Zhi Chen HW#17

1) $\phi(\vec{p}') = \langle \vec{p}' | \alpha \rangle$, what is the momentum-wave function for the time-reversed state $\Theta|\alpha\rangle$,

$$\Theta|\alpha\rangle = \Theta \, d^3p \, |\vec{p}\rangle\langle\vec{p}|\alpha\rangle$$

$$\overset{!}{=} UK \, d^3p \, |\vec{p}\rangle\langle\vec{p}|\alpha\rangle$$

$$\overset{!}{=} d^3p \, \langle\vec{p}|\alpha\rangle^* \, UK \, |\vec{p}\rangle$$

$$\overset{!}{=} d^3p \, \langle\vec{p}|\alpha\rangle^* \, |-\vec{p}\rangle$$

$$\overset{!}{=} d^3p \, \underbrace{\langle-\vec{p}|\alpha\rangle^*}_{\phi_\alpha(-\vec{p})^*} \, |\vec{p}\rangle$$

then $\langle\vec{p}'|\Theta|\alpha\rangle = d^3p \, \langle\vec{p}'|\vec{p}\rangle \, \langle-\vec{p}|\alpha\rangle^*$

$$= \langle-\vec{p}'|\alpha\rangle^*$$

$$\boxed{\langle\vec{p}'|\Theta|\alpha\rangle = \phi_\alpha^*(-\vec{p}')}$$

2) Spin 1 system: $H = AS_z^2 + B(S_x^2 - S_y^2)$

Find eigenvalue, eigenvectors. Is this H invariant under time reversal? How do eigenvectors transform under time reversal.

let $H = AS_z^2 + B(S_x^2 - S_y^2) = AS_z^2 + B(S_x^2 - (S^2 - S_x^2 - S_z^2))$

$$H = (A+B)S_z^2 - BS^2 + 2BS_x^2$$

For spin, $s=1$, $m=-1,0,+1$

We know $S_{x,y,z}$ for spin 1 in the basis $|m\rangle$, $m=0,\pm1$ using relation $s^2|s=1,m\rangle = \hbar^2(1+1)|s=1,m\rangle = 2\hbar^2|s=1,m\rangle$

$$S_z|s=1,m\rangle = \hbar m|s=1,m\rangle$$

$$\left.\begin{aligned} S_+|s=1,m\rangle &= \hbar\sqrt{(1-m)(1+m-1)}\,|s=1,m+1\rangle \\ S_-|s=1,m\rangle &= \hbar\sqrt{(1+m)(1-m-1)}\,|s=1,m-1\rangle \end{aligned}\right\} \quad S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

Quoting results from HW #12:

$$S_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \quad S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad S^2 = 2\hbar^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$S_y = \frac{\hbar}{i\sqrt{2}}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$$

then $S_z^2 = \hbar^2 \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ $\quad S_x^2 = \frac{\hbar^2}{2}\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{pmatrix}$

then $H = (A+B)S_z^2 - BS^2 + 2BS_x^2$

$$= \hbar^2 \begin{pmatrix}(A+B)-2B+B & 0 & B\\ 0 & -2B+2B & 0\\ B & 0 & (A+B)-2B+B\end{pmatrix}$$

$$H = \hbar^2 \begin{pmatrix}A & 0 & B\\ 0 & 0 & 0\\ B & 0 & A\end{pmatrix}$$

Find eigenvalue: $\hbar^2 \begin{pmatrix}A-E & 0 & B\\ 0 & -E & 0\\ B & 0 & A-E\end{pmatrix}$

$$\hbar^2\left[(A-E)^2(-E) + B^2E\right] = 0$$

$$-\hbar^2 E\left[(A-E)^2 - B^2\right] = -\hbar^2 E\left(\underbrace{E^2 - 2AE + A^2 - B^2}\right) = 0$$

$$= -\hbar^2 E\left(E-(A+B)\right)^2\left(E-(A-B)\right)^2$$

then we see $\boxed{E = 0,\ (A\pm B)\hbar^2}$

When $E = (A+B)\hbar^2$

$$H - EI = \hbar^2 \begin{pmatrix} -B & 0 & B \\ 0 & 0 & 0 \\ B & 0 & -B \end{pmatrix} \begin{pmatrix} E_{+,1} \\ E_{+2} \\ E_{+3} \end{pmatrix} = 0$$

$|E_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)$ with $E_+ = (A+B)\hbar^2$

When $E = (A-B)\hbar^2$:

$$H - EI = \hbar^2 \begin{pmatrix} B & 0 & B \\ 0 & 0 & 0 \\ B & 0 & B \end{pmatrix} \begin{pmatrix} E_{-1} \\ E_{-2} \\ E_{-3} \end{pmatrix} = 0$$

$|E_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle)$ with $E_- = (A-B)\hbar^2$

When $E = 0$.

$$H - EI = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix} \begin{pmatrix} E_{0,1} \\ E_{0,2} \\ E_{0,3} \end{pmatrix} = 0$$

$|E_0\rangle = |0\rangle$ with $E_0 = 0$

then We have

E	Eigenket in $\|m\rangle$ basis
$(A+B)\hbar^2$	$\|E_+\rangle = \frac{1}{\sqrt{2}}(\|1\rangle + \|-1\rangle)$
$(A-B)\hbar^2$	$\|E_-\rangle = \frac{1}{\sqrt{2}}(\|1\rangle - \|-1\rangle)$
0	$\|E_0\rangle = \|0\rangle$

If H is invariant under time reversal, then

$$[H, \Theta] = 0$$

$$\Theta\left(AS_z^2 + B(S_x^2 - S_y^2)\right) = \Theta\left(AS_z^2 + B(S_x^2 - S_y^2)\right)\Theta^{-1}\Theta$$
$$\stackrel{!}{=} \left[\Theta(AS_z^2)\Theta^{-1} + \Theta(BS_x^2)\Theta^{-1} - \Theta(BS_y^2)\Theta^{-1}\right]\Theta$$

Assume A and B are constant parameters,

and we know $\vec{S}$ undergoes $\Theta S_i \Theta^{-1} = -S_i$

then we have $\Theta S_i S_i \Theta^{-1} = \underbrace{\Theta S_i \Theta^{-1}}_{-S_i} \underbrace{\Theta S_i \Theta^{-1}}_{-S_i} = S_i^2$

So we see S_i^2 is even under time reversal, so H is also even under time reversal, so $[H, \Theta] = 0$

How does eigenket change under time reversal?

use property: $\Theta|l, m\rangle = (-1)^m |l, -m\rangle$

$$\Theta|E_+\rangle = \frac{1}{\sqrt{2}}\left(\Theta|m=1\rangle + \Theta|m=-1\rangle\right) = -\frac{1}{\sqrt{2}}\left(|-1\rangle + |1\rangle\right) = -|E_+\rangle$$

$$\Theta|E_-\rangle = \frac{1}{\sqrt{2}}\left(\Theta|m=1\rangle - \Theta|m=-1\rangle\right) = \frac{-1}{\sqrt{2}}\left(|-1\rangle - |1\rangle\right) = -|E_-\rangle$$

$$\Theta|E_0\rangle = \Theta|0\rangle = |0\rangle = |E_0\rangle$$

$\Rightarrow$ we see that $|E_0\rangle = |0\rangle$ is even under time reversal. while $|E_+\rangle = 1/\sqrt{2}(|1\rangle + |-1\rangle)$ and $|E_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle)$ are odd.