Laplace Equation Solutions: $\nabla^2 \varphi = 0$

→ Cartesian Symmetry: Rectangular Boundaries:

$\varphi(x, y, z) = X(x)\, Y(y)\, Z(z)$

$\nabla^2 \varphi = \partial_x^2 \varphi + \partial_y^2 \varphi + \partial_z^2 \varphi = 0$

$\hookrightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$

Since each component is only a function of one component, their sum can be zero only if each is equal to a constant.

$\frac{d^2}{dx^2} X = \alpha^2 X \quad , \quad \frac{d^2}{dy^2} Y = \beta^2 Y \; , \; \frac{d^2}{dz^2} Z = \gamma^2 Z$

so $\quad \alpha^2 + \beta^2 + \gamma^2 = 0$

$$-(\alpha^2 + \beta^2) = \gamma^2$$

then we have the following solutions:

$$X_\alpha = \begin{cases} A_0 + B_0 x & \alpha = 0 \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0 \end{cases}$$

$$Y_\beta = \begin{cases} C_0 + D_0 y & \beta = 0 \\ C_\beta e^{\beta y} + D_\beta e^{-\beta y} & \beta \neq 0 \end{cases}$$

$$Z_\gamma = \begin{cases} E_0 + F_0 z & \gamma = 0 \\ E_\gamma e^{\gamma z} + F_\gamma e^{-\gamma z} & \gamma \neq 0 \end{cases}$$

Cartesian

$$\boxed{\therefore \quad \varphi(x, y, z) = \sum_\alpha \sum_\beta \sum_\gamma X_\alpha(x)\, Y_\beta(y)\, Z_\gamma(z)\, \delta(\alpha^2 + \beta^2 + \gamma^2)}$$

Spherical Symmetry (spherical coordinate)

$$\rightarrow \nabla^2\varphi = \cancel{\frac{1}{r^2}} \underbrace{\partial_r\left(r^2\partial_r\varphi\right)}_{=l(l+1)} + \underbrace{\frac{1}{\cancel{r^2}\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\varphi\right) + \frac{1}{\cancel{r^2}\sin^2\theta}\partial_\phi^2\varphi}_{=\ -l(l+1)} = 0$$

use Trial solution: $\varphi(r,\theta,\phi) = R(r)Y(\theta,\phi)$

i) $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$

we get $\boxed{R_l(r) = A_l r^l + B_l r^{-(l+1)}}$

ii) $-\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta Y\right) - \frac{1}{\sin^2\theta}\partial_\phi^2 Y = l(l+1)Y$

$\hookrightarrow = L^2 Y_l^m(\theta,\phi) = l(l+1)Y_l^m(\theta,\phi)$

↑ spherical Harmonics.

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

Spherical Harmonic properties:

i) $-l \le m \le l$, $0 < l < \infty$, m, l both integers.

ii) $\int d\Omega\, Y_{l'}^{m'*} Y_l^m = \delta_{m'm}\delta_{l'l}$

iii) $Y_l^{-m}(\theta,\phi) = (-1)^m Y_l^{m*}(\theta,\phi)$

spherical

$$\boxed{\therefore\ \varphi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_l^m r^l + B_l^m r^{-(l+1)}\right]Y_l^m(\theta,\phi)}$$

Azimuthal Symmetry: Independent of angle ϕ.

→ special case of spherical coordinate.

$$\varphi(r,\theta,\phi) \to \varphi(r,\theta)$$

→ $\nabla^2\varphi = \frac{1}{r^2}\partial_r(r^2\partial_r\varphi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\varphi) = 0$

let $\cos\theta = x$, then $\varphi(r,x) = R(r)M(x)$

i) $\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) = \nu(\nu+1)R$

↳ $R_\nu(r) = A_\nu r^\nu + B_\nu r^{-(\nu+1)}$

Here ν can be any #, not necessarily integers.

ii) $\frac{d}{dx}\left((1-x^2)\frac{dM}{dx}\right) = -\nu(\nu+1)M$

Legendre's ODE:

$$(x^2-1)\frac{d^2}{dx^2}M + 2x\frac{d}{dx}M - \nu(\nu+1)M = 0$$

sol: $M(\theta) = C_\nu P_\nu(\cos\theta) + D_\nu Q_\nu(\cos\theta)$

P_ν : Legendre function of first kind. If ν = integer
$P_\nu = P_l$ = Legendre Polynomial

Q_ν: Legendre function of second kind.

property: i) $P_\nu(-1) = \infty$ if ν is not positive integer.
ii) $Q_\nu(\pm1) = \infty$
iii) if $\nu = l = 0, 1, 2 \dots$
then $P_l(x)$ is well behaved in $0 \le \theta \le \pi$
⟹ Becomes Legendre Polynomial.

So : If problem includes:

i) $0 < \theta < \pi$: want both P_ν and Q_ν

ii) $0 < \theta \leq \pi$: want P_ν

iii) $0 \leq \theta \leq \pi$: want $\nu = l = 0, 1, 2 \cdots$ and only $P_l(\cos\theta)$
$Q_l(\cos\theta)$ still singular at $\theta = \pi$.

Azimuthal Symmetry:

$$\boxed{\therefore\ \varphi(r,\theta) = \sum_\nu \left(A_\nu r^\nu + B_\nu r^{-(\nu+1)}\right)\left(C_\nu P_\nu(\cos\theta) + D_\nu Q_\nu(\cos\theta)\right)}$$

Cylindrical Symmetry / cylindrical coordinate:

$$\nabla^2 \varphi = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho \varphi) + \frac{1}{\rho^2}\partial_\phi^2 \varphi + \partial_z^2 \varphi = 0$$

Trial solution: $\varphi(\rho,\phi,z) = R(\rho)G(\phi)Z(z)$

$\hookrightarrow$ i) $\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (k^2\rho^2 - \alpha^2)R = 0$ ← Bessel's ODE

ii) $\frac{d^2G}{d\phi^2} + \alpha^2 G = 0$

iii) $\frac{d^2}{dz^2}Z - k^2 Z = 0$

If both α^2 and k^2 are positive:

$$G_\alpha(\phi) = \begin{cases} X_0 + Y_0\phi & \alpha = 0 \\ X_\alpha \exp(i\alpha\phi) + Y_\alpha \exp(-i\alpha\phi) & \alpha \neq 0 \end{cases}$$

$$Z_k(z) = \begin{cases} S_0 + t_0 z & k = 0 \\ S_k \exp(kz) + t_k \exp(-kz) & k \neq 0 \end{cases}$$

$$R_\alpha^k(\rho) = \begin{cases} A_0 + B_0 \ln\rho & k=0,\ \alpha=0 \ \text{(purely in } \rho) \\ A_\alpha \rho^\alpha + B_\alpha \rho^{-\alpha} & k=0,\ \alpha\neq 0 \ \text{(No } z\text{, polar)} \\ A_\alpha^k J_\alpha(k\rho) + B_\alpha^k N_\alpha(k\rho) & k^2 > 0 \ \text{(exp in } z) \\ A_\alpha^k I_\alpha(\kappa\rho) + B_\alpha^k K_\alpha(\kappa\rho) & k^2 < 0 \ \text{(oscillation in } z) \end{cases}$$

$\kappa = -ik$

→ Define Bessel Functions to have positive α and for $x \geq 0$

→ $J_\alpha(x)$ and $N_\alpha(x)$: Bessel functions of first and second kind

i) $J_\alpha(x)$ is regular everywhere, i.e. oscillatory

ii) $N_\alpha(x)$ diverges as $x \to 0$

→ $I_\alpha(x)$ and $K_\alpha(x)$: Modified Bessel Functions of first and second kind

i) $I_\alpha(x) = J_\alpha(-ix)$ is finite at the origin, but diverges exponentially as $x \to \infty$

ii) $K_\alpha(x) = N_\alpha(-ix)$ diverges as $x \to 0$, but goes 0 exponentially as $x \to \infty$

$$\boxed{\therefore \quad \varphi(\rho, \phi, z) = \sum_\alpha \sum_k R_\alpha^k(\rho)\, G(\phi)\, Z(z)}$$

Polar Coordinate: 2d:

$$\nabla^2\varphi = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho)\varphi + \frac{1}{\rho^2}\partial_\phi^2\varphi = 0$$

, i.e. $k=0$ case in cylindrical.

$$\varphi = \sum R(\rho)\,G(\phi)$$

$$= \underbrace{(A_0 + B_0\ln\rho)(x_0 + y_0\phi)}_{\alpha=0,\ k=0} + \sum_{\alpha\neq 0}\underbrace{(A_\alpha\rho^\alpha + B_\alpha\rho^{-\alpha})(x_\alpha\sin\alpha\rho + y_\alpha\cos\alpha\phi)}_{k=0,\ \alpha\neq 0}$$