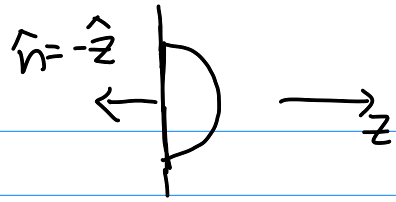
so $d^2S_j T_{jk} \propto \frac{1}{r^2} \to 0$ as $r \to \infty$

$$= \int d^2S\, \hat{n} \cdot T_{zk}$$

Since $T_{zk} = \varepsilon_0 \left(E_z E_k - \frac{1}{2} |E|^2 \delta_{zk} \right)$

at axis, $\theta = \frac{\pi}{2}$,

$$\vec{E}\Big|_{\theta=\frac{\pi}{2}} = \begin{cases} \frac{Q}{4\pi\varepsilon_0} \left(\frac{r}{R}\right)^3 \frac{1}{r^2} \hat{r} = \frac{Q}{4\pi\varepsilon_0} \left(\frac{r}{R}\right)^3 \frac{1}{r^2} \left[\cos\phi\, \hat{x} + \sin\phi\, \hat{y} + 0\, \hat{z} \right] & \text{for } r < R \\ \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2} \left[\cos\phi\, \hat{x} + \sin\phi\, \hat{y} + 0\, \hat{z} \right] & \text{for } r > R \end{cases}$$



Since $E_z = 0$

$$T_{zk} = \varepsilon_0 \left(\cancelto{0}{E_z} E_k - \frac{1}{2} |E|^2 \delta_{zk} \right)$$

need $k = z$

$$T_{zz} = -\frac{1}{2} \varepsilon_0 |E|^2 \hat{z}$$

so

$$F_z = \int d^2 s \left(-\hat{z}\right)\left(-\frac{1}{2}\varepsilon_0\right) |E|^2 \hat{z}$$

$$= \int_0^\infty 2\pi r\,dr \; \frac{1}{2} \varepsilon_0 |E|^2$$

$$= \pi\varepsilon_0 \left[\int_0^R r\,dr \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \left(\frac{r}{R}\right)^6 \frac{1}{r^4} + \int_R^\infty r\,dr \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4} \right]$$

$$= \pi\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \left[\frac{1}{R^6} \int_0^R r^3\,dr + \int_R^\infty \frac{1}{r^3}\,dr \right]$$

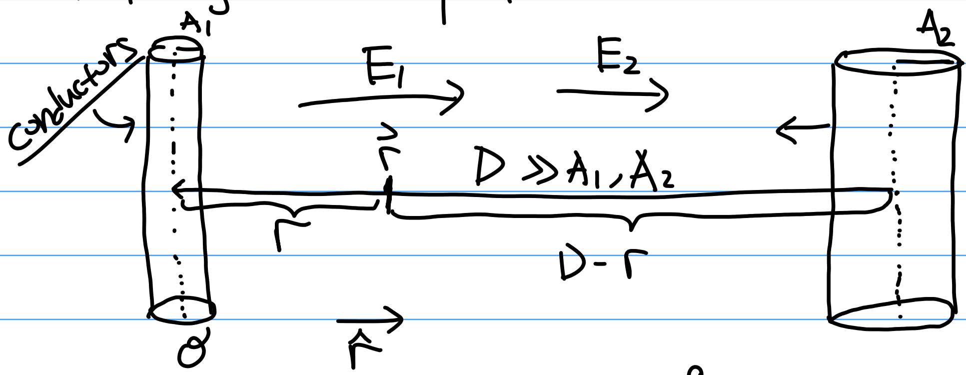
$$= \pi\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \left[\frac{1}{R^6} \frac{1}{4} R^4 + \frac{-1}{2} \frac{1}{r^2} \Big|_R^\infty \right]$$

$$= \pi\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{R^2} \left(\frac{1}{4} + \frac{1}{2} \right)$$

$$= \frac{3}{4} \pi\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{R^2}$$

$$\boxed{\vec{F} = \frac{3}{4} \pi\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{R^2} \hat{z}}$$

2) a) Calculating the Capacitance:



$$q = C\phi \quad \rightarrow \quad C = \frac{q}{\phi}$$

Imagine they have charge $+Q$, $-Q$:

Find $\vec{E}_1$: $$\int d^2\vec{S} \cdot \vec{E} = \frac{Q}{\varepsilon_0}$$

Since we have long rod, we expect $\vec{E} = E_r \hat{r}$
and $d^2S\,\hat{r} = dL_\theta \times dL_z = r\,d\theta\,dz\,\hat{r}$

$$\int_0^L \int_0^{2\pi} r\,d\theta\,dz\, E_{1,r} = \frac{Q}{\varepsilon_0}$$

$$2\pi r L E_{1,r} = \frac{Q}{\varepsilon_0}$$, here $r \gg A_1$

$$\vec{E}_1 = \frac{1}{2\pi\varepsilon_0 r} \frac{Q}{L} \hat{r}$$

Find $\vec{E}_2$: $$\int_0^L \int_0^{2\pi} r\,d\theta\,dz\,(-\hat{r}) \cdot \vec{E}_2 = -\frac{Q}{\varepsilon_0}$$

$$\hookrightarrow -2\pi(D-r)L\,E_{2,r} = -\frac{Q}{\varepsilon_0}$$

$$\hookrightarrow \vec{E}_2 = \frac{1}{2\pi\varepsilon_0(D-r)} \frac{Q}{L} \hat{r}$$

So $$\vec{E}_{tot} = \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\left(\frac{1}{r}+\frac{1}{D-r}\right)\hat{r}$$

$$\Delta\phi_{tot} = -\int_{D-A_2}^{A_1} E_r\hat{r}\cdot dr\,\hat{r}$$

$$= \int_{A_1}^{D-A_2} E_r\hat{r}\cdot dr\,\hat{r}$$

$$= \int_{A_1}^{D-A_2}\frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\left(\frac{1}{r}+\frac{1}{D-r}\right)dr$$

$$= \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\left[\ln(|r|)\Big|_{A_1}^{D-A_2} - \ln(|D-r|)\Big|_{A_1}^{D-A_2}\right]$$

$$= \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\left(\ln\left|\frac{D-A_2}{A_1}\right| - \ln\left|\frac{D-D+A_2}{D-A_1}\right|\right)$$

$$= \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\ln\left|\left(\frac{D-A_2}{A_1}\right)\left(\frac{D-A_1}{A_2}\right)\right|$$

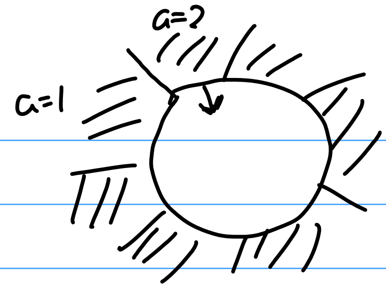
$$= \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\ln\left|\frac{D^2}{A_1A_2}\left[1-\frac{A_2}{D}\right]\left[1-\frac{A_1}{D}\right]\right|$$

Since $A_2 \ll D$, $A_1 \ll D$

$$\cong \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\ln\left|\left(\frac{D}{\sqrt{A_1A_2}}\right)^2\right|$$

$$= \frac{1}{\pi\varepsilon_0}\frac{Q}{L}\ln\left|\frac{D}{A}\right|$$

$$\boxed{\frac{C}{L} = \frac{Q}{\Delta\phi_{tot}}\frac{1}{L} = \pi\varepsilon_0 \Big/ \ln\left|\frac{D}{A}\right|}$$



b) Show $C = \varepsilon_0 \int_V d^3r \, |\vec{\nabla} \Phi|^2$

$$\overset{!}{=} \varepsilon_0 \int_{\text{vacuum}} d^3r \, |\vec{\nabla} \Phi|^2$$

↑ just integrate over vacuum since $\vec{\nabla}\Phi = -\vec{E}$ and $\vec{E} = 0$ inside conductor.

$$\overset{!}{=} \varepsilon_0 \int_{\text{Vac}} d^3r \left\{ \vec{\nabla} \cdot (\Phi \vec{\nabla} \Phi) - \Phi \nabla^2 \Phi \right\}$$

$$\overset{!}{=} \varepsilon_0 \int_{\substack{\text{surface}\\ \text{conductor}}} d^2\vec{S} \cdot \Phi \vec{\nabla} \Phi + \int_{\text{Vac}} d^3r \, \Phi \big(\underbrace{\vec{\nabla} \cdot \vec{E}}_{\rho/\varepsilon_0 = 0 \text{ in vacuum.}} \big)$$

since surface of conductor is equipotential

$$\overset{!}{=} \varepsilon_0 \sum_a \Phi_a \int d^2S_a \, (-\hat{n}) \underbrace{\vec{\nabla} \Phi_a}_{-\vec{E}}$$

$$\overset{!}{=} \varepsilon_0 \sum_a \phi_a \underbrace{\int d^3r \, \underbrace{\vec{\nabla} \cdot \vec{E}_a}_{\rho/\varepsilon_0}}_{q_a/\varepsilon_0}$$

$$C = \varepsilon_0 \int_V d^3r \, |\nabla \Phi|^2 \overset{!}{=} \sum_a \phi_a q_a$$

If $\phi_1 = 1$ and $\phi_{a \neq 1} = 0$

then $\boxed{C = q_1}$ ← From $C = \varepsilon_0 \int_V d^3r \, |\vec{\nabla} \Phi|^2$

Same result

We also know the definition of capacitance: $q_a = C_{ab} \phi_b$

$$\boxed{q_1 = C_{1b} \phi_b = C_{11}}$$ ← From $q_a = C_{ab} \phi_b$

4) Potential due to given charge distribution:

a) $$\rho(\vec{r}) = \begin{cases} (R^2 - r^2)\alpha & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases}$$

Find $\Phi(\vec{r})$

For $r \le R$

$$\int d^3r \, \vec{\nabla} \cdot \vec{E} = \int d^3r \frac{\rho}{\varepsilon_0}$$

$$\int d^2\vec{S} \cdot \vec{E} \stackrel{!}{=} \frac{1}{\varepsilon_0} \int_0^r r'^2 dr' \sin\theta d\theta d\phi \, (R^2 - r'^2)\alpha$$

$$E_r \underbrace{\int r^2 d\theta \sin\theta d\phi}_{r^2 4\pi} \stackrel{!}{=} 4\pi \frac{1}{\varepsilon_0} \alpha \left(R^2 \frac{r'^3}{3}\Big|_0^r - \frac{r'^5}{5}\Big|_0^r \right)$$

$$E_r \stackrel{!}{=} \frac{\alpha}{\varepsilon_0} \frac{1}{r^2} \left(R^2 \frac{r^3}{3} - \frac{r^5}{5} \right)$$

$$\boxed{E_r \hat{r} = \frac{\alpha}{\varepsilon_0} \left[\frac{1}{3} R^2 r - \frac{1}{5} r^3 \right] \hat{r} \quad \text{for} \quad r \le R}$$

For $r > R$:

$$\int d^2\vec{S} \cdot \vec{E} = \int d^3r \, \frac{\rho}{\varepsilon_0}$$

$$\hookrightarrow \cancel{4\pi} r^2 E_r = \frac{1}{\varepsilon_0} \cancel{\int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi} \left\{ \int_0^R r^2 dr \, \alpha (R^2 - r^2) + \int_R^\infty 0 \right\}$$

$$E_r = \frac{\alpha}{\varepsilon_0} \frac{1}{r^2} \left[R^2 \frac{1}{3} r^3 \Big|_0^R - \frac{r^5}{5}\Big|_0^R \right]$$

$$E_r = \frac{\alpha}{\varepsilon_0} \frac{1}{r^2} \left[\frac{R^5}{3} - \frac{R^5}{5} \right]$$

$$\boxed{E_r \hat{r} \stackrel{!}{=} \frac{\alpha}{\varepsilon_0} \frac{1}{r^2} \frac{2}{15} R^5 \hat{r} \quad \text{for} \quad r > R}$$

Now integrate to find $\phi(r)$:

For $\underline{r > R}$:

$$\phi = -\int_\infty^r \vec{E}\cdot d\vec{L}$$

$$\stackrel{!}{=} -\int_\infty^r dr' \, \frac{\alpha}{\varepsilon_0}\frac{2}{15}R^5\frac{1}{r'^2}$$

$$\stackrel{!}{=} \cancel{-}\frac{\alpha}{\varepsilon_0}\frac{2}{15}R^5\left(\cancel{-}\frac{1}{r'}\right)\Bigg|_\infty^r$$

$$\frac{1}{r}-\frac{1}{\infty}$$

$$\boxed{\phi \stackrel{!}{=} \frac{\alpha}{\varepsilon_0}\frac{2}{15}R^5\frac{1}{r} \quad \text{for} \quad r > R}$$

For $\underline{r \le R}$:

$$\phi = -\int_\infty^r \vec{E}\cdot d\vec{L}$$

$$\stackrel{!}{=} -\left[\int_\infty^R dr' \, \frac{\alpha}{\varepsilon_0}\frac{2}{15}R^5\frac{1}{r'^2} + \int_R^r dr' \, \frac{\alpha}{\varepsilon_0}\left[\frac{1}{3}R^2 r' - \frac{1}{5}r'^3\right]\right]$$

$$\stackrel{!}{=} -\left[\frac{\alpha}{\varepsilon_0}\frac{2}{15}R^5\left(\frac{-1}{R}\right) + \frac{\alpha}{\varepsilon_0}\left[\frac{1}{3}R^2\frac{r'^2}{2}\Bigg|_R^r - \frac{1}{5}\frac{r'^4}{4}\Bigg|_R^r\right]\right]$$

$$\stackrel{!}{=} \frac{\alpha}{\varepsilon_0}\frac{2}{15}R^4 + \frac{\alpha}{\varepsilon_0}\left[\frac{1}{6}R^2(r^2-R^2) - \frac{1}{20}(r^4-R^4)\right]$$

$$\stackrel{!}{=} \frac{\alpha}{\varepsilon_0}\left\{\frac{2}{15}R^4 + \frac{1}{6}R^2r^2 - \frac{1}{6}R^4 + \frac{1}{20}R^4 - \frac{1}{20}r^4\right\}$$

$$\boxed{\phi \stackrel{!}{=} \frac{\alpha}{\varepsilon_0}\left\{\frac{1}{60}R^4 + \frac{1}{6}R^2r^2 - \frac{1}{20}r^4\right\} \quad \text{for} \quad r \le R}$$

c) let $\Phi(r,\hat{n}) = \sum_{lm} \Phi_{lm}(r) Y_{lm}(\hat{n})$

and $\rho(r,\hat{n}) = \sum_{lm} \rho_{lm} Y_{lm}(\hat{n})$

$$-\nabla^2 \Phi(r,\hat{n}) = \frac{1}{\varepsilon_0} \rho(r,\hat{n})$$

$$\hookrightarrow -\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}L^2\right]\sum_{lm} \Phi_{lm}(r) Y_l^m(\hat{n}) = \frac{1}{\varepsilon_0}\sum_{lm} \rho_{lm} Y_l^m(\hat{n})$$

we know $L^2 Y_l^m = -l(l+1)$

$$\sum_{l,m}\left[-\frac{1}{r^2}\partial r(r\partial r)\Phi_{lm}\right] Y_l^m(\hat{n}) + \frac{1}{r^2} l(l+1)\Phi_{lm}(r) Y_l^m(\hat{n}) = \frac{1}{\varepsilon_0}\sum_{l,m}\rho_{lm} Y_l^m(\hat{n})$$

→ Multiply both side by $Y_{l'}^{*m'}(\hat{n})$ and integrate over solid angle, $d^2\hat{n}$.

since $\int d^2\hat{n}\; Y_{l'}^{*m'}(\hat{n}) Y_l^m(\hat{n}) = \delta_{mm'}\delta_{ll'}$

then:

$$\hookrightarrow \sum_{l',m'} -\frac{1}{r^2}\partial r(r\partial r)\Phi_{l'm'} + \frac{1}{r^2} l'(l'+1)\Phi_{l'm'} = \frac{1}{\varepsilon_0}\sum_{l'm'}\rho_{l'm'}(r)$$

$$\hookrightarrow \sum_{l',m'} \underbrace{\left\{-\frac{1}{r^2}\partial r(r\partial r) + \frac{1}{r^2} l'(l'+1)\right\}}_{\text{Linear operator } \mathcal{L}} \Phi_{l'm'} = \frac{1}{\varepsilon_0}\sum_{l'm'}\rho_{l'm'}(r)$$

Ordinary since derivative w.r.t one variable, r.

Linear operator $\mathcal{L}$ so this ODE is linear

Inhomogeneous due to this source-term as a function of r

Second derivative in r (2nd order)

Note: we observe that we can separate out different (l', m') pairs since they're independent from each other as the source term i.e. inhomogenity, $f_{l'm'}(r)$ also depend on $l'm'$. So they are separated into different (l', m') channels. In the end we just construct the final solution via $\sum_{l'm'}$.

We can add solutions of Poisson's Equation channel by channel because Poisson's Equation is linear in both the potential and the source term. The Laplace operator is also spherically symmetric, which allows for partial diagonalization using spherical Harmonics.