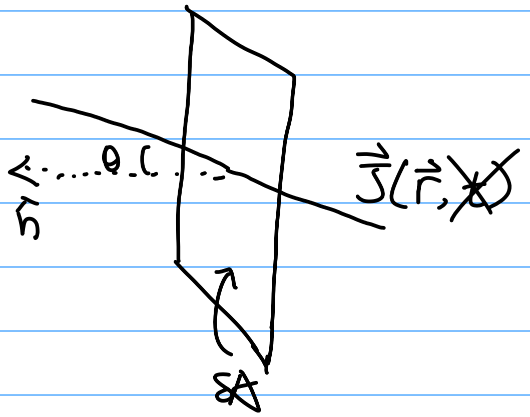
Steady electric current:

$$\vec{J} = \sigma \vec{E}$$ ← Ohm's Law.

current density ↗ $\vec{J}$ $\qquad$ ↑ σ electrical conductivity.



Charge Through Window:

$$\delta q = \delta A \, \delta t \, (\vec{J} \cdot \hat{n})$$

$$[\vec{J}] = C s^{-1} m^{-2}$$

Microscopically: $\vec{J}(\vec{r}) = \sum_{n=1}^{N} q_n \dot{\vec{r}}(t) \delta(\vec{r} - \vec{r}_n(t))$

Implication of <u>steady</u> and charge conservation:

↳ $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ (continuity Equation)

static case

$$\boxed{\begin{array}{l} \text{in steady condition: } \frac{\partial \rho}{\partial t} = 0 \\ \hookrightarrow \quad \vec{\nabla} \cdot \vec{J} = 0 \quad \text{amount flowing in} = \text{amount flowing out} \end{array}}$$

→ Now allow current flow in conductor: so there is $\vec{E}$ in conductor.

Implication for Electrodynamics:

$$\vec\nabla \times \vec E = -\cancel{\partial_t \vec B} = 0$$ ← same as electrostatic.

then we allow to choose a potential.

$$\vec E = -\vec\nabla \phi$$

Summary: for static.

$\vec\nabla \times \vec E = 0$

$\vec\nabla \cdot \vec J = 0 \longrightarrow \vec\nabla \cdot (\sigma \vec E) = 0$

Similarly

$\vec\nabla \times \vec E = 0$

$\vec\nabla \cdot \vec D = \vec\nabla \cdot (\varepsilon \vec E) = \rho$

Estimation of σ:

$$\underset{\substack{\uparrow \\ \# e^- \\ \text{per } V}}{n}\;\;\sigma = n \frac{e^2}{m} \tau$$ ← velocity relaxation time

$\vec J = \sigma \vec E$ → assume linearity

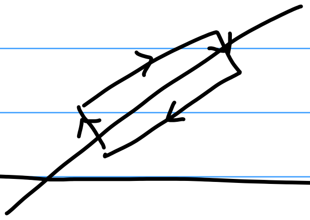
→ assume isotropy. $J_\mu = \sigma_{\mu\nu} E_\nu$ (anisotropy)

→ inhomogeneity: $\sigma(\vec r)$

→ locality: $\vec J(\vec r) = \int d^3\vec r' \, \sigma(\vec r - \vec r') E(\vec r')$

If homogeneous $\vec\nabla \cdot \vec J = \vec\nabla \cdot (\sigma \vec E) = \sigma \vec\nabla \cdot \vec E = 0.$

Boundary Condition:



cond / cond.

i) $\vec{\nabla} \times \vec{E} = 0$

↳ $\oint d\vec{L} \cdot \vec{E} = 0$ → $\vec{E}_{\parallel}$ is continuous → E is not necessarily 0 when there is current.

→ ϕ is continuous

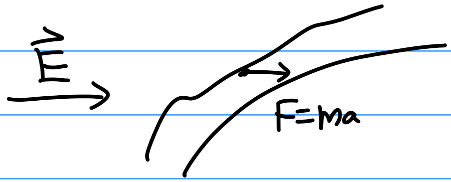
ii) $\vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot (\sigma \vec{E}) = 0$

↳ $\oint d^2\vec{s} \cdot \sigma \vec{E} = 0$ → $\sigma E_{\perp}$ is continuous

→ $\sigma \hat{n} \cdot \vec{\nabla}\phi$ is continuous

Conductor carrying steady Current	Polarized Dielectric.
$\vec{\nabla} \times \vec{E} = 0 \rightarrow \vec{E} = -\vec{\nabla}\phi$	$\vec{\nabla} \times \vec{E} = 0 \rightarrow \vec{E} = -\vec{\nabla}\phi$
$\vec{\nabla} \cdot (\sigma \vec{E}) = S$ ← source of current	$\vec{\nabla} \cdot (\varepsilon \vec{E}) = \rho_f$
BC: $E_{\parallel}$, $\sigma E_{\perp}$ continuous	BC: $E_{\parallel}$, $\varepsilon E_{\perp}$ continuous.

Joule heating:



$$\delta W = \underbrace{J\,\delta t}_{\text{displacement}} \cdot E\,\delta V$$

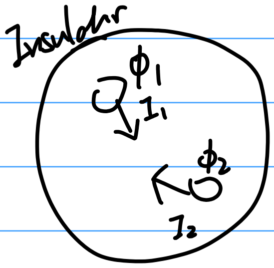
must be positive.

$$\frac{dQ}{dt} = T\frac{dS}{dt} = \boxed{J \cdot E = \sigma|E|^2}$$ ← Joule Heating per volume.

↑ rate of heating per volume per time.

$$\boxed{\text{Joule Heating} = \int d^3r\, \vec{J} \cdot \vec{E}}$$

Joule Heating : continuum → lumped.



$$\int d^3r\, \vec{J}\cdot\vec{E} = -\int d^3r\, J\cdot\vec{\nabla}\phi$$

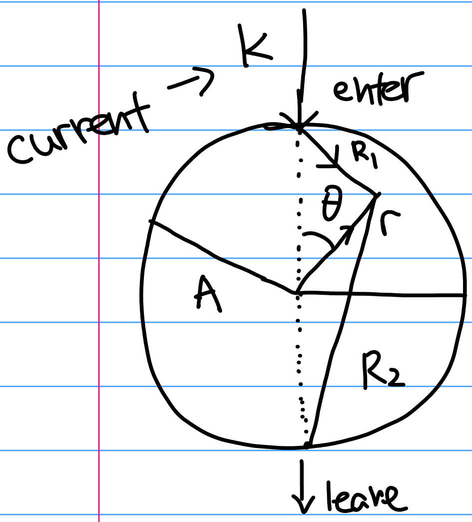
$$= -\int d^3r\, \vec{\nabla}\cdot(J\phi) + \int d^3r\, \phi\, \overset{0}{\cancel{\vec{\nabla}\cdot\vec{J}}}$$

$$= -\int d^2\vec{S}\cdot\phi\vec{J}$$

$$= \phi - \int d^2\vec{S}\cdot\vec{J}$$

$$= \sum_a I_a \phi_a$$

Example: Determine the potential and current distribution in a conducting sphere, with current K entering at the North-Pole and leaving at the South Pole. Determine Joule heating rate.



properties:

- → $\nabla^2\phi = 0$ inside conductor
- → No flow through boundary except North and South Pole.
- → Azimuthal symmetry, depends on r, θ.

Near the poles, we have a <u>source</u> and a <u>sink</u>,

North Pole: $\phi_N \approx + \frac{K}{2\pi\sigma}\frac{1}{R_1}$ + small correction (for small R_1)

South Pole $\phi_S \approx - \frac{K}{2\pi\sigma}\frac{1}{R_2}$ + small correction (for small R_2)

At North Pole : $\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\sigma\vec{E} = \frac{K}{V}$

$$\oint d^2\vec{S}\cdot\sigma\vec{E} = \sigma\oint d^2\vec{S}\cdot -\vec{\nabla}\phi_N = K$$

$$= \sigma\oint_{\text{Hemisphere.}} d^2\vec{S}\cdot\frac{K}{2\pi\sigma}\frac{1}{R_1^2}$$

$$\overset{!}{=} \sigma\ \underset{\uparrow}{2\pi R_1^2}\cdot\frac{K}{2\pi\sigma}\frac{1}{R_1^2}$$

$$\overset{!}{=} K$$

↑ we get back K.

Then write solution as:

$$\phi(r,\theta)= \frac{k}{2\pi\sigma}\Big(\underbrace{\frac{1}{R_1} - \frac{1}{R_2}}_{\text{source terms}} + \underset{\text{correction term.}}{\Psi}\Big)$$

→ In the bulk, we have $\nabla^2\phi = 0$, so $\nabla^2\Psi = 0$

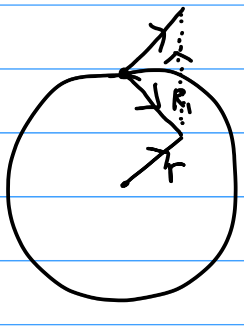
→ At boundary $r = A$: $E_\perp = 0$ except a $\theta = 0, \pi$.
where there are current entering and leaving,

Then: $$\left.\frac{\partial\phi}{\partial n}\right|_{r=A} = \frac{k}{2\pi\sigma}\left.\frac{\partial}{\partial r}\right|_{r=A}\left\{\frac{1}{R_1} - \frac{1}{R_2} + \Psi\right\} = 0$$

↳ $$\left.\frac{\partial\Psi}{\partial r}\right|_{r=A} = \left\{\frac{1}{R_1^2}\frac{\partial R_1}{\partial r} - \frac{1}{R_2^2}\frac{\partial R_2}{\partial r}\right\}$$

→ we recognize that $\vec{r} = A\hat{z} + \vec{R}_1$

$\vec{r} = -A\hat{z} + \vec{R}_2$



So $\vec{R}_1 = \vec{r} - A\hat{z}$, $R_1^2 = r^2 - 2Ar\cos\theta + A^2$

$$\partial_r R_1^2 = 2R_1\partial_r R_1 = 2r - 2A\cos\theta$$

then $$\partial_r \frac{1}{R_1} = -\frac{1}{R_1^2}\partial_r R_1 = -\frac{r - A\cos\theta}{R_1^3} \xrightarrow{r=A} -\frac{A}{R_1^3}(1-\cos\theta)$$

Similarly $$\partial_r \frac{1}{R_2} = -\frac{A}{R_2^3}(1+\cos\theta)$$

Therefore: $$\left.\frac{\partial \Psi}{\partial r}\right|_{r=A} = \left\{ \frac{1}{R_1^2}\frac{\partial R_1}{\partial r} - \frac{1}{R_2^2}\frac{\partial R_2}{\partial r} \right\}\Bigg|_{r=A}$$

$$\hookrightarrow \left.\frac{\partial \Psi}{\partial r}\right|_{r=A} = \frac{A}{R_1^3}\underbrace{(1-\cos\theta)}_{R_1^2/2A^2} - \frac{A}{R_2^3}\underbrace{(1+\cos\theta)}_{R_2^2/2A^2}$$

$$\boxed{\left.\frac{\partial \Psi}{\partial r}\right|_{r=A} = \frac{1}{2A}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\Bigg|_{r=A}}$$

Now, the general solution of $\nabla^2\phi = 0$ in spherical coordinate:

$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\alpha_{lm} r^{l} + \beta_{lm} r^{-l-1}\right) Y_l^m(\theta,\phi)$$

with azimuthal symmetry: $\boxed{\underline{\underline{m=0}}}$

$$\phi = \sum_{l=0}^{\infty}\left(\alpha_l r^{l} + \beta_l r^{-l-1}\right) P_l(\cos\theta)$$

→ Since we want regularity at $r=0$, so $\boxed{\beta_l = 0}$ for all l.

→ we observe $l=0$ gives a constant solution, which is arbitrary since we have Neumann Boundary Condition, so set $\boxed{\alpha_0 = 0}$

→ Now a special trick:

→ If $f(r,\theta)$ is a solution to Laplace's Equation, then

$$\int_0^r \frac{d\rho}{\rho} f(\rho,\theta) \text{ is also a solution.}$$

<u>prof</u>:

$$\int_0^r \frac{d\rho}{\rho} \underbrace{\sum_{l=1}^{\infty} \gamma_l \rho^l P_l(\cos\theta)}_{\text{a solution}}$$

$$\hookrightarrow = \sum_{l=1}^{\infty} \gamma_l \frac{1}{l} \rho^l P_l(\cos\theta)$$

↑ same solution but with a multiplicative constant.

Now suppose: $\Phi(r,\theta) = \int_0^r \frac{d\rho}{\rho} \tilde{\Phi}(\rho,\theta)$ ← takes this form

then $$\frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}\left(\int_0^r \frac{d\rho}{\rho} \tilde{\Phi}(\rho,\theta)\right) = \frac{1}{r}\tilde{\Phi}(r,\theta)$$

↖ fundamental theorem of calculus

So $$\left.\frac{\partial \Phi}{\partial r}\right|_{r=A} = \left.\frac{1}{2A}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right|_{r=A}$$

$$\hookrightarrow \cancel{\frac{1}{A}} \tilde{\Phi}(r=A,\theta) = \frac{1}{2\cancel{A}}\left(\frac{1}{R_1} - \frac{1}{R_2}\right|_{r=A}$$

but we note that $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is harmonic, i.e. applying ∇^2, it gives zero.

So by uniqueness: $$\boxed{\tilde{\Phi}(r,\theta) = \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

Now we need to evaluate $\mathcal{I}$:

$$\mathcal{I} = \int_0^r \frac{d\rho}{\rho}\left\{\frac{1}{\sqrt{\rho^2-2A\rho\cos\theta+A^2}} - \frac{1}{\sqrt{\rho^2+2A\rho\cos\theta+A^2}}\right\}\frac{1}{2}$$

$$= -\frac{1}{2a}\left\{\sinh^{-1}\left(\frac{A-r\cos\theta}{r\sin\theta}\right) - \sinh^{-1}\left(\frac{A+r\cos\theta}{r\sin\theta}\right)\right\}$$

Putting terms together:

$$\boxed{\begin{aligned}\phi(r,\theta) = \frac{K}{2\pi\sigma}\Bigg\{&\frac{1}{R_1} - \frac{1}{R_2} \\ &- \frac{1}{2a}\left(\sinh^{-1}\left(\frac{A-r\cos\theta}{r\sin\theta}\right) - \sinh^{-1}\left(\frac{A+r\cos\theta}{r\sin\theta}\right)\right)\Bigg\}\end{aligned}}$$

Then we can find $\vec{E}$ and $\vec{J}$, then

Joule Heating Rate $= \vec{E}\cdot\vec{J}$

Magnetostatics:

Focus on $\vec{J}$ and $\vec{A}$

× $\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$

✓ $\vec{B} = \vec{\nabla} \times \vec{A}$

↖ ↑

Time Independent.

$\vec{\nabla} \cdot \vec{E} = \cancel{\frac{1}{\varepsilon_0}\rho}$	$\vec{\nabla} \cdot \vec{B} = 0$
$\vec{\nabla} \times \vec{B} = \cancel{\frac{1}{c^2}\partial_t \vec{E}} + \mu_0 \vec{J}$	$\vec{\nabla} \times \vec{E} = \cancel{-\partial_t \vec{B}}$

Electrostatic vs. Magnetostatic

→ $\vec{\nabla} \times \vec{E} = 0$

→ $\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi \underset{\uparrow}{=} \frac{1}{\varepsilon_0}\rho$

Poisson

$$\boxed{\begin{aligned} &\rightarrow \vec{\nabla} \cdot \vec{B} = 0 \\ &\rightarrow \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \end{aligned}}$$

↖ Equation of Magnetostatic.

Charge Conservation and Gauge Invariant:

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B} = \mu_0 \vec{J})$

↳ $0 = \vec{\nabla} \cdot \vec{J} = -\partial_t \rho$

Vector Calculus:

If in Cartesian Coordinate:

$$\boxed{\vec{\nabla}\times\vec{\nabla}\times\vec{C} = \vec{\nabla}(\vec{\nabla}\cdot\vec{C}) - \nabla^2\vec{C}}$$

$$\partial_\nu(\vec{\nabla}\cdot\vec{A}) - \nabla^2 A_\nu = \mu_0 J_\nu$$ ← In Cartesian coordinate.

Gauge Invariant: $A \to A' = \vec{A} + \vec{\nabla}\chi(r,t)$

$\phi \to \phi' = \phi - \partial_t \chi\ (r,t)$

To maintain static: $\chi(r,t) = \alpha(r) + \cancel{\beta(t)}$ constant.

Choose $\alpha(\vec{r})$ so that $\vec{\nabla}\cdot\vec{A}' = 0$

$$= \vec{\nabla}\cdot\vec{A} + \nabla^2\alpha = 0$$

↳ $\boxed{\nabla^2\alpha = -\vec{\nabla}\cdot\vec{A}}$ choose $\alpha(\vec{r})$ that satisfy this.

The Gauge Invariant that A' such that $\boxed{\vec{\nabla}\cdot\vec{A}' = 0}$

↑ Culomb Gauge.

*

Then

$$\boxed{\begin{array}{l}\vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}'}}_{=0}) - \nabla^2\vec{A}' = \mu_0 J \\ \vec{\nabla}\cdot\vec{A}' = 0 \quad \leftarrow \text{Culomb gauge condition.}\end{array}}$$

Compared to electrostatic:

$$-\nabla^2\phi(\vec{r}) = \frac{1}{\varepsilon_0}\rho(\vec{r}) \longrightarrow \phi(\vec{r}) = \int d^3r' \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}\rho(\vec{r}')$$

For Magnetostatic

$$-\nabla^2\vec{A} = \mu_0\vec{J} \rightarrow \boxed{\begin{gathered}\vec{A}(\vec{r}) = \int d^3r' \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}\vec{J}(\vec{r}') \\ \uparrow \text{Cartesian and Coulomb gauge}\end{gathered}}$$

Check whether condition give $\vec{\nabla}\cdot\vec{A} = 0$

$$A_\mu(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}J_\mu(\vec{r}')$$

$$\vec{\nabla}\cdot\vec{A} = \partial_\mu A_\mu = \frac{\mu_0}{4\pi}\int d^3r'\,\partial_\mu\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)J_\mu(\vec{r}')$$

$$\partial_\mu\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\partial_{r'_\mu}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) \hookrightarrow = -\frac{\mu_0}{4\pi}\int d^3r'\,\partial_{r'_\mu}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)J_\mu(\vec{r}')$$

$$= -\frac{\mu_0}{4\pi}\int d^3r'\left\{\frac{1}{|\vec{r}-\vec{r}'|}J_\mu(\vec{r}')\right\} - \frac{1}{|\vec{r}-\vec{r}'|}\underbrace{\nabla'\cdot J(\vec{r}')}_{=0}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = -\frac{\mu_0}{4\pi}\oint d^2\vec{S}'\frac{1}{|\vec{r}-\vec{r}'|}J_\mu(\vec{r}') \rightarrow 0}$$

Multi-Pole Expansion:

Electrostatic

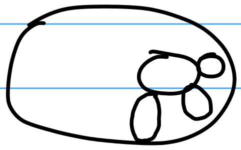
$$\phi(\vec r) \cong \frac{1}{4\pi\varepsilon_0}\left\{ \frac{q}{r} + \frac{\vec p \cdot \hat r}{r^2} + \mathcal{O} \right\} \quad \begin{array}{l} \nearrow q = \int d^3r\ \rho \\ \rightarrow \vec p = \int d^3r\ \vec r\, \rho \end{array}$$

Magnetostatic

$$\boxed{\begin{array}{l} A(\vec r) \cong \frac{\mu_0}{4\pi}\left\{ \frac{0}{r} + \frac{\vec \mu \times \hat r}{r^2} + \mathcal{O} \right\} \\ \\ \quad \hookrightarrow 0 = \int d^3r\ J(\vec r) \leftarrow \text{No monopole term.} \\ \\ \quad \vec \mu = \frac{1}{2}\int d^3r\ \vec r \times \vec J(\vec r) \end{array}}$$

↑ Magnetic Dipole Moment of the Current Distribution $\vec J$.

Problem: Wire with variational area; does current flow depend on area.



1)

$$0 = \frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$$

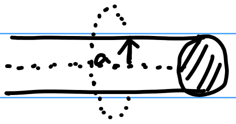
$$0 = \int d^3r \, \vec{\nabla}\cdot\vec{J}$$

$$\overset{!}{=} \oint d^2\vec{S}\cdot\vec{J}$$

$$\overset{!}{=} \underbrace{\int_L d^2S \; J}_{-I_L} + \underbrace{\int_R d^2S \; J}_{I_R} = 0$$

→ Independent of surface.

2) $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \quad \rightarrow \quad \oint d\vec{r}\cdot\vec{B} = \int d^2S \, \mu_0 \vec{J} = \mu_0 I.$



$\rightarrow \vec{J} = \frac{I}{\pi a^2}\hat{x}$

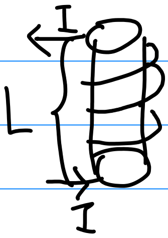
$\vec{B} = B_0\hat{\phi} \quad \rightarrow \quad B_0 \oint 2\pi dr = \mu_0 I$

↳ $B = \frac{\mu_0 I}{2\pi r}\hat{\phi} \qquad r > a$

$B_0 \, 2\pi r = \pi r^2 \frac{I}{\pi a^2}$

↳ $B = \frac{1}{2\pi r} I \frac{r^2}{a^2} = \frac{1}{2\pi a^2} I r \hat{\phi} \qquad r < a$

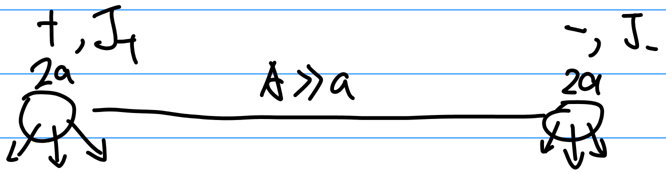
Solenoid:



$BL = \mu_0 N L I$

$B = \mu_0 N I$

↳ Turnsper unit length.



$$W = \int d^3r\, \vec{E} \cdot \vec{J}$$ ← Heating.

$$\overset{!}{=} \frac{1}{2} \frac{1}{\sigma} \int |J_+ + J_-|^2 d^3r$$

only $J_+ J_-$ term vary with separation.

$$\overset{!}{\simeq} \frac{1}{2} \sigma \int d^3r\, E_+ \underset{\downarrow \, -\nabla\phi_-}{E_-}\; 2$$

Leibniz ↳ $= \frac{1}{2} \sigma \int (\vec{\nabla} \cdot E_+) \phi_-$

↳ $$\frac{\partial R}{\partial A}_{tot} = \frac{1}{\pi\sigma} \frac{1}{D^2}$$

Multi-pole Expansion:

$$A(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\vec{J}(\vec{r}')$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \, \vec{J}(\vec{r}') \frac{1}{r}\left(1+\left(\frac{r'}{r}\right)r'\cdot r + \mathcal{O}\left(\frac{r'}{r}\right)^2\right)$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\underbrace{\int d^3r' \vec{J}(\vec{r}')}_{=0} + \frac{\mu_0}{4\pi}\frac{1}{r^2}\int d^3r \, r' J(\vec{r}')\hat{r}\cdot\hat{r}' + \mathcal{O}$$

$$= 0 + \frac{\mu_0}{4\pi}\frac{1}{r^2}(\vec{\mu}\times\hat{r}) \quad \text{where} \quad \vec{\mu} = \frac{1}{2}\int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')$$

Then $\vec{\nabla}\times\vec{A} = -\frac{\mu_0}{4\pi}\vec{\nabla}\left(\frac{\vec{\mu}\cdot\hat{r}}{r^2}\right) + \vec{c}\,\delta(\vec{r})$

↑ extra delta in origin.

Calculate: Monopole Term:

$\int d^3r' \vec{J}(\vec{r}')$, assume $\lim\limits_{r\to\infty} r^3\vec{J}(\vec{r}) \doteq 0$

evaluate: $\int d^3r \, \vec{\nabla}\cdot(r_\nu \vec{J}(\vec{r})) = \int d^3r \left[J_\nu + r_\nu \underbrace{\cancel{\vec{\nabla}\cdot\vec{J}(\vec{r})}}_{0}\right] \doteq \int d^3r \, J_\nu$

$$\|$$

$$\int_S \underset{R^2}{d^2\vec{S}}\cdot \underset{R}{r_\nu}\vec{J}(\vec{r}) = 0$$

but assumption $\lim\limits_{r\to\infty} r^3\vec{J}(\vec{r}) = 0$

Calculate dipole term:

$$\hat{r}_\nu \int d^3r' \underbrace{r_\nu' J_\rho(\vec{r}')} \qquad \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

If this is skew symmetric (flip index give negative)

Now assume $\lim_{r\to\infty} r^4 J(r) = 0$

then evaluate $\int d^3r\, \vec{\nabla}\cdot(r_\nu r_\lambda J) = \int d^3r \left[r_\lambda J_\nu + r_\nu J_\lambda + r_\lambda r_\nu \overset{0}{\cancel{\nabla}} J \right]$

$$\|$$

$$\int d^2S\, J r_\nu r_\lambda = 0$$

$$\boxed{r_\lambda J_\nu = - r_\nu J_\lambda}$$

↳ skew symmetric!

then $r_\lambda' J_\nu(\vec{r}') = \varepsilon_{\lambda\nu\rho}\, u_\rho$

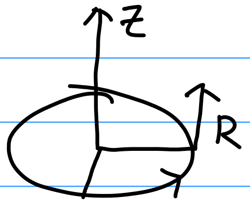
then we can multiply both side by $\varepsilon_{\lambda\nu\sigma}$

$$\varepsilon_{\lambda\nu\sigma} r_\lambda' J_\nu = \underbrace{\varepsilon_{\lambda\nu\sigma}\varepsilon_{\lambda\nu\rho}}_{2} u_\rho$$

$$\frac{1}{2}\varepsilon_{\lambda\nu\sigma} r_\lambda' J_\nu = u_\rho$$

then $\boxed{\vec{u} = \frac{1}{2}\int d^3r\ \vec{r}\times\vec{J}(\vec{r})}$

Ex:

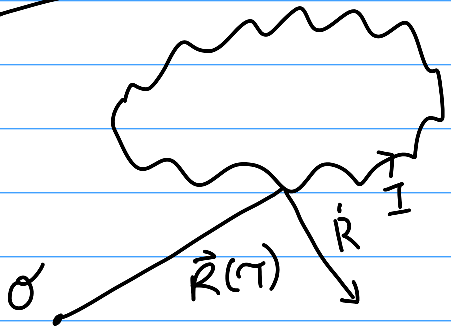


$$R(\theta) = a\hat{x}\cos\theta + a\hat{y}\sin\theta$$

$$J(\vec{r}) = I\int d\theta\,\dot{R}(\theta)\,\delta(r-R(\theta))$$

$$\vec{\mu} = I\pi a^2\,\hat{z}$$

Ex 2:



$$J(\vec{r}) = I\int d\underset{\uparrow\text{ parameterization.}}{\tau}\,\dot{\vec{R}}(\tau)\,\delta(r-R(\tau))$$

$$\mu = \frac{I}{2}\int d\tau\,\vec{R}(\tau)\times\frac{d\vec{R}}{d\tau}(\tau)$$

$$\overset{!}{=}\frac{I}{2}\oint\vec{R}\times d\vec{R}$$

$$\overset{!}{=} I\,\vec{\alpha} \qquad \text{where } \vec{\alpha} = \frac{1}{2}\oint\vec{R}\times d\vec{R}$$

If we apply any vector $\vec{m}\cdot\vec{\alpha}$

$$\vec{m}\cdot\vec{\alpha} = \oint\frac{1}{2}m\cdot(\vec{R}\times d\vec{R})$$

$$\overset{!}{=}\oint\frac{1}{2}d\vec{R}\cdot(\vec{m}\times\vec{R}) \quad \downarrow \text{ scalar triple product}$$

$$\overset{!}{=}\int d^2\vec{S}\,\underbrace{\frac{1}{2}\vec{\nabla}\times(\vec{m}\times\vec{R})}_{=\vec{m}} \quad \downarrow \text{ Stokes}$$

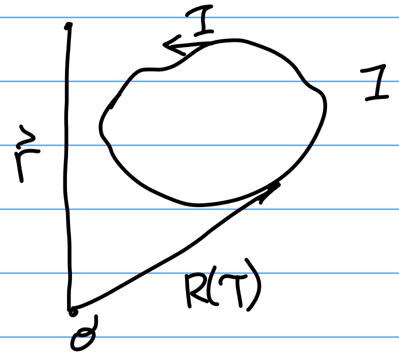
$$\overset{!}{=}\int d^2\vec{S}\cdot\vec{m}$$

Divide by $\vec{m}$ again: $\boxed{\vec{\alpha} = \int_S d^2S\,\hat{n}}$, then $\boxed{\vec{\mu} = I\int_S d^2S\,\hat{n}}$

With $A(\vec r) = \frac{\mu_0}{4\pi}\int d^3r' \frac{J(\vec r')}{|\vec r - \vec r'|}$

$$\boxed{\vec B = \vec\nabla \times \vec A = \frac{\mu_0}{4\pi}\int d^3r'\, \vec J(\vec r') \times \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3}}$$

ex:



$$A(\vec r) = \frac{\mu_0}{4\pi}\int d^3r'\, I \int d\tau\, \dot R(\tau)\, \delta(\vec r' - \vec R(\tau)) \frac{1}{|\vec r - \vec r'|}$$

$$\overset{!}{=} \frac{\mu_0 I}{4\pi}\int d\tau\, \dot R(\tau)\, \frac{1}{\vec r - \vec R(\tau)}$$

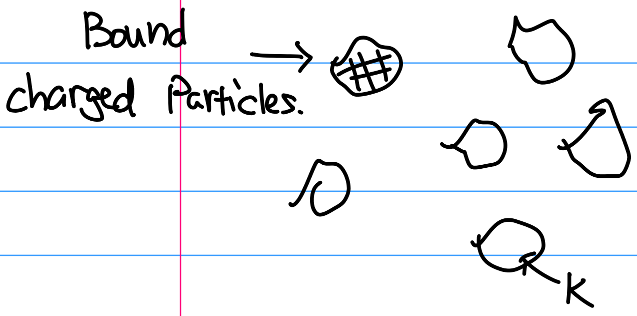
$$\overset{!}{=} \frac{\mu_0 I}{4\pi}\oint \frac{d\vec R}{|\vec r - \vec R|}$$

$$\boxed{B(\vec r) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec R \times [\vec r - \vec R]}{|\vec r - \vec R|^3}}$$

← Biot-Savart Law.

← Magnetic Field due to current loop.

Magnetic Media:



consider Lorentz average length which is much larger than atoms/molecules

Think of them as atoms, molecules, or ions.

Dielectric electric media: $\overset{\text{micro}}{{}^{\mu}}\overset{\text{dipole}}{p_k} \rightarrow {}^{\mu}\overset{\text{dipole density.}}{P_k} = \sum_k {}^{\mu}p_k \delta(r-r_k)$

after Lorentz average: $P(\vec{r}) = \sum_k p_k f(\vec{r}-\vec{r}')$

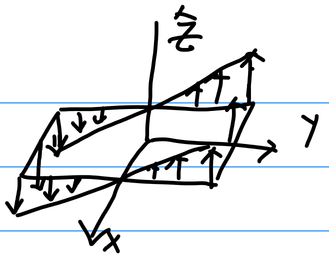
Magnetic Charge: micro-polarization: $\overset{\text{magnetic dipole.}}{{}^{\mu}m_k} = \frac{1}{2}\int d^3r' \, \vec{r} \times {}^{\mu}\vec{J}_k(\vec{r}')$

$$\left.\begin{aligned} {}^{\mu}m_k &\rightarrow \sum_k {}^{\mu}m_k \, \delta(\vec{r}-\vec{r}') \\ \vec{M}(\vec{r}) &= \sum_k m_k f(\vec{r}-\vec{r}') \end{aligned}\right] \text{LA}$$

Then

$$\boxed{\begin{aligned} &\vec{\nabla}\times\vec{M} = \vec{J}_p && \text{Bulk Magnetization current} \\ &\vec{M}\times\hat{n} = \vec{K} && \text{surface Magnetization current} \end{aligned}}$$

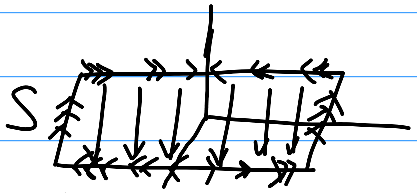
Analogy to $\sigma_p = \hat{n}\cdot\vec{P}$ and $\rho_p = -\vec{\nabla}\cdot\vec{P}$

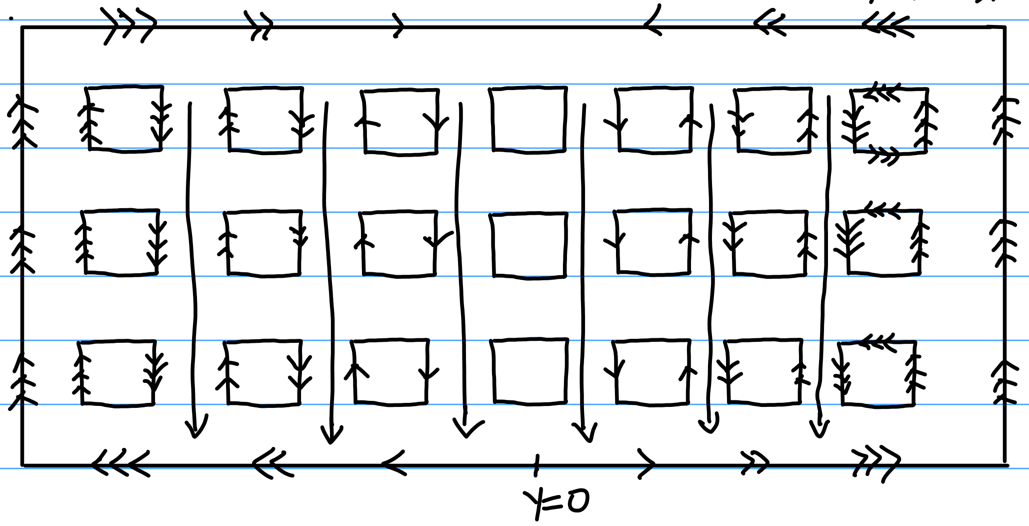


Ex: given $M(\vec r) = S y \hat z$

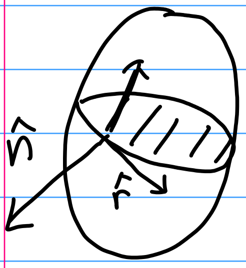
use $\vec J(\vec r) = \vec \nabla \times \vec M$

$\overset{!}{=} S \hat x$





Abstract Derivation:



$$I = \int d^2\vec S \cdot \vec J$$

$$\hookrightarrow \underbrace{\int d^2\vec S \cdot \vec J_{cond}}_{\text{suppose } \vec J_{cond}=0} + \int d^2\vec S \cdot \vec J_{mag} + \oint (\vec n \times d\vec r) \cdot \vec K_{mag} = 0$$

so $\boxed{\begin{array}{l} \vec J = \vec \nabla \times \vec M \\ \vec K = M \times \hat n \end{array}}$

$$\frac{1}{2}\int d^3r\ \vec r \times \vec J = \frac{1}{2}\int d^3r\ \vec r \times (\vec \nabla \times \vec M) = \underset{\text{terms}}{\text{surface}} + \int d^3r\ M(\vec r)$$

Maxwell :

$\vec{B} \rightarrow$ Magnetic field or Magnetic Induction.

$\vec{B} \rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} = \mu_0(\vec{J}_{cond} + \vec{\nabla}\times\vec{M})$

Magnetization Density.

$$H = \frac{1}{\mu_0}\vec{B} - \vec{M} \rightarrow \vec{\nabla}\times\vec{H} = \vec{J}_{cond}$$

Magnetic Field

$$H = \frac{1}{\mu}\vec{B}$$

Magnetic permeability of medium.

then $M = \frac{1}{\mu_0}\vec{B} - \vec{H} = \left(\frac{\mu}{\mu_0} - 1\right)\vec{H} = \left(\frac{\mu}{\mu_0} - 1\right)\frac{1}{\mu}\vec{B}$

let $\frac{\mu}{\mu_0} = 1 + \chi$

succeptibility.

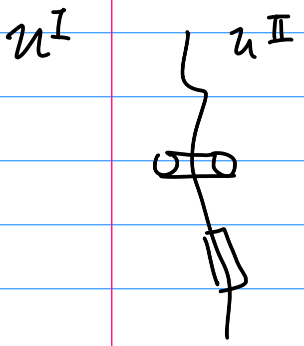
then $M = \chi\vec{H}$

when $\chi > 0$ paramagnetism

$\chi < 0$ diamagnetism

Magnetic Boundary Conditions:

$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{H} = J = 0$



$$\hat{n}\cdot\vec{B}(+) - \hat{n}\cdot\vec{B}(-) = \mu^{I}H^{I}_{\perp} - \mu^{II}H^{II}_{\perp} = 0$$

$$H_{\parallel}(+) - H_{\parallel}(-) = \frac{1}{\mu^{I}}B^{I}_{\parallel} - \frac{1}{\mu^{II}}B^{II}_{\parallel} = \vec{J}_{cond}.$$

Magnetic field → steady current.

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{B}=0\\ \vec{\nabla}\times\vec{H}=\vec{J}\end{array}\right\}\quad \boxed{\vec{\nabla}\times\left(\frac{1}{\mu}\vec{\nabla}\times\vec{A}\right)=\vec{J}}$$

Suppose μ is constant

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}=\mu\vec{J}$$

In Cartesian:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2 A=\mu\vec{J}$$

assume Coulomb gauge: $\vec{\nabla}\cdot\vec{A}=0$

$$-\nabla^2 A=\mu\vec{J}$$

<u>ex</u>: Collinear Current: $\vec{J}(\vec{r})=F(\rho)\hat{z}$

$$\int \underbrace{d^2\vec{S}}_{r\,dr\,d\theta\,\hat{z}}\cdot\vec{\nabla}\times\vec{H}=\oint \underbrace{d\vec{r}}_{r\,d\theta\,\hat{\theta}}\cdot\underbrace{\vec{H}}_{H_0\,\hat{\theta}}=\int\vec{J}\cdot d^2\vec{S}=J\hat{z}\cdot\underbrace{d^2S}_{r\,dr\,d\theta}\hat{z}$$

$H_0\,\hat{\theta}$ → by rH rule. thumb in $\hat{z}$.

$$I(\rho)=\int d^2\vec{S}\cdot\vec{J}=\int_0^{\rho}2\pi r\,dr\,F(r)$$

Interaction energy, force and Torque between currents.

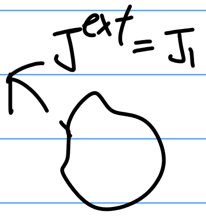
$$\mathcal{E} = \frac{1}{2\mu_0}\int d^3r\, |B(\vec r)|^2$$

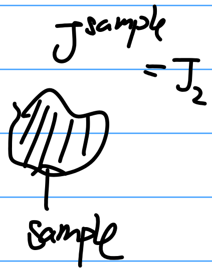
$$\overset{!}{=} \frac{1}{2\mu_0}\int d^3r\, \vec B\cdot(\vec\nabla\times\vec A)$$

use $\vec\nabla\cdot(\vec A\times\vec B) = \vec A\cdot(\vec\nabla\times\vec B) + \vec B\cdot(\vec\nabla\times\vec A)$

$$\overset{!}{=} \frac{1}{2\mu_0}\int d^3r\left[\cancel{\vec\nabla\cdot(\vec A\times\vec B)} - \vec A\cdot(\vec\nabla\times\vec B)\right]$$

$$\boxed{\mathcal{E} \overset{!}{=} \frac{-1}{2\mu_0}\int d^3r\, \vec A\cdot\vec J}$$





$$\mathcal{E}_{int} = \frac{1}{\mu_0}\int d^3r\, \vec B_1\cdot\vec B_2$$

$$\overset{!}{=} \int d^3r\, \vec A_1\cdot\vec J_2 = \int d^3r\, \vec A_2\cdot\vec J_1$$

$$\overset{!}{=} \int_{\text{sample.}} d^3r\, \vec A^{ext}(\vec r)\cdot\vec J(\vec r)$$

$$\overset{!}{=} \int d^3r\left[A_a^{ext}(0) + r_b\left(\partial_b A_a^{ext}(\vec r)\right)\Big|_{r=0} + \cdots\right]\cdot\vec J_a(\vec r)$$

$$\overset{!}{\approx} A_a^{ext}\underbrace{\int d^3r\, \vec J_a(\vec r)}_{=0} + \left(\partial_b A_a^{ext}(\vec r)\right)\Big|_{r=0}\underbrace{\int d^3r\, r_b J_a(\vec r)}_{\varepsilon_{bac}\, m_c}$$

$$\overset{!}{\approx} \left(\vec\nabla\times\vec A^{ext}\Big|_{r=0}\right)\cdot\vec m$$

$$\boxed{\mathcal{E}_{int} \overset{!}{\approx} \vec B^{ext}(\vec r=0)\cdot\vec m}$$

analogy to electrostatic: $\mathcal{E}_{int} \approx -\vec E^{ext}(0)\cdot\vec p$

energies → Forces

Force density: $\vec{f} = \vec{J} \times \vec{B}$

self-force: $\int_{\substack{\text{all}\\ \text{sample.}}} d^3r \; \vec{J}(\vec{r}) \times \vec{B}(\vec{r}) = 0$

Assume 2-current / patches:

$$\vec{F} = \int_{\text{sample}} d^3\vec{r} \; \vec{J} \times \Big[\overset{\text{self-force: 0}}{\cancel{\vec{B}^{\text{sample}}}} + \vec{B}^{\text{ext}} \Big]$$

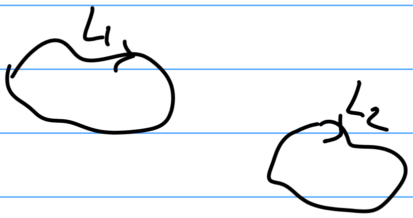
$$\boxed{\vec{F} = (\vec{m} \cdot \vec{\nabla}) \Big[\vec{B}^{\text{ext}}(\vec{r}) \Big|_{\vec{r}=0} \Big]}$$

$$\boxed{\text{Torque:} \quad T = \vec{m} \times B^{\text{ext}}}$$

→ analogous to capacitance.

Mutual Inductance of 2 current loops:

$$\boxed{M_{12} = \frac{\mu_0}{4\pi} \underbrace{\oint_{L_1} \oint_{L_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{\ell}_2 - \vec{\ell}_1|}}_{\text{geometry of current.}}}$$



$$\boxed{\mathcal{E}_{int} = I_1 I_2 M_{12}}$$

proof:

$$A_1(\vec{r}) = \left(\frac{\mu_0}{4\pi}\right) I_1 \oint_{L_1} \frac{d\ell_1}{|\vec{r} - \vec{\ell}_1|} \quad , \quad A_2(\vec{r}) = \frac{\mu_0}{4\pi} I_2 \oint_{L_2} \frac{d\ell_2}{|\vec{r} - \vec{\ell}_2|}$$

$$\mathcal{E}_{int} = \frac{1}{\mu_0} \int d^3\vec{r}\ \vec{B}_1 \cdot \vec{B}_2$$

$$\overset{!}{=} \int d^3r\ \vec{J}_2 \cdot \vec{A}_1$$

$$\hookrightarrow \vec{J}(\vec{r}) = \int dt\ \dot{R}(t)\, \delta(r - R(t)) = \oint d\ell\ \delta(r - \ell)$$

$$= \int d^3r\ I_2 \frac{\mu_0}{4\pi} \int d\ell_2\ \delta(\vec{r} - \vec{\ell}_2)\ A_1(\vec{r})$$

$$= I_2 \frac{\mu_0}{4\pi} \oint d\vec{\ell}_2 \cdot \vec{A}_1(\ell_2)$$

$$\overset{!}{=} I_2 \frac{\mu_0}{4\pi} \oint_{L_1} d\vec{\ell}_2 \cdot I_1 \frac{\mu_0}{4\pi} \oint_{L_2} d\ell_1 \frac{1}{|\vec{\ell}_2 - \vec{\ell}_1|}$$

$$\mathcal{E}_{int.} \overset{!}{=} I_1 I_2 M_{12}$$