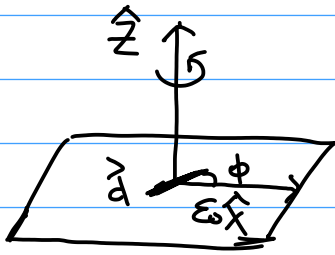


1)



→ choose coordinate system such that plane rotates around \hat{z} , so $L^2 = L_z^2$

→ \vec{E}_0 is lying on the x-y plane, so choose \vec{E}_0 to be in the x-axis, $\vec{E}_0 = E_0 \hat{x}$

$$H_0 = \frac{L_z^2}{2I} \quad \leftarrow \text{unperturbed Hamiltonian}$$

then since $L_z = -i\hbar \partial_\phi$, $L_z^2 = -\hbar^2 \partial_\phi^2$

$$-\frac{\hbar^2}{2I} \partial_\phi^2 \psi(\phi) = E \psi$$

$$\partial_\phi^2 \psi + \frac{2IE}{\hbar^2} \psi = 0$$

$$\text{let } \frac{2IE}{\hbar^2} = m^2 \quad \text{so } \boxed{E = \frac{\hbar^2}{2I} m^2}$$

$$\text{then } \partial_\phi^2 \psi + m^2 \psi = 0 \quad \text{or } \psi \propto e^{im\phi}$$

due to periodic boundary: $\psi(\phi) = \psi(\phi + 2\pi)$, $\Rightarrow e^{im\phi} = e^{im(2\pi + \phi)}$

$$\hookrightarrow e^{i2\pi m} = 1 \quad \text{or } m = 0, \pm 1, \pm 2, \dots$$

with normalization $|A|^2 \int d\phi = 1 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$

$$\boxed{\psi_m^{(0)} = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{with } E_m^{(0)} = \frac{\hbar^2}{2I} m^2, \quad m=0, \pm 1, \dots}$$

Now consider ground state, $m=0$, which is not degenerate.

$$|0\rangle = |0^{(0)}\rangle + \sum_k \frac{\langle k|V|0\rangle}{E_0^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

$$V = -\vec{d} \cdot \vec{E}_0 = -\vec{d} \cdot \epsilon_0 \hat{x} = -\epsilon_0 d \cos\phi = -\epsilon_0 d \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right)$$

first compute $\langle k|V|m\rangle$

$$\hookrightarrow \int_0^{2\pi} d\phi \frac{1}{2\pi} e^{-ik\phi} \left(\frac{-\epsilon_0 d}{2} \right) (e^{i\phi} + e^{-i\phi}) e^{im\phi}$$

$$= \int_0^{2\pi} d\phi \frac{-\epsilon_0 d}{4\pi} \left(e^{i\phi(m-k+1)} + e^{i\phi(m-k-1)} \right)$$

These terms integrate to 0 unless exponent is 0.

i.e. $m-k = \pm 1$

and integrates to 2π if $m-k = \pm 1$

$$\langle k|V|m\rangle = -\frac{\epsilon_0 d}{2} \delta_{k,m\pm 1}$$

then when $m=0$ we have:

$$\text{then } |0\rangle = |0^{(0)}\rangle + \frac{-\epsilon_0 d}{\cancel{E_0^{(0)}} - E_1^{(0)}} |1\rangle + \frac{-\epsilon_0 d}{\cancel{E_0^{(0)}} - E_{-1}^{(0)}} |-1\rangle$$

$$= |0^{(0)}\rangle + \frac{\epsilon_0 d}{\cancel{2\frac{\hbar^2}{R^2 I}} - E_1^{(0)}} (|1\rangle + |-1\rangle)$$

$$\boxed{|0\rangle = |0^{(0)}\rangle + \frac{\epsilon_0 I d}{\hbar^2} (|1^{(0)}\rangle + |-1^{(0)}\rangle)}$$

First order perturbation state.

Now evaluate $\alpha = \frac{d}{d\epsilon_0} \langle d \rangle$

$$\langle d \rangle = \left[\langle 0^{(0)} | + \frac{\epsilon_0 I d}{\hbar^2} (\langle 1 | + \langle -1 |) \right] \vec{d} \left[| 0^{(0)} \rangle + \frac{\epsilon_0 I d}{\hbar^2} (| 1 \rangle + | -1 \rangle) \right]$$

$$= \cancel{\langle 0^{(0)} | \vec{d} | 0^{(0)} \rangle} + \frac{\epsilon_0 I d}{\hbar^2} (\langle 0^{(0)} | \vec{d} (| 1 \rangle + | -1 \rangle) + (\langle 1 | + \langle -1 |) \vec{d} | 0 \rangle) + \left(\frac{\epsilon_0 I d}{\hbar^2} \right)^2 (\langle 1 | + \langle -1 |) \vec{d} (| 1 \rangle + | -1 \rangle)$$

0 by selection rule

= 0 by selection rule.

with $\vec{d} = d (\cos \phi \hat{x} + \sin \phi \hat{y}) = d \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \hat{x} + \frac{e^{i\phi} - e^{-i\phi}}{2i} \hat{y} \right)$

$$= \frac{\epsilon_0 I d^2}{\hbar^2} \left\{ \left(\langle 0^{(0)} | \frac{e^{i\phi} + e^{-i\phi}}{2} (| 1 \rangle + | -1 \rangle) + (\langle 1 | + \langle -1 |) \frac{e^{i\phi} + e^{-i\phi}}{2} | 0 \rangle \right) \hat{x} \right.$$

$$\left. + \left(\langle 0^{(0)} | \frac{e^{i\phi} - e^{-i\phi}}{2i} (| 1 \rangle + | -1 \rangle) + (\langle 1 | + \langle -1 |) \frac{e^{i\phi} - e^{-i\phi}}{2i} | 0 \rangle \right) \hat{y} \right\}$$

0

= 0

$$= \frac{\epsilon_0 I d^2}{\hbar^2} \left(4 \frac{1}{2} \hat{x} \right)$$

$$= - \frac{2 \epsilon_0 I d^2}{\hbar^2} \hat{x}$$

so

$$\alpha = \frac{d \langle \vec{d} \rangle}{d \epsilon_0} = \frac{2 I d^2}{\hbar^2}$$

Applicable when perturbation

theory holds which means
 $\langle k | V | n \rangle < (E_n^{(0)} - E_k^{(0)})$
 for $n=0, k=1$

$$\hookrightarrow \frac{\epsilon_0 d}{2} < \frac{\hbar^2}{2I}$$

$$\hookrightarrow \boxed{\epsilon_0 < \frac{\hbar^2}{I d}}$$

2) Find approximate wavefunction to first order.

$$|m\rangle = |m^{(0)}\rangle + \sum_k' \frac{\langle k^{(0)}|V|m^{(0)}\rangle}{E_m^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

For $m > 0$, we have 2-fold degeneracy, $E_m = E_{-m}$

but we have the selection rules $\langle k^{(0)}|V|m^{(0)}\rangle \propto \delta_{k,m\pm 1}$

So k is never $-m$.

$$\begin{aligned} |m\rangle &= |m^{(0)}\rangle + \frac{1}{E_m - E_{m\pm 1}} \left(\langle m+1^{(0)}|V|m^{(0)}\rangle |m+1^{(0)}\rangle \right. \\ &\quad \left. + \langle m-1^{(0)}|V|m^{(0)}\rangle |m-1^{(0)}\rangle \right) \\ &= |m^{(0)}\rangle + \frac{-\epsilon_0 I d}{\frac{\hbar^2}{2}(m^2 - (m+1)^2)} \langle m+1^{(0)}|e^{i\phi} + e^{-i\phi}|m^{(0)}\rangle |m+1^{(0)}\rangle \\ &\quad + \frac{-\epsilon_0 I d}{\frac{\hbar^2}{2}(m^2 - (m-1)^2)} \langle m-1^{(0)}|e^{i\phi} + e^{-i\phi}|m^{(0)}\rangle |m-1^{(0)}\rangle \end{aligned}$$

$$\begin{aligned} |m\rangle &= |m^{(0)}\rangle + \left(\frac{-\epsilon_0 I d}{\frac{\hbar^2}{2}(-2m-1)} \frac{1}{\sqrt{2}} |m+1^{(0)}\rangle \right) + \left(\frac{-\epsilon_0 I d}{\frac{\hbar^2}{2}(2m-1)} \frac{1}{\sqrt{2}} |m-1^{(0)}\rangle \right) \\ &= |m^{(0)}\rangle + \frac{\epsilon_0 I d}{\hbar^2} \left(\frac{1}{2m+1} |m+1^{(0)}\rangle - \frac{1}{2m-1} |m-1^{(0)}\rangle \right) \end{aligned}$$

Normalize: $\langle m|m\rangle A^2 = A^2 \left(1 + \left(\frac{\epsilon_0 I d}{\hbar^2} \right)^2 \left[\left(\frac{1}{2m+1} \right)^2 + \left(\frac{1}{2m-1} \right)^2 \right] \right) = 1$

then $A = \left(1 + \left(\frac{\epsilon_0 I d}{\hbar^2} \right)^2 \left[\left(\frac{1}{2m+1} \right)^2 + \left(\frac{1}{2m-1} \right)^2 \right] \right)^{-1/2}$

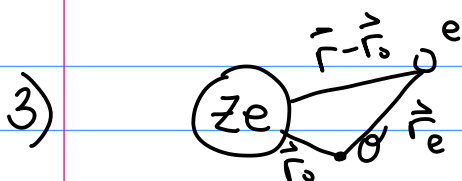
$$A \approx 1 - \frac{1}{2} \left(\frac{\epsilon_0 I d}{\hbar^2} \right)^2 \left[\left(\frac{1}{2m+1} \right)^2 + \left(\frac{1}{2m-1} \right)^2 \right]$$

then:

$$|m\rangle = \left(1 - \frac{1}{2} \left(\frac{\epsilon_0 I d}{\hbar^2} \right)^2 \left[\left(\frac{1}{2m+1} \right)^2 + \left(\frac{1}{2m-1} \right)^2 \right] \right) \left\{ |m^{(0)}\rangle + \frac{\epsilon_0 I d}{\hbar^2} \left(\frac{1}{2m+1} |m+1^{(0)}\rangle - \frac{1}{2m-1} |m-1^{(0)}\rangle \right) \right\}$$

then the wave function $\langle \phi | m \rangle$:

$$\Rightarrow \psi(\phi) = \left(1 - \frac{1}{2} \left(\frac{\epsilon_0 I d}{\hbar^2} \right)^2 \left[\left(\frac{1}{2m+1} \right)^2 + \left(\frac{1}{2m-1} \right)^2 \right] \right) \left\{ \frac{1}{\sqrt{2\pi}} e^{im\phi} + \frac{\epsilon_0 I d}{\hbar^2} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2m+1} e^{i(m+1)\phi} - \frac{1}{2m-1} e^{i(m-1)\phi} \right) \right\}$$



$$a) H = \frac{p_{nuc}^2}{2M} + \frac{p_e^2}{2m} - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_e - \vec{r}_n|}$$

$$= -\frac{\hbar^2}{2M} \nabla_{nuc}^2 - \frac{\hbar^2}{2m} \nabla_e^2 - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_e - \vec{r}_n|}$$

b) define $\vec{R}_{cm} = \frac{M\vec{r}_n + m\vec{r}_e}{M+m}$ and $\vec{r} = \vec{r}_e - \vec{r}_n$

$$\Rightarrow \frac{\partial}{\partial \vec{r}_n} = \frac{\partial}{\partial \vec{R}_{cm}} \frac{\partial \vec{R}_{cm}}{\partial \vec{r}_n} + \frac{\partial}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial \vec{r}_n}$$

$$= \frac{\partial}{\partial \vec{R}_{cm}} \left(\frac{M}{M+m} \right) + \frac{\partial}{\partial \vec{r}} (-1)$$

and

$$\frac{\partial}{\partial \vec{r}_e} = \frac{\partial}{\partial \vec{R}_{cm}} \frac{\partial \vec{R}_{cm}}{\partial \vec{r}_e} + \frac{\partial}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial \vec{r}_e}$$

$$= \frac{\partial}{\partial \vec{R}_{cm}} \left(\frac{m}{M+m} \right) + \frac{\partial}{\partial \vec{r}} (+1)$$

then:

$$\nabla_n^2 = \frac{\partial}{\partial \vec{r}_n} \cdot \frac{\partial}{\partial \vec{r}_n} = \left(\frac{M}{M+m} \right)^2 \left(\frac{\partial}{\partial \vec{R}_{cm}} \right)^2 - 2 \left(\frac{M}{M+m} \right) \frac{\partial}{\partial \vec{r}} \frac{\partial}{\partial \vec{R}_{cm}} + \left(\frac{\partial}{\partial \vec{r}} \right)^2$$

$$\nabla_e^2 = \frac{\partial}{\partial \vec{r}_e} \cdot \frac{\partial}{\partial \vec{r}_e} = \left(\frac{m}{M+m} \right)^2 \left(\frac{\partial}{\partial \vec{R}_{cm}} \right)^2 + 2 \left(\frac{m}{M+m} \right) \frac{\partial}{\partial \vec{r}} \frac{\partial}{\partial \vec{R}_{cm}} + \left(\frac{\partial}{\partial \vec{r}} \right)^2$$

then
$$-\frac{\hbar^2}{2M} \nabla_n^2 - \frac{\hbar^2}{2m} \nabla_e^2 - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_e - \vec{r}_n|}$$

$$= -\frac{\hbar^2}{2M} \left[\left(\frac{M}{M+m} \right)^2 \left(\frac{\partial}{\partial \vec{R}_{cm}} \right)^2 - 2 \left(\frac{M}{M+m} \right) \frac{\partial}{\partial \vec{r}} \frac{\partial}{\partial \vec{R}_{cm}} + \left(\frac{\partial}{\partial \vec{r}} \right)^2 \right]$$

$$-\frac{\hbar^2}{2m} \left[\left(\frac{m}{M+m} \right)^2 \left(\frac{\partial}{\partial \vec{R}_{cm}} \right)^2 + 2 \left(\frac{m}{M+m} \right) \frac{\partial}{\partial \vec{r}} \frac{\partial}{\partial \vec{R}_{cm}} + \left(\frac{\partial}{\partial \vec{r}} \right)^2 \right] - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}|}$$

recognize $\frac{\partial}{\partial \vec{R}_{cm}} = \vec{\nabla}_{R_{cm}}$ and $\frac{\partial}{\partial \vec{r}} = \vec{\nabla}_r$

$$= -\frac{\hbar^2}{2} \nabla_{R_{cm}}^2 \left(\frac{M}{(M+m)^2} + \frac{m}{(M+m)^2} \right) - \frac{\hbar^2}{2} \nabla_r^2 \left(\frac{1}{M} + \frac{1}{m} \right) - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}|}$$

$$= \frac{1}{M+m} \quad \quad \quad = \left[\frac{M+m}{Mm} = \frac{1}{\mu} \right], \text{ reduced mass}$$

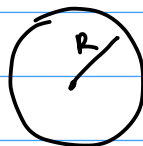
$$H = \underbrace{-\frac{\hbar^2}{2(M+m)} \nabla_{R_{cm}}^2}_{\text{Center of mass motion}} \underbrace{-\frac{\hbar^2}{2\mu} \nabla_r^2 - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}|}}_{\text{interaction terms, which depends on } \vec{r}.}$$

Center of mass motion

which only depends on \vec{R}_{cm}

interaction terms, which depends on \vec{r} .

so it decouples from center of mass motion

4) a)  $Q_{tot} = Ze$ $\rho_{uniform} = \frac{Ze}{\frac{4}{3}\pi R^3}$

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\hookrightarrow \int d^3r \vec{\nabla} \cdot \vec{E} = \int \frac{1}{\epsilon_0} \frac{Ze}{\frac{4}{3}\pi R^3} r^2 dr d\theta d\phi \sin\theta$$

divergence theorem

$$\hookrightarrow \int d^3r \vec{\nabla} \cdot \vec{E} = E_r \underbrace{\int \sin\theta d\theta \int d\phi}_{4\pi r^2} = \int \frac{1}{\epsilon_0} \frac{Ze}{\frac{4}{3}\pi R^3} r^2 dr 4\pi$$

For $r > R$:

$$E_r 4\pi r^2 = \int_0^R r'^2 dr' \frac{3Ze}{\epsilon_0 R^3}$$

$$\vec{E}_{r>R} = \frac{Ze}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \quad r > R$$

$$\phi = - \int_{\infty}^r \vec{E} \cdot d\vec{r} = \frac{Ze}{4\pi\epsilon_0} \frac{1}{r} \quad r > R$$

drop $4\pi\epsilon_0$ for Gaussian

For $r < R$:

$$E_r 4\pi r^2 = \int_0^r r'^2 dr' \frac{3Ze}{\epsilon_0 R^3}$$

$$\vec{E} = \frac{Ze}{4\pi\epsilon_0} \left(\frac{r^3}{R^3} \right) \frac{1}{r^2} \hat{r} \quad r < R$$

$$\phi = - \int_{\infty}^r \vec{E} \cdot d\vec{r} = - \int_{\infty}^R \vec{E}_{r>R} \cdot d\vec{r} - \int_R^r \vec{E}_{r<R} \cdot d\vec{r}$$

$$= \frac{Ze}{4\pi\epsilon_0} \frac{1}{R} - \frac{Ze}{4\pi\epsilon_0} \frac{1}{R^3} \frac{r^2}{2} \Big|_R^r$$

$$\phi = \frac{Ze}{4\pi\epsilon_0} \frac{1}{R^2} \frac{3}{2} - \frac{Ze}{4\pi\epsilon_0} \frac{1}{R^3} \frac{r^2}{2} \quad r < R$$

$$b) \quad V_0 = -e\phi = -\frac{ze^2}{r}$$

New potential:

$$V = -e\phi = \begin{cases} -\frac{3}{2} \frac{ze^2}{R} + \frac{1}{2} \frac{ze^2}{R^3} r^2 & r < R \\ -\frac{ze^2}{r} & r > R \end{cases}$$

the difference comes in $r < R$:

$$\Delta V = V_{r < R} - V_0$$

$$\Delta V = -\frac{3}{2} \frac{ze^2}{R} + \frac{1}{2} \frac{ze^2}{R^3} r^2 + \frac{ze^2}{r} \quad \text{for } r < R$$

Consider ground state of hydrogen atom: $\psi_{100} = R_{10} Y_0^0$, nondegenerate

$$\langle 100 | \Delta V | 100 \rangle = \int_0^R r^2 dr \underbrace{\sin^2 \theta d\phi}_{=1} |Y_0^0|^2 \underbrace{R_{10}^*(r) R_{10}(r)}_{=4\left(\frac{z}{a_0}\right)^3 e^{-\frac{2zr}{a_0}}} \left\{ -\frac{3}{2} \frac{ze^2}{R} + \frac{1}{2} \frac{ze^2}{R^3} r^2 + \frac{ze^2}{r} \right\}$$

$$R_{10} = 2\left(\frac{z}{a_0}\right)^{3/2} e^{-\frac{zr}{a_0}}$$

using
Mathematica

$$= \frac{2}{R} ze^2 \left(\frac{z}{a_0}\right)^3 \int_0^R r^2 dr \left(-3 + \left(\frac{r}{R}\right)^2 + \frac{2R}{r} \right) e^{-\frac{2zr}{a_0}}$$

$$E_0^{(1)} = \frac{2}{R} ze^2 \left(\frac{z}{a_0}\right)^3 \left\{ \frac{-3a_0^3 (a_0 + Rz)^2 e^{-\frac{2Rz}{a_0}}}{4R^2 z^5} + 3a_0^5 - 3a_0^3 R^2 z^2 + 2R a_0^2 z^3 \right\}$$

exact solution

If we assume $\frac{z}{a_0} \ll 1$, then $e^{\frac{-2zr}{a_0}} \sim 1$

then

$$E_0^{(1)} \approx \frac{2}{R} z e^2 \left(\frac{z}{a_0}\right)^3 \int_0^R r^2 dr \left(-3 + \left(\frac{r}{R}\right)^2 + \frac{2R}{r}\right)$$

$$\approx \frac{2}{R} z e^2 \left(\frac{z}{a_0}\right)^3 \left[\cancel{\frac{-3R^3}{3}} + \frac{R^5}{5} \frac{1}{R^2} + \cancel{\frac{2R^3}{2}} \right]$$

$$\approx \frac{2}{R} z^4 e^2 \frac{1}{a_0^3} \left[\frac{R^3}{5} \right]$$

$$E_{n=1}^{(1)} \approx \frac{2}{5} z^4 e^2 \left(\frac{R}{a_0}\right)^2 \frac{1}{a_0} \quad \leftarrow \text{approximate solution.}$$

using $E_0^{(0)} = -\frac{1}{2} \frac{z^2 e^2}{a_0}$

$$= -\frac{4}{5} \left(\frac{zR}{a_0}\right)^2 E_{n=1}^{(0)}$$

c) Consider relativistic effect:

$$T = \sqrt{p^2 c^2 + m_e^2 c^4} - m_e c^2$$

$$= m_e c^2 \sqrt{1 + \left(\frac{pc}{m_e c^2}\right)^2} - m_e c^2$$

assume $pc \ll m_e c^2$, use binomial expansion: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2$

$$= m_e c^2 \left(1 + \frac{1}{2} \left(\frac{pc}{m_e c^2}\right)^2 - \frac{1}{8} \left(\frac{pc}{m_e c^2}\right)^4 \right) - m_e c^2$$

$$T \approx \frac{p^2}{2m_e} - \frac{p^4}{8m_e^3 c^2}$$

then $H = \underbrace{\frac{p^2}{2m_e} - \frac{ze^2}{r}}_{H_0} - \underbrace{\frac{p^4}{8m_e^3 c^2}}_{\Delta V}$

$$\Delta V = -\frac{p^4}{8m_e^3 c^2} = -\frac{1}{2m_e c^2} \left(\frac{p^2}{2m_e}\right)^2 = \frac{-1}{2m_e c^2} \left(H_0 + \frac{ze^2}{r}\right)^2$$

then $\langle 100 | \Delta V | 100 \rangle = -\frac{1}{2m_e c^2} \langle 100 | H_0^2 + 2H_0 \frac{ze^2}{r} + \left(\frac{ze^2}{r}\right)^2 | 100 \rangle$

$$= -\frac{1}{2m_e c^2} \left(E_{n=0}^2 + 2E_{n=0} ze^2 \langle 100 | \frac{1}{r} | 100 \rangle + (ze^2)^2 \langle 100 | \frac{1}{r^2} | 100 \rangle \right)$$

Calculate: $\langle 100 | \frac{1}{r} | 100 \rangle = \int_0^\infty r^2 dr \cdot 4 \left(\frac{z}{a_0}\right)^3 e^{-\frac{2zr}{a_0}} \frac{1}{r}$

$$= \frac{1}{4} \left(\frac{a_0}{z}\right)^2 \cdot 4 \left(\frac{z}{a_0}\right)^3$$

$$\langle 100 | \frac{1}{r} | 100 \rangle = \frac{z}{a_0}$$

$$\begin{aligned} \text{Calculate: } \langle 100 | \frac{1}{r^2} | 100 \rangle &= \int_0^\infty r^2 dr \, 4 \left(\frac{z}{a_0} \right)^3 e^{-\frac{2zr}{a_0}} \frac{1}{r^2} \\ &= \frac{1}{2} \frac{a_0}{z} 4 \left(\frac{z}{a_0} \right)^3 \\ &= 2 \left(\frac{z}{a_0} \right)^2 \end{aligned}$$

$$= -\frac{1}{2m_e c^2} \left(E_{n=1}^2 + 2E_{n=1} z e^2 \frac{z}{a_0} + (ze^2)^2 2 \left(\frac{z}{a_0} \right)^2 \right)$$

$$\begin{aligned} &\text{know } E_{n=1}^{(0)} = -\frac{1}{2} \frac{z^2 e^2}{a_0} \\ &= -\frac{1}{2m_e c^2} \left(E_{n=1}^2 - 4E_{n=1}^2 + 8E_{n=1}^2 \right) \end{aligned}$$

$$= -\frac{5}{2} \frac{(E_{n=1}^{(0)})^2}{m_e c^2}$$

$$\boxed{E_{n=1}^{(1)} = -\frac{5}{8} \left(\frac{z^2 e^2}{a_0} \right)^2 \frac{1}{m_e c^2}}$$

← relativistic correction.

c) compare with correction term that we get from finite size nucleus:

$$\begin{aligned} \text{finite size } E_{n=1}^{(1)} &\approx \frac{2}{5} z^4 e^2 \left(\frac{R}{a_0} \right)^2 \frac{1}{a_0} \sim \frac{z^4 e^2 R^2}{a_0^3} \quad \begin{array}{l} R=a_0 \text{ for } z=1 \\ \uparrow \\ \text{incorrect,} \\ R \gg a_0 \end{array} \frac{e^2}{a_0} \\ \text{rel } E_{n=1}^{(1)} &= -\frac{5}{8} \frac{z^4 e^4}{a_0^2} \frac{1}{m_e c^2} \quad z=1 \quad \sim \frac{e^4}{a_0^2} \frac{1}{m_e c^2} \end{aligned}$$

$$\frac{\text{rel } E_{n=1}^{(1)}}{\text{finite size } E_{n=1}^{(1)}} \approx \frac{\frac{e^4}{a_0^2} \frac{1}{m_e c^2}}{\frac{e^2}{a_0}} \sim \frac{e^2}{m_e c^2 a_0} \quad \begin{array}{l} \text{other way} \\ \text{around shield} \\ R \gg a_0 \end{array} \quad \text{X } 1$$

so comparing case $z=1$, finite size perturbation is much more significant.