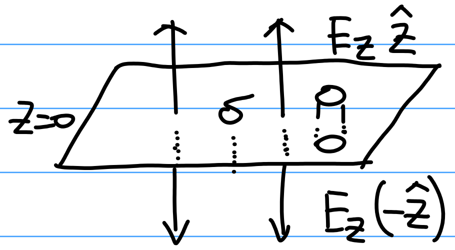
1) Sheets of charges and dipoles, with/without nearby dielectric slab.

a)



$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\hookrightarrow \int d^3r\, \vec{\nabla}\cdot\vec{E} = \int d^3r\, \rho/\varepsilon_0$$

$$\hookrightarrow \oint d^2S\,\hat{n}\cdot\vec{E} = \int d^2S\,\frac{\sigma}{\varepsilon_0}$$

By geometry, we expect $\vec{E}$ to be in the direction normal to the plane.

so
$$\vec{E} = \begin{cases} E_0\hat{z} & \text{for } z>0 \\ E_0(-\hat{z}) & \text{for } z<0 \end{cases}$$

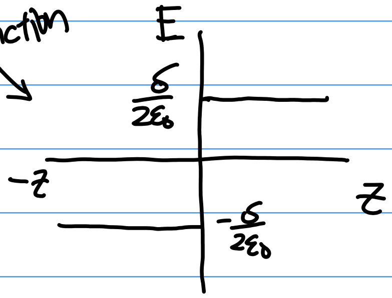
$$\hookrightarrow \oint d^2S\,\hat{n}\cdot\vec{E} = \int d^2S\,\frac{\sigma}{\varepsilon_0}$$

$$\hookrightarrow \int d^2S\left\{\underbrace{(+\hat{z})\cdot E_0\hat{z}}_{\text{upper plane}} + \underbrace{(-\hat{z})\cdot E_0(-\hat{z})}_{\text{lower plane.}}\right\} = \int d^2S\,\frac{\sigma}{\varepsilon_0}$$

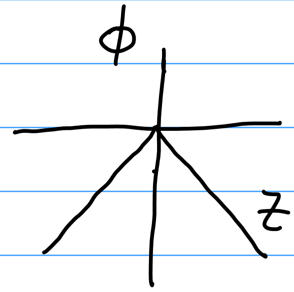
$$\hookrightarrow E_0 = \frac{\sigma}{2\varepsilon_0}$$

then
$$\boxed{\vec{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{z} & \text{for } z>0 \\ \frac{\sigma}{2\varepsilon_0}(-\hat{z}) & \text{for } z<0 \end{cases}}$$

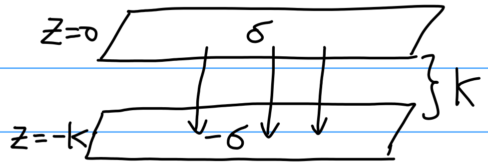
step function



then
$$\boxed{\phi = -\int_0^z \vec{E}\cdot d\vec{L} = \begin{cases} -\int_0^z \frac{\sigma}{2\varepsilon_0}\,dz' = -\frac{\sigma}{2\varepsilon_0}z & \text{for } z>0 \\ -\int_0^z -\frac{\sigma}{2\varepsilon_0}\,dz' = \frac{\sigma}{2\varepsilon_0}z & \text{for } z<0 \end{cases}}$$



b) i)



let sheet with $+\sigma$ at $z=0$
let sheet with $-\sigma$ at $z=-k$

We found $\vec{E}$ and ϕ of $+\sigma$ at $z=0$ from part a:

Then we can already infer $\vec{E}$ and ϕ of $-\sigma$ at $z=-k$:

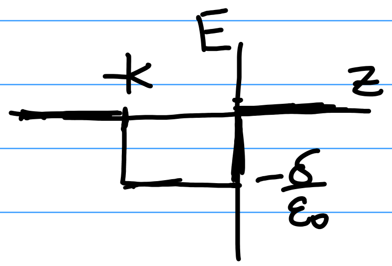
$$\vec{E}^{(-)} = \begin{cases} \frac{-\sigma}{2\varepsilon_0}\hat{z} & \text{for } z > -k \\ \frac{\sigma}{2\varepsilon_0}\hat{z} & \text{for } z < -k \end{cases}$$

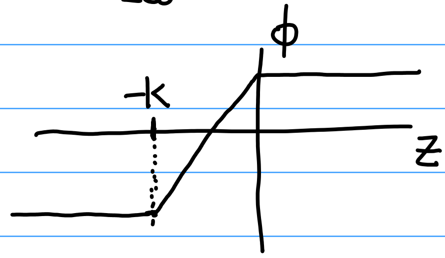
$$\phi^{(-)} = -\int_{-k}^{z} \vec{E}\cdot d\vec{L} = \begin{cases} -\int_{-k}^{z} -\frac{\sigma}{2\varepsilon_0}\,dz' = \frac{\sigma}{2\varepsilon_0}(z+k) & \text{for } z > -k \\ -\int_{-k}^{z} \frac{\sigma}{2\varepsilon_0}\,dz' = -\frac{\sigma}{2\varepsilon_0}(z+k) & \text{for } z < -k \end{cases}$$

By superposition:

$$\vec{E}^{tot} = \vec{E}^{(+)} + \vec{E}^{(-)} = \begin{cases} \left(\frac{\sigma}{2\varepsilon_0} + -\frac{\sigma}{2\varepsilon_0}\right)\hat{z} = 0 & \text{for } z > 0 \\ \left(\frac{-\sigma}{2\varepsilon_0} + -\frac{\sigma}{2\varepsilon_0}\right)\hat{z} = -\frac{\sigma}{\varepsilon_0}\hat{z} & \text{for } 0 > z > -k \\ \left(\frac{-\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0}\right)\hat{z} = 0 & \text{for } z < -k \end{cases}$$

$$\phi^{tot} = \phi^{(+)} + \phi^{(-)} = \begin{cases} \frac{-\sigma}{2\varepsilon_0}z + \frac{\sigma}{2\varepsilon_0}(z+k) = \frac{\sigma}{2\varepsilon_0}k & \text{for } z > 0 \\ \frac{\sigma}{2\varepsilon_0}z + \frac{\sigma}{2\varepsilon_0}(z+k) = \frac{\sigma}{\varepsilon_0}z + \frac{\sigma}{2\varepsilon_0}k & \text{for } 0 > z > -k \\ \frac{\sigma}{2\varepsilon_0}z + -\frac{\sigma}{2\varepsilon_0}(z+k) = -\frac{\sigma}{2\varepsilon_0}k & \text{for } z < -k \end{cases}$$





ii) $(\sigma, k) \to \lim_{\eta\to 0} (\sigma/\eta, k\eta)$

From part b) we have results for ϕ^{tot}, we just need to replace $k \to \lim_{\eta\to 0} k\eta$ and $\sigma \to \lim_{\eta\to 0} \sigma/\eta$.

$\Rightarrow$ For ϕ, region in between, $\phi = \lim_{\eta\to 0} \frac{1}{\varepsilon_0}\frac{\sigma}{\eta}\left(z + \frac{k\eta}{2}\right)$ for $0 > z > -k\eta$

Here we note that z takes value from $-k\eta$ to 0, i.e. negative.

So $\lim_{\eta\to 0} \phi(z = -k\eta) = -\frac{1}{\varepsilon_0}\frac{\sigma}{\eta}\frac{k\eta}{2} = -\frac{\sigma k}{2\varepsilon_0}$

and $\lim_{\eta\to 0} \phi(z=0) = \frac{1}{\varepsilon_0}\frac{\sigma}{\eta}\frac{k\eta}{2} = \frac{\sigma k}{2\varepsilon_0}$

And for the region outside of the sheet, $\lim_{\eta\to 0} \pm \frac{\sigma/\eta}{2\varepsilon_0} k\eta \to \pm \frac{\sigma k}{2\varepsilon_0}$

So ϕ increases steeply from $z=-k\eta$ to $z=0$. As $\lim_{\eta\to 0} z = -k\eta = 0$

so ϕ is like a step function.

As $\frac{\sigma}{2\varepsilon_0} k \to \lim_{\eta\to 0} \frac{1}{2\varepsilon_0}\frac{\sigma}{\eta} k\eta = \frac{\sigma k}{2\varepsilon_0}$

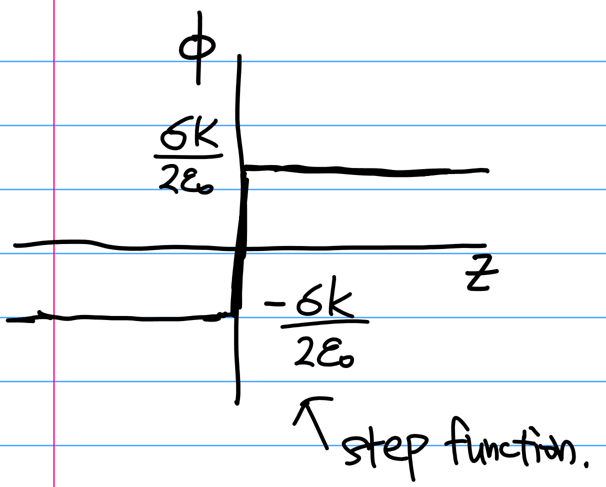
$$\phi^{tot} = \begin{cases} \lim_{\eta\to 0} \frac{1}{2\varepsilon_0}\left(\frac{-\sigma}{\eta} z + \frac{\sigma}{\eta}(z + k\eta)\right) = \frac{\sigma k}{2\varepsilon_0} & \text{for } z > 0 \\ \lim_{\eta\to 0} \frac{1}{2\varepsilon_0}\left(\frac{\sigma}{\eta} z - \frac{\sigma}{\eta}(z + k\eta)\right) = -\frac{\sigma k}{2\varepsilon_0} & \text{for } z < 0 \end{cases}$$

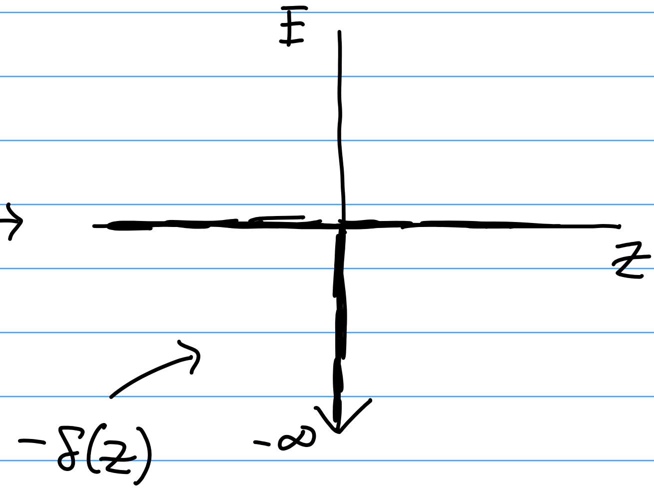
or $\boxed{\phi^{tot} = \frac{\sigma k}{2\varepsilon_0}\,\mathrm{sign}(z) \quad \text{where} \quad \begin{matrix} \mathrm{sign}(z) = +1 \text{ for } z>0 \\ \mathrm{sign}(z) = -1 \text{ for } z<0. \end{matrix}}$

By know ϕ^{tot}, $\vec{E} = -\vec{\nabla}\phi^{tot} = -\partial_z\left\{\frac{\sigma k}{2\varepsilon_0}\,\mathrm{sign}(z)\right\}$

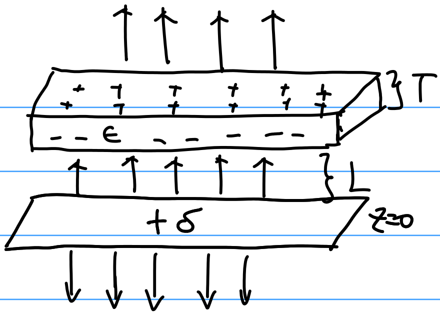
$$\boxed{\vec{E} \overset{!}{=} -\frac{\sigma k}{2\varepsilon_0}\,\delta(z)}$$

Here we use that $\partial_z\,\mathrm{sign}(z) = \delta(z)$





c)



Determine and sketch, $\vec{E}, \vec{D}, \vec{P}, \sigma_p$

$$\vec{\nabla}\cdot\vec{D} = \rho_{ext}$$

we know for $z \neq 0$, $\rho_{ext} = 0$

by symmetry ϕ is only in z-direction, so $\vec{\nabla}\cdot\vec{D} = -\varepsilon \nabla^2\phi = 0$

$\hookrightarrow \phi^{slab} = Az + B$, so E is constant in each region.

For $z < L$: $\phi^{slab} = Az + B$
$L < z < T+L$: $\phi^{slab} = Cz + D$
$z > T+L$: $\phi^{slab} = Ez + F$

Since $\vec{E}$ is constant in each region, so $\vec{\nabla}\cdot\vec{E} = 0$ in each region.

then $\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P}) = \varepsilon_0 \cancel{\vec{\nabla}\cdot\vec{E}} + \vec{\nabla}\cdot\vec{P} = 0$

so $\vec{\nabla}\cdot\vec{P} = \rho_P = 0$, this means induced charge all go to boundary.
Then we simply have a capacitor in the slab where

$\sigma_P = \hat{n}\cdot\vec{P} = -\sigma_p$ at $z = L$ and $+\sigma_p$ at $z = T+L$.
From part b) we see two infinite sheets with opposite charge only give nonzero $\vec{E}$-field between the sheets.

$\rightarrow$ So: $z < L$: $\phi^{slab} = \cancel{Az} + B = B$
$L < z < T+L$: $\phi^{slab} = Cz + D = Cz + D$
$z > T+L$: $\phi^{slab} = \cancel{Ez} + F = F$

We also know from part a) $\phi^{sheet} = \begin{cases} -\frac{\sigma}{2\varepsilon_0} z & \text{for } z>0 \\ \frac{\sigma}{2\varepsilon_0} z & \text{for } z<0. \end{cases}$

So by superposition, $\phi^{tot} = \phi^{slab} + \phi^{sheet}$.

$$z<L \begin{cases} z<0: & \phi^{tot} = B + \frac{\sigma}{2\varepsilon_0} z \\ 0<z<L: & \phi^{tot} = B - \frac{\sigma}{2\varepsilon_0} z \end{cases}$$

$$L<z<T+L: \quad \phi^{tot} = Cz + D - \frac{\sigma}{2\varepsilon_0} z = \left(C - \frac{\sigma}{2\varepsilon_0}\right) z + D$$

$$z>T+L: \quad \phi^{tot} = F - \frac{\sigma}{2\varepsilon_0} z$$

Now use Boundary Condition $D_\perp^{I} - D_\perp^{II} = \sigma_{ext}$

$$\hookrightarrow -\epsilon^{I} \frac{\partial \phi^{I}}{\partial n} + \epsilon^{II} \frac{\partial \phi^{II}}{\partial n} = \sigma_{ext}$$

At $z=L$: $$-\frac{\sigma}{2\varepsilon_0}\varepsilon_0 = \varepsilon\left(C - \frac{\sigma}{2\varepsilon_0}\right)$$

$$C = -\frac{\sigma}{2\varepsilon_0}\frac{\varepsilon_0}{\varepsilon} + \frac{\sigma}{2\varepsilon_0}$$

$$\boxed{C = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)}$$

Then:

$$L<z<T+L: \quad \phi^{tot} = \left(C - \frac{\sigma}{2\varepsilon_0}\right) z + D = -\frac{\sigma}{2\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right) z + D$$

Also use condition $\phi^{I} = \phi^{II}$

at $z = L$: $B - \frac{\sigma}{2\varepsilon_0} L = -\frac{\sigma}{2\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right) L + D$

$$\boxed{B = \frac{\sigma}{2\varepsilon_0} L \left(1 - \frac{\varepsilon_0}{\varepsilon}\right) + D}$$

at $z = T+L$: $F - \frac{\sigma}{2\varepsilon_0}(T+L) = -\frac{\sigma}{2\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)(T+L) + D$

$$\boxed{F = \frac{\sigma}{2\varepsilon_0}(T+L)\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) + D}$$

Lastly, we observe an overall constant D, which we set to 0

$$\boxed{D = 0}$$

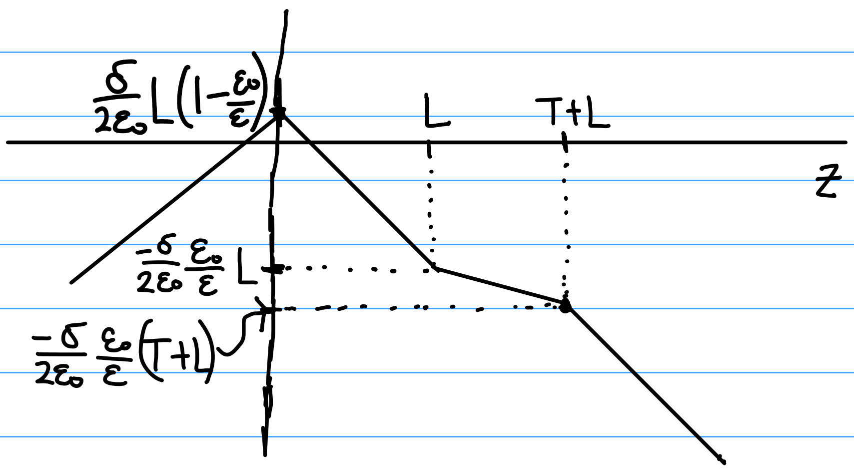
Finally:

$z < 0$: $\phi^{tot} = \frac{\sigma}{2\varepsilon_0}\left(z + L\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\right)$

$0 < z < L$: $\phi^{tot} = -\frac{\sigma}{2\varepsilon_0}\left(z - L\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\right)$

$L < z < T+L$: $\phi^{tot} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right) z$

$z > T+L$: $\phi^{tot} = -\frac{\sigma}{2\varepsilon_0}\left(z - (T+L)\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\right)$



And $\vec{E} = -\vec{\nabla}\phi = -\partial_z \phi$

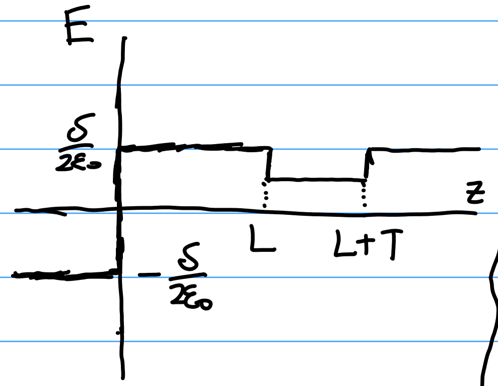
So

$z<0$: $E^{tot} = -\frac{\sigma}{2\varepsilon_0}\hat{z}$

$0<z<L$: $E^{tot} = \frac{\sigma}{2\varepsilon_0}\hat{z}$

$L<z<T+L$: $E^{tot} = \frac{\sigma}{2\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)\hat{z}$

$z>T+L$: $E^{tot} = \frac{\sigma}{2\varepsilon_0}\hat{z}$



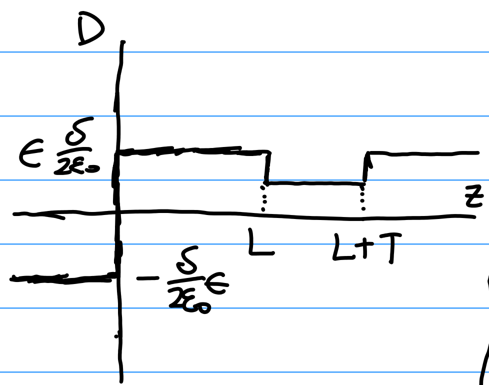
$\vec{D} = \varepsilon\vec{E}$:

$z<0$: $D^{tot} = -\frac{\sigma}{2}\hat{z}$

$0<z<L$: $D^{tot} = \frac{\sigma}{2}\hat{z}$

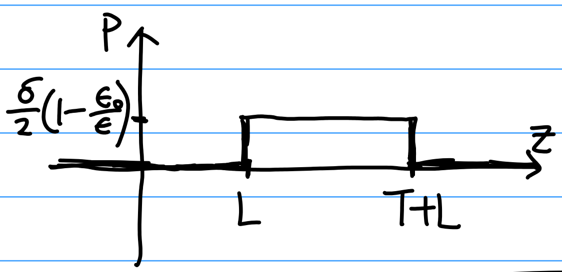
$L<z<T+L$: $D^{tot} = \frac{\sigma}{2}\hat{z}$

$z>T+L$: $D^{tot} = \frac{\sigma}{2}\hat{z}$



$\vec{P} = \vec{D} - \epsilon_0\vec{E}$, So $\vec{P} = 0$ everywhere except in dielectric:

For $L<z<T+L$, $\vec{P} = (\varepsilon - \varepsilon_0)\frac{\sigma}{2\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)\hat{z} = \frac{\sigma}{2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{z}$



Since $\vec{P} = \frac{\sigma}{2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{z}$, $\boxed{-\vec{\nabla}\cdot\vec{P} = \rho_p = 0}$

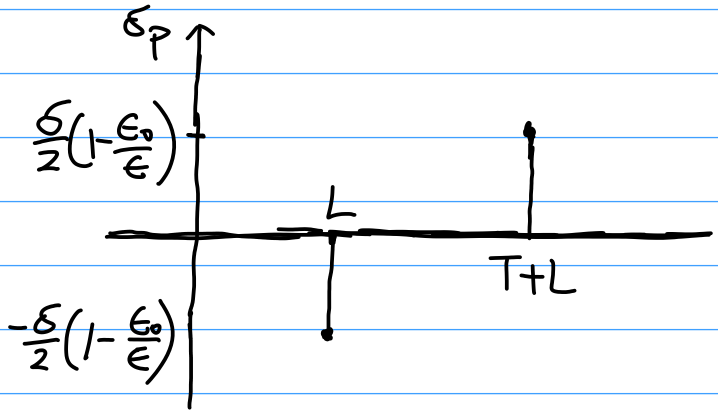
$$\sigma_p = \hat{n}\cdot\vec{P}$$

For $z = L$, $\hat{n} = -\hat{z}$,

$$\boxed{\sigma_p(z=L) = -\hat{z}\cdot\frac{\sigma}{2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right) = -\frac{\sigma}{2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)}$$

For $z = T+L$, $\hat{n} = +\hat{z}$

$$\boxed{\sigma_p(z=T+L) = \hat{z}\cdot\frac{\sigma}{2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\hat{z} = \frac{\sigma}{2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)}$$



$\Rightarrow$ If vacuum $\varepsilon/\varepsilon_0 \to 1$, then we go back to the case in part a) where we only have infinite sheet with $+\sigma$

Vacuum

$\phi = \begin{cases} \frac{\sigma}{2\varepsilon_0}z & \text{for } z<0 \\ -\frac{\sigma}{2\varepsilon_0}z & \text{for } z>0 \end{cases}$	$\vec{D} = \begin{cases} -\frac{\sigma}{2}\hat{z} & \text{for } z<0 \\ \frac{\sigma}{2}\hat{z} & \text{for } z>0 \end{cases}$
$\vec{E} = \begin{cases} -\frac{\sigma}{2\varepsilon_0}\hat{z} & \text{for } z<0 \\ \frac{\sigma}{2\varepsilon_0}\hat{z} & \text{for } z>0 \end{cases}$	$\vec{P} = \rho_p = \sigma_p = 0$

If <u>conductor</u>, $\frac{\varepsilon}{\varepsilon_0} \to \infty$, and $\frac{\varepsilon_0}{\varepsilon} \to 0$, then

$z<0$: $\phi^{tot} = \frac{\delta}{2\varepsilon_0}(z+L) \to \vec{E}^{tot} = -\frac{\delta}{2\varepsilon_0}\hat{z}$, $\vec{D}^{tot} = -\frac{\delta}{2}\hat{z}$

$0<z<L$: $\phi^{tot} = -\frac{\delta}{2\varepsilon_0}(z-L) \to \vec{E}^{tot} = \frac{\delta}{2\varepsilon_0}\hat{z}$, $\vec{D}^{tot} = \frac{\delta}{2}\hat{z}$

$L<z<T+L$: $\phi^{tot} = 0 \to \vec{E}^{tot} = 0\hat{z}$, $\vec{D}^{tot} = \frac{\delta}{2}\hat{z}$

$z>T+L$: $\phi^{tot} = -\frac{\delta}{2\varepsilon_0}(z-(T+L)) \to \vec{E}^{tot} = \frac{\delta}{2\varepsilon_0}\hat{z}$, $\vec{D}^{tot} = \frac{\delta}{2}\hat{z}$

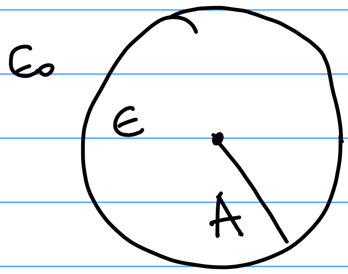
$$\vec{P} \begin{cases} = \frac{\delta}{2}\left(1-\cancel{\frac{\varepsilon_0}{\varepsilon}}\right)\hat{z} = \frac{\delta}{2}\hat{z} & \text{for } L<z<T+L \\ = 0 & \text{Elsewhere} \end{cases}$$

$\delta_p(z=L) = -\frac{\delta}{2}$ $\qquad$ $\delta_p(z=T+L) = \frac{\delta}{2}$

$\rho_p = 0$

Basically we recover the condition that $E=0$ inside conductor since the polarization completely canceled the external field caused by ∞ sheet with $+\delta$.

3) Dielectric Media in an uniform external $\vec{E}$-field:



$\vec{E}^{(0)} = E_z \hat{z}$ $\longrightarrow$

B.C.: $D_\perp^{I} = D_\perp^{II}$, $E_\parallel^{I} = E_\parallel^{II}$

$\vec{\nabla} \cdot \vec{D} = \rho_{ext} = 0$

b) i) Find ϕ, $\vec{E}$, $\vec{P}$ around / in sphere, $r \leq A$.

we know $\vec{\nabla} \cdot \vec{D} = \rho_{ext} = 0$ since no excess charge.

Homogeneous sphere $\rightarrow$ $\varepsilon(\vec{r}) \rightarrow \epsilon = $ const.

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\varepsilon \vec{E}) = \varepsilon \vec{\nabla} \cdot \vec{E} = -\varepsilon \nabla^2 \phi = 0$$

Know $\nabla^2 \phi = 0$ has solution

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(C_{ml} r^{l} + B_{ml} r^{-l-1} \right) Y_l^m(\theta, \phi)$$

we also expect $\boxed{m=0}$ since problem is symmetric in azimuth direction.

So $$\phi = \sum_{l=0}^{\infty} \left(C_l r^{l} + B_l r^{-l-1} \right) Y_l^0(\theta, \phi)$$

choose $\vec{E}^{(0)} = E_0 \hat{z}$,

then $\phi^{(0)} = -\int_0^z \vec{E} \cdot \vec{dL} = -E_0 z = -E_0 r\cos\theta$

$$\phi^{(0)}(r,\theta) = -E_0 r\cos\theta$$

For $r > A$ (outside sphere):

as $r \to \infty$: $\phi \to \phi^{(0)} = -E_0 r\cos\theta$

since $\lim_{r\to\infty} \phi(r) = \lim_{r\to\infty} (C_l r^l + \cancel{B_l r^{-l-1}}) Y_l^0 = -E_0 r\cos\theta$

$$\stackrel{!}{=} C_l r^l Y_l^0 = -E_0 r\cos\theta$$

by matching $\boxed{l = 1}$, and $Y_1^0 \sim \cos\theta$

$\Rightarrow$ so $\phi^{out}(r) = -E_0 r\cos\theta + B r^{-2} \cos\theta$ for $r > A$

For $r < A$ inside sphere: we don't expect ϕ diverge as $r \to 0$

$\hookrightarrow$ $\lim_{r\to 0} \phi = \lim_{r\to 0} \sum_{l=0}^{\infty} (C_l r^l + B_l r^{-l-1}) Y_l^0(\theta,\phi) = \text{finite}$

So $\boxed{B_l = 0}$ for $r < A$

$\Rightarrow$ $\phi^{in}(r) = \sum_{l=0}^{\infty} C_l r^l Y_l^0(\theta,\phi)$ for $r < A$

$\rightarrow$ Due to continuity of Φ, $\phi_{in}(r=A) = \phi_{out}(r=A)$

$\hookrightarrow \phi_{out}(A) = -E_0 A\cos\theta + BA^{-2}\cos\theta = (-E_0 A + BA^{-2})\cos\theta$

$\hookrightarrow \Phi_{in}(A) = \sum_{l=0}^{\infty} C_l A^l Y_l^0(\theta,\phi)$

by matching $\phi_{out}(A) = \phi_{in}(A)$, $\boxed{l=1}$ for ϕ_{in}

then $CA = -E_0 A + BA^{-2}$

or $\boxed{C = -E_0 + BA^{-3}}$

$\Rightarrow$ then $\phi_{out}(r) = (-E_0 r + Br^{-2})\cos\theta \qquad r > A$

$\phi_{in}(r) = (-E_0 + BA^{-3})\, r\cos\theta \qquad r < A$

We also know $D_\perp^{I} = D_\perp^{II} \rightarrow D_\perp^{in} = D_\perp^{out}$

$$E_\perp^{in} = -\hat{n}\cdot\vec{\nabla}\phi^{in} = -\hat{r}\cdot\vec{\nabla}\phi^{in} = -\partial_r\phi^{in} = (E_0 - BA^{-3})\cos\theta$$

$$E_\perp^{out} = -\hat{n}\cdot\vec{\nabla}\phi^{out} = -\hat{r}\cdot\vec{\nabla}\phi^{out} = -\partial_r\phi^{out} = -(-E_0 - 2Br^{-3})\cos\theta$$

$$D_\perp^{in} = D_\perp^{out}$$

$$\varepsilon E_\perp^{in}\Big|_{r=A} = \varepsilon_0 E_\perp^{out}\Big|_{r=A}$$

$\hookrightarrow \varepsilon(E_0 - BA^{-3})\cancel{\cos\theta} = \varepsilon_0(E_0 + 2BA^{-3})\cancel{\cos\theta}$

$$(\varepsilon - \varepsilon_0)E_0 = (2\varepsilon_0 + \varepsilon)A^{-3}B$$

$$\boxed{B = \left(\frac{\varepsilon - \varepsilon_0}{2\varepsilon_0 + \varepsilon}\right)E_0 A^3 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0}\right)E_0 A^3}$$

then
$$\boxed{\begin{aligned}\phi_{out}(r) &= E_0 r\cos\theta\left(-1+\left(\frac{\varepsilon/\varepsilon_0-1}{2+\varepsilon/\varepsilon_0}\right)\left(\frac{A}{r}\right)^3\right) && r>A\\ \phi_{in}(r) &= E_0 r\cos\theta\left(-1+\frac{\varepsilon/\varepsilon_0-1}{2+\varepsilon/\varepsilon_0}\right) && r<A\\ &= -E_0 r\cos\theta\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right) && r<A\end{aligned}}$$

Now find $\vec{E} = -\vec{\nabla}\phi$

$$\vec{E}^{in} = -\vec{\nabla}\phi^{in} = -\left(\partial_r\phi\,\hat{r}+\frac{1}{r}\partial_\theta\phi\,\hat{\theta}\right)$$

$$= E_0\cos\theta\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)\hat{r} - E_0\sin\theta\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)\hat{\theta}$$

$$\vec{E}^{in} = E_0\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)\underbrace{\left(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\right)}_{\hat{z}} \qquad r<A$$

$$\boxed{\vec{E}^{in} = E_0\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)\hat{z} \quad \text{for} \quad r<A}$$

$$\boxed{\vec{P} = (\varepsilon-\varepsilon_0)\vec{E}^{in} = (\varepsilon-\varepsilon_0)E_0\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)\hat{z} \quad \text{for} \quad r<A}$$

$$\vec{E}^{out} = -\vec{\nabla}\phi^{out} = -\left(\partial_r\phi^{out}\,\hat{r}+\frac{1}{r}\partial_\theta\phi^{out}\,\hat{\theta}\right) \quad \text{for} \quad r>A$$

$$= -\left\{E_0\cos\theta\left(-1-2\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)\left(\frac{A}{r}\right)^3\right)\hat{r} - E_0\sin\theta\left(-1+\left(\frac{\varepsilon/\varepsilon_0+1}{\varepsilon/\varepsilon_0+2}\right)\left(\frac{A}{r}\right)^3\right)\hat{\theta}\right\}$$

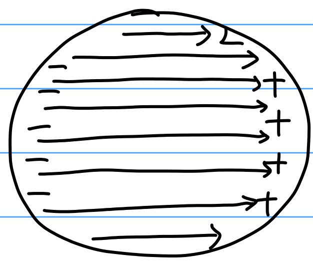
$$\boxed{\begin{aligned}&\vec{E}^{out} = E_0\left\{\cos\theta\left(1+2\,\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)\left(\frac{A}{r}\right)^3\hat{r}+\sin\theta\left(-1+\left(\frac{\varepsilon/\varepsilon_0+1}{\varepsilon/\varepsilon_0+2}\right)\left(\frac{A}{r}\right)^3\right)\hat{\theta}\right\} \text{ for } r>A\\ &\text{And } \vec{P}^{out} = 0 \quad \text{since} \quad \varepsilon=\varepsilon_0 \quad \text{outside.}\end{aligned}}$$

i) $\sigma_p = \hat{n} \cdot \vec{P} = \hat{n} \cdot (\varepsilon - \varepsilon_0)\vec{E}$

$$\sigma_p = \hat{r} \cdot (\varepsilon - \varepsilon_0) E_0 \frac{3}{2 + \varepsilon/\varepsilon_0} (\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta})$$

$$\boxed{\sigma_p = E_0 \frac{3(\varepsilon - \varepsilon_0)}{2 + \varepsilon/\varepsilon_0} \cos\theta}$$

ii) Inside sphere, the polarization look like:



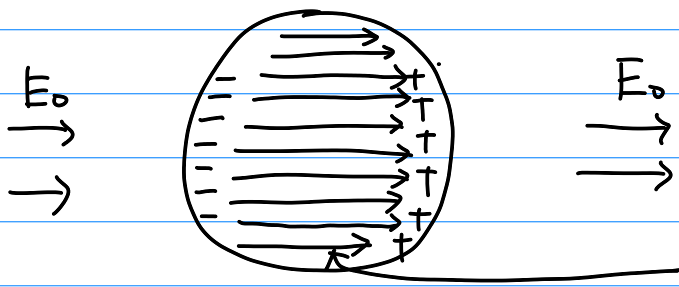
$\vec{P}$, pointing from (−) to (+)

then the induce $\vec{E}$ caused by polarization look like:

Electric-field induced by polarization that counteracts the external field. However, it is not strong enough to cancel the external field entirely. So the net field inside the sphere is still along the external field, but weakened.

Since $E^{in} = E_0 \left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right)$ and $\varepsilon/\varepsilon_0 > 1$, so $\frac{3}{2 + \varepsilon/\varepsilon_0} < 1$.

⇒ Then the overall E-field:



overall E field inside sphere is still in $+\hat{z}$, but reduced in magnitude compared to E_0

→ If we have a conductor, then it is as if $\epsilon \to \infty$.

then $\lim_{\varepsilon\to\infty} E^{in} = \lim_{\varepsilon\to\infty} E_0 \dfrac{3}{2+\varepsilon/\varepsilon_0} = 0$

and $\lim_{\varepsilon\to\infty} \phi^{out} = \lim_{\varepsilon\to\infty} E_0 r\cos\theta \left(-1 + \left(\dfrac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\left(\dfrac{A}{r}\right)^3\right)$

$\stackrel{!}{=} -E_0 r\cos\theta \left(1 - \left(\dfrac{A}{r}\right)^3\right)$

then induced field cancels external field entirely
and we get our ϕ^{out} for conductor.

→ We see that outside the sphere,

$$\phi^{out}(r) = \underbrace{-E_0 r\cos\theta}_{\text{term due to external field}} + \underbrace{E_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right) A^3 \frac{1}{r^2}\cos\theta}_{\text{extra term due to the sphere}}$$

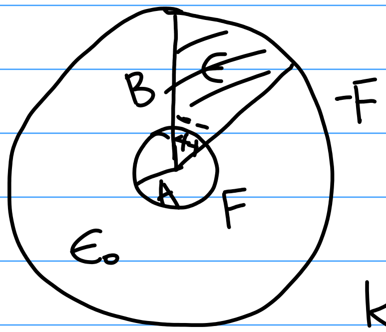
extra term due to the sphere,
since $\phi_{dipole} = \dfrac{1}{4\pi\epsilon_0} \dfrac{P\cos\theta}{r^2}$ for point charge

then we can recognize $P = 4\pi\epsilon_0 E_0 A^3 \left(\dfrac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)$

↑ dipole moment.

So the term due to sphere is like a dipole.

4) Partially dielectric-filled cavity between conducting shells.



Known Conditions:

since ε is const

know: $\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\varepsilon\vec{E}) = -\epsilon\nabla^2\phi = \rho = 0$

know dielectric / conductor Boundary Condition:

i) $E_{\parallel}^{die} = E_{\parallel}^{cond} = 0$ ii) $D_{\perp}^{die} = D_{\perp}^{cond} = \sigma^{cond.}$

a) Find ϕ between shells.

know $\vec{\nabla}\cdot\vec{D} = -\varepsilon\nabla^2\phi = \rho = 0$

or $\nabla^2\phi = \vec{\nabla}\cdot\vec{E} = 0$

since $\nabla^2\phi$ carries no information about ε or ε_0, and geometry is spherically symmetric, we expect ϕ to be only a function of radius.

For spherical coordinate: $\nabla^2\phi$ has general solution

$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{ml}r^{l} + B_{ml}r^{-l-1}\right)Y_l^m(\theta,\phi)$$

since spherically symmetric, expect $m=l=0$

then $\phi = A_{00} + B_{00}r^{-1}$

With Boundary Condition:

$$① \quad \phi(r=A) = F = A_\infty + B_\infty \frac{1}{A}$$

$$② \quad \phi(r=B) = -F = A_\infty + B_\infty \frac{1}{B}$$

$$①-②: \quad \phi(A) - \phi(B) = 2F = B_\infty \left(\frac{1}{A} - \frac{1}{B} \right) = B_\infty \left(\frac{B-A}{AB} \right)$$

then $\boxed{B_\infty = 2F \left(\frac{AB}{B-A} \right)}$

$$①+②: \quad \phi(A) + \phi(B) = 0 = 2A_\infty + 2F \left(\frac{\cancel{AB}}{B-A} \right) \left(\frac{A+B}{\cancel{AB}} \right)$$

$$\boxed{A_\infty = F \left(\frac{A+B}{A-B} \right)}$$

$$\boxed{\phi(r) = F \left\{ \left(\frac{A+B}{A-B} \right) + 2 \frac{AB}{B-A} \frac{1}{r} \right\}}$$

b) Find σ on both conductors.

we know $D_\perp = \varepsilon E_\perp = \varepsilon_0 E_\perp + P_\perp = \sigma$

For $\underline{\underline{r=A}}$:

$$E_\perp = -\hat{n}\cdot\vec{\nabla}\phi = -\hat{r}\cdot\left(\partial_r \phi \hat{r}\right) = 2F\frac{AB}{B-A}\frac{1}{r^2}$$

$$\boxed{E_\perp\Big|_{r=A} = 2F\frac{B}{A(B-A)}}$$

For $\underline{\underline{r=B}}$: $E_\perp = -\hat{n}\cdot\vec{\nabla}\phi = -\hat{r}\cdot\partial_r\phi\hat{r} = 2F\frac{AB}{B-A}\frac{1}{r^2}$

$$\boxed{E_\perp\Big|_{r=B} = 2F\frac{A}{B(B-A)}}$$

For $0<\theta<\Lambda$, $D_\perp = \varepsilon E_\perp = \sigma$

So
$$\boxed{\left.\begin{aligned}\sigma\Big|_{r=A} &= 2\varepsilon F\frac{B}{A(B-A)}\\ \sigma\Big|_{r=B} &= 2\varepsilon F\frac{A}{B(B-A)}\end{aligned}\right\}\text{ For } 0<\theta<\Lambda}$$

For $\Lambda<\theta<\pi$, $D_\perp = \varepsilon E_\perp = \varepsilon_0 E_\perp + \overset{0}{\cancel{P_\perp}} = \sigma$

So
$$\boxed{\left.\begin{aligned}\sigma\Big|_{r=A} &= 2\varepsilon_0 F\frac{B}{A(B-A)}\\ \sigma\Big|_{r=B} &= 2\varepsilon_0 F\frac{A}{B(B-A)}\end{aligned}\right\}\text{ For } \Lambda<\theta<\pi}$$

c) Find $\sigma_p = \hat{n} \cdot \vec{P}$ $\rho_p = -\vec{\nabla} \cdot \vec{P}$

→ $\vec{P}$ is not zero only when $A < r < B$ and $\underline{0 < \theta < \Lambda}$:

→ Find $\vec{P}$ first: , $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E} \Rightarrow \vec{P} = (\varepsilon - \varepsilon_0) \vec{E}$

we know $\sigma = \varepsilon E_\perp = \varepsilon_0 E_\perp + P_\perp$

so $\sigma_p = \hat{n} \cdot \vec{P} = (\varepsilon - \varepsilon_0) E_\perp$

then $\sigma_p \big|_{r=A} = = (\varepsilon - \varepsilon_0) E_\perp \big|_{r=A}$

$$\boxed{\sigma_p \big|_{r=A} = (\varepsilon - \varepsilon_0)\, 2F \frac{B}{A(B-A)}}$$

and $\boxed{\sigma_p \big|_{r=B} = (\varepsilon - \varepsilon_0)\, 2F \frac{A}{B(B-A)}}$

$\rho_p = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot (\varepsilon - \varepsilon_0 \vec{E}) = -(\varepsilon - \varepsilon_0) \vec{\nabla} \cdot \vec{E}$

$= (\varepsilon - \varepsilon_0) \underbrace{\nabla^2 \phi}_{=0}$

$\boxed{\rho_p = 0}$ everywhere.