Electrostatics:

We restrict ourselves to the case of static : time independent

i.e. $\partial_t \phi = \partial_t \vec{A} = \partial_t \rho = \partial_t \vec{j} = 0$

↳ Time - Independent Maxwell Equation:

	Coulomb / Gauss	No Monopole
constraint	$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$	$\vec{\nabla} \cdot \vec{B} = 0$
Dynamics	$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \cancelto{0}{\partial_t \vec{E}} + \mu_0 \vec{j}$	$\vec{\nabla} \times \vec{E} = -\cancelto{0}{\partial_t \vec{B}}$
	Maxwell, Ampere	Faraday
	Inhomogeneous	Homogeneous

1	3
2	4

We observe that for time-independent, $\vec{E}$ and $\vec{B}$ decouple with each other, i.e. one can address $\vec{E}$ without worrying about $\vec{B}$, vice-versa.

↳ Electro static : $\vec{A} = \vec{j} = 0$

↳ Magnetostatic : $\phi = \rho = 0$

Lets focus on electrostatic:

Question: Are the solutions to electrostatic problems unique?

Since $\vec{E} = -\vec{\nabla}\phi - \partial_t A = -\vec{\nabla}\phi$

then $\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi = \frac{1}{\varepsilon_0}\rho$

$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\phi) = 0$

Therefore, we focus on the single equation:

$$-\nabla^2\phi = \frac{1}{\varepsilon_0}\rho \qquad \text{(Poisson's Equation)}$$

$\vec{r}$ dependent scalar potential

$\vec{r}$-dependent charge density, the source that is presumed known.

Poisson Equation Properties:

① Linear: the coefficient of the derivatives in differential equation are not function of the dependent variable.

2 variable linear PDE can be written as:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

if
$B^2 - 2AC < 0 \Rightarrow$ Elliptic
$B^2 - 2AC > 0 \Rightarrow$ Hyperbolic
$B^2 - 2AC = 0 \Rightarrow$ Parabolic.

② Elliptic PDE

③ Inhomogeneous if $u \neq 0$, term that dependents of the

Gauge Invariance in Electrostatic:

We previously saw that:

$$\boxed{\begin{aligned} \phi &\to \phi - \partial_t \chi \\ \vec{A} &\to \vec{A} + \vec{\nabla} \chi \end{aligned}}$$

← general gauge trans for Maxwell's Eqs.

leaves the field unchanged.

But in electrostatic: $A = 0$, $\partial_t \phi = 0$
the freedom to choose $\chi(\vec{r}, t)$ is limited to the following:

① since $\vec{A} \to \vec{A} + \vec{\nabla}\chi$, to keep $\vec{A} = 0$, χ must be independent of $\vec{r}$.

② Since $\phi \to \phi - \partial_t \chi$, to keep $\partial_t \phi = 0$, χ can only be at most linear in t or $\chi = at + b$, or $\partial_t \chi = a$

Due to the restrictions above:

Gauge Trans for electrostatic

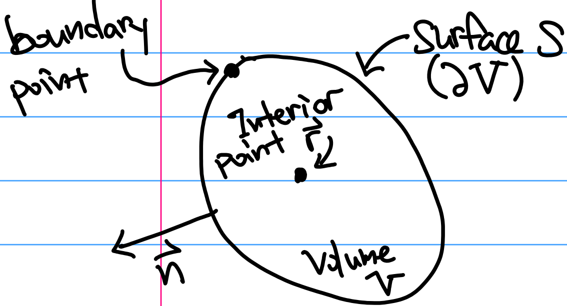
$$\boxed{\phi \to \phi + \text{const.} \qquad \vec{A} = 0}$$

The question of uniqueness for Poisson's Eq:

$$-\nabla^2 \phi = \frac{1}{\varepsilon_0} \rho$$

Suppose we found a solution, called ϕ_1, can there be others?

$\Rightarrow$ <u>Case 1:</u>



- Finite Region
- Dirichlet Boundary Conditions: on the boundary. | ϕ are specified

$\hookrightarrow$ $\phi(\vec{a}) = \Phi(\vec{a})$

solution, boundary points, specified boundary terms

With solution $\phi_1(\vec{r})$, we have:

$$-\nabla^2 \phi_1 = \frac{1}{\varepsilon} \rho \quad \text{in volume.}$$

$$\phi_1 = \Phi \quad \text{on surface (boundary)}$$

Consider a new potential $\phi_2 \equiv \phi_1 + \psi$

Suppose ϕ_2 obey the same conditions as ϕ_1.

then in Volume: $\nabla^2 \psi = \nabla^2 \phi_2 - \nabla^2 \phi_1 = \frac{-1}{\varepsilon_0} \rho + \frac{1}{\varepsilon} \rho = 0$

So $\nabla^2 \psi = 0$, or ψ obeys Laplace Equation in V.

On the boundary:

$$\psi(\vec{\alpha}) = \phi_2(\vec{\alpha}) - \phi_1(\vec{\alpha}) = \Phi(\vec{\alpha}) - \Phi(\vec{\alpha}) = 0 \quad \text{on } S$$

therefore $\psi(\vec{\alpha})$ obeys Laplace Eq with vanishing Dirchlet boundary Conditions

To see the consequence of this effect:

Evaluate : $\int_V d^3r \, \psi \nabla^2 \psi$

$$\hookrightarrow \underbrace{\int_V d^3r \, \psi \nabla^2 \psi}_{=0 \text{ since}} = \int_V d^3r \left\{ \vec{\nabla} \cdot (\psi \vec{\nabla} \psi) - |\vec{\nabla}\psi|^2 \right\}$$

Laplace : $\nabla^2 \psi = 0$

$$= \underbrace{\int d\vec{S} \cdot \psi \vec{\nabla} \psi}_{=0 \text{ since we have vanishing boundary}} - \int d^3r \, |\vec{\nabla}\psi|^2$$

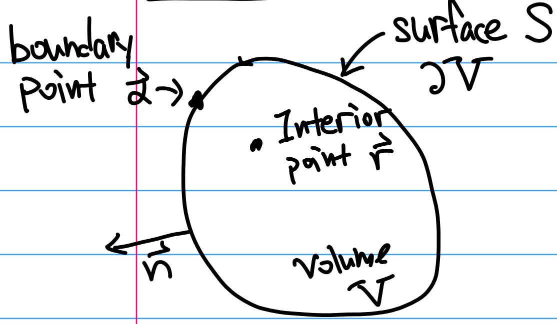
$$0 = -\int d^3r \underbrace{|\vec{\nabla}\psi|^2}_{\text{Non-negative.}}$$

$\hookrightarrow$ So $\vec{\nabla}\psi = 0$ for equality $\rightarrow$ ψ = constant in volume.

$\hookrightarrow$ But due to vanishing boundary condition, $\psi = 0$ on ∂V

$$\boxed{\therefore \; \psi = 0 \; \rightarrow \; \phi_2 = \phi_1}$$ i.e. solution is unique.

Case 2:



- Finite Region
- Neumann Boundary Condition: derivative $\hat{n}\cdot\vec{\nabla}\phi$ is specified on boundary:

$$\hat{n}\cdot\vec{\nabla}\phi(\underset{\substack{\uparrow \\ \text{boundary points}}}{\vec{\alpha}}) = \underset{\substack{\uparrow \\ \text{specified points}}}{P}(\vec{\alpha}) = -\vec{E}\cdot\hat{n}$$

Note that since we're working with second-order PDE, B.C. is at most first derivative.

$\hat{n}\cdot\vec{\nabla}\phi$ is the rate of change along $\hat{n}$:

$$\hat{n}\cdot\vec{\nabla}\phi = \frac{\partial}{\partial s}\phi(\vec{\alpha}+s\hat{n})\Big|_{s=0}$$

Again, we consider $\phi_2 \equiv \phi_1 + \psi$

then we again have

① $$\nabla^2\psi = \nabla^2\phi_2 - \nabla^2\phi_1 = -\frac{1}{\varepsilon_0}\rho - \frac{1}{\varepsilon_0}\rho = 0$$

② $$\hat{n}\cdot\vec{\nabla}\psi = \hat{n}\cdot\vec{\nabla}\phi_2 - \hat{n}\cdot\vec{\nabla}\phi_1 = \frac{\partial}{\partial s}\phi_2(\vec{\alpha}+s\hat{n})\Big|_{s=0} - \frac{\partial}{\partial s}\phi_1(\vec{\alpha}+s\hat{n})\Big|_{s=0} = 0$$

Again, we evaluate: $\int d^3r\, \psi\nabla^2\psi$

$$0 = \int d^3r\, \psi\underbrace{\nabla^2\psi}_{=0} = \int d^3r\, \vec{\nabla}\cdot(\psi\vec{\nabla}\psi) - |\vec{\nabla}\psi|^2$$

$$= \underbrace{\int d^2\vec{S}\cdot\psi\vec{\nabla}\psi}_{\substack{\text{since } \hat{n}\cdot\vec{\nabla}\psi = 0 \\ \text{at boundary}}} - \int d^3r\, |\vec{\nabla}\psi|^2$$

$$0 = -\int d^3r\, |\vec{\nabla}\psi|^2$$

↳ This again requires $\vec{\nabla}\psi = 0$, or $\psi =$ constant.

Since we don't require $\psi = 0$ as boundary condition

"Almost Unique" $\Rightarrow$ $\boxed{\phi_2 = \phi_1 + \text{const.}}$ For Neumann B.C.

Note that this is consistent with gauge invariant.

Case 3: All space, no boundary

lets call this asymptotic conditions:

Given $\rho(\vec{r})$, there is at most one solution to Poisson's Equation under the condition:

$$\phi(\vec{r}) \to 0 \quad \text{as} \quad |\vec{r}| \to \infty$$

If this is the case:

i) $r\phi$ remains bounded

ii) $r|\vec{\nabla}\phi| \to 0$

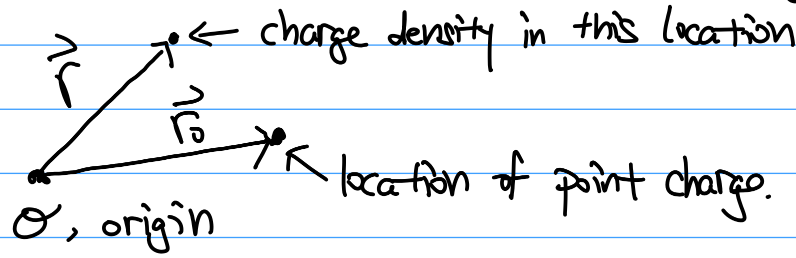
Note that not just $\phi \to 0$ but ϕ and $|\vec{\nabla}\phi| \to 0$ sufficiently fast.

Point Charges and Green's Function:

Charge density due to a point charge at $\vec{r}_0$:

$$\rho(\vec{r}) = \underset{\uparrow \text{ charge}}{q}\ \overset{\text{3D-Dirac-Delta}}{\delta^{(3)}}(\vec{r} - \underset{\uparrow \text{location}}{\vec{r}_0})$$

Then superposition allows for a distribution of charges



Key properties of Dirac-Delta function (Distribution):

i) $\int_V d^3r\ \delta(\vec{r} - \vec{a})\, f(\vec{r}) = f(\vec{a})$ ← sufficiently well-behaved

If $\vec{a}$ is within V

ii) $\int_V d^3r\ \delta(\vec{r} - \vec{a}) = 1$

iii) A continuum version of the identity matrix or kronecker-Delta:

$$\overleftrightarrow{I} \cdot \vec{V} = \vec{V}$$

$$\hookrightarrow \quad \sum_j \delta_{jk} V_k = V_j$$

$$\int dV_r\ \delta(\vec{s} - \vec{r})\, v(\vec{r}) = v(\vec{s})$$

iv) various representation that give $\delta(\vec{r})$ as $\lambda \to 0$

ⓐ $$\delta_\lambda(\vec{r}) = \begin{cases} 1/\lambda^3 & \left(-\frac{\lambda}{2} < x, y, z < \frac{\lambda}{2}\right) \\ 0 & \text{otherwise.} \end{cases} \quad \text{as } \lambda \to 0$$

A cube of volume λ^3 and density λ^{-3} centered at the origin. So the total weight is $\lambda^{-3}\lambda^3 = 1$

ⓑ $$f_\lambda(\vec{r}) = (2\pi\lambda^2)^{-3/2} \exp\left\{\frac{-|\vec{r}|^2}{2\lambda^2}\right\} \quad \text{as } \lambda \to 0$$

3D Gaussian function normalized to 1

* ⓒ $$\delta(\vec{r}-\vec{r}\,') = \int \frac{d^3k}{(2\pi)^3} \exp\left(i\vec{k}\cdot(\vec{r}-\vec{r}\,')\right)$$

Fourier Representation: For any nonzero $\vec{r}-\vec{r}\,'$, the $\vec{k}$ integration give complete destructive interference among other values of the complex phase factor.

$$\int d^3\vec{r}\,' \, \delta(\vec{r}-\vec{r}\,') f(\vec{r}\,') = \int d^3\vec{r}\,' \frac{d^3k}{(2\pi)^3} \exp(i\vec{k}\cdot(\vec{r}-\vec{r}\,')) f(\vec{r}\,')$$

$$\stackrel{!}{=} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \tilde{f}(\vec{k})$$

$$\stackrel{!}{=} f(\vec{r})$$

ⓓ Informally: $\delta(\vec{r})$ is ∞ at $\vec{r}=0$ and 0 for $\vec{r}\neq 0$ such that integrating over any region containing $\vec{r}=0$ give unity.

Now if we have point charge, $\rho = q\,\delta(\vec{r}-\vec{r}_0)$
then:

$$-\vec{\nabla}^2 \phi = \frac{1}{\varepsilon_0} q\,\delta(\vec{r}-\vec{r}_0)$$

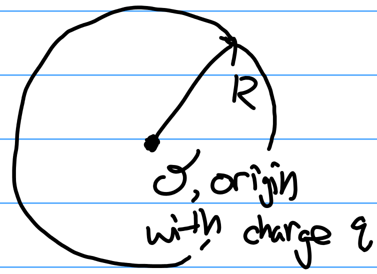
Lets assume $\vec{r}_0 = 0$, and use translational invariant later.

$\hookrightarrow$ $-\nabla^2 \phi = \frac{1}{\varepsilon_0} q\,\delta(\vec{r})$ with no boundary conditions (free space)

$\Rightarrow$ Given the rotational invariance of ∇^2 and $\delta(\vec{r})$, then any solution of the equation and the condition $\phi \to 0$ as $|r| \to \infty$ must be <u>spherically symmetric</u> because we would have multiple solutions under rotation, but the uniqueness theorem says there is only one solution. So solution itself must be rotationally symmetric.

To solve: $-\nabla^2 \phi = \frac{1}{\varepsilon_0} q\,\delta(\vec{r})$

$\hookrightarrow$ $\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} q\,\delta(\vec{r})$



integrate over volume and apply divergence theorem.

$$\int_{r \le R} d^3r\, \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} q \int_{r \le R} \delta(\vec{r})\, d^3r$$

$$\hookrightarrow \int_{r=R} d\vec{S} \cdot \vec{E} = \frac{1}{\varepsilon_0} q$$

Due to Rotational symmetry, $\vec{E} = E(|\vec{r}|)\hat{r}$ ← must only be in $\hat{r}$.

Thus: $$E(|r|)\int_{r=R} d\vec{S} \cdot \hat{r} = \frac{1}{\varepsilon_0} q$$

$$d\vec{S} = r\sin\theta d\theta \hat{\theta} \times r d\phi \hat{\phi} = r^2 \sin\theta d\theta d\phi \hat{r}$$

↳ $$E(|R|)\ 4\pi R^2 = \frac{1}{\varepsilon_0} q$$

then $$\boxed{\vec{E} = E(|r|)\hat{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{|r|^2} \hat{r}}$$

Now calculate ϕ, with assumption $\phi(r \to \infty) = 0$.

Since $E = -\vec{\nabla}\phi$, $\phi(\vec{r}) = \phi(\vec{r}) - \overbrace{\phi(\infty)}^{0}$

$$\phi = \int_{\infty\hat{r}}^{r} d\vec{r}' \cdot \vec{\nabla}'\phi = -\int_{\infty\hat{r}}^{r} d\vec{r}' \cdot \vec{E} = -\int_{\infty\hat{r}}^{r} \frac{1}{4\pi\epsilon_0} \frac{q}{|r'|^2} d\vec{r}' \cdot \hat{r}'$$

We integrate from $\vec{r}' = \infty\hat{r}$ to $\vec{r}' = r\hat{r}$

↳ $$\phi = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r'}\Big|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

Thus $$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}|}}$$ with $\phi \to 0$ as $|\vec{r}| \to \infty$

Relocation: Translating the charge from $\vec{0}$ to $\vec{r}_0$,
Poisson's Equation:

$$-\nabla_r^2 \phi(\vec{r}) = \frac{1}{\varepsilon_0} q\, \delta(\vec{r}-\vec{r}_0)$$

Then we have solution:

$$\boxed{\begin{aligned} \phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}-\vec{r}_0|} \\ \vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}-\vec{r}_0|^2} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|} \end{aligned}}$$

Superposition: By the linearity of the ∇^2, and the homogenity of the vanishing as $|\vec{r}| \to \infty$

A charge density due to a set of J-particles q_j, static at $\vec{r}_j$, i.e.

For Discrete Point Charges.

$$\rho(\vec{r}) = \sum_{j=1}^{J} q_j\, \delta(\vec{r}-\vec{r}_j)$$

is associated with superposition:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{J} \frac{q_j}{|\vec{r}-\vec{r}_j|}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{J} \frac{q_j}{|\vec{r}-\vec{r}_j|^2} \frac{\vec{r}-\vec{r}_j}{|\vec{r}-\vec{r}_j|}$$

Continuous - Distribution:

$$\sum_{j=1}^{J} q_j \to \int d^3r' \, \rho(\vec{r}')$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Note, to converge,

$\rightarrow$ $|\rho(\vec{r})| \leq \frac{C}{r^{2+\varepsilon}}$ $(C, \varepsilon > 0)$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3} \underbrace{(\vec{r} - \vec{r}')}_{\text{distance between charge and position}}$$

$\vec{r}'$ $\rightarrow$ position of charge

So, if we know $\rho(\vec{r})$, then we can integrate and get $\phi(\vec{r})$.

Green's function for electrostatic is:

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}'|}$$

$\leftarrow$ fundamental or translationally invariant Greens Function.

So that $\phi(\vec{r}) = \int d^3r' \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$

Check that Green's function works:

$$-\nabla^2 \phi(\vec{r}) = -\nabla^2 \int d^3r' \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

$$\overset{!}{=} \int d^3r' \, \underbrace{\left(-\nabla^2 G(\vec{r}, \vec{r}')\right)}_{\frac{1}{\epsilon_0}\delta(\vec{r}-\vec{r}')} \rho(\vec{r}')$$

$$= \frac{1}{\epsilon_0} \rho(\vec{r})$$

Green's Function in general:

If $G(\vec{r},\vec{r}')$ obeys:

$$\mathcal{L}_r\, G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

Linear Differential operator

and it also obeys suitable boundary or asymptotic conditions, then

a general inhomogenity

$$\mathcal{L}_r\, f(\vec{r}) = R(\vec{r})$$

is solved by $\quad f(\vec{r}) = \int d^3r'\, G(\vec{r},\vec{r}')\, R(\vec{r}')$

then G is called the Green's function of the problem

proof: $\quad \mathcal{L}_r f(\vec{r}) = \mathcal{L}_r \int d^3r'\, G(\vec{r},\vec{r}')\, R(\vec{r}')$

$$\stackrel{!}{=} \int d^3r'\, \underbrace{\mathcal{L}_r G(\vec{r},\vec{r}')}_{\delta(\vec{r}-\vec{r}')}\, R(\vec{r}')$$

$$\stackrel{!}{=} R(\vec{r})$$

Remarks: Think $\mathcal{L}$ and G are inverses. i.e. $\mathcal{L}^{-1} = G$

Analogy with Matrix / Vector:

If G obeys $\quad L \cdot G = I \quad \Rightarrow \quad L^{-1} = G$

then $\quad L \cdot F = R \quad$ is solved by

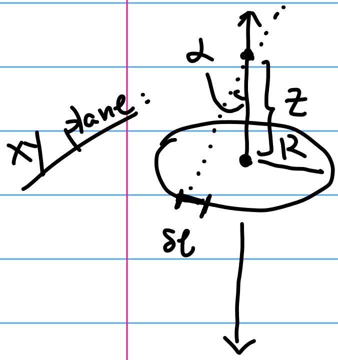
$$L^{-1}(L \cdot F = R)$$

$$\hookrightarrow \quad F = GR$$

And $\quad L \cdot F = L(GR) = IR = R$

Ex: Compute the on-axis electric field and potential due to uniform ring of charge Q and radius R at a distance z, from the center of the ring.

use result to compute the on-axis electric field due to a disk of radius R and charge per unit area σ.



Qualitatively: 1) $\vec{E}$ points along z.

2) $\vec{E}$ is odd in z, $\vec{E}(-\hat{z}) = -\hat{E}(\hat{z})$

3) $E^z = \frac{Q}{4\pi\epsilon_0 R^2} f\left(\frac{z}{R}\right)$

Quantitative: $\delta E = \frac{1}{4\pi\epsilon_0} \frac{f(r')}{|r-r'|^2}$

$\delta\ell = R\,d\phi$

$$\delta E^z = \frac{1}{4\pi\epsilon_0} Q \frac{\delta\ell}{2\pi R} \underbrace{\frac{1}{z^2+R^2}}_{\frac{1}{r^2}} \underbrace{\frac{z}{\sqrt{z^2+R^2}}}_{\cos\alpha} \hat{z}$$

$\frac{1}{r^2}$ ↳ distance from charge to position.

$$\hookrightarrow E^z = \frac{1}{4\pi\epsilon_0} Q \frac{z}{(z^2+R^2)^{3/2}} \hat{z}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \frac{z/R}{\left[1+\left(\frac{z}{R}\right)^2\right]^{3/2}}$$

as $z \ll R$: $E \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{z}{R} \operatorname{sgn}(z)$

as $z \gg R$: $E \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{z^2} \operatorname{sgn}(z)$

Then ϕ = integrate E since $-\vec{\nabla}\phi = \vec{E}$

$$\phi(z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{z^2+R^2}}$$ or use superposition

Now working with disk

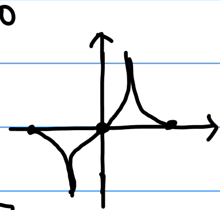
$$\int \delta E = \int_{r=0}^{R} \frac{1}{4\pi\epsilon_0}\, 2\pi r\, dr\, \sigma \frac{z}{(z^2+r^2)^{3/2}}\,\hat{z}$$

$$E = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{1+(R/z)^2}}\right] \underbrace{\mathrm{sgn}(z)}$$

$= 1$ if $z > 0$

$= -1$ if $z < 0$

as $|z| \ll R$, $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z} \rightarrow$



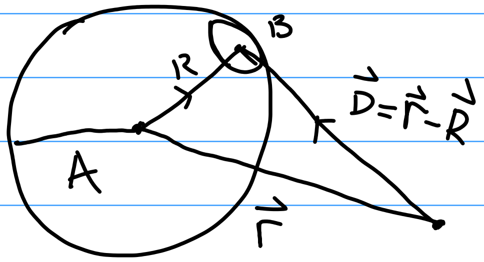
as $|z| \gg R$ $\quad E \approx \frac{\sigma}{2\epsilon_0}\left[1 - \left(1 - \frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right]$

$$\approx \frac{\sigma}{4\epsilon_0}\left(\frac{R}{z}\right)^2$$

$$= \frac{\pi R^2 \sigma}{4\pi\epsilon_0}\frac{1}{z^2}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{z^2}\,\mathrm{sign}(z)$$

Ex: Spherical shell of radius A, and uniform charge per unit area σ. A small cap of radius B ($\ll A$) is removed together with its charge. Compute the $\vec{E}$ far away from cap.

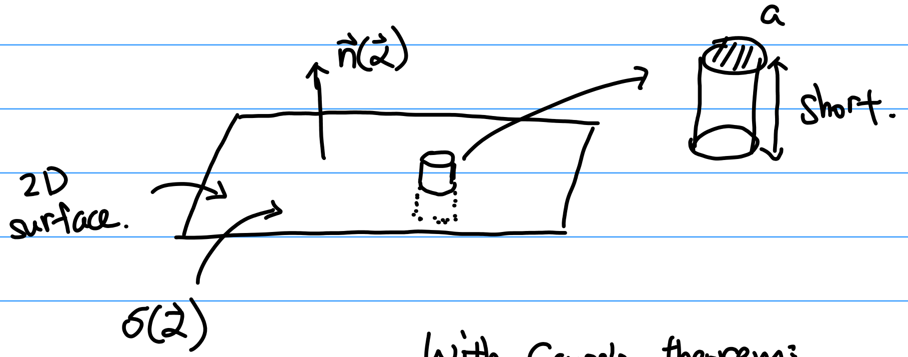


$$\vec{E}(\vec{r}) = \vec{E}_{\text{complete sphere}} + \vec{E}_{\text{cap-removed}}$$

$$= \frac{4\pi A^2\sigma}{4\pi\epsilon_0}\frac{\vec{r}}{|r|^3}\Theta(\vec{r}-\vec{A}) - \frac{\sigma\pi B^2}{4\pi\epsilon_0}\frac{\vec{r}-\vec{R}}{|\vec{r}-\vec{R}|^3}$$

$$E_{\text{sphere}} = \begin{cases} 0 & r < A \\ \frac{4\pi A^2\sigma}{4\pi\epsilon_0}\frac{\vec{r}}{|r|^3} & r > A \end{cases}$$

General matching conditions across charged interfaces:

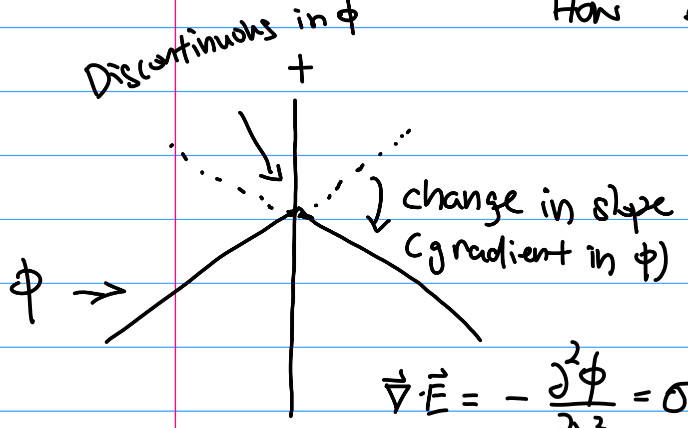


With Gauss's theorem:

$$\int \vec{\nabla}\cdot E \, dV = \int \frac{1}{\varepsilon_0} \rho \, dV$$

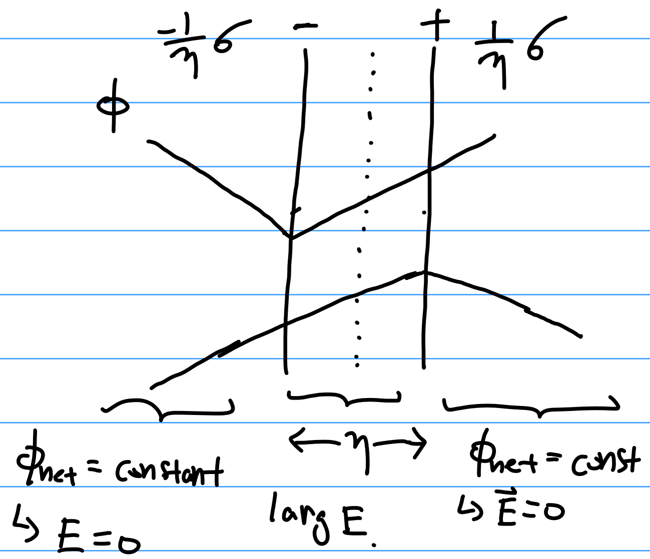
$$\underbrace{\int \vec{E}\cdot d\vec{s}^{\,2}}_{\vec{E}\cdot n(\vec{x})\, \cancel{a}} = \sigma(\vec{x}) \cancel{a} \frac{1}{\varepsilon_0}$$

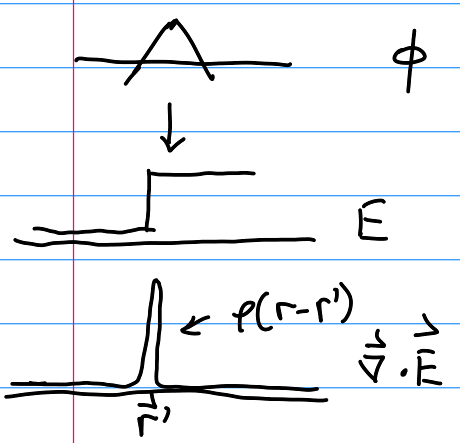
$$\underbrace{\lim_{\eta \to 0} \left[E(\vec{x}+\eta n(\vec{x})) - E(\vec{x}-\eta n(\vec{x})) \right]\cdot \vec{n}(\vec{x})}_{\text{How electric field change across surface.}} = \frac{1}{\varepsilon_0} \sigma(\vec{x})$$

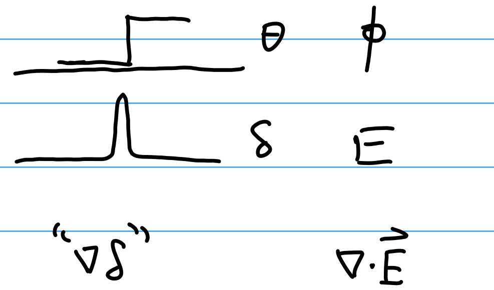


$$\vec{\nabla}\cdot\vec{E} = -\frac{\partial^2 \phi}{\partial x^2} = \sigma$$

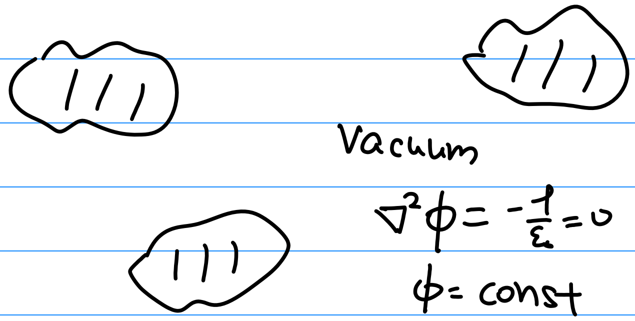
flat surface







Maximum Principle / Earnshaw's Theorem



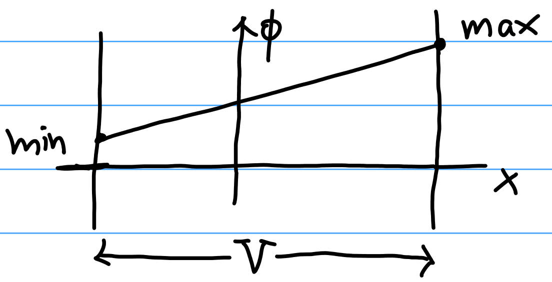
$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} = 0$$

$$\phi = \text{const}$$

$\ast$ No points in $\vec{r}$ have ϕ a local max or a local min.

Consequence: An infinitesimal-strength point charge inserted into a pre-dominated electrostatic potential cannot be in a position of equilibrium.

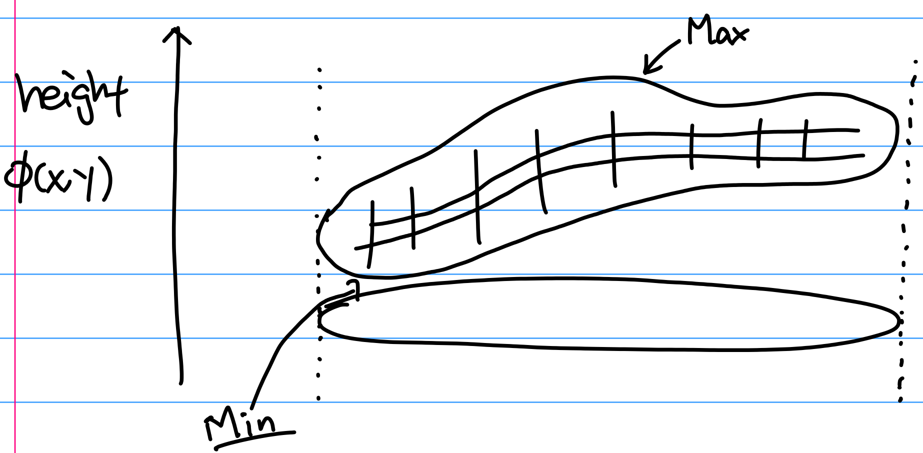
1D ex: $\partial_x^2\phi = 0$



$\phi(x) = A + Bx$, i.e. no local max or min. So no stable points.

2D ex: $\nabla^2\phi = \partial_x^2\phi + \partial_y^2\phi = 0$

Consider ϕ as the height, in mechanical equilibrium, of a drumhead, stretched across



Every point inside the region follows Laplace's Equation:

$$\partial_x^2 \phi = -\partial_y^2 \phi$$

i.e. curvature in x, y directions add to zero at every point. Since we can choose the orientation of the axes, this adding to zero must hold for every pair of orthogonal direction.

For $\phi(x,y) \sim \sin qx \, e^{-ky}$, oscillating in one-direction and decaying or growth in another, for Laplace equation to hold true:

$$\nabla^2 \phi = 0 \quad \Rightarrow \quad q^2 = k^2$$

Now suppose we have a minimum $\phi(x,y)$, such that $\vec{\nabla}\phi = 0$. We can expand $\phi(\vec{r})$ around $r = 0$:

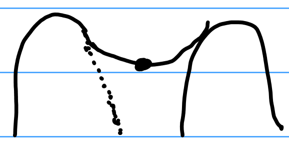
$$\phi(\vec{r}) = A + B_a r_a + C_{ab} r_a r_b + \cdots$$

↳ Require $B_a = 0$ for r_a to be stationary, $\vec{\nabla}\phi = 0$

↳ Require $C_{ab} = 0$ for ϕ to obey $\nabla^2 \phi = 0$

If diagonalize C: since eigenvalues add to 0 since $\nabla^2\phi = 0$

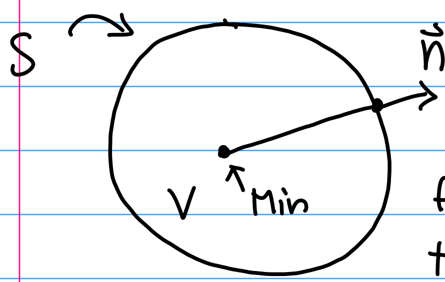
$$\phi(x,y) = \phi(0,0) + \underbrace{\lambda}_{\text{saddle point}} (x^2 - y^2)$$



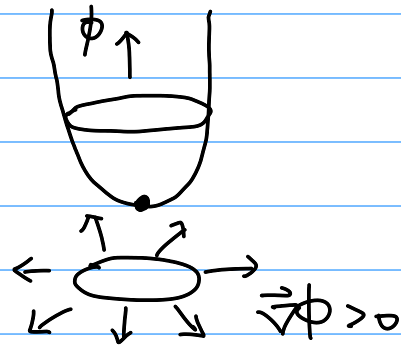
So any stationary point of $\phi(x,y)$ is a saddle point, <u>not</u> local min nor max.

Another example of Earnshaw's Theorem:

Consider a small sphere around a purtative minimum of ϕ. Since ϕ is a minimum, ϕ increases in all directions of departure from that point, i.e. $\hat{n}\cdot\vec{\nabla}\phi$ is positive at all points over the sphere.



$\vec{n}(\vec{2})\cdot\vec{\nabla}\phi > 0$ for the center to be minimum



Then collecting all positive values:

$$\int d^2s\ \hat{n}\cdot\vec{\nabla}\phi > 0$$

$$\hookrightarrow \int d^3r\ \vec{\nabla}\cdot\vec{\nabla}\phi = \int d^3r\ \underbrace{\nabla^2\phi}_{=0} \not> 0$$

so it cannot be true that $\hat{n}\cdot\vec{\nabla}\phi > 0$, hence there is not a local minimum. The same strategy applies for max.

Interaction energy, force and torque.

$$\boxed{\mathcal{E} = \frac{\varepsilon_0}{2}\int d^3r\ \vec{E}\cdot\vec{E}}$$ ← energy stored in electric field.

with Maxwell: $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$ and $-\nabla^2\phi = \frac{1}{\varepsilon_0}\rho$

$$\rightarrow\ \mathcal{E} = -\frac{\varepsilon_0}{2}\int d^3r\ \vec{E}\cdot\vec{\nabla}\phi$$

$$\overset{!}{=} -\frac{\varepsilon_0}{2}\Big[\underbrace{\int d^3r\ \vec{\nabla}(\vec{E}\,\phi)}_{\cancel{\int d^2\vec{S}\cdot(\vec{E}\,\phi)}} - \underbrace{\int d^3r\ \phi\,\vec{\nabla}\cdot\vec{E}}_{\int d^3r\ \phi\,\frac{\rho}{\varepsilon_0}}\Big]$$

→ as $d^2\vec{S} \to \infty$, $\vec{E} \to 0$ faster than $d^2S \to \infty$.

$$\boxed{\mathcal{E} = -\frac{1}{2}\int d^3r\ \rho\phi}$$

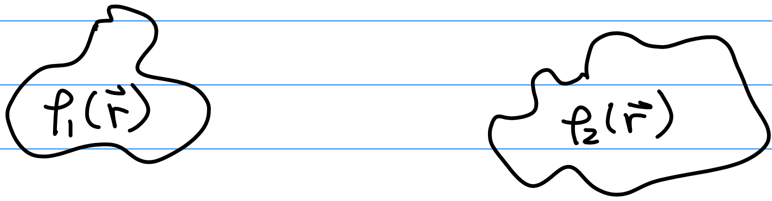
all space integration

→ given ϕ, get ρ by plug into $-\nabla^2\phi = \rho/\varepsilon_0$

or

→ given ρ, solve $-\nabla^2\phi = \frac{\rho}{\varepsilon_0}$ to get ϕ.

Interaction between two disjoint charge distribution:



Consider two relatively charged distribution that are disjoint from one another:

Alone: ρ_1 gives rise to ϕ_1
ρ_2 gives rise to ϕ_2

By superposition: $\rho_1 + \rho_2 \to \phi_1 + \phi_2 \to E_1 + E_2 = -\vec{\nabla}(\phi_1 + \phi_2)$

$$\mathcal{E}_{total} = \frac{1}{2}\varepsilon_0 \int d^3r \; |\vec{E}_1 + \vec{E}_2|^2$$

$$\overset{!}{=} \frac{1}{2}\varepsilon_0 \int d^3r \; |E_1|^2 + |E_2|^2 + 2 E_1 \cdot E_2$$

$$\overset{!}{=} \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_{int}$$

← interaction energy.

Identify interaction energy as:

$$\mathcal{E}_{int} = \varepsilon_0 \int d^3r \; \vec{E}_1 \cdot \vec{E}_2$$

$$\overset{!}{=} \int d^3r \; \rho_1 \phi_2 \quad \text{or} \quad \int d^3r \; \rho_2 \phi_1 = \int d^3r \; \underbrace{\rho}_{\text{nearby charge}} \; \underbrace{\phi^{ext}}_{\text{due to distant charges}}$$

Important setting: These are distant (but not ∞ distant) charges held at fixed position that give rise to external potential, ϕ^{ext}.
If let $\rho_2 = \rho$, $\phi_1 = \phi^{ext}$, then there are nearby charges ρ that are located in ϕ^{ext}. Vice-Versa. Since there is no overlap between charge distribution, where we have $\rho \neq 0$, ρ_{ext} must $= 0$, so $\nabla^2 \phi^{ext} = 0$

Green's Reciprocal Relation:

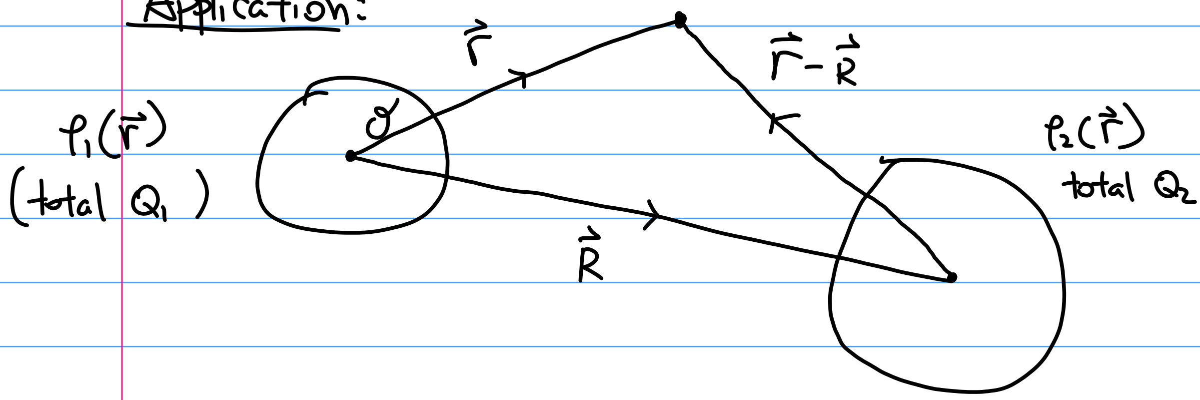
Suppose $\rho_1(\vec{r})$ causes $\phi_1(\vec{r})$
$\rho_2(\vec{r})$ causes $\phi_2(\vec{r})$

then $$\boxed{\int d^3r \; \rho_1(\vec{r})\,\phi_2(\vec{r}) = \int d^3r \; \rho_2(\vec{r})\,\phi_1(\vec{r})}$$

Both expressions lead to this

$$= \int \frac{d^3r_1 \, d^3r_2}{4\pi\epsilon_0} \frac{\rho_1(\vec{r}_1)\,\rho_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

Application:



Two spherically charged distribution ρ_1 and ρ_2. Total charges Q_1 and Q_2. Find interaction energy.

$$\mathcal{E} = \epsilon_0 \int d^3r \; \rho(\vec{r})\,\phi^{ext}(\vec{r})$$

$$\overset{!}{=} \epsilon_0 \int \frac{d^3r_1 \, d^3r_2}{4\pi\epsilon_0} \frac{\rho_1(\vec{r}_1)\,\rho_2(\vec{r}_2)}{\underbrace{|\vec{R} + \vec{r}_2 - \vec{r}_1|}_{|\vec{r}_1 - \vec{r}_2|}}$$ ← Difficult to integrate.

Instead use Green's Reciprocal Relation:

Consider ϕ_2 outside sphere #2, by Gauss's Law:

$$\phi_2(\vec r) = \frac{Q_2}{4\pi\epsilon_0}\frac{1}{|\vec r - \vec R|}$$ ← the same as all charge centrated at R.

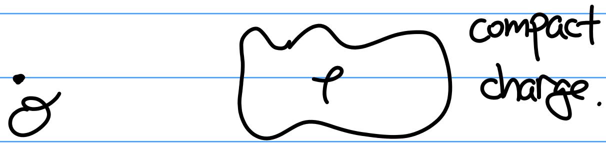
then $\mathcal{E}_{int} = \int d^3r\, \rho_1(\vec r)\,\phi_2(\vec r)$ ← so it is same as $\tilde\rho_2 \equiv Q\,\delta(\vec r - \vec R)$

$$\overset{!}{=} \int d^3r\, \phi_1(\vec r)\,\tilde\rho_2(\vec r)$$

$$\overset{!}{=} \int d^3r\, \frac{Q_1}{4\pi\epsilon_0}\frac{1}{|\vec r|}\, Q_2\,\delta(\vec r - \vec R)$$

$$\mathcal{E}_{int} \overset{!}{=} \frac{Q_1 Q_2}{4\pi\epsilon_0}\frac{1}{R}$$ ← Coulomb potential if the spheres were points.

<u>Expansions of $\mathcal{E}_{int}$ in powers of:</u>



Taylor expand ϕ around O, assume that ϕ^{ext} varies on a scale that is large compared with the size of region containing charge. → i.e. slowly varying ϕ.

$$\phi_{ext}(\vec r) = \phi_{ext}(\vec r)\Big|_{\vec r=0} + \vec r\cdot\vec\nabla\phi_{ext}\Big|_{r=0}$$

$$+\frac{1}{2}\sum_{k,l=1}^{3} \underbrace{r_k r_l \frac{\partial^2\phi_{ext}}{\partial r_k\,\partial r_l}\Big|_{r=0}}_{=0 \text{ since } \nabla^2\phi=0} + \dots$$

Inserting into $\mathcal{E}_{int} = \int d^3r\ \rho\ \phi_{ext}$

$$\mathcal{E}_{int} = \phi_{ext,o} \int d^3r\ \rho(\vec{r})$$
$$+ \vec{\nabla}\phi_{ext,o} \cdot \int d^3r\ \vec{r}\ \rho(\vec{r})$$
$$+ \frac{1}{2}\sum_{k,l=1}^{3} \phi^{kl}_{ext,o} \int d^3r\ r_k r_l\ \rho(\vec{r}) + \cdots$$

Noting that terms in $\mathcal{E}_{int}$ are characterized by moments of the charge distribution:

(couples to potential ϕ_{ext})

0^{th}: $q \equiv \int d^3r\ \rho(\vec{r})$: the total electric charge 1#

(couples to gradient, E_{ext})

1^{st}: $\vec{p} \equiv \int d^3r\ \vec{r}\rho(\vec{r})$: the electric dipole moment. 3#

(couples to curvature of ϕ_{ext}) 5#

2^{nd}: $Q_{kl} = \int d^3r\ [3r_k r_l - \underbrace{r^2\delta_{kl}}]\ \rho(\vec{r})$: quadrupole moment.

subtracts contribut in $\mathcal{E}_{int}$ as a result of $\nabla^2\phi$, trace-free.

Then $$\mathcal{E}_{int} = q\phi_{ext,o} + \vec{p}\cdot\vec{\nabla}\phi_{ext,o} + \frac{1}{6}\sum_{k,l=1}^{3} Q_{kl}\frac{\partial^2\phi_{ext,o}}{\partial r_k\,\partial r_l} + \cdots$$

Here we keep ρ fixed and vary ϕ.

We can do a similar construct by keeping ϕ fixed and vary ρ.

Force and Torque on $\rho(\vec{r})$ due to $Q^{ext}(\vec{r})$

For a distant charge that creates $\Phi_{ext}(\vec{r})$ and nearby charge ρ, the force on the body carrying ρ is:

$$\boxed{\vec{F} = \int d^3r \, \rho(\vec{r}) \, \vec{E}(\vec{r})}$$

due to all ρ → $\vec{E} = \vec{E}_{self} + \vec{E}_{ext}$

We now show that $\int d^3r \, \rho(\vec{r}) E_{self}(\vec{r}) = 0$, so that only E_{ext} contributes to $\vec{F}$.

$$\int d^3r \, \rho(\vec{r}) \vec{E}_{self}(\vec{r}) = \int d^3r \, \rho(\vec{r}) \underbrace{\frac{1}{4\pi\epsilon_0} \int d^3r' \, \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}_{E_{self}(\vec{r})}$$

$$= \frac{1}{4\pi\epsilon_0} \int \underbrace{d^3r \, \rho(\vec{r}) \, d^3r' \, \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}$$

(anti) Skew-symmetric: if we change $\vec{r}$ and $\vec{r}'$, they differ by negative sign, so it must $= 0$

$$= 0$$

Therefore, in electrostatic, self-force vanishes. But it is not always in electrodynamics.

$$\boxed{\vec{F} = \int d^3r \, \rho(\vec{r}) \, \vec{E}_{ext}(\vec{r})}$$

for electrostatic.

With variation in the $\vec{E}_{ext}$ field,

$$\vec{F} = q E_{ext,0} + \vec{p}\cdot\vec{\nabla}\,\vec{E}_{ext,0} + \frac{1}{6}\sum_{k,l=1}^{3} Q_{kl}\frac{\partial^2}{\partial r_k \partial r_l}E_{ext,0} + \cdots$$

→ It's important to leave out $\vec{E}_{self}$, since we don't expect nearby charge to vary slowly.

→ We also note the leading term in $\vec{F}$ is the total charge, q, and external field $\vec{E}$. And if $q \to 0$, the the leading terms become $\vec{p}$ and Q_{kl}.

Torque:

$$\tau = \int d^3r\ \vec{r}\times\vec{f} = \int d^3r\ \vec{r}\times\left(\rho(\vec{r})\vec{E}(\vec{r})\right)$$

and similarly, only $\vec{E}_{ext}$ contributes to τ.

Now consider slow spatially varying field:

$$\tau = \vec{p}\times\vec{E}_{ext,0} + \cdots$$

Dipole moment is the leading term since $\tau \sim \vec{r}\times\vec{E}$

Point Charge Idealization:

For a point charge q located at origin:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{|r|^2}$$

$$\hookrightarrow \quad \varepsilon = \frac{\epsilon_0}{2}\int d^3r\, |\vec{E}|^2 = \frac{\epsilon_0}{2}\int d^3r\, \frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}$$

$$\overset{!}{=} \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{r^2 dr}{r^4} = \infty$$

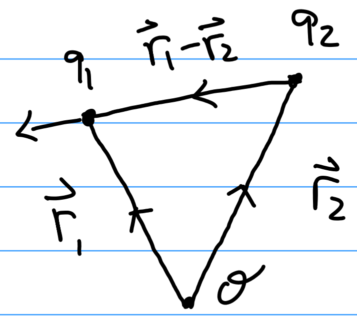
point charges are unphysical idealization that sometimes present obstacles

But we can consider interaction between particles:

$$\varepsilon_{int} = \int d^3r\, \underbrace{\rho_1}_{q_1\delta(\vec{r}-\vec{r}_1)}\, \underbrace{\Phi_2}_{\frac{q_2}{4\pi\epsilon_0}\frac{1}{|\vec{r}_1-\vec{r}_2|}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|} = \text{coulomb potential.}$$

then $$\vec{F} = -\partial_{r_1}\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|}$$

$$\vec{F}_{Coulomb} \overset{!}{=} q_1\frac{q_2}{4\pi\epsilon_0}\frac{(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}$$



Generally: $\vec{F} = q\,\vec{E}_{ext}$

charge of the body on which the force is acted on

electric field due to other charges.

Multipole Expansion of the fundamental Green function for Poisson Equation:

$$\boxed{G(\vec{r}, \vec{r}\,') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}\,'|}}$$

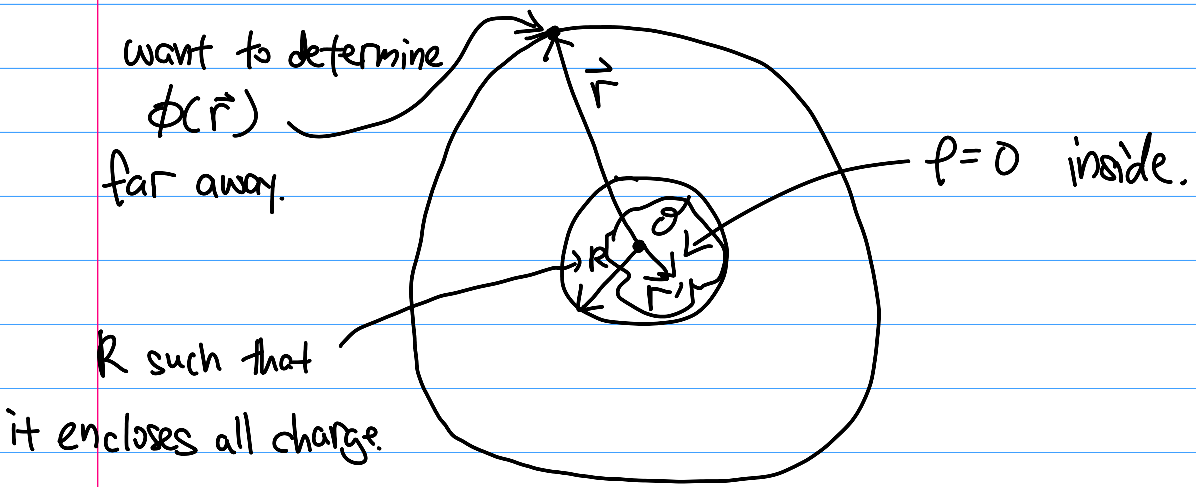
The fundamental Green function.

as $\quad -\nabla_r^2 G(\vec{r}, \vec{r}\,') = \frac{1}{\epsilon_0} \delta(\vec{r}-\vec{r}\,')$

we have seen that for ρ decaying rapidly enough at ∞, Poisson's Equation has the unique solution:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Now lets construct the multiple expansion of ρ; it is useful when we want to construct ϕ at regions away from ρ, say $\vec{r}$:



Focus on $\frac{1}{|\vec{r}-\vec{r}\,'|}$, note that $|\vec{r}\,'| < R < |\vec{r}|$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2}}$$

$$= \frac{1}{r}\frac{1}{\sqrt{1-2\left(\frac{r'}{r}\right)\hat{r}\cdot\hat{r}' + \left(\frac{r'}{r}\right)^2}}$$

$$= \frac{1}{r}\left\{1 + \left(\frac{r'}{r}\right)\hat{r}\cdot\hat{r}' + \frac{1}{2}\left(\frac{r'}{r}\right)^2\left[3(\hat{r}\cdot\hat{r}')^2 - 1\right] + \cdots\right\}$$

Insert expansion back to $\phi(r) = \frac{1}{4\pi\epsilon_0}\int d^3r'\, \rho(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\rho(r')\frac{1}{r}\left\{1 + \left(\frac{r'}{r}\right)\hat{r}\cdot\hat{r}' + \frac{1}{2}\left(\frac{r'}{r}\right)^2\left[3(\hat{r}\cdot\hat{r}')^2 - 1\right] + \cdots\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\Bigg[\frac{1}{r}\underbrace{\int d^3r'\,\rho(\vec{r}')}_{q} + \frac{\hat{r}}{r^2}\cdot\underbrace{\int d^3r'\,\vec{r}'\rho(\vec{r}')}_{\vec{p}\ \text{(dipole)}}$$

$$+\frac{1}{2}\frac{1}{r^3}\sum_{k,l=1}^{3}\Big(\hat{r}_k\hat{r}_l \underbrace{-\frac{1}{3}\delta_{kl}}_{\text{can be subtracted since } r_k r_l \text{ traceless}}\Big)\underbrace{\int d^3r'\,\rho(\vec{r}')\,3r_k' r_l'}_{Q_{kl}\ \text{(quadrupole)}} + \cdots\Bigg]$$

$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\hat{r}\cdot\vec{p}}{r^2} + \frac{1}{6}\frac{1}{r^3}\sum_{k,l=1}^{3}\hat{r}_k\hat{r}_l Q_{kl} + \cdots\right]}$$

Multipole Expansion for the fundamental Green Function:

Consider:
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 - 2rr'\hat{n}\cdot\hat{n}' + r'^2}}$$

$\vec{r} = r\hat{n}$ $\vec{r}\,' = r'\hat{n}'$

Assume $r' < r$, and write $\frac{1}{r}\frac{1}{\sqrt{1-2(r'/r)\hat{n}\cdot\hat{n}' + (r'/r)^2}}$

There is a standard generating function for Legendre polynomials:

It generates the RHS by taking derivative of LHS

generating function

$$\frac{1}{\sqrt{1-2u\hat{n}\cdot\hat{n}'+u^2}} = \sum_{l=0}^{\infty} u^l P_l(\hat{n}\cdot\hat{n}'), \quad u<1.$$

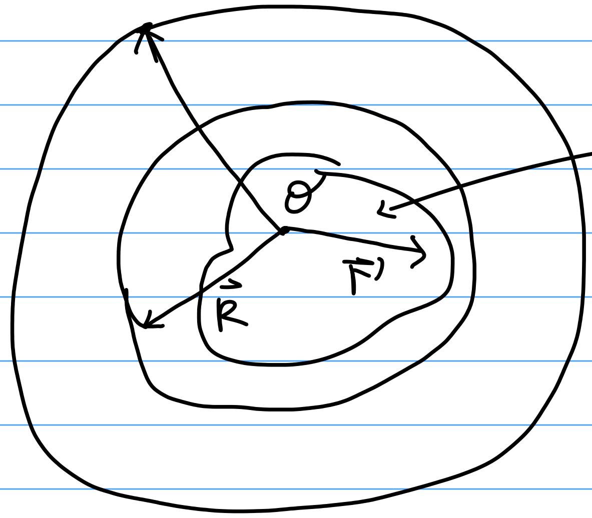
with
$$P_l(\hat{n}\cdot\hat{n}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\hat{n}')^* Y_{lm}(\hat{n})$$

Altogether: for $r' < r$:

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\hat{n}') Y_{lm}(\hat{n})}$$

Applying Multipole Expansion:

Given a charge distribution ρ, we can use this expansion to organize the solution:



$\rho \neq 0$ inside.

we have $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\rho(\vec{r}')$$

$$\hookrightarrow = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int d^3r' \sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\hat{n}')^* Y_{lm}(\hat{n})\rho(\vec{r}')$$

$$\hookrightarrow = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\frac{1}{r^{l+1}}\sum_{m=-l}^{l} Y_{lm}(\hat{n})\underbrace{\int d^3r' (r')^l Y_{lm}^*(\hat{n}')\rho(\vec{r}')}_{\substack{\text{spherical Multipole expansion} \\ q_{lm}}}$$

$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\frac{1}{r^{l+1}}\sum_{m=-l}^{l} Y_{lm}(\hat{n})\, q_{lm}}$$

where $q_{lm} = \int d^3r'\, r'^l\, Y_{lm}^*(\hat{n}')\rho(\vec{r}')$ ← spherical multi-pole moment of ρ.

$d^3r = r^2 dr\, d^2n$

$$= \int dr'\, r'^{l+2}\int d^2n\, Y_{lm}^*(\hat{n})\rho(r\hat{n})$$

↑ probes angular variation

↗ suppresses charge near O, also amplifies remote charge.

Ex: $l=0$, $q_{00} = \frac{1}{\sqrt{4\pi}} \underbrace{\int d^3r \, \rho(\vec{r})}_{\text{net charge.}}$

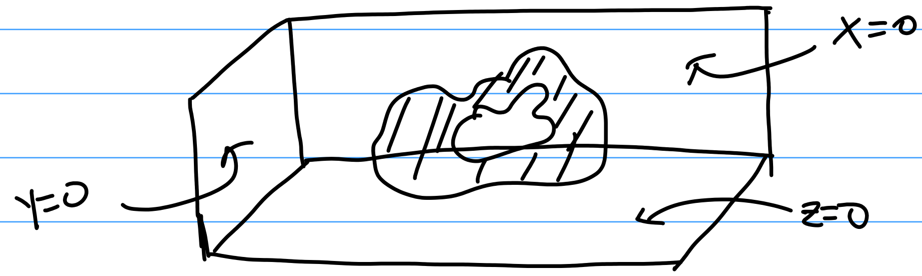
Dipole $l=1$ $q_{10} = \sqrt{\frac{3}{4\pi}} \, p_z$

$$q_{11} = -q^*_{1-1} = -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

noting that $\vec{p} = \int d^3r \, \vec{r} \, \rho(\vec{r})$

Quadrupole $l=2$

Examples of using Multipole - Expansion:



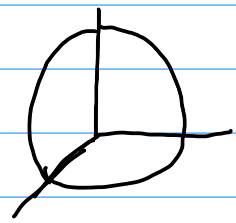
A spatial distribution of charge, total charge Q, is constrained within "all positive coordinates", $x>0, y>0, z>0$.

Show:

$$\int_0^\infty dy\,dz\, E^x(x,y,z)\Big|_{x=0} + \int_0^\infty dz\,dx\, E^y(x,y,z)\Big|_{y=0} + \int_0^\infty dx\,dy\, E^z(x,y,z)\Big|_{z=0} = -\frac{7}{8}\frac{Q}{\varepsilon_0}$$

Method: use Gauss's Law:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$



RHS: $\frac{1}{\varepsilon_0}\int d^3r\,\rho(\vec{r}) = \frac{Q}{\varepsilon_0}$

negative because point inside

LHS: $\int d^3r\,\vec{\nabla}\cdot\vec{E} = \int d^2\vec{S}\cdot\vec{E} =$

$$-\left\{\int_0^\infty dy\,dz\, E^x(x,y,z)\Big|_{x=0} + \int_0^\infty dz\,dx\, E^y(x,y,z)\Big|_{y=0} + \int_0^\infty dx\,dy\, E^z(x,y,z)\Big|_{z=0}\right\}$$

Integrate over quarter Quadrant.

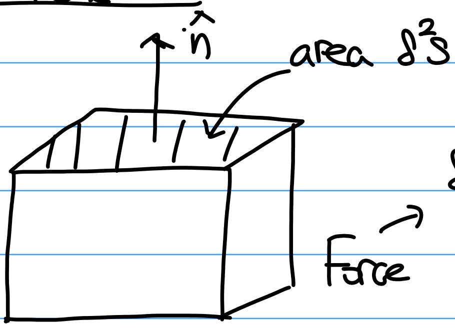
$l=0$ of multipole expansion.

$$+\underbrace{\int_0^{\frac{\pi}{2}}\underbrace{\int_0^{\pi/2} r^2\sin\theta\,d\theta}_{\frac{\pi}{2}}\,d\phi\left(\frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\right)\cdot\hat{r}}_{1/8\ Q/\varepsilon_0} + O = \frac{Q}{\varepsilon_0},$$

so the cartesian faces add up to $-\frac{7}{8}Q/\varepsilon_0$

Electrostatic Stress Tensor:

Consider a cube:



$$\delta \vec{F} = (-p)\,\hat{n}\,\delta^2 S$$

Force $\nearrow$ $\uparrow$ pressure $\uparrow$ differential area.

Then the total force on the volume is given by:

$$\vec{F} = \int_S -p\,\hat{n}\,d^2 S = \int d^2\vec{S}\,(-p)$$

or in Cartesian:

$$F_k = \int_S d^2 S_k\,(-p) = \int_S dS_j\,(-p\,\delta_{jk})$$

Now introduce stress tensor: $t_{jk} = -p\,\delta_{jk}$

then we have $F_k = \int_S dS_j\, t_{jk}$

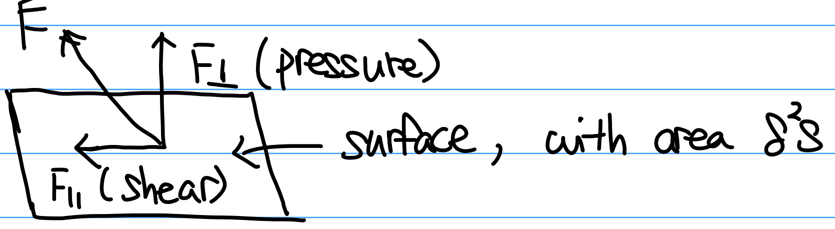
With divergence theorem:

$$F_k = \int_V d^3 r\,\partial_j t_{jk} = \int_V d^3 r\,(-\partial_j p\,\delta_{jk}) = \int d^3 r\,(-\partial_k p)$$

or $\vec{F} = \int d^3 r\,\underbrace{(-\vec{\nabla} p)}$

we see $-\vec{\nabla} p$ is analogous to "force density"

Stress is a generalization of pressure, developed to allow the force acting on the surface to have a parallel (shear) or normal (pressure) component:



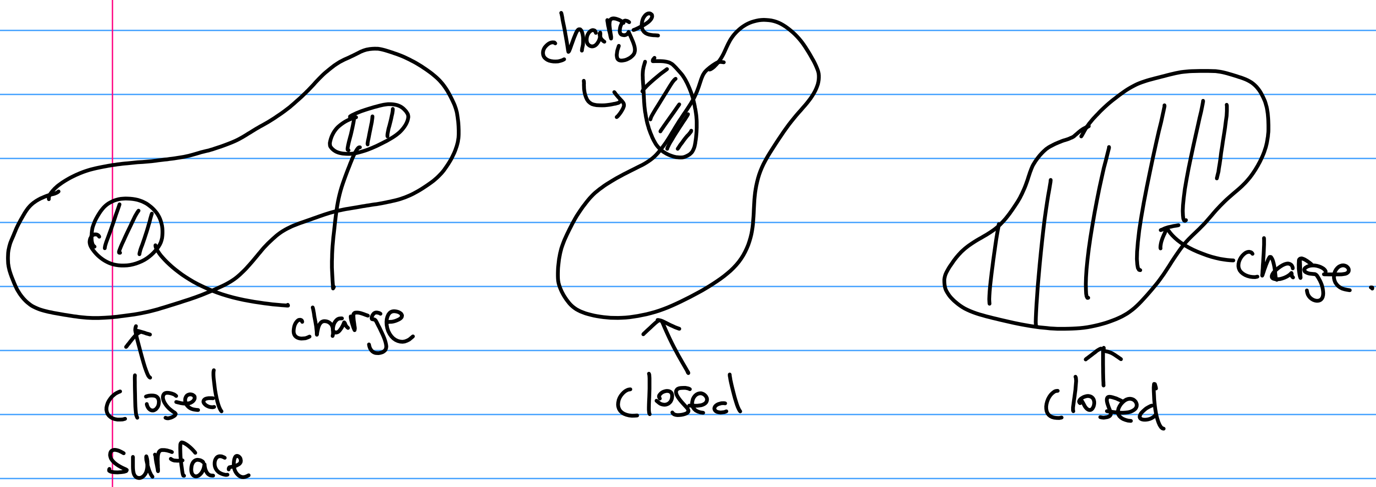
$$\delta F_k = \underbrace{\delta^2 S}_{\text{area element}} \; \underbrace{n_j}_{\text{surface Normal}} \; \underbrace{T_{jk}}_{\text{Stress Tensor, like pressure (same as energy density)}}$$

<u>Remarks:</u> i) $\underbrace{\hat{n} \cdot \delta\vec{F}}_{\text{component of } \delta F \text{ parallel to } \hat{n}} = n_j T_{jk} n_k \delta^2 S$

ii) The remainder: $\delta\vec{F} - \hat{n}(\hat{n} \cdot \overrightarrow{\delta F})$ = components of $\vec{F}$ that are not in $\hat{n}$ direction

$\hookrightarrow \left[\delta\vec{F} - \hat{n}(\hat{n} \cdot \delta\vec{F})\right]_k = n_j T_{jl} \underbrace{\left[\delta_{lk} - n_l n_k\right]}_{\text{effectively shear the volume, i.e. change shape but total V stays constant.}} \delta^2 S$

Lets introduce a closed surface that encloses some charge. The surface may coincide with the surface of a body, but it does not have to.



lets compute the total force due to an electrostatic field on all the charge inside the surface.

$$\vec{F} = \int d^3r \, \rho(\vec{r}) \, E(\vec{r})$$

volume enclosed by the surface.

$$F_k = \int d^3r \, \underbrace{\epsilon_0 (\partial_j E_j)}_{\text{Gauss's Law}} E_k$$

$$= \epsilon_0 \int d^3r \left[\partial_j (E_j E_k) - E_j \partial_j E_k \right]$$

since $\vec{\nabla} \times \vec{E} = \epsilon_{ijk} \partial_i E_j = -\partial_t B = 0$

then $\partial_j E_k - \partial_k E_j = 0$

$$= \epsilon_0 \int d^3r \left[\partial_j (E_j E_k) - E_j \partial_k E_j \right]$$

$$= \epsilon_0 \int d^3r \left[\partial_j (E_j E_k) - \frac{1}{2} \partial_k (\vec{E} \cdot \vec{E}) \right]$$

$$\stackrel{!}{=} \epsilon_0 \int d^3r \, \partial_j \left[E_j E_k - \frac{1}{2} E_j^2 \delta_{jk} \right]$$

$$\boxed{F_k \stackrel{!}{=} \int d^2S_j \, T_{jk}}$$

F_k: total of the electrostatic field acting on the charges

T_{jk}: Momentum flux through surface.

j: sum of all surfaces.

k: direction of force

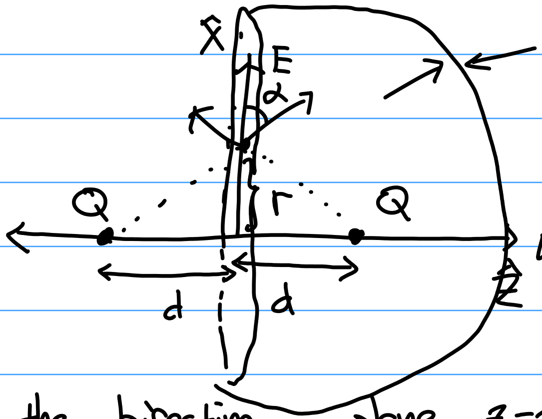
where $T_{jk} = \epsilon_0 \left[E_j E_k - \frac{1}{2} E_j^2 \delta_{jk} \right]$

T_{jk}: electrostatic stress-tensor.

Interpretation :

↳ The charge inside V (whether or not touching S) may be thought of as being subject to a force that is acting through the elements of the surrounding surface. It is as if vaccum has a kind of solidness that enables it to convey stress.

Ex: use the stress-tensor to compute the force on a charge Q, distance $2d$ from a charge Q.



ignore boundary since $A \sim r^2$ but $T \sim E^2 \sim \frac{1}{r^4}$ so $F \to 0$ as $r \to \infty$

Rotational symmetry in z.

Focus on the bisecting plane, $z=0$. Compute $\vec{E}$ on that plane, where symmetry dictates that it lies in the plane.

By coulomb's law applied to the two charges:

$$|\vec{E}| = 2\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2+d^2}\underbrace{\cos\alpha}_{\frac{r}{\sqrt{r^2+d^2}}}$$

$$|\vec{E}| = 2\frac{Q}{4\pi\epsilon_0}\frac{r}{(r^2+d^2)^{3/2}}$$

→ Imagine rotating the point at which we are considering $\vec{E}$ around in the xy-plane

To the point: $(x, y, z) = (r\cos\phi, r\sin\phi, 0)$

At that point, $\vec{E}$ has component:

$$(E_x, E_y, E_z) = |\vec{E}|(\cos\phi, \sin\phi, 0)$$

Then the Cartesian components of the stress tensor are:

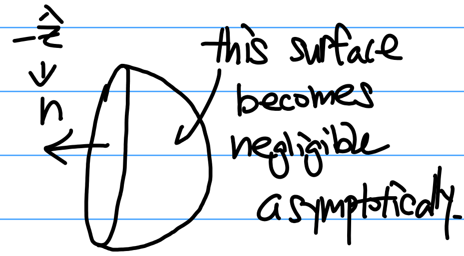
$$T_{jk} = \epsilon_0 \left(E_j E_k - \tfrac{1}{2} \delta_{jk} |\vec{E}|^2 \right)$$

$$= \epsilon_0 |\vec{E}|^2 \left(\begin{bmatrix} cc & cs & 0 \\ sc & ss & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$T_{jk} = \epsilon_0 |\vec{E}|^2 \begin{pmatrix} cc - \frac{1}{2} & cs & 0 \\ sc & ss - \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Now we can use T_{jk} to compute the force transmitted into the large hemisphere containing the right-hand charge.

By symmetry only the z-component is non-zero: so only find



$$F_z = \int_S \underbrace{d^2 S_j}_{\text{outward normal}} T_{jz}$$

$$= \int_{z=0} d\phi \, r dr \; \underbrace{\hat{n}_j}_{= -\hat{z}} \; T_{jz}$$

$$= -\int_{z=0} d\phi \, r dr \; T_{33}$$

$$F_z = -\int d\phi\, r\,dr \underbrace{\left(-\frac{1}{2}\right)\epsilon_0 \underbrace{\left[2\frac{Q}{4\pi\epsilon_0}\frac{r}{(r^2+d^2)^{3/2}}\right]^2}_{|\vec{E}|^2}}_{T_{33} = -\frac{1}{2}\varepsilon_0 |\vec{E}|^2}$$

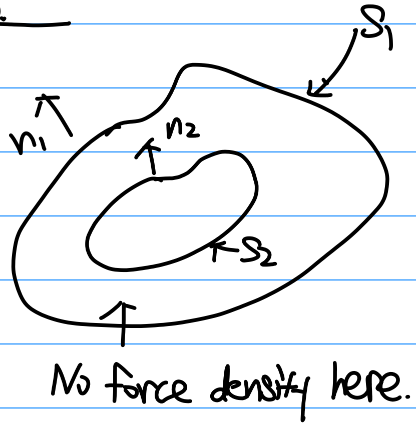
$$= \frac{Q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\frac{2}{\pi}\int_0^\infty \frac{r\,dr}{d^2}\frac{r^2 d^4}{(r^2+d^2)^3}\int_0^{2\pi} d\phi$$

let $t = r/d$

$$= \frac{Q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\, 4\underbrace{\int_0^\infty \frac{t^3\,dt}{(1+t^2)^3}}_{=\frac{1}{4}}$$

$$\boxed{F_z = \frac{Q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}}$$ ← recovers coulomb Law

Stress Transmission through the vacuum.



Force density $= \rho E_k = -\partial_j T_{jk}$

Suppose the force density vanishes in the region between surfaces S_1 and S_2. Then the force computed using one surface equals that computed using the other.

$$F_k^{(1)} - F_k^{(2)} = \int_{S_1} d^2S_j\, T_{jk} - \int_{S_2} d^2S_j\, T_{jk}$$

$$= \int_{V_1} d^3r\, \partial_j T_{jk} - \int_{V_2} d^3r\, \partial_j T_{jk}$$

$$= \int_{V_1 - V_2} d^3r\, \underbrace{\partial_j T_{jk}}_{\substack{\text{vanishes inside } V_1 \\ \text{but outside } V_2}} = 0$$

→ We may use any convenient surface, provided a vanishing force density (e.g. a vanishing ρ). Answers insensitive to the choice of the surface.

Spherical Harmonic Functions:

$Y_{\ell m}(\theta,\phi)$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1-1} = +\sqrt{3/8\pi}\ \sin\theta\, e^{i\phi} \quad \propto \frac{1}{r}(x-iy)$$

$$Y_{10} = \sqrt{3/4\pi}\cos\theta \quad \propto \frac{1}{r} z$$

$$Y_{11} = -\sqrt{3/8\pi}\ \sin\theta\, e^{-i\phi} \quad \propto \frac{1}{r}(x+iy)$$

Orthogonality:

$$\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\bar{\ell}\bar{m}}(\theta,\phi) = \delta_{\ell\bar{\ell}}\ \delta_{m\bar{m}}$$

Completeness:

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}(\theta,\phi)$$

and

$$f^{R}(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} D^{(\ell)}_{m\bar{m}}\, f_{\ell m}\, Y_{\ell m}(\theta,\phi)$$

Extraction:

$$f_{\ell m} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\ Y^*_{\ell m}\, f(\theta,\phi)$$

Laplace Eq in spherical coordinate:

$$\left[\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}\frac{1}{\sin\theta}\partial_\theta \sin\theta\, \partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\right]\Phi(r,\theta,\phi)=0$$

let $\Phi(r,\theta,\phi) = R(r)\, Y(\theta,\phi)$ $\quad (m=0)\ P_\ell^{(m=0)} \rightarrow$ Legendre-Poly

$$Y_{\ell m} = \begin{cases} (-1)^m \\ 1 \end{cases} \sqrt{\frac{2\ell+1}{2}}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\ P_\ell^{(m)}(\cos\theta)\ \frac{e^{im\phi}}{\sqrt{2\pi}} \qquad \begin{matrix} m\geq 0 \\ m<0 \end{matrix}$$

associated Legendre.