Essential Ingredient:

		Micro	Macro
charge	extrinsic, free, stuck	$\rho_{ext}(\vec{r})$	
	intrinsic, get polarized, or displaced	$\rho_{int}(\vec{r})$ $\rightarrow$	$\rho_p(\vec{r})$
			$\sigma_p(\vec{r})$

surface p-charge density.

Create new field: $\vec{P}(\vec{r})$, dielectric polarization

$$* \quad \boxed{\begin{aligned} -\rho_p(\vec{r}) &= \vec{\nabla} \cdot \vec{P} \\ \sigma_p(\vec{r}) &= \vec{n} \cdot \vec{P} \end{aligned}}$$

$\vec{p} = \int d^3r \, \vec{P}$

dipole moment density.

Implications: $\vec{\nabla} \times \vec{e} = 0$ $\xrightarrow{\text{Lorentz Avg}}$ $\vec{\nabla} \times \vec{E} = 0$

$\vec{\nabla} \cdot \vec{e} = \rho / \varepsilon_0$

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} (Q(\vec{r}) - \vec{\nabla} \cdot \vec{P})$

$\hookrightarrow \vec{\nabla} \cdot \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{= \vec{D}} = Q(\vec{r})$

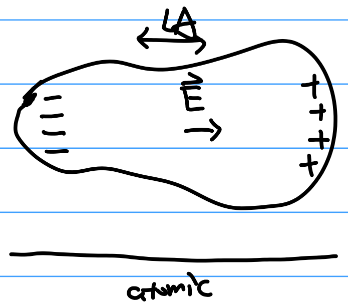
$\vec{\nabla} \cdot \vec{e} = \frac{1}{\varepsilon_0} (\rho_{int} + \rho_{ext})$, $\quad \rho_{int} \rightarrow -\vec{\nabla} \cdot \vec{P}$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

$\vec{\nabla} \cdot \vec{D} = Q(\vec{r})$

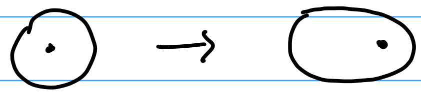
extrinsic charge density.

(Thought)
Gedanken Experiment:



i) $\vec{E}_{ext} = 0 \rightarrow \vec{p} = 0 \rightarrow \rho_p = 0 \ , \ \sigma_p = 0$

ii) $\vec{E}_{ext} = 0 \ , \ \vec{p} \neq 0 \ , \ \rho_p \neq 0 \ , \ \sigma_p \neq 0$

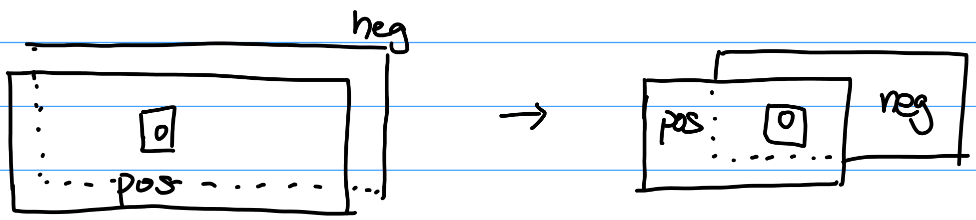


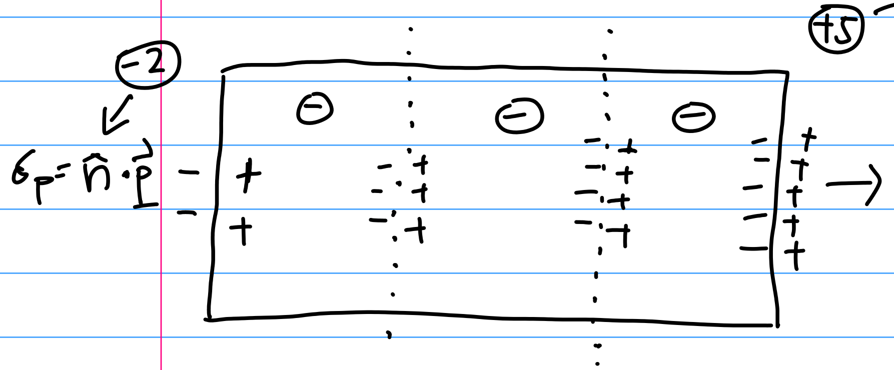
Note that: no new charge is created.

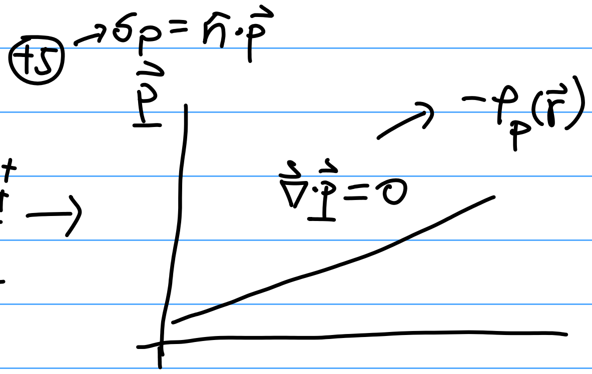
$$\int d^3r \, \rho_p + \int d^2s \, \sigma_p = -\int d^3r \, \vec{\nabla} \cdot \vec{p} + \int d^2s \, \hat{n} \cdot \vec{p}$$

$$\overset{!}{=} -\int_S d^2\vec{s} \cdot \vec{p} + \int d^2s \, \hat{n} \cdot \vec{p}$$

$$\overset{!}{=} 0$$







Energy / Length Scales : $\hbar$, m_e, e, ε_0

i) length scale: $a_B \cong \frac{1}{2}$ Å $\rightleftharpoons$ $\frac{e^2}{4\pi\varepsilon_0 a_B} = \frac{\hbar^2}{2m a_B^2}$

ii) energy scale: $\mathcal{E}_R = \frac{1}{32\pi^2} \frac{m_e 4}{\varepsilon_0^2 \hbar^2} = 2\times 10^{-18}$ J

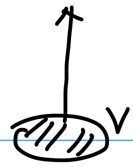
iii) Electric field scale: $\mathcal{E}_R = a_B e E$

$$E = \frac{1}{128\pi^3} \frac{m^2 e^5}{\varepsilon_0^3 \hbar^4} = 2.6\times 10^{11} \text{ V/m}$$

iv) Voltage Scale:

$E a_B \cong 13.6$ V

Problem:



$\nabla^2 \Phi = 0$ in cylindrical polar coordinate:

↳ $\left(\frac{1}{r}\partial_r r \partial_r + \cancel{\frac{1}{r^2}\partial_\theta^2} + \partial_z^2\right)\Phi(r,\cancel{\theta},z) = 0$

↳ let $\Phi(r,z) = W(r)\, Z(z)$

$$\underbrace{\frac{1}{rW}\partial_r r \partial_r W}_{\substack{\text{oscillation} \\ -\alpha^2}} + \underbrace{\frac{\partial_z^2 Z}{Z}}_{\substack{\text{decay/growth} \\ +\alpha^2}} = 0$$

↳ $\frac{1}{r}\partial_r r \partial_r W + \alpha^2 W = 0$ ↳ $Z(z) = \cancel{e^{\alpha z}}, e^{-\alpha z}$

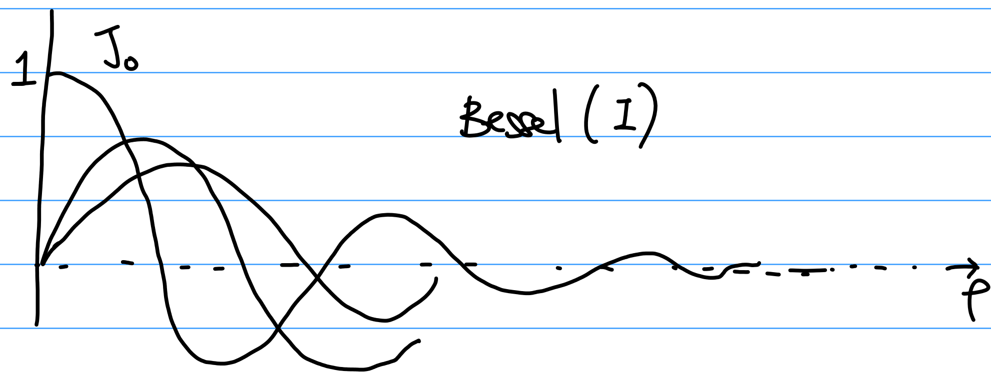
↳ $W'' + \frac{1}{r}W' + \alpha^2 W = 0$

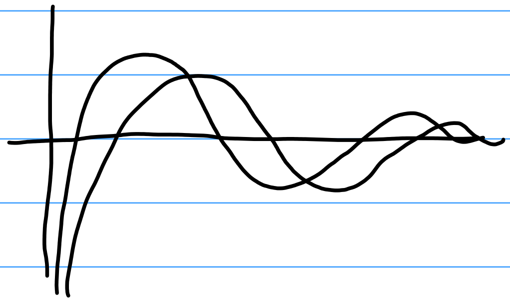
Bessel's equation

let $\left.\begin{aligned} w(r) &= S(\rho) \\ \alpha r &= \rho \end{aligned}\right\}$ $\boxed{S'' + \frac{1}{\rho}S' + \left[1 - \frac{m^2}{\rho^2}\right]S = 0}$

With solutions: $J_m(\rho) \sim \cos\rho$ (Regular) Bessel Function of first kind.

$N_{|m|}(\rho)$: Neumann, Bessel Function of second kind



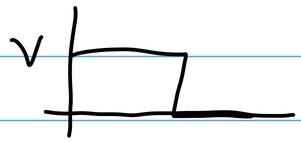


At lage x:

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$\Phi(r,z) = \int_0^\infty d\alpha \; \alpha \, A(\alpha) \, e^{-\alpha z} J_0(\alpha r)$$



$$V H(R-r) = \Phi(r, z=0) = \int_0^\infty d\alpha \; \alpha \, A(\alpha) \, J_0(\alpha r)$$

FT: $$\hat{f}(q) = \int_{-\infty}^{\infty} dx \; f(x) \, e^{-iqx}$$

$$f(x) = \int_{-\infty}^{\infty} dq \; \hat{f}(q) \, e^{iqx}$$

FBT: $$\hat{g}(q) = \int_0^\infty dx \; x \; g(x) \, J_0(qx)$$

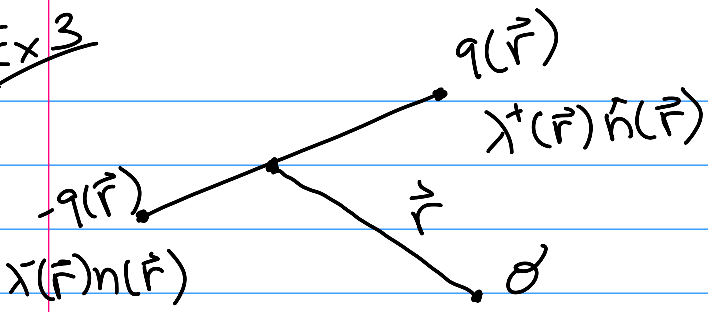
$$g(x) = \int_0^\infty dq \; q \; \hat{g}(q) \, J_0(qx)$$

$$A(\alpha) = V \int_0^R dr \; r \, J_0(\alpha r) = \frac{VR}{\alpha} J_1(\alpha R)$$

$$\boxed{\Phi(r,z) = \int_0^\infty d\alpha \; \cancel{\alpha} \, \frac{VR}{\cancel{\alpha}} J_1(\alpha R) \, e^{-\alpha z} J_0(\alpha r)}$$

on-axis, $\Phi(0,z) = VR \int_0^\infty d\alpha \; e^{-\alpha z} J_1(\alpha R) = V \left\{ 1 - \frac{z}{\sqrt{z^2+R^2}} \right\}$

Ex 3



Here, $\lambda^{\pm}$ are much shorter than lorentz averaging length scale. So they're dipoles above the averaging length scale, $\rho(\vec R)=0$

$$\rho_p(\vec R) = q(\vec R) - q(\vec R) = \int d^3r\, q(\vec r)\left[\delta(\vec R-\vec r) - \delta(\vec R-\vec r)\right]$$

$$\hookrightarrow = \int d^3r\, q(\vec r)\left[\delta(\vec R - (\vec r + \lambda^+(\vec r)\hat n(\vec r))) - \delta(\vec R - (\vec r - \lambda^-(\vec r)\hat n(\vec r)))\right]$$

$$= \int d^3r\, q(\vec r)\Big[\cancel{\delta(\vec R-\vec r)} - \lambda^+(\vec r)\hat n(\vec r)\vec\nabla_R\,\delta(\vec R-\vec r) - \cancel{\delta(\vec R-\vec r)} - \lambda^-(\vec r)\hat n(\vec r)\vec\nabla_R\,\delta(\vec R-\vec r)\Big)$$

$\delta(x)=\delta(-x)$

$$= -\vec\nabla_R\cdot\int d^3r\, \underbrace{\{q(\vec r)[\lambda^+(\vec r)+\lambda^-(\vec r)]\hat n(\vec r)\}}_{\vec p(\vec r)}\, \delta(\vec r-\vec R)$$

$$= -\vec\nabla_R\cdot\vec P(\vec r)$$

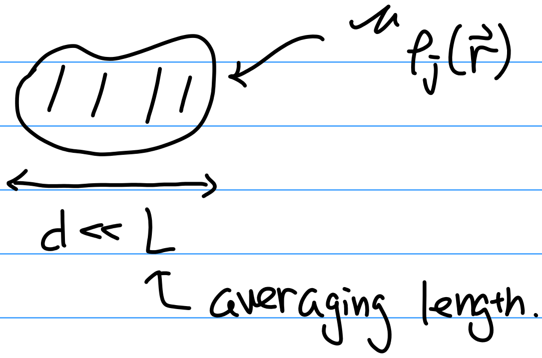
Macroscopic dipole-moment per unit volume.

Ex 4: Coarse-graining the charge density:

Assume we have microscopic localized charge distribution:

$$\{{}^{\mu}\rho_j(\vec r)\} \qquad (j=1,2,3\ldots)$$

means microscopic

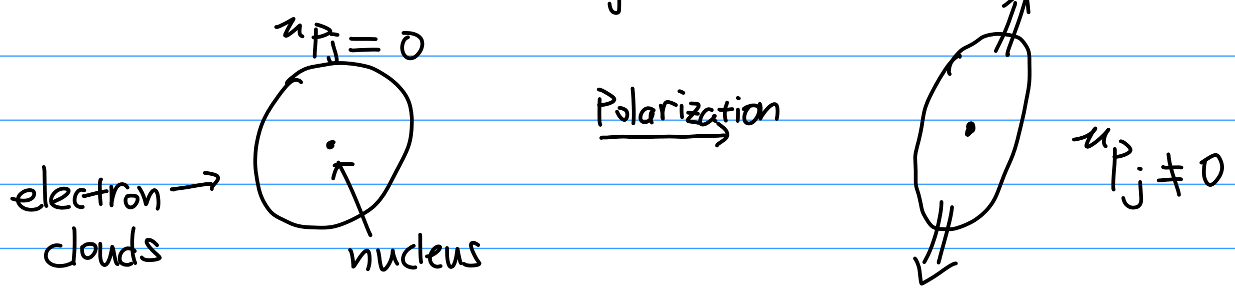


Each j describe an atom, molecule or charged ion.

The total charge from nucleus and atoms is q_j:

$$q_j \equiv \int d^3r \; {}^{\mu}\rho_j(\vec{r})$$

Now consider case $q_j = 0$:



How does such atom contribute to the average charge density?

$$\overline{{}^{\mu}\rho_j} = \int d^3r' \; {}^{\mu}\rho_j(\vec{r}\,') \, f(\vec{r} - \vec{r}\,')$$

averaged

microscopic, localized near $\vec{r}_j$, scale d

Normalized Lorentz Averaging function, scale d.

Since the weight is concentrated around $\vec{r}\,' = \vec{r}_j$, so we expand f around $r' = \vec{r}_j$:

$$f(\vec{r} - \vec{r}\,') = f\Big((\vec{r} - \vec{r}_j) + (\vec{r}_j - \vec{r}\,') \Big)$$

$$\approx f(\vec{r} - \vec{r}_j) + \underbrace{\underbrace{(\vec{r}_j - \vec{r}\,')}_{\mathcal{O}(d)} \cdot \underbrace{\vec{\nabla}_r f(\vec{r} - \vec{r}_j)}_{\mathcal{O}(1/L)}}_{\sim \mathcal{O}(d/L)} + \cdots$$

Then $\overline{{}^{u}\rho_j(\vec{r})} \approx \int d^3r' \, {}^{u}\rho_j(\vec{r}') \left[f(\vec{r}-\vec{r}_j) + (\vec{r}_j - \vec{r}') \cdot \vec{\nabla}_r f(\vec{r}-\vec{r}_j) \right]$

$$= \underbrace{\int d^3r' \, {}^{u}\rho_j(\vec{r}')}_{q_j} f(\vec{r}-\vec{r}_j)$$

$$+ \underbrace{\int d^3r' \, {}^{u}\rho_j(\vec{r}')(\vec{r}_j - \vec{r}')}_{-\vec{p}_j} \cdot \vec{\nabla}_r f(\vec{r}-\vec{r}_j)$$

$$\boxed{\overline{{}^{u}\rho_j(\vec{r})} = \sum_j q_j f(\vec{r}-\vec{r}_j) - \vec{p}_j \cdot \vec{\nabla}_r f(\vec{r}-\vec{r}_j)}$$

charge on site.   electric dipole moment about $\vec{r}_j$

Now let's introduce: microscopic variables:

1) $${}^{u}q(\vec{r}) \equiv \sum_j q_j \delta(\vec{r}-\vec{r}_j) \xrightarrow{\text{average}} Q(\vec{r})$$

2) $${}^{u}p(\vec{r}) \equiv \sum_j p_j \delta(\vec{r}-\vec{r}_j) \xrightarrow{\text{average}} \vec{P}(\vec{r})$$

Then $${}^{u}\rho(\vec{r}) = \int d^3r' \, \rho(\vec{r}') f(\vec{r}-\vec{r}')$$

$$\approx \overline{{}^{u}\rho_{ext}}(\vec{r}) - \vec{\nabla} \cdot {}^{u}\vec{p}(\vec{r})$$

$$\boxed{\overline{{}^{u}\rho(\vec{r})} \doteq Q(\vec{r}) - \underbrace{\vec{\nabla} \cdot \vec{P}(\vec{r})}}$$

dipole moment density.

extrinsic charge density   intrinsic charge density

Microscopic polarization density

$$\int d^3r \, r_\mu \rho_p(\vec{r}) = -\int d^3r \, r_\mu (\partial_\nu P_\nu)$$

$$\overset{!}{=} -\int_V d^3r \left[\partial_\nu (r_\mu P_\nu) - P_\nu \overbrace{\partial_\nu r_\mu}^{\delta_{\mu\nu}}\right]$$

$$\overset{!}{=} -\int_S d^2S_\nu \, P_\nu r_\mu + \int_V d^3r \, P_\mu(\vec{r})$$

$$\overset{!}{=} \underbrace{-\int_S d^2S (\hat{n}\cdot\vec{P}) r_\mu}_{\text{surface. term}} + \underbrace{\int_V d^3r \, P_\mu(\vec{r})}_{\text{Bulk Term.}}$$

Boundary between two dielectric $(Q=0)$:

i) $\vec{\nabla} \times \vec{E} = 0 \rightarrow \oint d\vec{L} \cdot \vec{E} = 0$

$$\boxed{E_{\parallel}^{I} = E_{\parallel}^{II}}$$

Tangential component of $\vec{E}$ is continuous across the surface.

ii) $\vec{\nabla} \cdot \vec{D} = 0 \rightarrow \int_S d^2\vec{S} \cdot \vec{D} = 0$

$$\boxed{D_{\perp}^{I} = D_{\perp}^{II}}$$

$$\boxed{\Phi_I = \Phi_{II}}$$

The normal component of $\underline{\vec{D}}$ is continuous across the interface. So, $\underline{\underline{E_n}}$ is discontinuous, due to surface polarization charge.

→ Dielectric - Conductor interface:

Dielectric	Metal
$\vec{E} = ?$ $\vec{P} = ?$ } can be nonzero	$\vec{E} = 0$ $\vec{P} = 0$

$$\boxed{\begin{array}{l} E_{\parallel}^{\text{Dielectric}} = 0 \\ D_{\perp}^{\text{Dielectric}} = \sigma_{\text{cond}} \end{array}}$$

How does $\vec{E}$ determine $\vec{p}$?

$$\boxed{\begin{matrix} D(\vec{r}) = \underset{\substack{\uparrow \\ \text{Dielectric Permitivity}}}{\varepsilon(\vec{r})} E(\vec{r}) \end{matrix}}$$

← Linear, isotropic and local.

← How medium adjust volume.

→ If Linear, local, but anisotropic, e.g. due to crystalline structure:

$$D_u(\vec{r}) = \varepsilon_{uv}(\vec{r})\, E_v(\vec{r})$$

→ If Linear, isotropic, but nonlocal:

$$\vec{D}(\vec{r}) = \int d^3r\ \varepsilon(\vec{r}-\vec{r}')\, \vec{E}(\vec{r}')$$

↑ a convolution.

For $\vec{P}$, it means

→ $$\boxed{\vec{P} = \varepsilon\vec{E} - \varepsilon_0\vec{E} = (\varepsilon-\varepsilon_0)\vec{E} = K\vec{E}}$$

Homogeneous Dielectric (No free charge)

$\rightarrow$ $\varepsilon(\vec{r}) \rightarrow \varepsilon$ $\leftarrow$ ε free of position.

$$\vec{\nabla}\cdot\vec{D}=0 \quad \rightarrow \quad \vec{\nabla}\cdot\varepsilon\vec{E}=0$$

Boundary condition and $\sigma_P \neq 0$

$$\left.\begin{aligned} -\vec{\nabla}\cdot\vec{P} &= \rho_P = 0 \\ \vec{P}\cdot\hat{n} &= \sigma_P \neq 0 \end{aligned}\right\}$$

If homogeneous $\varepsilon(\vec{r}) = \varepsilon$, then we have no bulk term, $\rho_P = 0$ but only $\underline{\sigma_P} \neq 0$.

For Inhomogeneous dielectric:

How does $\varepsilon(\vec{r})$ imply $\rho_P(\vec{r})$

$$\rho_P = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}(\vec{D}-\varepsilon_0\vec{E})$$

$$\stackrel{!}{=} -\vec{\nabla}\cdot\left(-\frac{\varepsilon_0}{\varepsilon}\vec{E}\right)$$

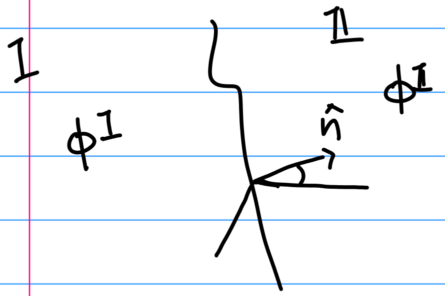
$$\stackrel{!}{=} -\left(\vec{E}\cdot\vec{\nabla}\varepsilon(\vec{r})\right)\frac{\varepsilon_0}{\varepsilon}$$

Generalization to Laplace Eq:

$$\vec{\nabla}\times\vec{E}=0 \quad \rightarrow \quad \vec{E} = -\vec{\nabla}\phi$$

$\hookrightarrow$ $\vec{\nabla}\cdot\vec{D}=0 \rightarrow \vec{\nabla}\cdot(\varepsilon\vec{E}) \rightarrow \boxed{\vec{\nabla}\cdot\left(\varepsilon(\vec{r})\vec{\nabla}\phi\right)=0}$

Two Dielectric Interface:



$$D_\perp^{I} = D_\perp^{II}$$

$$\varepsilon^{I} E_\perp^{I} = \varepsilon^{II} E_\perp^{II}$$

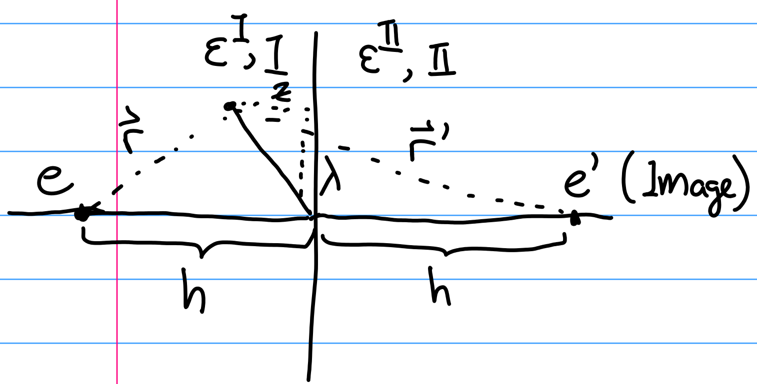
$$\boxed{\begin{aligned} \varepsilon^{I}\partial_n \phi^{I} &= \varepsilon^{II}\partial_n \phi^{II} \\ \phi^{I} &= \phi^{II} \end{aligned}}$$ ← Boundary Condition.

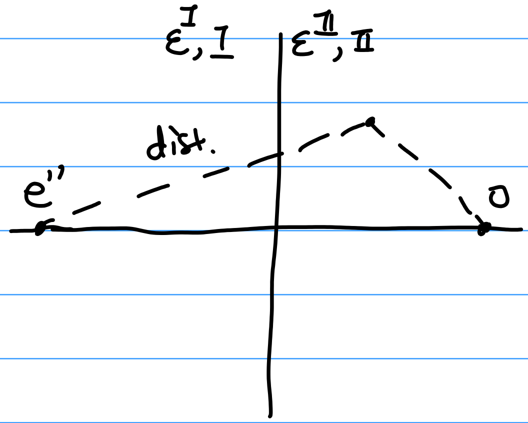
Strategy for Boundary-Value Problem for linear, isotropic, dielectric

$$\phi \to \vec{E},\ \vec{D},\ \vec{P},\ \rho_P,\ \sigma_P$$

Example Problem:

- Determine electrostatic potential due to a point charge a distance h from the plane boundary separating two homogeneous dielectric media:





when solving Φ_I when solving Φ_{II}

Field in I, $\Phi^{I}(\vec{r}) = \frac{1}{4\pi}\left[\frac{e}{\varepsilon_I r} + \frac{e'}{\varepsilon_I r'}\right]$

Field in II, $\Phi^{I}(\vec{r}) = \frac{1}{4\pi}\left[\frac{e''}{\varepsilon^{II} dist} + 0\right]$

Boundary Condition: $\Phi^{I}(\text{Boundary}) = \Phi^{II}(\text{Boundary})$

$$\hookrightarrow \frac{e}{\varepsilon^{I}} + \frac{e'}{\varepsilon^{I}} = \frac{e''}{\varepsilon^{II}} \qquad ①$$

let $\frac{1}{r} \to \frac{1}{\sqrt{\lambda^2 + (h-z)^2}}$, $\frac{1}{r'} \to \frac{1}{\sqrt{\lambda^2 + (h+z)^2}}$

then by BC. $\varepsilon^{I} \frac{\partial}{\partial n} \Phi^{I} = \varepsilon^{II} \frac{\partial}{\partial n} \Phi^{II}$

$$\hookrightarrow \varepsilon^{I}\left(\frac{e}{\varepsilon^{I}} - \frac{e'}{\varepsilon^{I}}\right) = \varepsilon^{II}\left(\frac{e''}{\varepsilon^{II}} + 0\right) \qquad ②$$

using ① and ②:

$$\hookrightarrow \quad e' = e \, \frac{\varepsilon^{I} - \varepsilon^{II}}{\varepsilon^{I} + \varepsilon^{II}}$$

$$e'' = e \, \frac{2\varepsilon^{II}}{\varepsilon^{I} + \varepsilon^{II}}$$

$$\hookrightarrow \quad F = \frac{e^2}{4\pi\varepsilon^{I}} \frac{1}{(2h)^2} \left(\frac{\varepsilon^{I} - \varepsilon^{II}}{\varepsilon^{I} + \varepsilon^{II}}\right)$$