Electrodynamics:

ρ and $\vec{J}$ time-dependent

Take ϕ and A :

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Gauge Invariant:

$$\phi \to \phi' = \phi - \partial_t \chi(\vec{r},t)$$
$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi(\vec{r},t)$$

$$\vec{E} \to \vec{E}' = \vec{E}$$
$$\vec{B} \to \vec{B}' = \vec{B}$$

Energy, momentum density and stress-tensor:

$$\mathcal{E} = \tfrac{1}{2}\left[\varepsilon_0 |E|^2 + \frac{1}{\mu_0}|B|^2\right] \quad \leftarrow \text{energy density}$$

$$\vec{P} = \varepsilon_0 \vec{E} \times \vec{B} \quad \leftarrow \text{momentum density}$$

$$\vec{L} = \vec{r} \times (\varepsilon_0 \vec{E} \times \vec{B}) \quad \leftarrow \text{Angular Momentum density}$$

$$\Theta_{ab} = T_{ab} = \varepsilon_0 E_a E_b - \frac{1}{\mu_0} B_a B_b - \frac{1}{2}\delta_{ab}\left[\varepsilon_0 |E|^2 + \frac{1}{\mu_0}|B|^2\right] \quad \leftarrow \text{stress tensor.}$$

$$\vec{S} = c^2 \vec{P} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad \leftarrow \text{energy Flux.}$$

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} \quad \leftarrow \text{force density.}$$

energy conservation : matter and field sectors.

$$\frac{d}{dt}\int_V d^3r\ \varepsilon_{field} = \frac{d}{dt}\int_V d^3r\ \frac{1}{2}\left(\varepsilon_0 |E|^2 + \frac{1}{\mu_0}|B|^2\right)$$

$$= \int d^3r \left(\varepsilon_0 \underbrace{\vec{E}\cdot\dot{\vec{E}}}_{c^2\vec{\nabla}\times\vec{B} - \frac{1}{\varepsilon_0}\vec{J}} + \frac{1}{\mu_0}\vec{B}\cdot\underbrace{\dot{\vec{B}}}_{-\vec{\nabla}\times\vec{E}}\right)$$

$$= -\int d^3r\ \vec{E}\cdot\vec{J} + \frac{1}{\mu_0}\int d^3r\ \underbrace{\left[\vec{E}(\vec{\nabla}\times\vec{B}) - \vec{B}(\vec{\nabla}\times\vec{E})\right]}_{-\vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

$$\frac{d}{dt}\int_V d^3r\ \varepsilon_{field} = -\int d^3r\ \vec{E}\cdot\vec{J} - \int d^2\vec{S}\cdot\underbrace{\vec{S}}_{\text{energy flux}}$$

$$\boxed{\frac{d}{dt}\int_V d^3r\ \varepsilon_{field} + \int d^3r\ \vec{E}\cdot\vec{J} = -\int d^2\vec{S}\cdot\vec{S}}$$

energy conservation.

Dynamics in ϕ and $\vec{A}$

1) $\vec{\nabla}\cdot\underbrace{(-\vec{\nabla}\phi-\partial_t\vec{A})}_{\vec{E}}=\rho/\varepsilon_0 \rightarrow -\vec{\nabla}^2\phi-\partial_t\vec{\nabla}\cdot\vec{A}=\rho/\varepsilon_0$

2) $\vec{\nabla}\times\underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}}=\frac{1}{c^2}\partial_t\overbrace{\left(-\vec{\nabla}\phi-\partial_t\vec{A}\right)}^{\vec{E}}+\mu_0\vec{J}$

work in Cartesian $\hookrightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2A=\frac{1}{c^2}\partial_t\left(-\vec{\nabla}\phi-\partial_t\vec{A}\right)+\mu_0\vec{J}$

Use Gauge Invariance to simplify:

$* \quad \boxed{\frac{1}{c^2}\partial_t\phi'+\vec{\nabla}\cdot\vec{A}'=0} \leftarrow$ Lorenz Gauge. New potentials obey this.

① $\partial_t\left(\frac{1}{c^2}\partial_t\phi'+\vec{\nabla}\cdot\vec{A}'\right)=0 \rightarrow \partial_t(\vec{\nabla}\cdot\vec{A})=-\frac{1}{c^2}\partial_t^2\phi$

② $\vec{\nabla}\left(\frac{1}{c^2}\partial_t\phi'+\vec{\nabla}\cdot\vec{A}'=0\right)\rightarrow\frac{1}{c^2}\partial_t\vec{\nabla}\phi=-\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

$-\vec{\nabla}^2\phi-\partial_t\vec{\nabla}\cdot\vec{A}=\rho/\varepsilon_0 \xrightarrow{①} \boxed{\vec{\nabla}^2\phi-\frac{1}{c^2}\partial_t^2\phi=-\rho/\varepsilon_0} \quad *$

$\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2A=\frac{1}{c^2}\partial_t\left(-\vec{\nabla}\phi-\partial_t\vec{A}\right)+\mu_0\vec{J}$

② $\hookrightarrow \boxed{\nabla^2\vec{A}-\frac{1}{c^2}\partial_t^2\vec{A}=-\frac{1}{\mu_0}\vec{J}} \quad *$

Green functions for temporal problems:

$$\vec{\Box}\psi = -f$$ ← known.

↑unknown

↳ $$\Box_{t,\vec r}\, G(t,\vec r, t', \vec r') = -\delta(t-t')\,\delta(\vec r - \vec r')$$

then
$$\boxed{\psi(\vec r, t) = \int d^3r' \int dt'\, G(t,\vec r, t', \vec r')\, f(\vec r', t) \quad \oplus \text{ any solution of the homogeneous wave eq.}}$$

Side note:

Solve $\ddot y + p(t)\,\dot y + q(t)\,y = f(t)$

Variation of parameters:

→ Guaranteed to work if you know $\ddot y + p\dot y + q y = 0$ with solution y_1, y_2.

General sol of homo eq: $a_1 y_1(t) + a_2 y_2(t) = y(t)$

instead, vary a_1, a_2: $a_1(t)\,y_1(t) + a_2(t)\,y_2(t) = y(t)$

with constraint: $\frac{d}{dt}(\dot a_1 y_1 + \dot a_2 y_2 = 0) \rightarrow \boxed{\ddot a_1 y_1 + \dot a_1 \dot y_1 + \ddot a_2 y_2 + \dot a_2 \dot y_2 = 0}$

constraint

$$(\ddot a_1 y_1 + \cancel{2}\dot a_1 \dot y_1 + \cancel{a_1 \ddot y_1}) + (\ddot a_2 y_2 + \cancel{2}\dot a_2 \dot y_2 + \cancel{a_2 \ddot y_2})$$
$$+ p\left([\cancel{\dot a_1 y_1} + \cancel{a_1 \dot y_1}] + [\cancel{\dot a_2 y_2} + \cancel{a_2 \dot y_2}]\right)$$
$$+ q\,(\cancel{a_1 y_1} + \cancel{a_2 y_2}) = f(t)$$

with homogeneous eq:
$$a_1(\ddot y_1 + p\dot y_1 + q y_1 = 0)$$
$$a_2(\ddot y_2 + p\dot y_2 + q y_2 = 0)$$

$$\begin{array}{r} \text{actual eq} \rightarrow \\ \text{constraint eq.} \rightarrow \end{array} \begin{pmatrix} \dot{y}_1 & \dot{y}_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \frac{1}{\underbrace{\dot{y}_1 y_2 - y_1 \dot{y}_2}_{=-W}} \begin{pmatrix} y_2 & -\dot{y}_2 \\ -y_1 & \dot{y}_1 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

then $$a_1 = -\int_{\alpha_1}^{t} d\tau \, \frac{y_2(\tau) f(\tau)}{W}$$

some constant

$$a_2 = -\int_{\alpha_2}^{t} d\tau \, \frac{-y_1(\tau) f(\tau)}{W}$$

some constant.

$$y(t) = y_1(t) \underbrace{\left(-\int_{\alpha_1}^{t} d\tau \, \frac{y_2(\tau) f(\tau)}{W(\tau)} \right)}_{a_1} + y_2(t) \underbrace{\left(-\int_{\alpha_2}^{t} d\tau \, \frac{-y_1(\tau) f(\tau)}{W(\tau)} \right)}_{a_2}$$

tuning to how much of homogeneous sol goes in, set by initial condition.

ex: $\left(\frac{d^2}{dt^2}+\Omega^2\right)y = f(t)$. with $y_1=\cos\Omega t$ $y_2=\sin\Omega t$

$$y(t) = \frac{-\cos\Omega t}{\Omega}\int_{\alpha_1}^{t} d\tau \sin\Omega\tau\, f(\tau)$$

$$+\frac{\sin\Omega t}{\Omega}\int_{\alpha_2}^{t} d\tau \cos\Omega t\, f(\tau)$$

with I.C.: $y(0)=0$ $\dot{y}(0)=0$

$$y(0)=0=\frac{-1}{\Omega}\int_{\alpha_1}^{t=0} d\tau \sin\Omega(\tau) f(\tau) \rightarrow \alpha_1=0$$

$$\dot{y}(0)=0=\int_{\alpha_2}^{t=0} d\tau \cos\Omega t\, f(\tau) - \frac{1}{\Omega}\sin\cancel{(\Omega t)}\, f(t) \rightarrow \alpha_2=0$$

$$y(t)=\int_0^t d\tau \sin(\Omega(t-\tau))\, f(\tau)\,\frac{1}{\Omega}$$

$$\stackrel{!}{=} \int_0^\infty d\tau \underbrace{\frac{1}{\Omega}\theta(t-\tau)\sin\Omega(t-\tau)}\, f(\tau)$$

then we see by comparing, this is just the Green's function.

Magnetostatic:

Without hole: $\vec{\nabla} \times \vec{H} = \vec{J}$

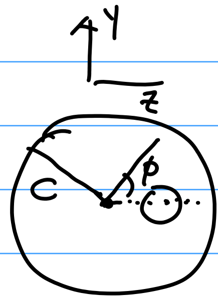
$$2\pi\rho \vec{H} = \vec{J} \pi \rho^2$$

large cylinder $\rightarrow$ $\vec{H} = \frac{J\rho}{2} \hat{\phi}$

$$\overset{!}{=} \rho \frac{J}{2}(-\sin\theta \hat{x} + \cos\theta \hat{y})$$

$$\overset{!}{=} \frac{J}{2}(-y\hat{x} + x\hat{y})$$

small cylinder: $\vec{H} = \frac{J}{2}(-y\hat{x} + (x-D)\hat{y})$



By superposition: $\vec{H}_{tot} = H_{large} - H_{small}$

$$\overset{!}{=} \frac{J}{2} D \hat{y} \quad , J = \frac{I}{\pi(C^2 - B^2)}$$

Solve for the Green's Function:

$$G(t,r;t',r') = g(t-t', r-r')$$

fundamental Green Function. (only depend on space and time difference)

then $\quad \Box_{t,r}\, g(t,r) = \delta(t)\,\delta(r)(-1)$

Solve by space/temporal Fourier transform:

$$g(t,r) = \int d^3q \int d\omega\, \hat{g}(\omega,q)\, e^{i\vec{q}\cdot\vec{r}}\, e^{-i\omega t}$$

then $\quad \Box_{t,r}\, g(t,r) = \int d^3q \int d\omega \left[-q^2 + \frac{\omega^2}{c^2}\right] \hat{g}(\omega,q)\, e^{i\vec{q}\cdot\vec{r} - i\omega t}$

$$\delta(t)\delta(\vec{r})(-1) = -\int d^3q \int d\omega\, e^{i\vec{q}\cdot\vec{r} - i\omega t}$$

by Comparing we see $\left[-|q|^2 + \frac{\omega^2}{c^2}\right] \hat{g}(\omega,q) = -1$

but if $\hat{g}(\omega,q) \neq \dfrac{1}{\frac{\omega^2}{c^2} - |q|^2}$ if $\omega \neq \pm c|q|$

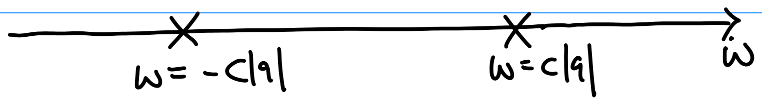
Note $\omega = \pm c|q|$ correspond to homogeneous solution which should be added later.

if $\hat{g}(\omega,q) = \dfrac{-1}{|q^2| - \frac{\omega^2}{c^2}}$, then

Homogeneous Solution.

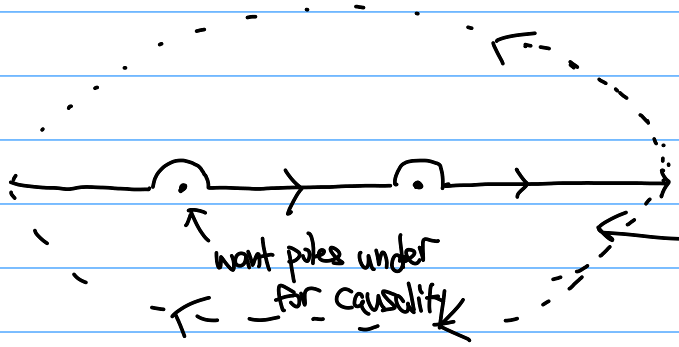
$$\boxed{g(t,\vec{r}) = -\int d^3q \int d\omega\, \frac{c^2}{\omega^2 - c^2|q|^2}\, e^{i\vec{q}\cdot\vec{r} - i\omega t} + \int d^3q \left\{ \alpha_+(q)\, e^{i\vec{q}\cdot\vec{r} - ic|q|t} + \alpha_-(q)\, e^{i\vec{q}\cdot\vec{r} + ic|q|t} \right\}}$$

This integral stops at poles, i.e. $\omega = \pm c|q|$



Side note: $\int dz \, \frac{1}{z} = \int_0^{2\pi} \frac{i d\theta \, \rho e^{i\theta}}{\rho e^{i\theta}} = 2\pi i$

Now integrate $\int d\omega \, \frac{e^{i\vec{q}\cdot\vec{r} - i\omega t}}{\omega^2 - c^2|q|^2} c^2$ in complex plane.



if $t<0$, we want $\omega>0$
so $e^{-i(\omega_R + i\omega_I)t} = e^{\omega_I t} + e^{-i\omega_R t}$.
so want $e^{\omega_I t} \to 0$

if $t>0$, we want $\omega<0$

$$g(t,q) = \begin{cases} 0 & t<0 \\ -\frac{1}{2\pi}(-1)c^2 \, 2\pi i \left\{ \lim_{\omega \to c|\vec{q}|} \frac{\omega - c|\vec{q}|}{(\omega - c|\vec{q}|)(\omega + c|\vec{q}|)} + \lim_{\omega \to -c|\vec{q}|} \frac{\omega + c|\vec{q}|}{(\omega - c|\vec{q}|)(\omega + c|\vec{q}|)} \right\} e^{-i\omega t} \end{cases}$$

clockwise

$$g(t,q) = \begin{cases} 0 & t<0 \\ \frac{c}{q} \sin cqt & t>0 \end{cases}$$

For $t>0$, $g(t,\vec{r}) = \int d^3q \, e^{i\vec{q}\cdot\vec{r}} \frac{c}{q} \sin c|\vec{q}| t$

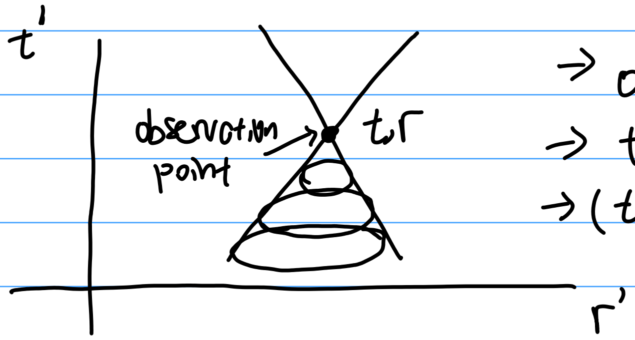
$t<0$, $g(t,\vec{r}) = 0$

then $\boxed{g(t,\vec{r}) = \theta(t) \frac{1}{4\pi|\vec{r}|} \delta\left(t - \frac{|\vec{r}|}{c}\right)}$

Now $\boxed{G(t,\vec{r},t',\vec{r}') = \frac{1}{4\pi|\vec{r}-\vec{r}'|} \theta(t-t') \, \delta\left([t-t'] - \frac{1}{c}|\vec{r}-\vec{r}'|\right)}$

Coulomb $\quad$ Causality $\quad$ light-cone condition.

where in $(t', \vec{r}')$ can contribute to ψ at (t, r)

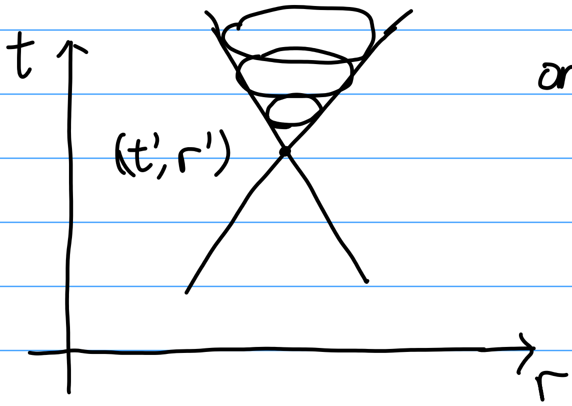


→ only points on the surface contribute

→ $t' < t$

→ $(t - t')c = |\vec{r} - \vec{r}'|$

where can (t, r) be influenced by (t', r')

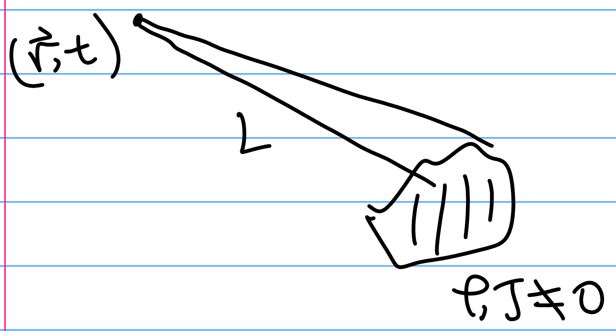


only influence future.

$$\psi(\vec{r}, t) = \int d^3r' \int dt' \frac{1}{4\pi|\vec{r} - \vec{r}'|} \theta(t - t') \delta\Big(\underbrace{[t - t'] - \tfrac{1}{c}|\vec{r} - \vec{r}'|}_{t' = t - \frac{1}{c}|\vec{r} - \vec{r}'|}\Big) f(\vec{r}'; t)$$

$$\boxed{\psi(\vec{r}, t) = \int d^3r' \frac{1}{4\pi|\vec{r} - \vec{r}'|} \underbrace{f\left(\vec{r}', t - \tfrac{1}{c}|\vec{r} - \vec{r}'|\right)}_{\equiv f(\vec{r}, t)_{ret}}}$$

Multipole Expansion for ϕ and A: non-relativistic source.



- Calculate $\phi, \vec{A} \to \vec{E}, \vec{B}$
- keep leading order term. $(1/r)$

Non-relativistic: $\dot{\rho} \sim T$ ← timescale of change in ρ

with L, length scale.

$\frac{L}{T} \ll c$: non relativistic condition (everything makes up ρ is slow compared to c)

$\frac{L}{T} \lesssim c$: relativistic.

$$\vec{A}(\vec{r},t) = \int d^3r' \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}-\vec{r}'|} J(\vec{r}, t-\frac{1}{c}|\vec{r}-\vec{r}'|)$$

$$\phi(\vec{r},t) = \int d^3r' \frac{1}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|} \underbrace{\rho(\vec{r}', t-\frac{1}{c}|\vec{r}-\vec{r}'|)}_{\equiv f(\vec{r},t)_{ret} = \rho_{ret}(\vec{r},t)}$$

With <u>multipole expansion:</u> $\boxed{r' \ll r}$ far away.

$$\hookrightarrow \phi(\vec{r},t) = \frac{1}{4\pi r}\int d^3r' \, \underbrace{\rho(\vec{r}', t-\frac{1}{c}\underbrace{|\vec{r}-\vec{r}'|}_{=\vec{r}-\hat{r}\cdot\vec{r}' + \mathcal{O}\left(\frac{r'^2}{r}\right)})}_{\rho(t-\frac{r}{c}+\underbrace{\frac{1}{c}\hat{r}\cdot\vec{r}'}_{\text{taylor}}, \vec{r}')}$$

$$\frac{R}{cT} \ll 1$$

$$\approx \rho(\tau,\vec{r}')\Big|_{\tau=t-\frac{r}{c}} + \frac{1}{c}\hat{r}\cdot\vec{r}' \partial_\tau \rho(\tau,\vec{r}')\Big|_{\tau=t-r/c} + \mathcal{O}$$

$$\hookrightarrow \phi(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\Big\{\underbrace{\int d^3r' [\rho(t,\vec{r}')]_{ret}}_{\text{total charge } q.}$$

$$+ \underbrace{c\,\hat{r}_a \int d^3r' \, r'_a \frac{\partial}{\partial\tau}\rho(\tau,\vec{r}')\Big|_{\tau=t-r/c}}_{\frac{d}{d\tau}p_a(\tau)\big|_{\tau=t-r/c}} = \dot{p}_a(\tau)\Big|_{t-r/c}$$

dipole variation.

$$\hookrightarrow \boxed{\phi(\vec{r},t) \approx \frac{1}{4\pi\varepsilon_0}\left\{ q/r + \frac{1}{rc}\,\hat{r}\cdot\dot{\vec{p}}(t)_{ret}\right\}}$$

Similarly:

$$\boxed{\vec{A}(t,\vec{r}) = \frac{\mu_0}{4\pi}\left\{\frac{0}{r} + \frac{1}{r}[\dot{\vec{p}}]_{ret}\right\}}$$

Now calculate $\vec{E}, \vec{B}$:

$$\boxed{\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A} \cong -\frac{\mu_0}{4\pi}\frac{1}{r}\frac{1}{c}\,\hat{r}\times\left[\ddot{\vec{p}}(t)\right]_{ret} \\ \vec{E} &= -\vec{\nabla}\phi - \partial_t\vec{A} \cong \frac{\mu_0}{4\pi}\frac{1}{r}\left\{\hat{r}\left(\hat{r}\cdot\left[\ddot{\vec{p}}(t)\right]_{ret}\right) - \left[\ddot{\vec{p}}(t)\right]_{ret}\right\}\end{aligned}}$$

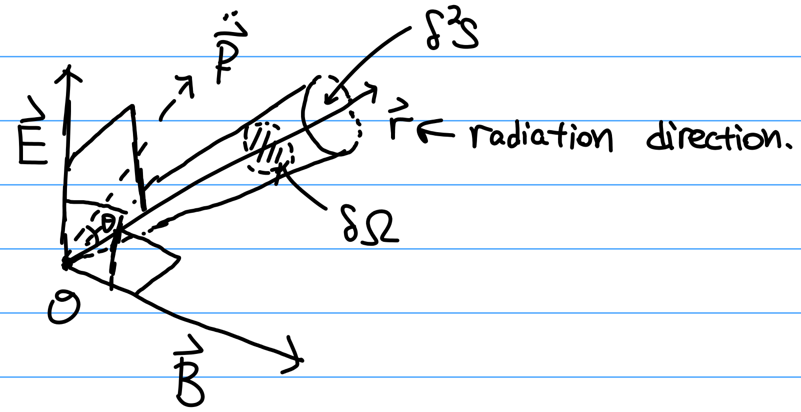
Qualitative Features:

1) leading order of $\vec{E}, \vec{B}$ is $\frac{1}{r}$, radiation field.

2) $\left[\ddot{\vec{p}}(t)\right]_{ret} \rightarrow$ radiation.

3) $\hat{r}, \vec{E}, \vec{B}$ form an orthonormal triad.

Start $\hat{r}$ and $\ddot{\vec{p}}$,



$$\boxed{\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} \\ &\text{energy flux density} \\ &\overset{!}{=} \frac{\mu_0}{16\pi^2 c}\frac{1}{r^2}\left|\left[\ddot{\vec{p}}\right]_{ret}\right|^2 \sin^2\theta\,\hat{r}\end{aligned}}$$

$$\underset{\text{power}}{\delta P} = \lim_{r\to\infty} \underbrace{\delta^2 S}_{r^2\delta\Omega}\,\vec{S} = \frac{\mu_0}{16\pi^2 c}\left|\left[\ddot{\vec{p}}(t)\right]_{ret}\right|^2 \sin^2\theta\,\delta\Omega$$

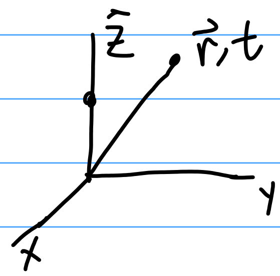
Larmor's Formula for dipole radiation

$$\boxed{P = \int d\Omega\,\frac{\delta P}{\delta\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta\,\frac{\delta P}{\delta\Omega} = \frac{1}{6\pi\varepsilon_0}\frac{1}{c^3}\left|\left[\ddot{\vec{p}}(t)\right]_{ret}\right|^2}$$

Comp exam question:

A charge Q in uniform rectlinear motion. What are $\vec{E}$ and $\vec{B}$?

$R(t) = vt\,\hat{z}$



$\rho(\vec{r},t) = Q\,\delta(\vec{r} - \vec{R}(t))$
$\vec{J}(\vec{r},t) = Qv\hat{z}\,\delta(\vec{r} - \vec{R}(t))$

$$\nabla^2\phi - \frac{1}{c^2}\partial_t^2\phi = -\frac{1}{\varepsilon_0}\rho(r,t) = -\frac{1}{\varepsilon_0}Q\,\delta(\vec{r} - \vec{R}(t))$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\partial_t^2\vec{A} = -\mu_0\vec{J}(r,t) = -\mu_0 Qv\hat{z}\,\delta(\vec{r} - \vec{R}(t))$$

Calculate Fourier transform:

$$\phi(\vec{r},t) = \int d^3k\; \hat{\phi}(\vec{k},t)\,e^{i\vec{k}\cdot\vec{r}}$$
$$\vec{A}(\vec{r},t) = \int d^3k\; \hat{A}(\vec{k},t)\,e^{i\vec{k}\cdot\vec{r}}$$

$$\hookrightarrow \int d^3k\; e^{i\vec{k}\cdot\vec{r}}\left[-|k|^2 - \frac{1}{c^2}\partial_t^2\right]\hat{\phi}(\vec{k},t) = -\frac{Q}{\varepsilon_0}\int d^3k\; e^{i\vec{k}\cdot(\vec{r}-\vec{R}(t))}$$

$$\hookrightarrow \left(|k|^2 + \frac{1}{c^2}\partial_t^2\right)\hat{\phi}(\vec{k},t) = \frac{Q}{\varepsilon_0}e^{-i\vec{k}\cdot\vec{R}(t)}$$

similarly $\left(|k|^2 + \frac{1}{c^2}\partial_t^2\right)\hat{A}(\vec{k},t) = Qv\mu_0\hat{z}\,e^{-i\vec{k}\cdot\vec{R}(t)}$

assume $\hat{\phi} \sim \beta e^{-i\vec{k}\cdot vt\hat{z}}$

then we find $\beta = \frac{Q}{\varepsilon_0}\frac{1}{|\vec{k}|^2 - \frac{1}{c^2}v^2k_z^2}$

$\hookrightarrow \hat{\phi}(\vec{k},t) = \frac{1}{\varepsilon_0}\frac{Qe^{-i\vec{k}\cdot vt\hat{z}}}{k_x^2 + k_y^2 + \gamma^2 k_z^2}$ where $\gamma^2 = 1 - \frac{v^2}{c^2}$

similarly:

$$\hat{A}(\vec{k},t) = \mu_0 \frac{Q v \hat{z}\, e^{-i\vec{k}\cdot v t \hat{z}}}{k_x^2 + k_y^2 + \gamma^2 k_z^2}$$

Inverse transform using $\iiint d\vec{k}^3 \frac{e^{i\vec{k}\cdot\vec{r}}}{|k|^2} = \frac{1}{4\pi}\frac{1}{|r|}$

and let $\tilde{k}_z = \gamma k_z$

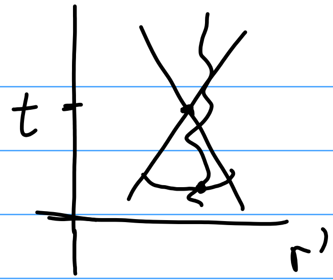
then $\phi(\vec{r},t) = \frac{Q}{\varepsilon_0}\frac{1}{4\pi}\frac{1}{\gamma}\frac{1}{\sqrt{x^2+y^2+\left(\frac{z-vt}{\gamma}\right)^2}}$

$$\vec{A}(\vec{r},t) = \mu_0 Q v \hat{z} \frac{1}{\sqrt{\gamma^2(x^2+y^2)+(z-vt)^2}}$$

$$\vec{E}(\vec{r},t) = \frac{Q}{4\pi\varepsilon_0}\frac{\gamma^2}{\gamma^2(x^2+y^2)+(z-vt)^2}\left(\vec{r}-\vec{R}(t)\right)$$

Lienard-Wiechart potential:

potential caused by single charge particle



We know:

$$\phi(t,\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(t,\vec{r}')_{ret}}{|\vec{r}-\vec{r}'|}$$

$$t_{ret} \to t - \frac{1}{c}|\vec{r}-\vec{r}'|$$

$$\vec{A}(t,\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(t,\vec{r})_{ret}}{|\vec{r}-\vec{r}'|}$$

$$\rho(t,\vec{r}) = q\,\delta(\vec{r}-\vec{R}(t))$$

$$\vec{J}(t,\vec{r}) = q\,\dot{\vec{R}}\,\delta(\vec{r}-\vec{R}(t))$$

Instead of doing dt'

$$\phi(t,\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int dt'\, d^3r' \, \frac{\Theta(t-t')}{|\vec{r}-\vec{r}'|}\,\delta\left([t-t'] - \frac{1}{c}|\vec{r}-\vec{r}'|\right) q\,\delta(\vec{r}-\vec{R}(t))$$

$$\vec{A}(t,\vec{r}) = \frac{\mu_0}{4\pi}\int dt'\, d^3r' \, \frac{\Theta(t-t')}{|\vec{r}-\vec{r}'|}\,\delta\left([t-t'] - \frac{1}{c}|\vec{r}-\vec{r}'|\right) q\,\dot{\vec{R}}\,\delta(\vec{r}-\vec{R}(t))$$

then

$$\phi(t,\vec{r}) = \frac{q}{4\pi\varepsilon_0}\int dt' \, \frac{1}{\vec{r}-\vec{R}(t)}\,\delta\left([t-t'] - \frac{1}{c}|\vec{r}-\vec{R}(t)|\right)$$

$$\vec{A}(t,\vec{r}) = \frac{q\mu_0}{4\pi}\int dt' \, \frac{\dot{\vec{R}}(t')}{\vec{r}-\vec{R}(t')}\,\delta\left([t-t'] - \frac{1}{c}|\vec{r}-\vec{R}(t')|\right)$$

<u>Aside</u>:

$$\delta(\alpha t) = \frac{\delta(t)}{|\alpha|}$$

$$\delta(f(t)) = \delta\left(f(t_{root}) + (t - t_{root}) f'(t_{root})\right) = \frac{\delta(t - t_{root})}{\left|\frac{df}{dt}\right|_{t_{root}}}$$ if $\frac{df}{dt} \neq 0$, i.e. not a minimum or maximum.

then $f(t') = (t - t') - \frac{1}{c}|\vec{r} - \vec{R}(t')|$ note $t_{root} = t_{ret} = t' - \frac{1}{c}|\vec{r} - \vec{R}(t')|$

Now taylor expand around $t' = t_{ret} = t_{root} = t - \frac{1}{c}|\vec{r} - \vec{R}|$

$$f(t') \approx \underbrace{f(t_{ret})}_{=0} + (t' - t_{ret}) \frac{d}{dt'} f(t) \Big|_{t' = t_{ret}}$$

$$= (t' - t_{ret}) \frac{d}{dt'} \left\{ t - t' - \frac{1}{c}\left[r^2 - 2\vec{r} \cdot \vec{R}(t') + R^2(t')\right]^{1/2} \right\} \Big|_{t' = t_{ret}}$$

$$= (t' - t_{ret}) \left[-1 - \frac{1}{c} \cancel{\frac{1}{2}} \frac{\left[-\cancel{2}\vec{r} \cdot \dot{\vec{R}}(t') + \cancel{2}\vec{R} \cdot \dot{\vec{R}}\right]}{|\vec{r} - \vec{R}(t')|} \Big|_{t' = t_{ret}} \right]$$

$$= (t' - t_{ret}) \left[-1 + \frac{1}{c} \frac{\vec{r} - \vec{R}(t_{ret})}{|\vec{r} - \vec{R}(t_{ret})|} \cdot \dot{\vec{R}}(t_{ret}) \right]$$

then $\delta\left([t - t'] - \frac{1}{c}|\vec{r} - \vec{R}(t)|\right) = \delta\left([t' - t_{ret}]\left\{-1 + \frac{1}{c}\dot{\vec{R}} \cdot \hat{n}\right\}\right)$

for $\hat{n} = \frac{\vec{r} - \vec{R}(t_{ret})}{|\vec{r} - \vec{R}(t_{ret})|}$

<u>then</u>: Lienard - Wiechart Potential

$$\phi(t, \vec{r}) = \frac{1}{4\pi\varepsilon_0} q \frac{1}{|\vec{r} - \vec{R}(t_{ret})|} \frac{1}{1 - \frac{1}{c}\dot{\vec{R}}(t_{ret}) \cdot \hat{n}}$$

$$\vec{A}(t, \vec{r}) = \frac{\mu_0}{4\pi} q \frac{1}{|\vec{r} - \vec{R}(t_{ret})|} \frac{\dot{\vec{R}}(t_{ret})}{1 - \frac{1}{c}\dot{\vec{R}}(t_{ret}) \cdot \hat{n}}$$

Then $\vec{E}(t,\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{[\hat{n}-\frac{1}{c}\dot{\vec{R}}][1-\frac{1}{c^2}\dot{\vec{R}}^2]}{\alpha^3\,|\vec{r}-\vec{R}|^2} + \frac{1}{c^2}\,\frac{\hat{n}\times[(\hat{n}-\frac{1}{c}\dot{\vec{R}})\times\ddot{\vec{R}}]}{\alpha^3\,|\vec{r}-\vec{R}|} \right]$

$$\vec{B}(t,\vec{r}) = \frac{1}{c}\,\hat{n}\times\vec{E}(t,\vec{r})$$

for $\alpha = 1 - \frac{1}{c}\hat{n}\cdot\dot{\vec{R}}$

$\hat{n} = \frac{\vec{r}-\vec{R}}{|\vec{r}-\vec{R}|}$ and $\vec{R}, \dot{\vec{R}}, \ddot{\vec{R}}$ all evaluated at t_{ret}

Plane Waves

↳ Solution to Maxwell's equation for ϕ and $\vec{A}$ when there are no source terms $\rho, \vec{J}$.

i.e. $\quad \Box \phi(t,\vec{r}) = 0 \qquad \Box \vec{A}(t,\vec{r}) = 0$

together with Lorenz Gauge: $\frac{1}{c^2}\partial_t \phi + \vec{\nabla}\cdot\vec{A} = 0$

and additional gauge freedom such that $\vec{\nabla}\chi = 0$

Plane Wave Solutions: $\boxed{\phi(t,\vec{r}) = 0 \quad \text{due to gauge choice.}}$

Now determine $\vec{A}(t,\vec{r})$:

Suppose: $$\underbrace{\vec{A}}_{\text{real}}(t,\vec{r}) = \int d^3q \; e^{i\vec{q}\cdot\vec{r}} \, \hat{\vec{A}}(t,\vec{q})$$

If $\vec{A}(t,\vec{r})$ is real, this means

↳ $$\vec{A}(t,\vec{r}) = \vec{A}(t,\vec{r})^*$$

↳ $$\vec{A}(t,-\vec{q}) = \vec{A}(t,\vec{q})^* \quad \text{with Fourier.}$$

Using Lorenz Gauge: $\vec{\nabla}\cdot\vec{A} = 0$

In Fourier ↳ $$\vec{q}\cdot\vec{A}(t,\vec{q}) = 0$$

This means $\vec{A}$ has no component along $\vec{q}$ direction, so $\vec{A}$ is transverse.

With $\Box \vec{A}(t,\vec{r}) = 0$

$\hookrightarrow -\frac{1}{c^2}\partial_t^2 \vec{A}(t,\vec{q}) - q^2 \vec{A}(t,\vec{q}) = 0$

Harmonic Oscillator-like solution.

→ Since $\vec{A}(t,\vec{q})$ is complex and 3 components → 6 eq

→ but due to the transverse property, $\vec{q}\cdot\vec{A}(t,\vec{q}) = 0$
we can eliminate 2 equations, for real and complex:
then $6 \to 4$ eq.

→ Also know $\vec{A}(t,\vec{q})$ is actually real, so $4 \to 2$ eq.

so general solution is:

$$\vec{A}(t,\vec{q}) = \vec{C}_1 e^{-icqt} + \vec{C}_2 e^{icqt}$$

$\omega = cq$ $\omega = -cq$.

polarizability is managed by $\vec{C}_1$ and $\vec{C}_2$

with transversalitity, $\vec{q}\cdot\vec{A}(t,\vec{q}) = 0$ for all t:

$\hookrightarrow \vec{q}\cdot\vec{C}_1 = 0 \qquad \vec{q}\cdot\vec{C}_2 = 0$

$\vec{A}$ is real requirement: $\vec{A}(t,-\vec{q}) = \vec{A}(t,\vec{q})^*$:

$$C_1(-\vec{q})\, e^{-icqt} + C_2(-\vec{q}) e^{+icqt} = \vec{C}_1(\vec{q})^* e^{+icqt} + \vec{C}_2^*(\vec{q}) e^{-icqt}$$

so $C_1(-\vec{q}) = C_2^*(\vec{q})$
$C_2(-\vec{q}) = C_1^*(\vec{q})$ } so $\vec{q}$ and $-\vec{q}$ are connected.

Now consider the entire Fourier Transform from +q to −q:

$$\left(\vec{C}_1(q)e^{-icqt}+\vec{C}_2(q)e^{+icqt}\right)e^{i\vec{q}\cdot\vec{r}}+\left(\vec{C}_1(-q)e^{+icqt}+\vec{C}_2(-q)e^{-icqt}\right)e^{-i\vec{q}\cdot\vec{r}}$$

use $C_1(-q)=C_2^*(q)$ and $C_2(-q)=C_1^*(q)$

$$\hookrightarrow =\left(\vec{C}_1(q)e^{-icqt}+\vec{C}_2(q)e^{+icqt}\right)e^{i\vec{q}\cdot\vec{r}}+\left(\vec{C}_2^*(q)e^{+icqt}+\vec{C}_1(q)^*e^{-icqt}\right)e^{-i\vec{q}\cdot\vec{r}}$$

$$\hookrightarrow = \vec{C}_1(\vec{q})e^{i\vec{q}\cdot\vec{r}-icqt}+\vec{C}_2(\vec{q})e^{i\vec{q}\cdot\vec{r}+icqt}+\text{c.c.}$$

$$\overset{!}{=}\boxed{\vec{C}(\vec{q})\,e^{i\vec{q}\cdot\vec{r}-icqt}}$$

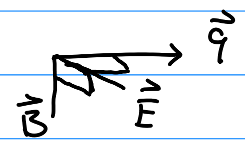
complex plane electromagnetic wave.

polarization is managed by $\vec{C}(\vec{q})$

then $$\vec{A}(t,\vec{r})=\mathrm{Re}\int d^3q\,\vec{C}(\vec{q})\exp\{i\vec{q}\cdot\vec{r}-icqt\}$$

4 pieces of information, amplitude and phase for the two directions normal to $\vec{q}$.

Then $\vec{E}=-\cancel{\vec{\nabla}\phi}-\partial_t\vec{A}$



$$\overset{!}{=}\mathrm{Re}\int d^3q\,\underbrace{\left[\vec{C}(\vec{q})icq\right]}\exp\{i\vec{q}\cdot\vec{r}-icqt\}$$

these are orthogonal to each other and orthogonal to $\vec{q}$.

$\vec{B}=\vec{\nabla}\times\vec{A}$

$$=\mathrm{Re}\int d^3q\,\overbrace{\left[i\vec{q}\times\vec{C}(\vec{q})\right]}\exp\{i\vec{q}\cdot\vec{r}-icqt\}$$

Notice:
$$\frac{[i\vec{q}\times\vec{C}(q)]\cdot[-i\vec{q}\times C(\vec{q})^*]}{[C(\vec{q})\,icq]\cdot[\vec{C}(\vec{q})^*(-i)\,cq]} = \frac{1}{c^2}$$

$\hookrightarrow$ speed of light.

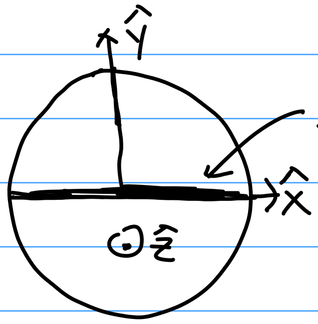
Now we choose $\vec{q}$ in $\hat{z}$

so $\vec{C} = C_x\,\hat{x} + C_y\,\hat{y}$

let $icq\vec{C} = \underbrace{\alpha_x}_{\text{amplitude}}\, e^{i\overbrace{\beta_x}^{\text{phase}}}\,\hat{x} + \alpha_y e^{i\beta_y}\,\hat{y}$

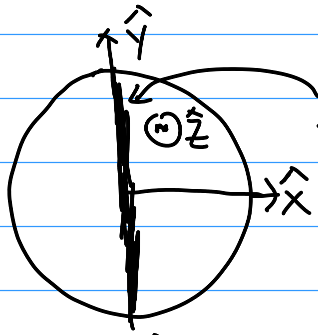
then
$$\vec{E} = \mathrm{Re}\,[\vec{C}(\vec{q})\,icq]\exp\{i\vec{q}\cdot\vec{r} - icqt\}$$
$$\stackrel{!}{=} \mathrm{Re}\,[\alpha_x e^{i\beta_x}\hat{x} + \alpha_y e^{i\beta_y}\hat{y}]\,e^{i\vec{q}\cdot\vec{r} - icqt}$$
$$\stackrel{!}{=} \alpha_x\cos(q(z-ct)+\beta_x)\,\hat{x} + \alpha_y\cos(q(z-ct)+\beta_y)\,\hat{y}$$

Polarization: linear



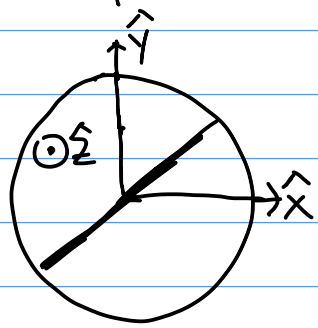
$\vec{E}$ oscillates back and forth in $\hat{x}$

$\Leftarrow \quad \alpha_x = 1 \quad \alpha_y = 0$



$\vec{E}$ oscillates back and forth in $\hat{y}$

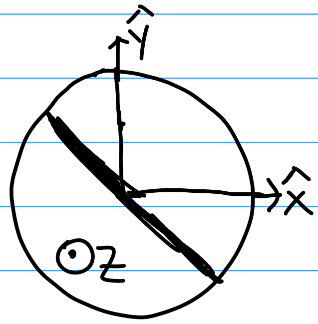
$\Leftarrow \quad \alpha_x = 0 \quad \alpha_y = 1$



$\Leftarrow$

$\alpha_x = \alpha_y = 1$

$\beta_x = \beta_y$

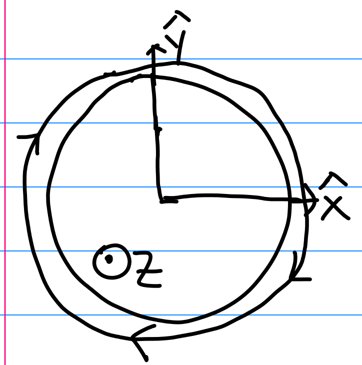


$\Leftarrow$

$\alpha_x = \alpha_y = 1$

$\beta_x = \beta_y \pm \pi$

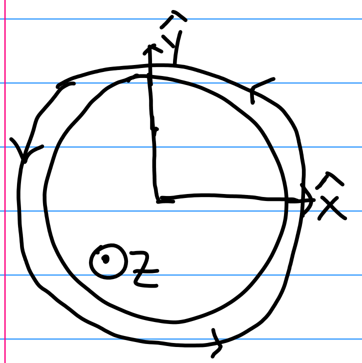
Polarization: Circular / Elliptical.



$\Leftarrow$ circular

$\alpha_x = \alpha_y = 1$

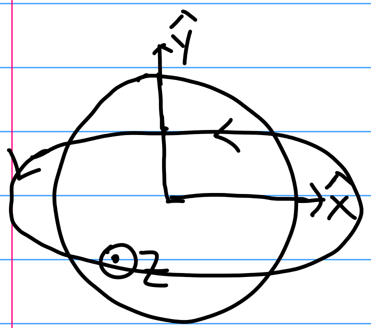
$\beta_x = \beta_y + \pi/2$



$\Leftarrow$ circular

$\alpha_x = \alpha_y = 1$

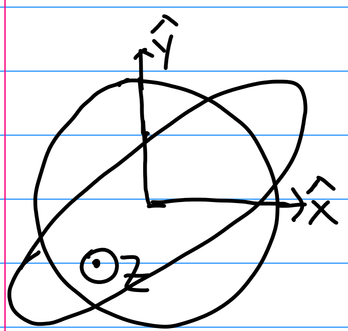
$\beta_x = \beta_y - \pi/2$



$\Leftarrow$ Elliptical

$\alpha_x = 1 \quad \alpha_y = 1/2$

$\beta_x = \beta_y + \pi/2$



$\Leftarrow$ Elliptical

$\beta_x - \beta_y \neq \pi/2, \pi$

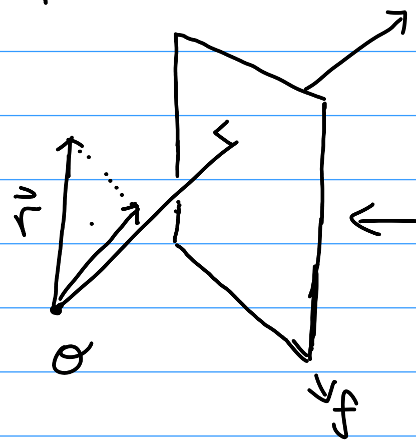
Coming back to complex plane EM wave:

$$\vec{C}(\vec{q})\, e^{i\vec{q}\cdot\vec{r} - icqt} \quad \text{with } \vec{C}(\vec{q})\cdot\vec{q} = 0$$

$\leftarrow \vec{\nabla}\cdot\vec{A} = 0$ condition.

At fixed t, for plane to be constant, $\vec{r}$ must lie on the surface:

$\hat{q}\cdot\vec{r} - ct = f$, const.



$\leftarrow$ plane wave.

component of $\vec{r}$ along $\vec{q}$ equals for all point on the constant surface.