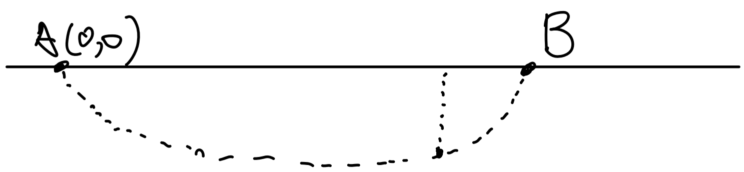
$F[f]$

Functional function

takes function
as an argument
and gives a number



time ⇒ T to minimum C(a,b)

Have trajectory : $y(x)$

Know $y(0) = 0$

$y(a) = b$

Total Energy Conservation:

$$m\frac{v_x^2+v_y^2}{2} = mgy$$

using $\frac{v_y}{v_x} = \frac{dy}{dx}$

$$\frac{1}{2}v_x^2\left(1+\frac{dy}{dx}^2\right) = gy \quad \Rightarrow \quad \frac{1}{v_x} = \sqrt{\frac{2gy}{1+\frac{dy}{dx}^2}}$$

then time of travel : $A \to C$

$$T[y(x)] = \int_0^T dt = \int_0^a \frac{dx}{v_x} = \int_0^a dx\sqrt{\frac{1+y'^2}{2gy}}$$

∵ So here we need to find $y(x)$ such that the functional T is minimum.

Suppose $y(x) = \underset{\text{stationary point}}{y^*(x)} + \underset{\epsilon \ll 1}{\epsilon\eta(x)}$

$$T[y(x)] = T[y^*(x) + \epsilon\eta(x)] > T[y^*(x)]$$

$$\stackrel{!}{=} T[y^*(x)] + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^2)$$

must be 0 since when expanding around stationary point. $\frac{\delta T}{\delta f(x)} = 0$

Local Functional in x: $T[y] = \int_{x_1}^{x_2} f(x, y, y', y'', \dots, y^{(n)})\, dx$

$$T[y] = \int dx\, f(x, y, y')$$

where $\delta y = \epsilon\eta(x)$, $\delta y' = \epsilon\eta'(x)$

$$T[y^* + \delta y] = \int dx\, f(x, y + \delta y, y' + \delta y')$$

$$\stackrel{!}{=} \int dx\, f(x, y, y') + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

$$T[y^* + \delta y] = T[y^*] + \epsilon \int dx\, \eta(x)\frac{\partial f}{\partial y} + \eta'(x)\frac{\partial f}{\partial y'}$$

$$\stackrel{!}{=} T[y^*] + \epsilon \int dx\, \eta(x)\frac{\partial f}{\partial y} + \frac{d}{dx}\left(\eta(x)\frac{\partial f}{\partial y'}\right) - \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$T[y^* + \delta y] - T[y^*] = \epsilon \int_0^a dx\, \eta(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] + \eta(x)\frac{\partial f}{\partial y'}\Big|_0^a$$

$$T[y^* + \delta y] - T[y^*] = \int_0^a dx \underbrace{\epsilon \eta(x)}_{\delta y} \underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]}_{\frac{\delta T}{\delta y}}$$

Since boundary are fixed, these vanish.

In discrete form:

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial y_i} \delta y_i$$

$\frac{\delta T}{\delta y} \equiv 0$ is a "sufficient" condition for $\delta T = 0$ is clear from

but to see it is the "necessary" condition, we must appeal to the assumed smoothness of $y(x)$.

proof:
Consider $y(x) = y^*(x)$ so that $T[y]$ is stationary, i.e. $\delta T = 0$, but $\frac{\delta T}{\delta y} \neq 0$ at some $x_0 \in [x_1, x_2]$. Because $f(x, y, y')$ is smooth, $\frac{\delta T}{\delta y}$ is also a smooth function of x. Therefore, by continuity, $\frac{\delta T}{\delta y}$ has the same sign throughout some interval containing x_0 (since $\frac{\delta T}{\delta y} \neq 0$ at $x_0 \in [x_1, x_2]$)
Let $\delta y(x) = \epsilon \eta(x)$ to be zero outside of $[x_1, x_2]$, and one sign within it, then we have $\delta T = \sum \frac{\partial T}{\partial y_i} \delta y_i \neq 0$ since $\delta y_i \neq 0$ which is a contradiction to stationarity.

If f depends on higher derivative in y:

$$f = f(x, y, y', y'', y''' \dots)$$

$$0 = \frac{\delta T}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right) + \cdots$$

To find minimum of T:

$$T[y] = \int_0^a dx \sqrt{\frac{1+y'^2}{2gy}} = \int_0^a dx\, f(x, y(x), y'(x))$$

use

$$\frac{\delta T}{\delta y} = 0$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

$$\frac{\partial f}{\partial y} = \frac{-1}{2y}\sqrt{\frac{1+y'^2}{2gy}}$$

$$\frac{\partial f}{\partial y'} = y' \frac{1}{\sqrt{2gy(1+y'^2)}}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = y'' \frac{1}{\sqrt{2gy(1+y'^2)}}$$

$$+ y'\left[-\frac{1}{2}\left(2gy(1+y'^2)\right)^{-3/2}\left[y'(1+y'^2) + 2yy'y''\right]\right]$$

$$\hookrightarrow \frac{1}{\sqrt{2gy(1+y'^2)}}\left(-\frac{1}{2y} - \frac{y''}{1+y'^2}\right)$$

$$\hookrightarrow = \frac{-g}{\left(2gy(1+y'^2)\right)^{3/2}}\left(\frac{1}{2}\left(1+y'^2\right)+yy''\right) = 0$$

let $x(\theta) = c(\theta - \sin\theta)$

$y(\theta) = c(1-\cos\theta)$

$$y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{c\sin\theta}{c(1-\cos\theta)} = \cot\frac{\theta}{2}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'/d\theta}{dx/d\theta} = \frac{\frac{1}{2}\left(-\frac{1}{\sin^2(\frac{\theta}{2})}\right)}{c(1-\cos\theta)}$$

$$1+y'^2 = 1+\cot^2\left(\frac{\theta}{2}\right) = \frac{1}{\sin^2\frac{\theta}{2}}$$

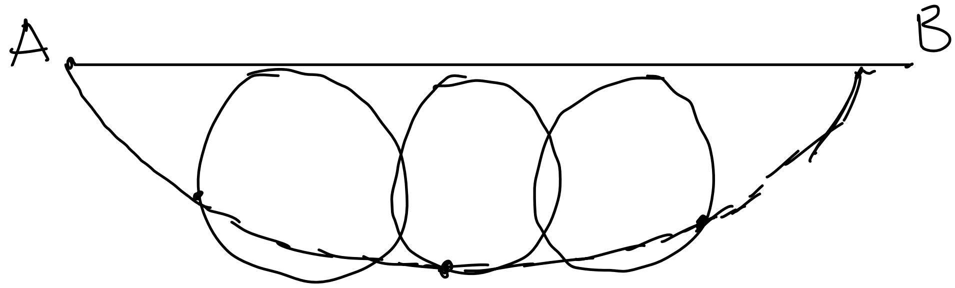
$$y \cdot y'' = c(1-\cos\theta)\left[\frac{\frac{1}{2}\left(-\frac{1}{\sin^2\frac{\theta}{2}}\right)}{c(1-\cos\theta)}\right] = \frac{-1}{2\sin^2\frac{\theta}{2}}$$

let $\theta = \omega t$, let $C = R$

$$x(\theta) \Rightarrow x(t) = R(\omega t - \sin\omega t) = vt - R\sin\omega t$$

$$y(\theta) \Rightarrow y(t) = R(1-\cos\omega t) = R - R\cos\omega t$$

circular motion.



After we found the trajectory, plug y into the functional T:

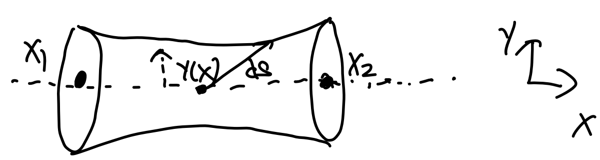
$$T = \int_0^a dx \sqrt{\frac{1+y'^2}{2gy}} = \int_0^{2\pi} d\theta\, C(1-\cos\theta) \sqrt{\frac{1}{\sin^2\frac{\theta}{2}} \frac{1}{g\frac{C(1-\cos\theta)}{2\sin^2\frac{\theta}{2}}}}$$

$$= \frac{1}{\sqrt{gC}} C \int_0^{2\pi} d\theta\, 2\sin^2\frac{\theta}{2} \frac{1}{2\sin^2\frac{\theta}{2}} = 2\pi\sqrt{\frac{C}{g}}$$

Ex 2: Soap film supported by a pair of coaxial rings.

The free energy of the soap film is equal to twice (one for each liquid-air interface) the surface tension σ of the soap times area of the film. So the film can minimize its free energy by minimizing the area, the axial symmetry suggests that the minimal surface will be a surface revolution about the x-axis.

So we need to find y such that $A(y(x), y'(x))$ is minimized



$$A[y] = \int y \, d\theta \, ds$$

$$\stackrel{!}{=} 2\pi \int y \sqrt{dy^2 + dx^2}$$

$$A[y] \stackrel{!}{=} 2\pi \int_{x_1}^{x_2} y \sqrt{1 + y'^2} \, dx$$

Find $y(x)$ such that $A[y]$ is minimized.

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

$$\Rightarrow = \sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{y y'}{\sqrt{1+y'^2}} \right)$$

$$\overset{!}{=} \sqrt{1+y'^2} - \left\{ \frac{y'^2}{\sqrt{1+y'^2}} + \frac{yy''}{\sqrt{1+y'^2}} - \frac{yy'^2y''}{(1+y'^2)^{3/2}} \right\}$$

$$\overset{!}{=} \frac{1+\cancel{y'^2} - \cancel{y'^2} - yy'' + yy'^2y''/(1+y'^2)}{\sqrt{1+y'^2}}$$

$$\overset{!}{=} \frac{1-yy''}{\sqrt{1+y'^2}} + \frac{yy'^2y''}{(1+y'^2)^{3/2}}$$

$$\overset{!}{=} \frac{1}{\sqrt{1+y'^2}}\left\{ \underbrace{1-yy''+\frac{yy'^2y''}{1+y'^2}}_{1-\frac{(yy'')(1+y'^2)}{(1+y'^2)}+\frac{yy'^2y''}{1+y'^2}} \right\}$$

$$= \frac{1}{\sqrt{1+y'^2}}\left\{1-\frac{yy''}{1+y'^2}\right\} = 0$$

multiply y' to both sides:

$$\frac{y'}{\sqrt{1+y'^2}} - \frac{yy''y'}{(1+y'^2)^{3/2}} = 0$$

$$\frac{d}{dx}\left\{\frac{y}{\sqrt{1+y'^2}}\right\} = 0$$

then
$$\frac{y}{\sqrt{1+y'^2}} = K$$

separating variables:
$$\sqrt{\left(\frac{y}{K}\right)^2 - 1} = \frac{dy}{dx}$$
$$\int dx = \int dy \left(\left(\frac{y}{K}\right)^2 - 1\right)^{-1/2}$$

if we let $y = K\cosh t$
$$dy = K\sinh t \, dt$$
$$\int dx = \int \left[\underbrace{\left(\frac{\cancel{K}\cosh t}{\cancel{K}}\right)^2 - 1}\right]^{-1/2} K\sinh t \, dt.$$

use $\cosh^2 t - \sinh^2 t = 1$.
$$\int_{x_0}^{x} dx = \int_{t_0}^{t} K \, dt.$$
$$x - x_0 = Kt$$
$$t = \frac{x - x_0}{K}$$

remember we still need $y = y(x)$

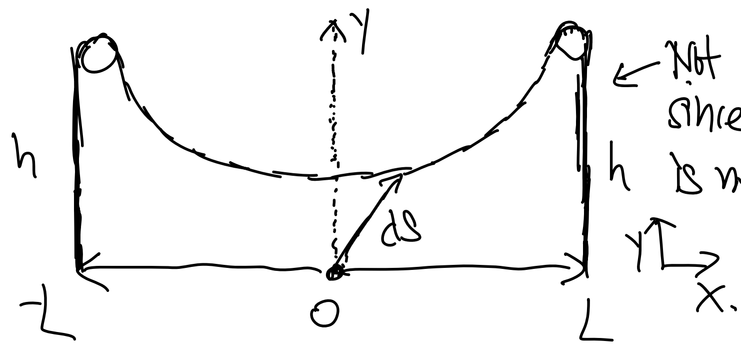
so use $t = \frac{x - x_0}{K}$

$$y = K \cosh\left(\frac{x - x_0}{K}\right)$$

$$y(x = x_1) = y_1 = K \cosh\left(\frac{x_1 - x_0}{K}\right)$$

$$y(x = x_2) = y_2 = K \cosh\left(\frac{x_2 - x_0}{K}\right)$$

Ex. 2:



← Not a catenary since length of rope is not fixed.

Determine the shape of the chain:

$$\mathcal{L} = \overset{0}{\cancel{T}} - V$$

$$V = mgy = \int \frac{M}{L_{tot}} g\, y\, ds$$

$\downarrow$

ρ = mass density of rope.

$$\overset{!}{=} \rho g \int y \sqrt{dx^2 + dy^2}$$

$$\overset{1}{=} \rho g \int_{-L}^{L} y\sqrt{1+y'^2}\,dx$$

Want to find y such that V is minimized.

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

$$\sqrt{1+y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] = 0$$

$$\frac{1+\cancel{y'^2}}{\sqrt{1+y'^2}} - \left[\frac{\cancel{y'^2}}{\sqrt{1+y'^2}} + \frac{yy''}{\sqrt{1+y'^2}} - \frac{yy'^2y''}{(1+y'^2)^{3/2}}\right] = 0$$

$$\hookrightarrow\ y'\left(\frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0\right)$$

$$\frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{y}{\sqrt{1+y'^2}} = K$$

$$\frac{dy}{dx} = \sqrt{\left(\frac{y}{K}\right)^2 - 1}$$

$$\int 1\left(\left(\frac{y}{K}\right)^2 - 1\right)^{-1/2} \quad \int 1$$

$$\int dy \left(\left(\frac{y}{k}\right) - 1 \right) = \int dx$$

let $y = k\cosh t$ $\quad dy = k\sinh t \, dt$

$$\int k \, dt = \int dx$$

So $$y = k\cosh\left(\frac{x - x_0}{k}\right)$$ same as before.

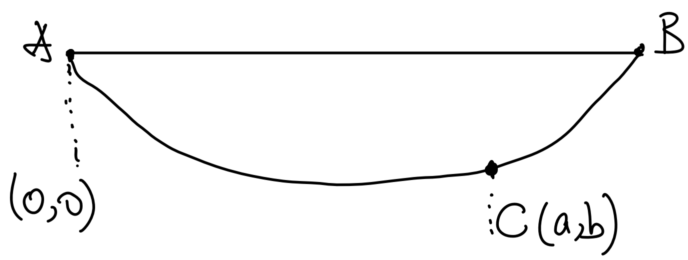
$$y(x=-L) = h = k\cosh\left(\frac{-L - x_0}{k}\right)$$
$$y(x=L) = h = k\cosh\left(\frac{L - x_0}{k}\right)$$

this requires $x_0 = 0$.

$$y(x=0) = y_{min} = k$$

$$y(x) = y_{min}\cosh\left(\frac{x}{y_{min}}\right)$$

Ex 3: Brachistochrone:



What is the minimum time takes for the cart to move from $A \to C$.

Want to minimize time

want to write T interms of $y(x)$.

$$T = \int_0^T dt = \int_0^a \frac{dx}{\frac{dx}{dt}}$$

Initially, $y=0$ and $v_x, v_y = 0$

energy conservation: $\frac{1}{2}\cancel{m}(v_x^2+v_y^2) = \cancel{m}gy.$

to find $\frac{dx}{dt}$:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2gy.$$

$$\left(\frac{dx}{dt}\right)^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = 2gy$$

$$\frac{dx}{dt} = \sqrt{\frac{2gy}{1+y'^2}}$$

then $$T[y(x)] = \int_0^a \underbrace{\sqrt{\frac{1+y'^2}{2gy}}}_{f}\, dx$$

To minimize T, $\frac{\delta T}{\delta y} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\hookrightarrow \frac{-1}{2y}\sqrt{\frac{1+y'^2}{y}} - \frac{d}{dx}\left(\frac{y'}{\sqrt{(1+y'^2)y}}\right) = 0$$

$$\hookrightarrow \frac{\frac{-1}{2y}(1+y'^2)}{\sqrt{(1+y'^2)y}} - \left\{\frac{y''}{\sqrt{(1+y'^2)y}} - \frac{1}{2}\frac{y'\left(y'[1+y'^2]+2yy'y''\right)}{\left((1+y'^2)y\right)^{3/2}}\right\}$$

$$\hookrightarrow \frac{1}{\quad} \left\{ \frac{-1}{\quad}\left(1+y'^2\right) - y'' + \frac{1}{\quad}\left(y'^2[1+y'^2] + 2yy'^2y''\right) \right.$$

$$\sqrt{(1+y'^2)y} \quad \big(2y(1+\ldots) \quad \ldots \quad \frac{2(\ldots)}{(1+y'^2)\,y}$$

$$\hookrightarrow \frac{1}{[(1+y'^2)y]^{3/2}}\Big\{\underbrace{-\tfrac{1}{2}(1+y'^2)^2 - y''(1+y'^2)y + \tfrac{1}{2}y'^2(1+y'^2) + yy'^2y''}_{-\frac{1}{2} - y'^2 - \cancel{\frac{1}{2}y'^4} - yy'' - \cancel{yy'^2y''} + \cancel{\frac{1}{2}y'^2} + \cancel{\frac{1}{2}y'^4} + \cancel{yy'^2y''}}\Big\}$$

$$= -\tfrac{1}{2} - \tfrac{1}{2}y'^2 - yy''$$

$$\stackrel{!}{=} -\left(\tfrac{1}{2}[1+y'^2] + yy''\right)$$

We would need to require.

$$yy'' + \tfrac{1}{2}[1+y'^2] = 0$$

multiply by y' since $f = f(y, y', y'')$, no explicit dependence on x.

$$y''y'y + \tfrac{1}{2}y'[1+y'^2] = 0$$

$$\hookrightarrow \frac{1}{2}\frac{d}{dx}\left(y(1+y'^2)\right) = 0$$

$$y(1+y'^2) = 2C$$

$$y' = \sqrt{\frac{2C}{y} - 1}$$

This differential equation has solution:

$$x = c(\theta - \sin\theta)$$
$$y = c(1 - \cos\theta)$$

First Integral:

Multiply by y' trick works if f is not explictely dependent on x.

$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x}$$

if $\frac{\partial f}{\partial x} = 0$. then no explicite dependent on x.

If $\frac{\partial f}{\partial x} = 0$:

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}\frac{dy}{dx} + \cancel{\frac{\partial f}{\partial y'}\frac{dy'}{dx}} - \cancel{\frac{dy'}{dx}\frac{\partial f}{\partial y'}} - \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$\overset{!}{=} \frac{dy}{dx}\underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)}_{\text{Euler-Lagrange}}$$

In otherwords, multiplying y' to E-L, we get

$$y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) = \frac{d}{dx}\Big(\underbrace{f - y'\frac{\partial f}{\partial y'}}_{I,\ \text{first integral.}}\Big) = 0$$

$I = f - y'\frac{\partial f}{\partial y'}$ is conserved with $\frac{\partial f}{\partial x} = 0$.

which also means we can multiply E-L both sides by y', to get $\frac{d}{dx}(I) = 0$ or $I =$ constant.

Ex: In soap-film:

$$I = f - y'\frac{\partial f}{\partial y'} = y\sqrt{1+y'^2} - y'y\frac{y'}{\sqrt{1+y'^2}}$$

$$= \frac{y(1+y'^2) - y\,y'^2}{\sqrt{1+y'^2}}$$

$$I = \frac{y}{\sqrt{1+y'^2}}$$

For multiple dependent variables y_i:

$$I = f - \sum_i y_i' \frac{\partial f}{\partial y_i'}$$

$$\frac{dI}{dx} = \sum_i \left(\frac{\partial f}{\partial y_i}\frac{dy_i}{dx} + \frac{\partial f}{\partial y_i'}\frac{dy_i'}{dx} - \frac{dy_i'}{dx}\frac{\partial f}{\partial y_i'} - \frac{dy_i}{dx}\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right)$$

$$\frac{dI}{dx} = \sum_i \frac{dy_i}{dx}\left(\underbrace{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)}_{=0}\right)$$

by Euler-Lagrange.