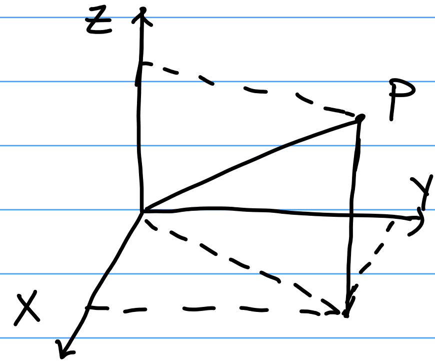
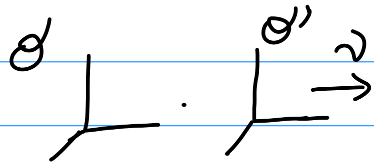
Special Relativity:

Inertial Reference Frame (constant velocity):



point P located via 3 coordinates.

$$(x,y,z) = (r^1, r^2, r^3)$$



Lorentz Transformation:

New inertial reference frame moving with velocity, v, in x-direction compared with the old one.

$$x \rightarrow x' = \frac{x - vt}{\sqrt{1-(v/c)^2}}$$

original frame

$$y \rightarrow y'$$

$$z \rightarrow z'$$

$$t \rightarrow t' = \frac{t - vx/c^2}{\sqrt{1-(v/c)^2}}$$

New frame moving along x-axis at speed v in the old frame. (With no rotating of the new frame)

Consider using hyperbolic functions. let $\boxed{\frac{v}{c} = \tanh\beta}$ *

$$\boxed{\frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\tanh^2\beta}} = \frac{\cosh\beta}{\sqrt{\cosh^2\beta-\sinh^2\beta}} = \cosh\beta}$$

and

$$\boxed{\frac{v/c}{\sqrt{1-(v/c)^2}} = \frac{\tanh\beta}{\sqrt{1-\tanh^2\beta}} = \frac{\sinh\beta}{\sqrt{\cosh^2\beta-\sinh^2\beta}} = \sinh\beta}$$

Thus:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice under <u>rotation</u>: it would read:

$$\begin{pmatrix} y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

with $(y')^2 + (z')^2 = y^2 + z^2$

However in 4-D space time:

$$(ct')^2 - (x'^2 + y'^2 + z'^2) = (ct\cosh\beta - x\sinh\beta)^2 - (x\cosh\beta - ct\sinh\beta)^2 - (y^2+z^2)$$

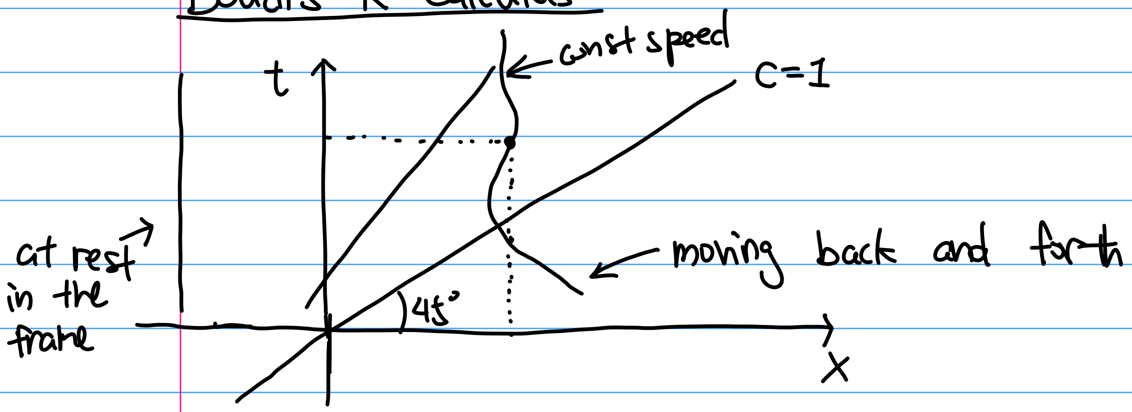
$$\hookrightarrow \boxed{(ct')^2 - (x'^2 + y'^2 + z'^2) = (ct)^2 - (x^2 + y^2 + z^2)} \leftarrow \underline{\text{conserved}}$$

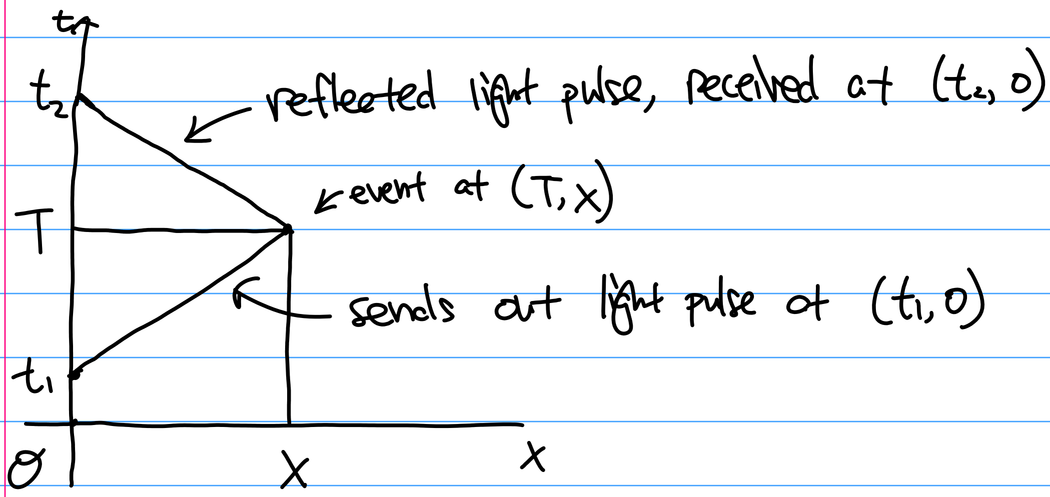
General form of Lorentz transformation:

The new frame move with constant velocity $\boxed{\vec{v} = v\hat{v} = c\tanh\beta\,\hat{v}}$ relative to the old frame.

$$\boxed{\begin{aligned} * \text{ then: } \quad ct \rightarrow ct' &= (ct)\cosh\beta - \hat{v}\cdot\vec{r}\sinh\beta \\ \vec{r} \rightarrow \vec{r}\,' &= \Big[\hat{v}\underbrace{(\hat{v}\cdot\vec{r})}_{\substack{\text{piece of } \vec{r} \text{ along} \\ \text{with } \hat{v}\text{, analogous to } x \text{ as before}}}\cosh\beta - (ct)\hat{v}\sinh\beta\Big] \Big\} \text{ longitudinal is transformed.} \\ &\quad + \underbrace{\Big[\vec{r} - \hat{v}(\hat{v}\cdot\vec{r})\Big]}_{\substack{\vec{r} \text{ with piece along} \\ \text{with } \vec{v} \text{ removed.}}} \Big\} \text{ transverse part is not transformed} \end{aligned}}$$

Bondi's k-Calculus





We can then locate the event at (T, X) as

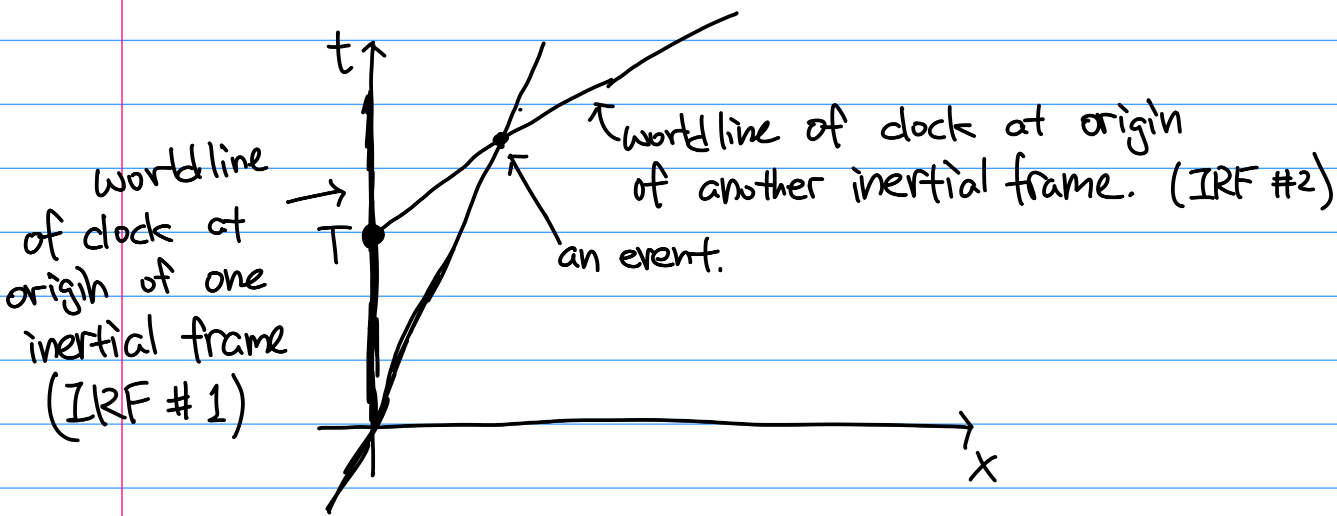
From the right bound pulse: $X = c(t_2 - t)$

From the left bound pulse: $X = c(t - t_1)$

$$\boxed{\left.\begin{aligned} T &= \frac{t_2 + t_1}{2} \\ X &= \frac{ct_2 - ct_1}{2} = c\left(\frac{t_2 - t_1}{2}\right) \end{aligned}\right\} \quad (T, X) = \left(\frac{t_2 + t_1}{2},\ c\left(\frac{t_2 - t_1}{2}\right)\right)}$$

Now consider two distinct inertial reference frames:

How to relate spacetime coordinates in the two frames?



Consider an event on the worldline of the clock at the origin of the second frame.

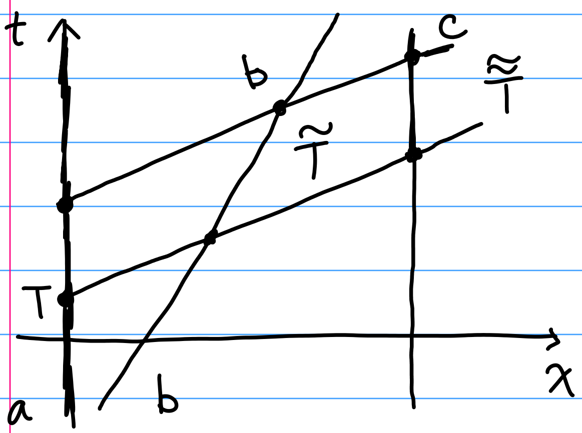
A light pulse from the first clock (IRF #1) reaches the event if it was emitted at time T.

let $\tilde{T}$ be the time on the second clock (IRF #2) when the light pulse arrives

then hypothesize:

$$\boxed{\tilde{T} = \underbrace{k}_{\text{Bondi's k factor.}} T}$$

Two distinct inertial reference frames:



T, $\tilde{T}$, and $\tilde{\tilde{T}}$: time interval measured on the respective clocks, a, b, c.

world lines:
- a: First clock
- b: second clock, recieving from first at speed v
- c: Third clock, at rest in first clock's frame, synchronized with first clock.

we then have $\overset{r}{\tilde{T}} = k(v)\overset{s}{T}$

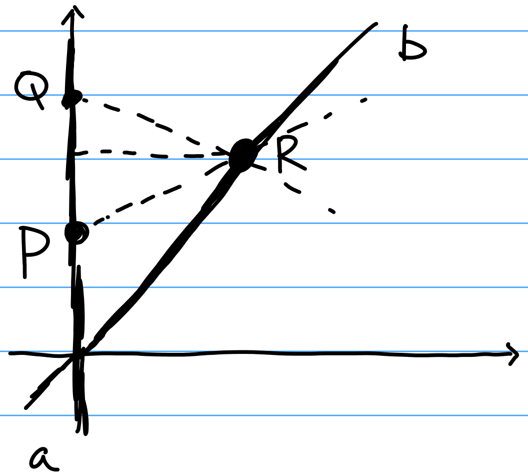
$\overset{r}{\tilde{\tilde{T}}} = k(-v)\overset{s}{\tilde{T}}$

ⓡ = receive. pulse
ⓢ = sends pulse.

we note that $T = \tilde{\tilde{T}}$, so $\boxed{k(-v) = \frac{1}{k(v)}}$

Now how does k depend on v:

suppose we have P, Q, R:



event coordinates in frame:

	IRF # a	IRF # b
Q :	$(t_2, 0)$	
R:	(t, x)	$(\tilde{t}_1, 0)$
P:	$(t_1, 0)$	

By inspection:

$$\left.\begin{aligned} t &= \tfrac{1}{2}(t_1 + t_2) \\ x &= \tfrac{1}{2}(t_2 - t_1)c \\ x &= vt \end{aligned}\right\} v = \frac{x}{t} = \frac{t_2 - t_1}{t_2 + t_1}$$

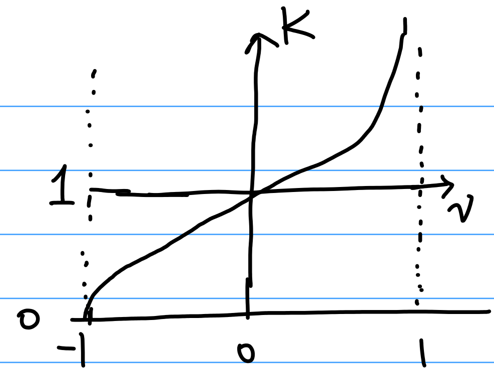
By definition of k-factor:

$$\left.\begin{aligned} t_2 &= k(v)\tilde{t}_1 \quad \rightarrow \text{send from R to Q} \\ \tilde{t}_1 &= k(v)\, t_1 \quad \rightarrow \text{send from P to R} \end{aligned}\right\} t_2 = k(v)^2 t_1$$

then $v = \dfrac{t_2 - t_1}{t_2 + t_1} = \dfrac{k^2 - 1}{k^2 + 1} \Rightarrow \boxed{k(v) = \sqrt{\dfrac{1+v}{1-v}}}$

Note:
$t_1, \tilde{t}_1, t_2$ are all measured on clocks at rest in their own IRFs.

Consequence of $k=\sqrt{\frac{1+v}{1-v}}$:



recall set $c=1$, otherwise $v \to v/c$

Current properties of k:

i) $k(-v) = \dfrac{1}{k(v)}$

ii) Near $v=0$ (i.e. $|v| \ll c$) $\quad k \approx 1+v$

iii) Near $v=1$ (i.e. $v \approx c$) $\quad k \approx \sqrt{\dfrac{2}{1-v}}$

Near $v=-1$ (i.e. $v \approx -c$) $\quad k \approx \sqrt{\dfrac{1+v}{2}}$

iv) if $v = \tanh\beta$ ← some parameterization

then $$k = \sqrt{\frac{1+\tanh\beta}{1-\tanh\beta}} = \sqrt{\frac{\sinh\beta+\cosh\beta}{\sinh\beta-\cosh\beta}} = \sqrt{\frac{2e^{\beta}}{2e^{-\beta}}} = e^{\beta}$$

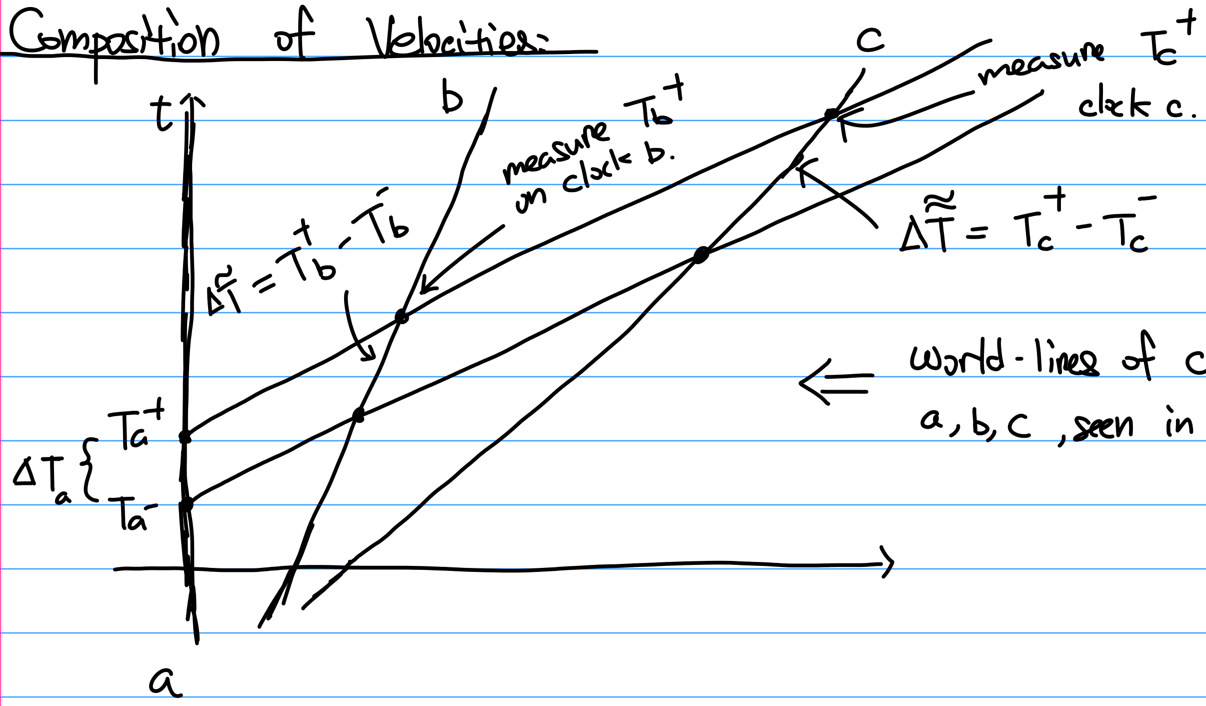
*

↳ $k = e^{\beta}$ with $v = \tanh\beta$

as β goes from $-\infty$ to ∞
k goes from 0 to ∞
v goes from -1 to $+1$

under $v \to -v$, $\beta \to -\beta$ and $k \to \dfrac{1}{k}$

Composition of Velocities:



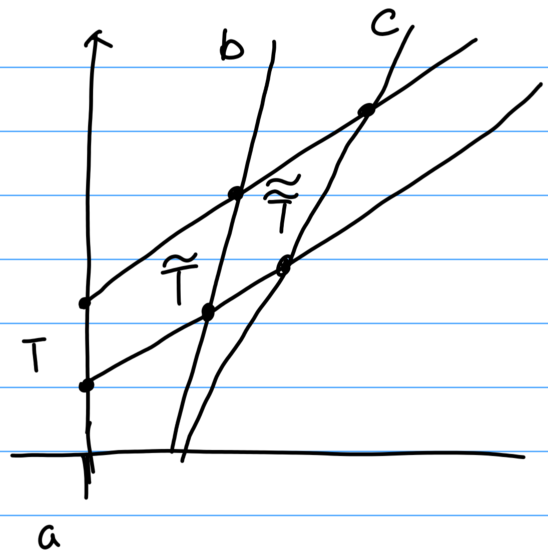
$\Leftarrow$ world-lines of clocks a, b, c, seen in IRF # a.

→ Frame b is receiving in frame a at speed v_{ba}
→ Frame c is receiving in frame b at speed v_{cb}

→ Frame c is receiving in frame a at speed v_{ca}

so how is v_{ca} determined by v_{cb} and v_{ba}

Composition of velocities, continued:



Introduce $k_{ba} = k(v_{ba})$
$k_{cb} = k(v_{cb})$
$k_{ca} = k(v_{ca})$

observe that

$$\left.\begin{aligned}\tilde{T} &= k_{ba} T\\ \tilde{\tilde{T}} &= k_{cb}\tilde{T}\end{aligned}\right\} \text{ so } \tilde{\tilde{T}} = k_{cb} k_{ba} T \qquad \text{where } k(v) = \sqrt{\frac{1+v}{1-v}}$$

$$\hookrightarrow \sqrt{\frac{1+v_{ca}}{1-v_{ca}}} = \sqrt{\frac{1+v_{cb}}{1-v_{cb}}}\sqrt{\frac{1+v_{ba}}{1-v_{ba}}}$$

After solving:

$$\boxed{v_{ca} = \frac{v_{cb}+v_{ba}}{1+v_{cb}v_{ba}}} \leftarrow \text{Einstein addition Formula}$$

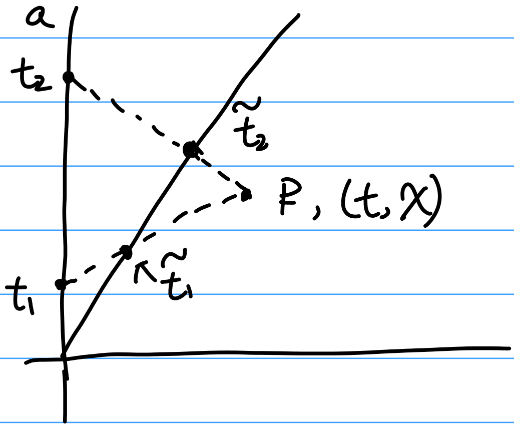
exchanging $v = \tanh\beta$:

$$\tanh\beta_{ca} = \frac{\tanh\beta_{cb} + \tanh\beta_{ba}}{1-\tanh\beta_{cb}\tanh\beta_{ba}} = \tanh(\beta_{cb}+\beta_{ba})$$

Addition Formula.

$$\hookrightarrow \boxed{\beta_{ca} = \beta_{cb} + \beta_{ba}}$$

Deriving Lorentz transformation:



In IRF #a

Knowing: $$t = \frac{1}{2}(t_2 + t_1)$$
$$x = \frac{1}{2}(t_2 - t_1)$$

$\Rightarrow$ or $$t_1 = t - x$$
$$t_2 = t + x$$

$\Rightarrow$ In IRF #b: $$\tilde{t} = \frac{1}{2}(\tilde{t}_2 + \tilde{t}_1)$$
$$\tilde{x} = \frac{1}{2}(\tilde{t}_2 - \tilde{t}_1)$$

Now using the k-factor idea: to relate $(t, x) \rightarrow (\tilde{t}, \tilde{x})$

$$\Rightarrow \quad \tilde{t}_1 = k\, t_1$$
$$t_2 = k\, \tilde{t}_2$$

then: $$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix}$$

using $\tilde{t}_1 = k t_1$, $t_2 = k \tilde{t}_2$

$$= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

using $t_1 = t - x$, $t_2 = t + x$

$$= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

let $\bar{k} = \frac{1}{k}$

$$= \begin{pmatrix} \frac{1}{2}[\bar{k} + k] & \frac{1}{2}[\bar{k} - k] \\ \frac{1}{2}[\bar{k} - k] & \frac{1}{2}[\bar{k} + k] \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

using $k = \exp\beta \;\rightarrow\; \frac{1}{2}[\bar{k} + k] = \frac{1}{2}(e^{-\beta} + e^{\beta}) = \cosh\beta$

$\rightarrow\; \frac{1}{2}[\bar{k} - k] = \frac{1}{2}(e^{-\beta} - e^{\beta}) = -\sinh\beta$

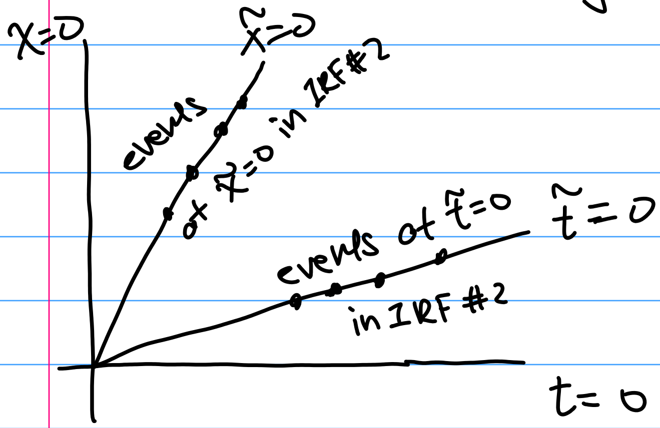
Lorentz Transformation

$$\boxed{\begin{aligned} \begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} &= \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\ \text{i.e.} \quad \tilde{t} &= t\cosh\beta - x\sinh\beta \\ \tilde{x} &= -t\sinh\beta + x\cosh\beta \end{aligned}}$$

← How event (t, x) in IRF # 1 appear on IRF # 2, $(\tilde{t}, \tilde{x})$ which moves at speed v relative to IRF # 1.

Lorentz Contraction:

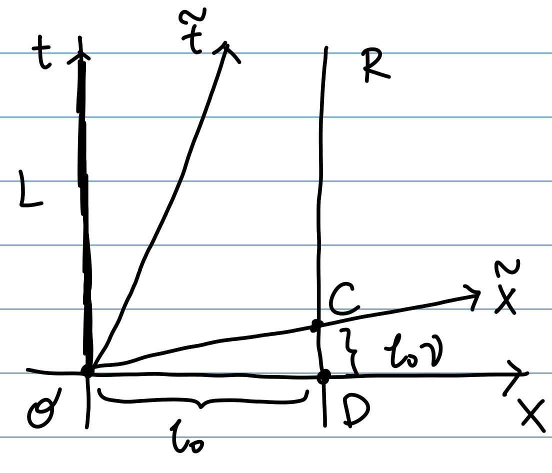
Take two frames moving apart at speed v:



First frame: (x, t)
Second frame: $(\tilde{x}, \tilde{t})$

How wide are things?

Suppose there are two rods, L and R, at the world lines of the ends of a rod at rest in the black frame.



$$\left.\begin{aligned} L &: (t, x) = (\lambda, 0) \\ R &: (t, x) = (u_0, l_0) \end{aligned}\right\} \lambda \text{ and } u_0 \text{ changes}$$

What is the length of rod in black frame (t, x)?

let $\lambda = 0$, then $u_0 = 0$ clearly from plot, so

length $= l_0$ in Black frame.

Now determine the length of rod in blue frame $(\tilde{t}, \tilde{x})$

let $\tilde{t} = 0$: $\quad \mathcal{O}: \; (\tilde{t}, \tilde{x}) = (0,0)$

$C: \; (\tilde{t}, \tilde{x}) = (0, ?)$

To find $\tilde{x}$, note that C is at $(t,x) = (l_0 v, l_0)$ in black frame.

Then use Lorentz Transformation:

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} l_0 v \\ l_0 \end{pmatrix}$$

$v = \tanh\beta$

$$= \begin{pmatrix} l_0 v \cosh\beta - l_0 \sinh\beta \\ l_0 \cosh\beta - l_0 v \sinh\beta \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ l_0 / \cosh\beta \end{pmatrix}$$

Lorentz contraction.

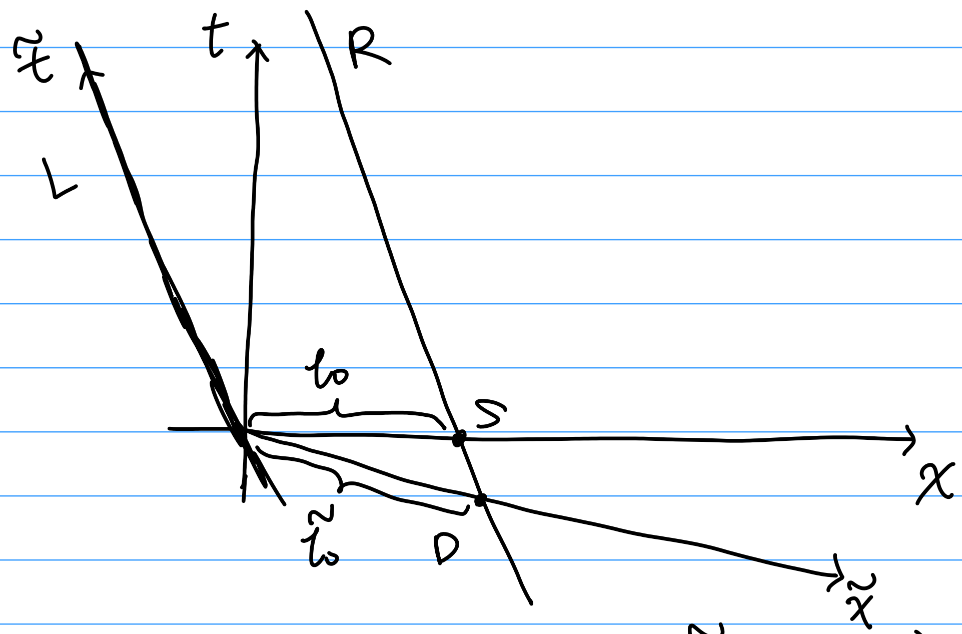
So $$\boxed{\tilde{l}_0 = \tilde{x}(\mathcal{O}) - \tilde{x}(C) = \frac{l_0}{\cosh\beta} = l_0\sqrt{1 - (v/c)^2}}$$

→ First determine $(\tilde{t}, \tilde{x})$ in the frame of (t, x)

→ then use Lorentz Transformation to $(t, x) \to (\tilde{t}, \tilde{x})$

Application: Length of moving rod as measured by a stationary observer.

→ Choose IRF # 2 to be moving with the rod.
→ Then rod length stays constant in IRF # 2.
→ Then perform inverse Lorentz transformation



then in IRF # 2, length is $\tilde{l}_0$, $(\tilde{t}, \tilde{x})_L = (0,0)$
$(\tilde{t}, \tilde{x})_R = (0, l_0)$

Now with Inverse Lorentz Transform:

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix}$$

	observer frame (t, x)		Rod frame $(\tilde{t}, \tilde{x})$
O	$(0,0)$		$(0,0)$
S	$(0, \ell_0)$	$\xrightarrow{\text{L.T.}}$	$(?, ?) = (\ell_0 \sinh\beta, \ell_0 \cosh\beta)$
D	$(0-\tau, \ell_0 + v\tau)$	$\xleftarrow{\text{I.L.T}}$	$(0, \tilde{\ell}_0)$

$\hookrightarrow = (-\tilde{\ell}_0 \sinh\beta, \tilde{\ell}_0 \cosh\beta)$

$\hookrightarrow$ so $\left.\begin{aligned} \tau &= \tilde{\ell}_0 \sinh\beta \\ \ell_0 + v\tau &= \tilde{\ell}_0 \cosh\beta \end{aligned}\right\}$ $\ell_0 + v\tilde{\ell}_0 \sinh\beta = \tilde{\ell}_0 \cosh\beta$

$\hookrightarrow \ell_0 = \tilde{\ell}_0(\cosh\beta - \tanh\beta \sinh\beta)$

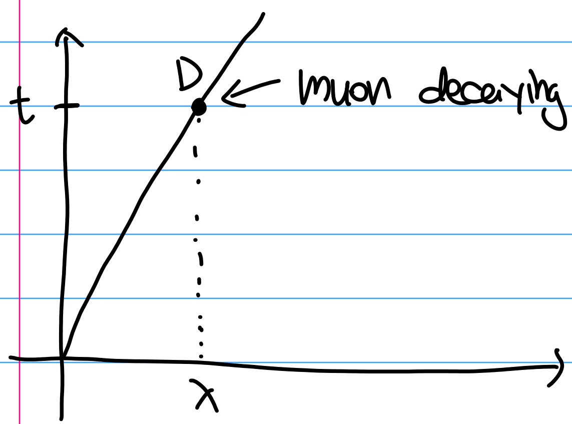
$\Rightarrow \hookrightarrow \ell_0 = \dfrac{\tilde{\ell}_0}{\cosh\beta} < \tilde{\ell}_0$ (contraction)

Ex: Dray 7.1:

lab frame.

A muon is observed to travel 800m before decaying. Life time, $\tau = 2\times10^{-6}$s . What is the speed of muon?

Note τ is measured in muon's rest frame.



D in lab frame: (t, x)

D in muon frame: $(\tau_u, 0)$

use frame invariant condition

$$t^2 - x^2 = \tau_u^2 - 0^2$$

In lab frame, muon move in speed v, so $t = \frac{cx}{v}$

$$\hookrightarrow \left(\frac{cx}{v}\right)^2 - x^2 = \tau_u^2$$

$$\hookrightarrow \frac{v}{c} = \frac{1}{\sqrt{1 + \left(\frac{c\tau_u}{x}\right)^2}}$$

then $\frac{v}{c} = \frac{1}{\sqrt{1+(3/4)^2}} = \frac{4}{5}$

Time Dilation.

Note that $t = \frac{x}{v} = \frac{5}{3}\tau_u$ ← so time it takes muon to decay is longer in lab fram.

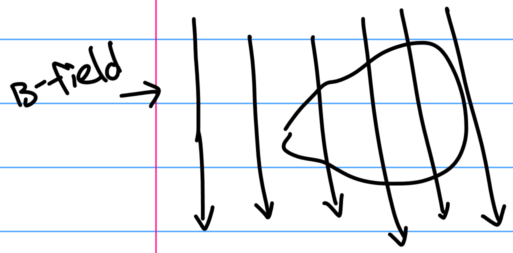
Since $(ct)^2 - (vt)^2 = (c\tau_u)^2$

Time Dilation.
t in lab frame.
τ in moving frame.

$$\hookrightarrow \boxed{t = \frac{\tau_u}{\sqrt{1-(v/c)^2}}} > \tau_u$$

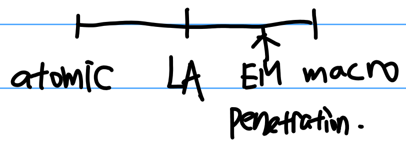
Superconductors:

$T > T_S$:



$T < T_S$:

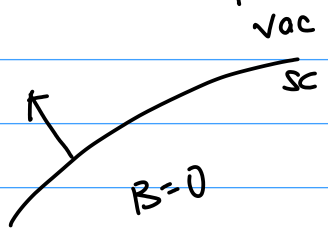
electric current
induced internal $\vec{B}$-field
that cancels external $\vec{B}$.



↳ two new length scales:
1) Coherence length
2) EM penetration depth.

Fundamental property $\vec{B}=0$:

B.C. on $\vec{B}$ near a superconductor:



$\vec{\nabla}\cdot\vec{B} = 0$

↳ $\vec{B}\cdot\hat{n} = 0$

so only tangential component.

Then the stress-tensor: $t_{ab} = \frac{1}{\mu_0}\left[B_a B_b - \frac{1}{2}\delta_{ab}|B|^2\right]$

then $F_b = \sum_a \int d^2S_a \, t_{ab}$

$$F_b = \int d^2S \; n_a \; \frac{1}{\mu_0}\left[\cancel{B_a B_b} - \frac{1}{2}\delta_{ab}\,|\vec{B}|^2\right]$$

due to B.C.

$$\stackrel{!}{=} \frac{-1}{2\mu_0}\int d^2\vec{S}\;|\vec{B}|^2$$

Now using ampere-law:

$$\vec{\nabla}\times\vec{B} = \mu_0 \rho \vec{v}$$

$$\int d^2S\; \vec{\nabla}\times\vec{B} = \int d^2\vec{S}\cdot \mu_0 \rho\,\vec{v}$$

$$\oint d\vec{L}\cdot\vec{B} = \mu_0 \vec{K}\cdot\tilde{\tilde{n}}$$

$$w\vec{B}\cdot\tilde{\tilde{n}} = \mu_0\vec{K}\cdot\tilde{\tilde{n}}$$

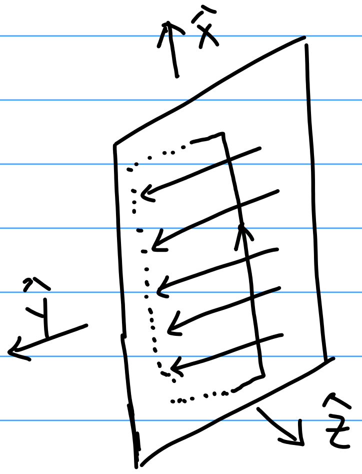
length in vaccum.

$\hookrightarrow \; w\,\vec{B}\cdot(\tilde{\tilde{n}}\times n) = \mu_0\vec{K}\cdot\tilde{\tilde{n}}$

$\hookrightarrow \; w\,\tilde{\tilde{n}}\cdot(n\times\vec{B}) = \mu_0\vec{K}\cdot\tilde{\tilde{n}}$

then $\quad \vec{K} = \frac{1}{\mu_0}\left(\hat{n}\times\vec{B}\right)$

ex 2: $\vec{B}$ due to an ∞ sheet of current:



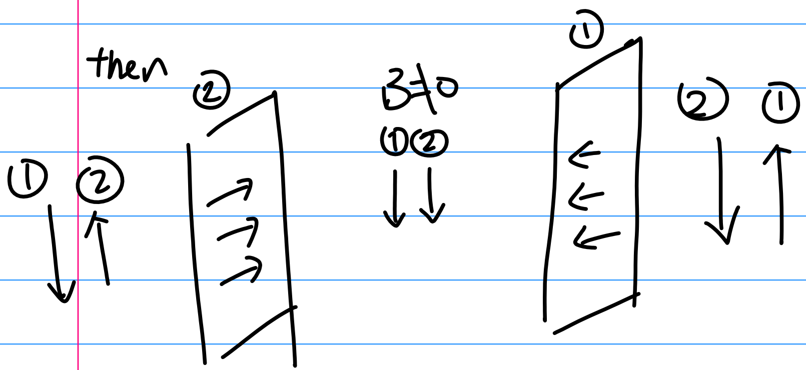
$$\oint d\vec{l} \cdot \vec{B} = B_x(L+L) = 2BL.$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad , \vec{J} = \alpha \, \hat{y}$$

$$2B = \mu_0 \alpha$$

$$\vec{B} = \frac{\mu_0}{2} \alpha \, \mathrm{sgn}(z) \, \hat{z}$$

then



but apply uniform external field such that in the middle it cancels exactly.

Formalism of special relativity:

vector: $\vec{x} \rightarrow (x^1, x^2, x^3)$

4-vector $\quad x \rightarrow (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

what quantity is independent of IRF ?

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2 = -\tau^2$$

Invariant interval (space) $\qquad$ proper time (invariant time)

ex: $\quad$ event $= (ct, 0, 0, 0) \rightarrow \tau^2 = (ct)^2$

$\qquad (0, l, 0, 0) \rightarrow s^2 = l^2$

Lorentz Transformation:

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} & & \\ & LT & \\ & & \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\tilde{x}^\mu = \sum_{\nu=0}^{3} L^\mu{}_\nu \, x^\nu$$

$$s^2 \rightarrow \tilde{s}^2 = s^2$$

ex: LT (v along x)

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Invariant interval:

$$s^2 = -(c\tilde{t})^2 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2$$

$$\overset{!}{=} -(ct\cosh\beta - x\sinh\beta)^2 + (x\cosh\beta - ct\sinh\beta)^2 + y^2 + z^2$$

$$\overset{!}{=} -(ct)^2 + x^2 + y^2 + z^2$$

Metric of Minkowski-space time. analogous to δ_{ij}

lower index

$$g_{\mu\nu} = \begin{cases} -1 & (0\,0) \\ +1 & 11, 22, 33 \\ 0 & \text{otherwise.} \end{cases}$$

raise index

$$g^{\mu\nu} = \begin{cases} -1 & (0\,0) \\ +1 & 11, 22, 33 \\ 0 & \text{otherwise.} \end{cases}$$

$$\underset{\text{contra-variant}}{x^\mu} \xrightarrow{g_{\mu\nu}} \underset{\text{covariant}}{x_\mu}$$

$\hookrightarrow$ $x_\mu = g_{\mu\nu} x^\nu \rightarrow (x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, x^3)$

then $s^2 = x_\mu x^\mu = (-ct)(ct) + x^2 + y^2 + z^2$

$$\stackrel{!}{=} g_{\mu\nu} x^\nu x^\mu$$

$$\stackrel{!}{=} g^{\mu\nu} x_\nu x_\mu$$

then s^2 must be same in different frames

$$g_{\mu\nu} x^\nu x^\mu \stackrel{?}{=} g_{\mu\nu} \tilde{x}^\nu \tilde{x}^\mu$$

So need:

$$g_{\mu\nu} x^\mu x^\nu = g_{\bar{\mu}\bar{\nu}} \tilde{x}^{\bar{\mu}} \tilde{x}^{\bar{\nu}}$$

$$\stackrel{!}{=} g_{\bar{\mu}\bar{\nu}} L^{\bar{\mu}}{}_{\mu} x^\mu L^{\bar{\nu}}{}_{\nu} x^\nu$$

with constraint for L ↳ $\boxed{g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} L^{\bar{\mu}}{}_{\mu} L^{\bar{\nu}}{}_{\nu}}$

$$g = L^T g L$$

Inner product is also invariant:

$$A^\mu B_\mu = \tilde{A}^\mu \tilde{B}_\mu$$

$$\stackrel{!}{=} A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$\stackrel{!}{=} -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

ex: $x^\mu = (ct, x, y, z)$

$$k^\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right)$$

$\hookrightarrow$ $$k_\mu x^\mu = \left(-\frac{\omega}{c}\right)(ct) + x k_x + y k_y + z k_z$$

$$= -\omega t + \vec{k}\cdot\vec{r} \quad \text{phase of wave.}$$

Ex 2: Taylor Theorem:

$$\phi(x+\delta x) \cong \phi(x) + \sum_\mu \delta x^\mu \left(? \right)_\mu$$

$$\approx \phi(x) + \delta x^\mu \underbrace{\frac{\partial \phi}{\partial x^\mu}}_{\partial_\mu \phi}$$

$\partial_\mu \phi$: covariant vector field.

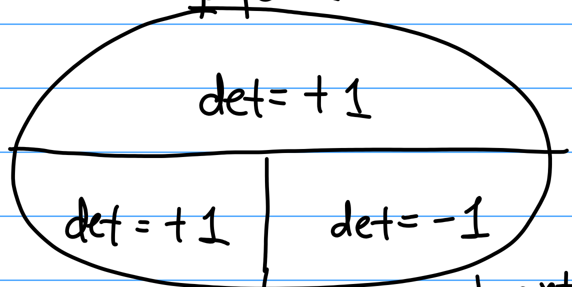
Few facts about LT:

1) $g = \Lambda^T g \Lambda \xrightarrow{\text{determinant}} \underbrace{\det(g)}_{(-1)} = \underbrace{\det(\Lambda^T)}_{\det(\Lambda)} \underbrace{\det(g)}_{(-1)} \det(\Lambda)$

$\hookrightarrow$ so $\boxed{\det(\Lambda) = \pm 1}$

2) Two pieces to LT:

proper $\leftarrow$ continuously connected to identity

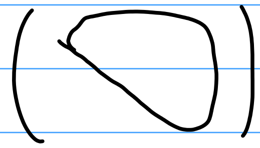


improper $\leftarrow$ not continuously connected to identity.

3) How many params?

$$g = \Lambda^T g \Lambda \quad : \quad 16 \text{ eqs}$$

$\nearrow$ 4×4

but Λ is anti-symmetric so 

only have 4 in diagonal and 6 in off-diagonal.

so we have 6 free variables.

Boost (velocity increase)

$$F = \begin{pmatrix} 0 & B_{10} & B_{20} & B_{30} \\ B_{10} & 0 & R_{12} & R_{13} \\ B_{20} & -R_{12} & 0 & R_{23} \\ B_{30} & -R_{13} & -R_{23} & 0 \end{pmatrix}$$

→ Rotated component

$$\Rightarrow \Lambda = \exp(F)$$

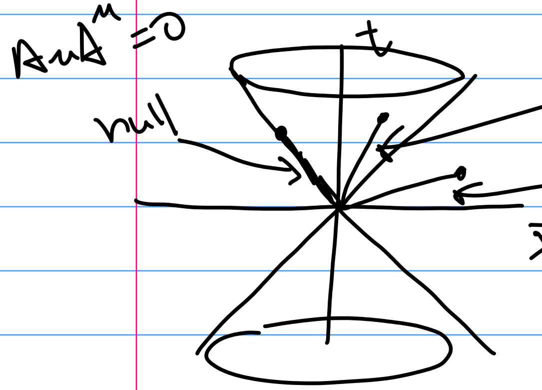
Lorentz Transformation form a group:

group needs to have the following property:

1) Identity
2) Inverse (undo)
3) closure (something something else = another thing)
4) associativity $\quad S''(S'\ S) = (S''\ S')\,S$

noncommutative: $\quad S'S \neq SS'$

Notation : spacelike / timelike / null.

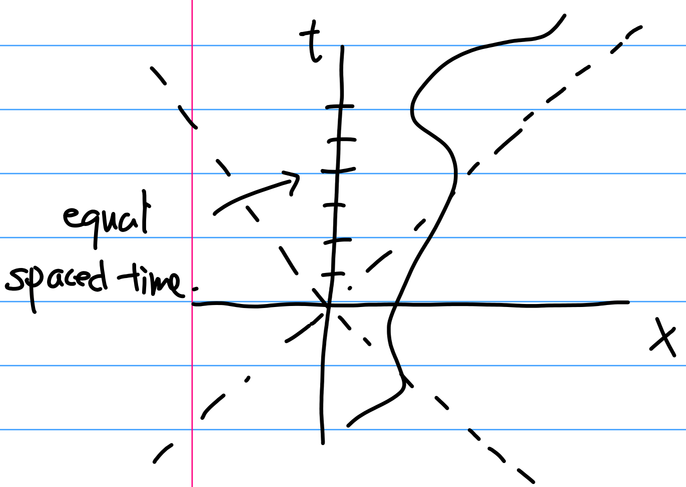


timelike : $A_\mu A^\mu < 0$

spacelike : $A_\mu A^\mu > 0$

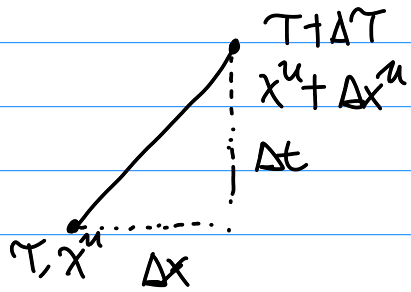
this is a frame invariant classification. .

4 - vectors in Kinematics:



use proper time , τ, as parameter for $x^\mu(\tau)$

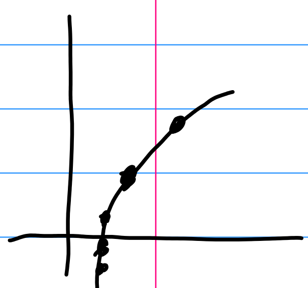
$\boxed{\tau :\ \text{time in the moving frame.}}$



$\hookrightarrow$ $\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta x)^2}$

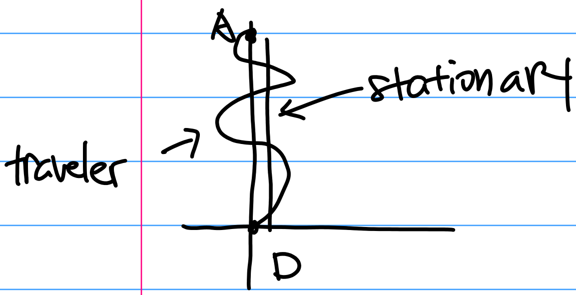
then $\underbrace{\tau_2 - \tau_1}_{\text{elapsed in moving frame.}} = \int_1^2 d\tau = \int_{T_1}^{T_2} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}$

moving velocity of the frame.

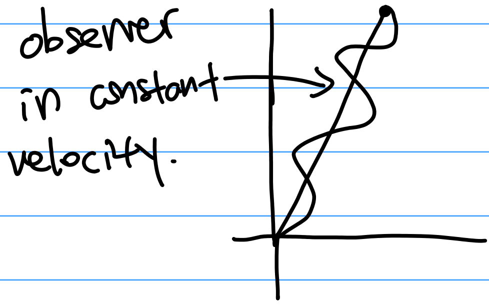


$T_2 - T_1 = \int_{T_1}^{T_2} dt$ $\leftarrow$ elapsed in reference frame

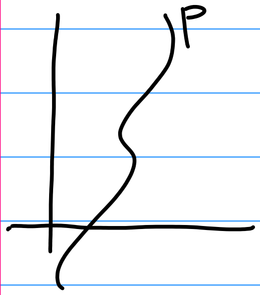
Twin Paradox : $\Delta \tau_{rest} > \Delta \tau_{traveler}$.



$\Rightarrow$



Action Principle : Particles:



$$S = -mc^2 \int_{T_1}^{T_2} dt \sqrt{1 - \frac{1}{c^2} \left| \frac{dx}{dt} \right|^2}$$

$$\boxed{S = -mc^2 \int_P d\tau}$$ ← action for special relativity. of free-particle.

Action for EM wave: $A_\mu = (\phi/c, \vec{A})$

$$\boxed{S = -mc^2 \int_P d\tau + q \int_P A_\mu \frac{dx^\mu}{d\tau} d\tau}$$

$$\delta S = \frac{d}{d\tau} P_\mu = q F_{\mu\nu} \frac{dx^\nu}{d\tau}$$

4-momentum

$$P_\mu = m \frac{dx^\mu}{d\tau}$$

$$F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$= \left(\begin{array}{c|ccc} 0 & \frac{1}{c}E^x & \frac{1}{c}E^y & \frac{1}{c}E^z \\ \hline -\frac{1}{c}E^x & 0 & B^z & -B^y \\ -\frac{1}{c}E^y & -B^z & 0 & B^x \\ -\frac{1}{c}E^z & B^y & -B^x & 0 \end{array} \right)$$

Initial conditions:

$\tau = 0 \quad : \quad (t_I, \vec{x}_I)$

only 3 out of 4 : $\frac{dx^\mu}{d\tau}$,

due to constraint eq: $\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = -1$

$$(\Delta\tau)^2 = (\Delta t)^2 - |\Delta \vec{x}|^2$$

4-velocity: $\frac{d}{d\tau} x^\mu = u(\tau)$

$u_\mu u^\mu = -1$ ← so the zeroth component is dependent on this.

$u^\mu = \gamma(1, \vec{v})$ where $\gamma = \frac{1}{\sqrt{1-|v|^2}}$

$u^\mu \rightarrow \boxed{m u^\mu = p^\mu} \rightarrow \boxed{p^0 = \gamma m = \frac{m}{\sqrt{1-v^2}}}$ ← energy of particle.

$\overset{(1,2,3)}{p^a} = \gamma m v$

then $p^\mu = (\varepsilon, \vec{p})$

For mass particles:

$$\boxed{P_\mu P^\mu = m^2 u_\mu u^\mu = -m^2}$$

$\hookrightarrow \quad -(p^0)^2 + |\vec{p}|^2 = -m^2$

$$-\varepsilon^2 + |\vec{p}|^2 = -m^2$$

$\hookrightarrow \quad \boxed{\varepsilon^2 = |\vec{p}|^2 c^2 + m^2 c^4}$ dispersion relation

For massless particles: $\boxed{P_\mu P^\mu = 0}$

$$\boxed{\varepsilon = |\vec{p}|\, c}$$

In general:

$$\boxed{\frac{d}{d\tau}(m u^\mu) = q F_{\mu\nu} \frac{dx^\nu}{d\tau}}$$

4-tensors: $F_{\mu\nu}$:

orbital angular momentum: $J^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu$

Levi civita : $\varepsilon^{\mu\nu\rho\sigma}$

Field:

scalar field Higgs:

$$\tilde{\psi}(\tilde{x}) = \psi(x) \quad \leftarrow \text{should be frame independent.}$$

$\tilde{x} = \Lambda x$ $\hookrightarrow$

$$\stackrel{!}{=} \psi(x(\tilde{x}))$$

Vector Field:

$$\tilde{A}^{\mu}(\tilde{x}) = \underbrace{\Lambda^{\mu}{}_{\nu}}_{\text{Lorentz Trans}} A_{\nu} \underbrace{(x(\tilde{x}))}_{\text{Lorentz Trans}}$$

From scalar field to vector Field via gradient:

$$B_{\mu} = \partial_{\mu}\phi$$
$$\stackrel{!}{=} \left(\frac{\partial}{\partial ct}\phi, \vec{\nabla}\phi\right)$$

Contraction: $\quad \partial_{\nu} B^{\mu} \rightarrow F_{\nu}{}^{\mu}$ 2 rank

$\partial_{\mu} B^{\mu} \rightarrow$ scalar , 0 rank.

Introduce : 4 - potential: $A^\mu = \left(\frac{1}{c}\phi, \vec{A}\right)$

4 - currents : $J^\mu = (c\rho, \vec{J})$

$$J^\mu A_\mu = -J^0 A^0 + \vec{J}\cdot\vec{A}$$

$$\overset{!}{=} -\rho\phi + \vec{J}\cdot\vec{A}$$

4 - wave vector: $K^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$

$$x^\mu = (ct, \vec{r})$$

$$K^\mu x_\mu = -\omega t + \vec{k}\cdot\vec{r}$$

Field strength 4-Tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Noting $\quad \partial_\mu = \left(\frac{\partial}{\partial ct}, \vec{\nabla}\right) \qquad \partial^\mu = \left(-\frac{\partial}{\partial ct}, \vec{\nabla}\right)$

$$\hookrightarrow F^{\mu\nu} = \left(\begin{array}{c:ccc} 0 & \frac{1}{c}E^x & \frac{1}{c}E^y & \frac{1}{c}E^z \\ -\frac{1}{c}E^x & 0 & +B^z & -B^y \\ -\frac{1}{c}E^y & -B^z & 0 & +B^x \\ -\frac{1}{c}E^z & +B^y & -B^x & 0 \end{array}\right)$$

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{c^2}|E|^2 + |B|^2$$

$$\overset{!}{=} -\frac{1}{c^2}\left|-\vec{\nabla}\phi - \partial_t\vec{A}\right|^2 + (\vec{\nabla}\times\vec{A})^2$$

In the end:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu$$

$\uparrow$ field $\qquad$ $\uparrow$ matter sector

$$S[A] = \frac{1}{c} \int d^4 r^\mu \left\{ -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right\}$$

with $\delta S = 0$

$$\hookrightarrow \boxed{\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu}$$

with

$$\boxed{\begin{array}{lll} \nu = 0 & \Rightarrow & \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \\ \nu = 1,2,3 & \Rightarrow & \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} + \mu_0 \vec{J} \end{array}}$$

use div and curl:

$$\partial^\sigma \underbrace{\varepsilon_{\rho\sigma\mu\nu} F^{\mu\nu}}_{{}^*F_{\rho\sigma} \text{ (dual of } F^{\mu\nu})} = 0 \quad \Rightarrow \quad \boxed{\begin{array}{l} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \end{array}}$$