Maxwell's Equations in Differential Form:

	Inhomogeneous / Source	Homogeneous / No source
Constraint	$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$ Gauss's Law	$\vec{\nabla} \cdot \vec{B} = 0$ No Magnetic Monopole.
Dynamics	Ampere's Law. $\vec{\nabla} \times \vec{B} = \underbrace{\frac{1}{c^2} \frac{d}{dt} E}_{\text{Maxwell Displacement current}} + \mu_0 \underbrace{j}_{\text{charge current densit}}$	Faraday's Law $\vec{\nabla} \times \vec{E} = - \frac{d}{dt} B$

Current Conservation:

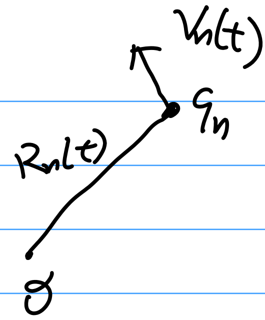
$$\frac{\partial}{\partial t}(\vec{\nabla} \cdot E) = \frac{1}{\varepsilon_0} \partial_t \rho$$

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{=0} = \frac{1}{c^2} \vec{\nabla} \cdot (\partial_t \vec{E}) + \mu_0 \vec{\nabla} \cdot \vec{j}$$

$$= \frac{1}{c^2} \frac{1}{\varepsilon_0} \partial_t \rho + \mu_0 \vec{\nabla} \cdot \vec{j}$$

$$\hookrightarrow \quad \frac{\partial}{\partial t} \rho = - \underbrace{\varepsilon_0 \mu_0 c^2}_{1} \vec{\nabla} \cdot j$$

$$\boxed{\text{Continuity Equation:} \quad \frac{\partial}{\partial t} \rho = - \vec{\nabla} \cdot j}$$



<u>For N-point particles</u> :

$$\rho(\vec{r},t) = \sum_{n=1}^{N} q_n \,\delta(\vec{r} - \vec{R}_n(t))$$

$$j(\vec{r},t) = \sum_{n=1}^{N} q_n V_n(t)\, \delta(\vec{r} - \vec{R}_n(t))$$

$$\hookrightarrow \quad V_n(t) = \dot{R}_n(t)$$

with $\int d^3r \,\delta(\vec{r} - \vec{R}_n(t)) = 1.$ $\leftarrow$ Dirac-Delta.

<u>Side Note</u>: <u>Einstein Summation</u>:

$$\rightarrow \boxed{\vec{\nabla} = e_u \partial_u}$$

$$\vec{\nabla}\cdot\vec{f} = (\vec{e}_u \partial_u)\cdot(f_v \vec{e}_v) \overset{!}{=} \vec{e}_u\cdot\vec{e}_v\, \partial_u f_v \overset{!}{=} \partial_u f_u$$

$$\vec{\nabla}\cdot(f\vec{g}) = e_u\partial_u(f\, g_v e_v) \overset{!}{=} (e_u\cdot e_v)\partial_u(f g_v) \overset{!}{=} \delta_{uv}\,\partial_u(f g_v) \overset{!}{=} (\partial_v f) g_v + f(\partial_v g_v) \overset{!}{=} \vec{\nabla} f\cdot\vec{g} + f(\vec{\nabla}\cdot g)$$

$$\boxed{\vec{e}_u \times \vec{e}_v = \epsilon_{uvp}\, \vec{e}_p}$$

$$\vec{A}\times\vec{B} = (A_u \vec{e}_u)\times(B_v \vec{e}_v) \overset{!}{=} (\vec{e}_u\times\vec{e}_v) A_u B_v \overset{!}{=} \epsilon_{uvp}\, \vec{e}_p A_u B_v$$

Dirac - Delta Recall:

$$\int d^3r \; \delta(\vec{r}-\vec{A})\, f(\vec{r}) = f(\vec{A})$$

$\Rightarrow$ Checking whether point - source particles described by Dirac - Delta Matches with Continuity Eq:

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$$

$$\frac{\partial}{\partial t}\rho = \sum_{n=1}^{N} q_n \frac{\partial}{\partial t}\delta(\vec{r}-\vec{R}_n(t))$$

$$\stackrel{!}{=} \sum_{n=1}^{N} q_n \frac{\partial}{\partial \vec{r}}\delta(\vec{r}-\vec{R}_n(t))\, \frac{\partial}{\partial t}(\vec{r}-\vec{R}_n(t))$$

$$\frac{\partial}{\partial t}\rho \stackrel{!}{=} \sum_{n=1}^{N} q_n \left(-\dot{\vec{R}}_n(t)\right) \vec{\nabla}_r\, \delta(\vec{r}-\vec{R}_n(t))$$

And $$\vec{\nabla}\cdot\vec{j} = \sum_{n=1}^{N} q_n \vec{\nabla}\cdot\left(\vec{v}_n(t)\,\delta(\vec{r}-\vec{R}_n(t))\right)$$

$$\stackrel{!}{=} \sum_{n=1}^{N} q_n\, e_\mu \partial_\mu \cdot V_{n,\nu}\, e_\nu\, \delta(\vec{r}-\vec{R}_n(t))$$

$$\stackrel{!}{=} \sum_{n=1}^{N} q_n\, \delta_{\mu\nu}\, \partial_\mu\left(V_{n,\nu}\,\delta(\vec{r}-\vec{R}_n(t))\right)$$

$$\stackrel{!}{=} \sum_{n=1}^{N} q_n\, \partial_\mu\left(V_{n,\mu}\,\delta(\vec{r}-\vec{R}_n(t))\right)$$

$$\stackrel{!}{=} \sum_{n=1}^{N} q_n \left[\underbrace{\partial_\mu(V_{n,\mu})}_{\frac{\partial^2 x}{\partial x \partial t}=0}\,\delta(\vec{r}-\vec{R}_n(t)) + V_{n,\mu}\,\partial_\mu\,\delta(\vec{r}-\vec{R}_n(t))\right]$$

$$\vec{\nabla}\cdot\vec{j} \stackrel{!}{=} \sum_{n=1}^{N} q_n\, \vec{V}_n\cdot\vec{\nabla}_r\,\delta(\vec{r}-\vec{R}_n(t))$$

$$\boxed{\therefore \quad \frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0}$$

Force Laws:

Coulomb Law — Lorentz Law

$$F(t) = \begin{cases} \text{Discrete:} & q\vec{E}(\vec{r},t) + q\dot{\vec{r}} \times \vec{B}(\vec{r},t) \\ \text{Continuous:} & \int d^3r \left[\rho^*(\vec{r},t)\vec{E}(\vec{r},t) + \vec{j}^*(\vec{r},t) \times \vec{B}(\vec{r},t) \right] \end{cases}$$

ρ^*: charge density — $\vec{j}^*$: current density.

E- and B- fields in scalar and vector potentials.

ϕ : scalar potential

A : vector potential

Then :

$$\boxed{\begin{aligned} E &= -\vec{\nabla}\phi - \partial_t \vec{A} \\ B &= \vec{\nabla} \times \vec{A} \end{aligned}}$$

Constants:

Fine Structure Constant:

Coupling of EM and matter

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137} = 2\pi \frac{e^2/4\pi\varepsilon_0 d}{hc/d}$$

$e^2/4\pi\varepsilon_0 d$ ← Coulomb energy

hc/d ← photon of wavelength d.

Compton Wavelength : $\lambda_c = \frac{h}{mc}$

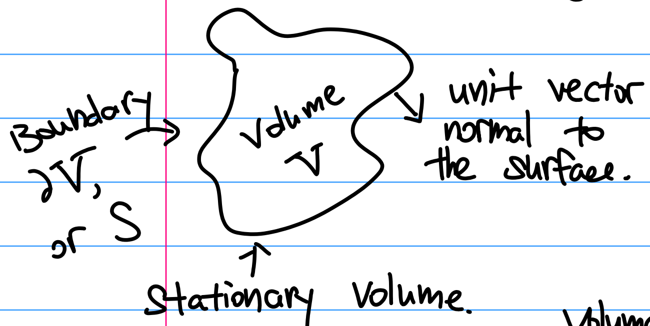
$p = \frac{h}{\lambda}$

energy $\approx \left(\frac{h}{\lambda}\right)^2 \frac{1}{2m}$, but as you increase energy, → smaller λ to resolve particle, you can cross over mc^2, creating particles.

$$\left.\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\ N/A^2 \\ \varepsilon_0 &\cong 8.8 \times 10^{-12} \frac{A^2 s^4}{kg\, m^3} \end{aligned}\right\} \quad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Two Important Theorem:

1) Gauss's (Divergence) Theorem:

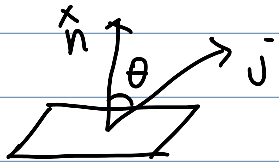


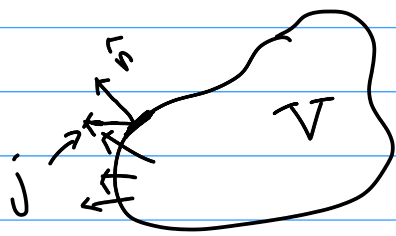
then:

$$\underbrace{\int_V d^3r}_{\text{Volume Integral}} \; \underbrace{\vec{\nabla}\cdot\vec{C}}_{\text{Divergence of } \vec{C}} = \underbrace{\int_S d^2r}_{\text{Surface Integral}} \; \underbrace{\hat{n}\cdot\vec{C}}_{\text{Component of } \vec{C} \text{ along } \hat{n}}$$

Application: obtaining the integral form of Continuity Equation from Differential form.

Take $\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$





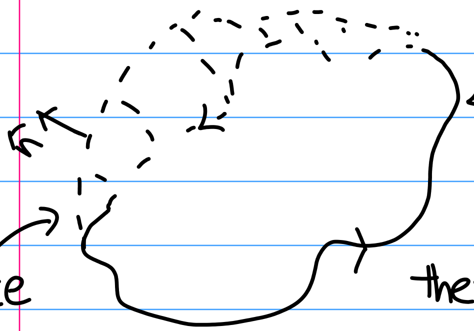
current density vector field.

$$\int d^3r \frac{\partial}{\partial t}\rho + \int d^3r\, \vec{\nabla}\cdot\vec{j} = 0$$

$$\hookrightarrow \underbrace{\frac{\partial}{\partial t} \underbrace{\int d^3r\, \rho}_{\text{total charge } Q}}_{\text{overall change of total charge within } V} + \underbrace{\int d^2r\, \hat{n}\cdot\vec{j}}_{\text{current through each surface element, summed over entire surface.}} = 0$$

$\therefore$ Rate of charge loss = Total Flux out.

2) Stokes' Theorem:



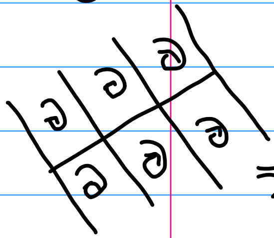
loop L, the boundary of a surface S

Surface S

then $\int_S d^2\vec{r}\; \vec{\nabla}\times\vec{C} = \oint_L d\vec{r}\cdot\vec{C}$

open surface integral

closed loop line integral



$\Rightarrow$ If choose opposite surface, which completes the entire surface, then $\int_{S_1} dS_1\cdot(\vec{\nabla}\times\vec{C}) + \int_{S_2} dS_2\cdot(\vec{\nabla}\times\vec{C}) = \int d^3r\, \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{C})}_{=0} = 0$

Gauss Theorem

Maxwell Equations in Integral Form:

Constraint:

Flux through surface

Total charge inside V

$$\int_S d^2S\; \hat{n}\cdot\vec{E} = \frac{1}{\varepsilon_0}\int_V d^3r\; \rho$$

Gauss

Total Flux Through Surface

$$\int_S d^2S\; \hat{n}\cdot\vec{B} = 0$$

No Magnetic Monopole.

Dynamics:

$$\oint_L d\vec{r}\cdot\vec{B} = \frac{1}{c^2}\frac{d}{dt}\int_S d^2S\;\hat{n}\cdot\vec{E} + \mu_0 \underbrace{\int_S d^2S\;\hat{n}\cdot\vec{j}}$$

Maxwell Displacement current

Electric flux through S

Ampere

Electric current through S

Faraday's Law.

$$\oint_L d\vec{r}\cdot\vec{E} = -\frac{d}{dt}\int_S d^2S\;\hat{n}\cdot\vec{B} = -\frac{d}{dt}\Phi$$

Flux

Electromotive Force around L

Rate of change of Magnetic flux through S.

Inhomogeneous / Source

Homogeneous / No source

Action Principle for Electrodynamics:

Action: $S[\phi, A] = \int dt\, L$

Lagrangian $L = \int d^3r\, \mathcal{L}$ ← Lagrangian Density.

For Electrodynamics:

$$\boxed{\begin{aligned}\mathcal{L} &= \frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2\mu_0} B^2 + [\vec{j}\cdot\vec{A} - \rho\phi] \\ &= \underbrace{\frac{1}{2}\varepsilon_0 \left|-\vec{\nabla}\phi - \partial_t \vec{A}\right|^2 - \frac{1}{2\mu_0}\left|\vec{\nabla}\times\vec{A}\right|^2}_{\text{pure electrodynamic.}} + \underbrace{[\vec{j}\cdot\vec{A} - \rho\phi]}_{\text{Matter / Electrodynamic coupling}}\end{aligned}}$$

we left out pure matter term.

Gauge Invariance of $\vec{E}$ and $\vec{B}$:

Suppose we shift ϕ and $\vec{A}$ via a single function $\chi(\vec{r},t)$.

$$\boxed{\begin{array}{ll} \phi \to \phi - \partial_t \chi & \text{(Time)} \\ \vec{A} \to \vec{A} + \vec{\nabla}\chi & \text{(Space)} \end{array}}$$

$\Rightarrow$ Then

$$E = -\vec{\nabla}\phi - \partial_t A$$
$$\overset{!}{=} -\vec{\nabla}(\phi - \partial_t \chi) - \partial_t(\vec{A} + \vec{\nabla}\chi)$$
$$\overset{!}{=} -\vec{\nabla}\phi - \partial_t \vec{A} + \partial_t \vec{\nabla}\chi - \partial_t \vec{\nabla}\chi$$
$$\vec{E} \overset{!}{=} -\vec{\nabla}\phi - \partial_t A \quad \leftarrow \text{unchanged.}$$

$\Rightarrow$

$$B = \vec{\nabla} \times \vec{A}$$
$$\overset{!}{=} \vec{\nabla} \times (\vec{A} + \vec{\nabla}\chi)$$
$$\overset{!}{=} \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times (\vec{\nabla}\chi)}$$

$$\vec{e}_i \partial_i \times (\vec{e}_j \partial_j \chi) = \varepsilon_{ijk} \vec{e}_k \partial_i \partial_j \chi$$
$$\overset{!}{=} \frac{1}{2}(\varepsilon_{ijk} \hat{e}_k \partial_i \partial_j \chi + \varepsilon_{jik} \hat{e}_k \partial_j \partial_i \chi)$$
$$\overset{!}{=} \frac{1}{2} \varepsilon_{ijk} \hat{e}_k (\partial_i \partial_j \chi - \partial_j \partial_i \chi)$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Examine the change in $\mathcal{L}$ due to shifts in Φ, A

$$\mathcal{L} = \frac{1}{2}\varepsilon_0 \left| -\vec{\nabla}\phi - \partial_t \vec{A} \right|^2 - \frac{1}{2\mu_0}\left|\vec{\nabla}\times\vec{A}\right|^2 + [\vec{A}\cdot\vec{j} - \rho\phi]$$

$$\overset{!}{=} \frac{1}{2}\varepsilon_0 \Big|\underbrace{-\vec{\nabla}(\phi - \partial_t \chi) - \partial_t(A + \vec{\nabla}\chi)}\Big|^2 - \frac{1}{2\mu_0}\Big|\underbrace{\vec{\nabla}\times(A+\vec{\nabla}\chi)}\Big|^2$$

$$+ [(\vec{A}+\vec{\nabla}\chi)\cdot\vec{j} - \rho(\phi - \partial_t\chi)]$$

we know E and B fields don't change under shift

$$\overset{!}{=} \frac{1}{2}\varepsilon_0 \left|-\vec{\nabla}\phi - \partial_t\vec{A}\right|^2 - \frac{1}{2\mu_0}\left|\vec{\nabla}\times\vec{A}\right|^2 + [\vec{A}\cdot\vec{j} - \rho\phi]$$

$$+ \underbrace{\left(\vec{j}\cdot\vec{\nabla}\chi + \rho\,\partial_t\chi\right)}$$

$$\left[\vec{\nabla}(\vec{j}\chi) - (\vec{\nabla}\cdot\vec{j})\chi\right] + \left[\partial_t(\rho\chi) - \chi\,\partial_t\rho\right]$$

$$\hookrightarrow \underbrace{\left[\vec{\nabla}(\vec{j}\chi) + \partial_t(\rho\chi)\right]}_{=0} - \chi\big[\underbrace{\partial_t\rho + \vec{\nabla}\cdot\vec{j}}_{=0}\big]$$

$=0$ by continuity Equation.

i.e. surface terms full derivative.

$=0$, since ρ, j vanish far away in space and time, these do not contribute in the action.

Q: why can't we use this argument for this as well?

$\therefore$ Action S, is also invariant with (charge conservation) continuity equation

Going Back to Action Principle:

If we make slight variation: $\phi \to \phi + \delta\phi$
$A \to A + \delta A$

Assume variation vanishes in the distant past and future, and far away in space.

$$S = \int dt \int d^3r \, \mathcal{L}$$

$$S(\phi, A) = \frac{1}{2}\varepsilon_0 \int dt \int d^3r \left[\left| -\vec{\nabla}\phi - \partial_t \vec{A} \right|^2 - c^2 \left| \vec{\nabla} \times \vec{A} \right|^2 \right]$$
$$+ \int dt \int d^3r \left[\vec{A} \cdot \vec{j} - \rho\phi \right]$$

$$\Rightarrow S(\phi + \delta\phi, A + \delta A) = \frac{1}{2}\varepsilon_0 \int dt \int d^3r \left[\left| -\vec{\nabla}(\phi + \delta\phi) - \partial_t(\vec{A} + \delta\vec{A}) \right|^2 \right.$$
$$\left. - c^2 \left| \vec{\nabla} \times (\vec{A} + \delta\vec{A}) \right|^2 \right] + \int dt \int d^3r \left[(\vec{A} + \delta\vec{A}) \cdot \vec{j} - \rho(\phi + \delta\phi) \right]$$

$$= \frac{1}{2}\varepsilon_0 \int dt \int d^3r \left[\left| \vec{\nabla}\phi + \vec{\nabla}\delta\phi + \partial_t \vec{A} + \partial_t \delta\vec{A} \right|^2 \right.$$
$$\left. - c^2 \left| \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \delta\vec{A} \right|^2 \right] + \int dt \int d^3r \left(\left[\vec{A} \cdot \vec{j} - \rho\phi \right] + \delta\vec{A} \cdot \vec{j} - \rho\delta\phi \right)$$

$$= \frac{1}{2}\varepsilon_0 \int dt \int d^3r \left| \vec{\nabla}\phi + \partial_t \vec{A} \right|^2 + 2\vec{\nabla}\phi \cdot (\vec{\nabla}\delta\phi + \partial_t \delta\vec{A})$$
$$+ 2\partial_t \vec{A} \cdot (\vec{\nabla}\delta\phi + \partial_t \delta\vec{A}) - c^2 \left(\left| \vec{\nabla} \times \vec{A} \right|^2 + 2(\vec{\nabla} \times \vec{A}) \cdot (\vec{\nabla} \times \delta\vec{A}) \right)$$
$$+ \int dt \int d^3r \left(\left[\vec{A} \cdot \vec{j} - \rho\phi \right] + \delta\vec{A} \cdot \vec{j} - \rho\delta\phi \right) + \mathcal{O}(\delta^2)$$

Then

$$\delta S = S(\phi+\delta\phi\,,\,A+\delta A) - S(\phi, A)$$

$$= \varepsilon_0 \int dt \int d^3r \Big[\big(\vec{\nabla}\phi + \partial_t \vec{A}\big)\cdot\big(\vec{\nabla}\delta\phi + \partial_t \delta\vec{A}\big)$$

$$- c^2 (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\delta\vec{A})\Big]$$

$$+ \int dt \int d^3r \,(\delta\vec{A}\cdot\vec{j} - \rho\,\delta\phi) + \mathcal{O}(\delta^2)$$

$$= \varepsilon_0 \int dt \int d^3r \,[\delta\vec{A}\cdot\vec{j} - \rho\,\delta\phi]$$

$$+ (\vec{\nabla}\phi + \partial_t\vec{A})\cdot\vec{\nabla}\delta\phi \rightarrow \vec{\nabla}\cdot[\delta\phi(\vec{\nabla}\phi + \partial_t\vec{A})] - \delta\phi\,\vec{\nabla}\cdot(\vec{\nabla}\phi + \partial_t\vec{A})$$

$$+ (\vec{\nabla}\phi + \partial_t\vec{A})\cdot\partial_t\delta\vec{A} \rightarrow \partial_t[\delta\vec{A}\cdot(\vec{\nabla}\phi + \partial_t\vec{A})] - \delta\vec{A}\cdot\partial_t(\vec{\nabla}\phi + \partial_t\vec{A})$$

$$- c^2 (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\delta\vec{A})$$

Note: $\vec{\nabla}\cdot[\delta\vec{A}\times(\vec{\nabla}\times\vec{A})] = \partial_j[\varepsilon_{jkl}\,\delta A_k(\varepsilon_{lmn}\,\partial_m A_n)]$

$$\hookrightarrow = \underbrace{\varepsilon_{jkl}\,(\partial_j\,\delta A_k)}(\varepsilon_{lmn}\,\partial_m A_n) + \delta A_k(\varepsilon_{jkl}\,\partial_j\,(\underbrace{\varepsilon_{lmn}\,\partial_m A_n}_{(\,)_l}))$$

(−) sign since $\varepsilon_{jkl}\,\partial_j\,(\,)_l$

$$= (\vec{\nabla}\times\delta\vec{A})_l\,(\vec{\nabla}\times\vec{A})_l - \delta A_k\,(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))_k$$

$$= (\vec{\nabla}\times\delta\vec{A})\cdot(\vec{\nabla}\times\vec{A}) - \delta\vec{A}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))$$

hence: $(\vec{\nabla}\times\delta\vec{A})\cdot(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\cdot[\delta\vec{A}\times(\vec{\nabla}\times\vec{A})] + \delta\vec{A}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))$

Then collect terms:

$$\delta S = \varepsilon_0 \int dt \int d^3r \left\{ \frac{-1}{\varepsilon_0} \rho\, \delta\phi + \frac{1}{\varepsilon_0} \delta\vec{A}\cdot\vec{j} - \delta\phi\, \vec{\nabla}\cdot(\vec{\nabla}\phi + \partial_t \vec{A}) - \delta\vec{A}\cdot\partial_t(\vec{\nabla}\phi + \partial_t\vec{A}) - c^2\, \delta\vec{A}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) \right\}$$

Non vanishing terms

$$=0 \left\{ + \varepsilon_0 \int dt \int d^3r \left\{ \vec{\nabla}\cdot[\delta\phi(\vec{\nabla}\phi + \partial_t\vec{A}) + \partial_t[\delta\vec{A}\cdot(\vec{\nabla}\phi + \partial_t\vec{A})]] - c^2\, \vec{\nabla}\cdot(\delta\vec{A}\times(\vec{\nabla}\times\vec{A})) \right\} \right.$$

Boundary or Surface terms that vanishes.

Now consider $\delta S = 0$

then if we group by $\delta\phi$ and δA:

$\underline{\delta\phi \text{ Terms:}}$

$$\hookrightarrow \left(-\vec{\nabla}\cdot(\vec{\nabla}\phi + \partial_t\vec{A}) - \frac{\rho}{\varepsilon_0}\right)\delta\phi = 0$$

$$\hookrightarrow \vec{\nabla}\cdot(-\vec{\nabla}\phi - \partial_t\vec{A}) = \frac{\rho}{\varepsilon_0}$$

$$\hookrightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

$\underline{\delta A \text{ Terms:}}$

$$-\delta\vec{A}\cdot\partial_t(\vec{\nabla}\phi + \partial_t\vec{A}) - c^2\,\delta\vec{A}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) + \frac{1}{\varepsilon_0}\delta\vec{A}\cdot\vec{j}$$

$$\delta\vec{A}\cdot\left[-\partial_t(\vec{\nabla}\phi + \partial_t\vec{A}) - c^2(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) + \frac{1}{\varepsilon_0}\vec{j}\right] = 0$$

$$\hookrightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{1}{\varepsilon_0 c^2}\vec{j} + \frac{1}{c^2}\partial_t(-\vec{\nabla}\phi - \partial_t\vec{A})$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\partial_t\vec{E}}$$

$\vec{E}, \vec{B}$ are gauge invariant.

From the definition of $\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$
$\vec{B} = \vec{\nabla} \times \vec{A}$

we see that $\vec{\nabla} \times \vec{E} = \underbrace{\vec{\nabla} \times (-\vec{\nabla}\phi)}_{=0} - \vec{\nabla} \times (\partial_t \vec{A})$

$$\boxed{\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}}$$

And likewise: $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

And the result of Inhomogeneous Equations from action principle:

$$\boxed{\begin{aligned} &\vec{\nabla} \cdot \overbrace{(-\vec{\nabla}\phi - \partial_t \vec{A})}^{\vec{E}} = \frac{1}{\varepsilon_0} \rho \\ &\vec{\nabla} \times \underbrace{(\vec{\nabla} \times \vec{A})}_{\vec{B}} - \frac{1}{c^2} \partial_t \underbrace{(-\vec{\nabla}\phi - \partial_t \vec{A})}_{\vec{E}} = \mu_0 \vec{j} \end{aligned}}$$

Therefore, if $\phi, \vec{A}$ both satisfy the equation above then their gauge transformed counter-part ϕ', A', given by

$$\phi \to \phi' = \phi - \partial_t \chi$$

$$\vec{A} \to \vec{A}' = A + \vec{\nabla} \chi$$

also satisfy the equation.

Proof:

$$\vec{\nabla}\cdot\left(-\vec{\nabla}(\phi-\partial_t\chi)-\partial_t(\vec{A}+\vec{\nabla}\chi)\right)=\frac{1}{\varepsilon_0}\rho$$

$$\hookrightarrow \vec{\nabla}\cdot(-\vec{\nabla}\phi-\partial_t\vec{A})-\cancel{\partial_t\vec{\nabla}\chi}+\cancel{\partial_t\vec{\nabla}\chi}=\frac{1}{\varepsilon_0}\rho$$

$$\hookrightarrow \vec{\nabla}\cdot(-\vec{\nabla}\phi-\partial_t\vec{A})=\frac{1}{\varepsilon_0}\rho$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times(\vec{A}+\vec{\nabla}\chi))-\frac{\partial_t}{c^2}\left(-\vec{\nabla}(\phi-\partial_t\chi)-\partial_t(\vec{A}+\vec{\nabla}\chi)\right)=\mu_0 j$$

$$\vec{\nabla}\times\Big(\vec{\nabla}\times\vec{A}+\underbrace{\vec{\nabla}\times\vec{\nabla}\chi}_{0}\Big)-\partial_t\left(-\vec{\nabla}\phi-\partial_t\vec{A}+\cancel{\partial_t\vec{\nabla}\chi-\partial_t\vec{\nabla}\chi}\right)=\mu_0 j$$

$$\hookrightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A})-\partial_t(-\vec{\nabla}\phi-\partial_t\vec{A})=\mu_0 j$$

Theorems of vector calculus:

Theorem 1:

a function of position $\vec{F}(r)$ can be decomposed into:

$$\boxed{\vec{F} = -\vec{\nabla} U + \vec{\nabla} \times \vec{w}}$$

However, the choice of U and $\vec{w}$ are not unique.

Say we have $\vec{F} = yz\,\hat{x} + zx\,\hat{y} + xy\,\hat{z}$, we note that

$$\vec{\nabla} \cdot \vec{F} = 0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{F} = 0.$$

However $\vec{F}$ <u>does not</u> vanish at ∞. Here we state:

$$\boxed{\text{the only vector function, } \vec{F}, \text{ with zero divergence and curl which also vanishes at } \infty \text{ is zero everywhere.}}$$

<u>Helmholtz Theorem:</u>

Suppose we don't know $\vec{F}(\vec{r})$, but given:

i) $\vec{\nabla} \cdot \vec{F} = D$ $\qquad$ ii) $\vec{\nabla} \times \vec{F} = \vec{C}$

iii) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \vec{\nabla} \cdot \vec{C} = 0$

then we can uniquely determine $\vec{F}$ if $D(\vec{r})$ and $\vec{C}(\vec{r})$ vanishes fast enough as $r \to \infty$.

$$\underset{\text{curl-free component } \vec{C}^{\parallel}}{\underbrace{}}\qquad \underset{\text{div-free component } \vec{C}^{\perp}}{\underbrace{}}$$

where $\vec{F} = -\vec{\nabla}U + \vec{\nabla}\times\vec{\omega}$

curl-free component $\vec{C}^{\parallel}$; div-free component $\vec{C}^{\perp}$

and $$U = \frac{1}{4\pi}\int_{\substack{\text{all}\\\text{space}}} \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\tau' \quad \leftarrow \text{volume}$$

and $$\vec{\omega} = \frac{1}{4\pi}\int_{\substack{\text{all}\\\text{space}}} \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\tau' \quad \leftarrow \text{volume.}$$

given $\vec{\nabla}\cdot\vec{F} = D$ and $\vec{\nabla}\times\vec{F} = \vec{C}$

__Proof:__ Take divergence of $\vec{F}$

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot\left(-\vec{\nabla}U + \vec{\nabla}\times\vec{\omega}\right) = -\nabla^2 U$$

let $$U = \frac{1}{4\pi}\int \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\tau'$$

$$-\nabla^2 U = -\frac{1}{4\pi}\int D(\vec{r}\,')\, \underbrace{\nabla_r^2 \frac{1}{|\vec{r}-\vec{r}\,'|}}_{-4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')}\, d\tau'$$

$$= \int D(\vec{r}\,')\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\vec{\nabla}\cdot\vec{F} = D(\vec{r}) \quad \checkmark$$

Now take the curl:

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\underbrace{-\vec{\nabla} U}_{=0} + \vec{\nabla} \times \vec{w})$$

$$= \vec{\nabla} \times (\vec{\nabla} \times \vec{w})$$

$$= -\nabla^2 w + \vec{\nabla}(\vec{\nabla} \cdot \vec{w})$$

Calculate term by term:

$$-\nabla^2 w = -\frac{1}{4\pi} \int \vec{C}(\vec{r}') \underbrace{\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|}}_{-4\pi \delta^3(\vec{r} - \vec{r}')} d\tau'$$

$$= C(\vec{r})$$

We then need to prove $\vec{\nabla}(\vec{\nabla} \cdot \vec{w}) = 0$

Scalar & gradient.

$$\vec{\nabla} \cdot \vec{w} = \frac{1}{4\pi} \int \vec{C}(\vec{r}') \cdot \vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) d\tau'$$

and since $\vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{r} - \vec{r}'|}\right)$

$$= \frac{-1}{4\pi} \int \vec{C}(\vec{r}') \cdot \vec{\nabla}'\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) d\tau'$$

Integration by parts

$$= \frac{-1}{4\pi} \int \left\{ \underbrace{\vec{\nabla}' \cdot \left(\vec{C}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|}\right)}_{\text{Surface term}} - \frac{1}{|\vec{r} - \vec{r}'|} \underbrace{\vec{\nabla}' \cdot \vec{C}(\vec{r}')}_{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0} \right\} d\tau'$$

Surface term by Gauss's Theorem

$\hookrightarrow \oint \vec{C}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \cdot d\vec{a}'$

$= 0$ if $\vec{C}(\vec{r}') \to 0$ at ∞

$$\vec{\nabla} \cdot \vec{w} = 0$$

$(\vec{F})$
These solutions are generally not unique since we can always add any vector functions that make divergence and curl to be zero.

But if we add constraint that $\vec{F} \to 0$ as $\vec{r} \to \infty$ then $\vec{F}$ is uniquely defined.

proof: prove that the only vector function, $\vec{K}(\vec{r})$, with zero divergence and zero curl and vanishes at ∞ is when $\vec{K}(\vec{r}) = 0$.

Since $\vec{\nabla} \times \vec{K}(\vec{r}) = 0$, then we can write

$$\vec{K}(\vec{r}) = \vec{\nabla}\phi$$

and since $\vec{\nabla} \cdot \vec{K}(\vec{r}) = 0$, $\Rightarrow \nabla^2 \phi = 0$ ← Laplace Equation

Uniqueness Theorem for Laplace's Theorem:
$\Rightarrow$ the solution to Laplace Equation in volume V is uniquely defined up to additive constant if the normal derivative of ϕ is specified

In our case, imagine sphere, $R \to \infty$, and $\vec{K}(\vec{r})$ vanishes on the boundary, when $\partial_r \phi = 0$ is when $\phi(r=R) = \text{const}$. Since $\phi = \text{const}$ is the solution to the Laplace equation and a unique one by the uniqueness theorem, then
$\vec{K}(\vec{r}) = \vec{\nabla}\phi = 0$ everywhere is also unique

$\Rightarrow$ Hence the only vector function (F) with zero divergence and zero curl which vanishes at ∞ is zero everywhere.

⇒ Since the only way to make solutions of $\vec{F}$ not unique is to add vector functions aside from U and $\vec{w}$ that would make $\vec{\nabla}\cdot\vec{F}=0$ $\vec{\nabla}\times\vec{F}=0$, but theorem above says that the only possible unique solution for that to happen is when that vector function $=0$, hence the solution $\vec{F}$ given by $\vec{F}=-\vec{\nabla}U+\vec{\nabla}\times\vec{w}$ is uniquely defined.

⇒ Now is U and $\vec{w}$ themselves uniquely defined, that is change U and $\vec{w}$ but $\vec{\nabla}U$ and $\vec{\nabla}\times\vec{w}$ remains unchanged.

⇒ We note that $\vec{\nabla}U$ is uniquely determined? up to an additive constant. If we add the condition that U vanishes at $r\to\infty$, then U is uniquely defined.

⇒ Similarly $\vec{\nabla}\times\vec{w}$ is also uniquely defined but can still add the gradient of any scalar function.

⇒ The invariance of $\vec{F}$ (field) while vector potential $(\vec{A})$ $\vec{w}$ changes by the gradient of scalar function is called gauge invariance.

⇒ We often choose the gauge $\vec{\nabla}\cdot\vec{w}=0$, and we ask whether we can change $\vec{w}$ but maintain its gauge and curl, or add changes such that they produce zero divergence and curl. Ans: If we require $w\to 0$ at $r\to\infty$, then only possible function is zero everywhere.

Ex: Electrostatic:

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} = 0$$

then $$\vec{E} = -\vec{\nabla}U + \vec{\nabla}\times\vec{w}$$

$$U = V = \frac{1}{4\pi}\int \underbrace{D(\vec{r}')}_{\rho/\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}\,d\tau' = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$w = \frac{1}{4\pi}\int \underbrace{\vec{C}(\vec{r}')}_{=0}\frac{1}{|\vec{r}-\vec{r}'|}\,d\tau' = 0$$

Magneto-static:

$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \cancel{\frac{1}{c^2}\partial_t E} + \mu_0\vec{j}$$

$$B = -\vec{\nabla}U + \vec{\nabla}\times\vec{w}$$

$$U = \frac{1}{4\pi}\int \underbrace{D(\vec{r}')}_{=0}\frac{1}{|\vec{r}-\vec{r}'|}\,d\tau' = 0$$

$$\vec{A} = \vec{w} = \frac{1}{4\pi}\int \underbrace{\vec{C}(\vec{r}')}_{\mu_0\vec{j}}\frac{1}{|\vec{r}-\vec{r}'|}\,d\tau' = \frac{\mu_0}{4\pi}\int\frac{\vec{j}}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$B = \vec{\nabla}\times\vec{A}$$