1) <u>Center of orbit:</u>

Since $H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 = \frac{m}{2}\vec{v}^2$

then $\vec{v} = \frac{\vec{P} - \frac{e}{c}\vec{A}}{m}$, $\vec{P} = -i\hbar\left(\partial_x\hat{x} + \partial_y\hat{y}\right)$
$\hat{A} = \vec{A}(\vec{x},\vec{y})$

then $v_x = \frac{P_x - \frac{e}{c}A_x}{m}$, $v_y = \frac{P_y - \frac{e}{c}A_y}{m}$

define: $W_B = \frac{eB}{mc}$, where $B = \partial_x A_y - \partial_y A_x$

Define operators:

$x_o = x + \frac{1}{W_B}v_y$ $y_o = y - \frac{1}{W_B}v_x$ where. $v_y = -W_B(x - x_o)$
$v_x = W_B(y - y_o)$

$r_o^2 = x_o^2 + y_o^2$ $R^2 = \frac{1}{W_B^2}(v_x^2 + v_y^2)$

Calculate different commutators:

① $[x_o, y_o] = \left[x + \frac{1}{W_B}v_x, \; y - \frac{1}{W_B}v_y\right]$

$\dot{=} \underbrace{[x, y]}_{=0} - \frac{1}{W_B}[x, v_y] + \frac{1}{W_B}[v_x, y] - \frac{1}{W_B^2}[v_x, v_y]$

Find Individual Commutators:

$[x, v_y] = \frac{1}{m}\left[x, \; P_y - \frac{e}{c}A_y\right] = \left(\underbrace{[x, P_y]}_{0} - \underbrace{[x, \frac{e}{c}A_y]}_{=0}\right)\frac{1}{m}$

$[x, v_y] \dot{=} 0$

Similarly, $[v_x, y] = -[y, v_x] = -[y, P_x - \frac{e}{c}A_x] \frac{1}{m}$

$$= \left(-[y, P_x] + [y, \frac{e}{c}A_x] \right) \frac{1}{m}$$

$$= 0$$

$$[v_x, v_y] = \frac{1}{m^2} \left(P_x - \frac{e}{c}A_x, P_y - \frac{e}{c}A_y \right)$$

$$= \left(\underbrace{[P_x, P_y]}_{=0} - [P_x, \frac{e}{c}A_y] - [\frac{e}{c}A_x, P_y] + \left(\frac{e}{c}\right)^2 \underbrace{[A_x, A_y]}_{=0} \right) \frac{1}{m^2}$$

Since $[P_x, \frac{e}{c}A_y] f(x) = -i\hbar\frac{e}{c} \left(\partial_x A_y - A_y \partial_x \right) f$

$$= -i\hbar \frac{e}{c} \left(f \partial_x A_y + \cancel{A_y \partial_x f} - \cancel{A_y \partial_x f} \right)$$

$$= -i\hbar \frac{e}{c} f \partial_x A_y$$

So $[P_x, \frac{e}{c}A_y] = -i\hbar \frac{e}{c} \partial_x A_y$

Similarly, $[\frac{e}{c}A_x, P_y] = i\hbar \frac{e}{c} \partial_y A_x$

then $[v_x, v_y] = i\hbar \frac{e}{m^2 c} \left(\partial_x A_y - \partial_y A_x \right) = i\hbar \frac{eB}{m^2 c} = \frac{i\hbar}{m} \omega_B$

Then: $[x_0, y_0] = -\frac{1}{\omega_B^2} \frac{i\hbar}{m} \omega_B$

① $\boxed{[x_0, y_0] = -\frac{i\hbar}{m\omega_B}}$ $\leftarrow$ x_0, y_0 not compatible

② $\quad [X_0, r_0^2] = [X_0, x_0^2 + y_0^2]$

$[X_0, Y_0] = \dfrac{-i\hbar}{m\omega_B}$

$$\stackrel{!}{=} [X_0, x_0^2] + [X_0, y_0^2]$$
$$\phantom{\stackrel{!}{=}} \qquad\qquad\quad =0$$
$$\stackrel{!}{=} [X_0, Y_0]\, Y_0 + Y_0\, [X_0, Y_0]$$

② $\boxed{[X_0, r_0^2] = \dfrac{-2i\hbar}{m\omega_B}\, Y_0}$ $\quad \leftarrow X_0, r_0^2$ not compatible

③ $\quad [Y_0, r_0^2] = [Y_0, x_0^2 + y_0^2]$

$[Y_0, X_0] = -[X_0, Y_0]$

$ = \dfrac{i\hbar}{m\omega_B}$

$$\stackrel{!}{=} [Y_0, x_0^2] + \cancel{[Y_0, y_0^2]}$$
$$\phantom{\stackrel{!}{=}[Y_0, x_0^2] + } =0$$
$$\stackrel{!}{=} [Y_0, X_0]\, X_0 + X_0\, [Y_0, X_0]$$

$\boxed{[Y_0, r_0^2] = \dfrac{2i\hbar}{m\omega_B}\, X_0}$ $\quad \leftarrow Y_0, r_0^2$ not compatible

④ $\quad [X_0, R^2] = \dfrac{1}{\omega_B^2}[X_0, V_x^2 + V_y^2]$

$$\stackrel{!}{=} \dfrac{1}{\omega_B^2}[X + \tfrac{1}{\omega_B} V_y, \ V_x^2 + V_y^2]$$

$$= \dfrac{1}{\omega_B^2}\left([X, V_x^2] + [X, V_y^2] + \tfrac{1}{\omega_B}[V_y, V_x^2] + \tfrac{1}{\omega_B}[V_y, V_y^2]\right)$$

$$\stackrel{!}{=} \dfrac{1}{\omega_B^2}\Big([X, V_x]V_x + V_x[X, V_x] + \underbrace{[X, V_y]}_{=0}V_y + V_y\underbrace{[X, V_y]}_{=0} \quad\text{use} \atop \leftarrow\text{results}$$
$$\dfrac{1}{\omega_B}\Big(\underbrace{[V_y, V_x]}_{\frac{-i\hbar}{m}\omega_B}V_x + V_x\underbrace{[V_y, V_x]}_{\frac{-i\hbar}{m}\omega_B} + 2\underbrace{[V_y, V_y]}_{=0}V_y\Big)\Big) \quad \text{calculated} \atop \text{previously.}$$

Previously we found $[x, v_y] = [y, v_x] = 0$

Need to find $[x, v_x]$ and $[y, v_y]$

$\Rightarrow [x, v_x] = \frac{1}{m}[x, P_x - \frac{e}{c}A_x]$

$\qquad = \frac{1}{m}\left(\underbrace{[x, P_x]}_{i\hbar} - \underbrace{[x, \frac{e}{c}A_x]}_{=0}\right)$

$\qquad = \frac{i\hbar}{m}$

Analogous to $[x, v_x]$:

$[y, v_y] = \frac{1}{m}[y, P_y - \frac{e}{c}A_y] = \frac{1}{m}\left([y, P_y] - [y, \frac{e}{c}A_y]\right)$

$\qquad\qquad\qquad\qquad = \frac{i\hbar}{m}$

Then $[x_0, R^2] = \frac{1}{\omega_B^2}\left(\underbrace{[x, v_x]}_{\frac{i\hbar}{m}}v_x + v_x\underbrace{[x, v_x]}_{\frac{i\hbar}{m}} + \frac{-2i\hbar}{m}v_x\right)$

$\boxed{④ \qquad [x_0, R^2] = 0} \qquad \leftarrow x_0, R^2 \text{ compatible}$

$$[Y_0, R^2] = \frac{1}{W_B^2}\left[y - \frac{1}{W_B}v_x, \; v_x^2 + v_y^2 \right]$$

$$= \frac{1}{W_B^2}\left(\underbrace{[y, v_x^2]}_{=0} + [y, v_y^2] - \frac{1}{W_B}\left(\underbrace{[v_x, v_x^2]}_{=0} + [v_x, v_y^2] \right) \right)$$

$$= \frac{1}{W_B^2}\left(\underbrace{[y, v_y]}_{\frac{i\hbar}{m}}v_y + v_y\underbrace{[y, v_y]}_{\frac{i\hbar}{m}} - \frac{1}{W_B}\left(\underbrace{[v_x, v_y]}_{\frac{i\hbar}{m}W_B}v_y + v_y\underbrace{[v_x, v_y]}_{\frac{i\hbar}{m}W_B} \right) \right)$$

⑤ $\boxed{[Y_0, R^2] = 0}$ ← Y_0, R^2 compatible

⑥ $[r_0^2, R^2] = [x_0^2 + y_0^2, \; R^2]$

$$= [x_0^2, R^2] + [y_0^2, R^2]$$

$$= x_0\underbrace{[x_0, R^2]}_{=0} + \underbrace{[x_0, R^2]}_{=0}x_0 + y_0\underbrace{[y_0, R^2]}_{=0} + \underbrace{[y_0, R^2]}_{=0}y_0$$

⑥ $\boxed{[r_0^2, R^2] = 0}$ ← r_0^2, R^2 compatible.

2) Heisenberg equation of motion:

For observable, $\mathcal{O}$, not explicitly function of time, then

$$\frac{d\mathcal{O}}{dt} = \frac{1}{i\hbar}[\mathcal{O}, H] = \frac{1}{i\hbar}\left[\mathcal{O}, \frac{m}{2}(V_x^2 + V_y^2)\right]$$
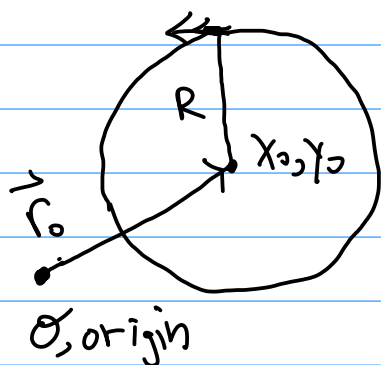
$$= \frac{m}{2i\hbar}[\mathcal{O}, R^2]$$

Use results from problem 1:

① $\frac{dX_o}{dt} = \frac{m}{2i\hbar}\underbrace{[X_o, R^2]}_{=0} = 0$

② $\frac{dY_o}{dt} = \frac{m}{2i\hbar}\underbrace{[Y_o, R^2]}_{=0} = 0$

Center of the orbit doesn't change as a function of time.

③ $\frac{d}{dt}r_o^2 = \frac{m}{2i\hbar}\underbrace{[r^2, R^2]}_{=0} = 0$

④ $\frac{d}{dt}R^2 = \frac{m}{2i\hbar}\underbrace{[R^2, R^2]}_{=0} = 0$ ⟹ The radius of the orbit also do not change.

It is very similar to classical case, where the particle moves in the circle.

3) **Integer Quantum Hall Effect:**

$$H = \frac{1}{2m}\left(-i\hbar \vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + V(x,y)$$

let $V(x,y) = \frac{1}{2}m\omega_0^2 y^2$

a) use Landau Gauge: $A_x = -By$, $A_y = 0$
and let $\psi(x,y) = \psi_k(y)e^{-ikx}$

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{eB}{c}y\hat{x}\right)^2 + \frac{1}{2}m\omega_0^2 y^2$$

$$= \frac{1}{2m}\left[\left(-i\hbar\partial_x + \frac{eB}{c}y\right)^2 + \left(-i\hbar\partial_y\right)^2\right] + \frac{1}{2}m\omega_0^2 y^2$$

$\Rightarrow$ Define: $\quad \ell^2 = \frac{\hbar c}{eB} \qquad \omega_B = \frac{eB}{mc}$

With $\psi(x,y) = \psi(y)e^{ikx}$, then $\partial_x \Rightarrow -ik$

$$H\psi(x,y) = \left(-\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2m}\left[-i\hbar(ik) + \frac{eB}{c}y\right]^2 + \frac{1}{2}m\omega_0^2 y^2\right)\psi(y)e^{-ikx}$$

$$H\psi_k(y) = \left(-\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\underbrace{\left(\frac{eB}{mc}\right)^2}_{\omega_B^2}\underbrace{\left[\left(\frac{c\hbar}{eB}\right)k + y\right]^2}_{\ell^2} + \frac{1}{2}m\omega_0^2 y^2\right)\psi_k(y)$$

$$= \frac{-\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\left(\omega_B^2\left[y - k\ell^2\right]^2 + \omega_0^2 y^2\right)\psi_k(y)$$

$$= \frac{-\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\left((\omega_B^2 + \omega_0^2)y^2 - 2\omega_B^2 yk\ell^2 + \omega_B^2 k^2\ell^4\right)\psi_k(y)$$

$$= \frac{-\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m(\omega_B^2 + \omega_0^2)\left[y^2 - 2\frac{\omega_B^2}{\omega_B^2 + \omega_0^2}yk\ell^2 + \frac{\omega_B^2}{\omega_B^2 + \omega_0^2}k^2\ell^4\right.$$

$$\left. + \left(\frac{\omega_B^2}{\omega_B^2 + \omega_0^2}k\ell^2\right)^2 - \left(\frac{\omega_B^2}{\omega_B^2 + \omega_0^2}k\ell^2\right)^2\right]\psi_k(y)$$

$$\hookrightarrow = \left(\frac{-\hbar^2}{2m} \partial_y^2 + \frac{1}{2}m\left(\omega_B^2 + \omega_o^2\right)\left[y - k\ell^2 \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right]^2 \right.$$

$$\left. + \frac{1}{2}m\omega_B^2 k^2 \ell^4 - \frac{1}{2}m\omega_B^2 k^2 \ell^4 \left(\frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right) \right) \psi_k(y)$$

$$H\psi_k(y) = \left(\frac{-\hbar^2}{2m} \partial_y^2 + \frac{1}{2}m\left(\omega_B^2 + \omega_o^2\right)\left[y - k\ell^2 \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right]^2 \right.$$

$$\left. + \frac{1}{2}m\omega_B^2 k^2 \ell^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right] \right) \psi_k(y)$$

b) We recognize the first part of Hamiltonian is a 1D Harmonic oscillator with frequency, $\omega^2 = \omega_B^2 + \omega_o^2$ so it has energy:

$$E_{n,k}^{HO} = \hbar\sqrt{\omega_B^2 + \omega_o^2}\left(n + \frac{1}{2}\right)$$

Then total Energy:

$$E_{n,k} = \hbar\sqrt{\omega_B^2 + \omega_o^2}\left(n + \frac{1}{2}\right) + \frac{1}{2}m\omega_B^2 k^2 \ell^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right]$$

c) Assume $n=0$,

$$E_{0,k} = \frac{1}{2}\hbar\sqrt{w_B^2 + w_0^2} + \frac{1}{2}mw_B^2 k^2 z^4 \left[1 - \frac{w_B^2}{w_B^2 + w_0^2}\right]$$

Since $\qquad E_{0,k} < u$

$\hookrightarrow \quad \frac{1}{2}mw_B^2 k^2 z^4 \left[1 - \frac{w_B^2}{w_B^2 + w_0^2}\right] < u - \frac{1}{2}\hbar\sqrt{w_B^2 + w_0^2}$

$\hookrightarrow \quad k < \pm\sqrt{\left(\frac{1}{2}mw_B^2\left[1 - \frac{w_B^2}{w_B^2 + w_0^2}\right]\right)^{-1}\left(u - \frac{1}{2}\hbar\sqrt{w_B^2 + w_0^2}\right)}$

$$\begin{cases} + \text{ sign for max } k \\ - \text{ sign for min } k \end{cases}$$

d) We can define a Y_0 in the Hamiltonian recognizing it in the Harmonic oscillator part,

$\rightarrow$ which is $\qquad Y_0 = k z^2 \dfrac{w_B^2}{w_B^2 + w_0^2}$

$\qquad$ and since $\qquad k = \dfrac{2\pi}{L_x}j$ for $j = 0, \pm1, \pm2\cdots, \pm\dfrac{L_x}{2\pi}k_{max}$

$$\boxed{Y_{0,j} = \frac{2\pi}{L_x}j\frac{w_B^2}{w_B^2 + w_0^2} \quad \text{where } j \text{ is bounded by } \frac{L_x}{2\pi}k_{max}}$$

e)

$$v_g = \frac{1}{\hbar} \frac{\partial E}{\partial k} = \frac{1}{\hbar} \frac{\partial}{\partial k} \left(\hbar \sqrt{\omega_B^2 + \omega_o^2} (n + \tfrac{1}{2}) + \tfrac{1}{2} m \omega_B^2 k^2 \ell^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right] \right)$$

$$= \frac{1}{\hbar} k \, m \omega_B^2 \ell^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right]$$

right
edge → $v_g \big|_{k=k_{max}} = \frac{k_{max}}{\hbar} m \omega_B^2 \ell^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right]$
state

Left
edge → $v_g \big|_{k=k_{min}} = \frac{k_{min}}{\hbar} m \omega_B^2 \ell^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_o^2} \right]$
State

$$= -v_g \big|_{k=k_{max}}.$$

4) a and b operator:

consider $H = -\frac{1}{2}\left(\partial_x + \frac{i}{2}y\right)^2 - \frac{1}{2}\left(\partial_y - \frac{i}{2}x\right)^2$

let $z = x+iy$, $\bar{z} = x-iy$, $\partial = \frac{1}{2}\left(\partial_x - i\partial_y\right)$, $\bar{\partial} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$

a) let $a = \sqrt{2}\left(-i\bar{\partial} - \frac{1}{4}z\right)$, $a^+ = \sqrt{2}\left(-i\partial + \frac{i}{4}\bar{z}\right)$

Show $[a, a^+] = 1$

$$[a, a^+] = \left[\sqrt{2}\left(-i\bar{\partial} - \frac{1}{4}z\right), \left(-i\partial + \frac{i}{4}\bar{z}\right)\sqrt{2}\right]$$

$$= -2\left[\bar{\partial} + \frac{1}{4}z, \partial - \frac{i}{4}\bar{z}\right]$$

$$= -2\left(\underset{①}{[\bar{\partial},\partial]} - \frac{i}{4}\underset{②}{[\bar{\partial},\bar{z}]} + \frac{1}{4}\underset{③}{[z,\partial]} - \frac{i}{16}\underset{④}{[z,\bar{z}]}\right)$$

Calculate individual terms:

① $[\bar{\partial},\partial] = \frac{1}{4}[\partial_x + i\partial_y, \partial_x - i\partial_y] = 0$ since no dependence in x or y.

④ $[z,\bar{z}] = [x+iy, x-iy] = 0$ positions commute.

② $[\bar{\partial}, \bar{z}] = \frac{1}{2}[\partial_x + i\partial_y, x-iy]$

$$= \frac{1}{2}\left([\partial_x, x] - i\underset{=0}{[\partial_x, y]} + i\underset{=0}{[\partial_y, x]} + [\partial_y, y]\right)$$

$\partial_x = -\frac{1}{i\hbar}P_x$, and $[P_x, x] = -i\hbar$

so $[\partial_x, x] = -\frac{1}{i\hbar}[P_x, x] = 1$

similarly $[\partial_y, y] = 1$

then $[\tilde{\partial}, \bar{z}] = \frac{1}{2}[1+1] = \underline{\underline{1}}$

③ $[z, \partial] = \frac{1}{2}[x+iy, \partial_x - i\partial_y]$

$= \frac{1}{2}\left(\underbrace{[x, \partial_x]}_{= -[\partial_x, x] = -1} - i\underbrace{[x, \partial_y]}_{=0} + i\underbrace{[y, \partial_x]}_{=0} + \underbrace{[y, \partial_y]}_{=-[\partial_y, y]=-1}\right)$

$= -1$

$$\boxed{\therefore \quad [a, a^\dagger] = -2\left[-\frac{1}{4}(1) + \frac{1}{4}(-1)\right] = 1}$$

b) Show $H = a^\dagger a + \frac{1}{2} = a^\dagger a + \frac{1}{2}(aa^\dagger - a^\dagger a) = \frac{1}{2}(\underset{①}{a^\dagger a} + \underset{②}{aa^\dagger})$

① $a^\dagger a = \sqrt{2}\left(-i\partial + \frac{1}{4}\bar{z}\right)\sqrt{2}\left(-i\bar{\partial} - \frac{i}{4}z\right)$

$= -2\left(\frac{1}{2}(\partial_x - i\partial_y) - \frac{1}{4}(x - iy)\right)\left(\frac{1}{2}(\partial_x + i\partial_y) + \frac{1}{4}(x+iy)\right)$

$= -2\left(\frac{1}{2}(\partial_x + \frac{i}{2}y) - \frac{i}{2}(\partial_y - \frac{i}{2}x)\right)\left(\frac{1}{2}(\partial_x + \frac{i}{2}y) + \frac{i}{2}(\partial_y - \frac{i}{2}x)\right)$

$= -\frac{1}{2}\left[(\partial_x + \frac{i}{2}y) - i(\partial_y - \frac{i}{2}x)\right]\left[(\partial_x + \frac{i}{2}y) + i(\partial_y - \frac{i}{2}x)\right]$

$= -\frac{1}{2}\left[(\partial_x + \frac{i}{2}y)^2 + (\partial_y - \frac{i}{2}x)^2 - i(\partial_y - \frac{i}{2}x)(\partial_x + \frac{i}{2}y)\right.$

$\left. \underbrace{+ i(\partial_x + \frac{i}{2}y)(\partial_y - \frac{i}{2}x)}_{\text{cross terms}}\right]$

② $a a^\dagger = \frac{1}{\sqrt{2}}\left(-i\bar{\partial} - \frac{i}{4}z\right)\frac{1}{\sqrt{2}}\left(-i\partial + \frac{i}{4}\bar{z}\right)$

$= -2\left(\frac{1}{2}(\partial_x + i\partial_y) + \frac{1}{4}(x+iy)\right)\left(\frac{1}{2}(\partial_x - i\partial_y) - \frac{1}{4}(x-iy)\right)$

$= -2\left(\frac{1}{2}\partial_x + \frac{i}{2}y\right) + \frac{i}{2}\left(\partial_y - \frac{i}{2}x\right)\right)\left(\frac{1}{2}(\partial_x + \frac{i}{2}y) - \frac{i}{2}(\partial_y - \frac{i}{2}x)\right)$

$= -\frac{1}{2}\Big[\underbrace{\left(\partial_x + \frac{i}{2}y\right)^2 + \left(\partial_y - \frac{i}{2}x\right)^2}_{\text{repeated terms}} + \underbrace{i\left(\partial_y - \frac{i}{2}x\right)\left(\partial_x + \frac{i}{2}y\right) - i\left(\partial_x + \frac{i}{2}y\right)\left(\partial_y - \frac{i}{2}y\right)}_{\text{cross-terms}}\Big]$

$H = \frac{1}{2}\{a, a^\dagger\} = \frac{1}{2}\left(aa^\dagger + a^\dagger a\right)$

plug in results

of $aa^\dagger$ and $a^\dagger a$

and realize cross-terms
cancel, then $= -\frac{1}{2}\left(\partial_x + \frac{i}{2}y\right)^2 - \frac{1}{2}\left(\partial_y - \frac{i}{2}x\right)^2$

c) Introduce $b = \sqrt{2}\left(-i\partial - \frac{i}{4}\bar{z}\right)$, $b^+ = \sqrt{2}\left(-i\bar{\partial} + \frac{i}{4}z\right)$

Show $[b, b^+] = 1$ $\quad$ $[b, a] = 0$ $\quad$ $[b, a^+] = 0$

$$[b, b^+] = \left[\sqrt{2}\left(-i\partial - \frac{i}{4}\bar{z}\right), \sqrt{2}\left(-i\bar{\partial} + \frac{i}{4}z\right)\right]$$

$$= -2\left[\partial + \frac{1}{4}\bar{z}, \bar{\partial} - \frac{1}{4}z\right]$$

$$= -2\left(\underbrace{[\partial, \bar{\partial}]}_{=0} - \frac{1}{4}\underbrace{[\partial, z]}_{=-[z,\partial]=1} + \frac{1}{4}\underbrace{[\bar{z}, \bar{\partial}]}_{=-[\bar\partial,\bar z]=-1} - \frac{1}{16}\underbrace{[\bar{z}, z]}_{=0}\right)$$

use results
calculated from part a)

$$\boxed{[b, b^+] = 1}$$

$$[b, a] = \left[\sqrt{2}\left(-i\partial - \frac{i}{4}\bar{z}\right), \sqrt{2}\left(-i\bar{\partial} - \frac{i}{4}z\right)\right]$$

$$= -2\left[\partial + \frac{1}{4}\bar{z}, \bar{\partial} + \frac{1}{4}z\right]$$

Again, use
commutator results
from part a)

$$= -2\left(\underbrace{[\partial, \bar{\partial}]}_{=0} + \frac{1}{4}\underbrace{[\partial, z]}_{=-[z,\partial]=1} + \frac{1}{4}\underbrace{[\bar{z}, \bar{\partial}]}_{=-[\bar\partial,\bar z]=-1} + \frac{1}{16}\underbrace{[\bar{z}, z]}_{=0}\right)$$

$$\underbrace{}_{=0}$$

$$\boxed{[b, a] = 0}$$

$$[b, a^\dagger] = \left[\sqrt{2}\left(-i\partial - \tfrac{i}{4}\bar{z}\right), \sqrt{2}\left(-i\partial + \tfrac{i}{4}\bar{z}\right)\right]$$

$$= -2\left[\partial + \tfrac{1}{4}\bar{z}, \partial - \tfrac{1}{4}\bar{z}\right]$$

$$= -2\left(\underbrace{[\partial, \partial]}_{=0} - \tfrac{1}{4}\underbrace{[\partial, \bar{z}]}_{=0} + \tfrac{1}{4}\underbrace{[\bar{z}, \partial]}_{=0} - \tfrac{1}{16}\underbrace{[\bar{z}, \bar{z}]}_{=0}\right)$$

Find $[\partial, \bar{z}] = \tfrac{1}{2}[\partial_x - i\partial_y, x - iy]$

$$= \tfrac{1}{2}\left(\underbrace{[\partial_x, x]}_{=1} - i\underbrace{[\partial_x, y]}_{=0} - i\underbrace{[\partial_y, x]}_{=0} - \underbrace{[\partial_y, y]}_{=1}\right)$$

From part a), we know
$[\partial_x, x] = [\partial_y, y] = 1$

$$= \tfrac{1}{2}(1-1)$$

$$= 0$$

So $\boxed{[b, a^\dagger] = 0}$

d) Assume $\quad a|0,0\rangle = 0 \quad , \quad b|0,0\rangle = 0$

then $\quad |n, m\rangle = (a^\dagger)^n (b^\dagger)^m |0,0\rangle$

Show $|n, m\rangle$ is eigenstate. of H, find its energy.

Know: $H = a^\dagger a + \tfrac{1}{2}$

then $\quad H|0,0\rangle = \overbrace{a^\dagger a |0,0\rangle}^{=0} + \frac{1}{2}|0,0\rangle$

$\qquad\qquad E|0,0\rangle \overset{!}{=} \frac{1}{2}|0,0\rangle$

So if $\quad a|0,0\rangle = 0$, then

$$H|0,0\rangle = \frac{1}{2}|0,0\rangle = E_0 |0,0\rangle$$

$$\hookrightarrow E_0 = \frac{1}{2}$$

Need to see whether $b^\dagger$ commutes with $a^\dagger$ and a.

$$[b^\dagger, a] = \left[\frac{1}{\sqrt{2}}\left(-i\bar{\partial} + \frac{i}{4}z\right), \frac{1}{\sqrt{2}}\left(-i\partial - \frac{i}{4}z\right)\right]$$

$$= -2\left[\bar{\partial} - \frac{1}{4}z, \partial + \frac{1}{4}z\right]$$

$$= -2\left(\underbrace{[\bar{\partial}, \partial]}_{=0} + \frac{1}{4}\underbrace{[\bar{\partial}, z]}_{=0} - \frac{1}{4}\underbrace{[z, \partial]}_{=0} - \frac{1}{16}\underbrace{[z, z]}_{=0}\right)$$

$$= 0$$

$$[b^\dagger, a^\dagger] = \left[\frac{1}{\sqrt{2}}\left(-i\bar{\partial} + \frac{i}{4}z\right), \sqrt{2}\left(-i\partial + \frac{i}{4}\bar{z}\right)\right]$$

$$= -2\left([\bar{\partial} - \frac{1}{4}z, \partial - \frac{1}{4}\bar{z}]\right)$$

$$= -2\left(\underbrace{[\bar{\partial}, \partial]}_{=0} - \frac{1}{4}\underbrace{[\bar{\partial}, \bar{z}]}_{1} - \frac{1}{4}\underbrace{[z, \partial]}_{-1} + \frac{1}{16}\underbrace{[z, \bar{z}]}_{=0}\right)$$

$$= 0$$

So $b^\dagger$ commutes with a, $a^\dagger$.

If we have $\quad |1,m\rangle = a^\dagger (b^\dagger)^m |0,0\rangle$

$$H|1,m\rangle = a^\dagger a\, a^\dagger (b^\dagger)^m |0,0\rangle + \tfrac{1}{2} a^\dagger (b^\dagger)^m |0,0\rangle$$

since $[a,a^\dagger] = aa^\dagger - a^\dagger a = 1$
$\quad \hookrightarrow aa^\dagger = 1 + a^\dagger a$

$$= a^\dagger (1 + a^\dagger a)(b^\dagger)^m |0,0\rangle + \tfrac{1}{2} a^\dagger (b^\dagger)^m |0,0\rangle$$

but $b^\dagger$ commutes with a, $a^\dagger$, so we can switch order.

$$= \left(a^\dagger (b^\dagger)^m |0,0\rangle + (b^\dagger)^m \underbrace{a^\dagger a\, |0,0\rangle}_{=0} + \tfrac{1}{2} a^\dagger (b^\dagger)^m |0,0\rangle \right)$$

$$= \left(1 + \tfrac{1}{2}\right) a^\dagger (b^\dagger)^m |0,0\rangle$$

$$= \tfrac{3}{2}\, a^\dagger (b^\dagger)^m |0,0\rangle$$

Now for $|2,m\rangle = (a^\dagger)^2 (b^\dagger)^m |0,0\rangle$

then $\quad H|2,m\rangle = a^\dagger a\, (a^\dagger)^2 (b^\dagger)^m |0,0\rangle + \tfrac{1}{2}(a^\dagger)^2 (b^\dagger)^m |0,0\rangle$

$$= \left(a^\dagger \underbrace{a\, a^\dagger}_{1+a^\dagger a} a^\dagger + \tfrac{1}{2}(a^\dagger)^2 \right)(b^\dagger)^m |0,0\rangle$$

$$= \left(a^\dagger (1 + a^\dagger a) a^\dagger + \tfrac{1}{2}(a^\dagger)^2 \right)(b^\dagger)^m |0,0\rangle$$

$$= \left((a^\dagger)^2 + a^\dagger a^\dagger \underbrace{a\, a^\dagger}_{1+a^\dagger a} + \tfrac{1}{2}(a^\dagger)^2 \right)(b^\dagger)^m |0,0\rangle$$

$$= \left((a^\dagger)^2 + (a^\dagger)^2 + a^\dagger a^\dagger a^\dagger a + \tfrac{1}{2}(a^\dagger)^2 \right)(b^\dagger)^m |0,0\rangle$$

$$= \tfrac{5}{2}(a^\dagger)^2 (b^\dagger)^m |0,0\rangle + (a^\dagger)^3 (b^\dagger)^m \underbrace{a\, |0,0\rangle}_{=0}$$

$$H|2,m\rangle = \tfrac{5}{2}(a^\dagger)^2 (b^\dagger)^m |0,0\rangle$$

If we continue, we see that

$$H|n,m\rangle = \underbrace{\left(n + \frac{1}{2}\right)}_{= E_n} |n,m\rangle$$

So $|n,m\rangle$ is an eigenstate with eigen energy E_n