waves in simple matter: transmission and reflection

simple: linear, isotropic, locality, no freq-dependant
translational invariant except at interface.

$\vec{\nabla}\cdot\vec{D} = \rho_f = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
$\vec{\nabla}\times\vec{H} = \dot{\vec{D}} + \cancel{\vec{J}_f}$	$\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$

let $\vec{D} = \varepsilon\vec{E}$ and $H = \frac{1}{\mu}\vec{B}$

assume monochromatic waves : ω

Plane waves : Locality $\left(\begin{smallmatrix} z>0 \\ z<0 \end{smallmatrix}\right)$

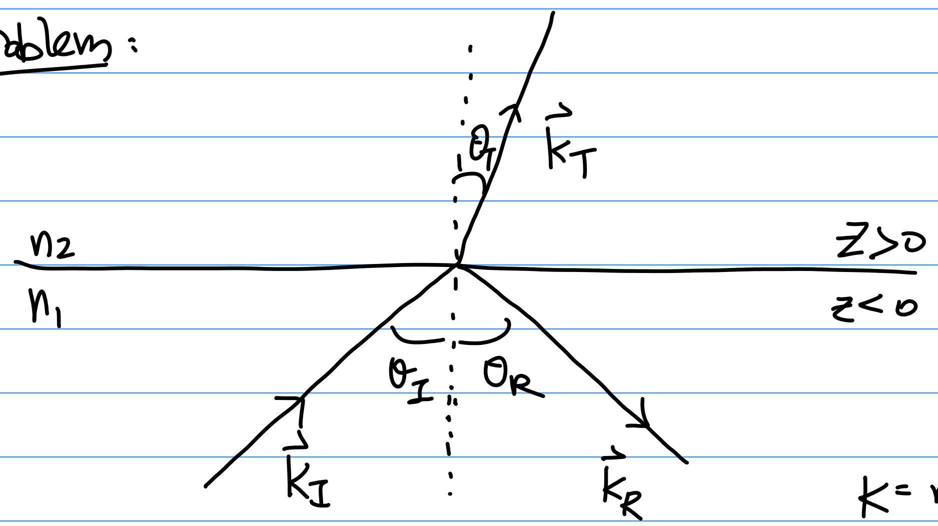
$$\vec{E}(\vec{r},t) = \underbrace{\vec{E}}_{\text{constant}} e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

$$\vec{H}(\vec{r},t) = \underbrace{\vec{H}}_{\text{constant.}} e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

then we have

$\vec{k}\cdot\vec{E} = 0$	$\vec{k}\cdot\vec{H} = 0$
$\vec{k}\times\vec{H} = -\omega\varepsilon\vec{E}$	$\vec{k}\times\vec{E} = \omega\mu\vec{H}$

problem:



$$k = n\frac{\omega}{c}$$

$$\vec{E}(\vec{r},t) \begin{cases} E_I e^{i\vec{k}_I\cdot\vec{r}-i\omega t} \\ E_R e^{i\vec{k}_R\cdot\vec{r}-i\omega t} \end{cases} \text{for } z<0 \Rightarrow \frac{\omega}{|k_I|} = \frac{c}{n_I} = \frac{\omega}{|k_R|}$$

$$E_T e^{i\vec{k}_T\cdot\vec{r}-i\omega t} \quad \text{for } z>0 \Rightarrow \frac{\omega}{|k_T|} = \frac{c}{n_2}$$

At interface:

$\hat{n}\cdot\vec{D}$ = continuous	$\hat{n}\cdot\vec{B}$ = continuous
$H_{\parallel}$ continuous	$D_{\parallel}$ continuous

At $z=0$, all phase should be the same.

1) $$k_I \cdot \vec{r}\big|_{z=0} = k_R \cdot \vec{r}\big|_{z=0} = k_T \cdot \vec{r}\big|_{z=0} = 0$$

2) $$k_I\big|_{\parallel} = k_R\big|_{\parallel} = k_T\big|_{\parallel}$$

Since all of them are co-planar, then choose axis so that there is no y-component, so $\parallel$ = x-axis

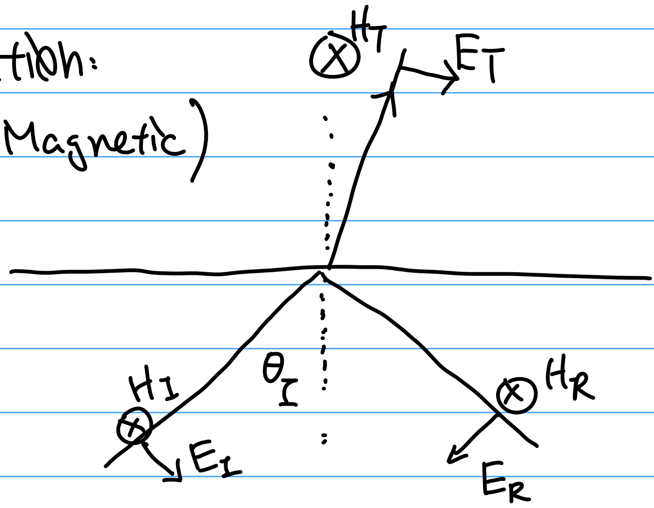
$$k_{I_\parallel} = \cancel{n_1 \frac{\omega}{c}} \sin\theta_I = k_{R_\parallel} = \cancel{n_1 \frac{\omega}{c}} \sin\theta_R$$

$$\hookrightarrow \boxed{\sin\theta_I = \sin\theta_R}$$

$$k_{I_\parallel} = n_1 \cancel{\frac{\omega}{c}} \sin\theta_I = k_{T_\parallel} = n_2 \cancel{\frac{\omega}{c}} \sin\theta_T$$

$$\hookrightarrow \boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$$

p- polarization:
(transverse - Magnetic)



$E_\parallel$ continuous: $E_I \cos\theta_I - E_R \cos\theta_R = E_T \cos\theta_T$

$H_\parallel$ continuous: $H_I + H_R = H_T$

$Z = \sqrt{\frac{\mu}{\varepsilon}}$

$$\hookrightarrow \frac{E_I}{Z_1} + \frac{E_R}{Z_1} = \frac{E_T}{Z_2}$$

$$\Rightarrow \frac{\frac{1}{E_I}}{\frac{1}{E_I}} \frac{E_I \cos\theta_1 - E_R \cos\theta_1}{\frac{E_I}{Z_1} + \frac{E_R}{Z_2}} = \frac{E_T \cos\theta_2}{E_T / Z_2}$$

let
$$\boxed{r_p = \frac{E_R}{E_I} = \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2}}$$

$$\boxed{t_p = \frac{E_T}{E_I} = \frac{2 Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2}}$$

Geometrical Optics from Maxwell Theory:

$\sim 10^{-7}$ m wavelength.

and aperture / lens size $\gg 10^{-7}$ m

↳ short wave approximation.

↳ geometrical optics decorated by diffraction and interference.

$$E(\vec{r},t) = E_0(\vec{r})\, e^{-i\omega t}$$
$$H(\vec{r},t) = H_0(\vec{r})\, e^{-i\omega t}$$

Intensity
phase
polarization.

In medium: $\varepsilon(\vec{r}),\ \mu(\vec{r}) \Rightarrow \boxed{n(\vec{r}) = \sqrt{\frac{\mu(\vec{r})}{\mu_0}\frac{\varepsilon(\vec{r})}{\varepsilon_0}}}$

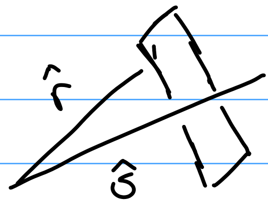
index of refraction

Impedence: $Z = \sqrt{\frac{\mu}{\varepsilon}}$

$$\frac{c}{n} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Consider no free charge: $\vec{E}(\vec{r},t) = E_0(\vec{r})e^{-i\omega t}$, $\vec{H}(\vec{r},t) = H_0 e^{-i\omega t}$

$\vec{\nabla}\cdot\vec{D} = \rho_f = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
$\hookrightarrow \vec{\nabla}\cdot(\varepsilon\vec{E}(\vec{r})) = 0$	$\vec{\nabla}\cdot(\mu\vec{H}_0) = 0$
$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \cancel{\vec{J}_f}$	$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$
$\hookrightarrow \vec{\nabla}\times\vec{H}_0 = -i\omega\varepsilon E_0$	$\vec{\nabla}\times\vec{E}_0 = +i\omega\mu H_0$



If medium is homogeneous: then μ, ε are constant.

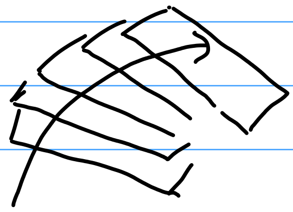
then $E_0(\vec{r}) = \vec{e}\exp\{ik_0 n(\hat{s}\cdot\vec{r})\}$ where $k_0 = \frac{\omega}{c}$

$\hookrightarrow$ unit-vector (propagation direction)

$H_0(\vec{r}) = \vec{h}\exp\{ik_0 n(\hat{s}\cdot\vec{r})\}$

$\hookrightarrow$ constant.

Now suppose medium is not homogeneous:



then $\vec{E}_0(\vec{r}) = e(\vec{r}) \exp(i k_0 S(\vec{r}))$

$\vec{H}_0(\vec{r}) = h(\vec{r}) \exp(i k_0 S(\vec{r}))$

then we have:

$\vec{e}\cdot\vec{\nabla}S = \frac{-1}{ik_0}\left[\vec{e}\cdot\vec{\nabla}\ln\varepsilon + \vec{\nabla}\cdot\vec{e}\right]$	$\vec{h}\cdot\vec{\nabla}S = \frac{-1}{ik_0}\left[\vec{h}\cdot\vec{\nabla}\ln\mu + \vec{\nabla}\cdot\vec{h}\right]$
$\vec{\nabla}S\times\vec{h} + \underbrace{\frac{\omega\varepsilon}{k_0}}_{\varepsilon c}\vec{e} = \frac{-1}{ik_0}\vec{\nabla}\times\vec{h}$	$\vec{\nabla}S\times\vec{e} - \underbrace{\frac{\omega_0\mu}{k_0}}_{\mu c}\vec{h} = \frac{-1}{ik_0}\vec{\nabla}\times\vec{e}$

$\Rightarrow$ Now due to short wavelength approximation, so k_0 = big. so ignore $\frac{1}{k_0}$ terms.

equations for optics

$\vec{e}\cdot\vec{\nabla}S = 0$	$\vec{h}\cdot\vec{\nabla}S = 0$
$\vec{\nabla}S\times\vec{h} + \varepsilon c\vec{e} = 0$	$\vec{\nabla}S\times\vec{e} - \mu c\vec{h} = 0$

These are redundant. we can take $(\vec{\nabla}S\cdot)$ on bottom two equations and get back first two

We can rewrite the bottom two equations as matrix:

$$\left(\begin{array}{c:c} 3\times3 & 3\times3 \\ \hdashline 3\times3 & 3\times3 \end{array}\right)\left(\begin{array}{c} e^x \\ e^y \\ e^z \\ \hdashline h^x \\ h^y \\ h^z \end{array}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\vec{\nabla} \cdot \vec{S}$

This has solution when it has no inverse.

i.e. we set determinant $= 0$ for that matrix

It turns out, det $= 0$ when $\boxed{|\vec{\nabla} S|^2 = n(\vec{r})^2}$

icond equation.

$$\hookrightarrow \boxed{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n(x,y,z)^2}$$

let's cross-check with plane-wave solution.

$$ik_0 S(\vec{r}) = ik_0 n \hat{s} \cdot \vec{r}$$

$$\hookrightarrow \vec{\nabla} S(\vec{r}) = n \hat{s}$$

$$|\vec{\nabla} S|^2 = n^2 \checkmark$$

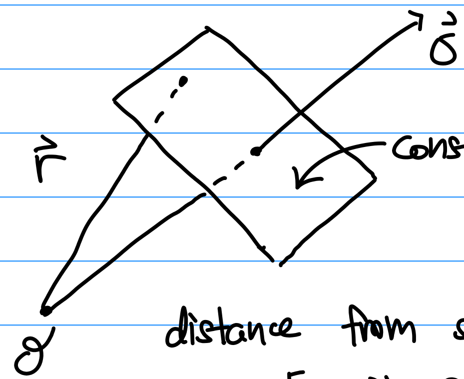
Recall that $n(\vec{r})$ influences S through the Eikonal equation

$$|\vec{\nabla} S|^2 = n(\vec{r})^2$$

For constant n:

$$k_0 S(\vec{r}) = k_0 n \vec{\sigma} \cdot \vec{r} + k_0 S(0)$$

then the surfaces of constant phase, $k_0 S(\vec{r}) = \text{constant}$ are infinite flat planes:



constant $k_0 S$ surface.

distance from surface $\vec{r} \cdot \vec{\sigma}$

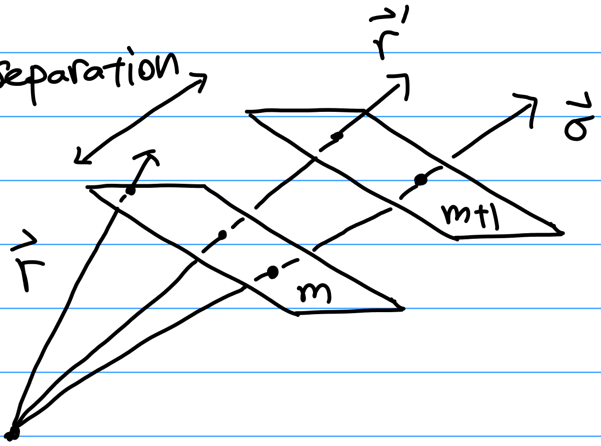
$= [S(\vec{r}) - S(0)]/n$

Planes on which differs by 2π:

$m = 0, \pm 1, \pm 2$

$$k_0 [S(\vec{r}\,') - S(\vec{r})] = k_0 n \vec{\sigma} \cdot (\vec{r}\,' - \vec{r}) = 2\pi m$$

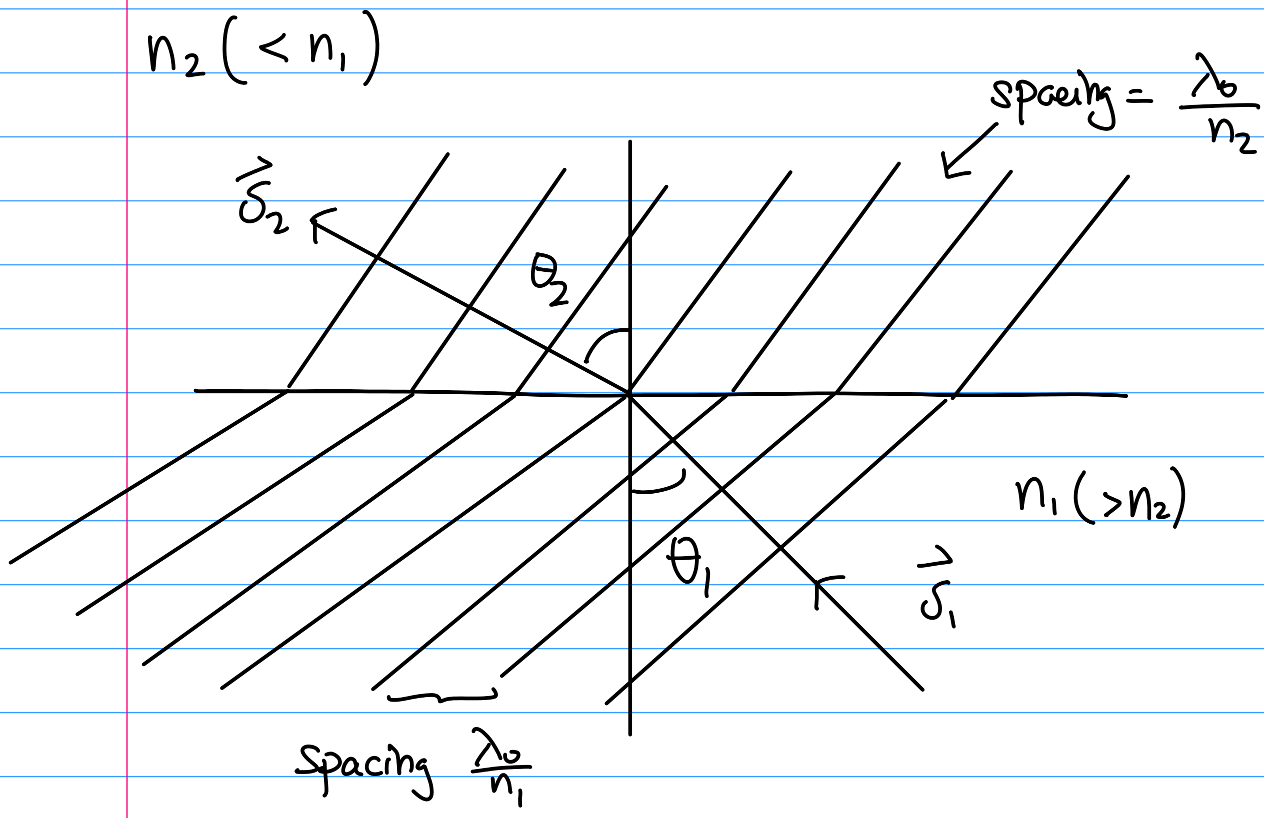
separation



$$\text{separation} = \frac{2\pi}{k_0 n} = \frac{\lambda_0}{n}$$

so they get closer if n increases

Flat interface between distinct Media



For the interface to be equal:

$$\frac{\lambda_0/n_2}{\sin\theta_2} = \text{Blue segment Length} = \frac{\lambda_0/n_1}{\sin\theta_1}$$

↳ $n_1 \sin\theta_1 = n_2 \sin\theta_2$ ← Snell's Law.

Now what is the correction terms when the wavelength is large, i.e. $\frac{1}{k_0}$ is large, then we have the RHS in maxwell equation.

use perturbation corrections:, in power of $\frac{1}{k_0}$

$$\left.\begin{aligned} E(\vec{r},t) &= e^{-i\omega t} e^{ik_0 S(\vec{r})} \sum_{n=0}^{\infty} \frac{1}{k_0^n} \vec{e}_n(\vec{r}) \\ H(\vec{r},t) &= e^{-i\omega t} e^{ik_0 S(\vec{r})} \sum_{n=0}^{\infty} \frac{1}{k_0^n} \vec{h}_n(\vec{r}) \end{aligned}\right\} \text{similar to WKB.}$$

<u>Light rays and the intensity law of geometrical optics:</u>

→ Electric energy: $\langle \tilde{\mathcal{E}}_e \rangle = \frac{1}{2}\varepsilon \langle |\vec{E}|^2 \rangle = \frac{1}{4}\varepsilon \vec{e}^* \cdot \vec{e}$

time average.

plug in e^*

↳ But $\vec{\nabla} S \times \vec{h}^* + \varepsilon c \vec{e}^* = 0$

so $$\boxed{\langle \tilde{\mathcal{E}}_e \rangle = \frac{1}{4}\varepsilon \vec{e}(-1)\frac{1}{\varepsilon c}(\vec{\nabla} S \times \vec{h}^*) = \frac{1}{4c}\vec{e} \cdot (\vec{h}^* \times \vec{\nabla} S)}$$

→ Magnetic energy: $\langle \tilde{\mathcal{E}}_m \rangle = \frac{1}{2}\mu \langle |\vec{H}|^2 \rangle = \frac{1}{4}\mu \vec{h}^* \cdot \vec{h}$

↳ but $\vec{\nabla} S \times \vec{e} - \mu c \vec{h} = 0$

so $$\boxed{\langle \tilde{\mathcal{E}}_m \rangle = \frac{1}{4}\mu \vec{h}^* \cdot \frac{1}{\mu c}(\vec{\nabla} S \times \vec{e}) = \frac{1}{4c}\vec{e} \cdot (\vec{h}^* \times \vec{\nabla} S)}$$

we note that $\langle \tilde{\mathcal{E}}_m \rangle = \langle \tilde{\mathcal{E}}_e \rangle = \frac{1}{2}\langle \tilde{\mathcal{E}} \rangle$, where $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_e + \tilde{\mathcal{E}}_m$

Time - averaged Poynting Vector $\langle \vec{S} \rangle$ (the energy flux)

$$\underset{\text{Poynting Vector}}{\langle \vec{S} \rangle} = \langle \vec{E} \times \vec{H} \rangle$$

$$\stackrel{!}{=} \langle \left[\tfrac{1}{2} \vec{e}(\vec{r}) e^{ik_0 S(\vec{r}) - i\omega t} + c.c. \right]$$

$$\times \left[\tfrac{1}{2} \vec{h}(\vec{r}) e^{ik_0 S(\vec{r}) - i\omega t} + c.c. \right] \rangle$$

$$\stackrel{!}{=} \tfrac{1}{4} \left[\vec{e} \times \vec{h}^* + \vec{e}^* \times \vec{h} \right]$$

eliminate using $[\vec{\nabla} S \times \vec{e} - \mu c \vec{h} = 0$ and its c.c.$]$

$$\stackrel{!}{=} \frac{1}{4} \frac{1}{\mu c} \left[\vec{e} \times (\vec{\nabla} S \times e^*) + c.c. \right]$$

$$\stackrel{!}{=} \frac{1}{4\mu c} \left[\left(\vec{\nabla} S \, \underbrace{(\vec{e} \cdot \vec{e}^*)} - e^* \underbrace{(\vec{e} \cdot \vec{\nabla} S)}_{=0} \right) + c.c. \right]$$

$$= \frac{2}{\varepsilon} \langle \tilde{\varepsilon} \rangle$$

$$\boxed{\underbrace{\langle \vec{S} \rangle}_{\text{time averaged energy flux}} = \underbrace{\frac{1}{n} \vec{\nabla} S}_{\text{unit-vector normal to } S\text{, ray tangent}} \underbrace{\langle \tilde{\varepsilon} \rangle}_{\text{energy density}} \underbrace{\frac{c}{n}}_{\text{speed of light in medium}}}$$

what equation does a ray obey?

$\vec{r}(\lambda)$: $\boxed{\frac{d\vec{r}}{d\lambda} = \frac{1}{n(\vec{r})}\vec{\nabla}S(\vec{r})}$

ray-trajectory

arclength

tangent vector

position dependent refractive index

if we know $n(\vec{r})$ and $S(\vec{r})$ and start at a point $\vec{r}_0$, then its evolution with "λ" is completely known.

Note that $\vec{e}(\vec{r})$ and $h(\vec{r})$ are orthogonal to $\dot{\vec{r}}$, since $\vec{e}$ and $\vec{h}$ are orthogonal to $\vec{\nabla}S$

$$\vec{e}\cdot\dot{\vec{r}} = 0 \qquad \text{and} \qquad \vec{h}\cdot\dot{\vec{r}} = 0$$

How much does S change when $\lambda \to \lambda + \delta\lambda$:

$$\delta S = S(\vec{r}(\lambda+\delta\lambda)) - S(\vec{r}(\lambda))$$

$$\stackrel{\downarrow}{=} S(\vec{r}(\lambda) + \delta\lambda\,\dot{\vec{r}}(\lambda)) - S(\vec{r}(\lambda))$$

$$\approx \delta\lambda\,\dot{\vec{r}}\cdot\vec{\nabla}S = \delta\lambda\, n(\vec{r})\,\cancel{|\dot{\vec{r}}|^2}$$

$$= \delta\lambda\left(\frac{c}{v}\right)$$

note $v = \frac{c}{n}$

Optical length of a ray:

consider a ray: $r(\lambda)$

then $\boxed{\text{optical length} = \int_{\lambda_1}^{\lambda_2} d\lambda \, n(\vec{r}(\lambda))}$

or $\quad c\delta t = \int d\lambda \; n(\vec{r}(\lambda))$

Intensity of the light: $\quad \boxed{I = |\langle \vec{S} \rangle|}$ ← Poynting Vector

Intensity of light

Introduce the unit-vector field $\vec{\sigma}(\vec{r})$

$$\boxed{\vec{\sigma}(\vec{r}) = \frac{1}{n} \vec{\nabla} S \quad \leftarrow \text{unit-vector field}}$$

which is the same as $\dot{\vec{r}}$

so $\vec{\sigma}(\vec{r}) = \dot{\vec{r}}$

so $\quad \langle \vec{S} \rangle = \vec{\sigma}(\vec{r}) \, \langle \tilde{\varepsilon} \rangle (\vec{r}) \; v(\vec{r})$

energy density

speed of light in vacuum $\quad v = \frac{c}{n}$

so $\quad \boxed{I = \langle \tilde{\varepsilon} \rangle \, v}$

or $\quad \boxed{\langle \vec{S} \rangle = \vec{\sigma}(\vec{r}) \, I}$

By conservation of energy: $\vec{\nabla}\cdot\langle\vec{S}\rangle = 0$

so $\quad \hookrightarrow \vec{\nabla}\cdot(I\vec{s}) = 0$

$\hookrightarrow \int d^3r\, \vec{\nabla}\cdot\left(I(\vec{r})\,\vec{s}(\vec{r})\right) = 0$

$\hookrightarrow \int d^2s\; I(\vec{r})\,\vec{s}(\vec{r}) = 0$

or $\quad \boxed{I_1\,\delta S_1 = I_2\,\delta S_2}$

i.e. if $\delta S_2 > \delta S_1$, then area increases, so intensity drops.
vice-versa.

How does intensity vary along $r(\lambda)$?

know $\quad \vec{s} = \frac{1}{n}\vec{\nabla}\mathcal{S}$

with conservation laws $\quad \vec{\nabla}\cdot\langle\vec{S}\rangle = \vec{\nabla}\cdot(I\vec{s}) = 0$

so $\quad \vec{\nabla}\cdot\left(\frac{I}{n}\vec{\nabla}\mathcal{S}\right) = 0$

$\hookrightarrow \vec{\nabla}\frac{I}{n}\cdot\vec{\nabla}\mathcal{S} + \frac{I}{n}\nabla^2\mathcal{S} = 0$

divide by I/n

$\hookrightarrow \vec{\nabla}\ln\left(\frac{I}{n}\right)\cdot\vec{\nabla}\mathcal{S} = -\nabla^2\mathcal{S}$

use $\vec{\nabla}\mathcal{S} = \dot{\vec{r}}\,n$

$\hookrightarrow n\,\dot{\vec{r}}\cdot\vec{\nabla}\ln\left(\frac{I}{n}\right) = -\nabla^2\mathcal{S}$

$\hookrightarrow \frac{d}{d\lambda}\ln\frac{I(\vec{r}(\lambda))}{n(\vec{r}(\lambda))} = -\frac{1}{n}\nabla^2\mathcal{S}\Big|_{\vec{r}=\vec{r}(\lambda)}$

$*\quad \hookrightarrow \boxed{\frac{I_2}{n_2} = \frac{I_1}{n_1}\exp\left\{-\int_{\lambda_1}^{\lambda_2} d\lambda\,\frac{1}{n}\nabla^2\mathcal{S}\Big|_{\vec{r}=\vec{r}(\lambda)}\right\}}$

how light ray propagates.