Zhi Chen HW # 15 QM

1) $Y(\theta,\phi) = Y_2^2(\theta,\phi)\, Y_1^{-1}(\theta,\phi)$

$$= \left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\, e^{2i\phi} \sin^2\theta\right)\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{-i\phi} \sin\theta\right)$$

$$= \frac{1}{8}\,\frac{1}{2\pi}\sqrt{45}\, e^{i\phi}\, \sin^3\theta$$

we observe that $m=1$

and $Y_3^1 = -\frac{1}{8}\sqrt{\frac{21}{\pi}}\, e^{i\phi}\underbrace{\sin\theta\,(5\cos^2\theta-1)}_{\sin\theta\,[5-5\sin^2\theta-1]}$

$$Y_3^1 = -\frac{1}{8}\sqrt{\frac{21}{\pi}}\, e^{i\phi}\left(4\sin\theta - 5\sin^3\theta\right)$$

So $e^{i\phi}\sin^3\theta = \left[Y_3^1 + \frac{1}{2}\sqrt{\frac{21}{\pi}}\, e^{i\phi}\sin\theta\right]\frac{8}{5}\sqrt{\frac{\pi}{21}}$

We also know $Y_1^1 = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\, e^{i\phi}\sin\theta$

so $e^{i\phi}\sin\theta = Y_1^1\,(-2)\sqrt{\frac{2\pi}{3}}$

then $e^{i\phi}\sin^3\theta = \left[Y_3^1 + \cancel{\frac{1}{2}}\sqrt{\frac{21}{\cancel{\pi}}}\,(\cancel{-2})\sqrt{\frac{2\cancel{\pi}}{3}}\, Y_1^1\right]\frac{8}{5}\sqrt{\frac{\pi}{21}}$

Then $Y(\theta,\phi) = \frac{1}{8}\,\frac{1}{2\pi}\sqrt{45}\, e^{i\phi}\,\sin^3\theta$

$$= \underbrace{\frac{1}{8}\,\frac{1}{2\pi}\sqrt{45}\;\frac{8}{5}\sqrt{\frac{\pi}{21}}}_{A}\left[Y_3^1 - \sqrt{\frac{42}{3}}\, Y_1^1\right]$$

$$= A\left[Y_3^1 - \sqrt{14}\, Y_1^1\right]$$

$$Y^* Y = A^2 (1 + 14) = A^2\, 15 = 1$$

so $A = \frac{1}{\sqrt{15}}$

so $\boxed{Y = \frac{1}{\sqrt{15}} \left(Y_3^1 - \sqrt{14}\, Y_1^1 \right)}$

we get $\ell = 1$, Y_1^1 with probability $\boxed{\frac{14}{15}}$

2) $[a, a^\dagger] = 1 \qquad [b, b^\dagger] = 1 \qquad$, others $= 0$

$$J_+ = \hbar a^\dagger b$$
$$J_- = \hbar b^\dagger a$$
$$J_z = \frac{\hbar}{2}(a^\dagger a - b^\dagger b)$$

a) $$[J_+, J_-] = \hbar^2 [a^\dagger b, b^\dagger a]$$
$$\overset{!}{=} \hbar^2 \left([a^\dagger b, b^\dagger] a + b^\dagger [a^\dagger b, a]\right)$$
$$\overset{!}{=} \hbar^2 \Big[\left(\cancel{[a^\dagger, b^\dagger]} b + a^\dagger \underbrace{[b, b^\dagger]}_{=1}\right) a + b^\dagger \left(\underbrace{[a^\dagger, a]}_{-1} b + a^\dagger \cancel{[b, a]}\right)\Big]$$
$$= \hbar^2 (a^\dagger a - b^\dagger b)$$

$J_z = \frac{\hbar}{2}(a^\dagger a - b^\dagger b)$

$$\boxed{[J_+, J_-] = 2\hbar J_z}$$

$$[J_+, J_z] = \frac{\hbar^2}{2} [a^\dagger b, a^\dagger a - b^\dagger b]$$
$$\overset{!}{=} \frac{\hbar^2}{2}\left([a^\dagger b, a^\dagger a] - [a^\dagger b, b^\dagger b]\right)$$
$$\overset{!}{=} \frac{\hbar^2}{2}\left\{[a^\dagger b, a^\dagger] a + a^\dagger [a^\dagger b, a] - [a^\dagger b, b^\dagger] b - b^\dagger [a^\dagger b, b]\right\}$$
$$= \frac{\hbar^2}{2}\Big\{\left(\cancel{[a^\dagger, a^\dagger]}^{0} b + a^\dagger \cancel{[b, a^\dagger]}^{0}\right) a + a^\dagger\left(\underbrace{[a^\dagger, a]}_{-1} b + a^\dagger \cancel{[b, a]}^{0}\right)$$
$$- \left(\cancel{[a^\dagger, b^\dagger]}^{0} b + a^\dagger \underbrace{[b, b^\dagger]}_{1}\right) b - b^\dagger\left(\cancel{[a^\dagger, b]} b + a^\dagger \cancel{[b, b]}\right)\Big\}$$

$$[J_+, J_z] = -\frac{\hbar^2}{2}\left(a^\dagger b + a^\dagger b\right)$$

$$= -\hbar^2 a^\dagger b \qquad \Big) \; J_+ = \hbar a^\dagger b$$

$$\boxed{[J_+, J_z] = -\hbar J_+}$$

$$[J_-, J_z] = \frac{\hbar^2}{2}\left[b^\dagger a, a^\dagger a - b^\dagger b\right]$$

$$= \frac{\hbar^2}{2}\left([b^\dagger a, a^\dagger a] - [b^\dagger a, b^\dagger b]\right)$$

$$= \frac{\hbar^2}{2}\left([b^\dagger a, a^\dagger]a + a^\dagger[b^\dagger a, a] - [b^\dagger a, b^\dagger]b - b^\dagger[b^\dagger a, b]\right)$$

$$= \frac{\hbar^2}{2}\Big\{\left(\cancel{[b^\dagger, a^\dagger]}a + b^\dagger\underbrace{[a, a^\dagger]}_{1}\right)a + a^\dagger\left(\cancel{[b^\dagger, a]}a + b^\dagger\cancel{[a, a]}\right)$$
$$- \left(\cancel{[b^\dagger, b^\dagger]}a + b^\dagger\cancel{[a, b^\dagger]}\right)b - b^\dagger\left(\underbrace{[b^\dagger, b]}_{-1}a + b^\dagger\cancel{[a, b]}\right)\Big\}$$

$$= \frac{\hbar^2}{2}\left(b^\dagger a + b^\dagger a\right)$$

$$\boxed{[J_-, J_z] = \hbar J_-}$$

we see that the commutators are the same as angular momentum.

b) $J^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+)$, $N = a^\dagger a + b^\dagger b$

$\rightarrow J_z^2 = \left(\frac{\hbar}{2}\right)^2 [a^\dagger a - b^\dagger b]^2$

$\overset{!}{=} \frac{\hbar^2}{4} [a^\dagger a a^\dagger a + b^\dagger b b^\dagger b - a^\dagger a b^\dagger b - b^\dagger b a^\dagger a]$

$\rightarrow N^2 = a^\dagger a a^\dagger a + b^\dagger b b^\dagger b + a^\dagger a b^\dagger b + b^\dagger b a^\dagger a$

$\rightarrow$ So $J_z^2 = \frac{\hbar^2}{4}(N^2 - 2a^\dagger a b^\dagger b - 2 b^\dagger b a^\dagger a)$

$\rightarrow \frac{1}{2}(J_+ J_- + J_- J_+) = \frac{1}{2}\hbar^2 (a^\dagger b b^\dagger a + b^\dagger a a^\dagger b)$

So $J^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+)$

$\overset{!}{=} \frac{\hbar^2}{4} N^2 - \frac{\hbar^2}{2}(a^\dagger a b^\dagger b + b^\dagger b a^\dagger a) + \frac{\hbar^2}{2}(a^\dagger b b^\dagger a + b^\dagger a a^\dagger b)$

since $[a, b] = [a, b^\dagger] = [b, a^\dagger] = 0$, then $ab = ba$, $ab^\dagger = b^\dagger a$, $ba^\dagger = a^\dagger b$.

and since $[a, a^\dagger] = 1 \rightarrow a a^\dagger = 1 + a^\dagger a$, $[b, b^\dagger] = 1 \rightarrow b b^\dagger = 1 + b^\dagger b$.

$\Rightarrow a^\dagger b b^\dagger a = a^\dagger b a b^\dagger = a^\dagger a b b^\dagger = a^\dagger a + a^\dagger a b^\dagger b$

$\Rightarrow b^\dagger a a^\dagger b = b^\dagger a b a^\dagger = b^\dagger b a a^\dagger = b^\dagger b + b^\dagger b a^\dagger a$

then $J^2 = \frac{\hbar^2}{4} N^2 - \frac{\hbar^2}{2}(\cancel{a^\dagger a b^\dagger b} + \cancel{b^\dagger b a^\dagger a} - a^\dagger a - \cancel{a^\dagger a b^\dagger b} - b^\dagger b - \cancel{b^\dagger b a^\dagger a})$

$\overset{!}{=} \frac{\hbar^2}{4} N^2 - \frac{\hbar^2}{2}(-a^\dagger a - b^\dagger b)$

$$\boxed{J^2 = \frac{\hbar^2}{2}\left(\frac{1}{2} N^2 + N\right)}$$

3) $|\psi\rangle = |+\rangle \otimes |-\rangle + 2\, |-\rangle \otimes |+\rangle + i\, |+\rangle \otimes |+\rangle$

First Normalize: $\langle\psi|\psi\rangle = A^2(1+4+1) = A^2 6 = 1 \Rightarrow A = \frac{1}{\sqrt{6}}$

$$|\psi\rangle = \frac{1}{\sqrt{6}} |+\rangle \otimes |-\rangle + \frac{2}{\sqrt{6}} |-\rangle \otimes |+\rangle + \frac{i}{\sqrt{6}} |+\rangle \otimes |+\rangle$$

a) $\rho_1 = Tr_2 \{ |\psi\rangle\langle\psi| \}$

$$\overset{!}{=} {}_2\langle + | \psi\rangle\langle\psi | +\rangle_2 + {}_2\langle - | \psi\rangle\langle\psi | -\rangle_2$$

$$\overset{!}{=} \left(\frac{2}{\sqrt{6}} |-\rangle + \frac{i}{\sqrt{6}} |+\rangle\right)\left(\frac{2}{\sqrt{6}} \langle -| - \frac{i}{\sqrt{6}} \langle +|\right) + \frac{1}{6} |+\rangle\langle +|$$

$$\rho_1 \overset{!}{=} \frac{4}{6} |-\rangle\langle -| - \frac{2i}{6} |-\rangle\langle +| + \frac{2i}{6} |+\rangle\langle -| + \frac{2}{6} |+\rangle\langle +|$$

$$\hookrightarrow \quad \rho_1 = \begin{pmatrix} \langle +|\rho_1|+\rangle & \langle +|\rho_1|-\rangle \\ \langle -|\rho_1|+\rangle & \langle -|\rho_1|-\rangle \end{pmatrix} = \begin{pmatrix} 1/3 & \frac{1}{3}i \\ -\frac{1}{3}i & \frac{2}{3} \end{pmatrix}$$

b) $S_{1,x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_{1,y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_{1,z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\langle S_{1,x} \rangle = \frac{\hbar}{2} Tr \{ \rho_1 \sigma_x \}$$

$$\overset{!}{=} \frac{\hbar}{2}\frac{1}{3} Tr \left\{ \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\langle S_{1,x} \rangle \overset{!}{=} \frac{\hbar}{6} Tr \left\{ \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \right\} = 0$$

$$\langle S_{1,y}\rangle = \frac{\hbar}{2}\,\mathrm{Tr}\{\rho_1 \sigma_y\}$$

$$\overset{!}{=} \frac{\hbar}{2}\frac{1}{3}\,\mathrm{Tr}\left\{\begin{pmatrix}1 & i\\ -i & 2\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right\}$$

$$\boxed{\langle S_{1,y}\rangle \overset{!}{=} \frac{\hbar}{6}\,\mathrm{Tr}\left\{\begin{pmatrix}-1 & -i\\ 2i & -1\end{pmatrix}\right\} = -\frac{\hbar}{3}}$$

$$\langle S_{1,z}\rangle = \frac{\hbar}{2}\,\mathrm{Tr}\{\rho_1 \sigma_z\}$$

$$\overset{!}{=} \frac{\hbar}{2}\frac{1}{3}\,\mathrm{Tr}\left\{\begin{pmatrix}1 & i\\ -i & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$$

$$\boxed{\langle S_{1,z}\rangle \overset{!}{=} \frac{\hbar}{6}\,\mathrm{Tr}\left\{\begin{pmatrix}1 & -i\\ -i & -2\end{pmatrix}\right\} = \frac{-\hbar}{6}}$$

c) Find entropy: $S = -k_B\,\mathrm{Tr}(\rho \ln \rho)$

$$\overset{!}{=} -k_B \sum_k \rho_{kk}^{(diag)} \ln \rho_{kk}^{(diag)}$$

Find eigenvalues: for $\rho_1 = \frac{1}{3}\begin{pmatrix}1 & i\\ -i & 2\end{pmatrix}$

$$\frac{1}{3}\left|\begin{pmatrix}1-\lambda & i\\ -i & 2-\lambda\end{pmatrix}\right| = \frac{1}{3}\left[(1-\lambda)(2-\lambda) - 1\right]$$

$$= \frac{1}{3}\left[\lambda^2 - 3\lambda + 1\right] = \frac{1}{3}(\lambda+2)(\lambda-1)$$

$$\lambda_{\pm} = \frac{1}{3}\left(\frac{3 \pm \sqrt{9-4}}{2}\right) = \left(\frac{3\pm\sqrt{5}}{2}\right)\frac{1}{3}$$

Then $S = -k_B \sum_i \lambda_i \ln \lambda_i$

$$\boxed{S \overset{!}{=} -k_B\left[\left(\frac{3+\sqrt{5}}{6}\right)\ln\left(\frac{3+\sqrt{5}}{6}\right) + \left(\frac{3-\sqrt{5}}{6}\right)\ln\left(\frac{3-\sqrt{5}}{6}\right)\right]}$$

4) a) suppose ensemble average of $[S_x]$ $[S_y]$, $[S_z]$ are known.

$$[S_x] = tr(\rho S_x) = tr\left(\begin{pmatrix}\rho_1 & \rho_2 \\ \rho_3 & \rho_4\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\right)\frac{\hbar}{2}$$

$$\stackrel{|}{=} Tr\begin{pmatrix}\rho_2 & \rho_1 \\ \rho_4 & \rho_3\end{pmatrix}\frac{\hbar}{2}$$

$$① \quad [S_x] \stackrel{|}{=} (\rho_2 + \rho_3)\frac{\hbar}{2}$$

$$[S_y] = Tr(\rho S_y) = \frac{\hbar}{2} Tr\left(\begin{pmatrix}\rho_1 & \rho_2 \\ \rho_3 & \rho_4\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\right)$$

$$\stackrel{|}{=} \frac{\hbar}{2} Tr\begin{pmatrix}i\rho_2 & -i\rho_1 \\ i\rho_4 & -i\rho_3\end{pmatrix}$$

$$② \quad [S_y] \stackrel{|}{=} \frac{\hbar}{2} i(\rho_2 - \rho_3)$$

using ②: $-\frac{2i}{\hbar}[S_y] + \rho_3 = \rho_2$

plug into ①: $[S_x] = \frac{\hbar}{2}\left(-\frac{2i}{\hbar}[S_y] + \rho_3 + \rho_3\right)$

$$\stackrel{|}{=} -i[S_y] + \hbar\rho_3$$

then $\boxed{\rho_3 = \frac{1}{\hbar}([S_x] + i[S_y])}$

then $\rho_2 = -\frac{2i}{\hbar}[S_y] + \frac{1}{\hbar}[S_x] + \frac{i}{\hbar}[S_y]$

$$\boxed{\rho_2 \stackrel{|}{=} \frac{1}{\hbar}([S_x] - i[S_y])}$$

Also know.

$$[S_z] = Tr(\rho S_z) = \frac{\hbar}{2} Tr\left(\begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\stackrel{!}{=} \frac{\hbar}{2} Tr\begin{pmatrix} \rho_1 & -\rho_2 \\ \rho_3 & -\rho_4 \end{pmatrix}$$

$$③ \quad [S_z] \stackrel{!}{=} \frac{\hbar}{2}(\rho_1 - \rho_4)$$

$$④ \quad 1 = Tr\,\rho = \rho_1 + \rho_4$$

Then using ③: $\frac{2}{\hbar}[S_z] + \rho_4 = \rho_1$

plug into ④: $1 = \frac{2}{\hbar}[S_z] + 2\rho_4$

or $\boxed{\rho_4 = \frac{1}{2} - \frac{1}{\hbar}[S_z]}$

Then $1 = \rho_1 + \frac{1}{2} - \frac{1}{\hbar}[S_z]$

$\hookrightarrow \boxed{\rho_1 = \frac{1}{2} + \frac{1}{\hbar}[S_z]}$

Then

$$\boxed{\rho = \begin{pmatrix} \frac{1}{2} + \frac{1}{\hbar}[S_z] & \frac{1}{\hbar}([S_x] - i[S_y]) \\ \frac{1}{\hbar}([S_x] + i[S_y]) & \frac{1}{2} - \frac{1}{\hbar}[S_z] \end{pmatrix}}$$

b) If ensemble is pure when $Tr(\rho^2) = 1$

$$Tr(\rho^2) = Tr\left[\begin{pmatrix} \frac{1}{2}+\frac{1}{\hbar}[S_z] & \frac{1}{\hbar}([S_x]-i[S_y]) \\ \frac{1}{\hbar}([S_x]+i[S_y]) & \frac{1}{2}-\frac{1}{\hbar}[S_z] \end{pmatrix}^2\right]$$

$$= \left(\frac{1}{2}+\frac{1}{\hbar}[S_z]\right)^2 + \left(\frac{1}{\hbar}\right)^2([S_x]-i[S_y])([S_x]+i[S_y])$$

$$+\left(\frac{1}{\hbar}\right)^2([S_x]-i[S_y])([S_x]+i[S_y]) + \left(\frac{1}{2}-\frac{1}{\hbar}[S_z]\right)^2$$

$$\stackrel{!}{=} \frac{1}{4} + \cancel{\frac{1}{\hbar}[S_z]} + \left(\frac{1}{\hbar}[S_z]\right)^2 + 2\left(\frac{1}{\hbar}\right)^2([S_x]^2+[S_y]^2)$$

$$+\frac{1}{4} - \cancel{\frac{1}{\hbar}[S_z]} + \left(\frac{1}{\hbar}[S_z]\right)^2$$

$$1 \stackrel{!}{=} \frac{1}{2} + 2\left(\frac{1}{\hbar}\right)^2\left([S_z]^2 + [S_x]^2 + [S_y]^2\right)$$

$$\hookrightarrow \boxed{\frac{1}{4}\hbar^2 = [S_z]^2 + [S_x]^2 + [S_y]^2}$$

c) Again find eigenvalues:

$$|\rho - \lambda \mathbb{I}| = \left| \begin{pmatrix} \rho_1 - \lambda & \rho_2 \\ \rho_3 & \rho_4 - \lambda \end{pmatrix} \right| = (\rho_1 - \lambda)(\rho_4 - \lambda) - \rho_2 \rho_3$$

$$\hookrightarrow \lambda^2 - \lambda(\rho_1 + \rho_4) + \rho_1 \rho_4 - \rho_2 \rho_3 = 0$$

$$\lambda_\pm = \frac{(\rho_1 + \rho_4) \pm \sqrt{(\rho_1 + \rho_4)^2 - 4(\rho_1 \rho_4 - \rho_2 \rho_3)}}{2}$$

$$\rho_1 + \rho_4 = \frac{1}{2} + \frac{1}{\hbar}[S_z] + \frac{1}{2} - \frac{1}{\hbar}[S_z] = 1$$

$$\rho_1 \rho_4 = \left(\frac{1}{2} + \frac{1}{\hbar}[S_z]\right)\left(\frac{1}{2} - \frac{1}{\hbar}[S_z]\right) = \frac{1}{4} - \left(\frac{1}{\hbar}[S_z]\right)^2$$

$$\rho_2 \rho_3 = \left(\frac{1}{\hbar}\right)^2 ([S_x] - i[S_y])([S_x] + i[S_y]) = \left(\frac{1}{\hbar}\right)^2 ([S_x]^2 + [S_y]^2)$$

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\left(\frac{1}{4} - \left(\frac{1}{\hbar}\right)^2 ([S_z]^2 + [S_x]^2 + [S_y]^2)\right)}$$

$$\stackrel{!}{=} \frac{1}{2} \pm \frac{1}{2}\sqrt{4\left(\frac{1}{\hbar}\right)^2 ([S_x]^2 + [S_y]^2 + [S_z]^2)}$$

$$\lambda_\pm \stackrel{!}{=} \frac{1}{2} \pm \sqrt{\left(\frac{1}{\hbar}\right)^2 ([S_x]^2 + [S_y]^2 + [S_z]^2)}$$

$$\stackrel{!}{=} \frac{1}{2} \pm \frac{1}{\hbar}[S] \qquad \text{define: } [S] \equiv \sqrt{[S_x]^2 + [S_y]^2 + [S_z]^2}$$

$$S = -k_B \sum_i \lambda_i \ln \lambda_i$$

$$\boxed{\begin{aligned} S &\overset{!}{=} -k_B \left\{ \left(\frac{1}{2} + \frac{1}{\hbar}[S] \right) \ln \left(\frac{1}{2} + \frac{1}{\hbar}[S] \right) \right. \\ &\qquad \left. + \left(\frac{1}{2} - \frac{1}{\hbar}[S] \right) \ln \left(\frac{1}{2} - \frac{1}{\hbar}[S] \right) \right\} \\ &\text{For } [S] = \sqrt{[S_x]^2 + [S_y]^2 + [S_z]^2} \end{aligned}}$$

We see that if $[S] = \frac{\hbar}{2}$, then $S = 0$

4) For a general 3x3 complex matrix, there are 9 matrix elements.

$$\rho = \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 \\ \rho_4 & \rho_5 & \rho_6 \\ \rho_7 & \rho_8 & \rho_9 \end{pmatrix}$$ each complex number has 2 real numbers,

so $2 \times 9 = 18$ real numbers.

We also know ρ is Hermitian, $\rho = \rho^\dagger$

so $$\begin{pmatrix} \rho_1 & \rho_2 & \rho_3 \\ \rho_4 & \rho_5 & \rho_6 \\ \rho_7 & \rho_8 & \rho_9 \end{pmatrix} = \begin{pmatrix} \rho_1^* & \rho_4^* & \rho_7^* \\ \rho_2^* & \rho_5^* & \rho_8^* \\ \rho_3^* & \rho_6^* & \rho_9^* \end{pmatrix}$$

→ This means ρ_{kk}, i.e. diagonal terms are real.
So there are 3 less numbers to consider, $18-3=15$

→ For off-diagonal terms, $\rho_4 = \rho_2^*$, $\rho_7 = \rho_3^*$, and $\rho_8 = \rho_6^*$,
So we can get rid of the lower triangle terms.
So : $15-(2\times3) = 9$.

→ We can finally use the constraint that $Tr(\rho) = \rho_1 + \rho_2 + \rho_3 = 1$ to get rid of a term in the diagonal.

So we finally have $9 - 1 = \boxed{8 \text{ real independent parameters}}$

In the end we have

$$\rho = \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 \\ \rho_2^* & \rho_5 & \rho_6 \\ \rho_3^* & \rho_6^* & 1-\rho_1-\rho_5 \end{pmatrix}$$

In addition to $[S_x]$, $[S_y]$, and $[S_z]$

we also need to know combination terms proportional to
$[S_x^2]$, $[S_y^2]$, $[S_z^2]$, $[S_xS_y]$, $[S_yS_z]$, and $[S_xS_z]$

Since we know $[S_x^2]+[S_y^2]+[S_z^2] = \hbar^2 \, l(l+1)\Big|_{l=1} = 2\hbar^2$

then we only need $\underline{\underline{2}}$ terms out of $[S_x^2]$, $[S_y^2]$ and $[S_z^2]$.

Therefore the 8 terms we need are:

$[S_x]$, $[S_y]$, $[S_z]$, $\underbrace{[S_x^2], [S_y^2], [S_z^2]}_{\text{Need 2 out of 3.}}$, $[S_xS_y]$, $[S_xS_z]$, $[S_yS_z]$

b) $\rho(t=0)$: pure

we know $\rho(t=0) = \sum_i w_i \, |\alpha^{(i)}(t=0)\rangle\langle\alpha^{(i)}(t=0)|$

$$\rho(t) = \sum_i w_i \, |\alpha^{(i)}(t)\rangle\langle\alpha^{(i)}(t)|$$

$$\rho(t) = \sum_i w_i \, U(t,t_0)|\alpha^{(i)}(t_0)\rangle\langle\alpha^{(i)}(t_0)| U^{\dagger}(t,t_0)$$

$$= U(t,t_0)\,\rho(t_0)\,U^{\dagger}(t,t_0)$$

If $\rho(t_0)$ is pure, $Tr(\rho(t_0)^2) = 1$

$$Tr(\rho(t)^2) = Tr\big(U(t,t_0)\rho(t_0)\underbrace{U^{\dagger}(t,t_0)\,U(t,t_0)}_{=1}\rho(t_0)U^{\dagger}(t,t_0)\big)$$

$$= Tr\big(U(t,t_0)\rho(t_0)^2 U^{\dagger}(t,t_0)\big)$$

know $Tr(ABC) = Tr(CAB) = Tr(BCA)$

$$= Tr\big(\underbrace{U^{\dagger}(t,t_0)\,U(t,t_0)}_{=1}\,\rho(t_0)^2\big)$$

$$\boxed{Tr(\rho(t)^2) = Tr(\rho(t_0)^2) = 1}$$

↳ since $Tr(\rho(t)^2) = 1$ always, $\rho(t)$ is stays pure.