## Essential Ingredient:

| | | Micro | Macro |
|---|---|---|---|
| charge | extrinsic, free, stuck | $\rho_{ext}(\vec{r})$ | |
| | intrinsic, get polarized, or displaced | $\rho_{int}(\vec{r})$ $\rightarrow$ | $\rho_p(\vec{r})$, $\sigma_p(\vec{r})$ |

surface p-charge density.

Create new field: $\vec{P}(\vec{r})$, dielectric polarization

$*$
$$\boxed{\begin{aligned} \rho_p(\vec{r}) &= -\vec{\nabla}\cdot\vec{P} \\ \sigma_p(\vec{r}) &= \hat{n}\cdot\vec{P} \end{aligned}}$$

$\vec{p} = \int d^3r \, \vec{P}$ ← $\vec{P}$: dipole moment density.

Implications: $\vec{\nabla}\times\vec{e} = 0$, $\vec{\nabla}\cdot\vec{e} = \rho/\varepsilon_0$

$\vec{\nabla}\cdot\vec{e} = \frac{1}{\varepsilon_0}(\rho_{int} + \rho_{ext})$, $\rho_{int} = -\vec{\nabla}\cdot\vec{p}$

$\xrightarrow{\text{Lorentz Avg}}$ $\vec{\nabla}\times\vec{E} = 0$

$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(Q(\vec{r}) - \vec{\nabla}\cdot\vec{p})$

↳ $\vec{\nabla}\cdot(\underbrace{\varepsilon_0\vec{E} + \vec{p}}_{=\vec{D}}) = Q(\vec{r}) = \rho_{ext}$

$$\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{p}}$$

$\vec{\nabla}\cdot\vec{D} = Q(\vec{r}) = \rho_{ext}$

$Q(\vec{r})$: extrinsic charge density.

## Maxwell Equations in Dielectric:

i) $\vec{\nabla}\times\vec{E} = 0$

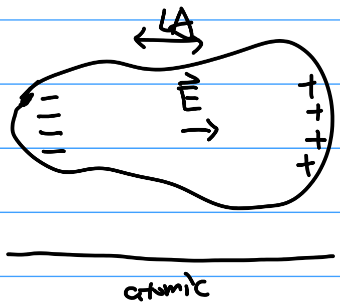
ii) $\vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_{int}$

↳ $\vec{\nabla}\cdot(\varepsilon\vec{E}) = \rho_{int} \rightleftharpoons -\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi) = \rho_{int}$

↳ $\vec{\nabla}\cdot\vec{D} = \rho_{int}$.

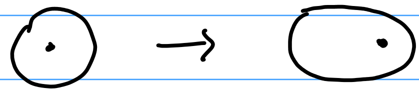
$\vec{P} = (\varepsilon - \varepsilon_0)\vec{E}$

(Thought)
## Gedanken Experiment :



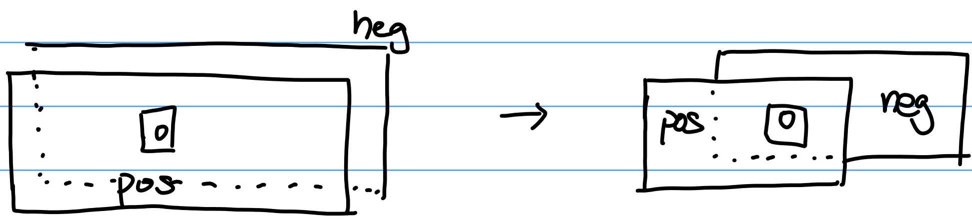

i) $\vec{E}_{ext} = 0 \rightarrow \vec{p} = 0 \rightarrow \rho_p = 0 \ , \ \sigma_p = 0$

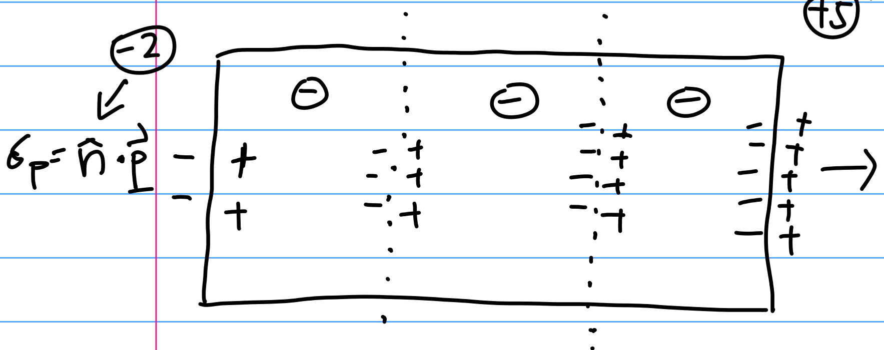
ii) $\vec{E}_{ext} = 0 \ , \ \vec{p} \neq 0 \ , \ \rho_p \neq 0 \ , \ \sigma_p \neq 0$

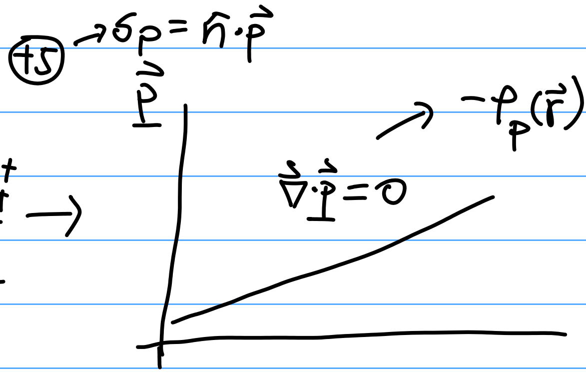


Note that : no new charge is created.

$$\int d^3r \ \rho_p + \int d^2s \ \sigma_p = -\int d^3r \ \vec{\nabla} \cdot \vec{p} + \int d^2s \ \hat{n} \cdot \vec{p}$$
$$= -\int_S d^2\vec{s} \cdot \vec{p} + \int d^2s \ \hat{n} \cdot \vec{p}$$
$$= 0$$



Energy / Length Scales : $\hbar$, $m_e$, $e$, $\varepsilon_0$

i) length scale: $a_B \cong \frac{1}{2}$ Å $\rightleftharpoons$ $\frac{e^2}{4\pi\varepsilon_0 a_B} = \frac{\hbar^2}{2m a_B^2}$

ii) energy scale: $\mathcal{E}_R = \frac{1}{32\pi^2} \frac{m_e{}^4}{\varepsilon_0^2 \hbar^2} = 2\times10^{-18}$ J

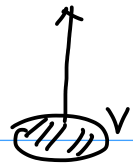
iii) Electric field scale: $\mathcal{E}_R = a_B e E$

$$E = \frac{1}{128\pi^3} \frac{m^2 e^5}{\varepsilon_0^3 \hbar^4} = 2.6\times10^{11} \text{ V/m}$$

iv) Voltage Scale:

$E a_B \cong 13.6$ V

## Problem:




$\nabla^2 \Phi = 0$ in cylindrical polar coordinate:

$$\hookrightarrow \left( \frac{1}{r} \partial_r r \partial_r + \cancel{\frac{1}{r^2} \partial_\phi^2} + \partial_z^2 \right) \Phi(r, \cancel{\phi}, z) = 0$$

$\hookrightarrow$ let $\Phi(r,z) = W(r)\, Z(z)$

$$\underbrace{\frac{1}{rW} \partial_r r \partial_r W}_{\substack{\text{oscillation} \\ -\alpha^2}} + \underbrace{\frac{\partial_z^2 Z}{Z}}_{\substack{\text{decay/growth} \\ +\alpha^2}} = 0$$

$\hookrightarrow \frac{1}{r} \partial_r r \partial_r W + \alpha^2 W = 0$ $\qquad \hookrightarrow Z(z) = \cancel{e^{\alpha z}}, e^{-\alpha z}$

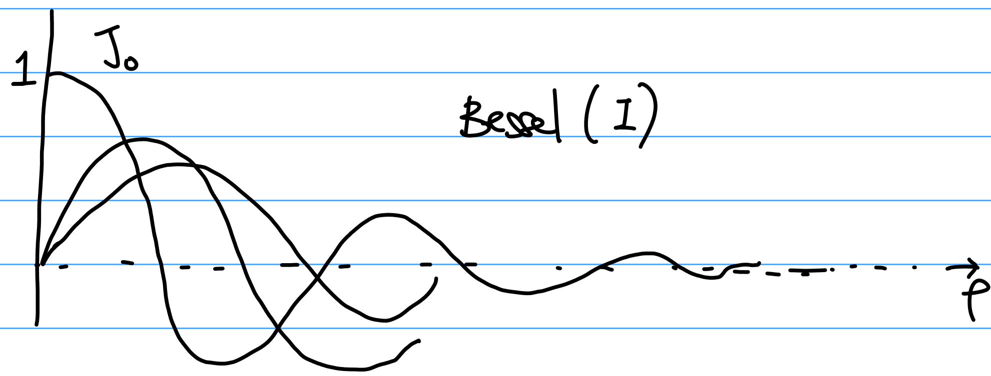
$\hookrightarrow W'' + \frac{1}{r} W' + \alpha^2 W = 0$

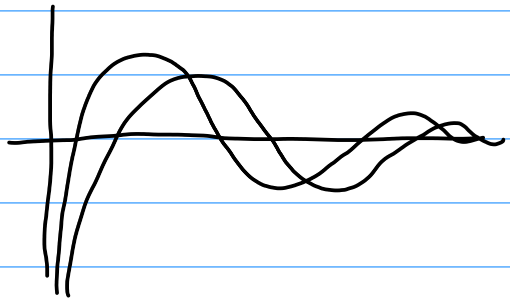
Bessel's equation

$$\left.\begin{aligned} \text{let } W(r) &= S(\rho) \\ \alpha r &= \rho \end{aligned}\right\} \boxed{S'' + \frac{1}{\rho} S' + \left[1 - \frac{m^2}{\rho^2}\right] S = 0}$$

With solutions: $J_m(\rho) \sim \cos\rho$ (Regular) Bessel Function of first kind.

$N_{|m|}(\rho)$ : Neumann, Bessel Function of second kind



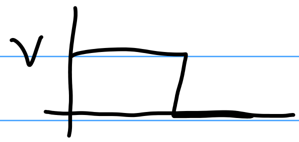


At lage $x$:

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$\Phi(r,z) = \int_0^\infty d\alpha \; \alpha \, A(\alpha) \, e^{-\alpha z} J_0(\alpha r)$$




$$V H(R-r) = \Phi(r, z=0) = \int_0^\infty d\alpha \; \alpha \, A(\alpha) J_0(\alpha r)$$

FT: $$\hat{f}(q) = \int_{-\infty}^{\infty} dx \, f(x) \, e^{-iqx}$$

$$f(x) = \int_{-\infty}^{\infty} dq \, \hat{f}(q) \, e^{iqx}$$

FBT: $$\hat{g}(q) = \int_0^\infty dx \, x \, g(x) J_0(qx)$$

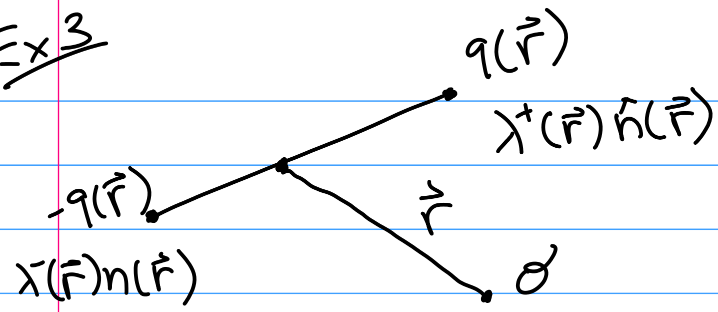
$$g(x) = \int_0^\infty dq \, q \, \hat{g}(q) J_0(qx)$$

$$A(\alpha) = V \int_0^R dr \, r \, J_0(\alpha r) = \frac{VR}{\alpha} J_1(\alpha R)$$

$$\boxed{\Phi(r,z) = \int_0^\infty d\alpha \; \cancel{\alpha} \, \frac{VR}{\cancel{\alpha}} J_1(\alpha R) \, e^{-\alpha z} J_0(\alpha r)}$$

on-axis, $\Phi(0,z) = VR \int_0^\infty d\alpha \, e^{-\alpha z} J_1(\alpha R) = V\left\{1 - \frac{z}{\sqrt{z^2+R^2}}\right\}$

Ex 3




Here, $\lambda^{\pm}$ are much shorter than lorentz averaging length scale. So they're dipoles above the averaging length scale, $\rho(\vec{R})=0$

$$\rho_p(\vec{R}) = q(\vec{R}) - q(\vec{R}) = \int d^3r\, q(\vec{r})\left[\delta(\vec{R}-\vec{r}) - \delta(\vec{R}-\vec{r})\right]$$

$$\hookrightarrow = \int d^3r\, q(\vec{r})\left[\delta(\vec{R}-(\vec{r}+\lambda^+(\vec{r})\hat{n}(\vec{r}))) - \delta(\vec{R}-(\vec{r}-\lambda^-(\vec{r})\hat{n}(\vec{r})))\right]$$

$$= \int d^3r\, q(\vec{r})\Big[\cancel{\delta(\vec{R}-\vec{r})} - \lambda^+(\vec{r})\hat{n}(\vec{r})\vec{\nabla}_R\,\delta(\vec{R}-\vec{r})$$
$$- \cancel{\delta(\vec{R}-\vec{r})} - \lambda^-(\vec{r})\hat{n}(\vec{r})\vec{\nabla}_R\,\delta(\vec{R}-\vec{r})\Big]$$

$\delta(x)=\delta(-x)$

$$= -\vec{\nabla}_R \cdot \int d^3r\, \underbrace{\{q(\vec{r})[\lambda^+(\vec{r}) + \lambda^-(\vec{r})]\,\hat{n}(\vec{r})\}}_{\vec{P}(\vec{r})}\, \delta(\vec{r}-\vec{R})$$

$$= -\vec{\nabla}_R \cdot \vec{P}(\vec{r})$$

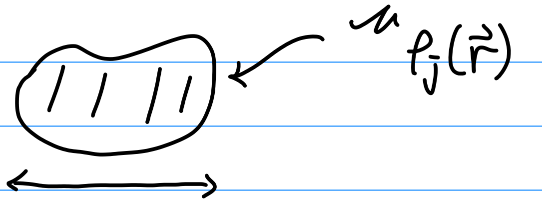
← Macroscopic dipole-moment per unit volume.

Ex 4: Coarse-graining the charge density:

Assume we have microscopic localized charge distribution:

$$\{{}^{\mu}\rho_j(\vec{r})\} \qquad (j = 1, 2, 3 \ldots)$$

means microscopic



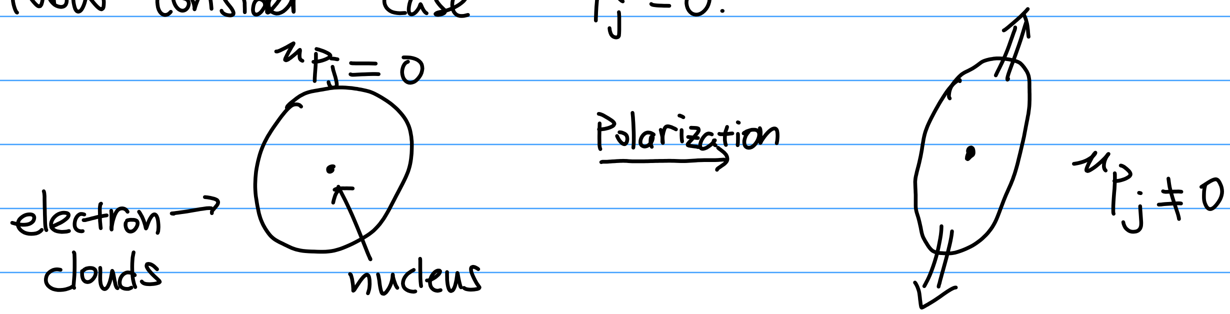

$d \ll L$
↳ averaging length.

Each $j$ describe an atom, molecule or charged ion.

The total charge from nucleus and atoms is $q_j$:

$$q_j \equiv \int d^3r \; {}^{\mu}\rho_j(\vec{r})$$

Now consider case $q_j = 0$:




How does such atom contribute to the average charge density?

$$\overline{{}^{\mu}\rho_j} = \int d^3r' \; {}^{\mu}\rho_j(\vec{r}\,') \; f(\vec{r} - \vec{r}\,')$$

averaged

microscopic, localized near $\vec{r}_j$, scale $d$

Normalized Lorentz Averaging function, scale $d$.

Since the weight is concentrated around $\vec{r}\,' = \vec{r}_j$, so we expand $f$ around $r' = \vec{r}_j$:

$$f(\vec{r} - \vec{r}\,') = f\Big((\vec{r} - \vec{r}_j) + (\vec{r}_j - \vec{r}\,')\Big)$$

$$\approx f(\vec{r} - \vec{r}_j) + \underbrace{\underbrace{(\vec{r}_j - \vec{r}\,')}_{\mathcal{O}(d)} \cdot \underbrace{\vec{\nabla}_r f(\vec{r} - \vec{r}_j)}_{\mathcal{O}(1/L)}}_{\sim \mathcal{O}(d/L)} + \cdots$$

Then $\overline{{}^{\mu}\rho_j(\vec{r})} \approx \int d^3r' \, {}^{\mu}\rho_j(\vec{r}') \left[ f(\vec{r}-\vec{r}_j) + (\vec{r}_j - \vec{r}') \cdot \vec{\nabla}_r f(\vec{r}-\vec{r}_j) \right]$

$$= \underbrace{\int d^3r' \, {}^{\mu}\rho_j(\vec{r}')}_{q_j} f(\vec{r}-\vec{r}_j)$$

$$+ \underbrace{\int d^3r' \, {}^{\mu}\rho_j(\vec{r}')(\vec{r}_j - \vec{r}')}_{-\vec{p}_j} \cdot \vec{\nabla}_r f(\vec{r}-\vec{r}_j)$$

$$\boxed{\overline{{}^{\mu}\rho_j(\vec{r})} = \sum_j q_j f(\vec{r}-\vec{r}_j) - \vec{p}_j \cdot \vec{\nabla}_r f(\vec{r}-\vec{r}_j)}$$

charge on site. electric dipole moment about $\vec{r}_j$

Now let's introduce: microscopic variables:

1) $${}^{\mu}q(\vec{r}) \equiv \sum_j q_j \delta(\vec{r}-\vec{r}_j) \xrightarrow{\text{average}} Q(\vec{r})$$

2) $${}^{\mu}p(\vec{r}) \equiv \sum_j \vec{p}_j \delta(\vec{r}-\vec{r}_j) \xrightarrow{\text{average}} \vec{P}(\vec{r})$$

Then $${}^{\mu}\rho(\vec{r}) = \int d^3r' \, \rho(\vec{r}') f(\vec{r}-\vec{r}')$$

$$\approx \overline{{}^{\mu}\rho_{ext}}(\vec{r}) - \vec{\nabla} \cdot {}^{\mu}\vec{p}(\vec{r})$$

$$\boxed{\overline{{}^{\mu}\rho(\vec{r})} \doteq Q(\vec{r}) \underbrace{- \vec{\nabla} \cdot \vec{P}(\vec{r})}}$$

dipole moment density.

extrinsic charge density

intrinsic charge density

Microscopic polarization density

$$\int d^3r\, r_\mu \rho_p(\vec{r}) = -\int d^3r\, r_\mu (\partial_\nu P_\nu)$$

$$\overset{!}{=} -\int_V d^3r \left[\partial_\nu (r_\mu P_\nu) - P_\nu \overbrace{\partial_\nu r_\mu}^{\delta_{\mu\nu}}\right]$$

$$\overset{!}{=} -\int_S d^2S_\nu P_\nu r_\mu + \int_V d^3r\, P_\mu(\vec{r})$$

$$\overset{!}{=} \underbrace{-\int_S d^2S (\hat{n}\cdot\vec{P}) r_\mu}_{\text{surface. term}} + \underbrace{\int_V d^3r\, P_\mu(\vec{r})}_{\text{Bulk Term.}}$$

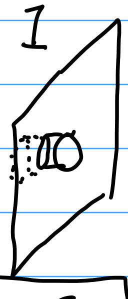
Boundary between two dielectric ($Q=0$):

i) $\vec{\nabla} \times \vec{E} = 0 \rightarrow \oint d\vec{L} \cdot \vec{E} = 0$ I II


$$\boxed{E_{\parallel}^{I} = E_{\parallel}^{II}}$$

Tangential component of $\vec{E}$ is continuous across the surface.

ii) $\vec{\nabla} \cdot \vec{D} = 0 \rightarrow \int_S d^2\vec{S} \cdot \vec{D} = 0$ I II



$$\boxed{D_{\perp}^{I} = D_{\perp}^{II}}$$

$$\boxed{\Phi_{I} = \Phi_{II}}$$

$$\boxed{D_{\perp}^{I} - D_{\perp}^{II} = \sigma_{ext}}$$

The normal component of $\vec{D}$ is continuous across the interface. So, $E_n$ is discontinuous, due to surface polarization charge.

→ Dielectric - Conductor interface:

Dielectric | Metal

$\vec{E} = ?$
$\vec{P} = ?$ } can be nonzero

$\vec{E} = 0$
$\vec{P} = 0$

$$\boxed{\begin{aligned} E_{\parallel}^{\text{Dielectric}} &= 0 \\ D_{\perp}^{\text{Dielectric}} &= \sigma_{cond} \end{aligned}}$$

# How does $\vec{E}$ determine $\vec{P}$?

$$\boxed{D(\vec{r}) = \varepsilon(\vec{r})\, E(\vec{r})}$$

← Linear, isotropic and local.

↑ Dielectric Permittivity ← How medium adjust volume.

→ If Linear, local, but anisotropic, e.g. due to crystalline structure:

$$D_u(\vec{r}) = \varepsilon_{uv}(\vec{r})\, E_v(\vec{r})$$

If Homogeneous $\varepsilon(\vec{r}) = \varepsilon$

→ If Linear, isotropic, but nonlocal:

$$\vec{D}(\vec{r}) = \int d^3r\ \varepsilon(\vec{r}-\vec{r}')\, \vec{E}(\vec{r}')$$

↑ a convolution.

For $\vec{P}$, it means

→ $$\boxed{\vec{P} = \varepsilon\vec{E} - \varepsilon_0\vec{E} = (\varepsilon-\varepsilon_0)\vec{E} = \kappa\vec{E}}$$

## Homogeneous Dielectric (No free charge)

$\rightarrow$ $\varepsilon(\vec{r}) \rightarrow \varepsilon$ $\leftarrow$ $\varepsilon$ free of position.

$$\vec{\nabla}\cdot\vec{D}=0 \quad \rightarrow \quad \vec{\nabla}\cdot\varepsilon\vec{E}=0$$

Boundary condition and $\sigma_P \neq 0$

$$\left.\begin{aligned} -\vec{\nabla}\cdot\vec{P} &= \rho_P = 0 \\ \vec{P}\cdot\hat{n} &= \sigma_P \neq 0 \end{aligned}\right\}$$

If homogeneous $\varepsilon(\vec{r}) = \varepsilon$, then we have no bulk term, $\rho_P = 0$ but only $\underline{\sigma_P} \neq 0$.

For Inhomogeneous dielectric:

How does $\varepsilon(\vec{r})$ imply $\rho_P(\vec{r})$

$$\rho_P = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\vec{D}-\varepsilon_0\vec{E})$$

$$= -\vec{\nabla}\cdot\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon}\vec{D}\right)$$

$$= -\vec{\nabla}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\cdot\vec{D} - \left(\frac{\varepsilon-\varepsilon_0}{\varepsilon}\right)\cancelto{0}{\vec{\nabla}\cdot\vec{D}} \quad \text{for } \rho_{ext}=0$$

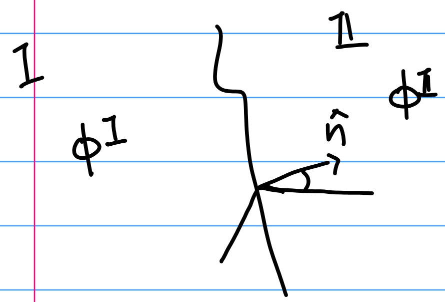
$$\rho_P = -\vec{E}\cdot\frac{\varepsilon_0}{\varepsilon}\vec{\nabla}\varepsilon$$

## Generalization to Laplace Eq:

$$\vec{\nabla}\times\vec{E}=0 \quad \rightarrow \quad \vec{E} = -\vec{\nabla}\phi$$

$\hookrightarrow$ $\vec{\nabla}\cdot\vec{D}=0 \rightarrow \vec{\nabla}\cdot(\varepsilon\vec{E}) \rightarrow \boxed{\vec{\nabla}\cdot\left(\varepsilon(\vec{r})\vec{\nabla}\phi\right)=0}$

## Two Dielectric Interface :




$D_\perp^{I} = D_\perp^{II}$

$\hookrightarrow \varepsilon^{I} E_\perp^{I} = \varepsilon^{II} E_\perp^{II}$

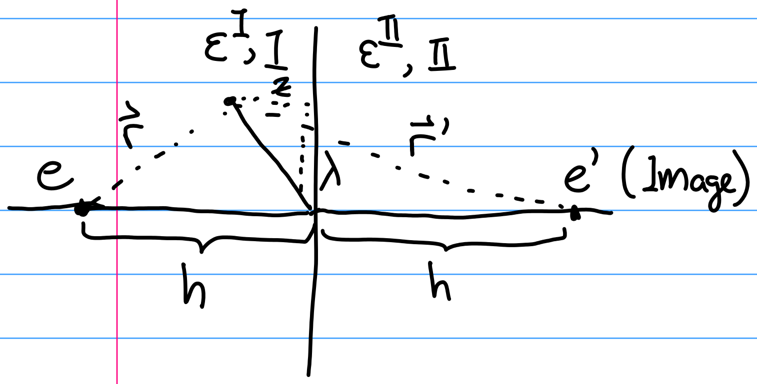
$\hookrightarrow \boxed{\begin{aligned} \varepsilon^{I} \partial_n \phi^{I} &= \varepsilon^{II} \partial_n \phi^{II} \\ \phi^{I} &= \phi^{II} \end{aligned}}$ ← Boundary Condition.

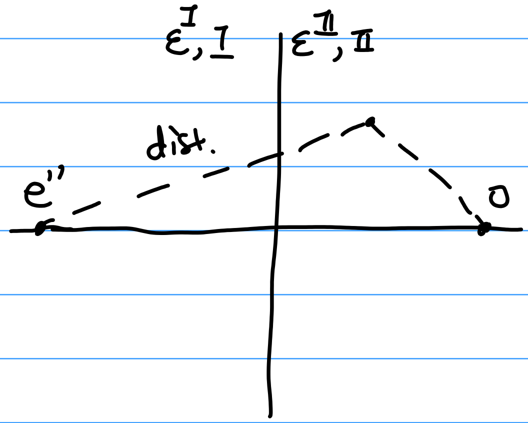
Strategy for Boundary - Value Problem for linear, isotropic, dielectric

$\phi \to \vec{E}, \vec{D}, \vec{P}, \rho_P, \sigma_P$

## Example Problem:

– Determine electrostatic potential due to a point charge a distance h from the plane boundary separating two homogeneous dielectric media:




when solving $\Phi_I$ when solving $\Phi_{II}$

Field in I, $\Phi^{I}(\vec{r}) = \frac{1}{4\pi}\left[\frac{e}{\varepsilon_I r} + \frac{e'}{\varepsilon_I r'}\right]$

Field in II, $\Phi^{II}(\vec{r}) = \frac{1}{4\pi}\left[\frac{e''}{\varepsilon^{II} \mathrm{dist}} + 0\right]$

Boundary Condition: $\Phi^{I}(\text{Boundary}) = \Phi^{II}(\text{Boundary})$

$$\hookrightarrow \quad \frac{e}{\varepsilon^{I}} + \frac{e'}{\varepsilon^{I}} = \frac{e''}{\varepsilon^{II}} \qquad (1)$$

let $\frac{1}{r} \to \frac{1}{\sqrt{\lambda^2 + (h-z)^2}}$ , $\frac{1}{r'} \to \frac{1}{\sqrt{\lambda^2 + (h+z)^2}}$

then by BC. $\varepsilon^{I} \frac{\partial}{\partial n} \Phi^{I} = \varepsilon^{II} \frac{\partial}{\partial n} \Phi^{II}$

$$\hookrightarrow \quad \varepsilon^{I}\left(\frac{e}{\varepsilon^{I}} - \frac{e'}{\varepsilon^{I}}\right) = \varepsilon^{II}\left(\frac{e''}{\varepsilon^{II}} + 0\right) \qquad (2)$$

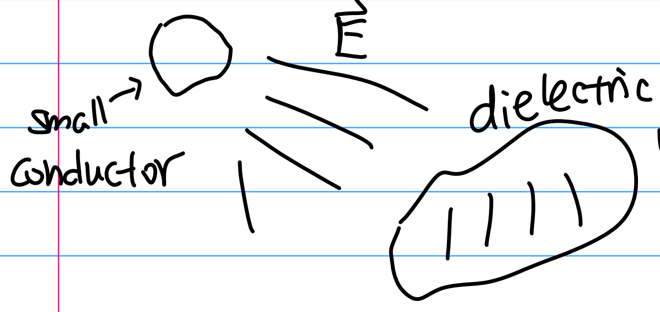
using (1) and (2):

$$\hookrightarrow \quad e' = e \, \frac{\varepsilon^{I} - \varepsilon^{II}}{\varepsilon^{I} + \varepsilon^{II}}$$

$$e'' = e \, \frac{2\varepsilon^{II}}{\varepsilon^{I} + \varepsilon^{II}}$$

$$\hookrightarrow \quad F = \frac{e^2}{4\pi\varepsilon^{I}} \frac{1}{(2h)^2} \left(\frac{\varepsilon^{I} - \varepsilon^{II}}{\varepsilon^{I} + \varepsilon^{II}}\right)$$

## Force on Dielectric Body:




Microscopic

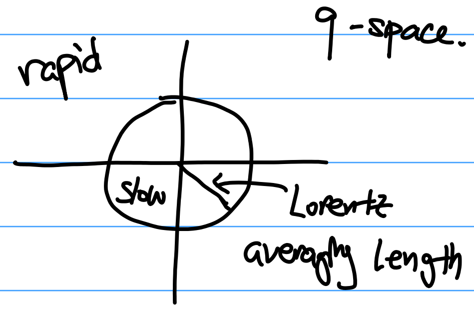
$$\vec{f} = \int d^3r\, \rho(r)\, \vec{E}^{ext}(\vec{r})$$

Macroscopic $V$ all the charge that matters.

$$\hookrightarrow = \int_V d^3r\, [Q(\vec{r}) - \vec{\nabla}\cdot\vec{P}]\, \vec{E}^{ext}(\vec{r})$$

Force $\rightarrow$ $\boxed{\vec{f} = \int d^3r\, Q(\vec{r})\vec{E}^{ext}(\vec{r}) + \int d^3r\, P(\vec{r})\,\vec{\nabla}_r \cdot \vec{E}^{ext}(\vec{r})}$

Imagine $\alpha(\vec{r})$, $\beta(\vec{r})$




$$\alpha(\vec{r}) = \int d^3q\, e^{i\vec{q}\cdot\vec{r}}\, \hat{\alpha}(q)$$

$$= \alpha_{slow}(\vec{r}) + \alpha_{rapid}(\vec{r})$$

Similarly: $\beta(\vec{r}) = \beta_{slow}(\vec{r}) + \beta_{rapid}(\vec{r})$

Overlap: $\int d^3r\, \alpha(\vec{r})\, \beta(\vec{r}) = \int d^3r\, \alpha_{fast}\beta_{fast} + \int d^3r\, \alpha_{slow}\beta_{slow}$

cross-terms go to 0.

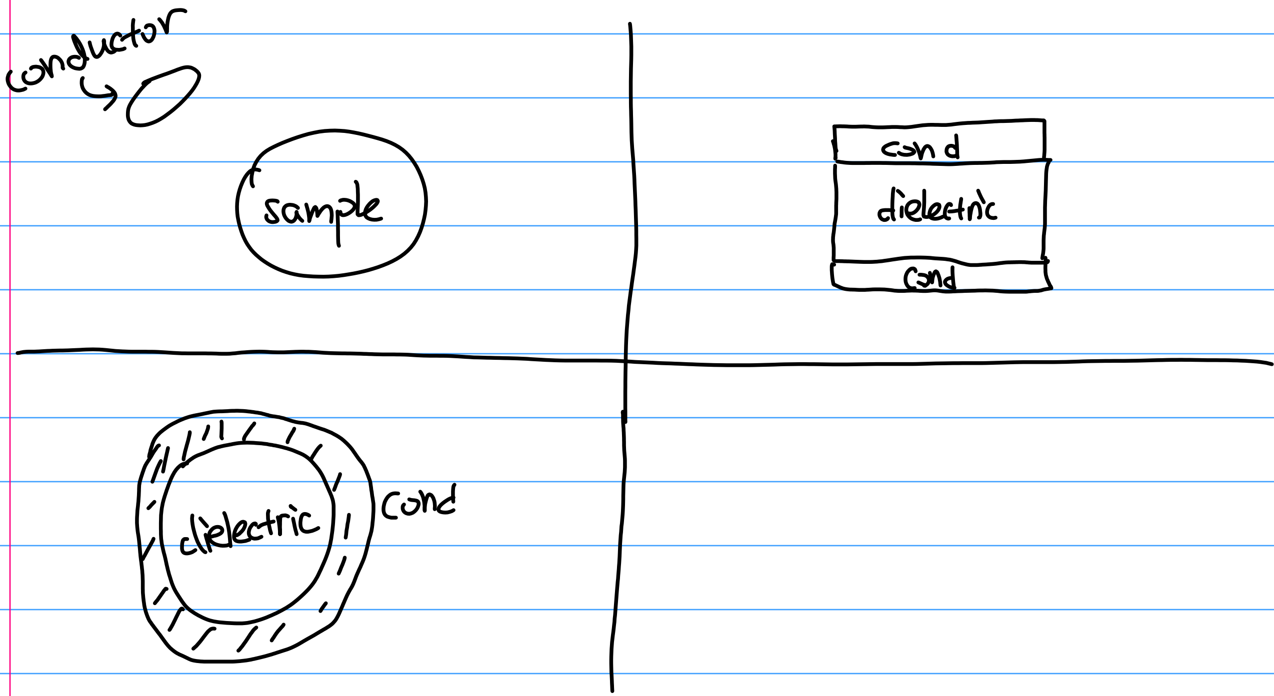
then $\vec{F} = \int d^3r\, (\rho_{fast} + \rho_{slow})(E^{ext}_{fast} + E^{ext}_{slow})$

$$\stackrel{!}{=} \int d^3r\, (\cancel{\rho_{fast} E^{ext}_{fast}} + \rho_{slow} E^{ext}_{slow})$$

$=0$, by arguing that E vary on the lengthscale of separation, which is ~ meters, not angstrong

$$\boxed{\vec{F} = \int d^3r\, [Q - \vec{\nabla}\cdot\vec{P}]\, \vec{E}^{ext} = \int d^3r\, Q\vec{E}^{ext} + \int d^3r\, \vec{P}\, \vec{\nabla}_r \cdot \vec{E}^{ext}}$$

# E-static energy in dielectric Media:




Bring $\delta q$ to conductor:

Before bring $\delta q$: $\phi$, $\vec{E}$, $\vec{D}$

After bring $\delta q$: $\phi \to \phi + \delta\phi$, $\vec{E} \to \vec{E} + \delta\vec{E}$, $\vec{D} \to \vec{D} + \delta\vec{D}$

change in energy: $\delta \mathcal{E} = \phi\,\delta q$

$$\stackrel{!}{=} \phi \int d^2s\ \delta\sigma \quad \leftarrow \text{change in } \sigma \text{ for cond.}$$

Bulk Term: $\vec{\nabla}\cdot\vec{D} = 0 \quad \to \quad \vec{\nabla}\cdot(\vec{D} + \delta\vec{D}) = 0 \quad \to \quad \vec{\nabla}\cdot\delta\vec{D} = 0$

$$\hookrightarrow \delta\mathcal{E} = \phi \int d^2s\ \delta D_\perp \quad \hookrightarrow \delta D_\perp = \delta\sigma$$

since $\phi$ is constant on surface of conductor

$$\delta\mathcal{E} = \int d^2\vec{S}\cdot(\phi\,\delta D)$$

$$\overset{!}{=} -\int_V d^3r\ \vec{\nabla}\cdot(\phi\,\delta D)$$

↳ everything except conductor

↳ (−) sign because we take $\hat{n}$ pointing inward to conductor.

$$= -\int_V d^3r\ \vec{\nabla}\phi\cdot\delta\vec{D} - \int_V d^3r\ \phi\,\cancel{\vec{\nabla}\cdot\delta\vec{D}}$$

$= 0$ outside conductor

$$\boxed{\delta\mathcal{E} = \int_{V,\text{ not cond}} d^3r\ \vec{E}\cdot\delta\vec{D}}$$

← change in energy after bringing $\delta q$ to the conductor.

Now add charge from 0: parameterize by $\tau$

$\tau$: $0 < \tau < 1$

↑ Not charged ↑ fully charged.

$$\mathcal{E}(\tau)\Big|_{\tau=1} = \cancel{\mathcal{E}(\tau)\Big|_{\tau=0}} + \int_0^1 d\tau \int d^3r\ \vec{E}(\vec{r},\tau)\,\frac{d\vec{D}(\vec{r},\tau)}{d\tau}$$

Assuming Linearity: $\vec{D} = \epsilon\vec{E}$, then $\frac{d\vec{D}}{d\tau} = \epsilon\frac{d\vec{E}}{d\tau}$

$$\mathcal{E}(\tau)\Big|_{\tau=1} = \int_0^1 d\tau \int d^3r\ \varepsilon\,\frac{d}{d\tau}\left[\frac{1}{2}E(\tau)^2\right]$$

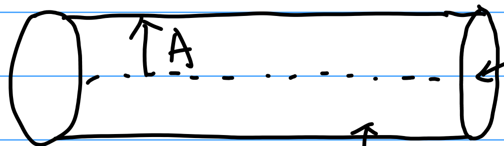
$$\overset{!}{=} \frac{1}{2}\int d^3r\ |E(\vec{r})|^2\,\varepsilon$$

$$\boxed{\mathcal{E} = \frac{1}{2}\int_V d^3r\ \vec{E}\cdot\vec{D}}$$

↳ outside conductor

if vacuum, $\mathcal{E} = \frac{1}{2}\int d^3r\ \varepsilon_0(\vec{E}\cdot\vec{E})$

problem:




want $\Phi$ in here.

$\Phi(r,\theta,z) = R(r)\,T(\theta)\,Z(z)$ given $\Phi(r=A) = \Gamma \neq 0$

i) $Z'' = -k^2 Z \quad \rightarrow \quad Z = e^{ikz}$

ii) $T'' = -m^2 T \quad \rightarrow \quad T = e^{im\theta}$

iii) $R'' + \frac{1}{r}R' - [k^2 + \frac{m^2}{r^2}]R = 0$

Bessels Eq: $S'' + \frac{1}{x}S' + [1 - \frac{m^2}{x^2}]S = 0 \rightarrow$ solution when

let $r \rightarrow x = kr$, $R(r) \rightarrow y(kr)$

$\hookrightarrow y'' + \frac{1}{x}y' - [1 + \frac{m^2}{x^2}]y = 0$ ← Modified Bessel Func. (grow / decay)

sign changed.

So $\Phi(r,z,\theta) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\; C_m(k)\, I_{|m|}(kr)\, e^{im\theta} e^{ikz}$

periodic in $\theta$, so $m$ is integer.

$z$ is not periodic, $k$ not integer

As $\Phi(A,\theta,z) = \Gamma(\theta,z) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\; C_m(k)\, I_{|m|}(kA)\, e^{im\theta} e^{ikz}$

Multiply Both sides by $\int_0^{2\pi} d\theta\, e^{-i\bar{m}\theta} \int_{-\infty}^{\infty} dz\, e^{-i\bar{k}z}$

$\Gamma_{\bar{m}}(\bar{k}) = \sum_m \int_{-\infty}^{\infty} dk\; C_m(k) \int d\theta\, e^{i(m-\bar{m})\theta} \int dz\; e^{i(k-\bar{k})z} = \sum_m \int dk\, C_m(k)\,\delta(m-\bar{m})\,\delta(k-\bar{k})(2\pi)^2$

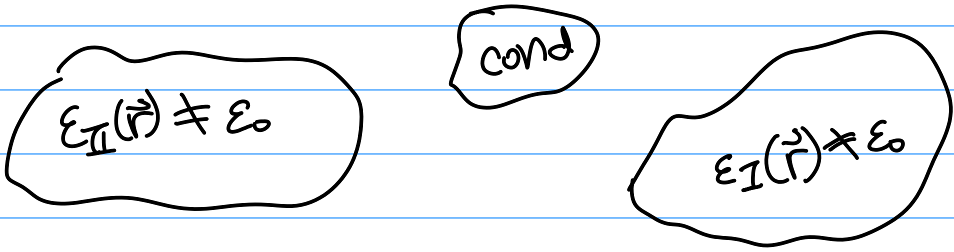
$\hookrightarrow \Gamma_m(k) = C_m(k)\, I_{|m|}(|k|A)\,(2\pi)^2$ ← Found $C_m(k)$

sub $C_m(k)$

$\hookrightarrow \Phi(r,\theta,z) = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\; \Gamma_m(k)\, \frac{I_{|m|}(|k|r)}{I_{|m|}(|k|A)}\, e^{im\theta} e^{ikz}$

## Application :

- Attraction of a dielectric to a region in which the electric field is nonzero:




- Fix total charge on the conductor
- Consider two location of dielectric body.
- Describe the location of the body through $\varepsilon(\vec{r})$.

  Inside: $\varepsilon(\vec{r}) > 1$, Outside $\varepsilon(\vec{r}) = \varepsilon_0$

$\mathcal{E}^{I}$ for location $I$ with dielectric function $\varepsilon^{I}(\vec{r})$
$\mathcal{E}^{II}$ for location $II$ with dielectric function $\varepsilon^{II}(\vec{r})$

Then, the energy difference:

energy

$$\mathcal{E}^{II} - \mathcal{E}^{I} = \frac{1}{2}\int_{V} d^3r \left[E^{II}\cdot D^{II} - E^{I}\cdot D^{I}\right]$$

↳ outside conductor

$$= \frac{1}{2}\int_{V} d^3r \left[E^{II}\cdot D^{I} - E^{I}\cdot D^{II}\right] \quad ①$$

$$+ \frac{1}{2}\int_{V} d^3r \left(E^{II}\cdot D^{II} - E^{I}\cdot D^{I}\right) - \left(E^{II}\cdot D^{I} - E^{I}\cdot D^{II}\right) \quad ②$$

Now let's focus on ②:

$$② : \frac{1}{2}\int_V d^3r \left[ \left( E^{II}\cdot D^{II} - E^{I}\cdot D^{I} \right) - \left( E^{II} D^{I} - E^{I}\cdot D^{II} \right) \right]$$

$$\overset{!}{=} \frac{1}{2}\int_V d^3r \left[ \underbrace{\left(E^{I}+E^{II}\right)}_{-\vec{\nabla}(\phi^{I}+\phi^{II})} \cdot \left(D^{II}-D^{I}\right) \right]$$

$$= -\frac{1}{2}\int_V d^3r \, \vec{\nabla}\cdot\left[ \left(\phi^{I}+\phi^{II}\right)\left(D^{II}-D^{I}\right) \right]$$

$$+ \frac{1}{2}\int_V d^3r \left(\phi^{I}+\phi^{II}\right) \underbrace{\vec{\nabla}\cdot\left(D^{II}-D^{I}\right)}_{=0 \text{ since } \vec{\nabla}\cdot D = 0 \text{ outside conductor.}}$$

$$= +\frac{1}{2}\int_V d^2\vec{S}\cdot\left(D^{II}-D^{I}\right)\underbrace{\left(\phi^{I}+\phi^{II}\right)}_{\text{Equipotential on surface.}}$$

$$= \frac{1}{2}\left(\phi^{I}+\phi^{II}\right) \underbrace{\int_S d^2\vec{S}\cdot\left(D^{II}-D^{I}\right)}_{= \int d^3r \, \vec{\nabla}\cdot\left(D^{II}-D^{I}\right) = 0}$$

$$② = 0$$

Therefore: $\mathcal{E}^{II} - \mathcal{E}^{I} = ① + \cancel{②} = \frac{1}{2}\int_V d^3r \left[ E^{II}\cdot D^{I} - E^{I}\cdot D^{II} \right]$

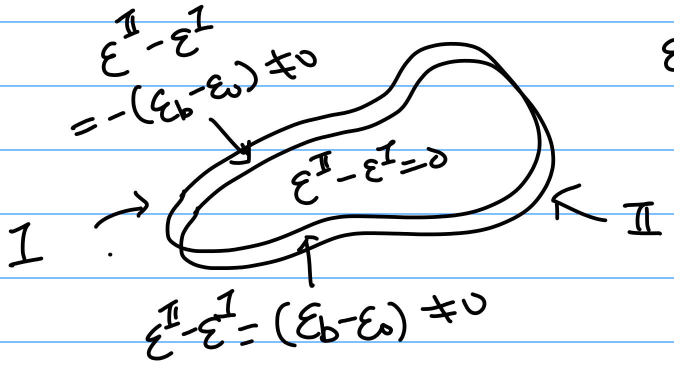
with $D^{I} = \varepsilon^{I} E^{I}$ and $D^{II} = \varepsilon^{II} E^{II}$

$$\hookrightarrow \boxed{\mathcal{E}^{II} - \mathcal{E}^{I} = -\frac{1}{2}\int_V d^3r \left(\varepsilon^{II}-\varepsilon^{I}\right) E^{I}\cdot E^{II}}$$

← energy between two dielectrics.

negative sign, if $\varepsilon^{II}-\varepsilon^{I} > 0$, then lower energy.

For example: if $\varepsilon^{I}(\vec{r})$ and $\varepsilon^{II}(\vec{r})$ only take out two values, $\varepsilon_b$ inside body and $\varepsilon_0$ outside, and if $\varepsilon^{II}$ and $\varepsilon^{I}$ differ by infinitesimal displacement then the integration is further restricted to the boundary.




$\varepsilon^{II}-\varepsilon^{I}=0$ in vacuum.

Lastly, consider situation when dielectric body II is brought in from far from charged conductor and it can be taken as not interacting with it, and not polarized by it, i.e. there is no dielectric body I.

then: $\mathcal{E}^{II} - \mathcal{E}^{I} = -\frac{1}{2}\int_{V_{II}} d^3r\, (\varepsilon^{II}-\varepsilon_0)\, E^{II}\cdot E^{cond.}$

where body II is present.

since $(\varepsilon^{II}-\varepsilon_0)E^{II} = \vec{P}^{II}$

then $$\boxed{\mathcal{E}^{II} - \mathcal{E}^{I} = -\frac{1}{2}\int_{V_{II}} d^3r\; \vec{P}^{II}\cdot\vec{E}^{cond}}$$