Zhi Chen HW#9

1) Show $\vec{\mu} = \int d^3r \, \vec{M}$

Know: $\vec{\mu} = \int d^3r \, \frac{1}{2} \vec{r} \times \vec{J}_{mag} + \int d^2S \, \frac{1}{2} \vec{r} \times \vec{K}_{mag}$

use $\vec{J}_{mag} = \vec{\nabla} \times \vec{M}$ and $\vec{K}_{mag} = \vec{M} \times \hat{n}$

$$\vec{\mu} = \int d^3r \, \frac{1}{2} \vec{r} \times (\vec{\nabla} \times \vec{M}) + \int d^2S \, \frac{1}{2} \vec{r} \times (\vec{M} \times \hat{n})$$

$$\overset{!}{=} \int d^3r \, \frac{1}{2} \varepsilon_{abc} r_b \varepsilon_{cde} \partial_d M_e$$

$$+ \int d^2S \, \frac{1}{2} \varepsilon_{abc} r_b \varepsilon_{cde} M_d \hat{n}_e$$

since $\varepsilon_{abc} \varepsilon_{cde} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}$

$$\overset{!}{=} \int d^3r \, \frac{1}{2} (\delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}) r_b \partial_d M_e$$

$$+ \int d^2S \, \frac{1}{2} (\delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}) r_b M_d \hat{n}_e$$

$$\overset{!}{=} \int d^3r \, \frac{1}{2} (r_b \partial_a M_b - r_b \partial_b M_a)$$

$$+ \int d^2S \, \frac{1}{2} (r_b M_a \hat{n}_b - r_b M_b \hat{n}_a)$$

$$\overset{!}{=} \int d^3r \, \frac{1}{2} \Big[\underset{\substack{\uparrow \\ \text{apply divergence} \\ \text{theorem}}}{\partial_a (r_b M_b)} - \underbrace{M_b \underbrace{\partial_a r_b}_{\delta_{ab}}}_{M_a} - \underset{\substack{\uparrow \\ \text{apply divergence} \\ \text{theorem}}}{\partial_b (r_b M_a)} + M_a \underbrace{\partial_b r_b}_{3} \Big]$$

$$+ \int d^2S \, \frac{1}{2} (r_b M_a \hat{n}_b - r_b M_b \hat{n}_a)$$

$$\overset{!}{=} \int d^3r\, M_a + \frac{1}{2}\int d^2S\, \hat{n}_a \cancel{r_b} M_b + \frac{1}{2}\int d^2S\, \hat{n}_b \cancel{r_b} M_a$$

$$+\int d^2S\, \frac{1}{2}\left(r_b \cancel{M_a} \hat{n}_b - r_b \cancel{M_b} \hat{n}_a\right)$$

$$\overset{!}{=} \int d^3r\, M_a$$

$$\boxed{\vec{u} \overset{!}{=} \int d^3r\, \vec{M}}$$

2) Show linear momentum is conserved.

a) $$\frac{d}{dt}\int d^3r\, \Pi_a + \int d^3r\, f_a = \int d^2S\, \hat{n}_b\, T_{ab}.$$

$$\hookrightarrow \frac{d}{dt}\int d^3r\, \varepsilon_0 \vec{E}\times\vec{B} = \int d^3r\, \varepsilon_0 \left(\frac{d}{dt}\vec{E}\times\vec{B}\right) + \left(\vec{E}\times\frac{d\vec{B}}{dt}\right)$$

since $\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \Rightarrow \frac{d\vec{B}}{dt} = -\vec{\nabla}\times\vec{E}$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\partial_t\vec{E} \Rightarrow \partial_t\vec{E} = c^2(-\mu_0\vec{J} + \vec{\nabla}\times\vec{B})$

$$\hookrightarrow = \int d^3r\, \varepsilon_0 \left[c^2(\vec{\nabla}\times\vec{B} - \mu_0\vec{J})\times\vec{B} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]$$

$$\hookrightarrow = \int d^3r \left\{\underbrace{-\varepsilon_0\mu_0 c^2}_{=-1}\vec{J}\times\vec{B} + \underbrace{\varepsilon_0 c^2}_{\frac{1}{\mu_0}}(\vec{\nabla}\times\vec{B})\times\vec{B} - \varepsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})\right\}$$

$$\hookrightarrow = \int d^3r\left(-\vec{J}\times\vec{B} - \frac{1}{\mu_0}\varepsilon_{abc}B_b\varepsilon_{cde}\partial_d B_e - \varepsilon_0\varepsilon_{ijk}E_j\varepsilon_{kmn}\partial_m E_n\right)$$

$$\stackrel{!}{=}\int d^3r\left\{-\vec{J}\times\vec{B} - \frac{1}{\mu_0}(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})B_b\partial_d B_e - \varepsilon_0(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})E_j\partial_m E_n\right\}$$

$$\stackrel{!}{=}\int d^3r\left\{-\vec{J}\times\vec{B} - \frac{1}{\mu_0}[B_b\partial_a B_b - B_b\partial_b B_a] - \varepsilon_0(E_j\partial_i E_j - E_j\partial_j E_i)\right\}$$

$$\stackrel{!}{=}\int d^3r\left\{-\vec{J}\times\vec{B} - \frac{1}{\mu_0}\left[\frac{1}{2}\partial_a(B_b^{\,2}) - \partial_b(B_bB_a) + B_a\partial_b B_b\right]\right.$$
$$\left. - \varepsilon_0\left[\frac{1}{2}\partial_i(E_j^{\,2}) - \partial_j(E_iE_j) + E_i\partial_j E_j\right]\right\}$$

use divergence thm

$$\hookrightarrow = \int d^3r - \left\{\vec{J}\times\vec{B} + \cancel{\frac{1}{\mu_0}B_a\underbrace{\partial_b B_b}_{\vec{\nabla}\cdot\vec{B}=0}} + \cancel{\varepsilon_0}E_i\underbrace{\partial_j E_j}_{\vec{\nabla}\cdot\vec{E} = \rho/\cancel{\varepsilon_0}}\right\}$$

$$+\int d^2S\left\{\hat{n}_a\underbrace{\left(-\frac{B_b^2}{2\mu_0}\right)}_{-\frac{1}{2\mu_0}|\vec{B}|^2} + \hat{n}_b\underbrace{\frac{1}{\mu_0}B_bB_a}_{\frac{1}{\mu_0}B_aB_b} - \hat{n}_i\underbrace{\frac{\varepsilon_0 E_j^2}{2}}_{-\frac{1}{2}\varepsilon_0|\vec{E}|^2} + \hat{n}_j\underbrace{\varepsilon_0 E_iE_j}_{\varepsilon_0 E_aE_b}\right\}$$

$$\hookrightarrow = \underbrace{\int d^3r \; -(\vec{J}\times\vec{B} + \rho\vec{E})}_{-f(\vec{r},t)}$$

$$+ \int d^2\vec{S} \; \underbrace{\varepsilon_0 E_a E_b + \frac{1}{\mu_0} B_a B_b - \frac{1}{2}\delta_{ab}\left(\frac{1}{\mu_0}|\vec{B}|^2 + \varepsilon_0|\vec{E}|^2\right)}_{T_{ab}}$$

rearrange:

$$\boxed{\frac{d}{dt}\int d^3r \; \varepsilon_0 \vec{E}\times\vec{B} + \int d^3r \, \underbrace{(\vec{J}\times\vec{B} + \rho\vec{E})}_{= f(\vec{r},t)} = \int d^2\vec{S} \; \underbrace{\varepsilon_0 E_a E_b + \frac{1}{\mu_0} B_a B_b - \frac{1}{2}\delta_{ab}\left(\frac{1}{\mu_0}|\vec{B}|^2 + \varepsilon_0|\vec{E}|^2\right)}_{T_{ab}}}$$

4) Quasistatic electrodynamics of conductors:

assume field vary slowly, i.e. $\omega \ll c\ell$, where ω is the timescale for field variations, and ℓ is the size scale of the conductor.

Suppose conductor has u and σ

Quasistatic obey: $\vec{\nabla}\cdot\vec{B}=0$ $\quad \vec{\nabla}\times\vec{H}=\vec{J}+\cancel{\frac{\partial\vec{D}}{\partial t}}$ $\quad \vec{\nabla}\times\vec{E}=-\partial_t\vec{B}$

$\vec{J}=\sigma\vec{E}$ $\quad \vec{H}=\frac{1}{u}\vec{B}$

a) Show $\vec{H}$ obey diffusion equation. Find diffusion constant.

$\hookrightarrow \vec{\nabla}\times\vec{H}=\vec{J}=\sigma\vec{E}$

then
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=\sigma\,\vec{\nabla}\times\vec{E}$$
$$=-\sigma\,\partial_t\vec{B}$$
$$=-\sigma u\,\partial_t\vec{H}$$

Use identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{H}})-\nabla^2 H$ in Cartesian Coord

Since $\vec{\nabla}\cdot\vec{H}=\frac{1}{u}\vec{\nabla}\cdot\vec{B}=0$

After rearranging : $\boxed{\frac{1}{\sigma u}\nabla^2\vec{H}=\partial_t\vec{H}}$

By comparing with diffusion equation $D\nabla^2\vec{T}=\partial_t\vec{T}$

we recognize diffusion const $\boxed{D=\frac{1}{\sigma u}}$

b) assume $\vec{H}(\vec{r},t) = \vec{h}(\vec{r}) \exp(-t/\tau)$

then a general sol: $\vec{H}(\vec{r},t) = \sum_n A^{(n)} \vec{h}^{(n)}(\vec{r}) \exp(-t/\tau^{(n)})$
and $A^{(n)}$ determines initial condition.

$$\frac{1}{\sigma\mu} \nabla^2 \vec{H} = \partial_t \vec{H}$$

$$\hookrightarrow \frac{1}{\sigma\mu} \sum_n A^{(n)} \left(\nabla^2 \vec{h}_{(n)}(\vec{r})\right) \exp(-t/\tau^{(n)}) = \sum_n -\frac{1}{\tau^{(n)}} A^{(n)} \vec{h}^{(n)}(\vec{r}) \exp(-t/\tau^{(n)})$$

with channel matching, for specific n:

$$\frac{1}{\sigma\mu} \nabla^2 \vec{h}_{(n)}(\vec{r}) = -\frac{1}{\tau^{(n)}} \vec{h}_n(\vec{r})$$

since $\nabla^2 \vec{h}_{(n)}(\vec{r})$ has two inverse length scale, and the length scale we have in the problem is, l, size of conductor, then expect

$$\frac{1}{\sigma\mu} \nabla^2 \vec{h}_{(n)}(\vec{r}) \sim \frac{1}{\sigma\mu} \frac{1}{l^2} \vec{h}_n(\vec{r}) \sim -\frac{1}{\tau^{(n)}} \vec{h}_n(\vec{r})$$

so $\boxed{\tau^{(0)} \sim \sigma\mu l^2}$

then plugging #'s for planetary molten core:

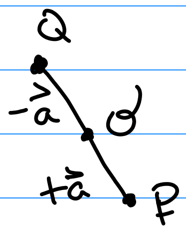
$$\tau \sim (1.5 \times 10^6\, m^{-1}\Omega^{-1})(4\pi \times 10^{-7}\, kg\, m\, s^{-2} A^{-2})(3{,}500 \times 10^3\, m)^2$$

$$\sim 2.3 \times 10^{13}\, s$$

or

$$\tau \sim 732{,}000 \text{ years}$$

5) Angular Momentum and Magnetic Monopoles, and electric charge quantization:



$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\vec{r}+\vec{a}|^3}(\vec{r}+\vec{a})$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{P}{|\vec{r}-\vec{a}|^3}(\vec{r}-\vec{a})$$

$$\vec{L} = \frac{1}{c}\int d^3r \; \vec{r} \times \left(\sqrt{\varepsilon_0}\,\vec{E} \times \frac{1}{\sqrt{\mu_0}}\vec{B}\right)$$

a) show $\vec{L} = \frac{1}{c}\frac{QP}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\hat{a}$ using $\int d^3r \frac{\vec{r}\times[(\vec{r}+\vec{a})\times(\vec{r}-\vec{a})]}{|\vec{r}+\vec{a}|^3\,|\vec{r}-\vec{a}|^3} = 4\pi\hat{a}$

$$\vec{L} = \frac{1}{c}\int d^3r \; \vec{r} \times \left(\sqrt{\varepsilon_0}\,\frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}+\vec{a}}{|\vec{r}+\vec{a}|^3} \times \frac{1}{\sqrt{\mu_0}}\frac{\mu_0}{4\pi}P\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3}\right)$$

$$= \frac{1}{c}\frac{Q}{4\pi\sqrt{\varepsilon_0}}\frac{\sqrt{\mu_0}\,P}{\cancel{4\pi}}\underbrace{\int d^3r \frac{\vec{r}\times[(\vec{r}+\vec{a})\times(\vec{r}-\vec{a})]}{|\vec{r}+\vec{a}|^3\,|\vec{r}-\vec{a}|^3}}_{\cancel{4\pi}\,\hat{a}}$$

$$\boxed{\vec{L} = \frac{1}{c}\frac{QP}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\hat{a}}$$

b) If $\vec{L}\cdot\hat{a} = \frac{n\hbar}{2} = \frac{1}{c}\frac{QP}{4\pi}\sqrt{\frac{\mu}{\varepsilon_0}}$

then $Q = n\,2\pi\hbar\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{c}{P}$

If n is an integer then we see that Q must be a multiple of $2\pi\hbar\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{c}{P}$