Zhi Chen HW # 11 QM

1) <u>Center of orbit:</u>

Since $H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \frac{m}{2}\vec{v}^2$

then $\vec{v} = \dfrac{\vec{p} - \frac{e}{c}\vec{A}}{m}$, $\vec{p} = -i\hbar\left(\partial_x \hat{x} + \partial_y \hat{y}\right)$

$\vec{A} = \vec{A}(\vec{x}, \vec{y})$

then $v_x = \dfrac{p_x - \frac{e}{c}A_x}{m}$, $v_y = \dfrac{p_y - \frac{e}{c}A_y}{m}$

define : $\omega_B = \frac{eB}{mc}$, where $B = \partial_x A_y - \partial_y A_x$

Define operators:

$x_0 = x + \frac{1}{\omega_B} v_y$ $\quad y_0 = y - \frac{1}{\omega_B} v_x$ where: $v_y = -\omega_B(x - x_0)$

$v_x = \omega_B(y - y_0)$

$r_0^2 = x_0^2 + y_0^2$ $\quad R^2 = \frac{1}{\omega_B^2}\left(v_x^2 + v_y^2\right)$

Calculate different commutators:

① $[x_0, y_0] = [x + \frac{1}{\omega_B} v_x,\ y - \frac{1}{\omega_B} v_y]$

$\stackrel{!}{=} \underbrace{[x, y]}_{=0} - \frac{1}{\omega_B}[x, v_y] + \frac{1}{\omega_B}[v_x, y] - \frac{1}{\omega_B^2}[v_x, v_y]$

Find Individual Commutators:

$[x, v_y] = \frac{1}{m}[x, p_y - \frac{e}{c}A_y] = \left(\underbrace{[x, p_y]}_{0} - \underbrace{[x, \frac{e}{c}A_y]}_{=0}\right)\frac{1}{m}$

$[x, v_y] \stackrel{!}{=} 0$

similarly, $[v_x, y] = -[y, v_x] = -[y, P_x - \frac{e}{c}A_x]\frac{1}{m}$

$$\overset{!}{=}\left(-[y, P_x] + [y, \tfrac{e}{c}A_x]\right)\frac{1}{m}$$

$$\overset{!}{=} 0$$

$$[v_x, v_y] = \frac{1}{m^2}\left[P_x - \frac{e}{c}A_x\ ,\ P_y - \frac{e}{c}A_y\right]$$

$$\overset{!}{=}\left(\underbrace{[P_x, P_y]}_{=0} - [P_x, \tfrac{e}{c}A_y] - [\tfrac{e}{c}A_x, P_y] + \left(\tfrac{e}{c}\right)^2 \underbrace{[A_x, A_y]}_{=0}\right)\frac{1}{m^2}$$

since $[P_x, \frac{e}{c}A_y]f(x) = -i\hbar\frac{e}{c}(\partial_x A_y - A_y\partial_x)f$

$$\overset{!}{=} -i\hbar\frac{e}{c}(f\partial_x A_y + \cancel{A_y\partial_x f} - \cancel{A_y\partial_x f})$$

$$\overset{!}{=} -i\hbar\frac{e}{c}f\partial_x A_y$$

So $[P_x, \frac{e}{c}A_y] = -i\hbar\frac{e}{c}\partial_x A_y$

similarly, $[\frac{e}{c}A_x, P_y] = i\hbar\frac{e}{c}\partial_y A_x$

then $[v_x, v_y] \overset{!}{=} i\hbar\frac{e}{m^2c}(\partial_x A_y - \partial_y A_x) = i\hbar\frac{eB}{m^2c} = \frac{i\hbar}{m}\omega_B$

Then: $[x_0, y_0] = -\frac{1}{\omega_B^{\cancel{2}}}\frac{i\hbar}{m}\cancel{\omega_B}$

1 $\boxed{[x_0, y_0] \overset{!}{=} \frac{-i\hbar}{m\omega_B}}$ ← x_0, y_0 not compatible

② $[x_0, r_0^2] = [x_0, x_0^2 + y_0^2]$

previously:

$[x_0, y_0] = \frac{-i\hbar}{m\omega_B}$

$$\stackrel{!}{=} \underbrace{[x_0, x_0^2]}_{=0} + [x_0, y_0^2]$$

$$\stackrel{!}{=} [x_0, y_0] y_0 + y_0 [x_0, y_0]$$

② $\boxed{[x_0, r_0^2] \stackrel{!}{=} \frac{-2i\hbar}{m\omega_B} y_0}$ $\Leftarrow$ x_0, r_0^2 not compatible

③ $[y_0, r_0^2] = [y_0, x_0^2 + y_0^2]$

$$\stackrel{!}{=} [y_0, x_0^2] + \cancel{\underbrace{[y_0, y_0^2]}_{=0}}$$

$[y_0, x_0] = -[x_0, y_0] \stackrel{!}{=} \frac{i\hbar}{m\omega_B}$

$$\stackrel{!}{=} [y_0, x_0] x_0 + x_0 [y_0, x_0]$$

$\boxed{[y_0, r_0^2] \stackrel{!}{=} \frac{2i\hbar}{m\omega_B} x_0}$ $\leftarrow$ y_0, r_0^2 not compatible

④ $[x_0, R^2] = \frac{1}{\omega_B^2} [x_0, v_x^2 + v_y^2]$

$$\stackrel{!}{=} \frac{1}{\omega_B^2} [x + \frac{1}{\omega_B} v_y, v_x^2 + v_y^2]$$

$$= \frac{1}{\omega_B^2} \left([x, v_x^2] + [x, v_y^2] + \frac{1}{\omega_B}[v_y, v_x^2] + \frac{1}{\omega_B}[v_y, v_y^2] \right)$$

$$\stackrel{!}{=} \frac{1}{\omega_B^2} \Big([x, v_x] v_x + v_x [x, v_x] + \underbrace{[x, v_y]}_{=0} v_y + v_y \underbrace{[x, v_y]}_{=0}$$

$\Leftarrow$ use results calculated previously.

$$\frac{1}{\omega_B} \Big(\underbrace{[v_y, v_x]}_{\frac{-i\hbar}{m}\omega_B} v_x + v_x \underbrace{[v_y, v_x]}_{\frac{-i\hbar}{m}\omega_B} + 2\underbrace{[v_y, v_y]}_{=0} v_y \Big)$$

Previously we found $[x, v_y] = [y, v_x] = 0$

Need to find $[x, v_x]$ and $[y, v_y]$

$$\Rightarrow [x, v_x] = \frac{1}{m}[x, P_x - \frac{e}{c}A_x]$$

$$= \frac{1}{m}\left(\underbrace{[x, P_x]}_{i\hbar} - \underbrace{[x, \frac{e}{c}A_x]}_{=0}\right)$$

$$= \frac{i\hbar}{m}$$

Analogous to $[x, v_x]$:

$$[y, v_y] = \frac{1}{m}[y, P_y - \frac{e}{c}A_y] = \frac{1}{m}\left([y, P_y] - [y, \frac{e}{c}A_y]\right)$$

$$= \frac{i\hbar}{m}$$

Then $[x_0, R^2] = \frac{1}{\omega_B^2}\left(\underbrace{[x, v_x]}_{\frac{i\hbar}{m}} v_x + v_x \underbrace{[x, v_x]}_{\frac{i\hbar}{m}} + \frac{-2i\hbar}{m} v_x\right)$

$$\boxed{④ \quad [x_0, R^2] = 0}$$ $\leftarrow$ x_0, R^2 compatible

$$[Y_0, R^2] = \frac{1}{\omega_B^2}\left[y - \frac{1}{\omega_B}v_x,\ v_x^2 + v_y^2\right]$$

$$= \frac{1}{\omega_B^2}\left(\underbrace{[y, v_x^2]}_{=0} + [y, v_y^2] - \frac{1}{\omega_B}\left(\underbrace{[v_x, v_x^2]}_{=0} + [v_x, v_y^2]\right)\right)$$

$$= \frac{1}{\omega_B^2}\left(\underbrace{[y, v_y]}_{\frac{i\hbar}{m}}v_y + v_y\underbrace{[y, v_y]}_{\frac{i\hbar}{m}} - \frac{1}{\omega_B}\left(\underbrace{[v_x, v_y]}_{\frac{i\hbar}{m}\omega_B}v_y + v_y\underbrace{[v_x, v_y]}_{\frac{i\hbar}{m}\omega_B}\right)\right)$$

$$\boxed{⑤\ [Y_0, R^2] = 0} \leftarrow Y_0, R^2 \text{ compatible}$$

⑥ $[r_0^2, R^2] = [X_0^2 + Y_0^2,\ R^2]$

$$= [X_0^2, R^2] + [Y_0^2, R^2]$$

$$= X_0\underbrace{[X_0, R^2]}_{=0} + \underbrace{[X_0, R^2]}_{=0}X_0 + Y_0\underbrace{[Y_0, R^2]}_{=0} + \underbrace{[Y_0, R^2]}_{=0}Y_0$$

$$\boxed{⑥\ [r_0^2, R^2] = 0} \leftarrow r_0^2, R^2 \text{ compatible.}$$

2) Heinsenberg equation of motion:

For observable, $\mathcal{O}$, not explicitely function of time, then

$$\frac{d\mathcal{O}}{dt} = \frac{1}{i\hbar}[\mathcal{O}, H] = \frac{1}{i\hbar}\left[\mathcal{O}, \frac{m}{2}(V_x^2+V_y^2)\right]$$
$$\overset{!}{=} \frac{m}{2i\hbar}[\mathcal{O}, R^2]$$

Use results from problem 1:

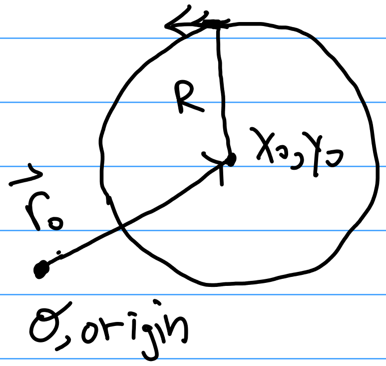
① $\frac{dx_0}{dt} = \frac{m}{2i\hbar}\underbrace{[x_0, R^2]}_{=0} = 0$

② $\frac{dy_0}{dt} = \frac{m}{2i\hbar}\underbrace{[y_0, R^2]}_{=0} = 0$

③ $\frac{d}{dt}r_0^2 = \frac{m}{2i\hbar}\underbrace{[r^2, R^2]}_{=0} = 0$

center of the orbit doesn't change as a function of time.

④ $\frac{d}{dt}R^2 = \frac{m}{2i\hbar}\underbrace{[R^2, R^2]}_{=0} = 0 \Rightarrow$ The radius of the orbit also do not change.



It is very similar to classical case, where the particle moves in the circle.

3) Integer Quantum Hall Effect:

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + V(x,y)$$

let $V(x,y) = \frac{1}{2}m\omega_0^2 y^2$

a) use Landau Gauge: $A_x = -By$, $A_y = 0$
and let $\psi(x,y) = \psi_k(y)e^{-ikx}$

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{eB}{c}y\hat{x}\right)^2 + \frac{1}{2}m\omega_0^2 y^2$$

$$= \frac{1}{2m}\left[\left(-i\hbar\partial_x + \frac{eB}{c}y\right)^2 + (-i\hbar\partial_y)^2\right] + \frac{1}{2}m\omega_0^2 y^2$$

$\Rightarrow$ Define: $\quad l^2 = \frac{\hbar c}{eB} \qquad \omega_B = \frac{eB}{mc}$

With $\psi(x,y) = \psi(y)\,e^{-ikx}$, then $\partial_x \Rightarrow -ik$

$$H\psi(x,y) = \left(-\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2m}\left[-i\hbar(-ik) + \frac{eB}{c}y\right]^2 + \frac{1}{2}m\omega_0^2 y^2\right)\psi(y)e^{-ikx}$$

$$H\psi_k(y) = \left(-\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\underbrace{\left(\frac{eB}{mc}\right)^2}_{\omega_B^2}\left[\underbrace{\left(\frac{c\hbar}{eB}\right)}_{l^2}k + y\right]^2 + \frac{1}{2}m\omega_0^2 y^2\right)\psi_k(y)$$

$$= -\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\left(\omega_B^2\left[y - kl^2\right]^2 + \omega_0^2 y^2\right)\psi_k(y)$$

$$= -\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\left((\omega_B^2+\omega_0^2)y^2 - 2\omega_B^2 ykl^2 + \omega_B^2 k^2 l^4\right)\psi_k(y)$$

$$= -\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m(\omega_B^2+\omega_0^2)\left[y^2 - 2\frac{\omega_B^2}{\omega_B^2+\omega_0^2}ykl^2 + \frac{\omega_B^2}{\omega_B^2+\omega_0^2}k^2l^4\right.$$

$$\left. + \left(\frac{\omega_B^2}{\omega_B^2+\omega_0^2}kl^2\right)^2 - \left(\frac{\omega_B^2}{\omega_B^2+\omega_0^2}kl^2\right)^2\right]\psi_k(y)$$

$$\hookrightarrow = \left(\frac{-\hbar^2}{2m} \partial_y^2 + \frac{1}{2} m (\omega_B^2 + \omega_0^2) \left[y - k l^2 \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} \right]^2 + \frac{1}{2} m \omega_B^2 k^2 l^4 - \frac{1}{2} m \omega_B^2 k^2 l^4 \left(\frac{\omega_B^2}{\omega_B^2 + \omega_0^2} \right) \right) \psi_k(y)$$

$$\boxed{H \psi_k(y) = \left(\frac{-\hbar^2}{2m} \partial_y^2 + \frac{1}{2} m (\omega_B^2 + \omega_0^2) \left[y - k l^2 \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} \right]^2 + \frac{1}{2} m \omega_B^2 k^2 l^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} \right] \right) \psi_k(y)}$$

b) we recognize the first part of Hamiltonian is a 1D Harmonic oscillator with frequency, $\omega^2 = \omega_B^2 + \omega_0^2$ so it has energy:

$$E_{n,k}^{HO} = \hbar \sqrt{\omega_B^2 + \omega_0^2} \left(n + \frac{1}{2} \right)$$

Then total Energy:

$$\boxed{E_{n,k} = \hbar \sqrt{\omega_B^2 + \omega_0^2} \left(n + \frac{1}{2} \right) + \frac{1}{2} m \omega_B^2 k^2 l^4 \left[1 - \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} \right]}$$

c) Assume $n = 0$,

$$E_{0,k} = \frac{1}{2}\hbar\sqrt{\omega_B^2+\omega_0^2} + \frac{1}{2}m\omega_B^2 k^2 \ell^4\left[1 - \frac{\omega_B^2}{\omega_B^2+\omega_0^2}\right]$$

Note that k is determined via periodic B.C. in x-direction.

$$k = \frac{2\pi}{L_x} j, \quad j = 0, \pm 1, \pm 2 \cdots$$

If we consider a box of size L_x, L_y, then the center of HO must be confined within the box,

$$0 < Y_0 < L_y$$

From looking at Hamiltonian, we observe $Y_0 = k\ell^2 \frac{\omega_B^2}{\omega_B^2+\omega_0^2}$

$$\hookrightarrow \quad \underbrace{0}_{k_{min}} < k < \frac{L_y}{\ell^2}\frac{\omega_B^2+\omega_0^2}{\omega_B^2} = \underbrace{\frac{L_y}{\ell^2}\left(1+\left(\frac{\omega_0}{\omega_B}\right)^2\right)}_{k_{max}}$$

d) Since $Y_0 = k\ell^2\frac{\omega_B^2}{\omega_B^2+\omega_0^2}$ and

$$0 < k = \frac{2\pi}{L_x} j < \frac{L_y}{\ell^2}\left(1+\left(\frac{\omega_0}{\omega_B}\right)^2\right)$$

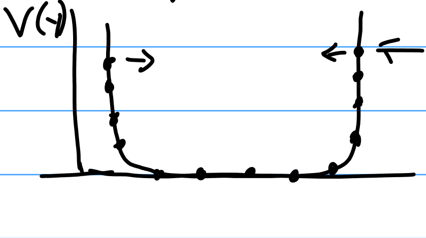
then $0 < j < \frac{L_x L_y}{2\pi\ell^2}\left(1+\left(\frac{\omega_0}{\omega_B}\right)^2\right)$

$$\boxed{\text{So } Y_{0,j} = \frac{2\pi}{L_x} j \ell^2 \frac{\omega_B^2}{\omega_B^2+\omega_0^2} \text{ for } j = 0, 1, 2 \cdots, j_{max} \text{ for } j_{max} < \frac{L_x L_y}{2\pi\ell^2}\left(1+\left(\frac{\omega_0}{\omega_B}\right)^2\right)}$$

e) Find velocity of edge state, $k_{min}=0$, $k_{max} = \frac{L_y}{\ell^2}\left(1+\left(\frac{\omega_0}{\omega_B}\right)^2\right)$

$$V_{eff} = \frac{1}{\hbar}\frac{\partial E}{\partial k}$$

If there is a confinement in y-direction, it must mean that there are steep potential near the edge



each point represent some k, with corresponding Y_0.

The local electric field $\vec{E} = -\vec{\nabla}\varphi = -\vec{\nabla}\left(\frac{V}{-e}\right) = v B$

So $$v = \frac{1}{eB}\partial_y V(y) = \frac{1}{eB}\partial_y H$$

$$\partial_y H = m(\omega_B^2+\omega_0^2)\left[y - k\ell^2\frac{\omega_B^2}{\omega_B^2+\omega_0^2}\right]$$

↳ For $k = k_{min} = 0$

Left edge $v_y(k=k_{min}) = \frac{1}{eB} m(\omega_B^2+\omega_0^2)\, y$

i.e. left edge state move to the right.

↳ For $k = k_{max} = \frac{L_y}{\ell^2}\left(1+\frac{\omega_0^2}{\omega_B^2}\right)$

Right Edge $v_y(k=k_{max}) = \frac{1}{eB} m(\omega_0^2+\omega_B^2)\underbrace{\left[y-L_y\right]}_{\text{negative.}}$ since $y<L_y$

so right edge move to the left.

4) a and b operator:

consider $\mathcal{H} = -\frac{1}{2}\left(\partial_x + \frac{i}{2}y\right)^2 - \frac{1}{2}\left(\partial_y - \frac{i}{2}x\right)^2$

let $z = x+iy$, $\bar{z} = x - iy$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$

a) let $a = \sqrt{2}\left(-i\bar{\partial} - \frac{i}{4}z\right)$, $a^{+} = \sqrt{2}\left(-i\partial + \frac{i}{4}\bar{z}\right)$

Show $[a, a^{+}] = 1$

$$[a, a^{+}] = \left[\sqrt{2}\left(-i\bar{\partial} - \frac{i}{4}z\right), \left(-i\partial + \frac{i}{4}\bar{z}\right)\sqrt{2}\right]$$

$$= -2\left[\bar{\partial} + \frac{1}{4}z, \partial - \frac{1}{4}\bar{z}\right]$$

$$= -2\left(\underset{①}{[\bar{\partial}, \partial]} - \frac{1}{4}\underset{②}{[\bar{\partial}, \bar{z}]} + \frac{1}{4}\underset{③}{[z, \partial]} - \frac{1}{16}\underset{④}{[z, \bar{z}]}\right)$$

Calculate individual terms:

① $[\bar{\partial}, \partial] = \frac{1}{4}[\partial_x + i\partial_y, \partial_x - i\partial_y] = 0$ since no dependence in x or y.

④ $[z, \bar{z}] = [x+iy, x-iy] = 0$ positions commute.

② $[\bar{\partial}, \bar{z}] = \frac{1}{2}[\partial_x + i\partial_y, x - iy]$

$$= \frac{1}{2}\left([\partial_x, x] - i\underbrace{[\partial_x, y]}_{=0} + i\underbrace{[\partial_y, x]}_{=0} + [\partial_y, y]\right)$$

$\partial_x = \frac{1}{-i\hbar}P_x$, and $[P_x, x] = -i\hbar$

so $[\partial_x, x] = \frac{1}{-i\hbar}[P_x, x] = 1$

similarly $[\partial_y, y] = 1$

then $[\bar{\partial}, \bar{z}] = \frac{1}{2}[1+1] = \underline{\underline{1}}$

③ $[z, \partial] = \frac{1}{2}[x+iy, \partial_x - i\partial_y]$

$$\overset{!}{=} \frac{1}{2}\Big(\underbrace{[x,\partial_x]}_{=-[\partial_x,x]=-1} - i\underbrace{[x,\partial_y]}_{=0} + i\underbrace{[y,\partial_x]}_{=0} + \underbrace{[y,\partial_y]}_{=-[\partial_y,y]=-1}\Big)$$

$$\overset{!}{=} -1$$

$$\boxed{\therefore \quad [a, a^\dagger] = -2\left[-\tfrac{1}{4}(1) + \tfrac{1}{4}(-1)\right] = 1}$$

b) Show $H = a^\dagger a + \frac{1}{2} = a^\dagger a + \frac{1}{2}(aa^\dagger - a^\dagger a) = \frac{1}{2}(\underset{①}{a^\dagger a} + \underset{②}{a a^\dagger})$

① $a^\dagger a = \sqrt{2}\left(-i\partial + \frac{i}{4}\bar{z}\right)\sqrt{2}\left(-i\bar{\partial} - \frac{i}{4}z\right)$

$$= -2\left(\tfrac{1}{2}(\partial_x - i\partial_y) - \tfrac{1}{4}(x - iy)\right)\left(\tfrac{1}{2}(\partial_x + i\partial_y) + \tfrac{1}{4}(x+iy)\right)$$

$$\overset{!}{=} -2\left(\tfrac{1}{2}\left(\partial_x + \tfrac{i}{2}y\right) - \tfrac{i}{2}\left(\partial_y - \tfrac{i}{2}x\right)\right)\left(\tfrac{1}{2}\left(\partial_x + \tfrac{i}{2}y\right) + \tfrac{i}{2}\left(\partial_y - \tfrac{i}{2}x\right)\right)$$

$$\overset{!}{=} -\frac{1}{2}\left[\left(\partial_x + \tfrac{i}{2}y\right) - i\left(\partial_y - \tfrac{i}{2}x\right)\right]\left[\left(\partial_x + \tfrac{i}{2}y\right) + i\left(\partial_y - \tfrac{i}{2}x\right)\right]$$

$$\overset{!}{=} -\frac{1}{2}\Big[\left(\partial_x + \tfrac{i}{2}y\right)^2 + \left(\partial_y - \tfrac{i}{2}x\right)^2 \underbrace{- i\left(\partial_y - \tfrac{i}{2}x\right)\left(\partial_x + \tfrac{i}{2}y\right) + i\left(\partial_x + \tfrac{i}{2}y\right)\left(\partial_y - \tfrac{i}{2}x\right)}_{\text{cross-terms}}\Big]$$

Similarly:

(2) $aa^\dagger = \sqrt{2}\left(-i\partial - \frac{i}{4}z\right)\sqrt{2}\left(-i\bar{\partial} + \frac{i}{4}\bar{z}\right)$

$$\stackrel{!}{=} -2\left(\frac{1}{2}(\partial_x + i\partial_y) + \frac{1}{4}(x+iy)\right)\left(\frac{1}{2}(\partial_x - i\partial_y) - \frac{1}{4}(x-iy)\right)$$

$$\stackrel{!}{=} -2\left(\frac{1}{2}\left(\partial_x + \frac{i}{2}y\right) + \frac{i}{2}\left(\partial_y - \frac{i}{2}x\right)\right)\left(\frac{1}{2}\left(\partial_x + \frac{i}{2}y\right) - \frac{i}{2}\left(\partial_y - \frac{i}{2}x\right)\right)$$

$$\stackrel{!}{=} -\frac{1}{2}\Big[\underbrace{\left(\partial_x + \frac{i}{2}y\right)^2 + \left(\partial_y - \frac{i}{2}x\right)^2}_{\text{repeated terms}} + \underbrace{i\left(\partial_y - \frac{i}{2}x\right)\left(\partial_x + \frac{i}{2}y\right) - i\left(\partial_x + \frac{i}{2}y\right)\left(\partial_y - \frac{i}{2}y\right)}_{\text{cross-terms}}\Big]$$

$H = \frac{1}{2}\{a, a^\dagger\} = \frac{1}{2}(aa^\dagger + a^\dagger a)$

plug in results of $aa^\dagger$ and $a^\dagger a$ and realize cross-terms cancel, then

$$= -\frac{1}{2}\left(\partial_x + \frac{i}{2}y\right)^2 - \frac{1}{2}\left(\partial_y - \frac{i}{2}x\right)^2$$

c) Introduce $b = \sqrt{2}\left(-i\partial - \frac{i}{4}\bar{z}\right)$, $b^{+} = \sqrt{2}\left(-i\bar{\partial} + \frac{i}{4} z\right)$

Show $[b, b^{+}] = 1$ $\quad [b, a] = 0$ $\quad [b, a^{+}] = 0$

$$[b, b^{+}] = \left[\sqrt{2}\left(-i\partial - \frac{i}{4}\bar{z}\right), \sqrt{2}\left(-i\bar{\partial} + \frac{i}{4} z\right)\right]$$

$$= -2\left[\partial + \frac{1}{4}\bar{z}, \bar{\partial} - \frac{1}{4} z\right]$$

$$= -2\Big(\underbrace{[\partial, \bar{\partial}]}_{=0} - \frac{1}{4}\underbrace{[\partial, z]}_{=-[z,\partial]=1} + \frac{1}{4}\underbrace{[\bar{z}, \bar{\partial}]}_{=-[\bar{\partial},\bar{z}]=-1} - \frac{1}{16}\underbrace{[\bar{z}, z]}_{=0}\Big)$$

use results calculated from part a)

$$\boxed{[b, b^{+}] = 1}$$

$$[b, a] = \left[\sqrt{2}\left(-i\partial - \frac{i}{4}\bar{z}\right), \sqrt{2}\left(-i\bar{\partial} - \frac{i}{4} z\right)\right]$$

$$= -2\left[\partial + \frac{1}{4}\bar{z}, \bar{\partial} + \frac{1}{4} z\right]$$

Again, use commutator results from part a)

$$= -2\Big(\underbrace{[\partial, \bar{\partial}]}_{=0} + \underbrace{\frac{1}{4}\underbrace{[\partial, z]}_{=-[z,\partial]=1} + \frac{1}{4}\underbrace{[\bar{z}, \bar{\partial}]}_{=-[\bar{\partial},z]=-1}}_{=0} + \frac{1}{16}\underbrace{[\bar{z}, \bar{z}]}_{=0}\Big)$$

$$\boxed{[b, a] = 0}$$

$$[b, a^\dagger] = \left[\sqrt{2}\left(-i\partial - \tfrac{i}{4}\bar{z}\right), \sqrt{2}\left(-i\partial + \tfrac{i}{4}\bar{z}\right)\right]$$

$$= -2\left[\partial + \tfrac{1}{4}\bar{z}, \partial - \tfrac{1}{4}\bar{z}\right]$$

$$= -2\left(\underbrace{[\partial, \partial]}_{=0} - \tfrac{1}{4}\underbrace{[\partial, \bar{z}]}_{=0} + \tfrac{1}{4}\underbrace{[\bar{z}, \partial]}_{=0} - \tfrac{1}{16}\underbrace{[\bar{z}, \bar{z}]}_{=0}\right)$$

Find $[\partial, \bar{z}] = \frac{1}{2}[\partial_x - i\partial_y, x - iy]$

$$= \frac{1}{2}\left(\underbrace{[\partial_x, x]}_{=1} - i\underbrace{[\partial_x, y]}_{=0} - i\underbrace{[\partial_y, x]}_{=0} - \underbrace{[\partial_y, y]}_{=1}\right)$$

From part a), we know $[\partial_x, x] = [\partial_y, y] = 1$

$$= \frac{1}{2}(1-1)$$

$$= 0$$

So $\boxed{[b, a^\dagger] = 0}$

d) Assume $a|0,0\rangle = 0$, $b|0,0\rangle = 0$

then $|n,m\rangle = (a^\dagger)^n (b^\dagger)^m |0,0\rangle$

Show $|n,m\rangle$ is eigenstate. of H, find its energy.

know: $H = a^\dagger a + \frac{1}{2}$

then $H|0,0\rangle = a^\dagger \overbrace{a|0,0\rangle}^{=0} + \frac{1}{2}|0,0\rangle$

$E|0,0\rangle \overset{!}{=} \frac{1}{2}|0,0\rangle$

So if $a|0,0\rangle = 0$, then

$H|0,0\rangle = \frac{1}{2}|0,0\rangle = E_0 |0,0\rangle$

$\hookrightarrow E_0 = \frac{1}{2}$

Need to see whether $b^\dagger$ commutes with $a^\dagger$ and a,

$$[b^\dagger, a] = \left[\sqrt{2}\left(-i\bar{\partial} + \frac{i}{4}z\right), \sqrt{2}\left(-i\bar{\partial} - \frac{i}{4}z\right)\right]$$

$$\overset{!}{=} -2\left(\left[\bar{\partial} - \frac{1}{4}z, \bar{\partial} + \frac{1}{4}z\right]\right)$$

$$= -2\left(\underbrace{[\bar{\partial}, \bar{\partial}]}_{=0} + \frac{1}{4}\underbrace{[\bar{\partial}, z]}_{=0} - \frac{1}{4}\underbrace{[z, \bar{\partial}]}_{=0} - \frac{1}{16}\underbrace{[z, z]}_{=0}\right)$$

$$= 0$$

$$[b^\dagger, a^\dagger] = \left[\sqrt{2}\left(-i\bar{\partial} + \frac{i}{4}z\right), \sqrt{2}\left(-i\partial + \frac{i}{4}\bar{z}\right)\right]$$

$$= -2\left(\left[\bar{\partial} - \frac{1}{4}z, \partial - \frac{1}{4}\bar{z}\right]\right)$$

$$\overset{!}{=} -2\left(\underbrace{[\bar{\partial}, \partial]}_{=0} - \frac{1}{4}\underbrace{[\bar{\partial}, \bar{z}]}_{1} - \frac{1}{4}\underbrace{[z, \partial]}_{-1} + \frac{1}{16}\underbrace{[z, \bar{z}]}_{=0}\right)$$

$$= 0$$

So $b^\dagger$ commutes with $a, a^\dagger$.

If we have $|1,m\rangle = a^\dagger (b^\dagger)^m |0,0\rangle$

$$H|1,m\rangle = a^\dagger a\, a^\dagger (b^\dagger)^m |0,0\rangle + \frac{1}{2} a^\dagger (b^\dagger)^m |0,0\rangle$$

since $[a,a^\dagger] = aa^\dagger - a^\dagger a = 1$

$\hookrightarrow aa^\dagger = 1 + a^\dagger a$

$$= a^\dagger (1 + a^\dagger a)(b^\dagger)^m |0,0\rangle + \frac{1}{2} a^\dagger (b^\dagger)^m |0,0\rangle$$

but $b^\dagger$ commutes with a, $a^\dagger$, so we can switch order.

$$= \left(a^\dagger (b^\dagger)^m |0,0\rangle + (b^\dagger)^m a^\dagger \underbrace{a |0,0\rangle}_{=0} + \frac{1}{2} a^\dagger (b^\dagger)^m |0,0\rangle \right.$$

$$= \left(1 + \frac{1}{2}\right) a^\dagger (b^\dagger)^m |0,0\rangle$$

$$= 3/2\ a^\dagger (b)^m |0,0\rangle$$

Now for $|2,m\rangle = (a^\dagger)^2 (b^\dagger)^m |0,0\rangle$

then $H|2,m\rangle = a^\dagger a (a^\dagger)^2 (b^\dagger)^m |0,0\rangle + \frac{1}{2} (a^\dagger)^2 (b^\dagger)^m |0,0\rangle$

$$= \left(a^\dagger \underbrace{a a^\dagger}_{1+a^\dagger a} a^\dagger + \frac{1}{2} (a^\dagger)^2 \right) (b^\dagger)^m |0,0\rangle$$

$$= \left(a^\dagger (1 + a^\dagger a) a^\dagger + \frac{1}{2} (a^\dagger)^2 \right) (b^\dagger)^m |0,0\rangle$$

$$= \left((a^\dagger)^2 + a^\dagger a^\dagger \underbrace{a a^\dagger}_{1+a^\dagger a} + \frac{1}{2} (a^\dagger)^2 \right) (b^\dagger)^m |0,0\rangle$$

$$= \left((a^\dagger)^2 + (a^\dagger)^2 + a^\dagger a^\dagger a^\dagger a + \frac{1}{2} (a^\dagger)^2\ (b^\dagger)^m \right) |0,0\rangle$$

$$= \frac{5}{2} (a^\dagger)^2 (b^\dagger)^m |0,0\rangle + (a^\dagger)^3 (b^\dagger)^m \underbrace{a |0,0\rangle}_{=0}$$

$$H|2,m\rangle = \frac{5}{2} (a^\dagger)^2 (b^\dagger)^m |0,0\rangle$$

If we continue, we see that

$$\boxed{H|n,m\rangle = \underbrace{(n+\tfrac{1}{2})}_{=E_n}|n,m\rangle}$$

so $|n,m\rangle$ is an eigenstate with eigen energy E_n