EM HW #1 Zhi Chen

1) Chern-Simons field and action:

New gauge field $A^i = (A^0, A^1, A^2)$

and position: $r^i = (r^0, r^1, r^2)$

$$\partial_i = (\partial_0, \partial_1, \partial_2) = (\partial/\partial r^0, \partial/\partial r^1, \partial/\partial r^2)$$

And $$A_i = (A_0, A_1, A_2) = (-A^0, A^1, A^2)$$

$$r_i = (r_0, r_1, r_2) = (-r^0, r^1, r^2)$$

$$\partial^i = (\partial^0, \partial^1, \partial^2) = (-\partial_0, \partial_1, \partial_2)$$

$$S_{CS} \equiv \frac{k}{4\pi}\int d^3r\, \vec{A}\cdot(\vec{\nabla}\times\vec{A}) = \frac{k}{4\pi}\int d^3r\, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$S_{FM} \equiv \int d^3r\, \vec{J}\cdot\vec{A} = \int d^3r\, J^\mu A_\mu$$

Field-Matter Coupling

Matter current.

$$J^i = (J^0, J^1, J^2) = (\rho, j^x, j^y)$$

a) Show that action $S_{CS} + S_{FM}$ is invariant under gauge transformation:

$$\vec{A}_\mu \rightarrow \vec{A}_\mu + \partial_\mu w(r)$$

$$S_{CS} + S_{FM} = \int d^3r \left[\frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + J^\mu A_\mu \right]$$

$$\hookrightarrow = \int d^3r \left[\frac{k}{4\pi} \epsilon^{\mu\nu\rho} (A_\mu + \partial_\mu w(r)) \partial_\nu (A_\rho + \partial_\rho w(r)) + J^\mu (A_\mu + \partial_\mu w(r)) \right]$$

$$= \int d^3r \left[\frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + J^\mu A_\mu \right.$$

Need $=0$ for invariant

$$\left. + \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \left(\overset{①}{A_\mu \partial_\nu \partial_\rho w(r)} + \overset{②}{\partial_\mu w(r) \partial_\nu A_\rho} + \overset{③}{\partial_\mu w(r) \partial_\nu \partial_\rho w(r)} \right) + \underset{④}{J^\mu \partial_\mu w(r)} \right]$$

$\Rightarrow$ Term ①:

$$A_\mu \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho w = \frac{1}{2} A_\mu \left(\epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho w + \epsilon^{\mu\rho\nu} \partial_\rho \partial_\nu w \right)$$

$$= \frac{1}{2} A_\mu \left(\epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho w - \epsilon^{\mu\nu\rho} \partial_\rho \partial_\nu w \right)$$

$$= \frac{1}{2} A_\mu \epsilon^{\mu\nu\rho} \left(\partial_\nu \partial_\rho w - \partial_\nu \partial_\rho w \right)$$

$$\vec{A} \cdot \underbrace{\vec{\nabla} \times (\vec{\nabla} w)}_{=0} = 0$$

$\Rightarrow$ Similarly Term ③ $\partial_\mu w \, \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho w = \vec{\nabla} w \cdot \left(\underbrace{\vec{\nabla} \times (\vec{\nabla} w)}_{=0} \right) = 0$

$\Rightarrow$ Term ②: $\partial_\mu \omega\, \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$

Note $\partial_\mu (\omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho) = \partial_\mu \omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \omega \partial_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$

So:

$$\partial_\mu \omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = \underbrace{\partial_\mu (\omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho)}_{\text{surface terms via } d^3r \text{ Assume } = 0} - \omega \underbrace{\underbrace{\partial_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho}_{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})}}_{=0 \text{ as we proved above.}}$$

$= 0$

Term 4: $$J^\mu \partial_\mu \omega = \underbrace{\partial_\mu (J^\mu \omega)}_{\text{surface term again, so } =0} - \omega \underbrace{\partial_\mu J^\mu}_{\text{since } J^\mu \text{ is conserved, } \vec{\nabla} \cdot \vec{J} = 0}$$

$= 0$

Since all 4 extra terms are 0, we recover the original action, so it is invariant under this gauge transformation.

$$\boxed{S = \int d^3r \left[\frac{k}{4\pi} A_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + J^\mu A_\mu \right]}$$

b) Make slight variation in A

$$A \to A + \delta A$$

$$S[A+\delta A] = \int d^3r \frac{k}{4\pi} (A_\mu + \delta A_\mu) \epsilon^{\mu\nu\rho} \partial_\nu (A_\rho + \delta A_\rho) + J^\mu (A_\mu + \delta A_\mu)$$

$$\stackrel{!}{=} \int d^3r \frac{k}{4\pi} A_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + J^\mu A_\mu \quad \leftarrow S[A]$$

$$\delta S \begin{cases} + \int d^3r \frac{k}{4\pi} \left[A_\mu \epsilon^{\mu\nu\rho} \partial_\nu \delta A_\rho + \delta A_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \mathcal{O}(\delta^2) \right] \\ + J^\mu \delta A_\mu \end{cases}$$

$$A_\mu \epsilon^{\mu\nu\rho} \partial_\nu \delta A_\rho = \partial_\nu (A_\mu \epsilon^{\mu\nu\rho} \delta A_\rho) - (\partial_\nu A_\mu \epsilon^{\mu\nu\rho}) \delta A_\rho$$

$\epsilon^{\mu\nu\rho} \to -\epsilon^{\nu\mu\rho}$

$\epsilon^{\mu\nu\rho} \to -\epsilon^{\rho\nu\mu}$

$$\stackrel{!}{=} -\partial_\nu (\epsilon^{\mu\rho\nu} A_\mu \delta A_\rho) + (\epsilon^{\rho\nu\mu} \partial_\nu A_\mu) \delta A_\rho$$

$$A \cdot (\vec{\nabla} \times \delta A) \stackrel{!}{=} \underbrace{-\vec{\nabla} \cdot (\vec{A} \times \delta \vec{A})}_{\text{surface term, } = 0} + (\vec{\nabla} \times \vec{A}) \cdot \delta \vec{A}$$

$$\text{Then } \delta S = \int d^3r \frac{k}{4\pi} \delta A_\mu \left[\underbrace{\epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \frac{4\pi}{k} J^\mu}_{=0} \right] = 0$$

EOM:

$$\cancel{2} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \frac{\overset{2}{\cancel{4}}\pi}{k} J^\mu = 0$$

$$\boxed{\epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \frac{2\pi}{k} J^\mu = 0}$$

For $\mu = 0$: $\quad \partial_1 A_2 - \partial_2 A_1 + \frac{2\pi}{k} J^0 = 0$

$$\hookrightarrow \partial_x A_y - \partial_y A_x + \frac{2\pi}{k} J^0 = 0$$

let $\boxed{\partial_x A_y - \partial_y A_x = B}$, and $J^0 = \rho$

then $\boxed{B + \frac{2\pi}{k} \rho = 0}$

$\mu = 1$: $\quad \partial_2 A_0 - \partial_0 A_2 + \frac{2\pi}{k} \vec{j}_x = 0$

Note that $\boxed{A_0 = -A^0 = -\phi}$

$$\hookrightarrow \underbrace{-\partial_y \phi - \partial_t A_y}_{} + \frac{2\pi}{k} \vec{j}_x = 0$$

Define $\equiv E_y \hat{y}$

and $\mu = 2$: $\quad \partial_0 A_1 - \partial_1 A_0 + \frac{2\pi}{k} \vec{j}_y = 0$

$$\hookrightarrow \underbrace{\partial_t A_x + \partial_x \phi}_{} + \frac{2\pi}{k} \vec{j}_y = 0$$

Define. $\equiv -E_x \hat{x}$

So $\quad \vec{E} = E_x \hat{x} + E_y \hat{y}$

$$\boxed{\vec{E} = (-\partial_x \phi - \partial_t A_x)\hat{x} + (-\partial_y \phi - \partial_t A_y)\hat{y}}$$

Then we see $\nu = 1, 2$ give

$$\boxed{\begin{array}{l} \epsilon_{mn} E_n + \frac{2\pi}{k} j_m = 0 \quad \text{for} \quad m, n = 1, 2 \\ \qquad\qquad\qquad\qquad \epsilon_{mn} \begin{cases} = 1 & \text{for } 12 \\ = -1 & \text{for } 21 \\ = 0 & \text{otherwise.} \end{cases} \end{array}}$$

and $\nu = 0$ give

$$\boxed{B + \frac{2\pi}{k} \rho = 0}$$

2) Helmholtz Theorem:

$$\hat{\vec{F}}(\vec{q}) = \int d^3r \exp(-i\vec{q}\cdot\vec{r})\vec{F}(\vec{r}) \quad , \quad \vec{F}(\vec{r}) = \int d^3q \exp(+i\vec{q}\cdot\vec{r})\hat{\vec{F}}(\vec{q})$$

a)

$$\vec{\nabla}\cdot\vec{F}(\vec{r}) = \partial_u\left(\int d^3q \exp(i\vec{q}\cdot\vec{r})\hat{F}_u(\vec{q})\right)$$

$$\stackrel{!}{=} \int d^3q \left\{\underbrace{\partial_u(\exp(i\vec{q}\cdot\vec{r}))}_{\exp(i\vec{q}\cdot\vec{r})\, iq_u}\hat{F}_u(\vec{q}) + \exp(i\vec{q}\cdot\vec{r})\underbrace{\cancel{\partial_u \hat{F}_u(\vec{q})}}_{0}\right\}$$

since $\hat{F}$ depends on $\vec{q}$.

$$\boxed{\vec{\nabla}\cdot\vec{F}(\vec{r}) = \underbrace{\int d^3q \exp(i\vec{q}\cdot\vec{r})(i\vec{q}\cdot\hat{\vec{F}}(\vec{q}))}_{\text{Inverse Fourier Transform of } i\vec{q}\cdot\hat{\vec{F}}(\vec{q})}}$$

$$\vec{\nabla}\times\vec{F}(\vec{r}) = \epsilon_{u\nu\rho}\,\partial_\nu \int d^3q \exp(i\vec{q}\cdot\vec{r})\hat{F}_\rho(\vec{q})$$

$$\stackrel{!}{=} \int d^3q \left\{\underbrace{\underbrace{\epsilon_{u\nu\rho}\,\partial_\nu(\exp(i\vec{q}\cdot\vec{r}))\hat{F}_\rho(\vec{q})}_{\epsilon_{u\nu\rho}\, iq_\nu \exp(i\vec{q}\cdot\vec{r})\hat{F}_\rho(\vec{q})}}_{=\exp(i\vec{q}\cdot\vec{r})(i\vec{q}\times\hat{\vec{F}}(\vec{q}))} + \underbrace{\epsilon_{u\nu\rho}\,\partial_\nu \hat{F}_\rho(\vec{q})\exp(i\vec{q}\cdot\vec{r})}_{=0 \text{ since } \hat{F} \text{ doesn't depend on } \vec{r}}\right\}$$

$$\boxed{\vec{\nabla}\times\vec{F} = \underbrace{\int d^3q \exp(i\vec{q}\cdot\vec{r})(i\vec{q}\times\hat{\vec{F}}(\vec{q}))}_{\text{Inverse Fourier transform of } i\vec{q}\times\hat{\vec{F}}(\vec{q})}}$$

b) Show $\hat{C}_a^{\parallel}(\vec{q}) = \hat{q}_a \hat{q}_b \hat{C}_b(\vec{q})$, $\hat{C}_a^{\perp}(\vec{q}) = (\delta_{ab} - \hat{q}_a \hat{q}_b)\hat{C}_b(\vec{q})$

Any vector field, $\vec{F}$, can be decomposed into two parts,

$\vec{C}^{\parallel} \to$ curl free $\Rightarrow \vec{\nabla} \times \vec{C}^{\parallel} = 0$

and

$\vec{C}^{\perp} \to$ div free $\Rightarrow \vec{\nabla} \cdot \vec{C}^{\perp} = 0$

know: $C_a = C_a^{\parallel} + C_a^{\perp}$

take div $\hookrightarrow \partial_a C_a = \partial_a C_a^{\parallel} + \cancel{\underbrace{\partial_a C_a^{\perp}}_{=0,\text{ because } C_a^{\perp} \text{ div-free.}}}$

From part a), replace $F(r) \to C(r)$ and $\hat{F}(q) \to \hat{C}(q)$

we find: $\partial_a C_a^{\parallel} = \int d^3q \exp(i\vec{q}\cdot\vec{r})(iq_a \hat{C}_a^{\parallel})$

$$\partial_b C_b = \int d^3q \exp(i\vec{q}\cdot\vec{r})(iq_b \hat{C}_b)$$

Since they are equal, then by comparing terms,

we see $iq_a \hat{C}_a^{\parallel} = iq_b \hat{C}_b$

let $q_a = |q|\hat{q}_a$ $\hookrightarrow \cancel{|q|}\hat{q}_a \hat{C}_a^{\parallel} = \cancel{|q|}\hat{q}_b \hat{C}_b$

multiply both side by $\hat{q}$ $\hookrightarrow \underbrace{\hat{q}_\gamma \hat{q}_a}_{\delta_{\gamma a}} \hat{C}_a^{\parallel} = \hat{q}_\gamma \hat{q}_b \hat{C}_b$

$\hookrightarrow \hat{C}_\gamma^{\parallel} = \hat{q}_\gamma \hat{q}_b \hat{C}_b$

rename $\gamma \to a$ $\hookrightarrow \boxed{\hat{C}_a^{\parallel} = \hat{q}_a \hat{q}_b \hat{C}_b}$

Similarly, since $C_a = C_a^{\parallel} + C_a^{\perp}$

take $\vec{\nabla}\times$ $\hookrightarrow$ $\epsilon_{u\nu a}\partial_\nu C_a = \underbrace{\epsilon_{u\nu a}\partial_\nu C_a^{\parallel}}_{=0 \text{ because } C_a^{\parallel} \text{ curl-free.}} + \epsilon_{u\nu a}\partial_\nu C_a^{\perp}$

from part a: replace F with C:

$$\epsilon_{u\nu a}\partial_\nu C_a^{\perp} = \int d^3q \exp(i\vec{q}\cdot\vec{r})\left(i\epsilon_{u\nu a} q_\nu \hat{C}_a^{\perp}\right)$$

and $$\epsilon_{u\gamma b}\partial_\gamma C_b = \int d^3q \exp(i\vec{q}\cdot\vec{r})\left(i\epsilon_{u\gamma b} q_\gamma \hat{C}_b\right)$$

by comparing terms,

we see $$\epsilon_{u\nu a} q_\nu \hat{C}_a^{\perp} = \epsilon_{u\gamma b} q_\gamma \hat{C}_b$$

$$\epsilon_{iju} q_j \epsilon_{u\nu a} q_\nu \hat{C}_a^{\perp} = \epsilon_{iju} q_j \epsilon_{u\gamma b} q_\gamma \hat{C}_b$$

change cyclic order: $\epsilon_{iju} \to \epsilon_{uij}$

$$\underbrace{\epsilon_{uij}\epsilon_{u\nu a}}_{\delta_{i\nu}\delta_{ja} - \delta_{ia}\delta_{j\nu}} q_j q_\nu \hat{C}_a^{\perp} = \underbrace{\epsilon_{uij}\epsilon_{u\gamma b}}_{\delta_{i\gamma}\delta_{jb} - \delta_{ib}\delta_{j\gamma}} q_j q_\gamma \hat{C}_b$$

$$\hookrightarrow \left(q_i q_a - \delta_{ia}|q|^2\right)\hat{C}_a^{\perp} = \left(q_i q_b - \delta_{ib}|q|^2\right)\hat{C}_b$$

$\Rightarrow$ Note that since $C_a^{\perp}$ is divergence free, this means

$$\vec{\nabla}\cdot\vec{C}^{\perp} = \int d^3q \exp(i\vec{q}\cdot\vec{r})\, i\underbrace{q_a\hat{C}_a^{\perp}}_{=0} = 0 \quad \text{or} \quad \boxed{q_a\hat{C}_a^{\perp} = 0}$$

then $\cancel{q_i \underbrace{q_a \hat{C}^\perp_a}_{=0}} - \delta_{ia} |q|^2 \hat{C}^\perp_a = (q_i q_b - \delta_{ib} |q|^2) \hat{C}_b$

Divide both sides by $|q|^2$

$\hookrightarrow \quad -\delta_{ia} \hat{C}^\perp_a = (\hat{q}_i \hat{q}_b - \delta_{ib}) \hat{C}_b$

$\hookrightarrow \quad \hat{C}^\perp_i = (\delta_{ib} - \hat{q}_i \hat{q}_b) \hat{C}_b$

Rename $i \to a$ $\quad \hookrightarrow \quad \boxed{\hat{C}^\perp_a = (\delta_{ab} - \hat{q}_a \hat{q}_b) \hat{C}_b}$

We can also get this very easily from realizing

$$C_a = C^\parallel_a + C^\perp_a$$

So $\hat{C}^\perp_a = \hat{C}_a - \hat{C}^\parallel_a = \hat{C}_a - \hat{q}_a \hat{q}_b \hat{C}_b = (\delta_{ab} - \hat{q}_a \hat{q}_b) \hat{C}_b.$

c) Recall: Convolution: $\int d^3q \, \exp(i\vec{q}\cdot\vec{r}) \hat{K}(\vec{q})\hat{L}(\vec{q}) = \int d^3r \, K(\vec{r}-\vec{r}')L(\vec{r}')$

Show if $\vec{\nabla}\cdot\vec{C}(r) = \delta(\vec{r})$ and $\vec{\nabla}\times\vec{C} = \vec{S}(\vec{r})$

then

$$\vec{C}^{\parallel}(\vec{r}) = -\vec{\nabla}\int d^3r' \frac{1}{4\pi|\vec{r}-\vec{r}'|}\delta(\vec{r}')$$

$$\vec{C}^{\perp}(\vec{r}) = \vec{\nabla}\times\int d^3r' \frac{1}{4\pi|\vec{r}-\vec{r}'|} S(\vec{r}')$$

We know

$$\vec{\nabla}\cdot\vec{C} = \vec{\nabla}\cdot\vec{C}^{\parallel} = \delta(\vec{r})$$

From previous part: $\vec{\nabla}\cdot\vec{C}^{\parallel} = \int d^3q \, \exp(i\vec{q}\cdot\vec{r}) \, iq_a \hat{C}_a^{\parallel} = \delta(\vec{r})$

then the Fourier Transform $\delta(r)$ is $\hat{\delta}(q) = iq_a \hat{C}_a^{\parallel}$

$$i|q|\hat{q}_a \hat{C}_a^{\parallel} = \hat{\delta}(q)$$

$$\hat{C}_a^{\parallel} \underbrace{\hat{q}_a \hat{q}_\gamma}_{\delta_{a\gamma}} = \frac{-i\hat{\delta}(q)}{|q|}\hat{q}$$

$$\hookrightarrow \hat{C}_\gamma^{\parallel} = -i\frac{\hat{\delta}(q)}{|q|}\frac{q_\gamma}{|q|}$$

$$\hookrightarrow \hat{C}_\gamma^{\parallel} = \frac{-i\hat{\delta}(q)}{|q|^2} q_\gamma$$

Then doing inverse fourier transform for $\hat{C}_\gamma^{\parallel}$

$$\hookrightarrow \vec{C}^{\parallel}(r) = \int d^3q \, \exp(i\vec{q}\cdot\vec{r}) \frac{-i\hat{\delta}(q)}{|q|^2} q_\gamma$$

but from part a) we saw $\vec{\nabla}$ give extra factor of iq_r from $\exp(i\vec{q}\cdot\vec{r})$

So $\hookrightarrow$ $\vec{C}^{\parallel}(\vec{r}) = -\vec{\nabla}\int d^3q \exp(i\vec{q}\cdot\vec{r}) \frac{1}{|q|^2}\hat{\delta}(q)$

and applying convolution theorem: $\int d^3q \exp(i\vec{q}\cdot\vec{r})\hat{K}(\vec{q})\hat{L}(\vec{q}) = \int d^3r' K(\vec{r}-\vec{r}')L(\vec{r}')$

let $\hat{K}(\vec{q}) = \frac{1}{|q|^2} \Rightarrow K(\vec{r}-\vec{r}') = \frac{1}{4\pi|\vec{r}-\vec{r}'|}$

and $\hat{L}(\vec{q}) = \hat{\delta}(q)$

then $$\boxed{\vec{C}^{\parallel}(\vec{r}) = -\vec{\nabla}\int d^3r' \frac{1}{4\pi|\vec{r}-\vec{r}'|}\delta(\vec{r}')}$$

Similarly for $\vec{C}^{\perp}(\vec{r})$:

$$\vec{\nabla}\times\vec{C} = \vec{\nabla}\times\vec{C}^{\perp}(\vec{r}) = \int d^3q \exp(i\vec{q}\cdot\vec{r})\, i\epsilon_{u\nu\rho} q_\nu \hat{C}^{\perp}_\rho = \vec{S}(\vec{r})$$

so $\hat{S}_u(\vec{q}) = i\epsilon_{u\nu\rho} q_\nu \hat{C}^{\perp}_\rho$

Multiply both sides by $\epsilon_{iju}q_j$ $\hookrightarrow$ $\epsilon_{iju}q_j\hat{S}_u(\vec{q}) = i\epsilon_{iju}q_j\epsilon_{u\nu\rho}q_\nu\hat{C}^{\perp}_\rho$

$$= i\epsilon_{uij}\epsilon_{u\nu\rho}q_j q_\nu \hat{C}^{\perp}_\rho$$

$$= i(\delta_{i\nu}\delta_{j\rho} - \delta_{i\rho}\delta_{j\nu})q_j q_\nu \hat{C}^{\perp}_\rho$$

$$= i(q_i q_\rho - \delta_{i\rho}|q|^2)\hat{C}^{\perp}_\rho$$

$$= i(q_i q_\rho \hat{C}^{\perp}_\rho - \delta_{i\rho}\hat{C}^{\perp}_\rho |q|^2)$$

Again using argument $C_p^{\perp}$ is div-free, so $q_p \hat{C}_p^{\perp} = 0$

then $\epsilon_{ijk} q_j \hat{S}_k = -i \, \delta_{ip} \hat{C}_p^{\perp} |q|^2$

$$\hookrightarrow \quad \frac{i \epsilon_{ijk} q_j \hat{S}_k}{|q|^2} = \hat{C}_i^{\perp}$$

Since $\vec{C}^{\perp}(\vec{r}) = \int d^3q \exp(i\vec{q}\cdot\vec{r}) \, \hat{C}^{\perp}$

$$\stackrel{\perp}{=} \int d^3q \exp(i\vec{q}\cdot\vec{r}) \frac{i \epsilon_{ijk} q_j \hat{S}_k(q)}{|q|^2}$$

And realize from part a) that taking curl, $\vec{\nabla}\times$ gives extra term $i\epsilon_{ijk} q_j$ in integral

$$\vec{\nabla}\times \leftrightarrows i\epsilon_{ijk} q_j$$

then $\vec{C}^{\perp}(\vec{r}) = \vec{\nabla}\times \int d^3q \exp(i\vec{q}\cdot\vec{r}) \frac{1}{|q|^2} \hat{S}_k(\vec{q})$

Apply convolution theorem again, then we have.

$$\frac{1}{|q|^2} \rightarrow \frac{1}{4\pi |\vec{r}-\vec{r}'|} \quad \text{and} \quad \hat{S}(q) \rightarrow S(\vec{r}')$$

$$\boxed{\vec{C}^{\perp}(\vec{r}) = \vec{\nabla}\times \int d^3r' \frac{1}{4\pi |\vec{r}-\vec{r}'|} S(\vec{r}')}$$