Zhi Chen HW# 10

1) Radiation from antennas:

a) Find charge density: ρ

$$\vec{J}(\vec{r},t) = \begin{cases} 0 & z < -\frac{l}{2} \\ I\delta(x)\delta(y)\sin\omega t\,\hat{z} & -\frac{l}{2} \leq z \leq \frac{l}{2} \\ 0 & z > \frac{l}{2} \end{cases}$$

or $$\vec{J}(\vec{r},t) = I\delta(x)\delta(y)\sin\omega t\left[\theta(z+\tfrac{l}{2}) - \theta(z-\tfrac{l}{2})\right]$$

Via charge conservation, continuity equation:

$$\partial_t \rho + \vec{\nabla}\cdot\vec{J} = 0$$

Since $\vec{J}$ only has component in $\hat{z}$, then:

$$\frac{\partial\rho}{\partial t} = -\partial_z J = -\partial_z\left(I\delta(x)\delta(y)\sin\omega t\left[\theta(z+\tfrac{l}{2}) - \theta(z-\tfrac{l}{2})\right]\right)$$

using $\frac{d}{dx}\theta(x) = \delta(x)$

$$\partial_t\rho = -I\delta(x)\delta(y)\sin\omega t\left[\delta(z+\tfrac{l}{2}) - \delta(z-\tfrac{l}{2})\right]$$

then $$\rho = \int -I\delta(x)\delta(y)\sin\omega t\left[\delta(z+\tfrac{l}{2}) - \delta(z-\tfrac{l}{2})\right]dt$$

$$\boxed{\rho = \frac{1}{\omega}I\delta(x)\delta(y)\cos\omega t\left[\delta(z+\tfrac{l}{2}) - \delta(z-\tfrac{l}{2})\right] + C}$$

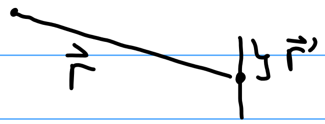
$C = 0$ since no static

given $\qquad \dfrac{dP}{d\Omega} = \dfrac{\mu_0}{16\pi^2 c}\left| \hat{n} \times \int d^3r' \left[\dot{\vec{J}}(\vec{r}', t) \right]_{ret} \right|^2$

b) Find angular distribution of radiated power.

$\rightarrow \quad \dot{\vec{J}}(\vec{r}, t) = \omega I \delta(x)\delta(y) \cos\omega t \left[\theta(z + \tfrac{l}{2}) - \theta(z - \tfrac{l}{2}) \right] \hat{z}$

with $\quad t_{ret} = t - \frac{1}{c}|\vec{r} - \vec{r}'|$



assume $\quad \left|\dfrac{r'}{r}\right| \ll 1$, then

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2}$$
$$= \sqrt{r^2\left(1 - 2\hat{r}\cdot\hat{r}'\left(\frac{r'}{r}\right) + \underbrace{\left(\frac{r'}{r}\right)^2}_{\text{ignore}}\right)} \quad \mathcal{O}\left(\left(\frac{r'}{r}\right)^2\right)$$
$$\approx r\left(1 - 2\hat{r}\cdot\hat{r}'\left(\frac{r'}{r}\right)\right)^{1/2}$$
$$\cong r\left(1 - \hat{r}\cdot\hat{r}'\left(\frac{r'}{r}\right)\right)$$
$$= r - \hat{r}\cdot\vec{r}'$$

replace $\hat{r}$ with $\hat{n}$

$$= r - \hat{n}\cdot\vec{r}'$$

then $\quad t_{ret} = t - \frac{1}{c}|\vec{r} - \vec{r}'| \cong t - \frac{1}{c}(r - \hat{n}\cdot\vec{r}')$

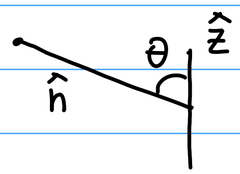
then

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\left| \hat{n} \times \int dx'dy'dz'\, \omega I \delta(x)\delta(y) \left[\theta(z + \tfrac{l}{2}) - \theta(z - \tfrac{l}{2})\right] \cos\left\{\omega\left[t - \frac{1}{c}(r - \hat{n}\cdot\vec{r}')\right]\right\} \hat{z} \right|$$

$$\hat{n}\cdot\vec{r}' = (x'\hat{n}\cdot\hat{x}' + y'\hat{n}\cdot\hat{y}' + z'\hat{n}\cdot\hat{z}')$$

we can integrate x' and y' easily due to $\delta(x)$ and $\delta(y)$, then
$x', y' \to 0$

so $\hat{n}\cdot\vec{r}' \to z'\hat{n}\cdot\hat{z}' = z\cos\theta$



$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\left|\hat{n}\times\int_{-l/2}^{l/2} dz'\,\omega I\cos\left\{\omega\left[t-\frac{r}{c}+\frac{z}{c}\cos\theta\right]\right\}\hat{z}\right|^2$$

with $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

then $\cos\left\{\omega t - \frac{\omega r}{c} + z'\frac{\omega}{c}\cos\theta\right\} = \cos\left(\omega t - \frac{\omega r}{c}\right)\cos\left(z'\frac{\omega}{c}\cos\theta\right) - \sin\left(\omega t - \frac{\omega r}{c}\right)\sin\left(z'\frac{\omega}{c}\cos\theta\right)$

then, $\int dz'\cos\left\{\omega\left[t-\frac{r}{c}+\frac{z}{c}\cos\theta\right]\right\}$

$$= \int_{-l/2}^{l/2} dz'\left[\cos\left(\omega t-\frac{\omega r}{c}\right)\cos\left(z'\frac{\omega}{c}\cos\theta\right) - \cancel{\sin\left(\omega t-\frac{\omega r}{c}\right)\underbrace{\sin\left(z'\frac{\omega}{c}\cos\theta\right)}_{\text{odd terms integrate to 0.}}}\right]$$

$$= \cos\left(\omega t - \frac{\omega r}{c}\right)\frac{2\sin\left(\frac{\omega l}{2c}\cos\theta\right)}{\frac{\omega}{c}\cos\theta}$$

so

$$\frac{dP}{d\Omega} = \frac{\mu_0(\omega I)^2}{16\pi^2 c}\left|(\hat{n}\times\hat{z})\right|^2\frac{4c^2}{\omega^2\cos^2\theta}\cos^2\left(\omega t - \frac{\omega r}{c}\right)\sin^2\left(\frac{\omega l}{2c}\cos\theta\right)$$

→ We know the magnitude of cross-product between 2 vectors
$|\vec{a}\times\vec{b}| = |a||b|\sin\theta$.

so $|\hat{n}\times\hat{z}|^2 = \sin^2\theta$

Lastly:

$$\hookrightarrow \frac{dP}{d\Omega} = \frac{\mu_0 (\cancel{\omega} I)^2}{16\pi^2 \cancel{c}} \sin^2\theta \frac{4c^{\cancel{2}}}{\cancel{\omega^2}\cos^2\theta} \cos^2\left(\omega t - \frac{\omega r}{c}\right) \sin^2\left(\frac{\omega l}{2c}\cos\theta\right)$$

$$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0 I^2 c}{4\pi^2} \tan^2\theta \cos^2\left(\omega t - \frac{\omega r}{c}\right) \sin^2\left(\frac{\omega l}{2c}\cos\theta\right)}$$

c) Find $\left\langle \frac{dP}{d\Omega} \right\rangle_t$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 I^2 c}{4\pi^2} \tan^2\theta \sin^2\left(\frac{\omega l}{2c}\cos\theta\right) \left\langle \cos^2\left(\omega t - \frac{\omega r}{c}\right)\right\rangle_t$$

One cycle has period $T = \frac{2\pi}{\omega}$

so $\left\langle \cos^2\left(\omega t - \frac{\omega r}{c}\right)\right\rangle_t = \frac{1}{T}\int_{r/c}^{T+r/c} dt \cos^2\left[\omega\left(t - \frac{r}{c}\right)\right]$

$$= \frac{\omega}{2\pi}\left(\frac{\sin[2\omega(t - r/c)]}{4\omega} + \frac{1}{2}t\right)\Bigg|_{r/c}^{\frac{2\pi}{\omega}+r/c}$$

$$= \frac{\cancel{\omega}}{\cancel{2\pi}} \frac{1}{2} \frac{\cancel{2\pi}}{\cancel{\omega}}$$

$$= 1/2$$

so $$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 I^2 c}{8\pi^2} \tan^2\theta \sin^2\left(\frac{\omega l}{2c}\cos\theta\right)}$$

d) i) let $\omega = \frac{2\pi c}{\lambda}$

then $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 I^2 c}{8\pi^2} \tan^2\theta \sin^2\left(\frac{l}{2c} \frac{2\pi c}{\lambda} \cos\theta \right)$

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 I^2 c}{8\pi^2} \tan^2\theta \sin^2\left(\frac{\pi l}{\lambda} \cos\theta \right)}$$

ii) $\theta = \frac{\pi}{2}$

then $\left\langle \frac{dP}{d\Omega} \right\rangle \Big|_{\theta = \frac{\pi}{2}} = \frac{\mu_0 I^2 c}{8\pi^2} \frac{\sin^2(\frac{\pi}{2})}{\cos^2(\frac{\pi}{2})} \sin^2\left(\frac{\pi l}{\lambda} \underbrace{\cos\frac{\pi}{2}}_{=0} \right)$

expand $\cos\theta$ around $\frac{\pi}{2}$ and $\sin\theta$ around 0 and $\frac{\pi}{2}$

$\cos\theta\big|_{\pi/2} = 0 - \sin\theta\big|_{\frac{\pi}{2}} \theta \approx -\theta$

$\sin\theta\big|_{\pi/2} \cong 1$

$\sin\theta\big|_0 \cong \theta$

$\left\langle \frac{dP}{d\Omega} \right\rangle \stackrel{!}{=} \frac{\mu_0 I^2 c}{8\pi^2} \frac{1}{\theta^2} \left(\frac{\pi}{\lambda} l \theta \right)^2$

$\cong \frac{\mu_0 I^2 c}{8\cancel{\pi^2}} \cancel{\pi^2} \left(\frac{l}{\lambda} \right)^2$

$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle \approx \frac{\mu_0 I^2 c}{8} \left(\frac{l}{\lambda} \right)^2}$ ← quadratic in length, $\sim l^2$

iii) $\left\langle \frac{dP}{d\Omega} \right\rangle$ if $l \ll \lambda$, $\sin^2\left(\frac{l}{\lambda}\pi\cos\theta\right) \approx \left(\frac{l}{\lambda}\pi\cos\theta\right)^2$

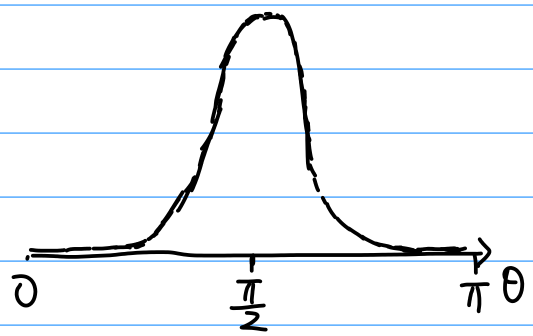
$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 I^2 c}{8\pi^2} \tan^2\theta \sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)$$

$$\cong \frac{\mu_0 I^2 c}{8\pi^2} \tan^2\theta \left(\frac{\pi l}{\lambda}\cos\theta\right)^2$$

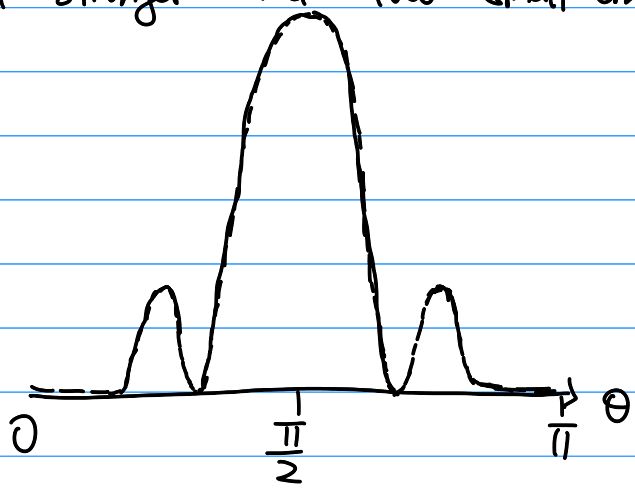
$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle \cong \frac{\mu_0 I^2 c}{8} \left(\frac{l}{\lambda}\right)^2 \sin^2\theta}$$

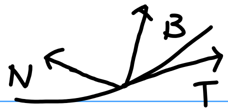
iv) If I plot $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 I^2 c}{8\pi^2} \tan^2\theta \sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)$

then if $l < \lambda$, then I see 1 lobe.



if $\lambda < l < 2\lambda$, then I see 3 lobes, the middle one get stronger and two small ones around it.





2) Coiled Optical Fibers

$$\frac{d}{ds}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} = \begin{pmatrix}0 & \overset{\text{arc length.}}{\kappa(s)} & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix}$$

a) Calculate $\vec{T}, \vec{N}, \vec{B}$, $\kappa(s)$ and $\tau(s)$

Since $\vec{R}(s) \propto \left(\cos qs\,\hat{x} + \sin qs\,\hat{y} + hqs\,\hat{z}\right)$, then

let $\vec{R}(s) = \underset{\text{proportional constant}}{A}\left(\cos qs\,\hat{x} + \sin qs\,\hat{y} + hqs\,\hat{z}\right)$

In Frenet-Serret Frame:

the tangent unit vector $\vec{T}$ (parallel with $\vec{R}$) is simply

$\vec{T} = \frac{d}{ds}\vec{R} = Aq\left(-\sin qs\,\hat{x} + \cos qs\,\hat{y} + h\hat{z}\right)$

since $\vec{T}$ is a unit vector, then $\vec{T}\cdot\vec{T} = \frac{d}{ds}\vec{R}\cdot\frac{d}{ds}\vec{R} = 1$

then we can determine proportion constant A:

as $\quad 1 = |\vec{T}|^2 = A^2q^2\left(\underbrace{\sin^2 qs + \cos^2 qs}_{=1} + h^2\right)$

so $\boxed{A = \frac{1}{q\sqrt{1+h^2}}}$

thus $\boxed{\vec{T} = \frac{1}{\sqrt{1+h^2}}\left(-\sin qs\,\hat{x} + \cos qs\,\hat{y} + h\hat{z}\right)}$

By looking at Frenet-Serret matrix, we see

$$\frac{d\vec{T}}{ds} = K(s)\,\vec{N}$$

and since $\vec{N}$ is unit vector, $K(s) = \sqrt{\frac{d\vec{T}}{ds}\cdot\frac{d\vec{T}}{ds}} = \left|\frac{d\vec{T}}{ds}\right|$

and $\vec{N} = \dfrac{\frac{d\vec{T}}{ds}}{K(s)}$

we see $\dfrac{d\vec{T}}{ds} = \dfrac{-q}{\sqrt{1+h^2}}\left(\cos qs\,\hat{x} + \sin qs\,\hat{y}\right)$

so $$\boxed{K = \left|\frac{d\vec{T}}{ds}\right| = \sqrt{\frac{q^2}{1+h^2}\left(\cos^2 qs + \sin^2 qs\right)} = \frac{q}{\sqrt{1+h^2}}}$$

Then $\vec{N} = \dfrac{\frac{d\vec{T}}{ds}}{\frac{q}{\sqrt{1+h^2}}} = \dfrac{\frac{-q}{\sqrt{1+h^2}}\left(\cos qs\,\hat{x} + \sin qs\,\hat{y}\right)}{\frac{q}{\sqrt{1+h^2}}}$

$\hookrightarrow$ $$\boxed{\vec{N} = -\left(\cos qs\,\hat{x} + \sin qs\,\hat{y}\right)}$$

Since we have $\vec{T}$ and $\vec{N}$, the third normal vector is just the cross product of the first two,

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{1+h^2}}\left(-\sin qs\,\hat{x} + \cos qs\,\hat{y} + h\,\hat{z}\right)$$
$$\times -\left(\cos qs\,\hat{x} + \sin qs\,\hat{y}\right)$$
$$= \frac{-1}{\sqrt{1+h^2}}\left(-h\sin qs\,\hat{x} + h\cos qs\,\hat{y} - \left(\sin^2 qs + \cos^2 qs\right)\hat{z}\right)$$
$$\boxed{\vec{B} = \frac{1}{\sqrt{1+h^2}}\left(h\sin qs\,\hat{x} - h\cos qs\,\hat{y} + \hat{z}\right)}$$

Then from matrix, we see

$$\frac{d}{ds}\vec{B} = -\tau\vec{N}$$

then $\frac{d}{ds}\vec{B} = \frac{q}{\sqrt{1+h^2}}\left(h\cos qs\,\hat{x} + h\sin qs\,\hat{y}\right) = -\tau\left(-\cos qs\,\hat{x} - \sin qs\,\hat{y}\right)$

By matching: $\boxed{\tau = \frac{qh}{\sqrt{1+h^2}}}$

b) The electric field in transverse direction is given by:

$$\vec{E}(\rho,s) = e^{iqs} f(\rho)\left[C_1(s)\vec{N}(s) + C_2(s)\vec{B}(s)\right]$$

since $\kappa(s)$ is comparable to the inverse of fiber bend length, and a gentle coiling means κ^2 is very small, so ignore κ^2 term.

then we're given: $i\frac{d}{ds}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 & iT(s) \\ -iT(s) & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

This is analogous to the Schrodinger equation:

$$i\hbar\frac{d}{dt}\psi(\vec{r},t) = \hat{H}\psi$$

where $\frac{d}{ds} = \hbar\frac{d}{dt}$ and $\hat{H} = \begin{pmatrix} 0 & iT(s) \\ -iT(s) & 0 \end{pmatrix}$

we know time-dependent schrodinger equation has solutions: if Hamiltonian of different times commute, then:

$$\psi(t) = \exp\left\{\frac{-i}{\hbar}\int_{t_0}^{t} dt'\,\hat{H}(t')\right\}\psi(t=t_0)$$

For our Hamiltonian, $\hat{H} = -T(s)\,\sigma_y$ and σ_y commutes with itself.
so by using this solution we then have:

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \exp\left\{-i\int_{-\infty}^{s} ds' \begin{pmatrix} 0 & iT(s) \\ -iT(s) & 0 \end{pmatrix}\right\}\begin{pmatrix} C_1(s=-\infty) \\ C_2(s=-\infty) \end{pmatrix}$$

$$= \exp\left\{-i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\underbrace{\int_{-\infty}^{s} ds'\,T(s')}_{\equiv A(s)}\right\}\begin{pmatrix} C_1(s=-\infty) \\ C_2(s=-\infty) \end{pmatrix}$$

$$L_s = \exp\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A(s) \right\} \begin{pmatrix} C_1(s=-\infty) \\ C_2(s=-\infty) \end{pmatrix}$$

We note that Taylor expansion of exponential:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

then

$$\exp\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A(s) \right\} = \sum_{n=0}^{\infty} \frac{A^n}{n!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n$$

Note that
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus we observe the following pattern for

$$\sum_{n=0}^{\infty} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{\substack{n=\text{even} \\ \downarrow 0,2,4\cdots}}^{\infty} (i)^n - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sum_{\substack{n=\text{odd} \\ \downarrow 1,3,5\cdots}}^{\infty} (i)^{(n+1)}$$

$n \to 2k$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{\infty} (i)^{2k} - i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} (i)^{2k+1}$$

$i^{2k} \to (-1)^k$

$i^{2k+1} \to (-1)^k i$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{\infty} (-1)^k - i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} (-1)^k i$$

Now all together

$$\exp\left\{\begin{pmatrix}0&1\\-1&0\end{pmatrix}A(s)\right\} = \sum_{n=0}^{\infty}\frac{A^n}{n!}\begin{pmatrix}0&1\\-1&0\end{pmatrix}^n$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\underbrace{\sum_{k=0}^{\infty}\frac{A^{2k}}{(2k)!}(-1)^k}_{\cos(A)} + \begin{pmatrix}0&1\\-1&0\end{pmatrix}\underbrace{\sum_{k=0}^{\infty}\frac{A^{2k+1}}{(2k+1)!}(-1)^k}_{\sin(A)}$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos A + \begin{pmatrix}0&1\\-1&0\end{pmatrix}\sin A$$

Therefore:

$$\begin{pmatrix}C_1(s)\\C_2(s)\end{pmatrix} = \begin{pmatrix}\cos A & \sin A\\ -\sin A & \cos A\end{pmatrix}\begin{pmatrix}\overbrace{C_1(s=-\infty)}^{\cos\alpha}\\ \underbrace{C_2(s=-\infty)}_{\sin\alpha}\end{pmatrix}$$

Knowing that we initially have linear polarization at $s=-\infty$ with angle α relative to $\vec{N}$, so $C_1(-\infty)=\cos\alpha$ and hence $C_2(-\infty)=\sin\alpha$

then
$$C_1(s) = \cos A\cos\alpha + \sin A\sin\alpha$$
$$C_2(s) = -\cos\alpha\sin A + \sin\alpha\cos A$$

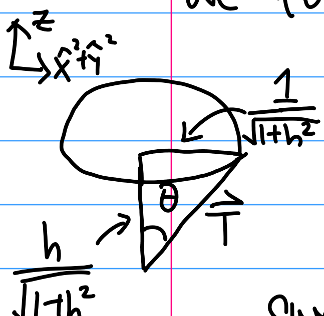
With Trig Identity:
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\boxed{\Rightarrow \left.\begin{aligned}C_1(s)&=\cos(\alpha-A(s))\\C_2(s)&=\sin(\alpha-A(s))\end{aligned}\right\}\text{ so }\frac{C_2(s)}{C_1(s)}=\tan(\alpha-A(s)) \quad \text{where } A(s)=\int_{-\infty}^{s}ds'\,T(s')}$$

c) Show a complete helical turn induces a polarization rotation equal to the solid angle swept out by the tangent to the curve.

know helix follow $\vec{R}(s) = \frac{1}{q\sqrt{1+h^2}}\left(\cos qs\ \hat{x} + \sin qs\ \hat{y} + hqs\hat{z}\right)$

we found the tangent vector in part a):



$$\vec{T} = \frac{d\vec{R}}{ds} = \frac{1}{\sqrt{1+h^2}}\left(-\sin qs\ \hat{x} + \cos qs\ \hat{y} + h\hat{z}\right)$$

as tangent vector rotates, $(-\sin qs\ \hat{x} + \cos qs\ \hat{y})$ sweeps out a circle with a constant vector in $\hat{z}$. with a constant θ.

with this geometry we know the solid angle that it swept out is:

$$\Omega = \int_0^{\theta}\int_0^{2\pi} \sin\theta d\theta d\phi$$

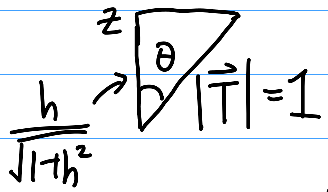
by looking at the geometry:

then

$$\hookrightarrow \Omega = \int_0^{\tan^{-1}\left(\frac{1}{h}\right)} \sin\theta' d\theta' 2\pi = -\cos\theta'\Big|_0^{\theta} 2\pi$$

$$= 2\pi(1-\cos\theta)$$

if we look at the geometry, we observe



$$\cos\theta = \vec{T}\cdot\hat{z} = \frac{h}{\sqrt{1+h^2}}$$

so

$$\boxed{\Omega = 2\pi\left(1 - \frac{h}{\sqrt{1+h^2}}\right)}$$

Now find the induced polarization

since $A(s) = \int ds' \,\tau(s')$

then $\Delta A = \Delta s \,\tau(s)$

we know $(-\sin qs\,\hat{x} + \cos qs\,\hat{y})$ completes a full turn

when $qs = 2\pi n \rightarrow \boxed{s = \frac{2\pi}{q} n}$

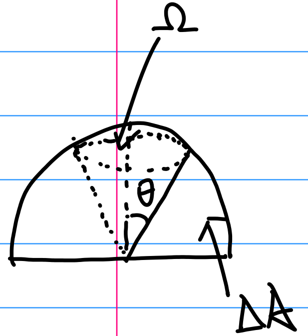
so the change of s over 1 full turn, i.e. $n=1$, is

$$\Delta s = \frac{2\pi}{q}$$

we also know $\tau = \frac{qh}{\sqrt{1+h^2}}$

so $\Delta A = \Delta s \,\tau = \frac{2\pi}{q} \frac{qh}{\sqrt{1+h^2}} = 2\pi \frac{h}{\sqrt{1+h^2}}$

so we see $\Omega = 2\pi\left(1 - \frac{h}{\sqrt{1+h^2}}\right) = 2\pi - \Delta A$



or $\boxed{\Omega + \Delta A = 2\pi}$

(cone-shape) area swept out by the tangent-vector

complement area.

This is the full half hemisphere.

3) Green functions:

a) $\mathcal{L}_t Y(t) = F(t) \quad , \quad \mathcal{L}_t = \frac{d^2}{dt^2} + W(t)^2$

i) Given $Y(t=t_0) = P$, $\dot{Y}(t=t_0) = V$, and $F(t)$

Green's Function follows:

$$\mathcal{L}_t G(t,t') = \delta(t-t')$$

First evaluate the follow expression

$$\int_{t_0}^{t} d\tau\, Y(\tau) \underbrace{\mathcal{L}_\tau G(\tau,t')}_{\delta(\tau-t')} - G(\tau,t') \underbrace{\mathcal{L}_\tau Y(\tau)}_{F(\tau)} = \int_{t_0}^{t} d\tau \Big\{ Y(\tau)\partial_\tau^2 G(\tau,t') + \cancel{W(\tau)^2 G(\tau,t') Y(\tau)}$$
$$- G(\tau,t')\partial_\tau^2 Y(\tau) - \cancel{G(\tau,t') W(\tau)^2 Y(\tau)} \Big\}$$

$$\hookrightarrow ① \quad Y(t') - \int_{t_0}^{t} d\tau\, G(\tau,t')\, F(\tau) = \int_{t_0}^{t} d\tau \left[Y(\tau)\partial_\tau^2 G(\tau,t') - G(\tau,t')\partial_\tau^2 Y(\tau) \right]$$

Then by Green's Theorem:

$$② \int_{t_0}^{t} d\tau \left[Y(\tau)\partial_\tau^2 G(\tau,t') - G(\tau,t')\partial_\tau^2 Y(\tau) \right] = \int_{t_0}^{t} d\tau\, \partial_\tau \left[Y(\tau)\partial_\tau G(\tau,t') - G(\tau,t')\partial_\tau Y(\tau) \right]$$
$$\overset{!}{=} \cancel{Y(t)\partial_\tau G(\tau,t')\big|_t} - \underbrace{Y(t_0)}_{P}\partial_\tau G(\tau,t')\big|_{t_0}$$
$$- \cancel{G(t,t')\partial_\tau Y(\tau)\big|_t} + G(t_0,t')\underbrace{\partial_\tau Y(\tau)\big|_{t_0}}_{V}$$

choose Boundary condition for $G(\tau,t')$
such that $G(t,t') = \partial_\tau G(\tau,t')\big|_t = 0$

By equating ① and ②:

$$Y(t') - \int_{t_0}^{t} d\tau\, G(\tau, t')\, F(\tau) = -P\, \partial_\tau G(\tau, t')\Big|_{t_0} + V\, G(t_0, t')$$

$$\hookrightarrow \; Y(t') = \underbrace{\int_{t_0}^{t} d\tau\, G(\tau, t') F(\tau)} - P\, \partial_\tau G(\tau, t')\Big|_{t_0} + V\, G(t_0, t')$$

$$\int_{t_0}^{t} d\tau\, G(\tau, t') F(\tau) = \int_{t_0}^{t'} d\tau\, G(\tau, t') F(\tau) + \cancel{\int_{t'}^{t} d\tau\, G(\tau, t') F(\tau)}$$

we expeel $G(t,t')=0$ for $t > t'$

Then switch dummy variable: $t' \to t$, $\tau \to t'$

$$\hookrightarrow \; \boxed{Y(t) = \int_{t_0}^{t} dt'\, G(t', t) F(t') - P\, \partial_{t_0} G(t_0, t) + V\, G(t_0, t)}$$

ii) Solve for G with case $W(t) = \Omega$, then express $Y(t)$

$$\Rightarrow \quad \partial_t^2 G(t,t_0) + \Omega^2 G(t,t_0) = \delta(t-t_0)$$

We know $G(t,t_0)=0$ for $t < t_0$ since there has not being an excitation.

Let's determine boundary condition:

We know G is continuous at $t=t_0$, but $G(t,t_0)\Big|_{t=t_0-\varepsilon} \overset{\text{before the excitation.}}{=} 0$

so by continuity: $\boxed{\lim_{t\to t_0^+} G(t,t_0) = 0}$ ①

We can determine second boundary condition by integrating around t_0

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} dt \left[\frac{d^2}{dt^2} G(t,t_0) + \Omega^2 G(t,t_0)\right] = \int_{t_0-\varepsilon}^{t_0+\varepsilon} dt\ \delta(t-t_0) = 1$$

$$\hookrightarrow \quad \frac{d}{dt} G(t,t_0)\Big|_{t_0-\varepsilon}^{t_0+\varepsilon} + \underbrace{\int_{t_0-\varepsilon}^{t_0+\varepsilon} dt\ \Omega^2 G(t,t_0)}_{=0} = 1$$

$=0$ due to causality, $\lim_{t\to t_0-\varepsilon} G(t,t_0)=0$
and due to confinuity: $\lim_{t\to t_0+\varepsilon} G(t,t_0)=0$
so integrand $= 0$

$$\hookrightarrow \quad \frac{d}{dt} G(t,t_0)\Big|_{t_0+\varepsilon} - \underbrace{\frac{d}{dt} G(t,t_0)\Big|_{t_0-\varepsilon}}_{=0 \text{ since } G(t,t_0)=0 \text{ for } t<t_0} = 1$$

$$\hookrightarrow \quad \boxed{\frac{d}{dt} G(t,t_0)\Big|_{t_0+\varepsilon} = 1} \leftarrow \text{boundary condition. ②}$$

We know for $t < t_0$, $G(t, t_0) = 0$ due to causality

Now for $\underline{t > t_0}$, $\delta(t - t_0) = 0$, so $G(t, t_0)$ obey:

$$\partial_t^2 G(t, t_0) + \Omega^2 G(t, t_0) = 0$$

which has solution of the form $G(t, t_0) = \alpha \sin \Omega t + \beta \cos \Omega t$ with 2 constants, α and β depending on initial condition.

Now with condition $\lim_{t \to t_0 + \varepsilon} G(t, t_0) = 0$

then $G(t, t_0)\Big|_{t = t_0 + \varepsilon} = \alpha \sin \Omega t_0 + \beta \cos \Omega t_0 = 0$

Then use the condition we derived before: $\frac{d}{dt} G(t, t_0)\Big|_{t = t_0 + \varepsilon} = 1$

$$\Rightarrow \quad \Omega \alpha \cos \Omega t_0 - \Omega \beta \sin \Omega t_0 = 1$$

$$\hookrightarrow \quad \alpha \cos \Omega t_0 - \beta \sin \Omega t_0 = \frac{1}{\Omega}$$

then we have: $$\begin{pmatrix} \sin \Omega t_0 & \cos \Omega t_0 \\ \cos \Omega t_0 & -\sin \Omega t_0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\Omega} \end{pmatrix}$$

solve by inverse matrix:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sin \Omega t_0 & \cos \Omega t_0 \\ \cos \Omega t_0 & -\sin \Omega t_0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\Omega} \end{pmatrix}$$

$$\hookrightarrow \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{\Omega} \cos \Omega t_0 \\ -\frac{1}{\Omega} \sin \Omega t_0 \end{pmatrix}$$

then $G(t, t_0) = \frac{1}{\Omega}\left(\cos\Omega t_0 \sin\Omega t - \sin\Omega t_0 \cos\Omega t\right)$

with trig identity $\boxed{G(t, t_0) = -\frac{1}{\Omega}\sin\left[\Omega(t - t_0)\right]}$

then plug into our formula from i)

$\hookrightarrow Y(t) = \int_{t_0}^{t} dt'\, G(t', t) F(t') - P\, \partial_{t_0} G(t_0, t) + V G(t_0, t)$

By matching the respective dummy index:

$$G(t', t) = -\frac{1}{\Omega}\sin\left[\Omega(t' - t)\right] = \frac{1}{\Omega}\sin\left[\Omega(t - t')\right]$$

$$G(t_0, t) = -\frac{1}{\Omega}\sin\left[\Omega(t_0 - t)\right] = \frac{1}{\Omega}\sin\left[\Omega(t - t_0)\right]$$

$$\partial_{t_0} G(t_0, t) = \frac{\partial}{\partial t_0}\left(\frac{1}{\Omega}\sin\left[\Omega(t - t_0)\right]\right) = -\cos\left[\Omega(t - t_0)\right]$$

$$\boxed{\therefore\ Y(t) = \int_{t_0}^{t} dt'\, \frac{1}{\Omega}\sin\left[\Omega(t - t')\right] F(t') + P\cos\left[\Omega(t - t_0)\right] + V\frac{1}{\Omega}\sin\left[\Omega(t - t_0)\right]}$$

iii) Variation of parameters method:

to solve: $\ddot{y} + p(t)\dot{y} + q(t)y = f(t)$

know homogeneous solution, i.e. when $f(t)=0$

↳ $a_1 y_1(t) + a_2 y_2(t) = y(t)$

let's vary a_1, a_2 so $a_1(t) y_1(t) + a_2 y_2(t) = y(t)$

with constraint: $\frac{d}{dt}\left(\dot{a}_1 y_1 + \dot{a}_2 y_2 = 0\right)$ $\ddot{a}_1 y_1 + \dot{a}_1 \dot{y}_1 + \ddot{a}_2 y_2 + \dot{a}_2 \dot{y}_2 = 0$

If we put solution into $\ddot{y} + p\dot{y} + qy = f(t)$

↳ $(\ddot{a}_1 y_1 + 2\dot{a}_1 \dot{y}_1 + \underline{a_1 \ddot{y}_1}) + (\ddot{a}_2 y_2 + 2\dot{a}_2 \dot{y}_2 + \underline{a_2 \ddot{y}_2})$

$+ p(\dot{a}_1 y_1 + \underline{a_1 \dot{y}_1} + \dot{a}_2 y_2 + \underline{a_2 \dot{y}_2})$

$+ q(\underline{a_1 y_1} + \underline{a_2 y_2}) = f(t)$

↳ $a_1 \underbrace{(\ddot{y}_1 + p\dot{y}_1 + q y_1)}_{=0 \text{ due to homogeneous}} + a_2 \underbrace{(\ddot{y}_2 + p\dot{y}_2 + q y_2)}_{=0 \text{ due to homogeneous}}$

$+ \underbrace{\ddot{a}_1 y_1 + \ddot{a}_2 y_2 + \dot{a}_1 \dot{y}_1 + \dot{a}_2 \dot{y}_2}_{=0 \text{ due to constraint}} + p\underbrace{(\dot{a}_1 y_1 + \dot{a}_2 y_2)}_{=0 \text{ due to constrain}}$

$+ \underbrace{\dot{a}_1 \dot{y}_1 + \dot{a}_2 \dot{y}_2 = f(t)}_{\text{only nonzero terms}}$

In the end: $\dot{a}_1 \dot{y}_1 + \dot{a}_2 \dot{y}_2 = f(t)$

along with constraint: $\dot{a}_1 y_1 + \dot{a}_2 y_2 = 0$

$$\begin{pmatrix} \dot{y}_1 & \dot{y}_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \frac{1}{\underbrace{\dot{y}_1 y_2 - \dot{y}_2 y_1}_{-\frac{1}{W}}} \begin{pmatrix} y_2 & -\dot{y}_2 \\ -y_1 & \dot{y}_1 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

then

$$\left.\begin{aligned} a_1 &= -\int_{\alpha_1}^{t} d\tau \, \frac{y_2(\tau) f(\tau)}{W} \\ a_2 &= -\int_{\alpha_2}^{t} d\tau \, \frac{-y_1(\tau) f(\tau)}{W} \end{aligned}\right\} y(t) = y_1(t)\left(-\int_{\alpha_1}^{t} d\tau \, \frac{y_2(\tau) f(\tau)}{W}\right) + y_2(t)\left(\int_{\alpha_2}^{t} d\tau \, \frac{y_1(\tau) f(\tau)}{W}\right)$$

For our problem: $\left(\frac{d^2}{dt^2} + \Omega^2\right) y = F(t)$

know homogeneous solution $y(t) = a_1 \underbrace{\cos \Omega t}_{y_1} + a_2 \underbrace{\sin \Omega t}_{y_2}$

then $\displaystyle \frac{1}{\dot{y}_1 y_2 - \dot{y}_2 y_1} = \frac{1}{\Omega(-\sin^2 \Omega t - \cos^2 \Omega t)} = -\frac{1}{\Omega}$

then $$Y(t) = -\frac{1}{\Omega}\left(\cos\Omega t \int_{\alpha_1}^{t} d\tau \; \sin(\Omega\tau) F(\tau) - \sin\Omega t \int_{\alpha_2}^{t} d\tau \; \cos(\Omega\tau) F(\tau)\right)$$

$$Y(t) = -\frac{1}{\Omega}\left(\cos\Omega t\left[\int_{t_0}^{t} d\tau \sin(\Omega\tau)F(\tau) + \underbrace{\int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau) F(\tau)}_{\text{constant.}}\right] - \sin\Omega t\left[\int_{t_0}^{t} d\tau \cos(\Omega\tau) F(\tau) + \underbrace{\int_{\alpha_2}^{t_0} d\tau \cos(\Omega\tau) F(\tau)}_{\text{some constant}}\right]\right)$$

Now re arrange:

$$Y(t) = -\frac{1}{\Omega}\left(\int_{t_0}^{t} d\tau \left(\overbrace{\cos\Omega t \sin\Omega\tau - \sin\Omega t \cos\Omega\tau}^{= -\sin[\Omega(t-\tau)]}\right) F(\tau) + \cos\Omega t \int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau) F(\tau) - \sin\Omega t \int_{\alpha_2}^{t_0} d\tau \cos(\Omega\tau) F(\tau)\right)$$

$$\Rightarrow Y(t) = \frac{1}{\Omega} \int_{t_0}^{t} d\tau \; \sin[\Omega(t-\tau)] \; F(\tau) - \frac{1}{\Omega} \cos\Omega t \int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau) F(\tau) + \frac{1}{\Omega} \sin\Omega t \int_{\alpha_2}^{t_0} d\tau \cos(\Omega\tau) F(\tau))$$

Now determine constants:

use initial condition $Y(t_0) = P \qquad \frac{d}{dt} Y \Big|_{t_0} = V$

$$Y(t_0) = P = -\frac{1}{\Omega}\left(\cos\Omega t_0 \int_{\alpha_1}^{t_0} d\tau \; \sin(\Omega\tau) F(\tau) - \sin\Omega t_0 \int_{\alpha_2}^{t_0} d\tau \; \cos(\Omega\tau) F(\tau)\right)$$

$$\frac{d}{dt}Y\Big|_{t_0} = V = \sin\Omega t_0 \int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau)F(\tau) + \cos\Omega t_0 \int_{\alpha_2}^{t_0} d\tau \cos(\Omega\tau)F(\tau)$$

$$\begin{pmatrix} -\frac{1}{\Omega}\cos\Omega t_0 & \frac{1}{\Omega}\sin\Omega t_0 \\ \sin\Omega t_0 & \cos\Omega t_0 \end{pmatrix}\begin{pmatrix} \int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau)F(\tau) \\ \int_{\alpha_2}^{t_0} d\tau \cos(\Omega\tau)F(\tau) \end{pmatrix} = \begin{pmatrix} P \\ V \end{pmatrix},$$

$$\begin{pmatrix} \int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau)F(\tau) \\ \int_{\alpha_2}^{t_0} d\tau \cos(\Omega\tau)F(\tau) \end{pmatrix} = \begin{pmatrix} -\Omega\cos\Omega t_0 & \sin\Omega t_0 \\ \Omega\sin\Omega t_0 & \cos\Omega t_0 \end{pmatrix}\begin{pmatrix} P \\ V \end{pmatrix}$$

Plug in constants:

$$Y(t) = \frac{1}{\Omega}\int_{t_0}^{t} d\tau \sin[\Omega(t-\tau)]F(\tau)$$
$$-\frac{1}{\Omega}\cos\Omega t \int_{\alpha_1}^{t_0} d\tau \sin(\Omega\tau)F(\tau) + \frac{1}{\Omega}\sin\Omega t\int_{\alpha_2}^{t_0} d\tau\cos(\Omega\tau)F(\tau))$$

$$= \frac{1}{\Omega}\int_{t_0}^{t} d\tau \sin(\Omega(t-\tau))F(\tau)$$
$$-\frac{1}{\Omega}\cos\Omega t[-P\Omega\cos\Omega t_0 + V\sin\Omega t_0] + \frac{1}{\Omega}\sin\Omega t[P\Omega\sin\Omega t_0 + V\cos\Omega t_0]$$

$$= \frac{1}{\Omega}\int_{t_0}^{t} d\tau \sin(\Omega(t-\tau))F(\tau)$$
$$+ P\underbrace{[\cos\Omega t\cos\Omega t_0 + \sin\Omega t\sin\Omega t_0]}_{\cos[\Omega(t-t_0)]} + \frac{V}{\Omega}\underbrace{[\sin\Omega t\cos\Omega t_0 - \cos\Omega t\sin\Omega t_0]}_{\sin[\Omega(t-t_0)]}$$

let $\tau \to t'$

$$\boxed{Y(t) = \frac{1}{\Omega}\int_{t_0}^{t} dt' \sin[\Omega(t-t')]F(t') + P\cos[\Omega(t-t_0)] + \frac{V}{\Omega}\sin[\Omega(t-t_0)]}$$

Same answer as part ii)

b) $$m\ddot{R}_a + \sum_{b=1}^{N} K_{ab} R_b = H_a(t)$$

Real, symmetric

lets try an ansatz for $G_a \sim E_a e^{-i\omega t}$

then $\ddot{G}_a = -\omega^2 E_a e^{-i\omega t}$ where E_a represent size N eigenvector for each oscillator.

N-normal modes

Now Green's function obey:

$$m\ddot{G}_a(t,t_0) + K_{ab} G_b(t,t_0) = \delta_a(t-t_0)$$

$$\hookrightarrow -m\omega^2 E_a e^{-i\omega t} + \sum_{b=1}^{N} K_{ab} E_b e^{-i\omega t} = \delta_a(t-t_0)$$

For $t \neq t_0$, $\delta_a(t-t_0) = 0$, we obey homogeneous eq:

$$m\omega^2 E_a \cancel{e^{-i\omega t}} = K_{ab} E_b \cancel{e^{-i\omega t}}$$

$$m\omega^2 E_a = K_{ab} E_b$$

we observe K_{ab} has eigenvalue of $m\omega^2 \delta_{ab}$ and eigenvector of E_b

by solving $\det\left(m\omega^2 \delta_{ab} - K_{ab}\right) = 0$ for G

we obtain corresponding $\omega_{\pm}$ for G

then we have a general solution for G:

$$G_a(t,t_0) = \sum_{j=1}^{N} C_j^{+} E_a^{j} e^{-i\omega^j (t-t_0)} + C_j^{-} E_a^{j} e^{+i\omega^j (t-t_0)}$$

with boundary condition.

$$\lim_{t \to t_0 - \varepsilon} G_a(t, t_0) = 0$$

$$\hookrightarrow \quad C_j^+ E_a^j + C_j^- E_a^j = 0 \qquad \leftarrow \text{first condition}$$

$$\text{let} \quad C_j^+ = -C_j^- \qquad ①$$

Integrate to get second boundary condition

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} dt \left(m \ddot{G}_a + K_{ab} G_b \right) = 1$$

$$\hookrightarrow \quad m \frac{dG_a}{dt}(t, t_0) \Big|_{t_0} = 1$$

$$\hookrightarrow -i m \omega^j \left(C_j^+ E_a^j - C_j^- E_a^j \right) = 1$$

using ①

$$\hookrightarrow \quad \left(-2i m \omega^j C_j^+ \right) E_a^j = 1$$

$$\hookrightarrow \quad C_j^+ = \frac{1}{-2im} \left[\omega^j \right]^{-1}$$

$$\text{then} \quad G_a(t, t_0) = \sum \frac{-1}{2im} E_a^j \left(e^{-i\omega^j (t - t_0)} - e^{i\omega^j (t - t_0)} \right) \left[\omega^j \right]^{-1}$$

$$\boxed{G_a(t, t_0) = \sum_{j=1}^{N} \frac{1}{m} \left[\omega^j \right]^{-1} E_a^j \sin\left[\omega^j (t - t_0) \right]}$$

Now use result from part a)

let $P_a = R_a(t=t_0)$ $\quad V_a = \dot{R}_a(t=t_0)$

Now let $E^j_a = V^j_{ab}$ where V^j_{ab} represent individual element in the matrix

then using solution found in part a), and plug in $G_a(t,t_0)$:

$$\boxed{\begin{aligned} R_a(t) = &\int_{t_0}^{t} dt' \, \frac{1}{m} [\omega^j]^{-1} E^j_a \sin[\omega^j (t'-t_0)] H_b(t') \\ &- P_a \frac{1}{m} E^j_a \cos[\omega^j (t-t_0)] + V_a \frac{1}{m} [\omega^j]^{-1} E^j_a \sin[\omega^j (t-t_0)] \end{aligned}}$$