$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,\hat{e}_\phi$

compare to differential geometry:

Basis vectors not normalized. $\partial r, \partial\theta, \partial\phi$

$$(dr\,\partial r + d\theta\,\partial\theta + d\phi\,\partial\phi) f(r,\theta,\phi)$$

$$(\partial r f\,dr + \partial\theta f\,d\theta + \partial\phi f\,d\phi) = df.$$

In orthonormal basis:

$$\vec{\nabla} f(r,\theta,\phi) = (\partial r f_r)\hat{e}_r + \frac{1}{r}(\partial_\theta f)\hat{e}_\theta + \frac{1}{r\sin\theta}(\partial\phi f)\hat{e}_\phi$$

Divergence: $S_r = h_\phi h_\theta = r^2\sin\theta \quad S_\theta = h_r h_\phi = r\sin\theta \quad S_\phi = h_r h_\theta = r$

$$\oint dS\,(\vec{g}\cdot\vec{n}) = \frac{\partial}{\partial r}(g_r S_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(g_\theta S_\theta) + \frac{\partial}{\partial\phi}(g_\phi S_\phi)$$

since $\int \operatorname{div} g\, dV \approx V \operatorname{div} g.$

and $\int \operatorname{div} g\, dV = \oint dS\,(\vec{g}\cdot\vec{n})$

$$\operatorname{div} g = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, g_r) + \frac{\partial}{\partial\theta}(r\sin\theta\, g_\theta) + \frac{\partial}{\partial\phi}(r g_\phi)\right]$$

$$\stackrel{!}{=} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, g_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} g_\phi$$

$$\nabla^2 f = \nabla\cdot(\vec{\nabla} f) = \frac{1}{r}\partial_r^2(rf) + \frac{1}{r^2}\underbrace{\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{\sin^2\theta}\partial_\phi^2 f\right]}_{=-\hat{L}^2}$$

<u>Solve Laplace Eq in 3D:</u>

$$\nabla^2 f(r,\theta,\phi) = 0$$

let $f(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

then
$$\begin{cases} \hat{L}^2 Y = l(l+1)Y \\ \frac{1}{r}\frac{\partial^2}{\partial r^2}(rR(r)) = \frac{1}{r^2}l(l+1)R \end{cases}$$

<u>Radial Part:</u> $r\,\partial_r^2(rR) = l(l+1)R$

two sols: $R(r) = r^{l}$ vs. $R(r) = r^{-l-1}$

for $r\to 0$ for $r\to\infty$

<u>Angular Part:</u>

Case 1: Axial Symmetric $\partial_\phi Y = 0$

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P_l) = l(l+1)P_l(\cos\theta)$$

Legendre Polynomial.

$$-1 \le x = \cos\theta \le 1 \qquad d\theta = \frac{-dx}{\sin\theta}$$

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}P_\ell(x)\right) = \ell(\ell+1)P_\ell(x)$$

Hypergeometric Equations with Rodrigue's Eq as general sol.

$$\sigma y'' + \tau y' + \lambda y = 0$$

$$\frac{1}{w}(w\sigma y')' + \lambda y = 0 \qquad (w\sigma)'' = w\tau$$

Rodrigue's formula: $V_n = \frac{1}{w}(w\sigma^n)^{(n)} \to (\sigma V_n')' = -\lambda_n V_n$

For Legendre: $\sigma = 1-x^2$

$w = 1$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left[(1-x^2)^\ell\right]^{(\ell)}$$

Normalized such that

$P_\ell(1) = 1$

$P_\ell(-1) = (-1)^\ell$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y\right)+\frac{1}{\sin^2\theta}\underbrace{\frac{\partial^2}{\partial\phi^2}}Y=l(l+1)Y$$

$$e^{im\phi}\,P_l^{|m|}(\cos\theta)$$

$$\frac{1}{\partial(\cos\theta)}\left[\sin^2\theta\frac{\partial}{\partial\cos\theta}P_l^{|m|}(\cos\theta)\right]-\frac{1}{\sin^2\theta}m^2P_l^{|m|}=l(l+1)P_l^{|m|}$$

Case $\underline{m=0}$:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_l(\cos\theta)\right]=l(l+1)P_l$$

$$(w\sigma)'=w\tau \qquad \sigma y''+\tau y'+\lambda y=0$$

$$\hookrightarrow (\sigma w y')'+\lambda_n w y=0$$

$$y_n=\frac{1}{w}(w\sigma^n)^{(n)}$$

$$P_l(x)=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(1-x^2)^l \quad \text{with } P_l(1)=1 \quad P_l(-1)=(-1)^l$$

For $-\nabla^2\phi=4\pi\delta^{(3)}(\vec{r})\Rightarrow\phi(\vec{r})=\frac{1}{|\vec{r}|}$

$$-\nabla^2\phi=4\pi\delta^{(3)}(\vec{r}-\vec{R})\Rightarrow\phi(\vec{r})=\frac{1}{|\vec{r}-\vec{R}|}$$

Multipole Expansion:

$\phi(\vec{r}) \nearrow$

$|\vec{r}|<|\vec{R}|$: $\phi(\vec{r})=\sum A_l r^l P_l(\cos\theta)$ $r\to 0$

$|\vec{r}|>|\vec{R}|$: $\phi(\vec{r})=\sum B_l r^{-l-1}P_l(\cos\theta)$ $r\to\infty$, $\phi\to 0$

Axial symmetry : $m=0$, independent of ϕ.

$$f(\vec{r}) \Rightarrow f(|\vec{r}|, \cos\theta)$$

$$f(\vec{r}, \cos\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta)$$

for $\cos\theta = 1$:

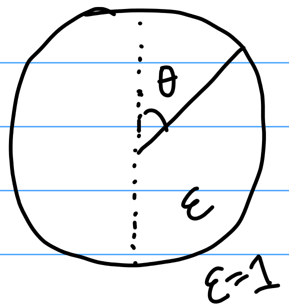
$$\frac{1}{|\vec{r}-\vec{R}|} \begin{cases} \to \frac{1}{R}\left(1 + \frac{r}{R} + \left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^3 + \cdots\right) & r<R \\ \to \frac{1}{r}\left(1 + \frac{R}{r} + \left(\frac{R}{r}\right)^2 + \cdots\right) & r>R \end{cases}$$

$$\frac{1}{|\vec{r}-\vec{R}|} = \begin{cases} r<R & \frac{1}{R}\sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^l P_l(\cos\theta) \\ r>R & \frac{1}{r}\sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^l P_l(\cos\theta) \end{cases}$$

Legendre Polynomial generating function:

$$\frac{R}{|\vec{r}-\vec{R}|} = \frac{1}{\sqrt{1-2\cos\theta \underset{t}{\underset{\downarrow}{\frac{r}{R}}} + \underset{t^2}{\underset{\downarrow}{\left(\frac{r}{R}\right)^2}}}}$$

$$= \frac{1}{\sqrt{1-2\cos\theta\, t + t^2}} = \sum_{l=0}^{\infty} t^l P_l(\cos\theta)$$



$$\vec{E}_0 = E_0\cos\theta\,\hat{e}_r - E_0\sin\theta\,\hat{e}_\theta$$

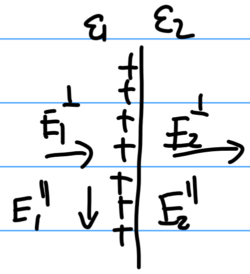
$$-\nabla^2\phi^{in}_{sphere} = 0 \qquad \phi_S^{in} = \sum_{l=0}^{\infty} P_l(\cos\theta)\, r^l \;\text{[finite]}$$

$$-\nabla^2\phi^{out}_{sphere} = 0 \qquad \phi_S^{out} = \sum_{l=0}^{\infty} P_l(\cos\theta)\, r^{-l-1} \xrightarrow[r\to\infty]{} 0$$

Since $\vec{E}_0$ only has $l=1$ component, so just consider $l=1$ case, dipole:

$$\phi_S^{in} = r\cos\theta\, A$$

$$\phi_S^{out} = \frac{1}{r^2}\cos\theta\, B$$



$$E_S^{in} = -\nabla\phi_S^{in} = -A\cos\theta\,\hat{e}_r - A\sin\theta\,\hat{e}_\theta$$

$$E_S^{out} = -\nabla\phi_S^{out} = \frac{2B}{r^3}\cos\theta\,\hat{e}_r - \frac{B}{r^3}\sin\theta\,\hat{e}_\theta$$

$$E_1^{\parallel} = E_2^{\parallel}$$

$$E_1^{\perp} \neq E_2^{\perp}$$

but

$$\underbrace{\varepsilon_1\varepsilon_0 E_1^{\perp}}_{D_1^{\perp}} = \underbrace{\varepsilon_2\varepsilon_0 E_2^{\perp}}_{D_2^{\perp}}$$

Since $E_1^{\parallel} = E_2^{\parallel}$ $(\hat{\theta})$

$$(-E_0 - A)\sin\theta = -E_0 - \frac{B}{R^3}\sin\theta$$

$$\Rightarrow B = AR^3$$

Now $\quad \varepsilon_1 \cancel{\varepsilon_0} E_1^{\perp} = \varepsilon_2 \cancel{\varepsilon_0} E_2^{\perp} \quad (\hat{r})$

$$\hookrightarrow \frac{\varepsilon_1}{\varepsilon_2} E_1^{\perp} = \varepsilon E_1^{\perp} = E_2^{\perp}$$

$$\hookrightarrow \varepsilon(E_0 - A)\cos\theta = \Big(E_0 + \underbrace{\frac{2B}{R^3}}_{2A}\Big)\cos\theta$$

$$A(2+\varepsilon) = E_0(\varepsilon - 1)$$

$$\Rightarrow A = \frac{E_0(\varepsilon-1)}{2+\varepsilon}$$

Then the total field inside:

$$E_{(z)}^{tot,in} = E_0 - A = E_0\left(1 - \frac{\varepsilon-1}{\varepsilon+2}\right) = E_0 \frac{3}{\varepsilon+2}$$

Compare to dipole field $\quad \phi_{dipole} = \frac{-\vec{p}\cdot\hat{r}}{r^2} = \frac{-p\cos\theta}{4\pi\varepsilon_0 r^2}$

$$\phi_B = \frac{B\cos\theta}{r^2}$$

then $\quad B = \frac{-p}{4\pi\varepsilon_0}$

$$p = -4\pi\varepsilon_0 B$$

should be $+$ since $p \parallel E$,

$$p = -4\pi\varepsilon_0 A R^3 = -\frac{\varepsilon-1}{\varepsilon+2} 4\pi R^3 \varepsilon_0 E_0$$

for $\varepsilon = 1+\chi$

$$= \frac{\chi}{1+\chi/3} \underbrace{\frac{4\pi}{3}R^3}_{\text{volume}} \varepsilon_0 E_0$$

then $P = \frac{p}{V} = \varepsilon_0 E_0 \frac{\chi}{1+\frac{\chi}{3}}$

for $\chi \ll 1$ $\quad = \varepsilon_0 E_0 \chi$