For V, vector space.

If $\vec{x}, \vec{y} \in V$, then linear combination, $x\vec{x} + y\vec{y} \in V$

$$\text{Span}\{v_1, v_2, v_3, v_4 \cdots v_n\} = \{X = \lambda_1 v_1 + \cdots \lambda_n v_n,\ \lambda_1 \cdots \lambda_n \in F\} \subset V$$
if $v_1 \ldots v_n \in V$

Linear independence: $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \cdots \lambda_n \vec{v}_n = 0$ iff $\lambda_1 \cdots \lambda_n = 0$.

Linear Dependent: $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \cdots \lambda_n \vec{v}_n = 0$ for $\lambda_1 \cdots \lambda_n \neq 0$.

which means we can eliminate S_i if $\lambda_i \neq 0$

for $S_i = -\frac{1}{\lambda_i}(\lambda_1 \vec{v}_1 + \cdots \lambda_n \vec{v}_n)$

<u>Basis:</u> $\{v_1 \cdots v_m\} \rightarrow$ Independent vectors, and $\{v_1 \cdots v_m\} \in V$

e.g. $\forall\ \vec{x} \in V$, $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 \cdots x_n\vec{v}_4$

For $\vec{x} = x_1\vec{e}_1 + \cdots x_n\vec{e}_m$

Not unique: $\{\vec{e}_1 \cdots \vec{e}_m\} \Rightarrow \vec{e}_i{}' = \vec{e}_j Q^j{}_i$

$$\vec{x} = x^i \vec{e}_i = x'^i \underset{\substack{\uparrow \\ \text{new basis}}}{\vec{e}_i{}'} = \vec{e}_j \underbrace{Q^j{}_i\, x'^i}_{x^j = Q^j{}_i\, x'^i}$$

Linear Mapping:

$X \in V, \quad Y \in W \qquad A(\vec{x}) = \vec{y} \quad : \quad A : V \to W$

any $\vec{x}_1, \vec{x}_2 \in V$ and $\alpha_1, \alpha_2 \in F$
then

$$A(\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2) = \alpha_1 A(\vec{x}_1) + \alpha_2 A(\vec{x}_2)$$

Null space or kernel A : $\{ x \in V : A(\vec{x}) = 0 \} = \ker A \subseteq V$

Range of A: $\{ \vec{y} \in W : \exists \vec{x} \in V : \vec{y} = A(\vec{x}) \} = \operatorname{Im} A \subseteq W$
(Image) range of A.

$$\dim(\operatorname{Im} A) = \operatorname{rank}(A)$$

Automorphism: $A : V \to V$

For $\{ \vec{v}_1 \dots \vec{v}_n \}$ = basis in V
$\{ \vec{w}_1 \dots \vec{w}_n \}$ = basis in W

$$A(\vec{v}_i) = \vec{w}_j A^i_j$$

$$A(x^i \vec{v}_i) = \vec{w}_j A^j_i x^i$$

$$A^j{}_i := \begin{array}{cc} \multicolumn{1}{c}{\to \text{col index } j} & \\ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix} & \downarrow \text{row index} \end{array}$$

rank A = # of linear independent cols of A

Linear Form (function)

$\{f\}$ = linear space.

$x \in V$: $\quad f(x) = f_1 \subseteq F$

$f: V \to F$

$f(\vec{e}_i x^i) = \sum_i f(\vec{e}_i) x^i = f_j \underbrace{e^{*j}(\vec{e}_i)}_{\delta^j_i} x^i$

δ^j_i, then e^{*j} and $\vec{e}_i$ are dual basis.

V^*: dual space of V

basis in dual space $\{\vec{e}^{*j}\}$: $\quad f = f_j e^{*j}$

Scalar product on V:

$\vec{x}, \vec{y} \in V$: $\quad \langle x, y \rangle \in F$

antilinear linear

$\langle x, \beta_1 \vec{y}_1 + \beta_2 \vec{y}_2 \rangle = \beta_1 \langle x, \vec{y}_1 \rangle + \beta_2 \langle x, \vec{y}_2 \rangle$

$\langle \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2, \vec{y} \rangle = \alpha_1^* \langle \vec{x}_1, \vec{y} \rangle + \alpha_2^* \langle \vec{x}_2, \vec{y} \rangle$

only matter if α is complex #.

non-degenerate:

Degenerate if: $\langle x, y\rangle = 0 \quad \forall y \Rightarrow x = 0.$

Null vector for x if $\langle x, x\rangle = 0$

$\langle x, y\rangle = \langle y, x\rangle^*$

Basis: $\{\vec{e}_1 \dots \vec{e}_n\}$ then $\langle \vec{e}_i, \vec{e}_j\rangle = g_{ij} = g^*_{ji}$

$$\langle \vec{x}, \vec{y}\rangle = \langle x^i \vec{e}_i, y^j \vec{e}_j\rangle = (x^i)^* y^j g_{ij}$$

(upper index) cotravariant — covariant (lower index)

$x'^i = Q^i{}_j x^j$ — $\vec{e}_j{}' = Q^i{}_j \vec{e}_i$

	Vector Space V	Dual Space V^*
basis	$\vec{e}_i$	w^j
component	x^i	f_j
	vector	form

$f(x) = f_i x^i$

For $x, y \in V$, $\vec{x} = x^i \hat{e}_i$ $\vec{y} = y^j \hat{e}_j$

$$\langle x, y \rangle = \langle y, x \rangle^*$$

ex:

$$\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1^* \langle x_1, y \rangle + \alpha_2^* \langle x_2, y \rangle$$

$$\langle \vec{x}, \vec{y} \rangle = (x^i)^* (y^j) \underbrace{\langle \vec{e}_i, \vec{e}_j \rangle}_{g_{ij}}$$

Scalar product in dual space:

$\omega = \omega_i \hat{e}^{i*} \rightarrow$ where $\hat{e}^{i*} \hat{e}_j = \delta^i_j$

$p = p_j \hat{e}^{j*}$ $\langle \omega, p \rangle = \omega_i^* p_j \underbrace{\langle \hat{e}^{i*}, \hat{e}^{j*} \rangle}_{g^{ij}}$

so $g^{ij} g_{jk} = \delta^i_k$

$$\underbrace{x^*(y)}_{\text{linear form}} = \underbrace{\langle x^*, y \rangle}_{\text{scalar product}} = x_i y^i$$

$\Downarrow$

covector components

$x_i = g_{ij} x^j$

Linear Maps

$$A: V \to W$$

$$A(\vec{x}) = \vec{y}$$

scalar product in V, W

Def Adjoint operator A^{+}

For $\forall x \in V$, $y \in W$

$$\langle \vec{y}, A\vec{x} \rangle = \langle A^{+}\vec{y}, \vec{x} \rangle$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{pmatrix}$$

$n = \dim(V)$
$m = \dim(W)$

$$(A^{+})_{ij} = (A_{ji})^{*}$$

only in orthonormal basis.

Use Gram-Schmidte to normalize:

$$W = (\vec{w}_1 \;\cdots\; \vec{w}_n) = Q\,R$$ ← QR decomposition.

$Q = (\vec{q}_1 \cdots \vec{q}_n)$

$$R = \begin{pmatrix} |w_1| & \cdots & \cdots \\ 0 & |w_2| & \cdots \\ 0 & 0 & \ddots \\ 0 & 0 & \ddots \\ \vdots & \vdots & \end{pmatrix}$$

Direct Sum:

For V, U and $V \cap U = \{0\}$ ← There is zero overlap.

Direct Sum: $V \oplus U$

any $x \in V \oplus U \overset{\text{unique}}{\Rightarrow} x_1 \in V,\ x_2 \in U,\ \vec{x} = \vec{x}_1 + \vec{x}_2$

For $W \to V \oplus U$

Quotient: pick V, $U = \{$ set of equivalence classes $\}$

$$\underset{\vec{y}}{W} \to \underset{\vec{x}}{V} \oplus \underset{\vec{z}}{V^{\perp}} \quad : \begin{cases} \vec{z} \perp \text{ any } \vec{x}_1 \in V \\ \vec{x} = \vec{y}\,\vec{z} \end{cases}$$

then $\dim(V \oplus U) = \dim(V) + \dim(U)$

$$\operatorname{Ker} A = \{ \forall \vec{x} \in V : A\vec{x} = \vec{0} \} \subseteq V$$

$$\operatorname{Im} A = \{ \forall y \in W : \exists \vec{x} \in V : A\vec{x} = \vec{y} \} \subseteq W$$

$$\boxed{\begin{array}{c} \underbrace{\dim(\ker A)}_{k = \{\bar{e}^{\circ}\}} + \dim(\operatorname{Im} A) = \dim V \\ \text{basis in } \ker A \qquad \{\bar{e}^{A}\} \text{ extra-linearly independent vectors} \\ A\bar{e}^{A} \neq 0 \end{array}}$$

if $m = \dim W < \dim V = n$

$$\dim \ker A = \underset{\overset{\parallel}{n}}{\dim V} - \underset{\leq m}{\dim(\operatorname{Im} A)} \geq (n-m)$$

<u>Linear Equations:</u>

$A \cdot \vec{x} = \vec{y}$ ← solution exists if $y \in \operatorname{Im} A$

↖ solution unique iff $\ker A = \{\bar{0}\}$ ← empty

If not unique

$$\left.\begin{array}{l} A x_1 = y \\ A x_2 = y \end{array}\right\} \quad A(x_1 - x_2) = 0$$

$x_1 - x_2$ then $\in \ker A$

since it makes it zero.

Fredholm Alternative: ①, ②, ③

① $A\vec{x} = \vec{y} \overset{\text{either}}{\leftrightarrows} \begin{cases} \bar{x} \text{ unique} \\ \exists \vec{x} : A\vec{x} = 0 \end{cases}$

$\vec{x} \in \ker A \subseteq V : A\vec{x} = 0$

$\forall \vec{y} \in W : 0 = \langle \vec{y}, A\vec{x} \rangle = \langle \underbrace{A^+ \vec{y}}, \vec{x} \rangle = 0$

$\{A^+ y\} = \mathrm{Im}\, A^+ \subseteq V$

$\vec{x} \perp \mathrm{Im} A^+ \Rightarrow x \in (\mathrm{Im} A^+)^\perp$

$$\Rightarrow \boxed{\begin{aligned} &\ker A = (\mathrm{Im} A^+)^\perp \subseteq V \\ &\ker A^+ = (\mathrm{Im} A)^\perp \subseteq W \\ &\dim \ker A = \dim \ker A^+ \end{aligned}}$$

② $A\vec{x} = \vec{y} \in \mathrm{Im}\, A \Leftrightarrow \bar{y} \perp (\mathrm{Im} A)^\perp$

$= \ker A^+ = \{\forall \bar{z} \in W : A^+ z = 0\}$

$\therefore \exists \vec{x} : A\vec{x} = \vec{y} \Leftarrow \vec{y} \perp \text{any } \bar{z} : \underbrace{A^+ \bar{z} = 0}_{\ker A^+}$

$\dim \ker A = \dim V - \dim \mathrm{Im} A = \dim (\mathrm{Im} A)^\perp = \dim \ker A^+$

Holds only for finite-dim , $\dim V = \dim W$

③ $A\vec{x} = 0$ has same # of solutions as $A^+ \vec{y} = 0$

For ∞ dimension vector space.

a: $\hat{a}\bar{e}_0 = 0$: $e_0 \in \ker A$, so $\dim \ker A = 1$

$\hat{a}\bar{e}_1 = \bar{e}_0$

$\hat{a}\bar{e}_2 = \bar{e}_1$

$\vdots$

ladder operator

$\hat{a}^+ =$ $\hat{a}^+\bar{e}_0 = \bar{e}_1$

$\hat{e}^+\bar{e}_1 = \bar{e}_2$

$\vdots$