Conductor Electrostatics:

Even though Maxwell's equation continues to hold at atomic levels, it is much simpler to consider a spatial averaging which smooths out length scales that are smaller than individual atoms and molecules. Hence, we consider "Lorentz - Coarse - graining".

Spatial Averaging:

Consider some microscopically defined quantity, i.e. $f(\vec{r})$, which varies at an atomic lengthscale:

Consider simple / discontinuous averaging: $\bar{f}(\vec{r}) = \frac{1}{\underset{\substack{\uparrow \\ \text{Sampling volume.}}}{\Omega}} \int_{\Omega} d^3 r' \, f(\vec{r} - \vec{r}\,')$

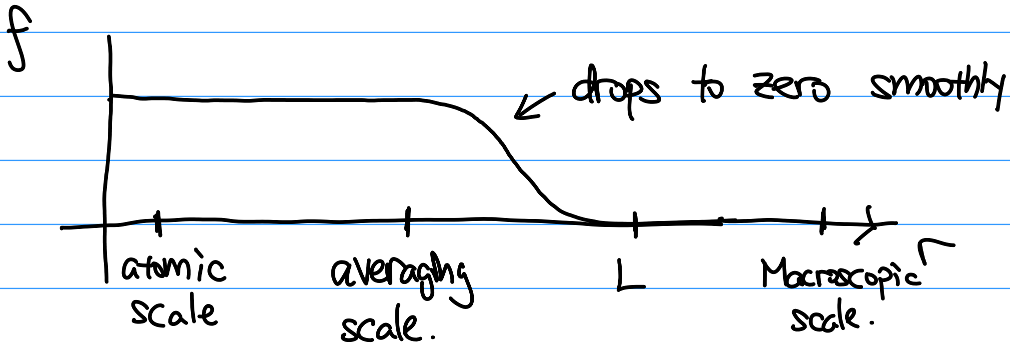
We can think of it in two ways:

i) we keep f fixed, go to the point of interest, $\vec{r}$, then average over the neighborhood of $\vec{r}$ by allowing $\vec{r}\,'$ to vary.

ii) Or, we create a new function, $f(\vec{r} - \vec{r}\,')$, i.e. f is shifted by $\vec{r}\,'$, then average over shifte $\vec{r}\,'$.

→ Discontinuous averaging is not good enough, as atoms move in-and-out of the circle created by $\vec{r}\,'$.

Now consider continuous averaging with a weighting function:



weighting function.

then $$\bar{\rho}(\vec{r}) = \int d^3r' \, f(\vec{r}') \, \rho(\vec{r}-\vec{r}')$$

with $\int d^3r' \, f(\vec{r}')$

This reduces jitter due to the smoothly decay curve.

With requirement: atomic scale $<<$ L $<<$ Macroscopic scale.

Microscopic Maxwell Equations:

Let $\vec{e}, \vec{b}, \gamma, \vec{g}$ for unaveraged field:

$$\vec{\nabla} \times \vec{e} = -\partial_t \vec{b}$$

$$\vec{\nabla} \cdot \vec{e} = \frac{1}{\varepsilon_0} \gamma(\vec{r})$$

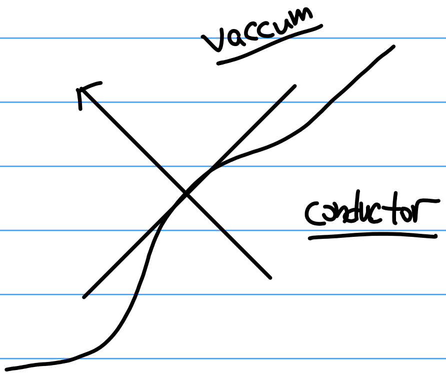
Now with coarse-graining: $\vec{e} \to \vec{E}$ $\quad \vec{b} \to \vec{B}$

$\gamma \to \rho$ $\quad \vec{g} \to \vec{j}$

Then we have:

$$\vec{\nabla} \times \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$$

Boundary Conditions for Conductors:



Claim: For electrostatic:
$\vec{E} = 0$ inside conductors

Reason: If $\vec{E} \neq 0 \to \vec{j} \neq 0$
then not electrostatic.

i) Consider $\vec{\nabla} \times \vec{E} = 0$

By stokes theorem $\int d^2\vec{S} \cdot \vec{\nabla} \times \vec{E} = \oint d\vec{r} \cdot \vec{E} = 0$

Consider geometry

$\vec{E} = 0$ in conductor

so $\boxed{\vec{E}_{\parallel} = 0}$

$$\int_1^2 d\vec{r} \cdot \vec{E} + \int_2^3 d\vec{r} \cdot \vec{E} + \int_3^4 d\vec{r} \cdot \vec{E} + \int_4^1 d\vec{r} \cdot \vec{E} = 0$$

Negligible.

Key Consequences:

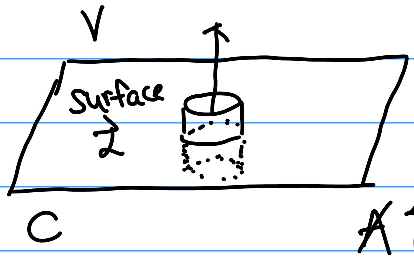
1) $\vec{E}$ is purely normal to the surface. $\vec{E} = \cancelto{0}{E_\parallel} + E_\perp$

2) $-\vec{\nabla}\phi$ is at every point normal to surface.

3) $-\vec{\nabla}\phi_\parallel = 0$ at the surface, the surface of a potential is equipotential.

ii) Now consider $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$ to give us:

$$\boxed{\frac{\sigma}{\varepsilon_0} = E_\perp = -\hat{n}\cdot\vec{\nabla}\phi.}$$ for conductors.



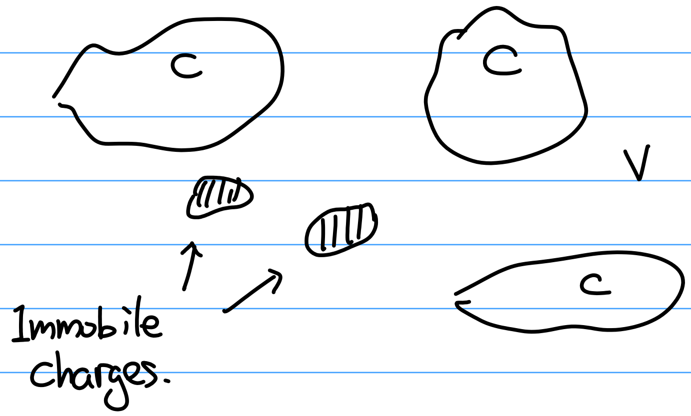
$$\int d^3r\, \vec{\nabla}\cdot\vec{E} = \oint d^2\vec{S}\cdot\vec{E} = \int d^3r \frac{\rho}{\varepsilon_0} = \int d^2S \frac{\sigma}{\varepsilon_0}$$

$$\cancel{A}\left\{\vec{E}(\vec{\alpha}+0\hat{n})\cdot(+\hat{n}) + \underbrace{\vec{E}(\vec{\alpha}-0\hat{n})}_{=0}\cdot(-\hat{n})\right\} = \frac{\sigma}{\varepsilon_0}\cancel{A}$$

So $E_\perp = \frac{\sigma}{\varepsilon_0}$

Summary:
1) $\vec{E} = 0$ in conductors. ← net field
2) $E_\parallel = 0$ on the surface of conductor
3) $E_\perp = \sigma/\varepsilon_0$
4) ϕ is equipotential along the surface.
5) Charges are pushed to the surface and freely to move on conductors.
6) ϕ is equipotential inside Conductor.

classic problems in conductors:



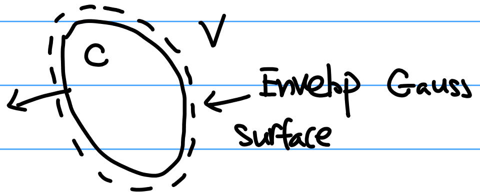
In vaccum: $-\nabla^2\phi = \frac{\rho}{\varepsilon_0}$ ← ρ: immobile charge density.

At surface: $E_\parallel = 0$

$E_\perp = \frac{\sigma}{\varepsilon_0}$

Asymptotically: $\phi \to 0$ at $\vec{r} \to \infty$

Relating $E_\perp$ at the surface to the total surface charge.



$$\vec\nabla\cdot\vec E = \frac{\rho}{\varepsilon_0}$$

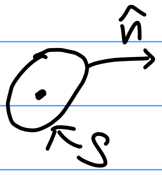
↳ $$\int d^2\vec S\cdot\vec E = \int d^2 s\, \frac{\sigma}{\varepsilon_0}$$

↳ $$\boxed{-\oint d^2 s\; \hat n\cdot\vec\nabla\phi = \frac{Q}{\varepsilon_0}}$$

⇒ Property: $\boxed{\text{No minimum nor maximum anywhere in V.}}$

Or Min / Max can only occur at the boundary.

Proof: Suppose there is a minimum at $\vec r$,



then $\int d^3 s\; \hat n\cdot\vec\nabla\phi > 0$

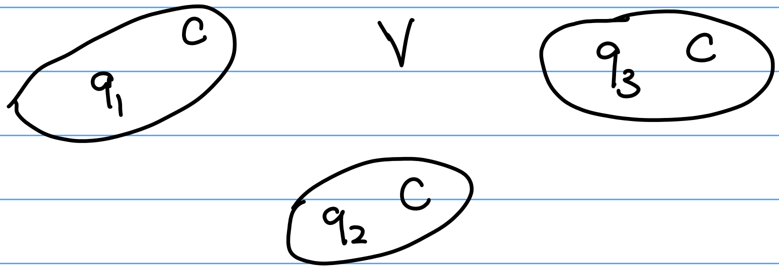
↑ ϕ increases as leaving S if at minimum

contradiction.

but $$\int d^2 s\, \hat n\cdot\vec\nabla\phi = -\int d^2\vec S\cdot\vec E = -\int d^3 r\, \underbrace{\vec\nabla\cdot\vec E}_{=\frac{\rho}{\varepsilon_0}=0 \text{ in vaccum}} = 0.$$

Energy of electrostatic field due to charged conductors

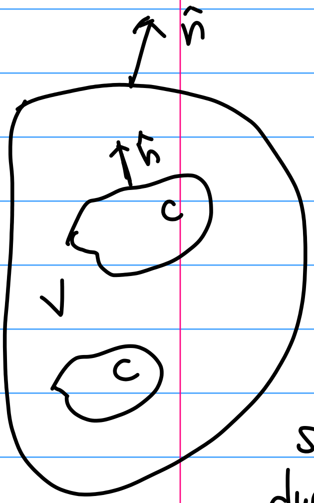
- → Conductors carry charge q_a
- → Location of conductors are fixed.



$$\mathcal{E} = \frac{\varepsilon_0}{2}\int_V d^3r\ |\vec{E}|^2$$

↑ over-all space, but techiquelly only vaccum since $\vec{E}$ inside conductor is zero.

$$\overset{|}{=} -\frac{1}{2}\varepsilon_0\int_{vac} d^3r\ \vec{\nabla}\phi\cdot\vec{E}$$



$$\overset{|}{=} -\frac{1}{2}\varepsilon_0\int_{vac} d^3r\ \vec{\nabla}\cdot(\phi\vec{E}) - \phi\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0}$$

$$\overset{|}{=} -\frac{1}{2}\varepsilon_0\int_{vac} d^2\vec{S}\cdot\phi\vec{E}$$

outward normal for conductors,

$$\overset{|}{=} \frac{1}{2}\varepsilon_0\sum_a\int_{S_a} d^2S\ \hat{n}\cdot\phi\vec{E} - \underbrace{\frac{1}{2}\varepsilon_0\int_{S\ at\ \infty} d^2\vec{S}\cdot\phi E}_{=0\ \text{since}\ \phi E\propto\frac{1}{r^3}\ \text{but}\ d^2S\propto r^2.}$$

↑ sign flip due to choice of $\hat{n}$

↑ conductor surface.

$$\overset{|}{=} \frac{1}{2}\sum_a\phi_a\ \underbrace{\varepsilon_0\int_{S_a} d^2\vec{S}\cdot\vec{E}}_{q_a}$$

$$\boxed{\mathcal{E} \overset{|}{=} \frac{1}{2}\sum_a\phi_a q_a}$$

Sine the field equations in the vaccum are linear and homogeneous, the geometry of the conductors and their placements must determine a matrix of constants:

$$\boxed{q_a = \sum_b C_{ab} \phi_b}$$

For C_{ab}:
- i) Diagonal elements: Coefficient of capacity
- ii) off-Diagonal, coefficient of electrostatic induction
- iii) for one conductor, $q = \underset{\substack{\uparrow \\ \text{Capacitance, or how much charge for a given voltage.}}}{c}\phi$.

$$\vec{\phi} = \overleftrightarrow{C}^{-1} \vec{q}$$

Now we ask how does $\mathcal{E}$ change when we $q \rightarrow q + \delta q$. Note that as charge changes, the potential on the conductors also change.

Start with: $$\mathcal{E} = \frac{1}{2} \varepsilon_0 \int_{vac} d^3r \, |\vec{E}|^2$$

as $q \rightarrow q + \delta q$, then $E \rightarrow \vec{E} + \delta\vec{E}$, so $\mathcal{E} \rightarrow \mathcal{E} + \delta\mathcal{E}$

$$\delta\mathcal{E} = \varepsilon_0 \int_{vac} d^3r \, \vec{E} \cdot \delta\vec{E}$$

Now there are two paths to take:

Path 1: let $-\vec{\nabla}\phi = \vec{E}$

$$\delta\mathcal{E} = -\varepsilon_0 \int_{vac} d^3r\ \vec{\nabla}\phi \cdot \vec{E}$$

$$\overset{!}{=} -\varepsilon_0 \int_{vac} d^3r \left[\vec{\nabla}\cdot(\phi\,\delta\vec{E}) - \phi \underbrace{\vec{\nabla}\cdot\delta\vec{E}}_{=0}\right]$$

$$\overset{!}{=} \varepsilon_0 \int d^2s\ \hat{n}\cdot \phi\,\delta\vec{E}$$

$$\overset{!}{=} \varepsilon_0 \sum_a \phi_a \int_{S_a} d^2\vec{S}\cdot\delta\vec{E} - \varepsilon_0 \int_{S,\infty} d^2\vec{s}\cdot\delta\vec{E}\,\phi \quad \rightarrow = 0.$$

$$\overset{!}{=} \varepsilon_0 \sum_a \phi_a \int_{S_a} \underbrace{d^2\vec{S}\cdot\left[(\vec{E}+\delta\vec{E})-\vec{E}\right)}_{d^3r\left(\frac{\rho+\delta\rho}{\varepsilon_0}-\frac{\rho}{\varepsilon_0}\right)}$$

$$\boxed{\delta\mathcal{E} \overset{!}{=} \sum_a \phi_a\,\delta q_a}$$

Path #2: let $\delta E = -\vec{\nabla}\delta\phi$

$$\delta\mathcal{E} = -\varepsilon_0 \int_{vac} d^3r \left[\vec{\nabla}\cdot(E\,\delta\phi) - \cancel{(\vec{\nabla}\cdot E)}\,\delta\phi\right]$$

$$\overset{!}{=} \varepsilon_0 \sum_a \int_{S_a} d^2S\ \hat{n}\cdot\vec{E}\ \delta\phi - \varepsilon_0 \int_{S,\infty} \cancelto{0}{d^2\vec{S}\cdot\vec{E}\,\delta\phi}$$

Move out of integral since equipotential on surface of conductor.

$$\boxed{\delta\mathcal{E} \overset{!}{=} \sum_a q_a\,\delta\phi_a}$$

Summary:

$$\delta\mathcal{E} = \begin{cases} \sum_a q_a\,\delta\phi_a & : \phi \text{ controlled, } q \text{ responds.} \\ \sum_a \phi_a\,\delta q_a & : q \text{ controlled, } \phi \text{ responds} \end{cases}$$

↳ Then $\boxed{\frac{\partial\mathcal{E}}{\partial q_a} = \phi_a}$ $\rightarrow$ $\mathcal{E} = \mathcal{E}(q)$

or $\boxed{\frac{\partial\mathcal{E}}{\partial\phi_a} = q_a}$ $\rightarrow$ $\mathcal{E} = \mathcal{E}(\phi)$

Since $$\frac{\partial \mathcal{E}}{\partial q_a} = \sum_b C^{-1}_{ab}\, q_b \rightarrow \frac{\partial^2 \mathcal{E}}{\partial q_a \partial q_b} = (C^{-1})_{ab}$$

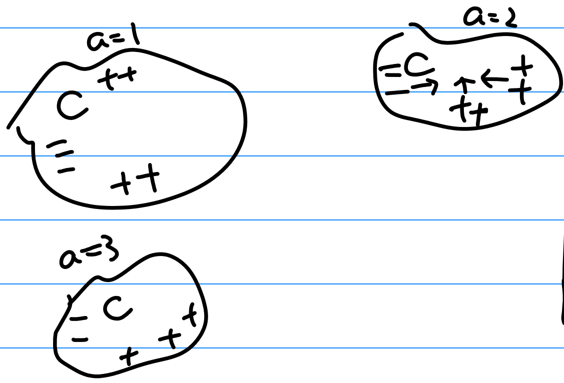
$$\frac{\partial \mathcal{E}}{\partial \phi_a} = \sum_b C_{ab}\, \phi_b \rightarrow \frac{\partial^2 \mathcal{E}}{\partial \phi_a \partial \phi_b} = C_{ab}$$

$$\boxed{\text{symmetrical } C_{ab}}$$

Since $$\mathcal{E} = \frac{1}{2} \sum_a q_a \phi_a$$

$$\boxed{\begin{aligned} \mathcal{E} &= \frac{1}{2} \sum_{ab} q_a (C^{-1})_{ab}\, q_b \\ \mathcal{E} &= \frac{1}{2} \sum_{ab} \phi_a\, C_{ab}\, \phi_b \end{aligned}} \left.\vphantom{\begin{aligned} a \\ b \\ c \end{aligned}}\right\} \begin{aligned} &\text{Diagonal terms are positive.} \\ &\text{off-Diagonal terms are negative.} \end{aligned}$$

Thomson's Stationality Theorem:



↳ $\mathcal{E}$[actual charge distribution, δ_a] is less than $\mathcal{E}$[rearrangement of total charge without any hopping.]

$$\delta\mathcal{E} = \varepsilon_0 \int_{\substack{\text{all}\\ \text{space}}} d^3r\, \vec{E}\cdot\delta\vec{E}$$

← here we consider all-space since $\vec{E}$ can be non-zero inside conductor

$$= -\varepsilon_0 \int d^3r\, \vec{\nabla}\phi\cdot\delta\vec{E}$$

$$= -\varepsilon_0 \int d^3r \left(\vec{\nabla}\cdot(\phi\,\delta E) - \phi\,\vec{\nabla}\cdot\delta E\right)$$

$$= \underbrace{-\varepsilon_0 \int_{\infty} d^2\vec{S}\cdot\phi\,\delta\vec{E}}_{=0} + \varepsilon_0 \int_{\substack{\text{all}\\ \text{space}}} d^3r\, \phi\, \underbrace{\vec{\nabla}\cdot\delta\vec{E}}_{\text{Nonzero in/on conductors.}}$$

$$= \sum_a \int_{V_a} d^3r\, \phi_a\, \delta\rho$$

Volume of conductors ↗ V_a

↑ ϕ_a in/on conductor a

$\delta\rho$ ↘ from $\vec{\nabla}\cdot\delta\vec{E}$

$$\delta\mathcal{E} = \sum_a \phi_a \underbrace{\int_{V_a} d^3r\, \delta\rho}_{=0} = 0$$

=0 charge is only rearranged, but total charge still conserved.

so $\boxed{\mathcal{E}\text{ is stationary (minimum)}}$

Now check if ε is at minimum or maximum.

$$\delta\mathcal{E} = \mathcal{E}[E+\delta E] - \mathcal{E}(E)$$

$=0$ by stationary

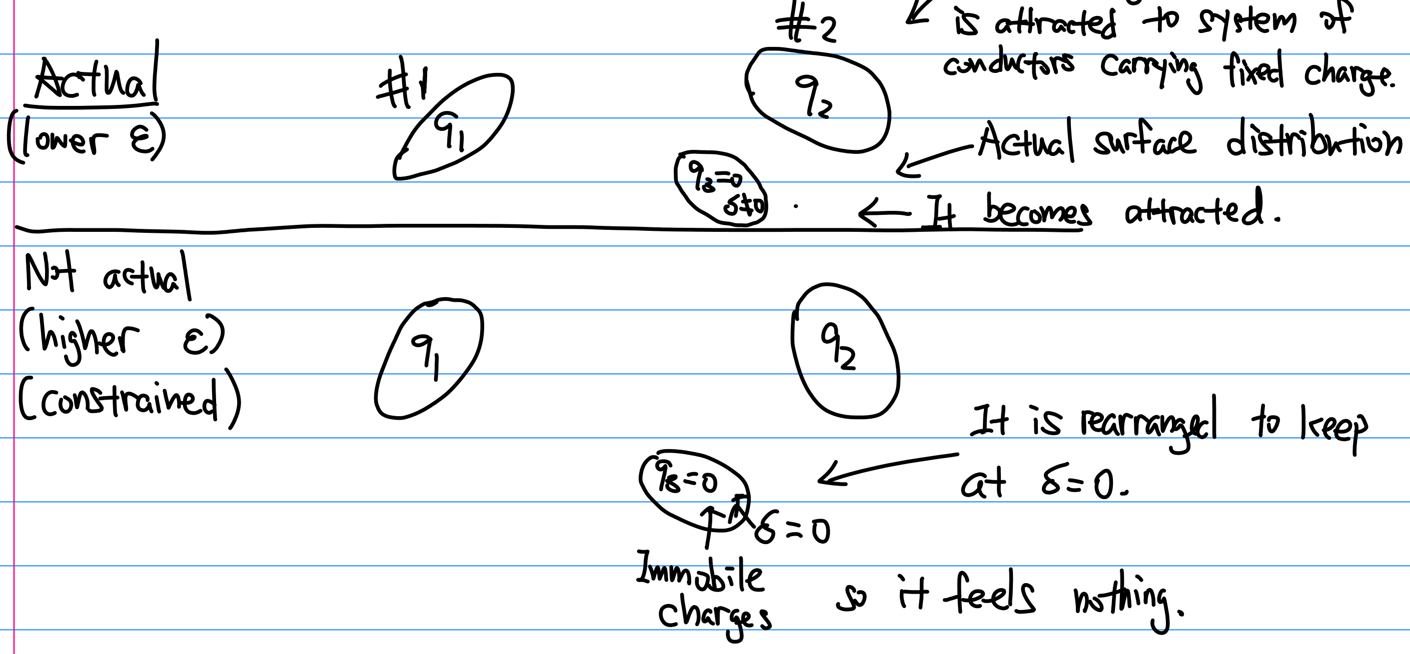
$$\overset{!}{=} \varepsilon_0 \int_{vac} d^3r \; E\cdot\delta E + \frac{1}{2}\varepsilon_0 \int_{vac} d^3r \; \underbrace{|\delta E|}_{\underline{\underline{>0}} \text{ if } \delta E \neq 0}{}^2$$

So $\delta^{(2)} E > 0$ means it is a minimum.

So Thomson Theorem:

ε [$\vec{E}$ due to actual surface charge distribution] $<$ ε [$\vec{E}$ due to any rearrangement of that charge in/on each conduc]

Application of Thomson Theorem:



An uncharged conductor is attracted to system of conductors carrying fixed charge.

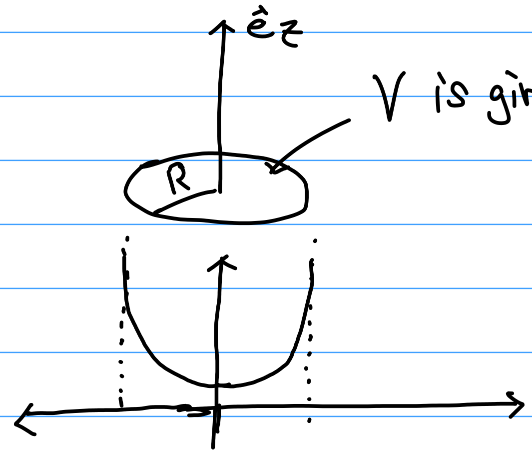
Actual surface distribution

It becomes attracted.

It is rearranged to keep at $\delta=0$.

so it feels nothing.

Jackson $\underline{3.3}$:



$$\delta = \frac{c}{\sqrt{R^2-r^2}}$$

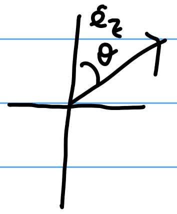
what is the total charge: $Q = \int_0^R 2\pi r\,dr \frac{c}{\sqrt{R^2-r^2}}$

$$r \to t = R\sin\psi$$

$$\overset{!}{=} \frac{R^{\cancel{2}}}{\cancel{R}} \int_0^1 2\pi t\,dt \frac{c}{\sqrt{1-t^2}}$$

$$\overset{!}{=} 2\pi R C$$

Compute potential everywhere.



$$\underline{\Phi}(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \rho(\vec{r}') \leftarrow r > R$$

$$\overset{!}{=} \frac{1}{4\pi\epsilon_0} \int_0^R r'dr' \int_0^{2\pi} d\phi' \frac{1}{|\vec{r}-\vec{r}'|} \frac{c}{\sqrt{R^2-r'^2}}$$

Use multipole expansion:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \left(\frac{r_<}{r_>}\right)^l \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi')$$

$$r_> = \max(r, r') \qquad r_< = \min(r, r')$$

For $r > R$, then $r_> = r$, $r_< = r'$

$$= \frac{c}{4\pi\epsilon_0}\frac{1}{r}\sum_l \frac{4\pi}{2l+1}\frac{1}{r^l}\int_0^R \frac{r'dr'r'^l}{\sqrt{R^2-r'^2}} \sum_m Y_{lm}(\theta,\phi) \underbrace{\int_0^{2\pi} d\phi' Y^*_{lm}\left(\theta=\frac{\pi}{2},\phi\right)}_{=0 \text{ unless } \boxed{m=0} \text{ since azimuthal symmetry.}}$$

$$= \frac{c}{4\pi\epsilon_0}\frac{1}{r}\sum_l\frac{4\pi}{2l+1}\frac{1}{r^l}\int_0^R\frac{r'dr'r'^l}{\sqrt{R^2-r'^2}}\sqrt{\frac{2l+1}{2}}\frac{1}{\sqrt{2\pi}}P_l(\cos\theta)\sqrt{\frac{2l+1}{2}}\frac{1}{\sqrt{2\pi}}P_l(0)2\pi$$

$$= \frac{c}{4\pi\epsilon_0}\frac{1}{2}4\pi\frac{1}{r}\sum_{l=0}^{\infty}\frac{1}{r^l}P_l(\cos\theta)P_l(0)\int_0^R\frac{dr'\,r'^{l+1}}{\sqrt{R^2-r'^2}}$$

let $t = r'/R$

$$= \frac{c}{4\pi\epsilon_0}\frac{1}{2}4\pi\frac{1}{r}\sum_{l=0}^{\infty}\frac{1}{r^l}P_l(\cos\theta)\underbrace{P_l(0)}_{\substack{P_l(0)=0\\ \text{for } l=\text{odd.}}}R^{l+1}\underbrace{\int_0^1\frac{dt\;t^{l+1}}{\sqrt{1-t^2}}}_{J_{2n+1}=\frac{(2^l l!)^2}{(2l+1)!}} \quad \text{let } t=\sin\psi$$

$$= \frac{c}{4\pi\epsilon_0}\frac{1}{2}4\pi\left(\frac{R}{r}\right)\sum_{n=0}^{\infty}J_{2n+1}\left(\frac{R}{r}\right)^l P_{2n}(\cos\theta)\,P_{2n}(0) \leftarrow \frac{(-1)^n(2l)!}{(2^l l!)^2}$$

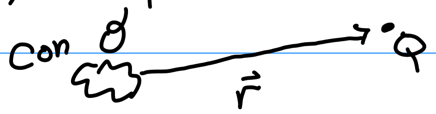
generating func: $P_l(x) = \frac{1}{l!}\frac{1}{2^l}\frac{d^l}{dx^l}(x^2-1)^l$

$Q = 2\pi RC$

$$\boxed{\Phi(r,\theta,\cancel{\phi}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{2n}P_{2n}(\cos\theta)\frac{(-1)^n}{2n+1} \qquad r>R}$$

Essentially uniform field due to extremely remote charge.

→ Away from its source charge, a potential $\Phi(r)$ obeys $\nabla^2 \Phi = 0$



→ Evidently, any linear function of Cartesian Coordinate

$$\Phi(\vec{r}) = \vec{C} \cdot \vec{r} = C_1 x + C_2 y + C_3 z \quad , \Phi(\vec{r}) = \text{Const} - \vec{E}_0 \cdot \vec{r}$$

is one family of solutions

since $\nabla^2 \vec{C} \cdot \vec{r} = \partial_u \partial_u C_v r_v = 0$

It is important, since it gives a constant-in-space $\vec{E}$-field:

$$\vec{E} = -\vec{\nabla} \Phi = -\vec{\nabla}(\vec{C} \cdot \vec{r}) = -\vec{C}$$

so $\Phi(\vec{r}) = -\vec{E}_0 \cdot \vec{r}$

$\hookrightarrow \vec{E}(\vec{r}) = \vec{E}_0$

This $\Phi(\vec{r}) = -\vec{E}_0 \cdot \vec{r}$ is an useful idealization of the consequences of suitably located far-away charge.

→ Now we consider the energy of an uncharged conductor in a uniform field $\vec{E}_0$.

→ We can consider $\vec{E}_0$ resulting from an infinitely remote charge instead of finite.

Then the interaction energy between conductor and the remote charge is:

$$\mathcal{E} = \frac{1}{2} e \, \phi$$

Field at the remote charge due to the charge on the conductor (not net charge but rearranged charge).

remote charge that causes E_0

The field, $\vec{E}_0$, causes conductor to have a dipole moment $\vec{\pi}$, which is the leading consequence at large distance of the charge rearrangement on the conductor.

potential of dipole. → $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{\pi}\cdot\vec{r}}{r^3}$ (conductor at origin)

then:

$$\mathcal{E} = \frac{1}{2} e \frac{1}{4\pi\epsilon_0} \frac{\vec{\pi}\cdot\vec{r}}{r^3} = -\frac{1}{2} \underbrace{\frac{1}{4\pi\epsilon_0} \frac{e(-\vec{r})}{r^3}}_{\text{Field at } O \text{ due to charge } (+e) \text{ at } \vec{r}. \text{ So } (-).} \cdot \vec{\pi}$$

charge at $\vec{r}$, far away from origin

$$\boxed{\mathcal{E} = -\frac{1}{2} \vec{E}_0 \cdot \vec{\pi}}$$

Interaction energy between far-away remote charge and conductor at origin.

Polarizability:

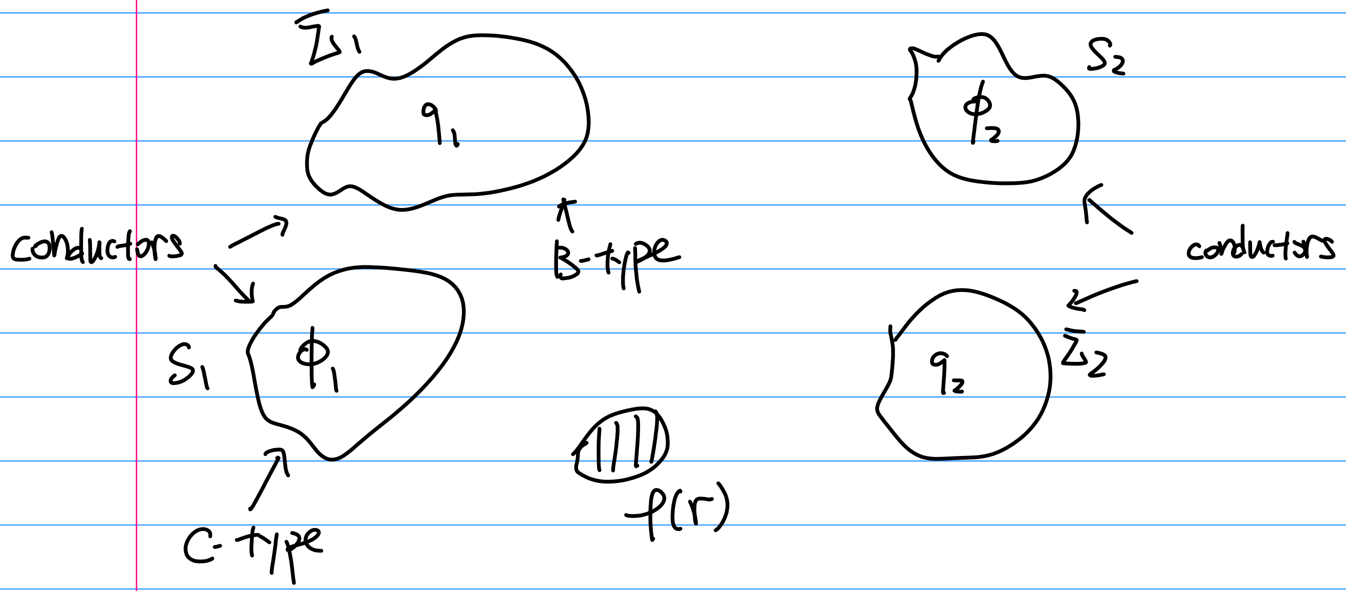
$$\Pi_a = 4\pi\varepsilon_0 \underset{\substack{\uparrow \\ \text{volume} \\ \text{of conductor} \\ \text{(geometry} \\ \text{of conductor)}}}{V} \overset{\text{Polarizability tensor (dimensionless)}}{\alpha_{ab}} (E_0)_b$$

Then

$$\varepsilon_{int} = -\frac{1}{2} \vec{E}_0 \vec{\Pi}$$

$$\boxed{\varepsilon_{int} \overset{!}{=} -\frac{1}{2} 4\pi\varepsilon_0 V (E_0)_a \alpha_{ab} (E_0)_b}$$

Construction of Green's Function:



Special potential problem:

i) $-\vec{\nabla}_r^2 \underset{\hookrightarrow \phi}{G}(\vec{r},\vec{r}\,') = \frac{1}{\varepsilon_0} \overset{\sim \rho}{\delta}(\vec{r}-\vec{r}\,')$ for $\vec{r},\vec{r}$ in $\overset{\text{Not conductor}}{\overline{\text{con}}}$

ii) C-conductors, S_c: $G(\vec{r},\vec{r}) = 0$

gives ϕ. ← on c-type conductor.

iii) For each of the $\overset{\text{give } q.}{\Sigma_b}$ surfaces $\rightarrow \int_{\Sigma_b} d^2\vec{S}_r \cdot \underbrace{\vec{\nabla}_r G(\vec{r},\vec{r}\,')}_{\sim E} = 0$

(Net charge for b-cond is zero)

Net charge = 0 since $\delta(\vec{r},\vec{r}\,')$, but we integrate over $d^2\vec{S}_r$ of Σ_b.

iv) $G(\vec{r},\vec{r}\,')$ = a constant on any Σ_b

Green's Theorem:

$$\boxed{\begin{aligned}&\int_V d^3r\,\left[A(\vec{r})\,\nabla_r^2 B(\vec{r}) - B(\vec{r})\,\nabla_r^2 A(\vec{r})\right]\\ &= \int_S d^2S\,\left[A(\vec{r})\,\vec{\nabla}_r B(\vec{r}) - B\,\vec{\nabla}_r A(\vec{r})\right]\end{aligned}}$$

Get via eval: $\int d^3r\,\vec{\nabla}\cdot\left[A\vec{\nabla}B - B\vec{\nabla}A\right]$ and use. div theorem.

Lets choose $A(\vec{r}) = \Phi(\vec{r})$, $B(\vec{r}) = G(\vec{r},\vec{r}\,')$

$$\int_{con} d^3r\left[\overset{-\frac{1}{\varepsilon_0}\Phi(\vec{r}\,')}{\overset{\uparrow}{\Phi(\vec{r})\underbrace{\nabla_r^2 G(\vec{r},\vec{r}\,')}_{=-\frac{1}{\varepsilon_0}\delta(\vec{r}-\vec{r}\,')}}} - \overbrace{\underbrace{G(\vec{r},\vec{r}\,')}_{\text{know}}\underbrace{\nabla_r^2\Phi}_{=-\frac{1}{\varepsilon_0}\rho(\vec{r})}}^{\int d^3r\,G(\vec{r},\vec{r}\,')\rho(\vec{r})\frac{1}{\varepsilon_0}}\right]$$

$$= -\sum_c \int_{S_c} d^2\vec{S}\cdot\Big[\underbrace{\Phi(\vec{r})}_{\text{equipotential, so pull out}}\vec{\nabla}_r G(\vec{r},\vec{r}\,') - \cancel{\underbrace{G(\vec{r},\vec{r}\,')}_{G=0\text{ on }C\text{-cond.}}}\,\vec{\nabla}_r\Phi(\vec{r})\Big]$$

then define $\int_{S_c} d^2\vec{S}_c\,\vec{\nabla}_r G(\vec{r},\vec{r}\,') = Q_c(\vec{r}\,')$

$$-\sum_b \int_{S_b} d^2\vec{S}\cdot\Big[\Phi(\vec{r})\vec{\nabla}_r G(\vec{r},\vec{r}\,') - \underbrace{G(\vec{r},\vec{r}\,')}_{=P_b(\vec{r}\,')}\,\vec{\nabla}_r\Phi(\vec{r})\Big]$$

$=P_b(\vec{r}\,')$ since equipotential on b-type conductor so shouldn't depend $\vec{r}$.

$$\underbrace{\int_{S_b} d^2\vec{S}\cdot\vec{\nabla}_r\Phi(\vec{r})}_{-\frac{1}{\varepsilon_0}q_b}$$

but $\int_{S_b} d^2\vec{S}\cdot\vec{\nabla}_r G(\vec{r},\vec{r}\,') \overset{\text{B.C.}}{=} 0$

In the end:

$$\Phi(\vec{r}\,') = \int d^3r\,G(\vec{r},\vec{r}\,')\rho(\vec{r}) - \sum_c \phi_c Q_c(\vec{r}\,') + \sum_b P_b(\vec{r}\,')\,q_b$$

where

$$Q_c(\vec{r}\,') = -\varepsilon_0\int_{S_c} d^2\vec{S}\cdot\vec{\nabla}_r G(\vec{r},\vec{r}\,')\ ,\quad P_b(\vec{r}\,') = G(\vec{r},\vec{r}\,')$$

Application Green's Function:

Case:
- → $\rho = 0$
- → No Σ_b conductors (q fixed)
- → only conductors with ϕ fixed.

Then $\Phi(\vec{r}\,') = -\sum_c \phi_c Q_c(\vec{r})$

↳ G dependent.

$$\mathcal{E} = \frac{1}{2}\varepsilon_0 \int_{\overline{\text{cond}}} d^3r \, |\vec{\nabla}\Phi|^2 = \frac{1}{2}\varepsilon_0 \int_{\overline{\text{con}}} d^3r \left(\vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi) - \Phi\nabla^2\Phi\right))$$

$$= -\frac{1}{2}\varepsilon_0 \sum_c \phi_c \int_{S_c} \underbrace{d^2\vec{S}_r \; \vec{\nabla}\Phi}_{q_c}.$$

$$\overset{!}{=} \frac{1}{2}\sum_{c\bar{c}} \bar{\bar{C}}_{c\bar{c}}\, \phi_c \phi_{\bar{c}}$$

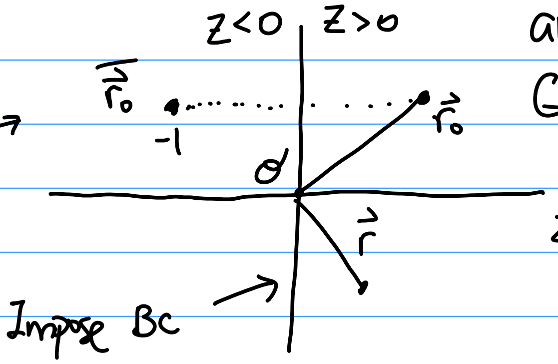
then $(\bar{\bar{C}})_{c\bar{c}} = -\varepsilon_0^2 \int_{S_{\bar{c}}} d^2\vec{S}_{r'} \cdot \vec{\nabla}_{r'} \int_{S_c} d^2\vec{S}_r \cdot \vec{\nabla}_r G(\vec{r}, \vec{r}\,')$

Method of Images :

Requires high symmetry

→ Scalar potential, electric field due to point charge when there are various types of boundary condition.

Ex : Find the potential $G(\vec{r},\vec{r}_0)$ due to a point charge in vaccum at $\vec{r}_0$. in region bounded by initial flat plane on which potential vanishes.



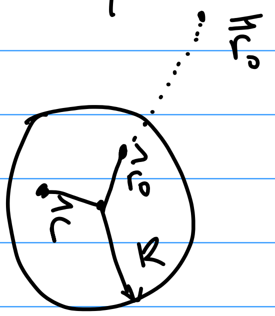
place charge, opposite charge, so that BC is not violated. Outside region of interest, so $\vec{\nabla}\cdot\vec{E}$ is not affected

Impose BC $G(\vec{r},\vec{r}_s)=0$

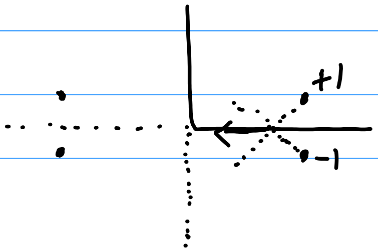
ans:

$$G(\vec{r},\vec{r}_0) = \frac{1}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{|\vec{r}-\vec{\bar{r}}_s|}\right]$$

Ex 2: Sphere Dirichlet BC for $G(\vec{r},\vec{r}_s)=0$ → on surface $\vec{r}=R\hat{n}$



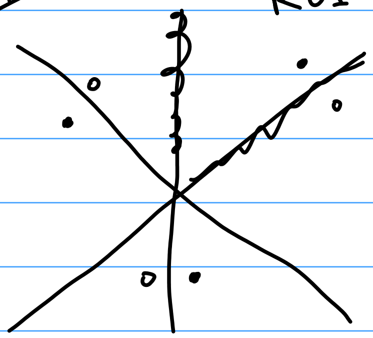
Ex3: Neumann BC. $\frac{\partial\phi}{\partial n}=0$



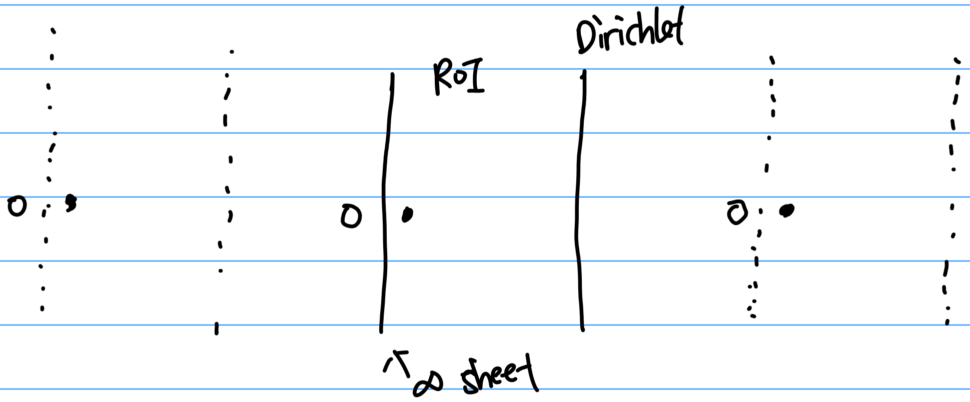
→ sum of 4 G.F.

Ex 4

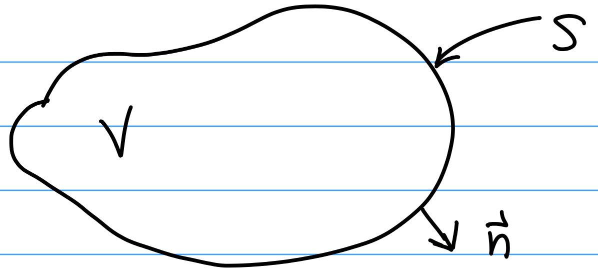
ROI , Dirichlet.



Ex 5.



⇒ For Dirichlet or Neumann boundary-value problems, using a Green function once you know it.



Recall that: $-\nabla_r^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$, $\vec{r}$ and $\vec{r}_0$ in V

for now leave boundary conditions unsatisfied.

Suppose you would like to use G to solve Poisson's eqn:

$$-\nabla_r^2 \Phi(\vec{r}) = \rho(r)\frac{1}{\varepsilon_0}$$

Then feed it into Green's Theorem:

$$\int d^3r \left[\underset{G}{A(\vec{r})}\, \nabla_r^2 \underset{\Phi}{B(\vec{r})} - \underset{\Phi}{B(\vec{r})}\, \nabla_r^2 \underset{G}{A(\vec{r})}\right] = \int d^3r\, \vec{\nabla}\cdot\left(A(\vec{r})\, \vec{\nabla}_r B(\vec{r}) - B(\vec{r})\vec{\nabla}_r A(\vec{r})\right)$$

$$= \int_S d^2\vec{S}\cdot\left[\underset{G}{A(\vec{r})}\cdot\vec{\nabla}_r \underset{\Phi}{B(\vec{r})} - \underset{\Phi}{B(\vec{r})}\cdot\vec{\nabla}_r \underset{G}{A(\vec{r})}\right]$$

$$\rightarrow \int d^3r \left[-G(\vec{r},\vec{r}_0)\underbrace{\rho(\vec{r})}_{-\nabla_r^2 \Phi = \rho(\vec{r})\frac{1}{\varepsilon_0}} + \Phi(\vec{r})\underbrace{\delta(\vec{r}-\vec{r}_0)}_{-\nabla_r^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)\frac{1}{\varepsilon_0}}\right]\frac{1}{\varepsilon_0}$$

$$= \int_S d^2\vec{S}_r \cdot \left[G(\vec{r},\vec{r}_0)\,\vec{\nabla}_r \Phi(\vec{r}) - \Phi(\vec{r})\,\vec{\nabla}_r G(\vec{r},\vec{r}_0)\right]$$

With Boundary Conditions

Neumann

$$\rightarrow \boxed{\Phi(\vec{r}_0) = \int_V d^3r\, G(\vec{r},\vec{r}_0)\, \rho(\vec{r}) + \varepsilon_0 \int_S d^2\vec{S}_r \cdot \left[G(\vec{r},\vec{r}_0) \vec{\nabla}_r \Phi(\vec{r}) - \Phi(\vec{r}) \vec{\nabla}_r G(\vec{r},\vec{r}_0) \right]}$$

Bulk Term

Surface terms

Dirichlet

This expresses : → what we want, Φ
→ in terms of what we know, $\rho(\vec{r})$
→ Plus pieces that we still need to settle with B.C. on Φ and G.

Examples with different boundary conditions:

1) Dirichlet B.C. on Φ.

$$\Phi(\vec{r}_0) = \int_V d^3r\, G(\vec{r},\vec{r}_0)\, \rho(\vec{r}) + \int_S d^2\vec{S}_r \cdot \left[G(\vec{r},\vec{r}_0) \vec{\nabla}_r \Phi(\vec{r}) - \Phi(\vec{r}) \vec{\nabla}_r G(\vec{r},\vec{r}_0) \right]$$

Neumann B.C. (don't know)

Dirichlet B.C. (we know)

Now since we don't know Neumann term, let's choose the B.C. for Green's function.

$$\rightarrow \quad G(\vec{r},\vec{r}_0) = 0 \quad (\vec{r} \text{ on } S)$$

Homogeneous Dirichlet B.C. on $G(\vec{r},\vec{r}_0)$

$$\boxed{\hookrightarrow \Phi(\vec{r}_0) = \int_V d^3r\, G(\vec{r},\vec{r}_0)\, \rho(\vec{r}) - \int_S d^2\vec{S}_r \cdot \Phi(\vec{r}) \vec{\nabla}_r G(\vec{r},\vec{r}_0)}$$

How Green Function feeds in the bulk and boundary inhomogenity into solution.

2) Neumann Boundary Condition on Φ.

$$\to \Phi(\vec r_0)=\int_V d^3r\; G(\vec r,\vec r_0)\,\rho(\vec r)+\int_S d^2\vec S_r\cdot\Big[\underset{\substack{\uparrow\\ \text{Neumann B.C.}\\ \text{(we know)}}}{G(\vec r,\vec r_0)\vec\nabla_r\Phi(\vec r)}-\underset{\substack{\uparrow\\ \text{Dirichlet B.C.}\\ \text{(don't know)}}}{\Phi(\vec r)\,\vec\nabla_r G(\vec r,\vec r_0)}\Big]$$

Now choose boundary information on G so that we eliminate Dirichlet BC on Φ, which is unknown.

$$\to \quad \hat n\cdot\vec\nabla_r G(\vec r,\vec r_0)\underset{\substack{\uparrow\\ \text{Homogeneous B.C.}}}{=0}\qquad (\vec r \text{ on } S)$$

$$\boxed{\hookrightarrow\ \Phi(\vec r_0)=\int_V d^3r\,\underset{\substack{\uparrow\\ \text{Bulk Term}}}{G(\vec r,\vec r_0)\,\rho(\vec r)}+\int_S d^2\vec S_r\cdot\underset{\substack{\uparrow\\ \text{Inhomogeneous Boundary term.}}}{G(\vec r,\vec r_0)\,\vec\nabla_r\Phi(\vec r)}}$$

Note that for Green's Function:

$$G(\vec r,\vec r_0)=G(\vec r_0,\vec r).$$

for Dirichlet and Neumann Boundary Condition

$$G(\vec r,\vec r_0)\big|_{\vec r=\vec R}=0$$

So we can swap $\vec r \rightleftarrows \vec r_0$.

$$\hat n\cdot\vec\nabla G(\vec r,\vec r_0)\big|_{\vec r=\vec R}=0$$

Symmetry of Green's Function under homogeneous Dirichlet or Neumann Boundary Condition:

Proof: With Green's Theorem:

$$\int d^3r\left[A(\vec r)\,\nabla_r^2 B(\vec r) - B(\vec r)\,\nabla_r^2 A(\vec r)\right]$$

$$= \int_S d^2\vec S\cdot\left[A(\vec r)\cdot\vec\nabla_r B(\vec r) - B(\vec r)\cdot\vec\nabla_r A(\vec r)\right]$$

let $A(\vec r) = G(\vec r, \vec r_1)$ $\quad B(\vec r) = G(\vec r, \vec r_2)$

↳ $$\int d^3r\left[G(\vec r,\vec r_1)\underbrace{\nabla_r^2 G(\vec r,\vec r_2)}_{\delta(\vec r-\vec r_2)} - G(\vec r,\vec r_2)\underbrace{\nabla_r^2 G(\vec r,\vec r_1)}_{\delta(\vec r-\vec r_1)}\right.$$

$$= \int_S d^2\vec S\cdot\left[\underset{\substack{\| \\ 0}}{G(\vec r,\vec r_1)}\,\underset{\substack{\| \\ 0}}{\vec\nabla_r G(\vec r,\vec r_2)} - \underset{\substack{\| \\ 0}}{G(\vec r,\vec r_2)}\,\underset{\substack{\| \\ 0}}{\vec\nabla_r G(\vec r,\vec r_1)}\right]$$

For homogeneous Dirichlet

For homogeneous Neumann

$$G(\vec r_2,\vec r_1) - G(\vec r_1,\vec r_2) = 0$$

$$\boxed{\therefore\ G(\vec r_1,\vec r_2) = G(\vec r_2,\vec r_1) \quad \text{For homogeneous Dirichlet / Neumann}}$$