## Symmetry in QM:

**Classical Physics:** $L(q_i, \dot{q}_i)$

$$q_i \to q_i + \delta q_i \quad , \quad \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial \dot{q}_i} = 0$$

so $\quad p_i = \text{const.}$

In Hamilton: $\quad \frac{dp_i}{dt} = \{H, p_i\} = \frac{\partial H}{\partial q_i}\{q_i, p_i\} = 0$

**Symmetry:** If $\{H, p_i\} = 0$, then $p_i = \text{const.}$

**QM:** symmetry operator (unitary): $\mathcal{L}$.

$$\mathcal{L}^{\dagger}\mathcal{L} = 1 \quad \leftarrow \text{unitary.}$$

$G$ is constant of motion.

$$\boxed{\mathcal{L}^{\dagger} H \mathcal{L} = H} \quad \to \quad i\hbar \frac{dG}{dt} = [G, H] = 0$$

Heinsenberg

If $\mathcal{L}$ depends on continuous parameter

$$\mathcal{L} \approx 1 - \frac{i}{\hbar}\varepsilon G$$

$G$ → generator of symmetry.

→ Hermitian, $G^{\dagger} = G$

$\approx$ ↑ approximate

$$\mathcal{L} = e^{-\frac{i}{\hbar}\varepsilon G}$$

Schrodinger:

$$G|g\rangle = g|g\rangle$$

↳ eigenket of $G$.

$$|g\rangle \to U(t,t_0)|g\rangle_{t_0}$$

↳ $= e^{-\frac{i}{\hbar}H(t-t_0)}$ if time independent V.

$$G|g\rangle_t = GU(t,t_0)|g\rangle_{t_0}$$

↳ $\stackrel{!}{=} U(t,t_0)\underbrace{G|g\rangle_{t_0}}_{g|g\rangle_{t_0}}$ ↙ $G(t) = UGU^{\dagger} = G$, $UG = GU$.

$$G|g\rangle_t = g|g\rangle_t$$

Suppose $[H, L] = 0$

$$H|n\rangle = E_n|n\rangle$$

↳ $H L|n\rangle = L H|n\rangle$

$$H L|n\rangle \stackrel{!}{=} E_n \underbrace{L|n\rangle}$$

is also an eigenket of $H$.

If $|n\rangle \neq L|n\rangle$, then $E_n$ is degenerate.

↑ allow phase difference.

Ex: $\left[\underset{\sim}{\mathcal{D}}(R), H\right] = 0$

$\hookrightarrow \left[\vec{J}, H\right] = 0$

G

$\hookrightarrow \left[J^2, H\right] = 0$

Introduce : $|n, j, m\rangle$ : eigenket of $H, J^2, J_z$

$\mathcal{D}(R)|n, j, m\rangle$ has the same $E_n$ for all $R$.

$$\hookrightarrow = \sum_{m'=-j}^{j} |n, j, m'\rangle \mathcal{D}^{(j)}_{m', m}(R)$$

$\uparrow$ $2j+1$ degenerate.

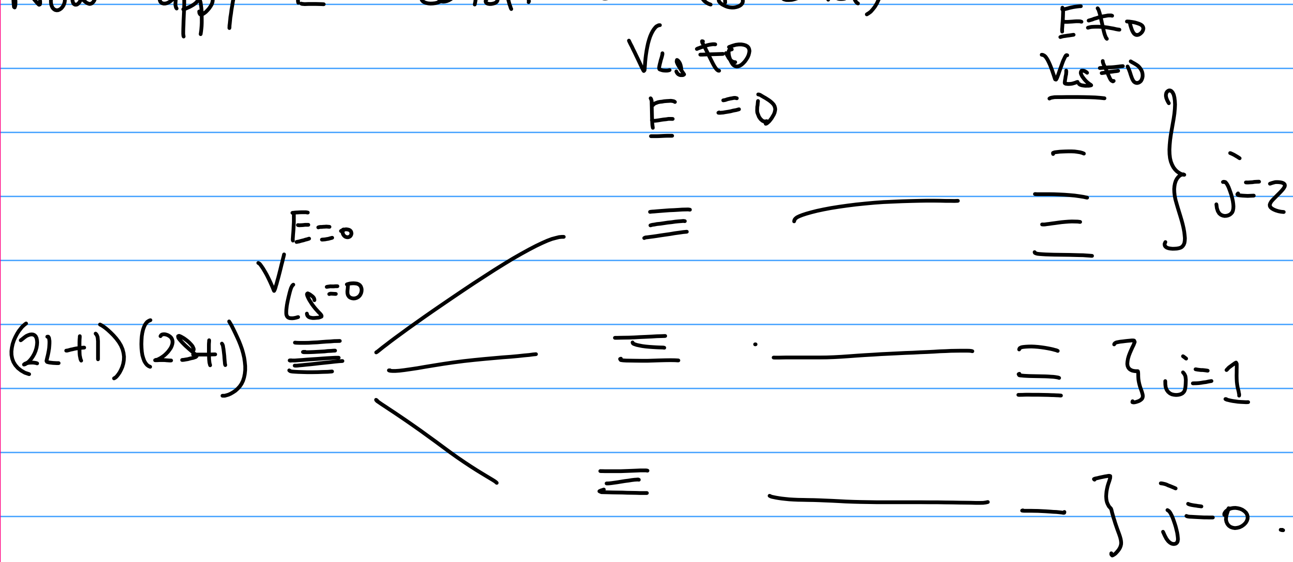
Application : $V = \underset{\substack{\uparrow \\ \text{central} \\ \text{potential}}}{V(r)} + \underbrace{V_{LS}\, \vec{L}\cdot\vec{S}}_{\text{spin-orbit interaction.}}$

If $V_{LS} = 0$ then we have $(2l+1)(2s+1)$ degenerate.

If $V_{LS} \neq 0$, $\left[V_{LS}(\vec{r})\, \vec{L}\cdot\vec{S}, \vec{J}\right] = 0$.

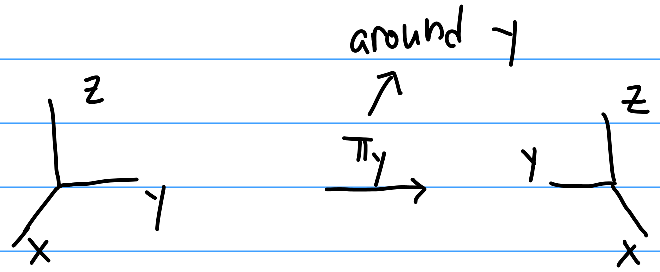
then $(2j+1)$ degenerate.

Now apply $E =$ const. or ($B =$ const)

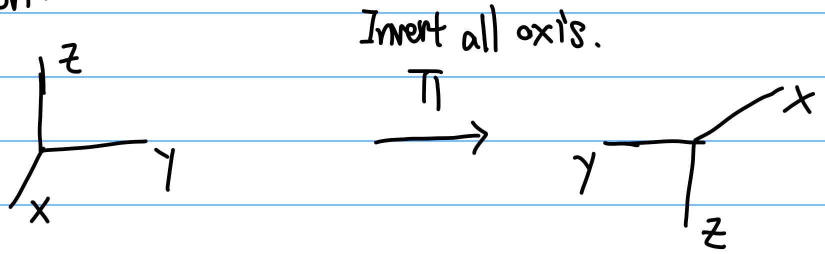


# Discrete Symmetries:

→ Parity :




→ Space Inversion:




→ Corresponding transformation of ket state:

$$|\alpha\rangle \rightarrow \underbrace{\Pi}_{\text{parity operator (unitary)}} |\alpha\rangle$$

$$\langle\alpha|\Pi^\dagger \vec{x}\, \Pi|\alpha\rangle = -\langle\alpha|\vec{x}|\alpha\rangle$$

$$\Pi^\dagger \vec{x}\, \Pi = -\vec{x}$$

$$\hookrightarrow \vec{x}\,\Pi = -\Pi\vec{x} \rightarrow \{\vec{x}, \vec{\Pi}\} = 0$$

odd operator.

| | |
|---|---|
| Odd operator if: | Even operator if: |
| $\{\Pi, \lambda\} = 0$ | $[\Pi, \lambda] = 0$ |

$|x'\rangle$ : position eigenket.

$$\vec{x}(\pi|\vec{x}'\rangle) = -\pi(\vec{x}|\vec{x}'\rangle) = -\vec{x}'\,\pi|\vec{x}'\rangle$$

$$\boxed{\pi|\vec{x}'\rangle = |-\vec{x}'\rangle}$$ ← convention, no phase $e^{i\phi}$

$$\boxed{\pi = \pi^{-1} = \pi^{+}}$$ ← unitary and Hermitian.

$$\pi^2|x'\rangle = \pi|-x'\rangle = |x'\rangle$$

Eigenvalue of $\pi = \pm 1$

## Translation:

$J(d\vec{x}')$ : translation by $d\vec{x}'$

$$\boxed{\pi\, J(d\vec{x}') = J(-d\vec{x}')\,\pi}$$

$$\pi J(d\vec{x}')|\vec{x}'\rangle = J(-d\vec{x}')\pi|x'\rangle = |-x' - d\vec{x}'\rangle$$

$$\pi\left(1 - \frac{i}{\hbar}\vec{p}\cdot d\vec{x}'\right)\pi^{+} = \left(1 + \frac{i}{\hbar}\vec{p}\cdot d\vec{x}'\right)$$

$$\left.\begin{aligned} \pi\vec{p}\pi^{+} &= -p \\ \pi^{+}\vec{p}\pi &= -\vec{p} \end{aligned}\right\}$$

$\{\pi, \vec{p}\} = 0$

So $p$ is odd under parity

Angular Momentum:

$$[\Pi, \vec{L}] = 0 \quad , \quad \vec{L} = \vec{r} \times \vec{p}$$

$\uparrow$ $\vec{L}$ is even.

Rotation of 3d space:

$$R^{(parity)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\hat{1}$$

$$R^{(parity)} R^{(rot)} = R^{(rot)} R^{(parity)}$$ $\leftarrow$ for matrix.

$\hookrightarrow$ $\Pi \, \underbrace{\mathcal{D}(R)}_{= 1 - \frac{i}{\hbar}\varepsilon \hat{n}\cdot\vec{J}} = \mathcal{D}(R)\, \Pi$ $\leftarrow$ postulate the same for operator.

$\hookrightarrow$ $\boxed{[\Pi, J] = 0 \quad \text{so} \quad \Pi^{\dagger} \vec{J}\, \Pi = \vec{J}}$

$\uparrow$ even operator.

| | Rotation ($J$) | Parity ($\Pi$) | |
|---|---|---|---|
| $\vec{x}, \vec{p}$ | vector | odd | Polar Vector |
| $\vec{J}, \vec{S}, \vec{L}$ | vector | even | axial vector (pseudovector) |
| $\vec{S}\cdot\vec{x}, \vec{S}\cdot\vec{p}$ | scalar | odd | pseudoscalar |
| $\vec{L}\cdot\vec{S}, \vec{x}\cdot\vec{p}$ | scalar | even | True Scalar. |

## Wave function under parity:

$$\psi_\alpha(\vec{x}) = \langle \vec{x} | \alpha \rangle$$

$$\langle \vec{x} | \pi | \alpha \rangle = \langle -\vec{x} | \alpha \rangle = \psi_\alpha(-\vec{x}) = \psi_{parity}(x)$$

$$\boxed{\psi(x) \xrightarrow{\pi} \psi(-x)}$$

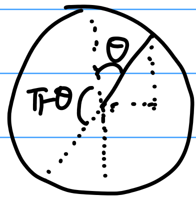
Eigenstate of parity: $\boxed{\pi | \alpha \rangle = \pm | \alpha \rangle}$

$$\langle \vec{x} | \pi | \alpha \rangle = \pm \langle \vec{x} | \alpha \rangle = \pm \psi_\alpha(x)$$

$$\langle -\vec{x} | \alpha \rangle = \boxed{\psi_\alpha(-\vec{x}) = \pm \psi_\alpha(x)} \rightarrow \begin{cases} + : \text{parity even} \\ - : \text{parity odd} \end{cases}$$

In spherical coordinate:

$r \to r$
$\theta \to \pi - \theta$
$\phi \to \pi + \phi$



$\cos\theta \to -\cos\theta$
$e^{im\phi} \to (-1)^m e^{im\phi}$

$$\pi | \alpha; l\, m \rangle = \delta | \alpha; l\, m \rangle \quad , \quad \text{note: } [\pi, L] = 0$$

so Apply $L_+$: $\quad \pi | \alpha; l\, m+1 \rangle = \delta | \alpha; l\, m+1 \rangle$

so $\delta$ doesn't depend on $m$

$$\langle \theta, \phi, r | \alpha; l\, m \rangle = R_{\alpha l}(r)\, Y_l^m(\theta, \phi)$$

$$\langle \theta, \phi | \pi | l, m \rangle = \langle \pi - \theta, \phi + \pi | l, m \rangle$$

$$\stackrel{!}{=} Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta, \phi)$$

so $\boxed{\pi | \alpha; l\, m \rangle = (-1)^l | \alpha; l\, m \rangle}$

Theorem: If $[H, \pi] = 0$, $H|n\rangle = E_n |n\rangle$ and $|n\rangle$ is non-degenerate.

then $|n\rangle$ is an eigenstate of $\pi$, so $|n\rangle$ must be either even or odd.

Take $|n_\pm\rangle = \frac{1\pm\Pi}{2}|n\rangle$ $\qquad \Pi\left(\frac{1\pm\Pi}{2}\right) = \frac{\Pi\pm1}{2} = \pm\left(\frac{1\pm\Pi}{2}\right)$

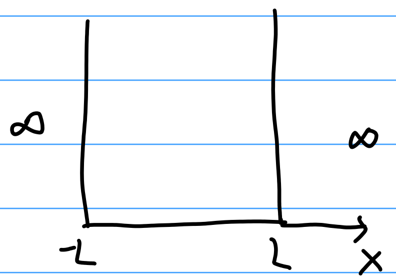
↳ $\Pi|n_\pm\rangle = \pm\left(\frac{1\pm\Pi}{2}\right)|n\rangle = \pm|n_\pm\rangle$

$H|n_\pm\rangle = E_n|n_\pm\rangle$

then $|n_+\rangle = |n_-\rangle$ ? $\qquad$ or $\qquad |n_+\rangle = 0$

$|n_-\rangle = 0 \qquad\qquad |n_-\rangle = |n_+\rangle$

Ex:




$\psi_n(x) = \frac{1}{\sqrt{L}}\sin\left(\frac{\pi}{2L}n(x+L)\right)$

$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{4L^2}$

When $n$ = odd, then its even under parity
$n$ = even, odd under parity.

odd: $\psi_{2k+1}(x) = \frac{(-1)^k}{\sqrt{L}}\cos\left(\frac{\pi}{L}(n+\tfrac{1}{2})x\right) \to \Pi\psi_{2k+1} \overset{\text{even}}{=} \psi_{2k+1}$

even: $\psi_{2k}(x) = \frac{(-1)^k}{\sqrt{L}}\sin\left(\frac{\pi}{L}kx\right) \to \Pi\psi_{2k} \underset{\text{odd}}{=} -\psi_{2k}$

Ex: free particle in 3d

$H = \frac{p^2}{2m}$ $[H, \Pi] = 0$ with eigenstate $|\vec{p}\rangle$

but $\Pi|\vec{p}\rangle = |-\vec{p}\rangle$ so $|\vec{p}\rangle$ is not eigenstate of $|\Pi\rangle$

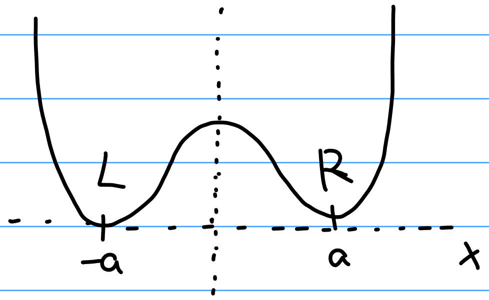
Also $E_p = \frac{p^2}{2m} = E_{-p}$, so it is degenerate.

So theorem doesn't apply.

We can use $\frac{|\vec{p}\rangle \pm |-\vec{p}\rangle}{\sqrt{2}}$ ← eigenstate of $\Pi$, but not for $\vec{p}$

$|\vec{p}\rangle$ ← eigenstate of $p$, but not $\Pi$.

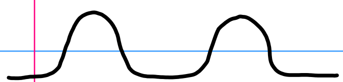
Ex: Symmetrical double-well potential:

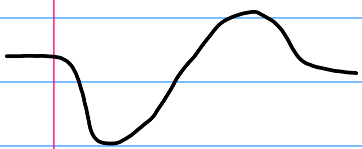



Since $U(x) = U(-x)$

$[\Pi, H] = 0$

So eigenstate have definite parity.

$\psi_0 =$  $= |S\rangle$

$\psi_1 =$  $= |A\rangle$

$$\left.\begin{aligned} \Pi|S\rangle &= |S\rangle \\ \Pi|A\rangle &= -|A\rangle \end{aligned}\right\} E_A > E_S$$

and $\Delta = E_A - E_S$ very small

then let
$$\left.\begin{aligned} |R\rangle &= \frac{1}{\sqrt{2}}\left(|S\rangle + |A\rangle\right) \\ |L\rangle &= \frac{1}{\sqrt{2}}\left(|S\rangle - |A\rangle\right) \end{aligned}\right\}$$

Not eigenstate of H
so its not stationary

overall phase.

$$|S\rangle \to e^{-\frac{i}{\hbar}E_S t}|S\rangle \quad , \quad |A\rangle \to e^{-\frac{i}{\hbar}E_A t}|A\rangle$$

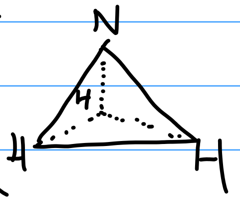
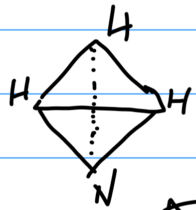
$$|R\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_S t}|S\rangle + e^{-\frac{i}{\hbar}E_A t}|A\rangle\right)$$ ← has extra interference so state changes.

$$\doteq \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}E_S t}\left(|S\rangle + e^{-\frac{i}{\hbar}\Delta t}|A\rangle\right)$$

period: $T = \frac{2\pi\hbar}{\Delta}$

After $t = T/2$ $\quad |R\rangle \xrightarrow{T/2} |L\rangle$

**Ex:** Ammonic Molecule: $NH_3$:

$\frac{1}{\sqrt{2}}\Big($  $\pm$  $\Big)$





$\sim |R\rangle$ $\qquad$ $\sim |L\rangle$

Not eigenstate of H

since after $\Pi$ it is not the same.

# Parity Selection Rule:

suppose $[H, \Pi] = 0$

eigenvalues of $\Pi$, $(\pm 1)$

$\Pi |\alpha\rangle = \varepsilon_\alpha |\alpha\rangle$

$\Pi |\beta\rangle = \varepsilon_\beta |\beta\rangle$

If $\vec{x}$ is odd $\rightarrow$ $\Pi^\dagger x \Pi = -x = \Pi x \Pi^\dagger$

$$\langle\beta| x |\alpha\rangle = \langle\beta| \Pi^\dagger \Pi x \Pi^\dagger \Pi |\alpha\rangle$$

$$= \varepsilon_\alpha \varepsilon_\beta \langle\beta| \Pi x \Pi^\dagger |\alpha\rangle$$

$$\langle\beta| x |\alpha\rangle = -\varepsilon_\alpha \varepsilon_\beta \langle\beta| x |\alpha\rangle$$

$\hookrightarrow$ $\boxed{\langle\beta| x |\alpha\rangle (1 + \varepsilon_\alpha \varepsilon_\beta) = 0}$

* parity selection rule.

$$\boxed{\text{So } \langle\beta| x |\alpha\rangle = 0 \text{ unless } \varepsilon_\alpha \varepsilon_\beta = -1 \text{ or } \varepsilon_\alpha = -\varepsilon_\beta}$$

$$\langle\beta| x |\alpha\rangle = \int \psi_\alpha^* x \psi_\beta \, d^3x.$$

$\rightarrow$ Implys non-degenerate energy state cannot posess a permanent dipole moment.

$$\langle\alpha| \vec{x} \cdot \vec{r} |\alpha\rangle = 0$$

$\rightarrow \quad \langle \alpha'; l' m' | \vec{X} | \alpha; l . m \rangle = 0 \quad$ unless $\varepsilon_\alpha \varepsilon_\beta = -1$.

$\hookrightarrow \varepsilon_\alpha = (-1)^{l'} \qquad \varepsilon_\beta = (-1)^{l}$

$\hookrightarrow \quad \varepsilon_\alpha \varepsilon_\beta = (-1)^{l}(-1)^{l'} \Rightarrow \boxed{l - l' = \text{odd}}$ ← parity selection rule.

Rotational selection rule:

$\hookrightarrow l = \begin{cases} l'-1 \\ l' \\ l+1 \end{cases} \xRightarrow{\text{with parity selection}} \boxed{l = l' \pm 1}$

Other discrete symetry:

- Point group
- Lattice translational.

# Time Reversal Symmetry:

Newton: If $m\ddot{x} = -\vec{\nabla} V$ : and $x(t)$ is a solution
then $x(-t)$ is also a solution.

Maxwell: $m\ddot{\vec{x}} = e(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})$

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$\vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{J}$

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t B$

$\vec{\nabla}\cdot\vec{B} = 0$

if $t \to -t$, then

$\vec{E} \to \vec{E}$

$\vec{B} \to -\vec{B}$

$\rho \to \rho$

$\vec{J} \to -\vec{J}$

$\vec{v} \to -\vec{v}$

Now Schrodinger: $i\hbar\,\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})\right]\psi$

If $\psi(x,t)$ is a solution, then
$\psi^*(x,-t)$ is a solution, notice that complex conjugate.

at $t=0$: $\psi = \langle x|\alpha\rangle$ - time reversal $\to \psi^* = \langle x|\alpha\rangle^*$

$\langle\alpha|\pi^\dagger\pi|\beta\rangle = \langle\alpha|\beta\rangle$

$\hookrightarrow \int \psi_\alpha^*(-x)\,\psi_\beta(-x)\,d\vec{x} = \int \psi_\alpha^*(x)\,\psi_\beta(x)\,d\vec{x}$ $\leftarrow$ parity.

but $\int dx\, \underbrace{(\psi_\alpha^*(x))^*}_{\langle\tilde{\alpha}|}\, \underbrace{(\psi_\beta(x)^*)}_{|\tilde{\beta}\rangle} = \left(\int dx\, \psi_\alpha^*(x)\psi_\beta(x)\right)^*$ $\leftarrow$ Time Reversal

$$\boxed{\hookrightarrow \langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle\alpha|\beta\rangle^*}$$

time reversal of $|\alpha\rangle$ and $|\beta\rangle$

implies time reversal is not unitary.

Def: $|\alpha\rangle \rightarrow |\tilde{\alpha}\rangle = \theta|\alpha\rangle$
$|\beta\rangle \rightarrow |\tilde{\beta}\rangle = \theta|\beta\rangle$ ) Transformation.

anti-unitary.

If i) $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$

ii) antlinear $\theta(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^*\theta|\alpha\rangle + c_2^*\theta|\beta\rangle$

complex conjugate.

$$\boxed{\begin{array}{l} \text{Write } \theta = \underset{\text{unitary}}{U}\,\underset{\text{complex conjugate.}}{K} \\ \text{where } K\,c|\alpha\rangle = c^* K|\alpha\rangle \end{array}}$$

let $|\alpha\rangle = \sum_a |a\rangle\langle a|\alpha\rangle$

$\xrightarrow{K}$ $K|\alpha\rangle = \sum_a \langle a|\alpha\rangle^* |a\rangle$

→ K: depends on basis , $K|a\rangle = |a\rangle$

Ex: check $\Theta = UK$ antiunitary holds:

$$|\alpha\rangle \xrightarrow{\Theta} |\tilde{\alpha}\rangle = \sum_a \langle a|\alpha\rangle^* \underbrace{UK}_{\Theta} |a\rangle$$

$$= \sum_a \langle a|\alpha\rangle^* U|a\rangle$$

$$|\tilde{\alpha}\rangle \overset{!}{=} \sum \langle\alpha|a\rangle U|a\rangle$$

Similarly $\langle\tilde{\beta}| = \sum_a \langle a|\beta\rangle \langle a|U^\dagger$

then $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \sum_{a_1 a_2} \langle a_2|\beta\rangle \underbrace{\langle a_2|U^\dagger U|a_1\rangle}_{\substack{\langle a_2|a_1\rangle \\ a_1 = a_2}} \langle\alpha|a_1\rangle$

$$= \sum_{a_2} \langle a_2|\beta\rangle\langle\alpha|a_2\rangle$$

$$\overset{!}{=} \langle\alpha|\beta\rangle$$

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle \overset{!}{=} \langle\beta|\alpha\rangle^*$$ ← proves anti-unitary.

but → $|\langle\tilde{\beta}|\tilde{\alpha}\rangle| = |\langle\beta|\alpha\rangle|$ absolute value don't change.

either unitary or antiunitary.

## Time reversal operator:

$$|\alpha\rangle \longrightarrow \Theta|\alpha\rangle = |\tilde{\alpha}\rangle$$

expect $\Theta|\vec{p}\rangle \rightarrow |-\vec{p}\rangle$

$$\vec{J} \xrightarrow{\Theta} -\vec{J}$$

$$|\alpha\rangle_{\delta t} = \left(1 - \frac{i}{\hbar} H \delta t\right)|\alpha\rangle$$

$$\left(1 - \frac{i}{\hbar} H \delta t\right)\Theta|\alpha\rangle = \Theta\left(1 - \frac{i}{\hbar} H(-\delta t)\right)|\alpha\rangle$$

$$-iH\Theta = \Theta iH$$

If $\Theta$ is unitary, so $i$ doesn't get conjugated.

$$-iH\Theta = i\Theta H \quad \rightarrow \quad H\Theta = -\Theta H$$

$$\Theta^{-1}\frac{p^2}{2m}\Theta = -\frac{p^2}{2m}$$

← can't be, so $\Theta$ is not unitary.

If $\Theta$ is anti-unitary:

$$-iH\Theta = \Theta iH$$

complex conjugate since anti-unitary.

$$-iH\Theta = -i\Theta H$$

↳ $\boxed{[H, \Theta] = 0}$ $\Theta$ commutes with $H$.

How does $\Theta$ act on operators:

$$\langle\beta|\Theta|\alpha\rangle = \langle\beta|\left(\Theta|\alpha\rangle\right)$$ , let $\Theta$ always act to the right.

$$|\tilde{\alpha}\rangle = \Theta|\alpha\rangle$$
$$|\tilde{\beta}\rangle = \Theta|?\rangle$$

Identity: $\boxed{\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A \Theta^{-1}|\tilde{\beta}\rangle}$ *

proof: $\langle\beta|A|\alpha\rangle = \left(\langle\beta|A\right)|\alpha\rangle = \langle\gamma|\alpha\rangle$

$$\langle\gamma|\alpha\rangle = \langle\tilde{\gamma}|\tilde{\alpha}\rangle^* = \langle\tilde{\alpha}|\tilde{\gamma}\rangle$$
$$\overset{!}{=} \langle\tilde{\alpha}|\Theta\underbrace{\left(A^\dagger|\beta\rangle\right)}_{|\gamma\rangle}$$
$$\overset{!}{=} \langle\tilde{\alpha}|\Theta A^\dagger|\beta\rangle$$
$$\overset{!}{=} \langle\tilde{\alpha}|\Theta A^\dagger \Theta^{-1}|\tilde{\beta}\rangle$$

$$\langle\beta|A|\alpha\rangle \overset{!}{=} \langle\tilde{\alpha}|\Theta A \Theta^{-1}|\tilde{\beta}\rangle \quad \text{if } A = A^\dagger$$

If $\tilde{A} = \Theta A \Theta^{-1} = \pm A$

then $\Theta A = \pm A\Theta$

$$\langle \beta | A | \alpha \rangle = \langle \tilde{\alpha} | \Theta A \Theta^{-1} | \tilde{\beta} \rangle$$

$$\stackrel{\downarrow}{=} \pm \langle \tilde{\alpha} | A | \tilde{\beta} \rangle$$

or $\langle \alpha | A | \alpha \rangle = \pm \langle \tilde{\alpha} | A | \tilde{\alpha} \rangle$ $\leftarrow$ restriction on exp value in time reversal state.

$$\Theta \vec{p} \Theta^{-1} = -\vec{p}$$

$\hookrightarrow$ $\langle \alpha | \vec{p} | \alpha \rangle = -\langle \tilde{\alpha} | \vec{p} | \tilde{\alpha} \rangle$

$$\vec{p}\, \Theta | \vec{p}' \rangle = -\Theta \vec{p} | \vec{p}' \rangle = -\Theta \vec{p}' | \vec{p}' \rangle = -p' (\Theta | \vec{p}' \rangle$$

$$\boxed{\Theta | \vec{p}' \rangle = | -\vec{p}' \rangle \quad \text{up to phase.}}$$

$$\Theta \vec{x} \Theta^{-1} = \vec{x} \quad \rightarrow \quad \Theta | \vec{x}' \rangle = | \vec{x}' \rangle$$