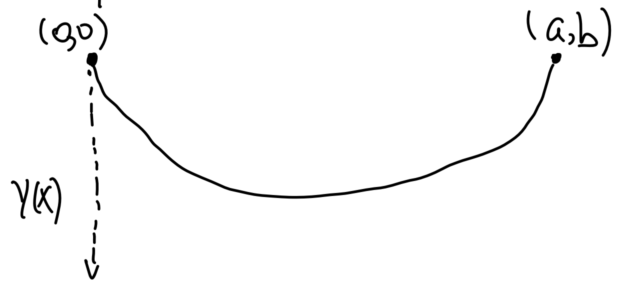
Catenary:



Need to minimize U(x) with const length. L(x).

$$L[y(x)] = \int ds = \int \sqrt{dx^2 + dy^2} = \int_0^a dx \sqrt{1+y'^2}$$

$$U[y(x)] = \int_0^a dx \sqrt{1+y'^2} \left(-g\, y\, \rho(x)\right)$$

← linear mass density.

stationary point ↳ $y^*(x) = \text{argmin}\left(U[y(x)]\right)$

$$\frac{\delta U}{\delta y(x)} = \text{gradient of } U \text{ w.r.t. } y(x)$$

$$\frac{\delta L}{\delta y} = \text{gradient of } L \text{ to } y(x).$$

$$\delta L = \int_0^a dx \frac{\delta L}{\delta y} \cdot \delta y = 0 \quad \text{only when } \vec{\delta y} \text{ is } \perp \text{ to } \frac{\delta L}{\delta y}$$

$$\delta U = \int_0^a dx \frac{\delta U}{\delta y} \cdot \delta y = 0 \quad \text{only when } \vec{\delta y} \text{ is } \perp \text{ to } \frac{\delta U}{\delta y}.$$

Since $\vec{\delta y} \perp \frac{\delta L}{\delta y}$ and $\vec{\delta y} \perp \frac{\delta U}{\delta y}$

So $\frac{\delta L}{\delta y}$ must be $\parallel \frac{\delta U}{\delta y}$.

So $\nabla_y U = \lambda \nabla_y L$

So $y^*(x) = \operatorname{argmin}\left(U - \lambda(L - L_0)\right)$ and $L(y) = L_0$

then $\frac{\delta}{\delta y}\left(U - \lambda(L - L_0)\right) = 0$

$\frac{\delta}{\delta \lambda}\left(U - \lambda(L - L_0)\right) = -(L - L_0) \Rightarrow 0$

then

$$\frac{\delta}{\delta y}\left(U - \lambda(L - L_0)\right) = \frac{\delta}{\delta y}\int_0^a dx\, \underbrace{(-\rho g y - \lambda)\sqrt{1+y'^2}}_{\mathcal{L}}$$

Find $\left[\frac{\partial}{\partial y} - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\right)\right]\left((-\rho g y - \lambda)\sqrt{1+y'^2}\right)$

$$\hookrightarrow = -\rho g\sqrt{1+y'^2} - \frac{d}{dx}\left([-\rho g y - \lambda]\frac{y'}{\sqrt{1+y'^2}}\right)$$

$$\overset{!}{=} -\rho g\sqrt{1+y'^2} - \left([-\rho g y']\frac{y'}{\sqrt{1+y'^2}} + [-\rho g y - \lambda]\frac{y''\sqrt{1+y'^2} - y'^2 y''/\sqrt{1+y'^2}}{(1+y'^2)}\right)$$

$$\overset{!}{=} \frac{-\rho g}{\sqrt{1+y'^2}}\left(1 + \cancel{y'^2} - \cancel{y'^2} - \left(y + \frac{\lambda}{\rho g}\right)\left(1 - \frac{y'^2}{1+y'^2}\right)y''\right)$$

$$\overset{!}{=} \frac{-\rho g}{(1+y'^2)^{3/2}}\left(1 + y'^2 - \left(y + \frac{\lambda}{\rho g}\right)\left(1 + \cancel{y'^2} - \cancel{y'^2}\right)y''\right) = 0$$

$$1+y'^2-\left(y+\frac{\lambda}{\rho g}\right)y''=0$$

guess $\quad y' \sim \sinh(B(x-x_0))$

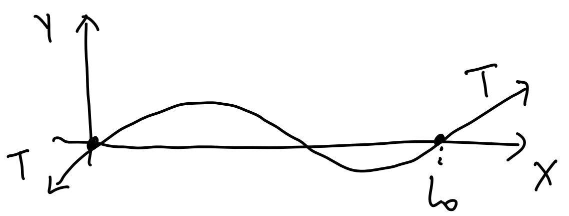
then $y'' \sim B\cosh(B(x-x_0))$

$y \sim \frac{1}{B}\cosh(B(x-x_0)) + y_0$

$$\underbrace{1+\left(\sinh(B(x-x_0))^2\right)}_{\cosh^2(B(x-x_0))}-\left(\frac{1}{B}\cosh(B(x-x_0))+y_0+\frac{\lambda}{\rho g}\right)B\cosh(B(x-x_0))=0$$

then $\quad y^* = \frac{1}{B}\cosh(B(x-x_0))+y_0$

For 1D vibrating string: (No constraint in this problem).



$y\left(\substack{x=0\\x=L},t\right)=0$

$y(x,t_1)=y_1(x)$

$y(x,t_2)=y_2(x)$

$$T=\int_0^{L_0}dx\,\rho\frac{1}{2}\dot{y}^2$$

$$U[y]\cong T\cdot(L[y]-L_0)$$

$$L(y)=\int_0^{L_0}dx\left(\sqrt{1+y'^2}-1\right)=\int_0^{L_0}dx\left(\cancel{1}+\frac{1}{2}y'^2\cancel{-1}\right)+O(y'^4)$$

$$\mathcal{L}_0 = T - U = \int_0^{L_0} dx \left[\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}T y'^2\right]$$

with condition $\quad y(t_1,x)=Y_1(x) \qquad y(t,0)=0$

$\qquad\qquad\qquad\quad y(t_2,x)=Y_2(x) \qquad y(t,L_0)=0$

$$S = \int_{t_1}^{t_2} dt \int_0^{L_0} dx \left[\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}T y'^2\right]$$

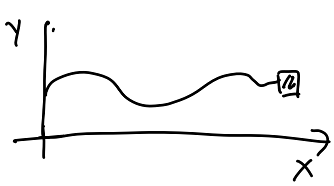
$$\delta S = \int_{t_1}^{t_2} dt \int_0^{L} dx \left[\overbrace{\frac{1}{2}\rho\dot{y}\,2\,\delta\dot{y}}^{\frac{\partial S}{\partial \dot{y}}\delta\dot{y}} - \overbrace{\frac{1}{2}T y'\,2\,\delta y'}^{\frac{\partial S}{\partial y'}\delta y'}\right]$$

$$= \int_{t_1}^{t_2} dt \int_0^{L} dx \underbrace{\left(-\rho\ddot{y} + Ty''\right)}_{\rho\ddot{y} - Ty'' = 0}\delta y + \int_0^{L} dx \left(\rho\dot{y}\cancel{\delta y}^{\,0}\right)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left(Ty'\cancel{\delta y}^{\,0}\right)\Big|_0^{L}$$

$$\rho\ddot{y} - Ty'' = 0$$

$$y = f_1(x - vt) + f_2(x + vt)$$

Suppose now:



then $\quad Y(x=0,t)=0$

$\qquad\quad Y(x,t_1)=Y_1$

$\qquad\quad Y(x,t_2)=Y_2$

$$L = \int_0^L \frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2 + \frac{1}{2}M\dot{y}^2 \, dx$$

upper boundary survives

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \underbrace{\left(-\rho\ddot{y} + Ty''\right)} \delta y + \int_0^L dx \, \overset{0}{\cancel{\left(\rho\dot{y}\,\delta y\right)}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left(Ty'\delta y\right)\Big|_{\cancel{0}}^{L}$$

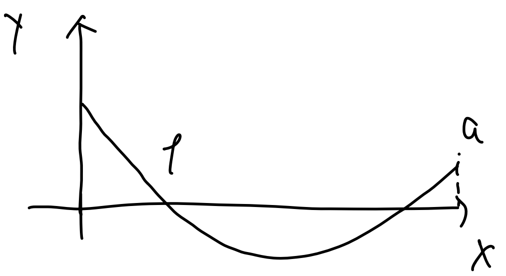
$$+ \frac{1}{2}m\dot{y}(L)\delta\dot{y}(L) - \frac{\delta U}{\delta y}\delta y(L)$$

$$\delta S = \int_{t_1}^{t_2} dt \left(\int_0^L dx \left[\left(-\rho\ddot{y} + Ty''\right) \delta y \right] + \left(-m\ddot{y}(L) - Ty''(L)\right) \delta y(L) \right) = 0$$

$$\rho\ddot{y} - Ty'' = 0 \quad \leftarrow \text{wave eq}$$

$$m\ddot{y}(L) + Ty' = 0$$

Weighted Catenary:



$$L[y] = \int_0^a dx \sqrt{1 + y'^2} = L_0$$

$$U[y] = \int_0^a dl \, \rho g y = \int_0^a dx \sqrt{1 + y'^2} \, \rho g y$$

but $\rho(l)$ instead of $\rho(x)$

Parametric $x \to l$

solve for $y(l)$ and $x(l)$, $l=0, L_0$

with constraint

$$dl^2 = dx^2 + dy^2 = \underbrace{(\dot{x}^2+\dot{y}^2)}_{=1 \text{ for each point, so infinite constraint.}} dl^2$$

$(\dot{x}dl)^2 + (\dot{y}\,dl)^2$

$\dot{x} = \frac{dx}{dl}$

$$\Lambda[x,y,\lambda] = \int_0^{L_0} d\lambda(l) \left(\sqrt{\dot{x}^2+\dot{y}^2} - 1\right)$$

$\hookrightarrow \frac{\delta\Lambda}{\delta\lambda} = \sqrt{\dot{x}^2+\dot{y}^2} - 1$

$$U = \int_0^{L_0} dl\, \rho(l)\, g\, y(l)$$

Find $x^*(l)$, $y^*(l)$, $\lambda^*(l) = \underset{x,y,\lambda}{\operatorname{argmin}}\underbrace{(U - \Lambda)}_{F = \int dl \left(\rho g y - \lambda\left(\sqrt{\dot{x}^2+\dot{y}^2} - 1\right)\right)}$

$\frac{\delta F}{\delta \lambda} = \sqrt{\dot{x}^2+\dot{y}^2} - 1$ ①

$$\frac{\delta F}{\delta x} = \frac{\partial F}{\partial x} - \frac{d}{dl}\left(\frac{\partial F}{\partial \dot{x}}\right) = -\frac{d}{dl}\left(-\lambda\left[\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right]\right) = 0$$

$$\lambda \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = C$$

with ①: $\sqrt{\dot{x}^2+\dot{y}^2} = 1$

$$\lambda = \frac{C}{\dot{x}}$$

$$\frac{\delta F}{\delta y} = \frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = \rho g + \frac{d}{dt}\left(\lambda \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0$$

$$\rho g + \frac{d}{dt}(\lambda \dot{y}) = 0$$

$$\rho g + C \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = 0$$

$$\frac{\rho g}{C} \pm \frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{1-\dot{y}^2}}\right) = 0 \qquad \dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'\dot{x}$$

$$\dot{x} = \frac{1}{\sqrt{1+y'^2}}$$

let $x = R\sin\frac{t}{R}$ $\qquad y = -R\cos\frac{t}{R}$

$$\dot{x} = \cos\frac{t}{R} \qquad \dot{y} = \sin\frac{t}{R} \qquad \frac{\dot{y}}{\dot{x}} = \tan\frac{t}{R}.$$

$$\frac{d}{dt}\frac{\dot{y}}{\dot{x}} = \frac{1}{R}\frac{1}{\cos^2 t/R}$$

$$\rho = -\frac{C}{g}\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)$$

$$\rho(t) = -\frac{C}{g}\frac{1}{R}\frac{1}{\cos^2 t/R}$$