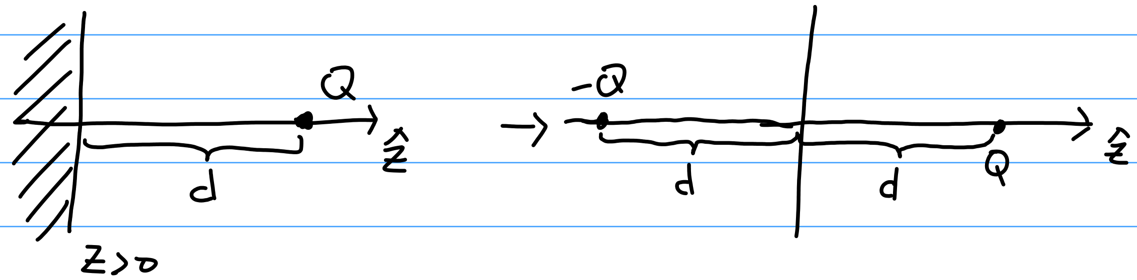
Zhi Chen HW#5 EM.

1) First consider ∞ half-plane by a conducting slab.

a)

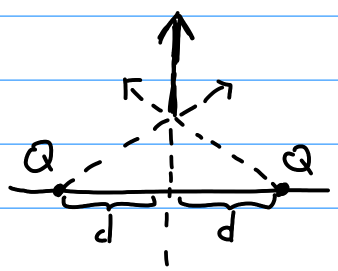


→ Due to the conducting slab, the boundary condition at $z=0$ must satisfy the property of conductors, which is $E_{\parallel} = 0$ and $E_{\perp} = E\hat{z} \neq 0$.
Therefore we seek for the placement of image charge that would replicate $E_{\parallel}=0$ and $E_z \neq 0$.

→ By symmetry, we should put the image charge that mirrors the physical charge over the $z=0$ axis. So $d(-\hat{z})$ away from origin.

→ Now to consider strength of mirror charge:

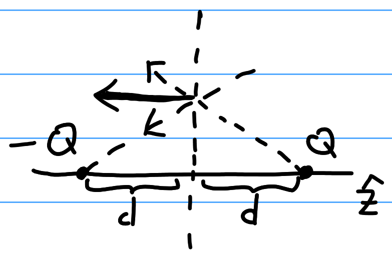
Case 1: Same sign and magnitude



This leads to $E_{\parallel} \neq 0$ but $E_{\perp} = 0$

So <u>not</u> what we want.

Case 2: opposite sign and same magnitude: $\boxed{\rho_{mirror} = -Q\,\delta(\vec{r}+d\hat{z})}$



leads to $E_{\parallel}=0$ and $E_z \neq 0$

so this configuration satisfies the B.C.

b) Find charge density induced on the conductor.

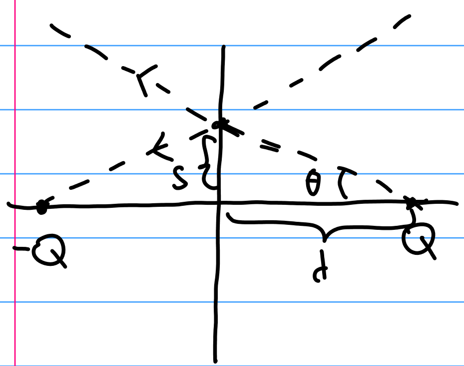
$$\int d^3r\ \vec{\nabla}\cdot\vec{E} = \int d^3r\ \frac{1}{\varepsilon_0}\rho$$

$$\int d^2\vec{S}\cdot\vec{E} = \int d^2S\ \frac{\sigma}{\varepsilon_0}$$

$$\left(E(+0\hat{n}) + \cancel{E(-0\hat{n})}\right)A = \frac{\sigma}{\varepsilon_0}A.$$

$$E_\perp = \frac{\sigma}{\varepsilon_0}$$

Now find $E_\perp$ = superposition between physical +Q charge and mirror −Q charge.



$$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}-d\hat{z}|^3}(\vec{r}-d\hat{z})$$

for $\vec{r} = s\hat{s} + 0\hat{z}$

$$\vec{E}_+ \overset{!}{=} \frac{1}{4\pi\varepsilon_0}\frac{Q}{(s^2+d^2)^{3/2}}\left(s\hat{s} - d\hat{z}\right)$$

similarly $\quad \vec{E}_- = \frac{1}{4\pi\varepsilon_0}\frac{-Q}{(s^2+d^2)^{3/2}}\left(s\hat{s} + d\hat{z}\right)$

then $\quad \vec{E}_- + \vec{E}_+ = E_\perp\hat{z} = -2\frac{1}{4\pi\varepsilon_0}\frac{Q}{(s^2+d^2)}\frac{d}{\sqrt{s^2+d^2}}\hat{z}$

Since $\quad E_\perp = \frac{\sigma}{\varepsilon_0}$

then $\quad \boxed{\sigma = \frac{-2}{4\pi}\frac{Q}{s^2+d^2}\frac{d}{\sqrt{s^2+d^2}}}$

c) Calculate the total charge induced using σ.

$$\text{charge induced} = \int d^2s \; \sigma$$

$$\stackrel{!}{=} \int_0^{2\pi} d\phi \int_0^{\infty} s\,ds \; \frac{-2}{4\pi} \; \frac{Q}{s^2+d^2} \; \frac{d}{\sqrt{s^2+d^2}}$$

$$\stackrel{!}{=} \underbrace{\frac{-4\pi}{4\pi} Q}_{-Q} \int_0^{\infty} s\,ds \; \frac{d}{(s^2+d^2)^{3/2}}$$

let $s = td$

so $ds = d\,dt$

$$\stackrel{!}{=} -Q \int_0^{\infty} \cancel{d^3}\,dt \; \frac{t}{(t^2d^2+d^2)^{3/2}}$$

$$\stackrel{!}{=} -Q \int_0^{\infty} dt \; \underbrace{\frac{t}{(t^2+1)^{3/2}}}_{= \frac{d}{dt}\left(\frac{-1}{\sqrt{t^2+1}}\right)}$$

$$\stackrel{!}{=} -Q \; \underbrace{\left(\frac{-1}{\sqrt{t^2+1}}\right)\Bigg|_0^{\infty}}_{=1}$$

$$\boxed{\text{charge induced.} = -Q}$$

As expected, the charge induced is the same as the mirror charge we put.

d) Compare energy of $\vec{E}$-field due to point charge,
i) in the absence and ii) presence of conducting slab.

$$\mathcal{E}_{No\text{-}cond} = \frac{1}{2}\varepsilon_0 \int_{\substack{all\\space}} d^3r \left|E_+\right|^2 = \frac{1}{2}\varepsilon_0\left(\int_{z>0} d^3r \left|E_+\right|^2 + \int_{z<0} d^3r \left|E_+\right|^2\right)$$

$$\mathcal{E}_{with\text{-}cond} = \frac{1}{2}\varepsilon_0 \int_{\substack{all\\space}} d^3r \left|E_{tot}\right|^2$$

since $\vec{E} = 0$ inside conductor, so $\left|E_{tot}\right|^2 = 0$ for $z<0$

$$\stackrel{!}{=} \frac{1}{2}\varepsilon_0 \int_{z>0} d^3r \left|E_{tot}\right|^2$$

$$\stackrel{!}{=} \frac{1}{2}\varepsilon_0 \int_{z>0} d^3r \left(\vec{E}_+ + \vec{E}_-\right)^2$$

$$\stackrel{!}{=} \frac{1}{2}\varepsilon_0 \int_{z>0} d^3r \left(E_+^2 + E_-^2 + 2\vec{E}_+\cdot\vec{E}_-\right)$$

Now find the difference between energies:

$$\Delta\mathcal{E} = \mathcal{E}_{with\text{-}cond} - \mathcal{E}_{no\text{-}cond}$$

$$\stackrel{!}{=} \frac{1}{2}\varepsilon_0 \int_{z>0} d^3r \left(\cancel{E_+^2} + \cancel{E_-^2} + 2\vec{E}_+\cdot\vec{E}_-\right) - \cancel{\frac{1}{2}\varepsilon_0 \int_{z>0} d^3r\, E_+^2} - \cancel{\frac{1}{2}\varepsilon_0 \int_{z<0} d^3r\, E_+^2}$$

we can also argue that $\int_{z>0} d^3r\, |E_-|^2 = \int_{z<0} d^3r\, |E_+|^2$

since
$$\vec{E}_+(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\vec{r} - d\hat{z}|^3} (\vec{r} - d\hat{z})$$
$$\stackrel{!}{=} \frac{1}{4\pi\varepsilon_0} \frac{Q}{|s\hat{s} + \underbrace{(z-d)}\hat{z}|^3} \left(s\hat{s} + (z-d)\hat{z}\right)$$

If $z<0$, $z-d$ goes from $[-d, -\infty)$

and
$$\vec{E}_-(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{-Q}{|s\hat{s} + (z+d)\hat{z}|^3} \left(s\hat{s} + (z+d)\hat{z}\right)$$

If $z>0$, $z+d$ goes from $[d, \infty)$.

Now upon taking square, the negative sign won't matter, so
$$|E_-|^2_{z>0} = |E_+|^2_{z<0}$$

So
$$\Delta\mathcal{E} = \varepsilon_0 \int_{z>0} d^3r\, \vec{E}_+ \cdot \vec{E}_-$$
$$\stackrel{!}{=} \frac{1}{2}\varepsilon_0 \int_{\substack{\text{all}\\\text{space}}} d^3r\, \vec{E}_+ \cdot (-\vec{\nabla}\phi_-)$$
$$\stackrel{!}{=} \frac{1}{2}\varepsilon_0 \int_{\substack{\text{all}\\\text{space}}} d^3r \left[-\vec{\nabla}\cdot(\vec{E}_+\phi_-) + \phi_- \underbrace{\vec{\nabla}\cdot\vec{E}_+}_{\frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0} Q\delta(\vec{r} - d\hat{z})}\right]$$
$$\stackrel{|}{=} \frac{1}{2}\varepsilon_0 \left\{ \cancel{-\oint_S \underbrace{d^2\vec{s}}_{\propto r^2} \cdot \underbrace{\vec{E}_+\phi_-}_{\propto \frac{1}{r^3}}}_{\propto \frac{1}{r} \to 0 \text{ as } r\to\infty} + \int_{\substack{\text{all}\\\text{space}}} d^3r\, \phi_-(\vec{r}) \frac{1}{\varepsilon_0} Q\delta(\vec{r} - d\hat{z}) \right.$$

$$\Delta\mathcal{E} \stackrel{|}{=} \frac{1}{2}\phi_-(\vec{r} = d\hat{z})\, Q$$

the potential of point charge: $\phi_{-}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{-Q}{|r\hat{r}+d\hat{z}|}$

$$\phi_{-}(\vec{r}=d\hat{z}) = \frac{1}{4\pi\varepsilon_0}\frac{-Q}{2d}$$

$$\hookrightarrow \Delta\mathcal{E} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{-Q}{2d}Q$$

$$\boxed{\Delta\mathcal{E} = \frac{1}{2}\frac{-Q^2}{4\pi\varepsilon_0}\frac{1}{2d}}$$ ← Difference in energy.

Find force $\vec{F} = -\frac{\partial\Delta\mathcal{E}}{\partial d}\hat{z} = -\frac{\partial}{\partial d}\left(\frac{1}{2}\frac{-Q^2}{4\pi\varepsilon_0}\frac{1}{2d}\right)\hat{z}$

$$= -\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{2d^2}\hat{z}$$

$$\boxed{\vec{F}_{surface} = -\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{(2d)^2}\hat{z}}$$

Using Coulomb Law, force between two point charges is:

$$\boxed{\vec{F}_{image} = \frac{1}{4\pi\varepsilon_0}\frac{-Q^2}{(2d)^2}\hat{z}}$$

We see that we get the same force term.

Now suppose we have Neumann Boundary Condition on Φ.

e) The differential equation that Green's Function has to solve is:

$$\boxed{-\nabla_r^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)}$$

If we are given Neumann Boundary Condition on Φ, then we want to choose homogeneous Neumann Boundary condition for $G(\vec{r},\vec{r}_0)$.

$$\boxed{\hat{n}\cdot\vec{\nabla}_r G(\vec{r},\vec{r}_0)\Big|_{\vec{r}=\vec{s}} = \left(\frac{\partial}{\partial z}G(\vec{r},\vec{r}_0)\right)\Big|_{z=0} = 0}$$

With Green's Theorem:

$$\int d^3r\left[A(\vec{r})\nabla_r^2 B(\vec{r}) - B(\vec{r})\nabla_r^2 A(\vec{r})\right] = \int_S d^2\vec{s}\cdot\left[A(\vec{r})\vec{\nabla}_r B(\vec{r}) - B(\vec{r})\vec{\nabla}_r A(\vec{r})\right]$$

By choosing $A(\vec{r}) = G(\vec{r},\vec{r}_0)$, $B(\vec{r}) = \Phi(\vec{r})$

$$\hookrightarrow \int d^3r\left[G(\vec{r},\vec{r}_0)\underbrace{\nabla_r^2\Phi(\vec{r})}_{-\rho(\vec{r})} - \Phi(\vec{r})\underbrace{\nabla_r^2 G(\vec{r},\vec{r}_0)}_{-\delta(\vec{r}-\vec{r}_0)}\right] = \int_S d^2\vec{s}\cdot\left[G(\vec{r},\vec{r}_0)\vec{\nabla}_r\Phi(\vec{r}) - \Phi(\vec{r})\vec{\nabla}_r G(\vec{r},\vec{r}_0)\right]$$

$$\hookrightarrow \Phi(\vec{r}_0) = \int d^3r\, G(\vec{r},\vec{r}_0)\rho(\vec{r}) + \int_S d^2\vec{s}\cdot\left[G(\vec{r},\vec{r}_0)\vec{\nabla}_r\Phi(\vec{r}) - \cancel{\Phi(\vec{r})\vec{\nabla}_r G(\vec{r},\vec{r}_0)}\right]$$

with Boundary Condition $\hat{n}\cdot\vec{\nabla}_r G(\vec{r},\vec{r}_0)\big|_{\vec{r}=S} = 0$, $d^2\vec{s}\cdot\vec{\nabla}_r G(\vec{r},\vec{r}_0) = 0$

$$\hookrightarrow \boxed{\therefore\ \Phi(\vec{r}_0) = \int d^3r\, G(\vec{r},\vec{r}_0)\rho(\vec{r}) + \int_S d^2\vec{s}\cdot G(\vec{r},\vec{r}_0)\vec{\nabla}_r\Phi(\vec{r})}$$

Bulk Term   Boundary Inhomogenity.

f) Suppose $-\frac{\partial}{\partial z}\Phi(\vec{r})\Big|_{z=0} = E_z(\vec{r})\,\hat{z}\Big|_{z=0}$

Need to put image charge such that

$$-\frac{\partial}{\partial z}\Phi_{tot}\Big|_{z=0} = E_0(s)\,\hat{z}$$

$$-\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left(\frac{nQ}{|\vec{r}-\vec{r}_0|} + \frac{Q}{|\vec{r}-d\hat{z}|}\right)\Bigg|_{z=0}\hat{z} = E_0(s)\,\hat{z}$$

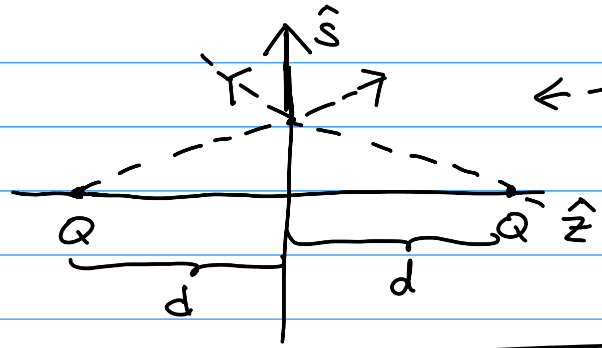
By symmetry we should put image charge along z-axis

$$\hookrightarrow \frac{\partial}{\partial z}\left(\frac{nQ}{\sqrt{s^2+(z-z_0)^2}} + \frac{Q}{\sqrt{s^2+(z-d)^2}}\right)\Bigg|_{z=0}\hat{z} = -4\pi\varepsilon_0 E_0(s)\,\hat{z}$$

$$\hookrightarrow \frac{nQ}{s^2+z_0^2} + \frac{Q}{s^2+d^2} = 4\pi\varepsilon_0 E_0(s)$$

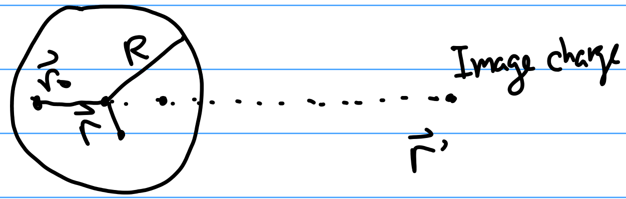
→ We can now choose z_0 (placement of image charge in plane $z<0$) and n (the strength of the image charge) so that it matches with $4\pi\varepsilon_0 E_0$.

→ If $\hat{z}\cdot\vec{\nabla}\Phi = 0$, i.e. homogeneous Neumann Boundary for Green's Function, the obvious choice is $n=1$ and $z_0\hat{z} = -d\hat{z}$



← then $\hat{z}$ component of $\vec{E}$ cancels only leaving with component parallel to the $\hat{s}$-axis.

9)



with homogeneous Dirichlet B.C. for point charge (Green's Function)

$$G(\vec{r}=\vec{R},\vec{r}_0)=0$$

Now we want to place the image charge so that the sum of the two potential caused by point charges vanish at $G(\vec{r}=\vec{R},\vec{r}_0)=0$

$$G(\vec{r},\vec{r}_0)=\frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|}-\frac{1}{4\pi\varepsilon_0}\frac{\Sigma}{|\vec{r}-\vec{r}\,'|}$$

$$=\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r_0^2-2rr_0\hat{r}\cdot\hat{r}_0}}-\frac{\Sigma}{\sqrt{r^2+r'^2-2rr'\hat{r}\cdot\hat{r}'}}\right)$$

Note that by symmetry, we should put image charge same direction as physical charge.

So $\boxed{\hat{r}_0=\hat{r}'}$

For $\vec{r}=\vec{R}$.

$$G(\vec{r}=\vec{R},\vec{r}_0)=\frac{1}{4\pi\varepsilon_0}\underbrace{\left(\frac{1}{\sqrt{R^2+r_0^2-2Rr_0\cos\theta}}-\frac{\Sigma}{\sqrt{R^2+r'^2-2Rr'\cos\theta}}\right)}_{=0}=0$$

$$\hookrightarrow\quad R^2+r'^2-2Rr'\cos\theta=\Sigma^2(R^2+r_0^2-2Rr_0\cos\theta)$$

Now we expect the part depending on the angle to match, and the terms don't depend on the angles to match.

So we have: (1) $R^2+r'^2=\Sigma^2(R^2+r_0^2)$

(2) $\cancel{-2R}r'\cancel{\cos\theta}=\Sigma^2(\cancel{-2R}r_0\cancel{\cos\theta})$

$$\frac{r'}{r_0}=\Sigma^2$$

From ①, we have $R^2 + r'^2 = \Sigma^2 (R^2 + r_0^2)$

substitute $\Sigma^2 = \frac{r'}{r_0}$ $\hookrightarrow R^2 + r'^2 = \left(\frac{r'}{r_0}\right)(R^2 + r_0^2)$

$$\hookrightarrow r'^2 r_0 + R^2 r_0 - r'(R^2 + r_0^2) = 0$$

$$\hookrightarrow r'^2 r_0 - r'(R^2 + r_0^2) + R^2 r_0 = 0$$

$$r' = \frac{(R^2 + r_0^2) \pm \sqrt{(R^2 + r_0^2)^2 - 4 r_0^2 R^2}}{2 r_0}$$

$$= \frac{(R^2 + r_0^2) \pm (R^2 - r_0^2)}{2 r_0}$$

$$\boxed{r' = \frac{R^2}{r_0}} \quad \text{or} \quad \cancel{r_0}$$

if $r' = r_0$, then image charge is just on top of the physical charge, then potential is 0 everywhere.

With $r' = \frac{R^2}{r_0}$

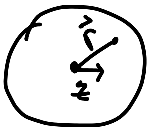
and $r' = \Sigma^2 r_0$

$$\hookrightarrow \frac{R^2}{r_0} = \Sigma^2 r_0$$

then $\Sigma = \pm \sqrt{\frac{R}{r_0}}$ $\Rightarrow$ $\boxed{\Sigma = \frac{R}{r_0}}$

Since we want $G(\vec{r} = \vec{R}, \vec{r}_0) = 0$, we choose sign

$$\boxed{\therefore\ G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r} - r_0 \hat{r}_0|} - \frac{(R/r_0)}{\left|\vec{r} - \frac{R^2}{r_0}\hat{r}_0\right|}\right)}$$



h) $\rho(\vec{r}) = -\lambda\,\hat{e}_z \cdot \vec{\nabla}\delta(\vec{r})$, this is a source term of a diple pointing at z-direction, where λ is the magnitude of the dipole.
i.e. $\boxed{\vec{p} = \lambda\hat{e}_z}$, so $\rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}\delta(\vec{r})$

we know density of point source $= Q\,\delta(\vec{r})$.

Now consider slight variation in $\vec{r}$ by $\vec{s}$:

$$Q\,\delta(\vec{r}-\vec{s}) = Q\,\delta(\vec{r}) - Q\vec{s}\cdot\vec{\nabla}\delta(\vec{r}) + \mathcal{O}(s^2)$$

then we observe that $-Q\vec{s}\cdot\vec{\nabla}\delta(\vec{r}) = Q\left[\delta(\vec{r}-\vec{s}) + \left(-\delta(\vec{r})\right)\right]$

therefore, $-\vec{s}\cdot\vec{\nabla}\delta(\vec{r})$ is simply two point charges that are opposite sign separated by vector $\vec{s}$, which forms a dipole.
Therefore, we see that $Q\vec{s} \equiv \vec{p}$, which is $\lambda\hat{e}_z$ stated in the problem.

Therefore we can rewrite $\rho = -\lambda\hat{e}_z\cdot\vec{\nabla}\delta(\vec{r}) = -\vec{p}\cdot\vec{\nabla}\delta(\vec{r})$

$\Rightarrow$ We find the solution to Poisson's equation by using results from part e, with Dirichlet B.C.

$$\Phi(\vec{r}_0) = \int d^3r\, G(\vec{r},\vec{r}_0)\,\rho(\vec{r}) + \int_S d^2\vec{s}\cdot\Big[\underbrace{\cancel{G(\vec{r},\vec{r}_0)\vec{\nabla}_r\Phi(\vec{r})}}_{\text{0 due to Homogeneous Dirichlet B.C. on } G} - \underbrace{\cancel{\Phi(\vec{r})\vec{\nabla}_r G(\vec{r},\vec{r}_0)}}_{\text{0 due to homogeneous Dirichlet B.C. on } \Phi \text{ specified by the problem.}}\Big]$$

$$\Phi(\vec{r}_0) = \int d^3r\, G(\vec{r},\vec{r}_0)\,\rho(\vec{r})$$

Since $G(\vec{r},\vec{r}_0)$ is symmetric under swapping $\vec{r} \rightleftharpoons \vec{r}_0$

we swap $\vec{r} \rightleftharpoons \vec{r}_0$, then: $\Phi(\vec{r}) = \int d^3r_0 \; G(\vec{r}_0, \vec{r})\,\rho(\vec{r}_0)$

$$\Phi(\vec{r}) = \int d^3r_0 \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}_0 - r\,\hat{r}|} - \frac{(R/r)}{|\vec{r}_0 - \frac{R^2}{r}\hat{r}|}\right)\Big[-\underbrace{\lambda\hat{z}}_{\vec{p}=\lambda\hat{z}}\cdot\vec{\nabla}\delta(\vec{r}_0)\Big]$$

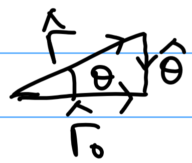
using property: $-f'(x)\delta(x) = f(x)\delta'(x)$

$$\overset{!}{=} \int d^3r_0 \frac{\vec{p}}{4\pi\varepsilon_0}\cdot\delta(\vec{r}_0)\,\vec{\nabla}_{r_0}\left(\frac{1}{|\vec{r}_0 - r\hat{r}|} - \frac{R/r}{|\vec{r}_0 - \frac{R^2}{r}\hat{r}|}\right)$$

$$\overset{!}{=} \frac{\vec{p}}{4\pi\varepsilon_0}\cdot\int d^3r_0\,\delta(\vec{r}_0)\,\vec{\nabla}_{r_0}\left(\frac{1}{\sqrt{r_0^2+r^2-2r_0 r\cos\theta}} - \frac{R/r}{\sqrt{r_0^2+\frac{R^4}{r^2}-2r_0\frac{R^2}{r}\cos\theta}}\right)$$

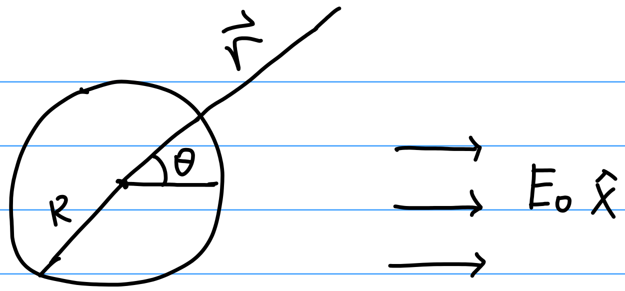
$$= \frac{\vec{p}}{4\pi\varepsilon_0}\cdot\int d^3r_0\,\delta(\vec{r}_0)\left(\frac{-(r_0 - r\cos\theta)\hat{r}_0 - r\sin\theta\,\hat{\theta}}{(r_0^2+r^2-2r_0 r\cos\theta)^{3/2}} + \frac{\frac{R}{r}\left\{(r_0-\frac{R^2}{r}\cos\theta)\hat{r}_0 + \frac{R^2}{r}\sin\theta\,\hat{\theta}\right\}}{\left(r_0^2+\frac{R^4}{r^2}-2r_0\frac{R^2}{r}\cos\theta\right)}\right)$$

$$\overset{!}{=} \frac{\vec{p}}{4\pi\varepsilon_0}\cdot\left[\frac{r\cos\theta\,\hat{r}_0 - r\sin\theta\,\hat{\theta}}{r^3} - \frac{\frac{R^3}{r^2}\{\cos\theta\,\hat{r}_0 - \sin\theta\,\hat{\theta}\}}{\left(\frac{R^4}{r^2}\right)^{3/2}}\right]$$

Notice that $\underline{\underline{\cos\theta\,\hat{r}_0 - \sin\theta\,\hat{\theta} = \hat{r}}}$ $\rightleftharpoons$ 

$$\boxed{\Phi(\vec{r}) \overset{!}{=} \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0}\frac{1}{r^2}\left[1 - \left(\frac{r}{R}\right)^3\right] \quad \text{for } \vec{p} = \lambda\,\hat{e}_z}$$

2 b)



For scalar potential outside the uncharged conducting cylinder, since net charge is 0, we expect:

$$\nabla^2 \Phi = 0$$

via $\Phi(r, \theta, z) = R(r)\, \Theta(\theta)\, Z(z)$

Due to symmetry, we don't expect variation in z, so $\Phi = R(r)\Theta(\theta)$

$$\frac{1}{r}\partial_r (r\, \partial_r) R\Theta + \frac{1}{r^2}\partial_\theta^2 R\Theta = 0$$

$$\hookrightarrow \quad \frac{1}{R}\frac{1}{r}\left[\partial_r R + r\partial_r^2 R\right] + \frac{1}{\Theta}\frac{1}{r^2}\partial_\theta^2 \Theta = 0$$

$$\hookrightarrow \quad \underbrace{r^2 \frac{1}{R}\left[\partial_r^2 R + \frac{1}{r}\partial_r R\right]}_{m^2} + \underbrace{\frac{1}{\Theta}\partial_\theta^2 \Theta}_{-m^2} = 0$$

then $\partial_\theta^2 \Theta + m^2 \Theta = 0$

$$\hookrightarrow \quad \Theta = A_m \cos m\theta + B_m \sin m\theta$$

And $r^2 \partial_r^2 R + r\,\partial_r R - m^2 R = 0$

$\hookrightarrow$ This has a solution of $R(r) = C_m r^m + D_m r^{-m}$

Therefore: $\Phi(r,\theta) = \sum_{m=0}^{\infty} \left(A_m \cos m\theta + B_m \sin m\theta\right)\left(C_m r^m + D_m r^{-m}\right)$

However we expect $B_m = 0$ since Φ should be symmetric about x-axis, but for $\sin\theta$, it changes sign as $\theta \to -\theta$.
So $\boxed{B_m = 0}$

then we have: $\Phi(r,\theta) = \sum_{m=0}^{\infty} \cos m\theta \left(C_m r^m + D_m r^{-m}\right)$

As $r \to \infty$, we know scalar potential is only dependent on the uniform external field, which is:

$$\phi_0 = -\int_0^x E_0 \hat{x} \cdot d\vec{L} = -E_0 x = -E_0 r\cos\theta$$

We also know that at $r \to \infty$, r^{-m} term should all vanish
then we have

$$\Phi(r \to \infty, \theta) = \sum_{m=0}^{\infty} C_m r^m \cos m\theta = -E_0 r\cos\theta$$

By comparing, we see that there is only $\boxed{m=1}$ term,
and $C_1 = -E_0$

Then $\Phi(r,\theta) = -E_0 r\cos\theta + D_1 r^{-1}\cos\theta$

We also know at $r = R$, $\Phi(r=R,\theta) = 0$

$$\text{So} \quad \Phi(R,\theta) = \left(-E_0 R + D_1 R^{-1}\right)\cos\theta = 0$$

$$\text{or} \quad \boxed{D_1 = E_0 R^2}$$

Then all together:

$$\Phi(r,\theta) = -E_0 r\cos\theta + E_0 R^2 \frac{1}{r}\cos\theta$$

$$\boxed{\Phi(r,\theta) \overset{!}{=} -E_0 r\cos\theta\left(1-\left(\frac{R}{r}\right)^2\right)}$$

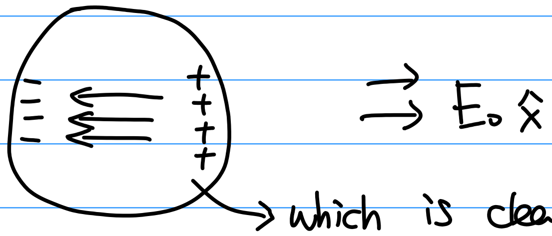
$$\vec{E} = -\vec{\nabla}\Phi(r,\theta)$$

$$\overset{!}{=} -\left[\partial_r \Phi(r,\theta)\,\hat{r} + \frac{1}{r}\partial_\theta \Phi(r,\theta)\,\hat{\theta}\right]$$

$$\overset{!}{=} -\left[-E_0\cos\theta\left[\left(1-\left(\frac{R}{r}\right)^2\right) - rR^2(-2)\frac{1}{r^3}\right]\hat{r} + E_0\sin\theta\left(1-\left(\frac{R}{r}\right)^2\right)\hat{\theta}\right]$$

$$\boxed{\vec{E} \overset{!}{=} E_0\cos\theta\left[1+\left(\frac{R}{r}\right)^2\right]\hat{r} - E_0\sin\theta\left(1-\left(\frac{R}{r}\right)^2\right)\hat{\theta}}$$

The induced field due to the cylinder is analogous to the field generated via dipoles. We can see that since the external electric rearranges the charge distribution inside the conductor as:



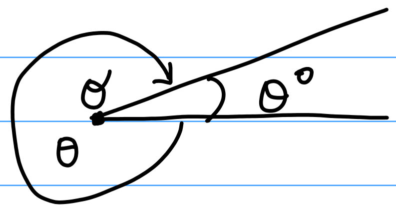
which is clearly dipole-like.

For conductors we know $\hat{n}\cdot\vec{E}\big|_{surface} = \frac{\sigma}{\varepsilon_0}$

$$\hat{r}\cdot\vec{E}(r=R)\,\varepsilon_0 = \sigma$$

$\hookrightarrow$ $\boxed{\sigma = 2\varepsilon_0 E_0\cos\theta}$ ← surface density.

c)



Since No excess charge present, we still have $\nabla^2 \Phi = 0$

and the same solution applies from part b:

$$\Phi(r,\theta) = \sum_{m=0}^{\infty} \left(A_m \cos m\theta + B_m \sin m\theta\right)\left(C_m r^m + D_m r^{-m}\right)$$

Since we don't want Φ to diverge at $r=0$, discard r^{-m} terms by $\boxed{D_m = 0}$

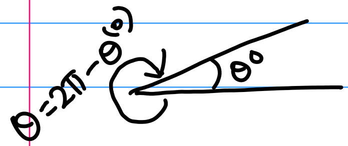
Since we require $\Phi(r, \theta=0) = 0$ due to Boundary Condition,

$$\Phi(r,\theta=0) = \sum_m A_m \cos(0) + \cancel{B_m \sin(0)} = 0$$

so $\boxed{A_m = 0}$

Then we have $\Phi(r,\theta) = \sum_m B_m r^m \sin m\theta$

Now, since we have a wedge with angle $\theta^{(0)}$, this means that we reach top of the wedge not at 2π, but $2\pi - \theta^{(0)}$



so we have B.C.: $\Phi(r, \theta = 2\pi - \theta^{(0)}) = 0$

$\hookrightarrow \Phi(r, \theta = 2\pi - \theta^{(0)}) = B_m r^m \sin\left(\underbrace{m(2\pi - \theta^{(0)})}_{=n\pi}\right) = 0$

then require $m(2\pi - \theta^{(0)}) = n\pi$ for $n = 1, 2, 3 \ldots$

or $\boxed{m = n \dfrac{\pi}{2\pi - \theta^{(0)}}}$

$$\therefore \quad \Phi(r,\theta) = \sum_{n=0}^{\infty} B_n \, r^{\frac{n\pi}{2\pi-\theta^{(0)}}} \sin\left(n \frac{\pi}{2\pi-\theta^{(0)}} \theta\right)$$

Since we're near the tip of the edge, so $r \ll 1$, then to consider the leading term, which is $\boxed{n=1}$

then $$\boxed{\Phi(r,\theta) \approx B r^{\frac{\pi}{2\pi-\theta^{(0)}}} \sin\left(\frac{\pi}{2\pi-\theta^{(0)}} \theta\right)}$$

Then $\vec{E} = -\vec{\nabla}\Phi = -[\partial_r \Phi \hat{r} + \frac{1}{r}\partial_\theta \Phi \hat{\theta}]$

$$\boxed{\vec{E} = -B \frac{\pi}{2\pi-\theta^{(0)}} r^{\frac{\pi}{2\pi-\theta^{(0)}}-1} \left[\sin\left(\frac{\pi}{2\pi-\theta^{(0)}} \theta\right)\hat{r} + \cos\left(\frac{\pi}{2\pi-\theta^{(0)}} \theta\right)\hat{\theta}\right]}$$

→ If $\theta^{(0)} < \pi$, then $2\pi - \theta^{(0)} > \pi$, then $\frac{\pi}{2\pi-\theta^{(0)}} < 1$,

then $\frac{\pi}{2\pi-\theta^{(0)}} - 1 < 0$, and since $E \propto r^{\frac{\pi}{2\pi-\theta^{(0)}}-1}$

$\vec{E}$ diverges as $\vec{r} \to 0$ since we have negative exponent in r.

→ As $\theta^{(0)} \ll 1$, $\frac{\pi}{2\pi-\theta^{(0)}} - 1 \approx \frac{\pi}{2\pi} - 1 = -\frac{1}{2}$

since $E \propto r^{\frac{\pi}{2\pi-\theta^{(0)}}-1}$, then $\boxed{\vec{E} \propto r^{-1/2}}$