Zhi Chen HW#14 QM

1) Calculate # of of linearly independent singlets:

a) $\underbrace{\tfrac{1}{2}\otimes\tfrac{1}{2}}\otimes\underbrace{\tfrac{1}{2}\otimes\tfrac{1}{2}}$

$$= (0\oplus 1)\otimes(0\oplus 1)$$

$$\overset{!}{=} \underbrace{0\otimes 0}\oplus\underbrace{0\otimes 1}\oplus\underbrace{1\otimes 0}\oplus\underbrace{1\otimes 1}$$

$$\overset{!}{=} 0\oplus 1\oplus 1\oplus(0\oplus 1\oplus 2)$$

$$\overset{!}{=} 0\oplus 1\oplus 1\oplus 0\oplus 1\oplus 2$$

$$\overset{!}{=} 2\cdot 0\oplus 3\cdot 1\oplus 2$$

Use notation $a\cdot b$ where a is the # of repeated occurrence of $j=b$ to save writing

$\boxed{2\ \ j=0 \text{ state (singlet)}}$

b) $\underbrace{\tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2}}\otimes\underbrace{\tfrac{1}{2}\otimes\tfrac{1}{2}}_{0\oplus 1}$

$$=(2\cdot 0\oplus 3\cdot 1\oplus 2)\otimes(0\oplus 1)$$

$$\overset{!}{=}(2\cdot 0\oplus 3\cdot 1\oplus 2)\otimes 0\oplus(2\cdot 0\oplus 3\cdot 1\oplus 2)\otimes 1$$

$$\overset{!}{=}[2\cdot 0\oplus 3\cdot 1\oplus 2]\oplus[2\cdot 1\oplus 3\cdot(0\oplus 1\oplus 2)\oplus(1\oplus 2\oplus 3)]$$

$$\overset{!}{=} 5\cdot 0\oplus 9\cdot 1\oplus 5\cdot 2\oplus 3$$

$\boxed{5 \text{ singlet states}}$

2) Possible l values

a) 4 p-electrons, $p \to l = 1$.

$$\underbrace{1 \otimes 1} \otimes \underbrace{1 \otimes 1}$$

$$= (0 \oplus 1 \oplus 2) \otimes (0 \oplus 1 \oplus 2)$$

$$= (0 \oplus 1 \oplus 2) \otimes 0 \oplus (0 \oplus 1 \oplus 2) \otimes 1 \oplus (0 \oplus 1 \oplus 2) \otimes 2$$

$$= (0 \oplus 1 \oplus 2) \oplus [1 \oplus (0 \oplus 1 \oplus 2) \oplus (1 \oplus 2 \oplus 3)]$$

$$\oplus [2 \oplus (1 \oplus 2 \oplus 3) \oplus (0 \oplus 1 \oplus 2 \oplus 3 \oplus 4)]$$

$$= 3 \cdot 0 \oplus 6 \cdot 1 \oplus 6 \cdot 2 \oplus 3 \cdot 3 \oplus 4$$

$\boxed{\text{It can have } l \text{ from } 0, 1, 2, 3, 4.}$

b) $$\underbrace{1 \otimes 1} \otimes \underbrace{1 \otimes 3}$$

$$= (0 \oplus 1 \oplus 2) \otimes (2 \oplus 3 \oplus 4)$$

$$= 0 \otimes (2 \oplus 3 \oplus 4) \oplus 1 \otimes (2 \oplus 3 \oplus 4) \oplus 2 \otimes (2 \oplus 3 \oplus 4)$$

$$= 2 \oplus 3 \oplus 4 \oplus 1 \oplus 2 \oplus 3 \oplus 2 \oplus 3 \oplus 4 \oplus 3 \oplus 4 \oplus 5 \oplus 0 \oplus 1 \oplus 2 \oplus 3 \oplus 4$$
$$\oplus 1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6$$

$$= 0 \oplus 3 \cdot 1 \oplus 6 \cdot 2 \oplus 7 \cdot 3 \oplus 6 \cdot 4 \oplus 3 \cdot 5 \oplus 6$$

$\boxed{l \text{ can have values from } 0, 1, 2, 3, 4, 5, 6}$

3) Symmetric Top: $E = \dfrac{L_x^2 + L_y^2}{2I_1} + \dfrac{L_z^2}{2I_3}$

Find eigenenergy, eigenfunctions.

$$L^2 = L_x^2 + L_y^2 + L_z^2 \rightarrow L^2 - L_z^2 = L_x^2 + L_y^2$$

$$E = \frac{L^2}{2I_1} + L_z^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)$$

We know the spherical Harmonics, $Y_l^m(\theta,\phi)$ are eigenfunctions of both L^2 and L_z^2,

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} E\, Y_l^m(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left\{\frac{L^2}{2I_1} + L_z^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)\right\} Y_l^m$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{\hbar^2}{2I_1} l(l+1) + \hbar^2 m^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)\right) Y_l^m$$

$\therefore$ Eigenfunctions are $\sum_{l,m} Y_l^m(\theta,\phi)$

With eigenenergies $\quad E_{lm} = \dfrac{\hbar^2}{2}\left(\dfrac{1}{I_1} l(l+1) + m^2\left(\dfrac{1}{I_3} - \dfrac{1}{I_1}\right)\right)$

4) Energy levels of a particle in spherical box, radius R with $l=0$.

If $l=0 \rightarrow m=0$.

A free particle in a box obeys:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) = E\psi(\vec{r}).$$

$$\hookrightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}_{-\frac{1}{\hbar^2 r^2}L^2}\right]\psi(\vec{r}) = E\psi(\vec{r})$$

$$\hookrightarrow = \frac{1}{2mr^2}\left[-\hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + L^2\right]\psi = E\psi$$

let $\psi = R_{nl}(r)\,Y_l^m(\theta,\phi)$

$$\hookrightarrow = \frac{1}{2mr^2}\left[-\hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_{nl}(r)Y_l^m(\theta,\phi) + \hbar^2 l(l+1)R_{nl}(r)Y_l^m(\theta,\phi)\right] = E R_{nl}(r)Y_l^m$$

To pick out $l=0$, $m=0$ term, multiply both sides by Y_0^{*0}, then integrate over $d\Omega$. $\int d\Omega\, Y_{l'}^{*m'} Y_l^m = \delta_{mm'}\delta_{ll'}$, then we have $l=0$, $m=0$.

$$\hookrightarrow \frac{1}{2mr^2}\left[-\hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_{n0}(r)\right] = E R_{n0}(r)$$

let $u = rR \rightarrow R = \frac{u(r)}{r}$

$$\hookrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(ru' - u\right) + \frac{2mE}{\hbar^2}\frac{u}{r} = 0$$

$$\hookrightarrow \frac{1}{r^2}\left(u''r + \cancel{u' - u'}\right) + \frac{2mE}{\hbar^2}\frac{u}{r} = 0$$

$$\hookrightarrow u'' + \frac{2mE}{\hbar^2}u = 0$$

let $k^2 = \frac{2mE}{\hbar^2}$

then $\partial_r^2 u + k^2 u = 0$

or $u = A\sin kr + B\cos kr.$

since $R_{n0}(r) = \frac{u(r)}{r} = \frac{A\sin kr + B\cos kr}{r}$

as $r \to 0$ $R \to \infty$ unless $\boxed{B=0}$

as $r \to R$, require $R = \frac{A\sin kR}{R} = 0$,

so $\boxed{kR = n\pi}$ for $n = 1, 2, 3\cdots$

therefore since $k^2 = \frac{2mE}{\hbar^2} = \left(\frac{n\pi}{R}\right)^2$

Energy Level $\rightarrow$ $\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mR}}$ for $l=0$

5) $H = \frac{L^2}{2I} + f\vec{L}\cdot\vec{S}$

a) Find spectrum, E_n, and degeneracy of eigenstates, $f=0$

→ We can describe the eigenstates using $|l,s;j,m\rangle$

$$H|l,s;m_l\ m_s\rangle = \frac{L^2}{2I}|l,s;j,m\rangle$$

$$\stackrel{!}{=} \underbrace{\frac{1}{2I}\hbar^2 l(l+1)}\ |l,s;j,m\rangle$$

$$\boxed{E_l = \frac{1}{2I}\hbar^2 l(l+1)}$$

For a given l and s, there are $(2l+1)$ different m_l and $(2s+1)$ different m_s. Therefore the total # of degeneracy for a given l and s is $\boxed{(2l+1)(2s+1)}$

→ If $s=1/2$, then total degeneracy is then $4l+2$.

b) $H = \frac{L^2}{2I} + f\vec{L}\cdot\vec{S}$

since $J^2 = (\vec{L}+\vec{S})^2 = L^2 + \vec{L}\cdot\vec{S} + \underbrace{\vec{S}\cdot\vec{L}}_{=\vec{L}\vec{S}} + S^2$

↳ $\vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{J}^2 - L^2 - S^2)$

then $H = \frac{L^2}{2I} + f\frac{1}{2}(J^2 - L^2 - S^2)$

$$H \stackrel{!}{=} \frac{1}{2}\left[fJ^2 + \left(\frac{1}{I} - f\right)L^2 - fS^2\right]$$

$$H|l,s;j,m\rangle = \frac{1}{2}\left[fJ^2+\left(\frac{1}{I}-f\right)L^2-fS^2\right]|l,s;j,m\rangle$$

$$= \boxed{\underbrace{\frac{\hbar^2}{2}\left[fj(j+1)+\left(\frac{1}{I}-f\right)l(l+1)-fs(s+1)\right]}_{E_j \text{ for a given } l,s}}|l,s;j,m\rangle$$

→ Now j can take values of $\boxed{j\in\{l+s,\ l+s-1,\cdots,|l-s|\}}$

⇒ Assuming l and s are fixed. In the representation of $|l\ s;j\ m\rangle$, we then have $(2j+1)$ different m for a given j. Therefore we have a degeneracy of $\boxed{2j+1}$ for each eigenenergy, E_j. For example when $j=l+s$, then $E_{j=l+s}$ has degeneracy of $2(l+s)+1$

→ Suppose we work with $s=1/2$, then $j=\{l+1/2,\ l-1/2\}$

→ when $j=l-1/2$, there is a degeneracy of $2(l-1/2)+1=\boxed{2l}$

$$E_{l-1/2,l} = \frac{\hbar^2}{2}\left[f(l-1/2)(l-1/2+1)+\left(\frac{1}{I}-f\right)l(l+1)-\frac{3}{4}f\right]$$

$$= \frac{\hbar^2}{2}\left[f(\cancel{l^2}-\tfrac{1}{4})+\left(\frac{1}{I}-f\right)(\cancel{l^2}+l)-\frac{3}{4}f\right]$$

$$\boxed{E_{l-1/2,l} = \frac{\hbar^2}{2}\left[-f(l+1)+\frac{1}{I}l(l+1)\right]}$$

When $j=l+1/2$, there is a degeneracy of $2(l+1/2)+1=\boxed{2l+2}$

$$E_{l+1/2,l} = \frac{\hbar^2}{2}\left[f(l+1/2)(l+1/2+1)+\left(\frac{1}{I}-f\right)l(l+1)-\frac{3}{4}f\right]$$

$$= \frac{\hbar^2}{2}\left[f(\cancel{l^2}+2l+\cancel{\tfrac{3}{4}})+\left(\frac{1}{I}-f\right)(\cancel{l^2}+l)-\cancel{\tfrac{3}{4}f}\right]$$

$$\boxed{E_{l+1/2,l} = \frac{\hbar^2}{2}\left[fl+\frac{1}{I}l(l+1)\right]}$$

c) With $f=0$, $E_l = \frac{1}{2I}\hbar^2 l(l+1)$ with $(2l+1)(2s+1)$ degeneracy

with $f\neq 0$, $E_j = \frac{\hbar^2}{2I} l(l+1) + \underbrace{\frac{\hbar^2}{2} f\big(j(j+1) - l(l+1) - s(s+1)\big)}_{\text{Additional term due to spin-orbit}}$

Initially, $f=0$, a state with given l and s have a single eigen energy and $(2l+1)(2s+1)$ degeneracy.

After we let $f\neq 0$, the eigen energy splits into finer energy levels labeled by their total angular momentum j, where j takes value from $\{l+s, l+s-1, \cdots |l-s|\}$. For each value of j, or for each E_j, there are $2j+1$ degeneracies. However, the total # of states $|l,s;j,m\rangle$ are consistent with $(2l+1)(2s+1)$. So essentially by turning $f\neq 0$, we can break some degeneracies, but not all.

<u>Suppose we work with $s=\frac{1}{2}$ again:</u>

We see that $E_l = \frac{\hbar^2}{2}\frac{1}{I} l(l+1)$

while $E_{l-\frac{1}{2},l} = \frac{\hbar^2}{2}\frac{1}{I} l(l+1) - \frac{\hbar^2}{2} f(l+1)$

$= E_l - \frac{\hbar^2}{2} f(l+1)$ ← lower energy compared to E_l

and $E_{l+\frac{1}{2},l} = \frac{\hbar^2}{2}\frac{1}{I} l(l+1) + \frac{\hbar^2}{2} f l$

$= E_l + \frac{\hbar^2}{2} f l.$ ← higher energy compared to E_l.

We see that $E_l = \frac{\hbar^2}{2}\frac{1}{I}l(l+1)$

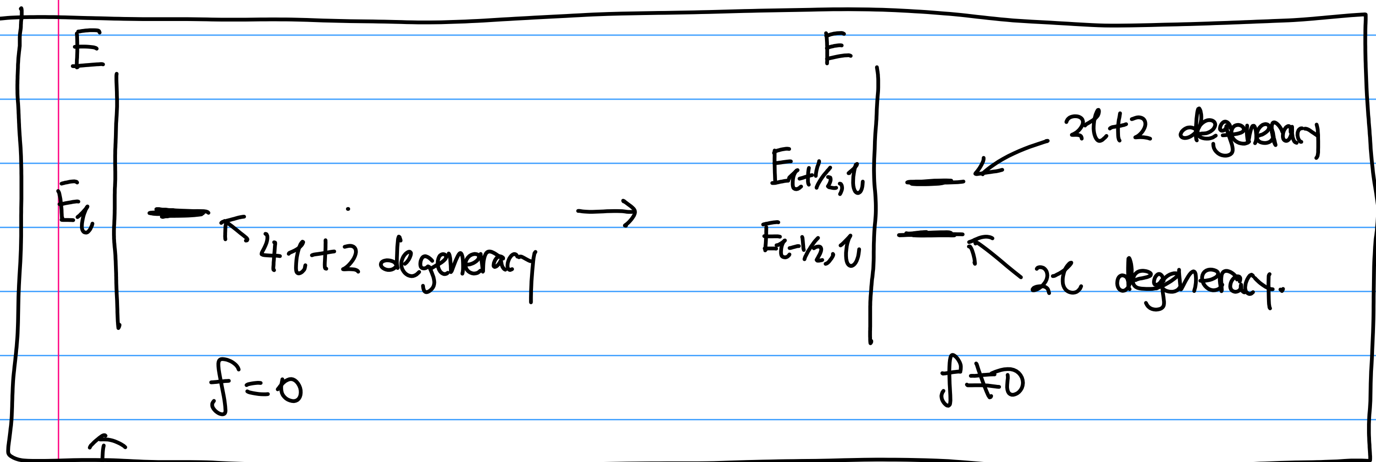
while $E_{l-1/2,l} = \frac{\hbar^2}{2}\frac{1}{I}l(l+1) - \frac{\hbar^2}{2}f(l+1)$

$\overset{!}{=} E_l - \frac{\hbar^2}{2}f(l+1)$ ← lower energy compared to E_l

and $E_{l+1/2,l} = \frac{\hbar^2}{2}\frac{1}{I}l(l+1) + \frac{\hbar^2}{2}fl$

$\overset{!}{=} E_l + \frac{\hbar^2}{2}fl.$ ← higher energy compared to E_l.

Therefore by turning $f \neq 0$, we split E_l with $4l+2$ degeneracy into 2 different energies, $E_{l-1/2,l}$ being the lower and $E_{l+1/2,l}$ being higher



$S=\frac{1}{2}$ case.

6) $|l_1, l_2; l, m\rangle = |1\,1; 1, -1\rangle$

Find values a probability of L_{1z}.

Need to convert $|l_1, l_2; l\, m\rangle \to |l_1, l_2; m_1, m_2\rangle$

$$|1\,1; m_1, m_2\rangle\langle 1\,1; m_1, m_2|1\,1; 1\,-1\rangle$$

Also know $m = m_1 + m_2$, so $-1 = m_1 + m_2$

$l_1 = 1 \to m_1 = \{-1, 0, 1\}$
$l_2 = 1 \to m_2 = \{-1, 0, 1\}$

So possible combination for $m_1 + m_2 = -1$ are ① $m_1 = -1$, $m_2 = 0$
② $m_1 = 0$, $m_1 = -1$.

$\hookrightarrow$ $|1\,1; 1\,-1\rangle = |1\,1; -1\,0\rangle\langle 1\,1; -1\,0|1\,1; 1\,-1\rangle$
$+ |1\,1; 0\,-1\rangle\langle 1\,1; 0\,-1|1\,1; 1\,-1\rangle$

Now just need to find CB coefficients:

$$\langle 1\,1; -1\,0|1\,1; 1\,-1\rangle = \sqrt{\tfrac{1}{2}}$$

$$\langle 1\,1; 0\,-1|1\,1; 1\,-1\rangle = \sqrt{\tfrac{1}{2}}$$

$$L_{1z}|1\,1; 1\,-1\rangle = L_{1z}\sqrt{\tfrac{1}{2}}\left(|1\,1; -1\,0\rangle + |1\,1; 0\,-1\rangle\right)$$

$$\overset{!}{=} \sqrt{\tfrac{1}{2}}\left(-\hbar\,|1\,1; -1\,0\rangle + 0\,|1\,1; 0\,1\rangle\right)$$

$$\langle 11;1\,-1|L_{1z}|11;1\,-1\rangle = \underbrace{\tfrac{1}{2}}_{\text{probability}}\underbrace{(-\hbar)}_{\text{value}} + \underbrace{\tfrac{1}{2}}_{\text{prob}}\underbrace{(0)}_{\text{value}}.$$

Therefore we can measure $\boxed{\begin{array}{l} -\hbar \text{ with probability } 1/2 \\ \text{or } 0 \text{ with probability } 1/2 \end{array}}$