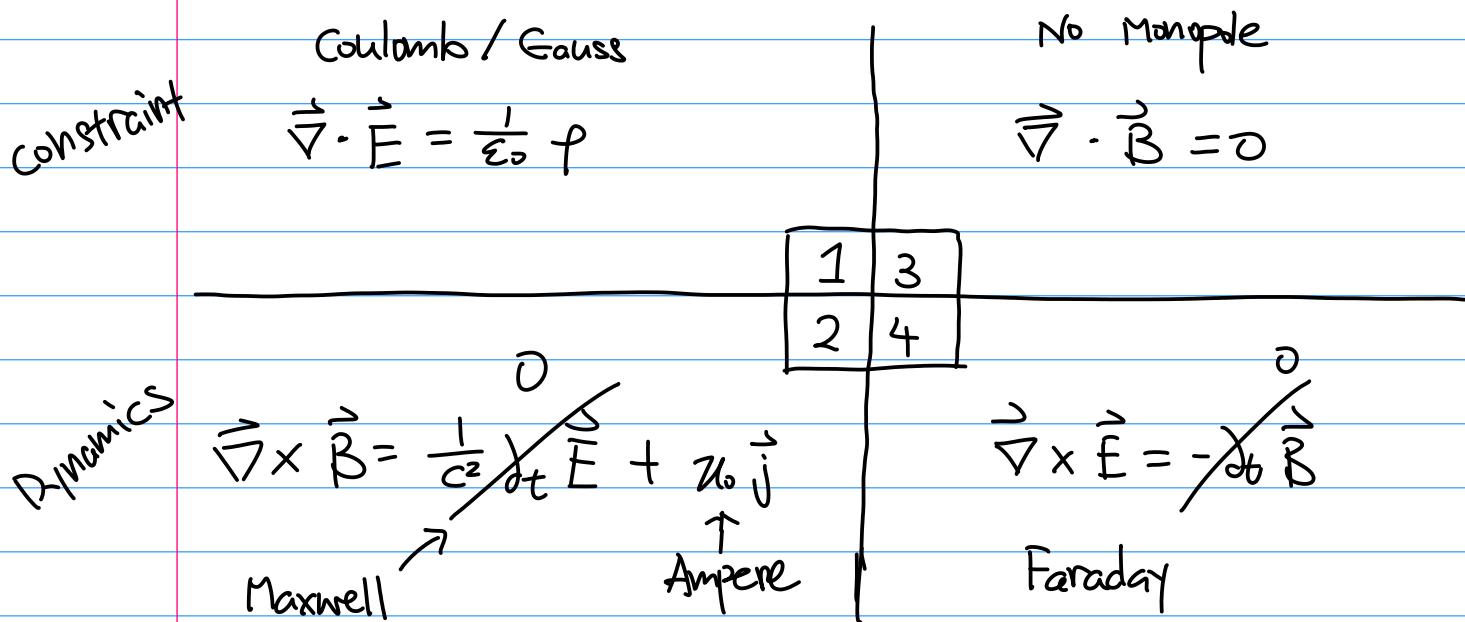


## Electrostatics:

We restrict ourselves to the case of static: time independent

$$\text{i.e. } \partial_t \phi = \partial_t \vec{A} = \partial_t \vec{v} = \partial_t \vec{j} = 0$$

↳ Time - Independent Maxwell Equation:



In homogeneous

Homogeneous

We observe that for time-independent,  $\vec{E}$  and  $\vec{B}$  decouple with each other, i.e. one can address  $\vec{E}$  without worrying about  $\vec{B}$ , vice-versa.

↳ Electro static :  $\vec{A} = \vec{j} = 0$

↳ Magnetostatic :  $\phi = v = 0$

Lets focus on electrostatic:

Question: Are the solutions to electrostatic problems unique?

Since  $\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} = -\vec{\nabla}\phi$

then  $\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi = \frac{1}{\epsilon_0} \rho$   
 $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} \phi) = 0$

Therefore, we focus on the single equation:

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \rho \quad (\text{Poisson's Equation})$$

$\vec{r}$  dependent scalar potential       $\vec{r}$ -dependent charge density, the source that is presumed known.

Poisson Equation Properties:

① Linear: the coefficient of the derivatives in differential equation are not function of the dependent variable.

2 variable linear PDE can be written as:

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$$

if  $B^2 - 2AC < 0 \Rightarrow$  Elliptic

$$B^2 - 2AC > 0 \Rightarrow$$
 Hyperbolic

$$B^2 - 2AC = 0 \Rightarrow$$
 Parabolic.

② Elliptic PDE

③ Inhomogeneous if  $G \neq 0$ , term that depends of the

## Gauge Invariance in Electrostatics

We previously saw that:

$$\boxed{\begin{aligned}\phi &\rightarrow \phi - \partial_t \chi \\ \vec{A} &\rightarrow \vec{A} + \vec{\nabla} \chi\end{aligned}}$$

← general gauge transform  
for Maxwell's Eqs.

leaves the field unchanged.

But in electrostatics:  $\vec{A} = 0$ ,  $\partial_t \phi = 0$   
the freedom to choose  $\chi(\vec{r}, t)$  is limited to the following:

- (1) since  $\vec{A} \rightarrow \vec{A} + \vec{\nabla} \chi$ , to keep  $\vec{A} = 0$ ,  $\chi$  must be independent of  $\vec{r}$ .
- (2) Since  $\phi \rightarrow \phi - \partial_t \chi$ , to keep  $\partial_t \phi = 0$ ,  $\chi$  can only be at most linear in  $t$  or  $\chi = at + b$ , or  $\partial_t \chi = a$

Due to the restrictions above:

Gauge Trans  
for electrostatics

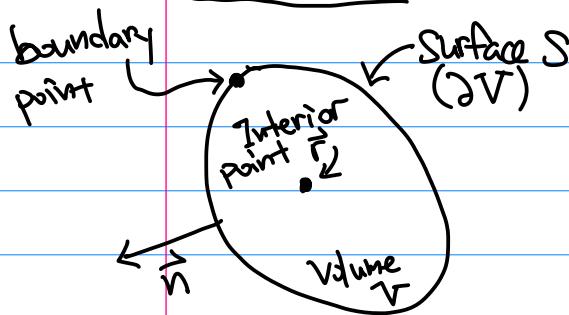
$$\boxed{\phi \rightarrow \phi + \text{const.} \quad \vec{A} = 0}$$

The question of uniqueness for Poisson's Eq:

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \rho$$

Suppose we found a solution, called  $\phi_1$ ,  
Can there be others?

$\Rightarrow$  Case 1:



- Finite Region
- Dirichlet Boundary Conditions:  $\phi$  are specified on the boundary.

$$\phi(\vec{r}) = \Phi(\vec{r})$$

↑ solution      ↑ boundary points      ↑ Specified boundary terms

With solution  $\phi_1(\vec{r})$ , we have:

$$-\nabla^2 \phi_1 = \frac{1}{\epsilon_0} \rho \quad \text{in volume.}$$

$$\phi_1 = \Phi \quad \text{on surface (boundary)}$$

Consider a new potential  $\phi_2 \equiv \phi_1 + \psi$

Suppose  $\phi_2$  obey the same conditions as  $\phi_1$ .

then in Volume:  $\nabla^2 \psi = \nabla^2 \phi_2 - \nabla^2 \phi_1 = \frac{1}{\epsilon_0} \rho + \frac{1}{\epsilon_0} \rho = 0$

so  $\nabla^2 \psi = 0$ , or  $\psi$  obeys Laplace Equation in V.

On the boundary:

$$\psi(\vec{r}) = \phi_2(\vec{r}) - \phi_1(\vec{r}) = \underline{\Phi}(\vec{r}) - \overline{\Phi}(\vec{r}) = 0 \quad \text{on } S$$

therefore  $\psi(\vec{r})$  obeys Laplace Eq with vanishing Dirichlet boundary conditions

To see the consequence of this effect:

Evaluate :

$$\int_V d^3r \psi \nabla^2 \psi$$

$$\begin{aligned} \hookrightarrow \underbrace{\int_V d^3r \psi \nabla^2 \psi}_{=0 \text{ since Laplace: } \nabla^2 \psi = 0} &= \int_V d^3r \left\{ \vec{\nabla} \cdot (\psi \vec{\nabla} \psi) - |\vec{\nabla} \psi|^2 \right\} \\ &= \underbrace{\int_V d\vec{s} \cdot \psi \vec{\nabla} \psi}_{=0 \text{ since we have vanishing boundary}} - \int_V d^3r |\vec{\nabla} \psi|^2 \\ 0 &= - \int_V d^3r \underbrace{|\vec{\nabla} \psi|^2}_{\text{Non-negative.}} \end{aligned}$$

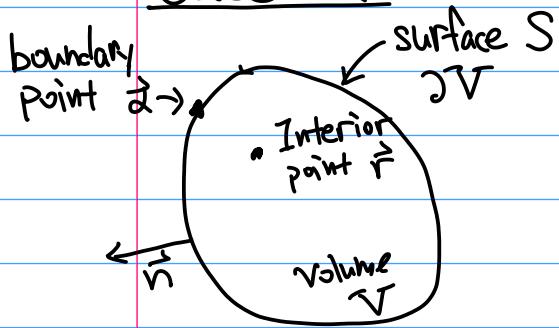
↪ So  $\vec{\nabla} \psi = 0$  for equality  $\rightarrow \psi = \text{constant}$  in volume.

↪ But due to vanishing boundary condition,  $\psi = 0$  on  $\partial V$

$$\boxed{\therefore \psi = 0 \rightarrow \phi_2 = \phi_1}$$

i.e. solution is unique.

## Case 2:



- Finite Region
- Neumann Boundary Condition: derivative  $\hat{n} \cdot \vec{\nabla} \phi$  is specified on boundary:

$$\hat{n} \cdot \vec{\nabla} \phi(z) = p(z) = -\frac{d}{E} \cdot \hat{n}$$

Note that since we're working with second-order PDE,  
B.C. is at most first derivative.

$\uparrow$   
boundary points.       $\uparrow$   
specified points

$\hat{n} \cdot \vec{\nabla} \phi$  is the rate of change along  $\hat{n}$ :

$$\hat{n} \cdot \vec{\nabla} \phi = \frac{\partial}{\partial s} \phi(z + s\hat{n}) \Big|_{s=0}$$

Again, we consider  $\phi_2 = \phi + \psi$

then we again have

$$① \quad \nabla^2 \psi = \nabla^2 \phi_2 - \nabla^2 \phi_1 = -\frac{1}{E} \phi - \frac{1}{E} \phi = 0$$

$$② \quad \hat{n} \cdot \vec{\nabla} \psi = \hat{n} \cdot \vec{\nabla} \phi_2 - \hat{n} \cdot \vec{\nabla} \phi_1 = \frac{\partial \phi_2}{\partial s}(z + s\hat{n}) \Big|_{s=0} - \frac{\partial \phi_1}{\partial s}(z + s\hat{n}) \Big|_{s=0} = 0$$

Again, we evaluate:  $\int d^3r \psi \nabla^2 \psi$

$$0 = \int d^3r \underbrace{\psi \nabla^2 \psi}_{=0} = \int d^3r \vec{\nabla} \cdot (\psi \vec{\nabla} \psi) - |\vec{\nabla} \psi|^2$$

$$= \underbrace{\int d^2S \cdot \psi \vec{\nabla} \psi}_{\text{since } \hat{n} \cdot \vec{\nabla} \psi = 0 \text{ at boundary}} - \int d^3r |\vec{\nabla} \psi|^2$$

$$0 = - \int d^3r |\vec{\nabla} \psi|^2$$

↳ This again requires  $\vec{\nabla}\psi = 0$ , or  $\psi = \text{constant}$ .

Since we don't require  $\psi=0$  as boundary condition

"Almost Unique"  $\Rightarrow$   $\boxed{\phi_2 = \phi_1 + \text{const.}}$  For Neumann B.C.  
Note that this is consistent  
with gauge invariant.

Case 3: All space, no boundary

lets call this asymptotic conditions:

Given  $\rho(\vec{r})$ , there is at most one solution to  
Poisson's Equation under the condition:

$$\phi(\vec{r}) \rightarrow 0 \quad \text{as} \quad |\vec{r}| \rightarrow \infty$$

If this is the case:

- i)  $r\phi$  remains bounded
  - ii)  $r|\vec{\nabla}\phi| \rightarrow 0$
- Note that not just  $\phi \rightarrow 0$   
but  $\phi$  and  $|\vec{\nabla}\phi| \rightarrow 0$   
sufficiently fast.

## Point Charges and Green's Function:

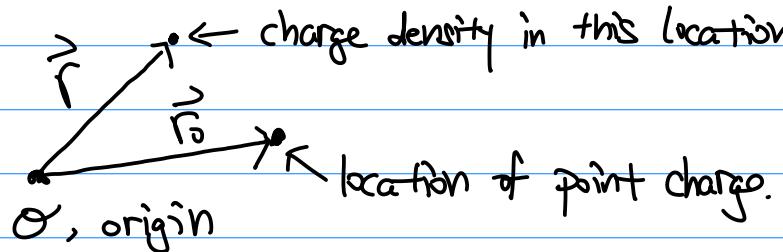
Charge density due to a point charge at  $\vec{r}_0$ :

$$\rho(\vec{r}) = q \delta^{(3)}(\vec{r} - \vec{r}_0)$$

↑                            ↑  
charge                      location

3D- Dirac-Delta.

Then superposition allows for a distribution of charges



Key properties of Dirac-Delta function (Distribution):

i)  $\int_V d^3r \delta(\vec{r} - \vec{a}) f(\vec{r}) = f(\vec{a})$

↑ sufficiently well-behaved.  
If  $\vec{a}$  is within  $V$

ii)  $\int_V d^3r \delta(\vec{r} - \vec{a}) = 1$

iii) A continuum version of the identity matrix  
or Kronecker-Delta:

$$\vec{\mathbb{I}} \cdot \vec{V} = \vec{V}$$

↓

$$\sum_j \delta_{jk} V_k = V_j$$

$$\int dV_r \delta(\vec{s} - \vec{r}) V(\vec{r}) = V(\vec{s})$$

iv) Various representation that give  $\delta(\vec{r})$  as  $\lambda \rightarrow 0$

$$\textcircled{a} \quad S_\lambda(\vec{r}) = \begin{cases} 1/\lambda^3 & (-\frac{\lambda}{2} < x, y, z < \frac{\lambda}{2}) \\ 0 & \text{otherwise.} \end{cases} \quad \text{as } \lambda \rightarrow 0$$

A cube of volume  $\lambda^3$  and density  $\lambda^3$   
centered at the origin. So the total  
weight is  $\lambda^3 \lambda^3 = 1$

$$\textcircled{b} \quad f_\lambda(\vec{r}) = (2\pi\lambda)^{-3/2} \exp\left\{-\frac{|\vec{r}|^2}{2\lambda^2}\right\} \quad \text{as } \lambda \rightarrow 0$$

↑  
3D Gaussian function normalized to 1

$$\textcircled{c} \quad \delta(\vec{r} - \vec{r}') = \int \frac{d^3k}{(2\pi)^3} \exp(i\vec{k} \cdot (\vec{r} - \vec{r}'))$$

Fourier Representation: For any nonzero  $\vec{r} - \vec{r}'$ ,  
the  $\vec{k}$  integration give complete destructive interference  
among other values of the complex phase factor.

$$\begin{aligned} \int d^3\vec{r}' \delta(\vec{r} - \vec{r}') f(\vec{r}') &= \int d^3\vec{r}' \frac{d^3k}{(2\pi)^3} \exp(i\vec{k} \cdot (\vec{r} - \vec{r}')) f(\vec{r}') \\ &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \hat{f}(\vec{k}) \\ &= f(\vec{r}) \end{aligned}$$

\textcircled{d} Informally:  $\delta(\vec{r})$  is  $\infty$  at  $\vec{r} = 0$  and 0 for  $\vec{r} \neq 0$   
such that integrating over any region containing  
 $\vec{r} = 0$  give unity.

Now if we have point charge,  $f = q \delta(\vec{r} - \vec{r}_0)$   
then:

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} q \delta(\vec{r} - \vec{r}_0)$$

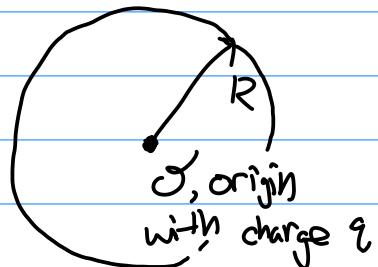
Let's assume  $\vec{r}_0 = 0$ , and use translational invariant later.

$$\hookrightarrow -\nabla^2 \phi = \frac{1}{\epsilon_0} q \delta(\vec{r}) \quad \text{with no boundary conditions (free space)}$$

$\Rightarrow$  Given the rotational invariance of  $\nabla^2$  and  $\delta(\vec{r})$ ,  
then any solution of the equation and the condition  
 $\phi \rightarrow 0$  as  $|r| \rightarrow \infty$  must be spherically symmetric  
because we would have multiple solutions under rotation,  
but the uniqueness theorem says there is only one  
solution. So solution itself must be rotationally symmetric.

To solve:  $-\nabla^2 \phi = \frac{1}{\epsilon_0} q \delta(\vec{r})$

$$\hookrightarrow \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} q \delta(\vec{r})$$



integrate over volume and apply divergence theorem.

$$\int_{r=R} d^3 r \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} q \int_{r=R} \delta(\vec{r}) d^3 r$$

$$\hookrightarrow \int_{r=R} dS \cdot \vec{E} = \frac{1}{\epsilon_0} q$$

Due to Rotational symmetry,  $\vec{E} = E(|\vec{r}|) \hat{r}$  ← must only be in  $\hat{r}$ .

Thus:  $E(R) \int_{r=R} d\vec{s} \cdot \hat{r} = \frac{1}{\epsilon_0} q$

$$d\vec{s} = r \sin\theta d\theta \hat{\theta} \times r d\phi \hat{\phi} = r^2 \sin\theta d\theta d\phi \hat{r}$$

$$E(R) 4\pi R^2 = \frac{1}{\epsilon_0} q$$

then  $\boxed{\vec{E} = E(|r|) \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{|r|^2} \hat{r}}$

Now calculate  $\phi$ , with assumption  $\phi(r \rightarrow \infty) = 0$ .

Since  $E = -\vec{\nabla} \phi$ ,  $\phi(\vec{r}) = \phi(\vec{r}) - \overset{\circ}{\phi}(\infty)$

$$\phi = \int_{\infty \hat{r}}^{\vec{r}} d\vec{r}' \cdot \vec{\nabla}' \phi = - \int_{\infty \hat{r}}^{\vec{r}} d\vec{r}' \cdot \vec{E} = - \int_{\infty \hat{r}}^{\vec{r}} \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}'|^2} d\vec{r}' \cdot \hat{r}'$$

We integrate from  $\vec{r}' = \infty \hat{r}$  to  $\vec{r}' = \vec{r} \hat{r}$

$$\hookrightarrow \phi = - \int_{\infty}^{\vec{r}} \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}'|^2} d\vec{r}' = \left. \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}'|} \right|_{\infty}^{\vec{r}} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}|}$$

Thus  $\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}|}}$  with  $\phi \rightarrow 0$  as  $|\vec{r}| \rightarrow \infty$

Relocation: Translating the charge from  $\vec{r}_0$  to  $\vec{r}$ ,  
 Poisson's Equation:

$$-\nabla_{\vec{r}}^2 \phi(\vec{r}) = \frac{1}{\epsilon_0} q \delta(\vec{r} - \vec{r}_0)$$

Then we have solution:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}_0|}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}_0|^2} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|}$$

Superposition: By the linearity of the  $\nabla^2$ , and the homogeneity of the vanishing as  $|\vec{r}| \rightarrow \infty$

A charge density due to a set of J-particles  $q_j$ , static at  $\vec{r}_j$ , i.e.

$$\rho(\vec{r}) = \sum_{j=1}^J q_j \delta(\vec{r} - \vec{r}_j)$$

is associated with superposition :

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^J \frac{q_j}{|\vec{r} - \vec{r}_j|}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^J \frac{q_j}{|\vec{r} - \vec{r}_j|^2} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|}$$

For  
Discrete  
Point  
charges.

Continuous - Distribution:

$$\sum_{j=1}^J q_j \rightarrow \int d\vec{r}' \rho(\vec{r}')$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad \text{Note, to converge, } |\rho(\vec{r}')| \leq \frac{C}{r'^{2+\varepsilon}} \quad (C, \varepsilon > 0)$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

↓ position of charge  
 ↓ distance between  
 charge and position

So, if we know  $\rho(\vec{r})$ , then we can integrate and get  $\phi(\vec{r})$ .

Green's function for electrostatic is:

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} \quad \leftarrow \begin{array}{l} \text{fundamental or} \\ \text{translationally invariant} \\ \text{Greens Function.} \end{array}$$

so that  $\phi(\vec{r}) = \int d^3r' G(\vec{r}, \vec{r}') \rho(\vec{r}')$

For no Boundary Condition

Check that Green's function works

$$\begin{aligned} -\nabla^2 \phi(\vec{r}) &= -\nabla^2 \int d^3r' G(\vec{r}, \vec{r}') \rho(\vec{r}') \\ &= \int d^3r' \underbrace{(-\nabla^2 G(\vec{r}, \vec{r}'))}_{\frac{1}{\epsilon_0} \delta(\vec{r} - \vec{r}')} \rho(\vec{r}') \\ &= \frac{1}{\epsilon_0} \rho(\vec{r}) \end{aligned}$$

## Green's Function in general:

If  $G(\vec{r}, \vec{r}')$  obeys:

$$\mathcal{L}_r G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}')$$

Linear Differential  
Operator

and it also obeys suitable boundary or asymptotic conditions,  
then  $\leftarrow$  a general inhomogeneity

$$\mathcal{L}_r f(\vec{r}) = R(\vec{r})$$

is solved by  $f(\vec{r}) = \int d^3 r' G(\vec{r}, \vec{r}') R(\vec{r}')$

then  $G$  is called the Green's function of the problem

Proof:  $\mathcal{L}_r f(\vec{r}) = \mathcal{L}_r \int d^3 r' G(\vec{r}, \vec{r}') R(\vec{r}')$

$$= \int d^3 r' \underbrace{\mathcal{L}_r G(\vec{r}, \vec{r}')}_{\delta(\vec{r} - \vec{r}')} R(\vec{r}')$$

$$= R(\vec{r})$$

Remarks: Think  $\mathcal{L}$  and  $G$  are inverses. i.e.  $\mathcal{L}^{-1} = G$

Analogy with Matrix / vector:

$$\text{If } G \text{ obeys } L \cdot G = I \Rightarrow L^{-1} = G$$

then  $L \cdot F = R$  is solved by

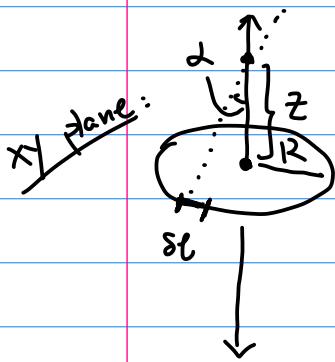
$$L^{-1} (L \cdot F = R)$$

$$\hookrightarrow F = GR$$

And  $L \cdot F = L(GR) = IR = R$

Ex: Compute the on-axis electric field and potential due to uniform ring of charge  $Q$  and radius  $R$  at a distance  $z$ , from the center of the ring.

use result to compute the on-axis electric field due to a disk of radius  $R$  and charge per unit area  $\sigma$ .



Qualitatively: 1)  $\vec{E}$  points along  $z$ .

2)  $\vec{E}$  is odd in  $z$ ,  $\vec{E}(-\hat{z}) = -\hat{E}(\hat{z})$

$$3) E^z = \frac{Q}{4\pi\epsilon_0 R^2} f\left(\frac{z}{R}\right)$$

Quantitative:

$$\delta E = \frac{1}{4\pi\epsilon_0} \frac{f(r)}{|r-r'|^2}$$

$$\delta E^z = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \frac{\delta l}{2\pi R} \frac{1}{z^2+R^2} \frac{z}{\sqrt{z^2+R^2}} \hat{z}$$

$$\hookrightarrow E^z = \frac{1}{4\pi\epsilon_0} Q \frac{z}{(z^2+R^2)^{3/2}} \hat{z}$$

distance from charge to position.

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \frac{z/2}{[1+(\frac{z}{R})^2]^{3/2}}$$

$$\text{as } z \ll R: E \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{z}{R} \text{sgn}(z)$$

$$\text{as } z \gg R: E \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{z^2} \text{sgn}(z)$$

Then  $\phi = \int \vec{E} \cdot d\vec{r}$  since  $-\nabla \phi = \vec{E}$

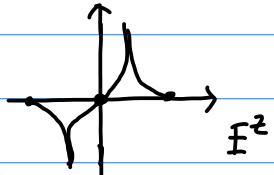
$$\phi(z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{z^2+R^2}} \quad \text{or use superposition}$$

Now working with disk

$$\int \delta E = \int_{r=0}^R \frac{1}{4\pi\epsilon_0} 2\pi r dr \sigma \frac{z}{(z+r^2)^{3/2}} \hat{z}$$

$$E = \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{1}{\sqrt{1+(R/z)^2}} \right] \underbrace{\text{sgn}(z)}_{=1 \text{ if } z>0} \\ = -1 \text{ if } z<0$$

$$\text{as } |z| \ll R \quad , \quad \vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z} \rightarrow$$



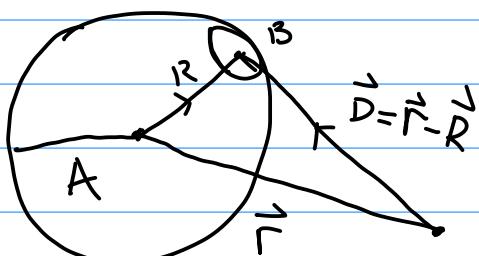
$$\text{as } |z| \gg R \quad E \approx \frac{\sigma}{2\epsilon_0} \left[ 1 - \left(1 - \frac{1}{2} \left(\frac{R}{z}\right)^2\right) \right]$$

$$\stackrel{!}{=} \frac{\sigma}{4\epsilon_0} \left(\frac{R}{z}\right)^2$$

$$\stackrel{!}{=} \frac{\pi R^2 \sigma}{4\pi\epsilon_0} \frac{1}{z^2}$$

$$\stackrel{!}{=} \frac{Q}{4\pi\epsilon_0} \frac{1}{z^2} \text{ sign}(z)$$

Ex: Spherical shell of radius A, and uniform charge per unit area  $\sigma$ . A small cap of radius B ( $\ll A$ ) is removed together with its charge. Compute the  $\vec{E}$  far away from cap.

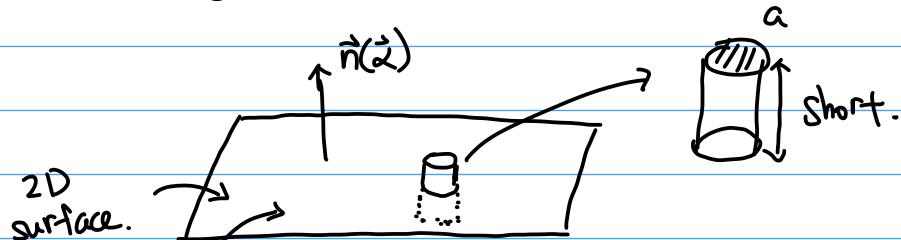


$$\vec{E}(\vec{r}) = \vec{E}_{\text{complete sphere}} + \vec{E}_{\text{cap-removed}}$$

$$= \frac{4\pi A^2 \sigma}{4\pi\epsilon_0} \frac{\vec{r}}{|\vec{r}|^3} \theta(\vec{r} - \vec{A}) - \frac{\sigma B^2}{4\pi\epsilon_0} \frac{\vec{r} - \vec{R}}{|\vec{r} - \vec{R}|^3}$$

$$E_{\text{sphere}} = \begin{cases} 0 & r < A \\ \frac{4\pi A^2 \sigma}{4\pi\epsilon_0} \frac{\vec{r}}{|\vec{r}|^3} & r > A \end{cases}$$

General matching conditions across charged interfaces:



charge per  
unit area

With Gauss's theorem:

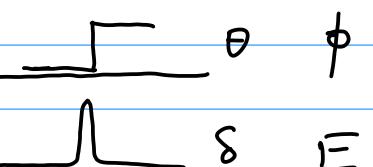
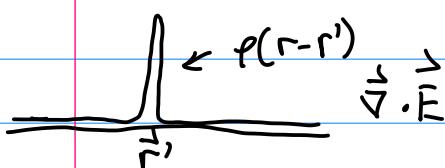
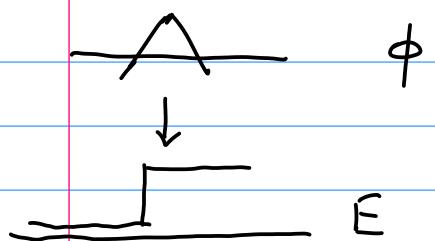
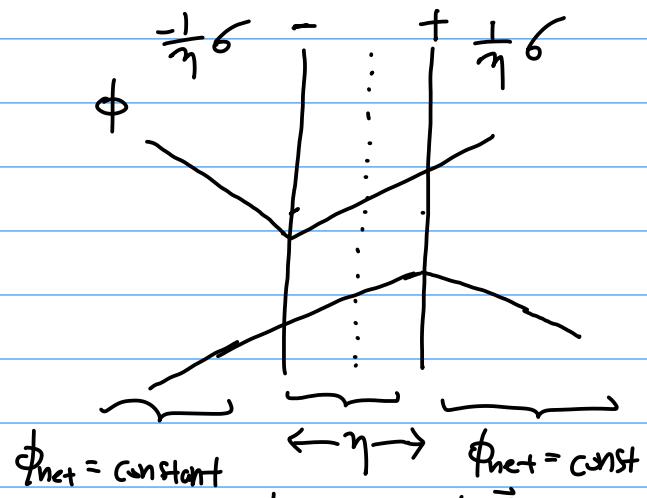
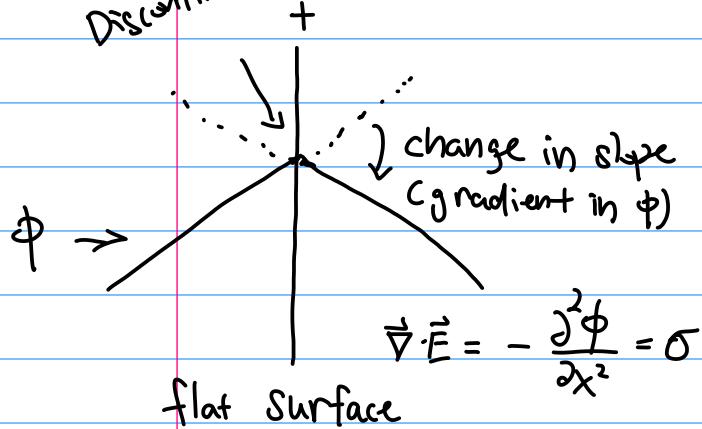
$$\int \vec{\nabla} \cdot \vec{E} dV = \int \frac{1}{\epsilon_0} \rho dV$$

$$\underbrace{\int \vec{E} \cdot d\vec{s}}_{\vec{E} \cdot \vec{n}(\vec{z})} = \sigma(\vec{z}) \cancel{d} \frac{1}{\epsilon_0}$$

$$\lim_{\eta \rightarrow 0} [E(\vec{z} + \eta \vec{n}(\vec{z})) - E(\vec{z} - \eta \vec{n}(\vec{z}))] \cdot \vec{n}(\vec{z}) = \frac{1}{\epsilon_0} \sigma(\vec{z})$$

Discontinuities in  $\phi$

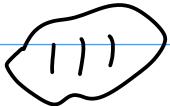
How electric field change across surface.



## Maximum Principle / Earnshaw's Theorem



Vacuum



$$\nabla^2 \phi = -\frac{1}{\epsilon_0} = 0$$

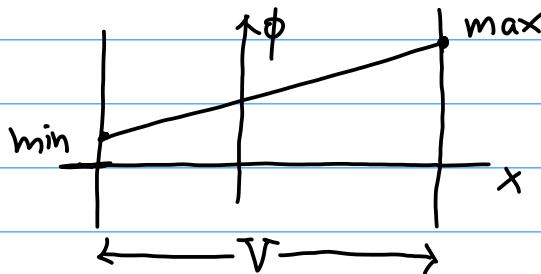
$$\phi = \text{const}$$



No points in  $\tilde{\Gamma}$  have  $\phi$  a local max or a local min.

Consequence: An infinitesimal-strength point charge inserted into a pre-dominated electrostatic potential cannot be in a position of equilibrium.

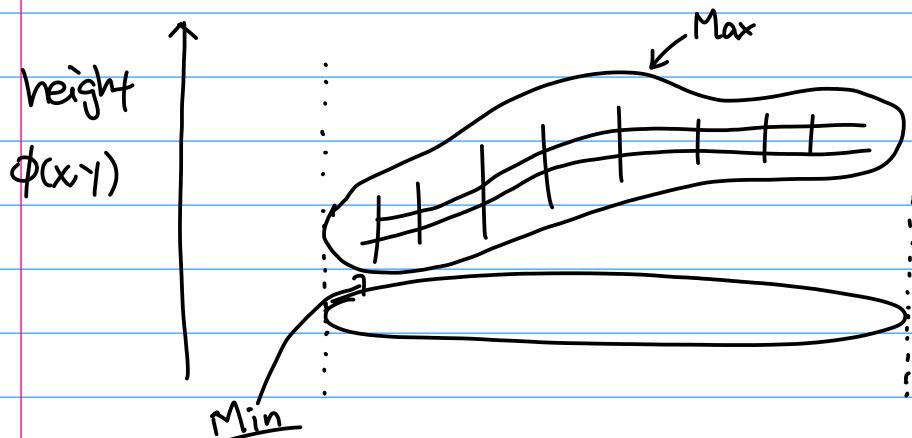
1D ex:  $\partial_x^2 \phi = 0$



$\phi(x) = Ax + Bx^3$ , i.e.  
no local max or min. So  
no stable points.

2D ex:  $\nabla^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi = 0$

Consider  $\phi$  as the height, in mechanical equilibrium, of a drum head, stretched across



Every point inside the region follows Laplace's Equation:

$$\partial_x^2 \phi = -\partial_y^2 \phi$$

i.e. curvature in  $x, y$  directions add to zero at every point. Since we can choose the orientation of the axes, this adding to zero must hold for every pair of orthogonal directions.

For  $\phi(x,y) \sim \sin qx e^{-ky}$ , oscillating in one-direction and decaying or growth in another, for Laplace equation to hold true:

$$\nabla^2 \phi = 0 \Rightarrow q^2 = k^2$$

Now suppose we have a minimum  $\phi(x,y)$ , such that  $\vec{\nabla} \phi = 0$ . We can expand  $\phi(\vec{r})$  around  $\vec{r} = 0$ :

$$\phi(\vec{r}) = A + B_a r_a + C_{ab} r_a r_b + \dots$$

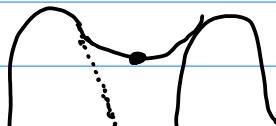
↪ Require  $B_a = 0$  for  $r_a$  to be stationary,  $\vec{\nabla} \phi = 0$

↪ Require  $C_{ab} = 0$  for  $\phi$  to obey  $\nabla^2 \phi = 0$

If diagonalize  $C$ : since eigenvalues add to 0 since  $\nabla^2 \phi = 0$

$$\phi(x,y) = \phi(0,0) + \lambda (x^2 - y^2)$$

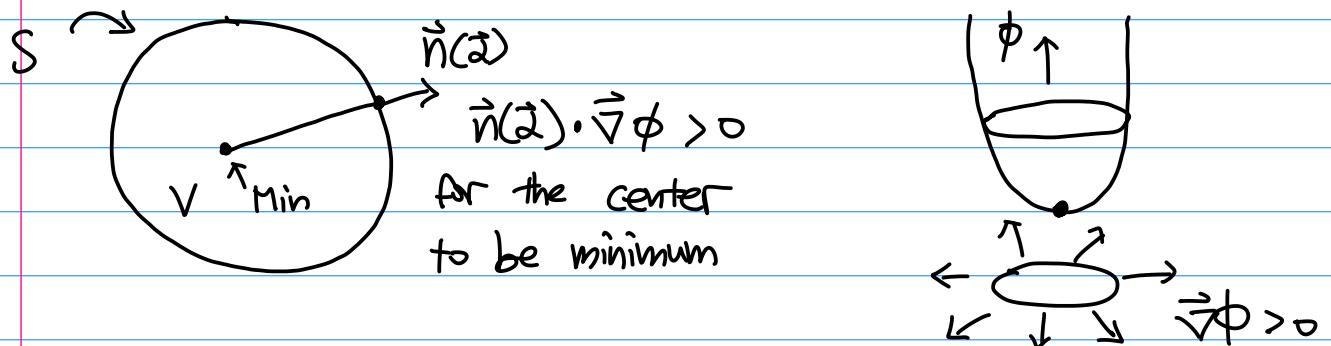
Saddle point



So any stationary point of  $\phi(x,y)$  is a saddle point, not local min nor max.

## Another example of Earnshaw's Theorem:

Consider a small sphere around a putative minimum of  $\phi$ . Since  $\phi$  is a minimum,  $\phi$  increases in all directions of departure from that point, i.e.  $\vec{n} \cdot \vec{\nabla} \phi$  is positive at all points over the sphere.



Then collecting all positive values:

$$\int d^2S \vec{n} \cdot \vec{\nabla} \phi > 0$$

$$\hookrightarrow \int d^3r \vec{\nabla} \cdot \vec{\nabla} \phi = \int d^3r \underbrace{\nabla^2 \phi}_{=0} \not> 0$$

so it cannot be true that  $\vec{n} \cdot \vec{\nabla} \phi > 0$ , hence there is not a local minimum. The same strategy applies for max.

## Interaction energy, force and torque.

$$\boxed{\epsilon = \frac{\epsilon_0}{2} \int d^3r \vec{E} \cdot \vec{E}}$$

< energy stored in electric field.

with Maxwell:  $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$  and  $-\nabla^2 \phi = \frac{1}{\epsilon_0} \rho$

$$\rightarrow \epsilon = -\frac{\epsilon_0}{2} \int d^3r \vec{E} \cdot \vec{\nabla} \phi$$

$$= -\frac{\epsilon_0}{2} \left[ \underbrace{\int d^3r \vec{\nabla}(\vec{E} \phi)}_{\int d^3r \vec{E} \phi} - \underbrace{\int d^3r \phi \vec{\nabla} \cdot \vec{E}}_{\int d^3r \phi \frac{\rho}{\epsilon_0}} \right]$$

as  $d^3S \rightarrow \infty$ ,  $\vec{E} \rightarrow 0$  faster than  $d^3S \rightarrow \infty$ .

$$\boxed{\epsilon = \frac{1}{2} \int d^3r \rho \phi}$$

all space  
integration

→ given  $\phi$ , get  $\rho$  by plug into  $-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$

or

→ given  $\rho$ , solve  $-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$  to get  $\phi$ .

## Interaction between two disjoint charge distribution:



Consider two relatively charged distribution that are disjoint from one another:

Alone:  $p_1$  gives rise to  $\phi_1$   
 $p_2$  gives rise to  $\phi_2$

By superposition:  $p_1 + p_2 \rightarrow \phi_1 + \phi_2 \rightarrow E_1 + E_2 = \vec{E}(\phi_1 + \phi_2)$

$$\begin{aligned}\epsilon_{\text{total}} &= \frac{1}{2} \epsilon_0 \int d^3r |\vec{E}_1 + \vec{E}_2|^2 \\ &= \frac{1}{2} \epsilon_0 \int d^3r |\vec{E}_1|^2 + |\vec{E}_2|^2 + 2 \vec{E}_1 \cdot \vec{E}_2 \\ &= \epsilon_1 + \epsilon_2 + \epsilon_{\text{int}}\end{aligned}$$

↪ interaction energy.

Identify interaction energy as:

$$\epsilon_{\text{int}} = \epsilon_0 \int d^3r \vec{E}_1 \cdot \vec{E}_2$$

due to  
distant charges

$$\epsilon_{\text{int}} = \int d^3r p_1 \phi_2$$

nearby  
charge

$$\text{or } \int d^3r p_2 \phi_1 = \int d^3r p \phi^{\text{ext}}$$

Important setting: These are distant (but not too distant) charges held at fixed position that give rise to external potential,  $\phi^{\text{ext}}$ .

If let  $p_2 = p$ ,  $\phi_1 = \phi^{\text{ext}}$ , then there are nearby charges  $p$  that are located in  $\phi^{\text{ext}}$ . Vice-Versa. Since there is no overlap between charge distribution, where we have  $p \neq 0$ ,  $\phi^{\text{ext}}$  must = 0, so  $\nabla^2 \phi^{\text{ext}} = 0$

## Green's Reciprocal Relation:

Suppose  $\rho_1(\vec{r})$  causes  $\phi_1(\vec{r})$   
 $\rho_2(\vec{r})$  causes  $\phi_2(\vec{r})$

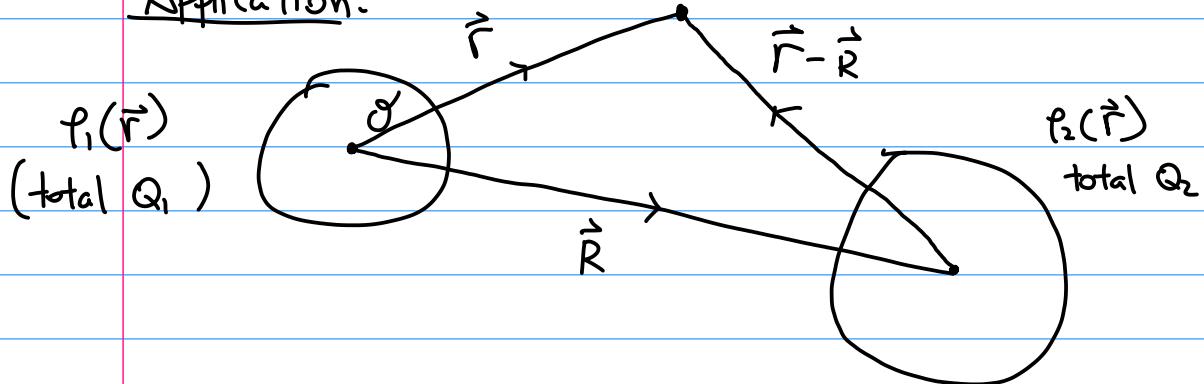
then

$$\int d^3r \rho_1(\vec{r}) \phi_2(\vec{r}) = \int d^3r \rho_2(\vec{r}) \phi_1(\vec{r})$$

Both expressions  
lead to this

$$= \int \frac{d^3r_1 d^3r_2}{4\pi\epsilon_0} \frac{\rho_1(\vec{r}_1) \rho_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

### Application:



Two spherically charged distribution  $\rho_1$  and  $\rho_2$ . Total charges  $Q_1$  and  $Q_2$ . Find interaction energy.

$$\mathcal{E} = \epsilon_0 \int d^3r \rho(\vec{r}) \phi^{\text{ext}}(\vec{r})$$

$$= \epsilon_0 \int \frac{d^3r_1 d^3r_2}{4\pi\epsilon_0} \underbrace{\frac{\rho_1(\vec{r}_1) \rho_2(\vec{r}_2)}{|\vec{R} + \vec{r}_2 - \vec{r}_1|}}_{|\vec{r}_1 - \vec{r}_2|} \quad \leftarrow \text{Difficult to integrate.}$$

Instead use Green's Reciprocal Relation:

Consider  $\phi_2$  outside sphere #2, by Gauss's Law:

$$\phi_2(\vec{r}) = \frac{Q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{R}|} \quad \leftarrow \text{the same as all charge concentrated at } R.$$

$$\text{then } E_{\text{int}} = \int d^3r \rho_1(\vec{r}) \phi_2(\vec{r}) \quad \leftarrow \text{so it is same as } \tilde{\rho}_2 \equiv Q_2 \delta(\vec{r}-\vec{R})$$

$$= \int d^3r \phi_1(\vec{r}) \tilde{\rho}_2(\vec{r})$$

$$= \int d^3r \frac{Q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{R}|} Q_2 \delta(\vec{r}-\vec{R})$$

$$E_{\text{int}} = \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{1}{R} \quad \leftarrow \text{Coulomb potential if the spheres were points.}$$

Expansions of  $E_{\text{int}}$  in powers of:



Taylor expand  $\phi$  around  $\vec{r}$ , assume that  $\phi_{\text{ext}}$  varies on a scale that is large compared with the size of region containing charge.  $\rightarrow$  i.e. slowly varying  $\phi$ .

$$\phi_{\text{ext}}(\vec{r}) = \phi_{\text{ext}}(\vec{r}) \Big|_{\vec{r}=0} + \vec{r} \cdot \vec{\nabla} \phi_{\text{ext}} \Big|_{\vec{r}=0}$$

$$+ \frac{1}{2} \sum_{k,l=1}^3 r_k r_l \underbrace{\frac{\partial^2 \phi_{\text{ext}}}{\partial r_k \partial r_l}}_{=0 \text{ since } \nabla^2 \phi = 0} \Big|_{\vec{r}=0} + \dots$$

Inserting into  $E_{int} = \int d^3r \varphi \phi_{ext}$

$$E_{int} = \phi_{ext,0} \int d^3r \varphi(\vec{r})$$

$$+ \vec{\nabla} \phi_{ext,0} \cdot \int d^3r \vec{r} \varphi(\vec{r})$$

$$+ \frac{1}{2} \sum_{k,l=1}^3 \partial_k \partial_l \phi_{ext,0}^{kl} \int d^3r r_k r_l \varphi(\vec{r}) + \dots$$

Noting that terms in  $E_{int}$  are characterized by moments of the charge distribution:

0<sup>th</sup>:  $q \equiv \int d^3r \varphi(\vec{r})$  : the total electric charge 1# (couples to potential  $\phi_{ext}$ )

1<sup>st</sup>:  $\vec{p} \equiv \int d^3r \vec{r} \varphi(\vec{r})$  : the electric dipole moment. 3# (couples to gradient,  $E_{ext}$ )

2<sup>nd</sup>:  $Q_{kl} = \int d^3r [3\vec{r}_k \vec{r}_l - \underbrace{r^2 \delta_{kl}}_{\text{subtracts contribution}}] \varphi(\vec{r})$  : quadrupole moment. 5# (couples to curvature of  $\phi_{ext}$ )  
5#

in  $E_{int}$  as a result  
of  $\nabla^2 \varphi$ , trace-free.

Then  $E_{int} = q \phi_{ext,0} + \vec{p} \cdot \vec{\nabla} \phi_{ext,0} + \frac{1}{2} \sum_{k,l=1}^3 Q_{kl} \frac{\partial^2 \phi_{ext,0}}{\partial r_k \partial r_l} + \dots$   
Here we keep  $\varphi$  fixed and vary  $\phi$ .

We can do a similar construct by keeping  $\phi$  fixed and vary  $\varphi$ .

## Force and Torque on $\rho(\vec{r})$ due to $Q^{\text{ext}}(\vec{r})$

For a distant charge that creates  $\phi_{\text{ext}}(\vec{r})$  and nearby charge  $\rho$ , the force on the body carrying  $\rho$  is:

$$\vec{F} = \int d^3r \rho(\vec{r}) \vec{E}(\vec{r})$$

due to  
all  $\rho \rightarrow \vec{E} = \vec{E}_{\text{self}} + \vec{E}_{\text{ext}}$

We now show that  $\int d^3r \rho(\vec{r}) \vec{E}_{\text{self}}(\vec{r}) = 0$ ,  
so that only  $E_{\text{ext}}$  contributes to  $\vec{F}$ .

$$\begin{aligned} \int d^3r \rho(\vec{r}) \vec{E}_{\text{self}}(\vec{r}) &= \int d^3r \rho(\vec{r}) \frac{1}{4\pi\epsilon_0} \underbrace{\int d^3r' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}}_{E_{\text{self}}(\vec{r})} \\ &= \frac{1}{4\pi\epsilon_0} \underbrace{\int d^3r \rho(\vec{r}) d^3r' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}}_{(\text{anti}) \text{ Skew-symmetric: if we change } \vec{r} \text{ and } \vec{r}', \text{ they differ by negative sign, so it must } = 0} \\ &= 0 \end{aligned}$$

Therefore, in electrostatic, self-force vanishes. But it is not always in electrostatics.

$$\vec{F} = \int d^3r \rho(\vec{r}) \vec{E}_{\text{ext}}(\vec{r})$$

for electrostatic.

(with variation in the  $\vec{E}_{\text{ext}}$  field)

$$\vec{F} = q \vec{E}_{\text{ext},0} + \vec{p} \cdot \vec{\nabla} \vec{E}_{\text{ext},0} + \frac{1}{6} \sum_{k=1}^3 Q_k \frac{\partial^2}{\partial r_k \partial r_l} E_{\text{ext},0} + \dots$$

- It's important to leave out  $\vec{E}_{\text{self}}$ , since we don't expect nearby charge to vary slowly.
- We also note the leading term in  $\vec{F}$  is the total charge,  $q$ , and external field  $\vec{E}$ . And if  $q \rightarrow 0$ , the the leading terms become  $\vec{p}$  and  $Q_k$ .

Torque:

$$\tau = \int d^3r \vec{r} \times \vec{f} = \int d^3r \vec{r} \times (\vec{p}(\vec{r}) \vec{E}(\vec{r}))$$

and similarly, only  $\vec{E}_{\text{ext}}$  contributes to  $\tau$ .

Now consider slow spatially varying field:

$$\tau = \vec{p} \times \vec{E}_{\text{ext},0} + \dots$$

↑ Dipole moment is the leading term since  $\tau \sim \vec{r} \times \vec{E}$

## Point Charge Idealization:

For a point charge  $q$  located at origin:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$

$$\hookrightarrow E = \frac{\epsilon_0}{2} \int d^3r |\vec{E}|^2 = \frac{\epsilon_0}{2} \int d^3r \frac{q^2}{(4\pi\epsilon_0)^2} \frac{1}{r^4}$$

$$= \frac{q^2}{8\pi\epsilon_0} \int_0^\infty \frac{r^2 dr}{r^4} = \infty$$

point charges are unphysical idealization  
that sometimes present obstacles.

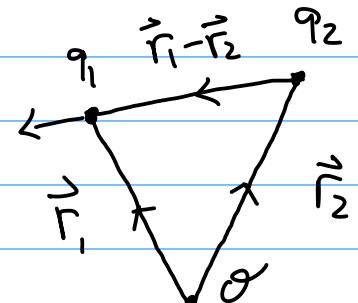
But we can consider interaction between particles:

$$E_{int} = \int d^3r_1 \phi_1 \phi_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} = \text{Coulomb potential.}$$

$\overset{q_1 \delta(\vec{r} - \vec{r}_1)}{\uparrow} \quad \overset{q_2}{\uparrow}$   
 $\frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|}$

then  $\vec{F} = -d\phi / dr_1 \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$

$$\vec{F}_{\text{Coulomb}} = q_1 \frac{q_2}{4\pi\epsilon_0} \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$



Generally:  $\vec{F} = q \vec{E}_{ext}$

↑ electric field due to  
charge of other charges.  
the body on which  
the force is acted on

Multipole Expansion of the fundamental Green function  
for Poisson Equation:

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}'|}$$

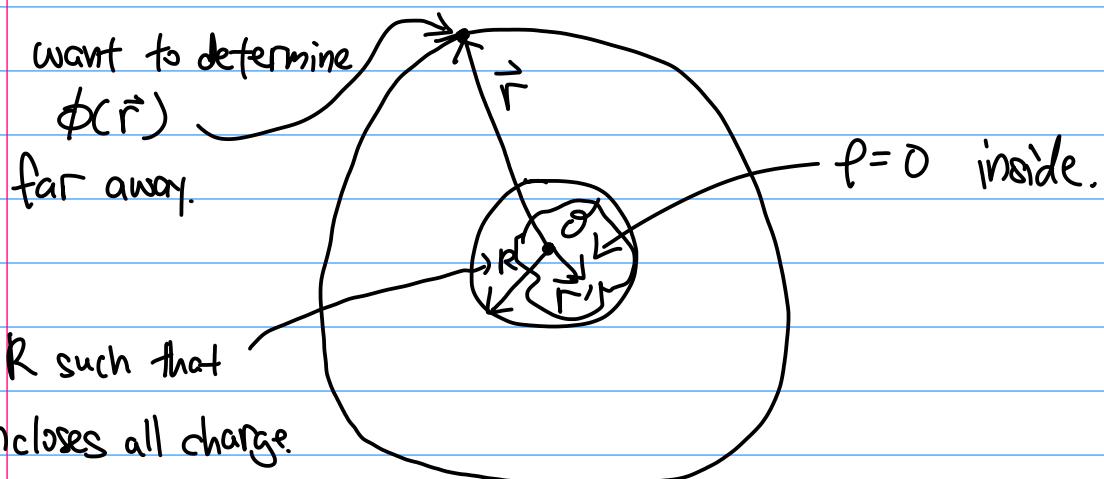
The fundamental  
Green function.

as  $-\nabla_{\vec{r}}^2 G(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} \delta(\vec{r}-\vec{r}')$

We have seen that for  $\rho$  decaying rapidly enough at  $\infty$ ,  
Poisson's Equation has the unique solution:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Now let's construct the multipole expansion of  $\phi$ ;  
it is useful when we want to construct  $\phi$  at regions  
away from  $\rho$ , say  $\vec{r}$ :



Focus on  $\frac{1}{|\vec{r}-\vec{r}'|}$ , note that  $|r'| < R < |\vec{r}|$

$$\begin{aligned}
 \frac{1}{|\vec{r} - \vec{r}'|} &= \frac{1}{\sqrt{\vec{r}^2 - 2\vec{r} \cdot \vec{r}' + \vec{r}'^2}} \\
 &= \frac{1}{\vec{r}} \frac{1}{\sqrt{1 - 2\left(\frac{\vec{r}'}{\vec{r}}\right)\hat{\vec{r}} \cdot \hat{\vec{r}'} + \left(\frac{\vec{r}'}{\vec{r}}\right)^2}} \\
 &= \frac{1}{\vec{r}} \left\{ 1 + \left(\frac{\vec{r}'}{\vec{r}}\right) \hat{\vec{r}} \cdot \hat{\vec{r}'} + \frac{1}{2} \left(\frac{\vec{r}'}{\vec{r}}\right)^2 [3(\hat{\vec{r}} \cdot \hat{\vec{r}'})^2 - 1] + \dots \right\}
 \end{aligned}$$

Insert expansion back to  $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3 r' f(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|}$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3 r' f(\vec{r}') \frac{1}{\vec{r}} \left\{ 1 + \left(\frac{\vec{r}'}{\vec{r}}\right) \hat{\vec{r}} \cdot \hat{\vec{r}'} + \frac{1}{2} \left(\frac{\vec{r}'}{\vec{r}}\right)^2 [3(\hat{\vec{r}} \cdot \hat{\vec{r}'})^2 - 1] + \dots \right\}$$

$$\begin{aligned}
 &= \frac{1}{4\pi\epsilon_0} \left[ \underbrace{\frac{1}{\vec{r}} \int d^3 r' f(\vec{r}')}_q + \underbrace{\frac{\vec{r}}{\vec{r}^2} \cdot \int d^3 r' \vec{f}(\vec{r}')}_p \right] \text{(dipole)} \\
 &\quad + \frac{1}{2} \frac{1}{\vec{r}^3} \sum_{k,l=1}^3 \left( \hat{\vec{r}}_k \hat{\vec{r}}_l - \frac{1}{3} \delta_{kl} \right) \underbrace{\int d^3 r' f(\vec{r}') 3 \vec{r}'_k \vec{r}'_l}_{\substack{\text{can be subtracted} \\ \text{since } \vec{r}_k \vec{r}_l \text{ traceless}}} + \dots
 \end{aligned}$$

$$+ \frac{1}{2} \frac{1}{\vec{r}^3} \sum_{k,l=1}^3 \left( \hat{\vec{r}}_k \hat{\vec{r}}_l - \frac{1}{3} \delta_{kl} \right) \underbrace{\int d^3 r' f(\vec{r}') 3 \vec{r}'_k \vec{r}'_l}_{Q_{kl}} + \dots$$

can be subtracted  
since  $\vec{r}_k \vec{r}_l$  traceless

$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{\vec{r}} + \frac{\vec{r} \cdot \vec{P}}{\vec{r}^2} + \frac{1}{6} \frac{1}{\vec{r}^3} \sum_{k,l=1}^3 \hat{\vec{r}}_k \hat{\vec{r}}_l Q_{kl} + \dots \right]}$$

## Multipole Expansion for the fundamental Green function:

Consider:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 - 2rr'\hat{n} \cdot \hat{n}' + r'^2}}$$

$\vec{r} = r\hat{n}$        $\vec{r}' = r'\hat{n}'$

Assume  $r' < r$ , and write  $\frac{1}{r} \sqrt{1 - 2(r'/r)\hat{n} \cdot \hat{n}' + (r'/r)^2}$

There is a standard generating function for Legendre polynomials:

generating function

↓

$$\frac{1}{\sqrt{1 - 2u\hat{n} \cdot \hat{n}' + u^2}} = \sum_{l=0}^{\infty} u^l P_l(\hat{n} \cdot \hat{n}'), \quad u < 1.$$

↓

It generates the RHS by taking derivative of LHS

With

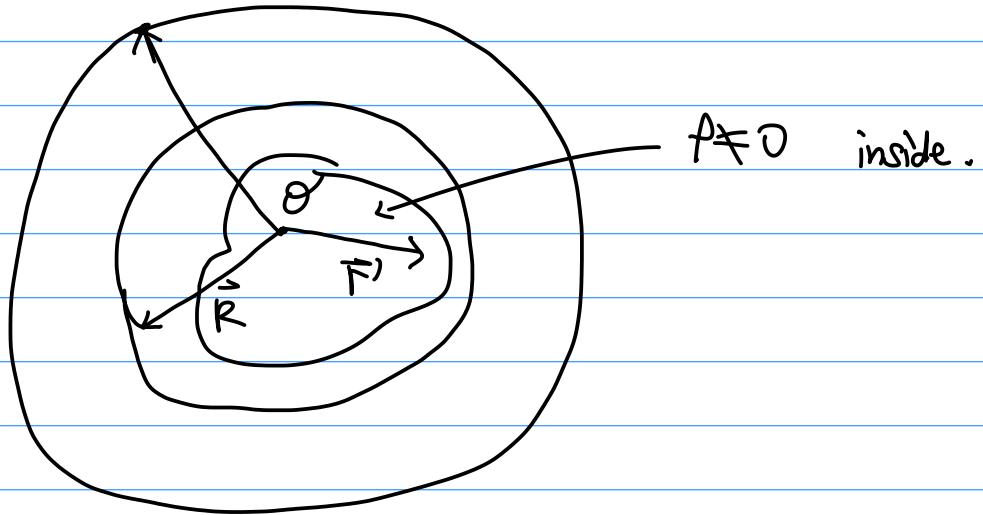
$$P_l(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{n}')^* Y_{lm}(\hat{n})$$

Altogether: for  $r' < r$ :

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\hat{n}') Y_{lm}(\hat{n})$$

## Applying Multipole Expansion:

Given a charge distribution  $\rho$ , we can use this expansion to organize the solution:



we have  $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \rho(\vec{r}')$

$$\hookrightarrow = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int d^3r' \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{n}')^* Y_{lm}(\hat{n}) \rho(\vec{r}')$$

$$\hookrightarrow = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \sum_{m=-l}^l Y_{lm}(\hat{n}) \underbrace{\int d^3r' (r')^l Y_{lm}(\hat{n}')^* \rho(\vec{r}')}_{\text{spherical multipole expansion}}$$

Spherical Multipole expansion

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \sum_{m=-l}^l Y_{lm}(\hat{n}) q_{lm}$$

where  $q_{lm} = \int d^3r' r'^l Y_{lm}(\hat{n}')^* \rho(\vec{r}')$  ← spherical multipole moment of  $\rho$ .

$d^3r = r^2 dr d^2n$        $= \int dr' r'^{l+2} \int d^2n Y_{lm}(\hat{n}) \rho(r' \hat{n})$   
↑ probes angular variation

suppresses charge near  $\theta$ ,  
also amplifies remote charge.

$$\text{Ex: } l=0, \quad q_{00} = \frac{1}{\sqrt{4\pi}} \underbrace{\int d^3r \rho(\vec{r})}_{\text{net charge}}$$

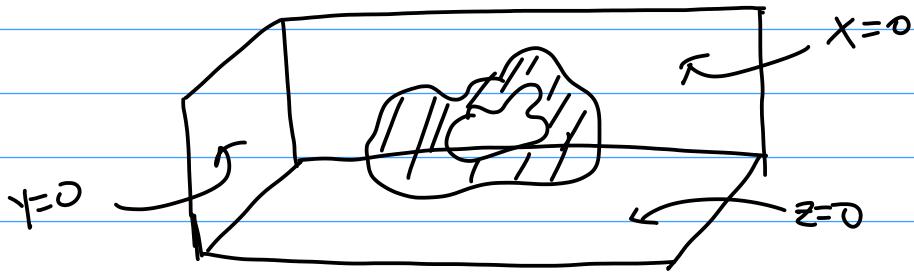
$$\underline{\text{Dipole}} \quad l=1 \quad q_{10} = \sqrt{\frac{3}{4\pi}} P_z$$

$$q_{11} = -q_{1-1}^* = -\sqrt{\frac{3}{8\pi}} (P_x - iP_y)$$

noting that  $\vec{P} = \int d^3r \vec{r} \rho(\vec{r})$

Quadrupole  $l=2$

## Examples of using Multiple - Expansion:



A spatial distribution of charge, total charge  $Q$ , is constrained within "all positive coordinates",  $x > 0, y > 0, z > 0$ .

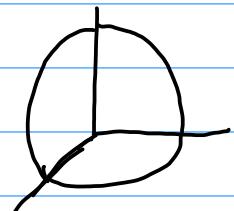
Show:

$$\int_0^\infty dy dz E^x(x, y, z) \Big|_{x=0} + \int_0^\infty dz dx E^y(x, y, z) \Big|_{y=0} + \int_0^\infty dx dy E^z(x, y, z) \Big|_{z=0} = -\frac{1}{8} \frac{Q}{\epsilon_0}$$

Method: use Gauss's Law:

$$\vec{\nabla} \cdot \vec{E} = \frac{Q}{\epsilon_0}$$

$$\text{RHS: } \frac{1}{\epsilon_0} \int d^3 r \rho(\vec{r}) = \frac{Q}{\epsilon_0}$$



$$\text{LHS: } \int d^3 r \vec{\nabla} \cdot \vec{E} = \int dS \cdot \vec{E} =$$

negative  
because  
point inside

$$-\left\{ \int_0^\infty dy dz E^x(x, y, z) \Big|_{x=0} + \int_0^\infty dz dx E^y(x, y, z) \Big|_{y=0} + \int_0^\infty dx dy E^z(x, y, z) \Big|_{z=0} \right\}$$

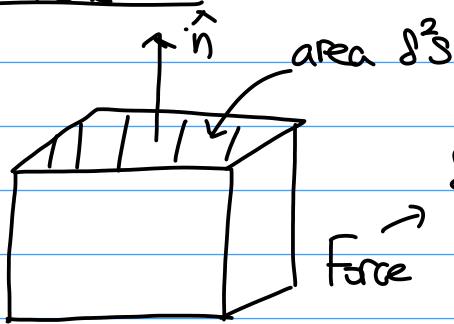
Integrate over quarter quadrant.

$\iint_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\theta d\phi d\theta \left( \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \right) \cdot \hat{r} + \phi = \frac{Q}{\epsilon_0}$  so the cartesian faces add up to  $-\frac{1}{8} \frac{Q}{\epsilon_0}$ .

$$+ \underbrace{\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\theta d\phi d\theta d\theta}_{\frac{1}{8} \frac{Q}{\epsilon_0}}$$

## Electrostatic Stress Tensor:

Consider a cube:



$$\vec{F} = (-P) \hat{n} \delta^2 S$$

↑  
pressure      ↑  
differential  
area.

Then the total force on the volume is given by:

$$\vec{F} = \int_S -P \hat{n} d\vec{S} = \int d^3S (-P)$$

or in Cartesian:

$$F_k = \int d^3S_k (-P) = \int_S dS_j (-P \delta_{jk})$$

Now introduce stress tensor:  $t_{jk} = -P \delta_{jk}$

$$\text{then we have } F_k = \int_S dS_j t_{jk}$$

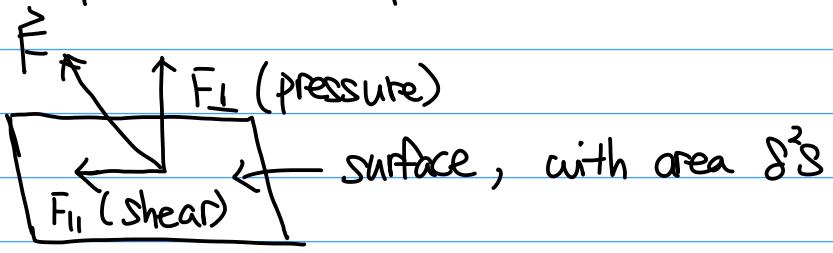
With divergence theorem:

$$F_k = \int_V d^3r \partial_j t_{jk} = \int_V d^3r (-\partial_j P \delta_{jk}) = \int_V (\partial_k P)$$

$$\text{or } \vec{F} = \int dV \underbrace{(-\nabla P)}$$

we see  $-\nabla P$  is analogous to "force density"

Stress is a generalization of pressure, developed to allow the force acting on the surface to have a parallel (shear) or normal (pressure) component:



$$\delta F_k = S^2 S \underbrace{n_j}_{\text{area element}} T_{jkl} \underbrace{n_k}_{\text{Surface Normal}}$$

Stress Tensor, like pressure  
(same as energy density)

Remarks:

$$i) \hat{n} \cdot \delta \vec{F} = n_j T_{jkl} n_k S^2 S$$

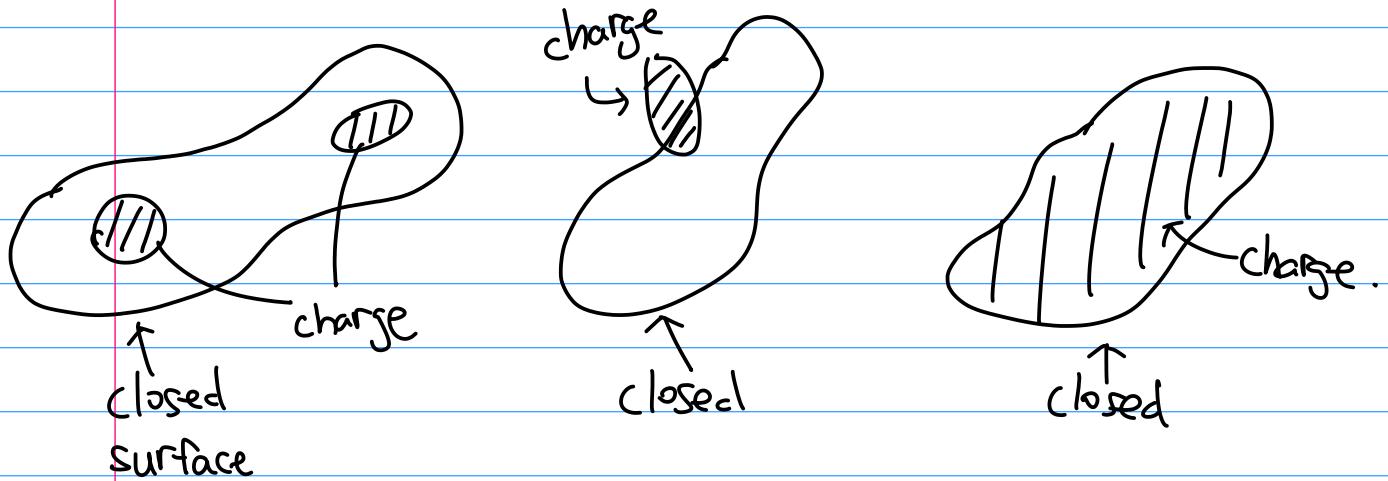
$\tau$   
component of  $\delta \vec{F}$  parallel to  $\hat{n}$

ii) The remainder:  $\hat{\delta \vec{F}} - \hat{n}(\hat{n} \cdot \hat{\delta \vec{F}})$  = components of  $\vec{F}$  that

$$\hookrightarrow [\hat{\delta \vec{F}} - \hat{n}(\hat{n} \cdot \hat{\delta \vec{F}})]_k = n_j T_{jkl} [\underbrace{\delta_{lk} - n_l n_k}_{\text{effectively shear the volume, i.e. change shape}}] S^2 S$$

but total  $V$  stays constant.

Let's introduce a closed surface that encloses some charge. The surface may coincide with the surface of a body, but it does not have to.



Let's compute the total force due to an electrostatic field on all the charge inside the surface.

$$\vec{F} = \int d^3r f(\vec{r}) E(\vec{r})$$

volume enclosed by the surface.

$$F_k = \int d^3r \underbrace{\epsilon_0 (\partial_j E_j)}_{\text{Gauss's Law}} \hat{E}_k$$

$$= \epsilon_0 \int d^3r [\partial_j (E_j E_k) - E_j \partial_j E_k]$$

Since  $\vec{\nabla} \times \vec{E} = \epsilon_{ijk} \partial_i E_j = -\partial_t B = 0$   
then  $\partial_j E_k - \partial_k E_j = 0$

$$= \epsilon_0 \int d^3r [\partial_j (E_j E_k) - E_j \partial_k E_j]$$

$$= \epsilon_0 \int d^3r [\partial_j (E_j E_k) - \frac{1}{2} \partial_k (\vec{E} \cdot \vec{E})]$$

$$= \epsilon_0 \int d^3r \partial_j [E_j E_k - \frac{1}{2} E_j^2 \delta_{jk}]$$

$F_k = \int d^2S_j T_{jk}$   
 ↑ direction of force  
 total of the electrostatic field acting on the charges  
 sum of all surfaces.  
 Momentum flux through surface.

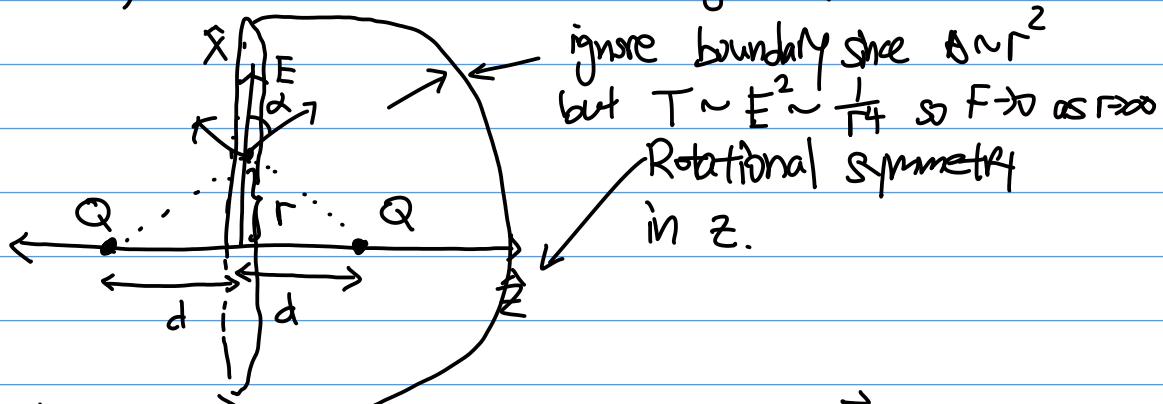
where  $T_{jk} = \epsilon_0 [E_j E_k - \frac{1}{2} E_j^2 \delta_{jk}]$

↑ electrostatic stress-tensor.

### Interpretation :

↪ The charge inside  $V$  (whether or not touching  $S$ ) may be thought of as being subject to a force that is acting through the elements of the surrounding surface. It is as if vacuum has a kind of solidness that enables it to convey stress.

Ex: use the stress-tensor to compute the force on a charge  $Q$ , distance  $2d$  from a charge  $Q$ .



Focus on the bisection plane,  $z=0$ . Compute  $\vec{E}$  on that plane, where symmetry dictates that it lies in the plane.

By coulomb's Law applied to the two charges:

$$|\vec{E}| = 2 \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2+d^2} \underbrace{\cos \alpha}_{\frac{r}{\sqrt{r^2+d^2}}}$$

$$|\vec{E}| = 2 \frac{Q}{4\pi\epsilon_0} \frac{r}{(r^2+d^2)^{3/2}}$$

→ Imagine rotating the point at which we are considering  $E$  around in the  $xy$ -plane

To the point:  $(x, y, z) = (r \cos \phi, r \sin \phi, 0)$

At that point,  $\vec{E}$  has components:

$$(E_x, E_y, E_z) = |\vec{E}| (\cos \phi, \sin \phi, 0)$$

Then the Cartesian components of the stress tensor are:

$$T_{jk} = \epsilon_0 (E_j E_k - \frac{1}{2} \delta_{jk} |\vec{E}|^2)$$

$$= \epsilon_0 |\vec{E}|^2 \left( \begin{bmatrix} CC & CS & 0 \\ SC & SS & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

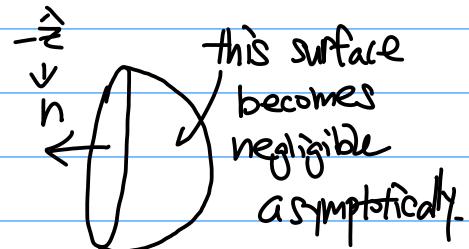
$$T_{jk} = \epsilon_0 |\vec{E}|^2 \begin{pmatrix} CC - \frac{1}{2} & CS & 0 \\ SC & SS - \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Now we can use  $T_{jk}$  to compute the force transmitted into the large hemisphere containing the right-hand charge.

By symmetry only the  $z$ -component is non-zero:  
so only find

$$F_z = \int_S d^2 S_j T_{jz}$$

↑ outward  
normal



$$= \int_{z=0} d\phi r dr \hat{n}_j T_{jz}$$

$$= -\hat{z}$$

$$= - \int_{z=0} d\phi r dr T_{33}$$

$$F_z = - \int d\phi \, r dr \left( -\frac{1}{2} \right) \epsilon_0 \underbrace{\left[ 2 \frac{Q}{4\pi\epsilon_0} \frac{r}{(r^2+d^2)^{3/2}} \right]}_{|\vec{E}|^2}^2$$

$$= \frac{Q^2}{4\pi\epsilon_0} \frac{1}{(2d)^2} \frac{2}{\pi} \int_0^\infty \frac{r dr}{d^2} \frac{r^2 d^4}{(r^2+d^2)^3} \int_0^{2\pi} d\phi$$

let  $t = r/d$

$$= \frac{Q^2}{4\pi\epsilon_0} \frac{1}{(2d)^2} 4 \int_0^\infty \frac{t^3 dt}{(1+t^2)^3}$$

$$= \frac{1}{4}$$

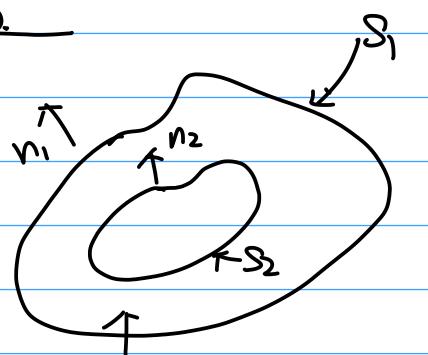
$$\boxed{F_z = \frac{Q^2}{4\pi\epsilon_0} \frac{1}{(2d)^2}}$$

$\leftarrow$  recovers coulomb Law

## Stress Transmission through the vacuum.

$$\text{Force density} = \rho F_k = -\partial_j T_{jk}$$

Suppose the force density vanishes in the region between surfaces  $S_1$  and  $S_2$ . Then the force computed using one surface equals that computed using the other.



No force density here.

$$\begin{aligned} F_k^{(1)} - F_k^{(2)} &= \int_{S_1} d^2 S_j T_{jk} - \int_{S_2} d^2 S_j T_{jk} \\ &= \int_{V_1} d^3 r \partial_j T_{jk} - \int_{V_2} d^3 r \partial_j T_{jk} \\ &\stackrel{!}{=} \int_{V_1 - V_2} d^3 r \underbrace{\partial_j T_{jk}}_{=0} = 0 \end{aligned}$$

Vanishes inside  $V_1$   
but outside  $V_2$

→ We may use any convenient surface, provided a vanishing force density (e.g. a vanishing  $\rho$ ). Answers insensitive to the choice of the surface.

## Spherical Harmonic Functions:

$$Y_{lm}(\theta, \phi) \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1-1} = +\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \propto \frac{1}{r} (x - iy)$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \propto \frac{1}{r} z$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \propto \frac{1}{r} (x + iy)$$

Orthogonality:

$$\int d\Omega Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

Completeness:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm} Y_{lm}(\theta, \phi)$$

and

$$f^R(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l D_{mm}^{(l)} f_{lm} Y_{lm}(\theta, \phi)$$

Extraction:

$$f_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y_{lm}^* f(\theta, \phi)$$

Laplace Eq in spherical coordinate:

$$\left[ \frac{1}{r^2} \partial_r r^2 \partial_r + \frac{1}{r^2 \sin\theta} \partial_\theta \sin\theta \partial_\theta + \frac{1}{r^2 \sin^2\theta} \partial_\phi^2 \right] \bar{\Phi}(r, \theta, \phi) = 0$$

$$\text{let } \bar{\Phi}(r, \theta, \phi) = R(r) Y(\theta, \phi)$$

$\stackrel{(m=0)}{\rightarrow} P_l^{(m=0)} \rightarrow \text{Legendre-Poly}$

$$Y_{lm} = \begin{cases} (-1)^m \sqrt{\frac{2l+1}{2}} \sqrt{\frac{(1-|m|)!}{(l+|m|)!}} P_l^{(m)}(\cos\theta) \frac{e^{im\phi}}{\sqrt{2\pi}} & m \geq 0 \\ 1 & m < 0 \end{cases}$$

associated Legendre.

## Vaccum Maxwell Electrodynamics

$$\left. \begin{array}{l} \vec{E} \rightarrow \vec{e} \\ \vec{B} \rightarrow \vec{b} \\ \rho \rightarrow \gamma \\ \vec{j} \rightarrow \vec{g} \\ \uparrow \quad \uparrow \end{array} \right\} \text{Still work with electrostatic}$$

No  $\vec{j}$ ,  $\vec{B}$

$$\left. \begin{array}{l} \vec{\nabla} \cdot \vec{e} = \frac{\gamma}{\epsilon_0} \rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{e} = -\partial_t \vec{b} \rightarrow \vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \end{array} \right\} \begin{array}{l} \text{Microscopic} \\ \text{Macroscopic} \end{array}$$

Coarse-grained "true" microscopic

Now focus on conductors:

\*  $\vec{E} = 0$  inside a conductor for electrostatic.  
 ↑ Macroscopic.

Constitutive Relation:

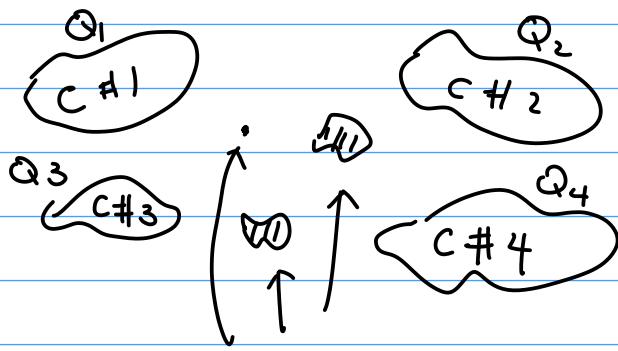
Ohm's Law:  $\vec{j}(\vec{r}) = \sigma(\vec{r}) \vec{E}(\vec{r})$  conductivity  $\leftarrow$  Local isotropic

$\vec{j}(\vec{r}) = \sigma(\vec{r}) \vec{E}(\vec{r})$   $\leftarrow$  Local anisotropic.

$\vec{j}(\vec{r}) = \int d^3 r' K(\vec{r}-\vec{r}') \vec{E}(\vec{r}')$   $\leftarrow$  nonlocal isotropic response.

$\hookrightarrow \vec{j}(q) = \hat{K}(q) \hat{E}(q)$

## Conductor Boundary Condition:



Find  $\bar{\Phi}(\vec{r})$  in the vacuum.

immobile charge.

wiggles larger than Lorentz averaging.

$$\int d\vec{r} \cdot \vec{E} = \int d\vec{S} \cdot \vec{\nabla} \times \vec{E} = 0$$

$$\left( \int_{1 \rightarrow 2} + \int_{2 \rightarrow 3} + \int_{3 \rightarrow 4} + \int_{4 \rightarrow 1} \right) d\vec{r} \cdot \vec{E} = 0$$

$= 0 \quad = 0 \quad = 0 \quad = 0$

since  $E = 0$  in conductor we can shrink  $d\vec{r} \rightarrow 0$

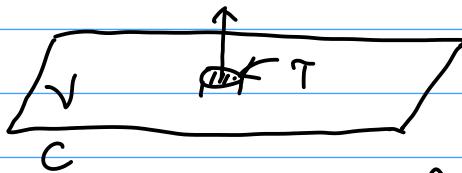
then  $\int_{3 \rightarrow 4} d\vec{r} \cdot \vec{E} = 0$

$E_{||} = 0$

→ Since  $E_{||} = 0$ , there is only  $E_{\perp}$  on the surface.

→ This leads to equipotential on the surface of conductor

Since  $-\vec{\nabla} \phi = \vec{E}$



$$\int d^2\vec{s} \cdot \vec{E} = \frac{1}{\epsilon_0} \int d^3r \rho$$

$$\nabla \cdot \vec{E}(\vec{x} + 0\hat{n}) = \frac{1}{\epsilon_0} \delta(x) \rho$$

$$\vec{E}_\perp = \frac{\rho}{\epsilon_0}$$

Conductor B.C.:

$$\textcircled{1} E_{||} = 0$$

$$\textcircled{2} E_\perp = \frac{\rho}{\epsilon_0}$$

\textcircled{3} Equi-potential on surface.

\textcircled{4}  $E = 0$  inside conductors.

\textcircled{5} charges are  $\Rightarrow$  on the surface.

Problem: a system of conductors:

$Q_1$

$Q_2$

$qQ_2$

$\rightarrow$  Find  $\vec{E}$  in vacuum.

$\rightarrow$  Find  $\sigma$  on the surface.

B.C.: In vacuum  $\nabla^2 \phi = 0 = \frac{1}{\epsilon_0}$   
 $\rightarrow$  At  $\infty$ ,  $\phi \rightarrow 0$   
 $\rightarrow E_{||} = 0$ ,  $E_\perp = \frac{\rho}{\epsilon_0}$

C

Apply Gauss:

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\hookrightarrow \int d^2\vec{s} \cdot \vec{E} = - \int d^2\vec{s} \cdot \vec{\nabla} \phi = \int \frac{1}{\epsilon_0} \rho d^3s = \frac{-\text{total charge}}{\epsilon_0}$$

$\uparrow$  relate  $E_\perp$  with  $\rho$

\* No min/max in  $\phi$  in region where  $\nabla^2 \phi = 0$ .

Now calculate  $\epsilon$ -field due to charged conductors:

$$\epsilon = \frac{\epsilon_0}{2} \int_{\text{vac}} d^3r |E|^2$$

$$= -\frac{\epsilon_0}{2} \int d^3r \vec{E} \cdot \vec{\nabla} \phi$$

$$= -\frac{\epsilon_0}{2} \int d^3r \vec{\nabla}(\vec{E} \phi) - (\vec{\nabla} \cdot \vec{E})\phi$$

$$= -\frac{\epsilon_0}{2} \int_{\text{at } \infty} d^2S \frac{\vec{E}}{r^2} \underbrace{\phi}_{\frac{1}{r^3}} - \left( \frac{\epsilon_0}{2} \sum_a \int_{S_a} d^2S \vec{E} \phi \right)$$

$\hookrightarrow$  equipotential on surface.

$$= \frac{1}{2} \epsilon_0 \sum_a \phi_a \underbrace{\int d^2S \cdot \vec{E}}_{\frac{q_a}{\epsilon_0}}$$

$$\boxed{\epsilon = \frac{1}{2} \sum_a \phi_a q_a}$$

Since  $-\nabla^2 \phi = \rho / \epsilon_0$ ,  $\phi$  and  $\psi$  are linear

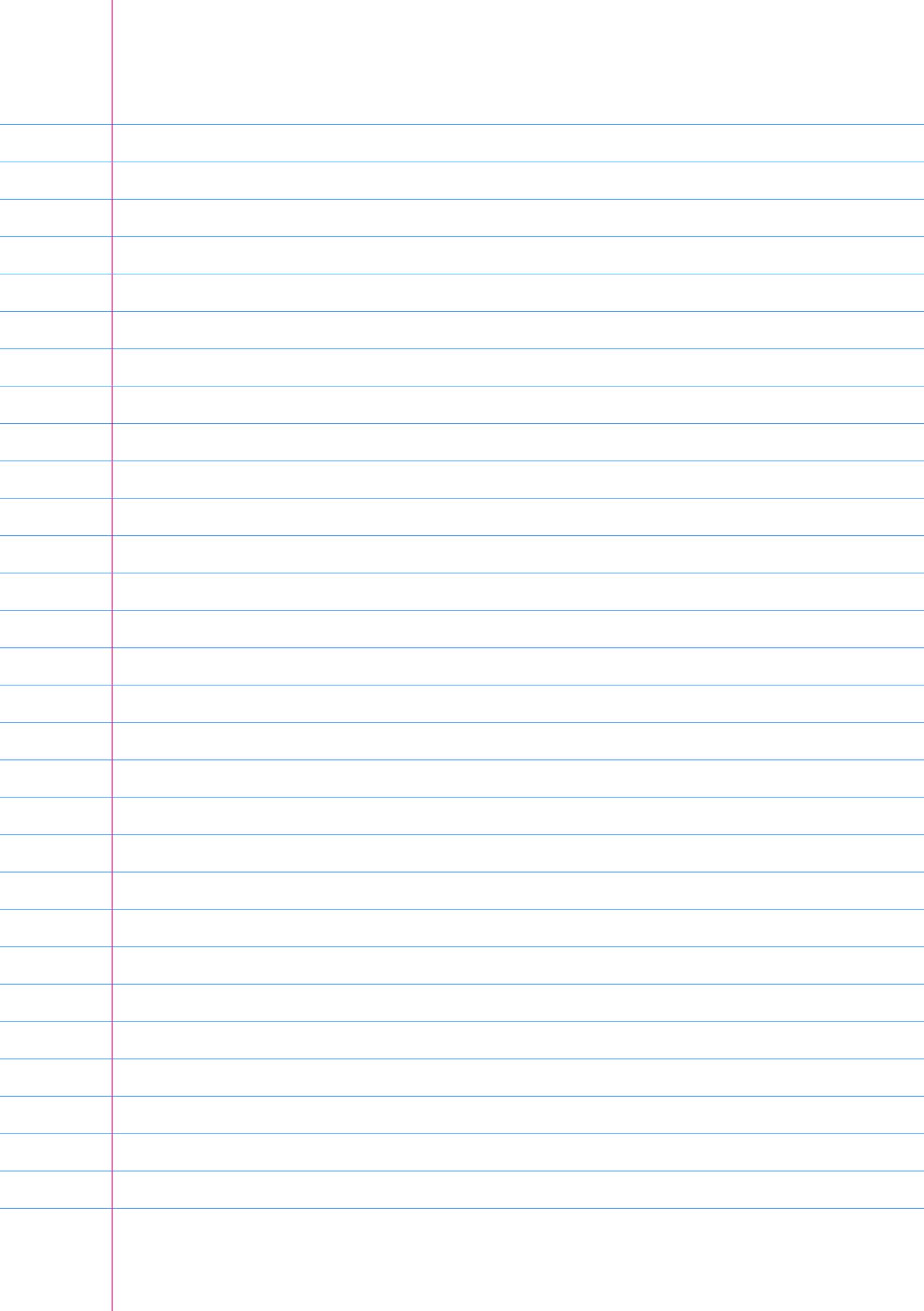
this suggests:  $\begin{cases} q_a = \sum_b C_{ab} \phi_b \\ \phi_a^{-1} = \sum_b (C^{-1})_{ab} q_b \end{cases}$

$\hookrightarrow$  coefficient of capacitance.

changing  $q$  or  $\phi$   
results changes in  
the other.

$C_{ab}$ : Diagonal : Coefficient of capacity

Off-Diag : Coeff. of electrostatic induction



What happens when we change

$$\vec{q} \rightarrow \vec{q} + \delta\vec{q}$$

$$\vec{E} \rightarrow \vec{E} + \delta\vec{E}$$

$$\epsilon \rightarrow \epsilon + \delta\epsilon.$$

$$\delta\epsilon = \epsilon_0 \int_{\text{vac}} d^3r \vec{E} \cdot \delta\vec{E}$$

Path 1: let  $\vec{E} = -\vec{\nabla}\phi$ .

$$= - \int d^3r \vec{\nabla}\phi \cdot \delta\vec{E}$$

$$= -\epsilon_0 \int_{\text{vac}} d^3r \vec{\nabla}(\phi \delta\vec{E}) - (\phi \vec{\nabla} \cdot \delta\vec{E})$$

$$= \epsilon_0 \sum_a \phi_a \int_S d^2\vec{s} \cdot \delta\vec{E}$$

$$= \epsilon_0 \sum_a \phi_a \int_V d^3r \underbrace{(\vec{\nabla} \cdot \delta\vec{E})}_{=0}$$

$$\vec{\nabla}(\vec{E} + \delta\vec{E}) - \vec{\nabla}\vec{E} = \frac{\delta P}{\epsilon_0}$$

$$\boxed{\delta\epsilon = \sum_a \phi_a \delta q_a}$$

