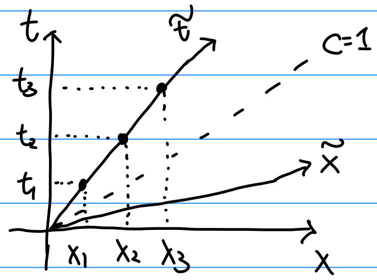
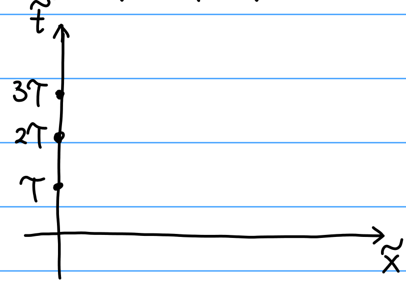
Zhi Chen HW# 11

1) a) In Lab Frame:



In Rocket Frame:



know
$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} n\tau \\ 0 \end{pmatrix}$$

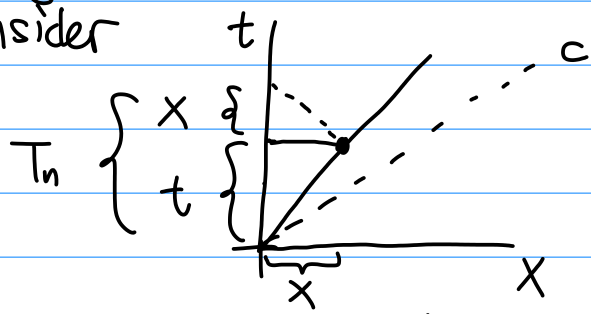
then

+ sign for going from moving to lab frame or $-V$

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\beta & +\sinh\beta \\ +\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} n\tau \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} n\tau \cosh\beta \\ n\tau \sinh\beta \end{pmatrix}$$

To get the time when the pulse returns, consider



since light pulse travels at $v=c$, so it makes $45°$, so the triangle has equal sides.

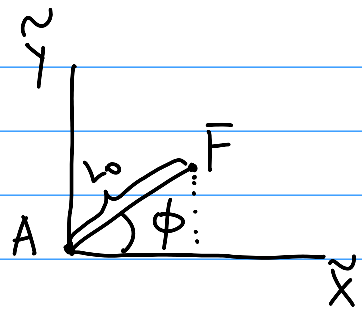
thus $T_n = t_n + x_n = n\tau(\cosh\beta + \sinh\beta)$

$$= n\tau\left(\frac{e^{\beta}+e^{-\beta}}{2} + \frac{e^{\beta}-e^{-\beta}}{2}\right)$$

knowing $\tanh\beta = V$

$$\boxed{T_n = n\tau e^{\beta} = n\tau\sqrt{\frac{1+V/c}{1-V/c}}}$$

b) In Rocket Frame:



let rocket move in x-direction with speed V.

then in Rocket frame we have

$$\tilde{A} = \begin{pmatrix} A_{\tilde{t}} \\ A_{\tilde{x}} \\ A_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} \qquad F = \begin{pmatrix} F_{\tilde{t}} \\ F_{\tilde{x}} \\ F_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} u \\ L_0\cos\phi \\ L_0\sin\phi \end{pmatrix}$$

Now use Lorentz Transform to go to Lab frame:

$$A = \begin{pmatrix} A_t \\ A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cosh\beta & +\sinh\beta & 0 \\ +\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda\cosh\beta \\ \lambda\sinh\beta \\ 0 \end{pmatrix}$$

$$F = \begin{pmatrix} F_t \\ F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cosh\beta & +\sinh\beta & 0 \\ +\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ L_0\cos\phi \\ L_0\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} u\cosh\beta + L_0\sinh\beta\cos\phi \\ u\sinh\beta + L_0\cosh\beta\cos\phi \\ L_0\sin\phi \end{pmatrix}$$

Now if we are measuring the length or orientation of the rod, we expect time of A and F to be the same:

i.e. $$A_t = F_t$$

$$\hookrightarrow \quad \lambda\cosh\beta = u\cosh\beta + L_0 \sinh\beta \cos\phi$$

or $$\lambda - u = L_0 \tanh\beta \cos\phi$$

Now define ψ to be the orientation angle of the rod relative to the rocket for lab frame observer.

$$\tan\psi = \frac{F_y - A_y}{F_x - A_x}$$

$$= \frac{L_0 \sin\phi}{u\sinh\beta + L_0\cosh\beta\cos\phi - \lambda\sinh\beta}$$

$$= \frac{L_0 \sin\phi}{-(\lambda-u)\sinh\beta + L_0\cosh\beta\cos\phi} \; \frac{\cosh\beta}{\cosh\beta}$$

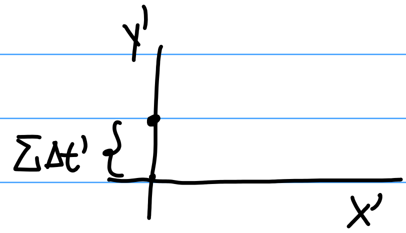
using $\lambda - u = L_0 \tanh\beta\cos\phi$

$$= \frac{\cancel{L_0}\cosh\beta\sin\phi}{\cancel{L_0}\underbrace{(\cosh^2\beta - \sinh^2\beta)}_{=1}\cos\phi}$$

$$\boxed{\tan\psi = \tan\phi\cosh\beta = \frac{\tan\phi}{\sqrt{1-(v/c)^2}}}$$

c) i) First determine particle position in rocket frame.

$$\begin{pmatrix} dt' \\ dx' \\ dy' \end{pmatrix} = \begin{pmatrix} \Delta t' \\ 0 \\ \sum_i \Delta t' \end{pmatrix}$$



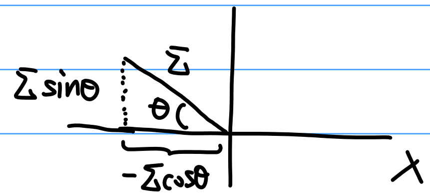
$$\begin{pmatrix} dt \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \cosh\beta & +\sinh\beta & 0 \\ +\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ 0 \\ \sum_i \Delta t' \end{pmatrix}$$

$$\begin{pmatrix} dt \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \cosh\beta \, \Delta t' \\ \sinh\beta \, \Delta t' \\ \sum_i \Delta t' \end{pmatrix}$$

then

$$\boxed{\begin{aligned} u_x &= \frac{dx}{dt} = \frac{\sinh\beta \, \Delta t'}{\cosh\beta \, \Delta t'} = \tanh\beta = V \\ u_y &= \frac{dy}{dt} = \frac{\sum_i \cancel{\Delta t'}}{\cosh\beta \, \cancel{\Delta t'}} = \sum_i \sqrt{1-\left(\frac{V}{c}\right)^2} \end{aligned}}$$

ii) Now with angle θ



$$\begin{pmatrix} dt' \\ dx' \\ dy' \end{pmatrix} = \begin{pmatrix} \Delta t' \\ -\Sigma\cos\theta\ \Delta t' \\ \Sigma\sin\theta\ \Delta t' \end{pmatrix}$$

$$\begin{pmatrix} dt \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \cosh\beta & +\sinh\beta & 0 \\ +\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ -\Sigma\cos\theta\,\Delta t' \\ \Sigma\sin\theta\ \Delta t' \end{pmatrix}$$

$$= \begin{pmatrix} \Delta t' \cosh\beta - \Sigma\sinh\beta\cos\theta\,\Delta t' \\ \Delta t' \sinh\beta - \Sigma\cosh\beta\cos\theta\,\Delta t' \\ \Sigma\sin\theta\ \Delta t' \end{pmatrix}$$

$$u_y = \frac{dy}{dt} = \frac{\Sigma\sin\theta\,\cancel{\Delta t'}}{[\cosh\beta - \Sigma\sinh\beta\cos\theta]\cancel{\Delta t'}}$$

$$\boxed{u_y = \frac{\Sigma\sin\theta}{\cosh\beta(1-\Sigma\cos\theta\tanh\beta)}}$$ ← always go in $+\hat{y}$

$$u_x = \frac{dx}{dt} = \frac{\cancel{\Delta t'}\sinh\beta - \Sigma\cosh\beta\cos\theta\,\cancel{\Delta t'}}{\cancel{\Delta t'}\cosh\beta - \Sigma\sinh\beta\ \cos\theta\,\cancel{\Delta t'}}$$

$$= \frac{\cancel{\cosh\beta}(\tanh\beta - \Sigma\cos\theta)}{\cancel{\cosh\beta}(1-\Sigma\cos\theta\tanh\beta)}$$

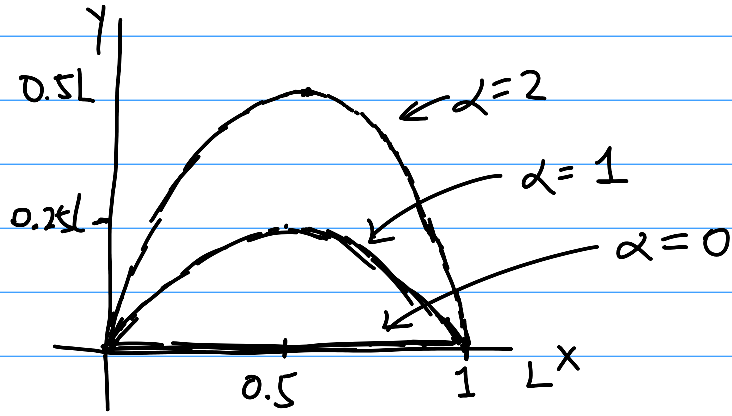
$$\boxed{u_x = \frac{\tanh\beta - \Sigma\cos\theta}{1-\Sigma\cos\theta\tanh\beta}}$$

if $\tanh\beta > \Sigma\cos\theta$ or the rocket speed is greater than particle speed, $\Sigma\cos\theta$, then it goes in $+\hat{x}$ direction, and vice-versa.

2) Twin Paradox:

a) i) $(x, y) = (\lambda, \alpha\lambda(1-\lambda))L$

let $\alpha = 0, 1, 2$:



ii) $s = \int\int\sqrt{(dx)^2+(dy)^2}$

$$\overset{!}{=} \int_0^1 \sqrt{\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2}\, d\lambda$$

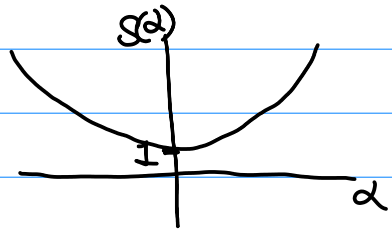
$$\frac{dx}{d\lambda} = L, \quad \frac{dy}{d\lambda} = \alpha L(1-2\lambda)$$

$$\overset{!}{=} \int_0^1 L\sqrt{1+\alpha^2(1-2\lambda)^2}\, d\lambda$$

$$\overset{!}{=} L\,\frac{\operatorname{arcsinh}(\alpha)+\alpha\sqrt{\alpha^2+1}}{2\alpha}$$

Not surprised that path length depends on α, since from sketch, we see as α increases, we have a "taller" path.

The above function looks like $\Rightarrow$



so by eye, we see it has minimum at $\boxed{\alpha = 0}$

we can also calculate $\frac{d}{d\alpha}S(\alpha)=0$ to minimize $S(\alpha)$:

$$\frac{d}{d\alpha}S(\alpha) = L\frac{d}{d\alpha}\frac{\operatorname{arcsinh}(\alpha)+\alpha\sqrt{\alpha^2+1}}{2\alpha}$$

$$\overset{!}{=} L\left\{\frac{1}{2\alpha\sqrt{\alpha^2+1}} - \frac{\operatorname{arcsinh}\alpha}{2\alpha^2} + \frac{\alpha}{2\sqrt{\alpha^2+1}}\Big)\right\}$$

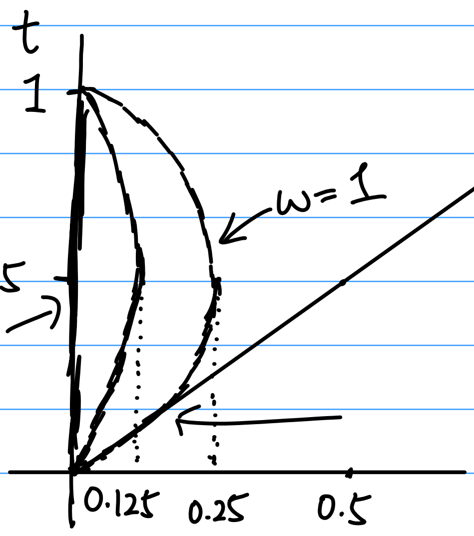
$$0 \overset{!}{=} L\left\{\frac{-\sqrt{\alpha^2+1}\operatorname{arcsinh}\alpha + \alpha^3 + \alpha}{2\alpha^2\sqrt{\alpha^2+1}}\right\}$$

so minimum when $\boxed{\alpha = 0.}$

By looking at the diagram from part i) no surprise that path length is minimize when we connect a straight line between $(x=0, y=0)$ and $(x=1, y=0)$

b) Now $\big(ct(u), x(u)\big) = \big(u, wu(1-u)\big)X$ with $|w|<1$

i)



Require $|w|<1$, because if $|w|>1$ then we would be over the world light of the photon, i.e. exceeding speed of light.

ii) Calculate proper time $\quad \left(\begin{array}{l} ct = u \\ x = wu(1-u) \end{array}\right)$

$$cT = \int ds$$

$$= \int \sqrt{(c\,dt)^2 - (dx)^2}$$

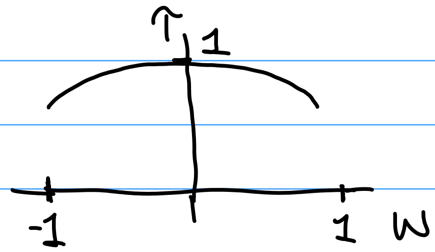
$$= \int \sqrt{\left(c\frac{dt}{du}\right)^2 - \left(\frac{dx}{du}\right)^2}\ du$$

$$= \int_0^1 \sqrt{1 - w^2(1-2u)^2}\ du$$

Mathematica

$$cT = \frac{w\sqrt{1-w^2} + \arcsin(w)}{2w}$$

the above function looks like



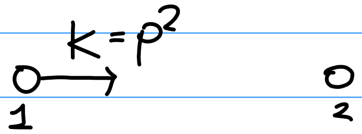
And since T is bounded by $|w| < 1$, we can simply find maximum by look at the diagram, and we see maximum at $\boxed{w=0}$

we see that $w=0$ corresponds to a straight line in space-time diagram.

It is related to the twin-paradox because then we know that the twin that moves inertially, i.e. $\alpha = 0$, will experience the maximum proper time, i.e. older. While the other twin, moving non-inertially, i.e. $\alpha \neq 0$, always experiences small proper time, i.e. younger.

5) Inelastic Collisions and decay:

a) i) before collision



Total kinetic energy : K since 2^{nd} particle at rest.

$E_1 = K + mc^2 \qquad E_2 = mc^2$

$E_{tot}^{before} = E_1 + E_2 = K + 2mc^2$

ii) Due to energy conservation, $E_{tot}^{before} = E_{tot}^{after} = K + 2mc^2$

iii) Find momentum before and after collision:

Before collision: $\quad p_1^2 = E_1^2 - m^2 \quad$ and $\quad E_1 = K + m$

so $\quad (p^1_{before})^2 = (K+m)^2 - m^2 = K^2 + 2Km$

so $\quad p^1_{before} = \sqrt{K^2 + 2Km}$

$p^2_{before} = 0$

$$\left.\begin{array}{l} p^1_{before} = \sqrt{K^2+2Km} \\ p^2_{before} = 0 \end{array}\right\} p^{tot}_{before} = \sqrt{K^2 + 2Km}$$

Due to conservation of momentum:

$$\boxed{P_{after} = P^{tot}_{before} = \sqrt{K^2 + 2Km}}$$

iv) Find mass after merge:

$$E_{after}^2 - P_{after}^2 = m_{after}^2$$

$$m_{after} = \sqrt{(K+2m)^2 - (K^2+2Km)}$$

$$= \sqrt{4Km + 4m^2 - 2Km}$$

$$m_{after} = \sqrt{4m^2 + 2Km} = \sqrt{4m^2 + 2\frac{Km}{c^2}}$$

v) if nonrelativistic $K \ll mc^2$

$$m_{after} = 2m\sqrt{1 + \frac{K}{2mc^2}}$$

$$\cong 2m\left(1 + \frac{K}{4mc^2} + \cdots\right)$$

first order correction.

$$\cong 2m$$ ← sum of two masses if K is very very small

If relativistic, $K \gg mc^2$

$$m_{after} = \sqrt{2}\sqrt{\frac{Km}{c^2}}\sqrt{1 + 2\frac{mc^2}{K}}$$

$$\cong \frac{\sqrt{2Km}}{c}\left(1 + \frac{mc^2}{K} + \cdots\right)$$

first order correction.

$$m_{after} \cong \frac{\sqrt{2Km}}{c}$$

upper bound, as $K \gg mc^2$, $\frac{mc^2}{K} \to 0$

b) For energy, we have: $E^2 - p^2c^2 = m^2c^4 \Rightarrow E^2 - p^2 = m^2$

For photon: $E = |p|$

Before: atom at rest $\rightarrow p = 0$

$$\hookrightarrow \begin{pmatrix} E \\ p \end{pmatrix}^{\text{atom}}_{\text{before}} = \begin{pmatrix} m \\ 0 \end{pmatrix}$$

After 

$$\begin{pmatrix} E \\ p \end{pmatrix}^{\text{atom}}_{\text{after}} = \begin{pmatrix} (m - \delta m)\cosh\beta \\ (m - \delta m)\sinh\beta \end{pmatrix}$$

let energy emitted by photon $= q$, and its momentum moving backwards, so $-q$

$$\begin{pmatrix} E \\ p \end{pmatrix}^{\text{photon}} = \begin{pmatrix} q \\ -q \end{pmatrix}$$

Due to energy conservation and momentum conservation.

$$① \quad m = (m - \delta m)\sinh\beta + q$$
$$② \quad 0 = (m - \delta m)\cosh\beta - q$$

$$① \quad (m - q)^2 = (m - \delta m)^2 \sinh^2\beta$$

$$② \quad q^2 = (m - \delta m)^2 \cosh^2\beta$$

② - ①: $(m-q)^2 - q^2 = (m-\delta m)^2\,\overset{1}{\cancel{(\cosh^2\beta - \sinh^2\beta)}}$

↳ $-2qm + m^2 = (m-\delta m)^2$

↳ $q = \dfrac{-2m\delta m + \delta m^2}{-2m}$

$$\boxed{q = \delta m\left(1 - \frac{\delta m}{2m}\right)}$$

c) before: $\longrightarrow$ $p = 3m_\pi \frac{c}{4}$

so $\begin{pmatrix} E \\ p \end{pmatrix} = \begin{pmatrix} m_\pi \cosh\beta \\ m_\pi \sinh\beta \end{pmatrix}$

After $\leftsquigarrow \quad \rightsquigarrow$

$\begin{pmatrix} E \\ p \end{pmatrix}_{right} = \begin{pmatrix} Q \\ Q \end{pmatrix} \qquad \begin{pmatrix} E \\ p \end{pmatrix}_{left} = \begin{pmatrix} q \\ -q \end{pmatrix}$

① $m_\pi \cosh\beta = q + Q$

② $m_\pi \sinh\beta = q - Q = \frac{3}{4} m_\pi c$

so $\sinh\beta = \frac{3}{4}$ or $\beta = \operatorname{arcsinh} \frac{3}{4}$

then $\cosh\left(\operatorname{arcsinh}\frac{3}{4}\right) = 1.25$

① + ② : $m_\pi(\cosh\beta + \sinh\beta) = 2Q$

$$Q = \frac{m_\pi}{2}(\cosh\beta + \sinh\beta)$$

$$= \frac{m_\pi}{2}(0.75 + 1.25)$$

$$\boxed{Q \stackrel{!}{=} m_\pi c^2}$$ ← put c^2 back

① − ② : $m_\pi(\cosh\beta - \sinh\beta) = 2q$

$$q = \frac{m_\pi}{2}(\cosh\beta - \sinh\beta)$$

$$\stackrel{!}{=} \frac{m_\pi}{2}(1.25 - 0.75)$$

$$\boxed{q \stackrel{!}{=} \frac{1}{4} m_\pi c^2}$$

d) Before pion at rest:

$$\begin{pmatrix} E \\ p \end{pmatrix}_\pi = \begin{pmatrix} m_\pi \cosh\beta \\ m_\pi \sinh\beta \end{pmatrix} = \begin{pmatrix} m_\pi \\ 0 \end{pmatrix}$$

After: m_u and neutrino. $\overset{\nu}{\longleftarrow}$ $\overset{u}{\longrightarrow}$

$$\begin{pmatrix} E \\ p \end{pmatrix}_{muon} = \begin{pmatrix} m_u \cosh\beta \\ m_u \sinh\beta \end{pmatrix}$$

$$\begin{pmatrix} E \\ p \end{pmatrix}_\nu = \begin{pmatrix} q \\ -q \end{pmatrix}$$

① $m_\pi = m_u \cosh\beta + q \Rightarrow (m_\pi - q)^2 = m_u^2 \cosh^2\beta$

② $0 = m_u \sinh\beta - q \Rightarrow q^2 = m_u^2 \sinh^2\beta$

then ① - ②: $(m_\pi - q)^2 - q^2 = m_u^2 \underbrace{(\cosh^2\beta - \sinh^2\beta)}_{=1}$

$\hookrightarrow m_\pi^2 - 2q m_\pi = m_u^2$

$\hookrightarrow q = \frac{1}{2} \frac{m_\pi^2 - m_u^2}{m_\pi}$

then $P_{muon} = m_u \sinh\beta = q = \frac{1}{2} \frac{m_\pi^2 - m_u^2}{m_\pi}$

add constant c back:

$$\boxed{P_{muon} = \frac{1}{2} \frac{m_\pi^2 - m_u^2}{m_\pi} c}$$

$$E_{muon} = m_\mu \cosh\beta = m_\pi - q$$

$$\stackrel{!}{=} m_\pi - \frac{1}{2}\frac{m_\pi^2 - m_\mu^2}{m_\pi}$$

$$E_{muon} \stackrel{!}{=} \frac{1}{2}\frac{m_\pi^2 + m_\mu^2}{m_\pi}$$

with constant c : $\boxed{E_{muon} = \frac{1}{2}\frac{m_\pi^2 + m_\mu^2}{m_\pi} c^2}$

Now $\frac{P_{muon}}{E_{muon}} = \frac{m_\mu \sinh\beta}{m_\mu \cosh\beta} = \tanh\beta = \frac{v}{c}$

$$\hookrightarrow \frac{\frac{1}{2}\frac{m_\pi^2 - m_\mu^2}{m_\pi}}{\frac{1}{2}\frac{m_\pi^2 + m_\mu^2}{m_\pi}} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} = \frac{v}{c}$$

So $\boxed{v = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} c}$

7) $$i\,\partial_t \Psi(\vec{r},t) = \Omega(-i\,\partial/\partial\vec{r})\,\Psi(\vec{r},t)$$

a) let $\Psi(\vec{r},t) = \alpha\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$

then $$i\partial_t\left(\alpha e^{i\vec{k}\cdot\vec{r}-i\omega t}\right) = \Omega\left(-i\,\partial/\partial\vec{r}\right)\alpha e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

↳ $$i(-i\omega)\alpha \cancel{e^{i\vec{k}\cdot\vec{r}-i\omega t}} = \Omega\left(-i(i\vec{k})\right)\cancel{\alpha e^{i\vec{k}\cdot\vec{r}-i\omega t}}$$

↳ $\boxed{\omega = \Omega(\vec{k})}$ ← dispersion relation

We see that plane-wave solution solves this equation with dispersion relation: $\omega = \Omega(\vec{k})$

Now using $\omega = \Omega(\vec{k})$:

$$\Psi(\vec{r},t) = \alpha\, e^{i\vec{k}\cdot\vec{r} - i\Omega(\vec{k})t}$$

With a superposition of different wave packets

$$\Psi(\vec{r},t) = \int d^3k\, g(k)\, e^{i\vec{k}\cdot\vec{r} - i\Omega(\vec{k})t}$$

then $$\Psi(\vec{r},t=0) = \Phi_0(\vec{r}) = \int d^3k\, g(k)\, e^{i\vec{k}\cdot\vec{r}}$$

so $g(k)$ is the fourier transform of $\Phi_0(\vec{r})$

Then do inverse Fourier Transform on $g(k)$:

$$g(k) = \int d^3r'\, e^{-i\vec{k}\cdot\vec{r}'}\,\Phi_0(\vec{r}') = \hat{\Phi}_0(k)$$

then $$\boxed{\begin{aligned}\Psi(\vec r,t) &= \int d^3k \int d^3r'\, e^{ik(\vec r-\vec r')}\,\Phi_0(\vec r')\, e^{-i\Omega(\vec k)t} \\ &= \int d^3k\ \hat\Phi_0(\vec k)\, e^{i\vec k\cdot\vec r - i\Omega(\vec k)t}\end{aligned}}$$

b) Since plane waves are concentrated near $\vec k = \bar k$

Taylor Expand: $\Omega(\bar k + (k-\bar k)) \cong \Omega(\bar k) + \underbrace{(k-\bar k)}_{\text{linearization}} \left.\frac{d\Omega}{dk}\right|_{k=\bar k} + \cdots$

$$= \Omega(\bar k) + (\vec k - \bar k)v_g \leftarrow \text{let } \left.\frac{d\Omega}{dk}\right|_{k=\bar k} = v_g$$

let $\hat\Phi_0(k) = \int d^3r'\, e^{-i\vec k\cdot\vec r'}\,\Phi_0(\vec r')$

then

$$\Psi(\vec r,t) = \int d^3k\ \hat\Phi_0(k)\ \exp\{i\vec k\cdot\vec r - i(\Omega(\bar k) + (\vec k-\bar k)v_g t)\}$$

$$= \int d^3k\ \hat\Phi_0(k)\exp\{i(\vec k - \bar k + \bar k)\cdot\vec r - i(\Omega(\bar k) + (\vec k - \bar k)v_g t)\}$$

$$= \underbrace{\exp\{i\bar k(\vec r - \tfrac{\Omega(\bar k)}{\bar k}t)\}}_{\substack{\text{plane-wave solution} \\ \text{which has phase-velocity}}}\ \underbrace{\exp\{-i\bar k\cdot(\vec r - v_g t)\}}_{\substack{\text{since wave move} \\ \text{with velocity, } v_g, \\ \vec r - v_g t \approx a, \\ \text{a constant phase.}}}\ \underbrace{\int d^3k\ \hat\Phi_0(k)\exp\{i\vec k\cdot(\vec r - v_g t)\}}_{=\Psi(\vec r - v_g t,\, t)}$$

$$\boxed{v_s = \frac{\Omega(\bar k)}{\bar k}}$$

envelope function, which is the origional $\Psi(\vec r,t)$ but modulated with group velocity

$$\boxed{v_g = \left.\frac{d\Omega(k)}{dk}\right|_{k=\bar k}}$$

we note that linearization holds when $\Omega(k)$ vary slowly in k, i.e. $\left.\frac{d\Omega(k)}{dk}\right|_{k=\bar k} \ll 1$

c) First expand $\Omega(k)$ to quadratic terms:

$$\Omega(k) = \Omega(\bar{k}) + (k-\bar{k})\underbrace{\frac{d\Omega}{dk}\Big|_{k=\bar{k}}}_{=v_g} + \frac{1}{2}(k-\bar{k})^2 \underbrace{\frac{d^2\Omega}{dk^2}\Big|_{k=\bar{k}}}_{=u}$$

$$\overset{!}{=} \Omega(\bar{k}) + (k-\bar{k})v_g + \frac{1}{2}(k-\bar{k})^2 u$$

then using results from part a):

$$\Psi(x,t) = \int dk\, \hat{\Phi}_0(k) \exp\left\{ikx - i\left[\Omega(\bar{k}) + (k-\bar{k})v_g + \tfrac{1}{2}(k-\bar{k})^2 u\right]t\right\}$$

$$\overset{!}{=} \int dk\, \hat{\Phi}_0(k) \exp\left\{i(k-\bar{k}+\bar{k})x - i\left[\Omega(\bar{k}) + (k-\bar{k})v_g + \tfrac{1}{2}(k-\bar{k})^2 u\right]t\right\}$$

$$= \exp\{i(\bar{k}x - \Omega(\bar{k})t)\} \int dk\, \Phi_0(k) \exp\{i(k-\bar{k})[x - v_g t]\} \exp\left\{\tfrac{-i}{2}(k-\bar{k})^2 ut\right\}$$

We know the initial condition to be a gaussian, $e^{-\frac{1}{2}\left(\frac{x}{w}\right)^2}$

then $$\Phi_0(k) = \int_{-\infty}^{\infty} dx\, e^{-i(k-\bar{k})x} e^{-\frac{1}{2}\left(\frac{x}{w}\right)^2}$$

$$\overset{!}{=} \sqrt{2\pi w^2}\, e^{-\frac{1}{2}w^2(k-\bar{k})^2}$$

$$\to = \sqrt{2\pi w^2} \exp\{i(\bar{k}x - \Omega(\bar{k})t)\} \underbrace{\int_{-\infty}^{\infty} dk\, \exp\{i(k-\bar{k})[x - v_g t]\} \exp\left\{-\tfrac{1}{2}(k-\bar{k})^2[w^2 + iut]\right\}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{w^2+iut}} \exp\left\{-\frac{1}{2}\frac{(x-v_g t)^2}{w^2+iut}\right\}$$

$$\boxed{\therefore \quad \Psi(x,t) = \sqrt{\frac{w^2}{w^2+iut}}\, \underbrace{\exp\{i\bar{k}x - i\Omega(\bar{k})t\}}_{\text{plane-wave}}\, \underbrace{\exp\left\{-\frac{1}{2}\frac{(x-v_g t)^2}{w^2+iut}\right\}}_{\text{envelope part.}}}$$

Examine the envelope function: $\exp\left\{ -\frac{1}{2}\frac{(x-V_g t)^2}{\omega^2 + i\alpha t} \right\}$

we see that at $t=0$, we recover the initial gaussian wave with width ω.

As t progresses, $x - V_g t$ where $V_g t \sim x_0(t)$, i.e. the center of the gaussian peak, changes. So it moves.
we see that the denominator which is analogous to the width of gaussian also increases. So the envelope moves and broadens as t increases.

To examine the width quantitatively, we can look at the modulus of Ψ.

$$|\Psi|^2 \cong \exp\left\{ -\frac{1}{2}(x-V_g t)^2 \left(\frac{1}{\omega^2 + i\alpha t} + \frac{1}{\omega^2 - i\alpha t} \right) \right\}$$

$$= \exp\left\{ -\frac{1}{2}(x-V_g t)^2 \left(\frac{\omega^2 - i\alpha t}{\omega^4 + (\alpha t)^2} + \frac{\omega^2 + i\alpha t}{\omega^4 + (\alpha t)^2} \right\}$$

$$= \exp\left\{ -\frac{1}{2}(x-V_g t)^2 \frac{2}{\omega^2 + \left(\frac{\alpha t}{\omega}\right)^2} \right\}$$

So we see $(\text{width})^2 \approx \frac{1}{2}\left[\omega^2 + \left(\frac{\alpha t}{\omega}\right)^2\right]$ ← So as t increases, the width increases.

For the initial gaussian wave, since $\Phi_0(x,t) = e^{-\frac{1}{2}\left(\frac{x}{\omega}\right)^2}$ but $\Phi_0(k,t) = e^{-\frac{1}{2}\omega^2 k^2}$ so the width is inverted. i.e. in real space width is ω, then in Fourier space width becomes $\frac{1}{\omega}$.

This means if ω is small, gaussian is narrow in real-space but broad in Fourier space. And vice-versa.