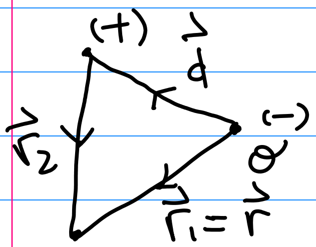
Zhi Chen EM HW#3

1) Electric Dipoles: Find the force $\vec{F}_{12}$ exerted by a dipole $\vec{p}_1$ on another dipole $\vec{p}_2$.



First calculate electric field of the dipole.

We know point charge has electric field:

$$E(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{|\vec{r}-\vec{r}'|^2} \hat{r}$$

Then we place origin at (−), and so

$$\vec{E}^{(-)} = \frac{1}{4\pi\epsilon_0} \frac{-Q}{|\vec{r}|^2} \hat{r}$$

and Electric field of (+) is then

$$\vec{E}^{(+)} = \frac{1}{4\pi\epsilon_0} \frac{Q}{|\vec{r}_2|^2} \hat{r}_2 = \frac{1}{4\pi\epsilon_0} \frac{Q}{|\vec{r}-\vec{d}|^3} (\vec{r}-\vec{d})$$

at limit $d \to 0$, $r \gg d$,

$$\vec{E}^{+} = \frac{1}{4\pi\epsilon_0} \frac{Q}{(r^2 - 2\vec{r}\cdot\vec{d} + d^2)^{3/2}} (\vec{r}-\vec{d})$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^3} \left[1 - 2\hat{r}\cdot\left(\frac{\vec{d}}{r}\right) + \mathcal{O}\left(\frac{d}{r}^2\right)\right]^{-3/2} (\vec{r}-\vec{d})$$

$$\simeq \frac{1}{4\pi\epsilon_0} \frac{Q}{r^3} \left[1 + 3\hat{r}\cdot\left(\frac{\vec{d}}{r}\right) + \cdots\cdots\right] (\vec{r}-\vec{d})$$

$$\simeq \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r^2}\hat{r} - \frac{Q}{r^3}\vec{d} + \frac{3Q(\hat{r}\cdot\vec{d})}{r^4}(\vec{r}-\vec{d})\right]$$

Now combine E-field :

$$\vec{E}_{tot} = \vec{E}^{(-)} + \vec{E}^{(+)}$$

$$= \frac{1}{4\pi\epsilon_0}\cancel{\frac{-Q}{r^2}\hat{r}} + \frac{1}{4\pi\epsilon_0}\left[\cancel{\frac{Q}{r^2}\hat{r}} - \frac{Q}{r^3}\vec{d} + \frac{3Q(\hat{r}\cdot\vec{d})}{r^4}(\vec{r}-\vec{d})\right]$$

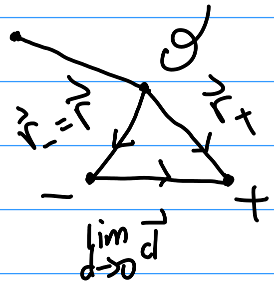
recognize $Q\vec{d} = \vec{p}$

$$= \frac{1}{4\pi\epsilon_0}\left[-\frac{\vec{p}}{r^3} + 3\frac{(\hat{r}\cdot\vec{p})(\hat{r}-\vec{d})}{r^3}\right]$$

$\lim_{d\to 0}$

$$\vec{E}_{tot} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^5}\left[3(\vec{r}\cdot\vec{p})\vec{r} - \vec{p}\,r^2\right]$$

Now calculate force exerted from p_2 to p_1.



Find how p_2 affect (+) and (−) separately then add together.

$$\Rightarrow \vec{F}_- = (-Q)\vec{E}_{p_2} = \frac{1}{4\pi\epsilon_0}\frac{-Q}{r^5}\left[3(\vec{r}\cdot\vec{p}_2)\vec{r} - \vec{p}_2\,r^2\right]$$

$$\Rightarrow \vec{F}_+ = Q\,\vec{E}_{p_2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_+^5}\left[3(\vec{r}_+\cdot\vec{p}_2)\vec{r}_+ - \vec{p}_2\,r_+^2\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{|\vec{r}+\vec{d}|^5}\left[3\left((\vec{r}+\vec{d})\cdot\vec{p}_2\right)(\vec{r}+\vec{d}) - \vec{p}_2\,|\vec{r}+\vec{d}|^2\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{(r^2+2\vec{r}\cdot\vec{d}+d^2)^{5/2}}\left[3\left[(\vec{r}+\vec{d})\cdot\vec{p}_2\right](\vec{r}+\vec{d}) - \vec{p}_2\left(r^2+2\vec{r}\cdot\vec{d}+\mathcal{O}(d^2)\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^5}\left[1+2\hat{r}\cdot\left(\frac{\vec{d}}{r}\right)+\mathcal{O}\left(\left(\frac{d}{r}\right)^2\right)\right]^{-5/2}\left\{3\left[(\vec{r}+\vec{d})\cdot\vec{p}_2\right](\vec{r}+\vec{d}) - \vec{p}_2\left(r^2+2\vec{r}\cdot\vec{d}\right)\right\}$$

$$\stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\frac{Q}{r^5}\left[1-5\hat{r}\cdot\frac{\vec{d}}{r}+\cdots\right]\left\{3\left[(\vec{r}+\vec{d})\cdot\vec{p}_2\right](\vec{r}+\vec{d}) - \vec{p}_2\left(r^2+2\vec{r}\cdot\vec{d}\right)\right\}$$

$$\hookrightarrow = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^5}\left[1-5\hat{r}\cdot\frac{\vec{d}}{r}\right]\left\{3(\vec{r}\cdot\vec{p}_2+\vec{d}\cdot\vec{p}_2)(\vec{r}+\vec{d})-\vec{p}_2(r^2+2\vec{r}\cdot\vec{d})\right\}$$

$$\vec{F}_+ = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^5}\left[1-5\hat{r}\cdot\frac{\vec{d}}{r}\right]\left[3\left\{(\vec{r}\cdot\vec{p}_2)\hat{r}+(\vec{r}\cdot\vec{p}_2)\vec{d}+(\vec{d}\cdot\vec{p}_2)\hat{r}+\mathcal{O}(d^2)\right\}-\vec{p}_2(r^2+2\vec{r}\cdot\vec{d})\right]$$

$$F_{tot} = F_- + F_+$$

$$\stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\frac{-Q}{r^5}\cancel{\left[3(\vec{r}\cdot\vec{p}_2)\vec{r}-\vec{p}_2 r^2\right]}$$

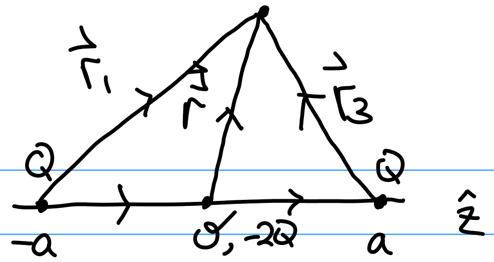
$$+\frac{1}{4\pi\epsilon_0}\frac{Q}{r^5}\left[1-5\hat{r}\cdot\frac{\vec{d}}{r}\right]\left[3\left\{\cancel{(\vec{r}\cdot\vec{p}_2)\vec{r}}+(\vec{r}\cdot\vec{p}_2)\vec{d}+(\vec{d}\cdot\vec{p}_2)\vec{r}\right]\cancel{-\vec{p}_2 r^2}-2\vec{p}_2(\vec{r}\cdot\vec{d})\right]$$

$$\hookrightarrow = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^5}\left[3(\vec{r}\cdot\vec{p}_2)\vec{d}+3(\vec{d}\cdot\vec{p}_2)\vec{r}-2\vec{p}_2(\vec{r}\cdot\vec{d})-15\left(\hat{r}\cdot\frac{\vec{d}}{r}\right)(\vec{r}\cdot\vec{p}_2)\vec{r}+5\left(\hat{r}\cdot\frac{\vec{d}}{r}\right)\vec{p}_2 r^2\right]$$

$$\stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\left[3(\hat{r}\cdot\vec{p}_2)\vec{p}_1+3(\vec{p}_1\cdot\vec{p}_2)\hat{r}-2\vec{p}_2(\hat{r}\cdot\vec{p}_1)-15(\hat{r}\cdot\vec{p}_1)(\hat{r}\cdot\vec{p}_2)\hat{r}+5(\hat{r}\cdot\vec{p}_1)\vec{p}_2\right]$$

$$\boxed{F_{tot} = \frac{1}{4\pi\epsilon_0}\frac{3}{r^4}\left[(\hat{r}\cdot\vec{p}_2)\vec{p}_1+(\vec{p}_1\cdot\vec{p}_2)\hat{r}-\frac{2}{3}\vec{p}_2(\hat{r}\cdot\vec{p}_1)-5(\hat{r}\cdot\vec{p}_1)(\hat{r}\cdot\vec{p}_2)\hat{r}+\frac{5}{3}(\hat{r}\cdot\vec{p}_1)\vec{p}_2\right]}$$

2) <u>Dipole and Quadrupole:</u>



Determine Asymptotic behavior of the far-field electrostatic potential

Point Charge Potential is: $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{|\vec{r}-\vec{r}'|}$

$$\phi^{(-)} = \frac{1}{4\pi\epsilon_0} \frac{-2Q}{|\vec{r}|} = \frac{1}{4\pi\epsilon_0} \frac{-2Q}{r} \leftarrow \text{cylindrical coord.}$$

$$\phi_r^{(+)} = \frac{1}{4\pi\epsilon_0} \frac{Q}{|\vec{r}-a\hat{z}|} = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{r^2-2az+a^2}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \left(1-2\frac{az}{r^2}+\left(\frac{a}{r}\right)^2\right)^{-1/2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \left(1+\frac{az}{r^2}-\frac{1}{2}\left(\frac{a}{r}\right)^2+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2}\left[-2\frac{az}{r^2}+\left(\frac{a}{r}\right)^2\right]^2\right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \left(1+\frac{az}{r^2}-\frac{1}{2}\left(\frac{a}{r}\right)^2+\frac{3}{8}\left[4\frac{z^2}{r^2}\frac{a^2}{r^2}+\mathcal{O}\left(\left(\frac{a}{r}\right)^3\right)\right]\right)$$

$$\phi_r^{(+)} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \left(1+\frac{a}{r}\frac{z}{r}+\left(\frac{a}{r}\right)^2\left[\frac{3}{2}\left(\frac{z}{r}\right)^2-\frac{1}{2}\right]\right)$$

Similarly

$$\phi_l^{(-)} = \frac{1}{4\pi\epsilon_0} \frac{Q}{(\vec{r}+a\vec{z})} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \left(1+2\frac{a}{r}\frac{z}{r}+\left(\frac{a}{r}\right)^2\right)^{-1/2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \left(1-\frac{a}{r}\frac{z}{r}+\left(\frac{a}{r}\right)^2\left[\frac{3}{2}\left(\frac{z}{r}\right)^2-\frac{1}{2}\right]+\mathcal{O}\left(\left(\frac{a}{r}\right)^3\right)\right)$$

$$\phi_{tot} = \phi^{(-)} + \phi_l^{(+)} + \phi_r^{(+)}$$

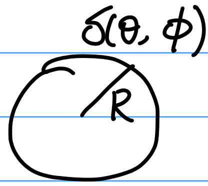
$$= \frac{1}{4\pi\epsilon_0}\cancel{\frac{-2Q}{r}} + \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\left(\cancel{1} + \cancel{\frac{a}{r}\frac{z}{r}} + \left(\frac{a}{r}\right)^2\left[\frac{3}{2}\left(\frac{z}{r}\right)^2 - \frac{1}{2}\right]\right)$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\left(\cancel{1} - \cancel{\frac{a}{r}\frac{z}{r}} + \left(\frac{a}{r}\right)^2\left[\frac{3}{2}\left(\frac{z}{r}\right)^2 - \frac{1}{2}\right]\right)$$

$$\boxed{\phi_{tot} = \frac{1}{4\pi\epsilon_0}\frac{Qa^2}{r^3}\left[3\left(\frac{z}{r}\right)^2 - 1\right]}$$

$$\frac{z}{r} = \cos\theta$$

3) Shell of immobile charge:



B.C.: $\Phi(r,\theta,\phi)\Big|_{r=R} = \gamma \sin\theta\cos\phi$

a) Find Φ for $r \le R$, $r > R$

we know $\nabla \cdot E = \dfrac{\rho}{\varepsilon_0}$

$$\hookrightarrow = -\nabla^2 \Phi(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

For $r<R$, $r>R$, $\rho = 0$

$$\hookrightarrow \quad \nabla^2 \Phi = 0$$

In spherical polar coordinate, using separation of variable we have solution:

$$\Phi(\vec{r}) = R(r)\,Y(\theta,\phi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm} r^{l} + B_{lm} r^{-l-1}\right) Y_l^m(\theta,\phi)$$

where $Y_l^m(\theta,\phi) = \epsilon \sqrt{\dfrac{2l+1}{2}}\sqrt{\dfrac{(l-|m|)!}{(l+|m|)!}}\, P_l^m(\cos\theta)\,\dfrac{e^{im\phi}}{\sqrt{2\pi}}$

$$\epsilon = \begin{cases} (-1)^m & \text{for } m \ge 0 \\ 1 & \text{for } m < 0 \end{cases}$$

B.C. to consider: $\Phi \to 0$ for $r > R$ as $r \to \infty$

$\Phi =$ finite for $r < R$

$\Phi(r,\theta,\phi)\Big|_{r=R} = \gamma\sin\theta\cos\phi.$

For $r > R$, we know $\Phi \to 0$ as $r \to \infty$

So $$\Phi_{out}(\vec{r}) = \sum_{lm} \left(\cancel{A_{lm} r^l} + B_{lm} r^{-l-1} \right) Y_l^m(\theta, \phi)$$

$$A_{lm} = 0 \quad \hookrightarrow \quad \stackrel{!}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} r^{-l-1} Y_l^m(\theta, \phi)$$

And $r < R$, we want $\Phi \to$ finite as $r \to 0$

so $$\Phi_{in}(\vec{r}) = \sum_{lm} \left(A_{lm} r^l + B_{lm} r^{-l-1} \right) Y_l^m(\theta, \phi)$$

$$B_{lm} = 0 \quad \hookrightarrow \quad \stackrel{!}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_l^m(\theta, \phi)$$

Now we use $\Phi(r=R, \phi, \theta) = \gamma \sin\theta \cos\phi$

We know $Y_l^m \propto e^{im\phi}$, so in order to reproduce $\cos\phi$,

we need $\boxed{m = \pm 1}$, so $\dfrac{e^{i\phi} + e^{-i\phi}}{2} = \cos\phi$.

And $Y_l^m(\theta, \phi) \propto \sin\theta$ when $\boxed{l = 1}$

So $$\Phi_{in}(\vec{r}) = r \left(A_{11} Y_1^1 + A_{1-1} Y_1^{-1} \right)$$

$$\Phi_{out}(\vec{r}) = r^{-2} \left(B_{11} Y_1^1 + B_{1-1} Y_1^{-1} \right)$$

using $\Phi(r=R, \theta, \phi) = \gamma \sin\theta \cos\phi$

$$\Phi_{in}(r=R) = R \left(A_{11} Y_1^1 + A_{1-1} Y_1^{-1} \right) = \gamma \sin\theta \cos\phi$$

$$\Phi_{out}(r=R) = \frac{1}{R^2} \left(B_{11} Y_1^1 + B_{1-1} Y_1^{-1} \right) = \gamma \sin\theta \cos\phi$$

Know $Y_1^1 = \overset{\swarrow \text{some constant.}}{C_1^1} e^{i\phi} \sin\theta$

$$Y_1^{-1} = C_1^{-1} e^{-i\phi} \sin\theta$$

Then $\Phi_{in}(r=R) = R\big(\underbrace{A_{11} C_1^1}_{=D} e^{i\phi} + \underbrace{A_{1-1} C_1^{-1}}_{=G} e^{-i\phi}\big)\sin\theta = \gamma \sin\theta\cos\phi$

$$\overset{!}{=} R\left[(D-G)e^{i\phi} + 2G\cos\phi\right]\sin\theta \qquad = \gamma\sin\theta\cos\phi$$

By matching, we see $D-G=0 \Rightarrow D=G$

so $\quad 2RG = \gamma \quad$ or $\quad 2G = \frac{\gamma}{R}$

so $\boxed{\Phi_{in}(\vec{r}) = \gamma \frac{r}{R} \sin\theta\cos\phi \qquad r<R}$

Similarly:

$$\Phi_{out}(r=R) = \frac{1}{R^2}\left(B_{1\,1} C_1^1 e^{i\phi} + B_{1\,-1} C_1^{-1} e^{-i\phi}\right)\sin\theta = \gamma\sin\theta\cos\phi$$

with the same argument for Φ_{in}, we find $B_{11} C_1^1 = B_{1-1} C_1^{-1} = B$

So $\quad \Phi_{out}(r=R) = \frac{2B}{R^2}\sin\theta\cos\phi = \gamma\sin\theta\cos\phi$

then $\quad 2B = R^2\gamma$

$\hookrightarrow \boxed{\Phi_{out}(\vec{r}) = \gamma\left(\frac{R}{r}\right)^2 \sin\theta\cos\phi \qquad r>R}$

b) since $-\nabla^2\Phi = \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

and $\int \vec{\nabla}\cdot\vec{E}\, dV = \int \frac{\rho}{\varepsilon_0}\, dV$

$\hookrightarrow \int \vec{E}\cdot d\vec{a} = \int \frac{1}{\varepsilon_0} \underbrace{\rho\, dl}_{\sigma} \underbrace{dA}_{A}$

$\hookrightarrow \vec{E}\cdot\hat{n} = \sigma/\varepsilon_0$

$\hookrightarrow -\vec{\nabla}\Phi\cdot\hat{n} = \frac{1}{\varepsilon_0}\sigma$

since $\hat{n} = \hat{r}$: $\left[-\frac{\partial\Phi}{\partial r}\bigg|_{out}\right] - \left[-\frac{\partial\Phi}{\partial r}\bigg|_{in}\right] = \sigma/\varepsilon_0$

$-\frac{\partial\Phi}{\partial r}\bigg|_{in} = -\frac{\gamma}{R}\sin\theta\cos\phi$

$-\frac{\partial}{\partial r}\Phi\bigg|_{out} = \gamma R^2\frac{2}{r^3}\sin\theta\cos\phi\bigg|_{r=R} = \gamma\frac{2}{R}\sin\theta\cos\phi$

then $\boxed{\sigma = \varepsilon_0\left[\frac{\gamma}{R} + \gamma\frac{2}{R}\right]\sin\theta\cos\phi = 3\frac{\gamma\varepsilon_0}{R}\sin\theta\cos\phi}$

c) Find total electrostatic energy. $\mathcal{E} = \frac{\varepsilon_0}{2}\int |E|^2 dV$

Find $\vec{E}$: $\vec{\nabla} = \partial_r \hat{r} + \frac{1}{r}\partial_\theta \hat{\theta} + \frac{1}{r\sin\theta}\partial_\phi \hat{\phi}$

For $r < R$:

$$\vec{E} = -\vec{\nabla}\Phi = -\frac{\gamma}{R}\left\{\sin\theta\cos\phi\,\hat{r} + \frac{1}{r} r\cos\theta\cos\phi\,\hat{\theta} + \frac{-1}{r\sin\theta} r\sin\theta\sin\phi\,\hat{\phi}\right\}$$

$$\vec{E} \stackrel{!}{=} -\frac{\gamma}{R}\left\{\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right\} \quad \text{for } r<R$$

For $r > R$:

$$\vec{E} = -\vec{\nabla}\Phi = -\gamma R^2\left\{\frac{-2}{r^3}\sin\theta\cos\phi\,\hat{r} + \frac{1}{r}\frac{1}{r^2}\cos\theta\cos\phi\,\hat{\theta} + \frac{-1}{r\sin\theta}\frac{1}{r^2}\sin\theta\sin\phi\,\hat{\phi}\right\}$$

$$\stackrel{!}{=} -\gamma\frac{R^2}{r^3}\left(-2\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right\} \quad \text{for } r>R$$

$$\mathcal{E} = \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\left\{\int_0^R r^2\sin\theta\, d\theta\, d\phi\, dr \left(\frac{\gamma}{R}\right)^2 \underbrace{\left[\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi\right]}_{=1}\right.$$

$$\left. + \int_R^\infty r^2\sin\theta\, d\theta\, d\phi\, dr\,(\gamma R^2)^2\frac{1}{r^6}\underbrace{\left[4\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi\right]}_{3\sin^2\theta\cos^2\phi + 1}\right\}$$

$$= \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\left\{\sin\theta\, d\theta\, d\phi\,\frac{R^3}{3}\frac{\gamma^2}{R^2} \underbrace{-\frac{1}{3}\frac{1}{r^3}\Big|_R^\infty}_{1/3R^3}\sin\theta\, d\theta\, d\phi\,(\gamma R^2)^2\left[3\sin^2\theta\cos^2\phi + 1\right]\right\}$$

$$\stackrel{!}{=} \frac{\varepsilon_0}{2}\frac{R}{3}\gamma^2\left[4\pi + 4\pi + 3\underbrace{\int_0^\pi \sin^3\theta}_{4/3}\ \underbrace{\int_0^{2\pi}\cos^2\phi}_{\pi}\right]$$

$$\stackrel{!}{=} \frac{\varepsilon_0}{2}\frac{R}{3}\gamma^2\left[8\pi + 3\,\frac{4}{3}\pi\right]$$

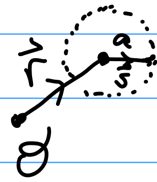
$$\boxed{\mathcal{E} \stackrel{!}{=} 2\pi\varepsilon_0 R\gamma^2}$$

4) Mean - value theorem and application:

-Mean value theorem: Φ (solution of $\nabla^2\Phi=0$) at any point $\vec{r}$ and the average over a sphere centered at $\vec{r}$ and equal to one another, regardless of radius of sphere.

$$\vec{\Phi}(\vec{r}) = \langle \Phi(\vec{r}) \rangle_a = \frac{1}{4\pi a^2}\int_{|\vec{s}|=a} d^2s\ \Phi(r+s)$$

b) general solution of $\Phi(r)$ in spherical polar:



It has a general solution:

$$\Phi(\vec{r}) = R(r)\,Y(\theta,\phi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \underbrace{(A_l r^l + B_l r^{-l-1})}_{R_l(|r|)} Y_l^m(\theta,\phi)$$

$$\Phi(\vec{r}) = \langle \Phi(\vec{r}) \rangle_a = \frac{1}{4\pi a^2}\int a^2 \sin\theta\, d\theta\, d\phi\ \Phi(\vec{r}+\vec{s})$$

$$\overset{!}{=} \frac{1}{4\pi}\int \sin\theta' d\theta' d\phi' \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_l(|\vec{r}+\vec{s}|)\,Y_l^m(\theta',\phi')$$

lets consider $\vec{r}=0$, i.e. at the origin:

$$\Phi(r=0) = \sum_{l,m} R^l(0)\ Y_l^m(\theta,\phi)$$

$$\overset{!}{=} \sum_{l,m}\Big[\underbrace{A_{lm}(0)^l}_{=\begin{cases}0 & l\neq 0\\ 1 & l=0\end{cases}} + \underbrace{B_{lm}(0)^{-l-1}}_{\text{gives } \infty,}\Big] Y_l^m(\theta,\phi)$$

but we want finite, so $\boxed{B_{lm}=0}$

since $l=0 \to m=0$

$$= A_{00} Y_0^0(\theta,\phi)$$

$$\boxed{\Phi(r=0) = \frac{1}{\sqrt{4\pi}} A_{00}}$$

$$\phi(0) = \langle \phi(0) \rangle_a = \frac{1}{4\pi} \int \sum_{l m} R^l(a)\, Y_l^m(\theta,\phi)\, d^2S$$

$$= \frac{1}{4\pi} \sum_{l,m} R^l(a) \int Y_l^m(\theta,\phi)\, d^2S$$

$$= \frac{1}{4\pi} \sum_{l,m} R^l(a) \sqrt{4\pi} \int Y_l^m(\theta,\phi) \underbrace{\frac{1}{\sqrt{4\pi}}}_{Y_0^0} d^2S$$

Due to orthogonality

$$\int Y_l^m Y_0^0\, d^2S = \delta_{m0}\delta_{l0}$$

$$= \frac{1}{4\pi} \sum_{l,m} R^l(a) \sqrt{4\pi}\, \delta_{m,0}\delta_{l,0}$$

$$= \frac{1}{\sqrt{4\pi}} R^{l=0}(a)$$

$$= \frac{1}{\sqrt{4\pi}} \left(A_{lm} a^l + \underbrace{B_{lm}} a^{-l-1} \right)\Big|_{l,m=0}$$

$B_{lm}=0$ as we saw previously.

$$\boxed{\langle \Phi(0) \rangle_a = \frac{1}{\sqrt{4\pi}} A_{00}}$$

We see that $\Phi(0) = \langle \Phi(0) \rangle_a$, and we can always have freedom of choosing the origin, this means

$$\Phi(r) = \langle \Phi(r) \rangle_a .$$