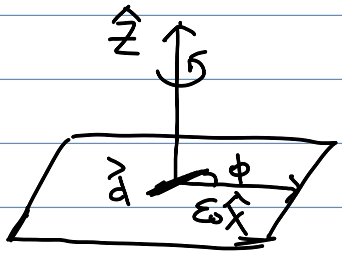
Zhi Chen HW#19

i)



→ choose coordinate system such that plane rotates around $\hat{z}$, so $L^2 = L_z^2$

→ $\vec{\mathcal{E}}_0$ is lying on the x-y plane, so choose $\vec{\mathcal{E}}_0$ to be in the x-axis, $\vec{\mathcal{E}}_0 = \mathcal{E}_0 \hat{x}$

$$H_0 = \frac{L_z^2}{2I} \quad \leftarrow \text{unperturbed Hamiltonian}$$

then since $L_z = -i\hbar\,\partial_\phi$, $L_z^2 = -\hbar^2 \partial_\phi^2$

$$-\frac{\hbar^2}{2I}\partial_\phi^2 \psi(\phi) = E\psi$$

$$\partial_\phi^2 \psi + \frac{2IE}{\hbar^2}\psi = 0$$

let $\frac{2IE}{\hbar^2} = m^2$ so $\boxed{E = \frac{\hbar^2}{2I} m^2}$

then $\partial_\phi^2 \psi + m^2 \psi = 0$ or $\psi \propto e^{im\phi}$

due to periodic boundary: $\psi(\phi) = \psi(\phi + 2\pi)$, $\Rightarrow e^{im\phi} = e^{im(2\pi+\phi)}$

$\hookrightarrow e^{i2\pi m} = 1$ or $m = 0, \pm 1, \pm 2, \cdots$

with normalization $|A|^2 \int d\phi = 1 \rightarrow A = \frac{1}{\sqrt{2\pi}}$

$$\boxed{\psi_m^{(0)} = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{with } E_m^{(0)} = \frac{\hbar^2}{2I} m^2, \quad m = 0, \pm 1 \cdots}$$

Now consider ground state, $\underline{m=0}$, which is not degenerate..

$$|0\rangle = |0^{(0)}\rangle + \sum_k \frac{\langle k|V|0\rangle}{E_0^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

$$V = -\vec{d}\cdot\vec{\mathcal{E}_0} = -\vec{d}\cdot \mathcal{E}_0 \hat{x} = -\mathcal{E}_0 d \cos\phi = -\mathcal{E}_0 d \left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)$$

first compute $\langle k|V|m\rangle$

$$\hookrightarrow = \int_0^{2\pi} d\phi \, \frac{1}{2\pi} \, e^{-ik\phi} \left(\frac{-\mathcal{E}_0 d}{2}\right)\left(e^{i\phi}+e^{-i\phi}\right) e^{im\phi}$$

$$= \int_0^{2\pi} d\phi \, \frac{-\mathcal{E}_0 d}{4\pi} \underbrace{\left(e^{i\phi(m-k+1)} + e^{i\phi(m-k-1)}\right)}$$

These terms integrate to 0 unless exponent is 0.
i.e. $m-k=\pm 1$
and integrates to 2π if $m-k=\pm 1$

$$\langle k|V|m\rangle = -\frac{\mathcal{E}_0 d}{2} \delta_{k,m\pm 1}$$

then when $\underline{\underline{m=0}}$ we have:

then $$|0\rangle = |0^{(0)}\rangle + \frac{-\frac{\mathcal{E}_0 d}{2}}{\cancel{E_0^{(0)}} - E_1^{(0)}} |1\rangle + \frac{-\frac{\mathcal{E}_0 d}{2}}{\cancel{E_0^{(0)}} - E_{-1}^{(0)}} |-1\rangle$$

$$= |0^{(0)}\rangle + \frac{\mathcal{E}_0 d}{\cancel{2}\frac{\hbar^2}{\cancel{2}I}} \left(|1\rangle + |-1\rangle\right)$$

$$\boxed{|0\rangle = |0^{(0)}\rangle + \underbrace{\frac{\mathcal{E}_0 I d}{\hbar^2}\left(|1^{(0)}\rangle + |-1^{(0)}\rangle\right)}_{\text{First order perturbation state.}}}$$

Now evaluate $\alpha = \frac{d}{d\varepsilon_0}|\langle d\rangle|$

$$\langle d\rangle = \left[\langle 0^{(0)}| + \frac{\varepsilon_0 I d}{\hbar^2}\left(\langle 1| + \langle -1|\right)\right] \vec{d} \left[|0^{(0)}\rangle + \frac{\varepsilon_0 I d}{\hbar^2}\left(|1\rangle + |-1\rangle\right)\right]$$

$$= \cancel{\langle 0^{(0)}|\vec{d}|0^{(0)}\rangle} + \frac{\varepsilon_0 I d}{\hbar^2}\left(\langle 0^{(0)}|\vec{d}\left(|1\rangle + |-1\rangle\right) + \left(\langle 1| + \langle -1|\right)\vec{d}|0\rangle\right)$$

0 by selection rule

$$+ \left(\frac{\varepsilon_0 I d}{\hbar^2}\right)^2 \underbrace{\cancel{\left(\langle 1| + \langle -1|\right)\vec{d}\left(|1\rangle + |-1\rangle\right)}}_{=0 \text{ by selection rule.}}$$

with $\vec{d} = d\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right) = d\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\hat{x} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\hat{y}\right)$

$$= \frac{\varepsilon_0 I d^2}{\hbar^2}\left\{\left(\langle 0^{(0)}|\frac{e^{i\phi}+e^{-i\phi}}{2}\left(|1\rangle + |-1\rangle\right) + \left(\langle 1| + \langle -1|\right)\frac{e^{i\phi}+e^{-i\phi}}{2}|0\rangle\right)\hat{x}\right.$$

$$\left. + \left(\underbrace{\langle 0^{(0)}|\frac{e^{i\phi}-e^{-i\phi}}{2i}\left(|1\rangle + |-1\rangle\right)}_{0} + \underbrace{\left(\langle 1| + \langle -1|\right)\frac{e^{i\phi}-e^{-i\phi}}{2i}|0\rangle}_{=0}\right)\hat{y}\right\}$$

$$= \frac{\varepsilon_0 I d^2}{\hbar^2}\left(4\,\frac{1}{2}\,\hat{x}\right)$$

$$\stackrel{!}{=} -\frac{2\varepsilon_0 I d^2}{\hbar^2}\hat{x}$$

so $\boxed{\alpha = \frac{d|\langle\vec{d}\rangle|}{d\varepsilon_0} = \frac{2 I d^2}{\hbar^2}}$

Applicable when perturbation theory holds which means $|\langle k|V|n\rangle| < \left(E_n^{(0)} - E_k^{(0)}\right)$ for $n=0,\ k=1$

$\hookrightarrow \frac{\varepsilon_0 d}{2} < \frac{\hbar^2}{2I}$

$\hookrightarrow \boxed{\varepsilon_0 < \frac{\hbar^2}{Id}}$

2) Find approximate wavefunction to first order.

$$|m\rangle = |m^{(0)}\rangle + \sum_k{}' \frac{\langle k^{(0)}|V|m^{(0)}\rangle}{E_m^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

For $m > 0$, we have 2-fold degeneracy, $E_m = E_{-m}$

but we have the selection rules $\langle k^{(0)}|V|m^{(0)}\rangle \propto \delta_{k,m\pm1}$

So k is never $-m$.

$$|m\rangle = |m^{(0)}\rangle + \frac{1}{E_m - E_{m\pm1}} \Big(\langle m+1^{(0)}|V|m^{(0)}\rangle |m+1^{(0)}\rangle + \langle m-1^{(0)}|V|m^{(0)}\rangle |m-1^{(0)}\rangle \Big)$$

$$= |m^{(0)}\rangle + \frac{-\varepsilon_0 I d}{\frac{\hbar^2}{2}(m^2-(m+1)^2)} \langle m+1^{(0)}| e^{i\phi} + e^{-i\phi} |m^{(0)}\rangle |m+1^{(0)}\rangle + \frac{-\varepsilon_0 I d}{\frac{\hbar^2}{2}(m^2-(m-1)^2)} \langle m-1^{(0)}| e^{i\phi} + e^{-i\phi} |m^{(0)}\rangle |m-1^{(0)}\rangle\Big)$$

$$|m\rangle = |m^{(0)}\rangle + \left(\frac{-\varepsilon_0 I d}{\frac{\hbar^2}{\cancel{2}}(-2m-1)} \cancel{\tfrac{1}{2}} |m+1^{(0)}\rangle\right) + \left(\frac{-\varepsilon_0 I d}{\frac{\hbar^2}{\cancel{2}}(2m-1)} \cancel{\tfrac{1}{2}} |m-1^{(0)}\rangle\right)$$

$$= |m^{(0)}\rangle + \frac{\varepsilon_0 I d}{\hbar^2}\left(\frac{1}{2m+1}|m+1^{(0)}\rangle - \frac{1}{2m-1}|m-1^{(0)}\rangle\right)$$

Normalize: $\langle m|m\rangle A^2 = A^2\left(1 + \left(\frac{\varepsilon_0 I d}{\hbar^2}\right)^2\left[\left(\frac{1}{2m+1}\right)^2 + \left(\frac{1}{2m-1}\right)^2\right]\right) = 1$

then $A = \left(1 + \left(\frac{\varepsilon_0 I d}{\hbar^2}\right)^2\left[\left(\frac{1}{2m+1}\right)^2 + \left(\frac{1}{2m-1}\right)^2\right]\right)^{-1/2}$

$$A \cong 1 - \frac{1}{2}\left(\frac{\varepsilon_0 I d}{\hbar^2}\right)^2 \left[\left(\frac{1}{2m+1}\right)^2 + \left(\frac{1}{2m-1}\right)^2\right]$$

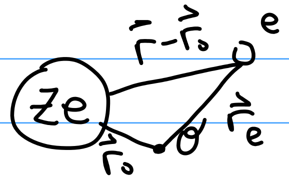
then:

$$|m\rangle = \left(1 - \frac{1}{2}\left(\frac{\varepsilon_0 I d}{\hbar^2}\right)^2 \left[\left(\frac{1}{2m+1}\right)^2 + \left(\frac{1}{2m-1}\right)^2\right]\right)\left\{|m^{(0)}\rangle + \frac{\varepsilon_0 I d}{\hbar^2}\left(\frac{1}{2m+1}|m+1^{(0)}\rangle - \frac{1}{2m-1}|m-1^{(0)}\rangle\right)\right\}$$

then the wave function $\underline{\langle\phi|m\rangle}$:

$$\boxed{\psi(\phi) = \left(1 - \frac{1}{2}\left(\frac{\varepsilon_0 I d}{\hbar^2}\right)^2 \left[\left(\frac{1}{2m+1}\right)^2 + \left(\frac{1}{2m-1}\right)^2\right]\right) \left\{\frac{1}{\sqrt{2\pi}} e^{im\phi} + \frac{\varepsilon_0 I d}{\hbar^2}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2m+1} e^{i(m+1)\phi} - \frac{1}{2m-1} e^{i(m-1)\phi}\right)\right\}}$$

3)



a) $$H = \frac{P_{nuc}^2}{2M} + \frac{P_e^2}{2m} - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{|\vec{r}_e - \vec{r}_n|}$$

$$= -\frac{\hbar^2}{2M}\nabla_{nuc}^2 - \frac{\hbar^2}{2m}\nabla_e^2 - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{|\vec{r}_e - \vec{r}_n|}$$

b) define $\vec{R}_{cm} = \dfrac{M\vec{r}_n + m\vec{r}_e}{M+m}$ and $\vec{r} = \vec{r}_e - \vec{r}_n$

$$\Rightarrow \quad \frac{\partial}{\partial \vec{r}_n} = \frac{\partial}{\partial \vec{R}_{cm}}\frac{\partial \vec{R}_{cm}}{\partial \vec{r}_n} + \frac{\partial}{\partial \vec{r}}\frac{\partial \vec{r}}{\partial \vec{r}_n}$$

$$= \frac{\partial}{\partial \vec{R}_{cm}}\left(\frac{M}{M+m}\right) + \frac{\partial}{\partial \vec{r}}(-1)$$

and

$$\frac{\partial}{\partial \vec{r}_e} = \frac{\partial}{\partial \vec{R}_{cm}}\frac{\partial \vec{R}_{cm}}{\partial \vec{r}_e} + \frac{\partial}{\partial \vec{r}}\frac{\partial \vec{r}}{\partial \vec{r}_e}$$

$$= \frac{\partial}{\partial \vec{R}_{cm}}\left(\frac{m}{M+m}\right) + \frac{\partial}{\partial \vec{r}}(+1)$$

then:

$$\nabla_n^2 = \frac{\partial}{\partial \vec{r}_n}\cdot\frac{\partial}{\partial \vec{r}_n} = \left(\frac{M}{M+m}\right)^2\left(\frac{\partial}{\partial \vec{R}_{cm}}\right)^2 - 2\left(\frac{M}{M+m}\right)\frac{\partial}{\partial \vec{r}}\frac{\partial}{\partial \vec{R}_{cm}} + \left(\frac{\partial}{\partial \vec{r}}\right)^2$$

$$\nabla_e^2 = \frac{\partial}{\partial \vec{r}_e}\cdot\frac{\partial}{\partial \vec{r}_e} = \left(\frac{m}{M+m}\right)^2\left(\frac{\partial}{\partial \vec{R}_{cm}}\right)^2 + 2\left(\frac{m}{M+m}\right)\frac{\partial}{\partial \vec{r}}\frac{\partial}{\partial \vec{R}_{cm}} + \left(\frac{\partial}{\partial \vec{r}}\right)^2$$

then $\frac{-\hbar^2}{2M}\nabla_n^2 - \frac{\hbar^2}{2m}\nabla_e^2 - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{|\vec{r}_e-\vec{r}_n|}$

$$= \frac{-\hbar^2}{2M}\left[\left(\frac{M}{M+m}\right)^2\left(\frac{\partial}{\partial\vec{R}_{cm}}\right)^2 \cancel{-2\left(\frac{M}{M+m}\right)\frac{\partial}{\partial\vec{r}}\frac{\partial}{\partial\vec{R}_{cm}}} + \left(\frac{\partial}{\partial\vec{r}}\right)^2\right]$$

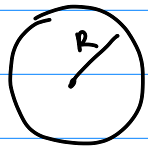
$$-\frac{\hbar^2}{2m}\left[\left(\frac{m}{M+m}\right)^2\left(\frac{\partial}{\partial\vec{R}_{cm}}\right)^2 \cancel{+2\left(\frac{m}{M+m}\right)\frac{\partial}{\partial\vec{r}}\frac{\partial}{\partial\vec{R}_{cm}}} + \left(\frac{\partial}{\partial\vec{r}}\right)^2\right] - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{|\vec{r}|}$$

recognize $\frac{\partial}{\partial\vec{R}_{cm}} = \vec{\nabla}_{R_{cm}}$ and $\frac{\partial}{\partial\vec{r}} = \vec{\nabla}_r$

$$= -\frac{\hbar^2}{2}\nabla_{R_{cm}}^2 \underbrace{\left(\frac{M}{(M+m)^2} + \frac{m}{(M+m)^2}\right)}_{=\frac{1}{M+m}} - \frac{\hbar^2}{2}\nabla_r^2 \underbrace{\left(\frac{1}{M} + \frac{1}{m}\right)}_{=\boxed{\frac{M+m}{Mm} = \mu}} - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{|\vec{r}|}$$

reduced mass

$$\boxed{H = \underbrace{\frac{-\hbar^2}{2(M+m)}\nabla_{R_{cm}}^2}_{\text{Center of mass motion which only depends on } \vec{R}_{cm}} \underbrace{- \frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{|\vec{r}|}}_{\text{interaction terms, which depends on } \vec{r}\text{. so it decouples from center of mass motion}}}$$

4) a)  $Q_{tot} = Ze$ $\qquad \rho_{uniform} = \dfrac{Ze}{\frac{4}{3}\pi R^3}$

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\hookrightarrow \int d^3r\, \vec{\nabla}\cdot\vec{E} = \int \frac{1}{\varepsilon_0}\frac{Ze}{\frac{4}{3}\pi R^3}\, r^2 dr\, d\theta\, d\phi \sin\theta$$

divergence theorem $\hookrightarrow \int d^2\vec{S}\cdot\vec{E} = E_r \underbrace{\int r\sin\theta\, d\theta\, r\, d\phi}_{4\pi r^2} = \int \frac{1}{\varepsilon_0}\frac{Ze}{\frac{4}{3}\pi R^3}\, r^2 dr\, 4\pi$

For $r > R$:

$$E_r 4\pi r^2 = \int_0^R r'^2 dr' \frac{3Ze}{\varepsilon_0 R^3}$$

$$\vec{E}_{r>R} = \frac{Ze}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r} \qquad r > R$$

$$\boxed{\phi = -\int_\infty^r \vec{E}\cdot d\vec{r} = \frac{Ze}{\cancel{4\pi\varepsilon_0}}\frac{1}{r} \qquad r > R}$$

drop $4\pi\varepsilon_0$ for Gaussian

For $r < R$:

$$E_r 4\pi r^2 = \int_0^r r'^2 dr' \frac{3Ze}{\varepsilon_0 R^3}$$

$$\vec{E} = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{r^3}{R^3}\right)\frac{1}{r^2}\hat{r} \qquad r < R$$

$$\phi = -\int_\infty^r \vec{E}\cdot d\vec{r} = -\int_\infty^R \vec{E}_{r>R}\cdot d\vec{r} - \int_R^r \vec{E}_{r<R}\cdot d\vec{r}$$

$$= \frac{Ze}{4\pi\varepsilon_0}\frac{1}{R} - \frac{Ze}{4\pi\varepsilon_0}\frac{1}{R^3}\frac{r^2}{2}\Big|_R^r$$

$$\boxed{\phi = \frac{Ze}{\cancel{4\pi\varepsilon_0}}\frac{1}{R}\frac{3}{2} - \frac{Ze}{\cancel{4\pi\varepsilon_0}}\frac{1}{R^3}\frac{r^2}{2} \qquad r < R}$$

b) $V_0 = -e\phi = -\dfrac{Ze^2}{r}$

New potential:

$$V = -e\phi = \begin{cases} -\dfrac{3}{2}\dfrac{Ze^2}{R} + \dfrac{1}{2}\dfrac{Ze^2}{R^3}r^2 & r < R \\ -\dfrac{Ze^2}{r} & r > R \end{cases}$$

the difference comes in $r < R$:

$$\Delta V = V_{r<R} - V_0$$

$$\Delta V = -\frac{3}{2}\frac{Ze^2}{R} + \frac{1}{2}\frac{Ze^2}{R^3}r^2 + \frac{Ze^2}{r} \quad \text{for } r < R$$

consider ground state of hydrogen atom: $\psi_{100} = R_{10} Y_0^0$, nondegenerate

$$\langle 100|\Delta V|100\rangle = \int_0^R r^2 dr \underbrace{\sin\theta d\theta d\phi |Y_0^0|^2}_{=1} \underbrace{R_{10}^*(r) R_{10}(r)}_{=4\left(\frac{Z}{a_0}\right)^3 e^{\frac{-2Zr}{a_0}}} \left\{-\frac{3}{2}\frac{Ze^2}{R} + \frac{1}{2}\frac{Ze^2}{R^3}r^2 + \frac{Ze^2}{r}\right\}$$

$R_{10} = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}}$

$$= \frac{2}{R}Ze^2\left(\frac{Z}{a_0}\right)^3 \underbrace{\int_0^R r^2 dr \left(-3 + \left(\frac{r}{R}\right)^2 + \frac{2R}{r}\right) e^{\frac{-2Zr}{a_0}}}$$

using Mathematica

$$\boxed{E_0^{(1)} = \frac{2}{R}Ze^2\left(\frac{Z}{a_0}\right)^3 \left\{\frac{-3a_0^3(a_0+RZ)^2 e^{\frac{-2RZ}{a_0}} + 3a_0^5 - 3a_0^3R^2Z^2 + 2R^3a_0^2Z^3}{4R^2Z^5}\right\}}$$

exact solution

If we assume $\frac{Z}{a_0} \ll 1$, then $e^{\frac{-2Zr}{a_0}} \sim 1$

then $$E_0^{(1)} \cong \frac{2}{R} Z e^2 \left(\frac{Z}{a_0}\right)^3 \int_0^R r^2 dr \left(-3 + \left(\frac{r}{R}\right)^2 + \frac{2R}{r}\right)$$

$$\cong \frac{2}{R} Z e^2 \left(\frac{Z}{a_0}\right)^3 \left[\cancel{\frac{-3R^3}{3}} + \frac{R^5}{5}\frac{1}{R^2} + \cancel{\frac{2R^3}{2}} \right]$$

$$\cong \frac{2}{R} Z^4 e^2 \frac{1}{a_0^3} \left[\frac{R^3}{5} \right]$$

$$\boxed{E_{n=1}^{(1)} \cong \frac{2}{5} Z^4 e^2 \left(\frac{R}{a_0}\right)^2 \frac{1}{a_0}} \leftarrow \text{approximate solution.}$$

using $E_0^{(0)} = -\frac{1}{2} Z^2 e^2 / a_0$

$$= -\frac{4}{5}\left(\frac{ZR}{a_0}\right)^2 E_{n=1}^{(0)}$$

c) Consider relativistic effect:

$$T = \sqrt{p^2c^2 + m_e^2c^4} - mc^2$$

$$= m_ec^2\sqrt{1+\left(\frac{pc}{m_ec^2}\right)^2} - m_ec^2$$

assume $pc \ll m_ec^2$, use binomial expansion: $(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2$

$$= m_ec^2\left(\cancel{1} + \frac{1}{2}\left(\frac{pc}{m_ec^2}\right)^2 - \frac{1}{8}\left(\frac{pc}{m_ec^2}\right)^4\right) - m_ec^2$$

$$T \simeq \frac{p^2}{2m_e} - \frac{p^4}{8m_e^3c^2}$$

then $$H = \underbrace{\frac{p^2}{2m_e} - \frac{Ze^2}{r}}_{H_0} - \underbrace{\frac{p^4}{8m_e^3c^2}}_{\Delta V}$$

$$\Delta V = -\frac{p^4}{8m_e^3c^2} = -\frac{1}{2m_ec^2}\left(\frac{p^2}{2m_e}\right)^2 = \frac{-1}{2m_ec^2}\left(H_0 + \frac{Ze^2}{r}\right)^2$$

then $$\langle 100|\Delta V|100\rangle = -\frac{1}{2m_ec^2}\langle 100|H_0^2 + 2H_0\frac{Ze^2}{r} + \left(\frac{Ze^2}{r}\right)^2|100\rangle$$

$$= -\frac{1}{2m_ec^2}\left(E_{n=0}^2 + 2E_{n=0}Ze^2\langle 100|\frac{1}{r}|100\rangle + (Ze^2)^2\langle 100|\frac{1}{r^2}|100\rangle\right)$$

Calculate: $$\langle 100|\frac{1}{r}|100\rangle = \int_0^\infty r^2dr\, 4\left(\frac{Z}{a_0}\right)^3 e^{-\frac{2Zr}{a_0}}\frac{1}{r}$$

$$= \frac{1}{4}\left(\frac{a_0}{Z}\right)^2 4\left(\frac{Z}{a_0}\right)^3$$

$$\langle 100|\frac{1}{r}|100\rangle = \frac{Z}{a_0}$$

Calculate: $\langle 100|\frac{1}{r^2}|100\rangle = \int_0^\infty r^2 dr\, 4\left(\frac{Z}{a_0}\right)^3 e^{-\frac{2Zr}{a_0}} \frac{1}{r^2}$

$$= \frac{1}{2}\frac{a_0}{Z}\, 4\left(\frac{Z}{a_0}\right)^3$$

$$= 2\left(\frac{Z}{a_0}\right)^2$$

$$= -\frac{1}{2m_e c^2}\left(E_{n=1}^2 + 2E_{n=1} Ze^2 \frac{Z}{a_0} + (Ze^2)^2\, 2\left(\frac{Z}{a_0}\right)^2\right)$$

know $\quad E^{(0)}_{n=1} = \frac{-1}{2}\frac{Z^2e^2}{a_0}$

$$= -\frac{1}{2m_e c^2}\left(E_{n=1}^2 - 4E_{n=1}^2 + 8E_{n=1}^2\right)$$

$$= -\frac{5}{2}\frac{\left(E^{(0)}_{n=1}\right)^2}{m_e c^2}$$

$$\boxed{E^{(1)}_{n=1} = -\frac{5}{8}\left(\frac{Z^2e^2}{a_0}\right)^2 \frac{1}{m_e c^2}}$$

← relativistic correction.

c) compare with correction term that we get from finite size nucleus:

$${}^{\text{finite size}}E^{(1)}_{n=1} \cong \frac{2}{5} Z^4 e^2 \left(\frac{R}{a_0}\right)^2 \frac{1}{a_0} \sim \frac{Z^4 e^2 R^2}{a_0^3} \underset{R=a_0 \text{ for } Z=1}{\sim} \frac{e^2}{a_0}$$

$${}^{\text{rel}}E^{(1)}_{n=1} = -\frac{5}{8}\frac{Z^4e^4}{a_0^2}\frac{1}{m_e c^2} \overset{Z=1}{\sim} \frac{e^4}{a_0^2}\frac{1}{m_e c^2}$$

$$\frac{{}^{\text{rel}}E^{(0)}_{n=1}}{{}^{\text{finite size}}E^{(0)}_{n=1}} \cong \frac{\frac{e^{\cancel{4}\,2}}{a_0^{\cancel{2}}}\frac{1}{m_e c^2}}{\frac{\cancel{e^2}}{\cancel{a_0}}} \sim \frac{e^2}{m_e c^2 a_0} \ll 1$$

So comparing case $Z=1$, finite size perturbation is much more significant.