Quantum Dynamics:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Ex: precession of spin $\frac{1}{2}$ in $\vec{B}$

use $|+\rangle$ and $|-\rangle$

$$S^z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = \frac{-e\hbar}{2m_e c}\vec{\sigma}\cdot\vec{B}$$

choose $\vec{B} = B_0\hat{z}$

$$H = \frac{-e\hbar}{2m_e c}B_0\sigma_z$$

$H|\pm\rangle = E_\pm|\pm\rangle$, $E_\pm = \mp\frac{\hbar e B_0}{2m_e c} = \pm\frac{1}{2}\hbar\omega$

then $\hat{H} = \omega\hat{\sigma}_z$ where $\omega = \frac{|e|B}{m_e c}$

let $|\psi(t)\rangle = C_+|+\rangle + C_-|-\rangle$

then: $i\hbar\frac{d}{dt}(\psi(t)\rangle = i\hbar\left(\frac{dC_+}{dt}|+\rangle + \frac{dC_-}{dt}|-\rangle\right)$

$$H|\psi(t)\rangle = C_+\frac{\hbar\omega}{2}|+\rangle - C_-\frac{\hbar\omega}{2}|-\rangle$$

then $i\hbar\frac{dC_+}{dt} = C_+\frac{\hbar\omega}{2}$

$\hookrightarrow C_\pm(t) = e^{\frac{-i\omega}{2}t}C_\pm(0)$

then $|\psi(t)\rangle = C_+(0)e^{\frac{-i\omega t}{2}}|+\rangle + C_-(0)e^{\frac{-i\omega t}{2}}|-\rangle$

Compute observable:

$$\langle S_z\rangle = \langle\psi|S_z|\psi\rangle$$

$$\overset{!}{=} |C_+|^2 \frac{\hbar}{2}\langle +|+\rangle - |C_-|^2\frac{\hbar}{2}\langle -|-\rangle$$

$$\overset{!}{=} |C_+|^2 - |C_-|^2\frac{\hbar}{2}$$

$$\langle S_x\rangle = \frac{\hbar}{2}C_+^* C_- e^{i\omega t} + \frac{\hbar}{2}C_-^* C_+ e^{-i\omega t}$$

$$\langle S_y\rangle = -i\frac{\hbar}{2}C_+^* C_- e^{i\omega t} + c.c.$$

let $(C_+, C_-) = e^{i\psi}\left(\cos\frac{\theta}{2}, e^{i\phi}\sin\frac{\theta}{2}\right)$

then $\langle S_z\rangle = \frac{\hbar}{2}\cos\theta$

$$\langle S_x\rangle = \frac{\hbar}{2}\cos(\omega t + \phi)\sin\theta$$

$$\langle S_y\rangle = \frac{\hbar}{2}\sin(\omega t+\phi)\sin\theta$$

General Procedure: time-independent Schrodinger:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

1) $\hat{H} | E_i \rangle = E_i | E_i \rangle$

2) $|\psi(t=0)\rangle = C_i | E_i \rangle$

3) $|\psi(t)\rangle = C_1 e^{-\frac{i}{\hbar} E_i t} | E_i \rangle$

$$U_t = \sum |E_i\rangle e^{-\frac{i}{\hbar} E_i t} \langle E_i|$$

$$\stackrel{!}{=} e^{-\frac{i}{\hbar} H t}$$

$$i\hbar \frac{d}{dt} \left(U_t | \psi (t=0) \rangle \right) = H U_t |\psi(t=0)\rangle$$

1D wave mechanics:

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$j-1 \quad j \quad j+1$

$|j\rangle$

$\langle j|k\rangle = \delta_{jk}$

$$\sum_j |j\rangle\langle j| = 1$$

$$\sum_{j-\Delta j}^{j+\Delta j} \langle j|k\rangle = 1$$

let $|\alpha\rangle = \sum_j |j\rangle \underbrace{\langle j|\alpha\rangle}_{\psi_j} \quad \psi_j = \langle j|\alpha\rangle$

$$\stackrel{!}{=} |j\rangle \psi_j$$

Now for $|x'\rangle$

$$\langle x'|x''\rangle = \delta_{x',x''}$$

$$\stackrel{!}{=} \int dx' \, |x'\rangle\langle x'| = 1$$

$$\int_{x'-\Delta x'}^{x'+\Delta x'} dx'' \langle x'|x''\rangle = 1$$

Dirac-Delta Func :
$$\delta(x'-x'') = \lim_{a\to\infty} \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$$
$$\overset{!}{=} \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$$

$$|\alpha\rangle = \int dx'\, |x'\rangle \underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')}$$

$$\int dx' \, \psi_\alpha(x')^2 = 1$$

$$\int dx' \underbrace{\langle\phi|x'\rangle}_{\psi^*(x')} \underbrace{\langle x'|\psi\rangle}_{\psi(x)}$$ ← wave function.

Projector:

$P_{[a,b]}$ = Projection on $x \in [a,b]$

$$P_{[a,b]} = \begin{cases} 0 & \text{if } x \notin [a,b] \\ 1 & \text{if } x \in [a,b] \end{cases}$$

Position operator with continuous spectrum.

$$X\,|x'\rangle = x'\,|x'\rangle$$

operator eigenstate eigenvalue.

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|\alpha\rangle$$

$$\stackrel{!}{=} \int_{x'-\Delta x'}^{x'+\Delta x'} dx'' \, |x'\rangle \langle x''|\alpha\rangle$$

$P=1$ with probability $\int_{x'-\Delta x'}^{x'+\Delta x'} dx'' \, |\psi_\alpha(x')|^2 \sim \underbrace{|\psi_\alpha(x')|^2}_{\sim \text{ density matrix}} 2\Delta x'$

3d: $\vec{x} = (x, y, z)$

$$\hat{\vec{x}}\,|\vec{x}'\rangle = \vec{x}'\,|\vec{x}'\rangle$$

$[x_i, x_j] = 0$ ← different directions are compatible.

Translation Operator:

$|x'\rangle \Rightarrow$ Define operator: $\mathcal{F}(dx')|x'\rangle \stackrel{\uparrow}{=} |x'+dx'\rangle$

translation operator. → $\mathcal{F}$ No phase.

$$\boxed{\text{Define. } \mathcal{F}(dx') = \int dx' \, |x'+dx'\rangle\langle x'| = \int dx' \, |x'\rangle\langle x'-dx'|}$$

$$\mathcal{F}(dx')\,|\alpha\rangle = \int dx' \, |x'+dx'\rangle \underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')}$$

$$\stackrel{!}{=} \int dx' \, |x'\rangle\langle x'-dx'|\alpha\rangle$$

$$\psi_\alpha(x') \xrightarrow{\mathcal{F}(dx')} \psi_\alpha(x'-dx')$$

Suppose $\langle\alpha|\alpha\rangle = 1$

then $\langle\alpha|\mathcal{F}^{\dagger}(dx')\,\mathcal{F}(dx)|\alpha\rangle = 1$

Properties of translation operator:

1) we see that $\mathcal{F}^{\dagger}(dx')\,\mathcal{F}(dx) = \mathbb{1}$

so $\mathcal{F}$ is unitary, or $\mathcal{F}^{\dagger} = \mathcal{F}^{-1}$

2) $\mathcal{F}(dx'') + \mathcal{F}(dx') = \mathcal{F}(dx' + dx'')$

3) $\mathcal{F}(-dx') = \mathcal{F}^{-1}(dx')$

4) as $dx' \to 0$ $\mathcal{F}(dx') = \mathbb{1}$

$\mathcal{F}[dx'] = 1 - i\vec{K}\cdot\vec{dx'}$ where $\vec{dx'}$ is infinitesimal small step

K is a Hermitian Operator.

$\vec{K}$ is the generator of translation.

Check $\mathcal{F}^{\dagger}\mathcal{F} = (1 + i\vec{K}^{\dagger}\cdot\vec{dx'})(1 - i\vec{K}\cdot\vec{dx'})$

$$\overset{!}{=} 1 + \underbrace{(\vec{K}^{\dagger} - \vec{K})}\,dx' + \mathcal{O}(dx'^2)$$

K must also be Hermitian operator if $\mathcal{F}^{\dagger}\mathcal{F} = \mathbb{1}$

check: $\mathcal{F}(dx'')\mathcal{F}(dx') = 1 - i\vec{K}(\vec{dx''} + \vec{dx'}) = \mathcal{F}(\vec{dx''} + \vec{dx'})$

① $\vec{X}\,\mathcal{F}(d\vec{x}')\,|\vec{x}'\rangle = \vec{X}\,|\vec{x}'+d\vec{x}'\rangle = (\vec{x}'+d\vec{x}')\,|\vec{x}'+d\vec{x}'\rangle$

② $\mathcal{F}\vec{X}|\vec{x}'\rangle = \mathcal{F}(d\vec{x}')\,\vec{x}'|\vec{x}'\rangle = \vec{x}'\,|\vec{x}'+d\vec{x}'\rangle$

$$[\vec{X}, \mathcal{F}(d\vec{x}')]\,|\vec{x}'\rangle = ① - ②$$

$$\overset{1}{=} d\vec{x}'\,|\vec{x}'+d\vec{x}'\rangle$$

$$\overset{\checkmark}{\cong} d\vec{x}'\,|\vec{x}'\rangle$$

$$[\vec{X}, 1 - i\vec{K}\cdot d\vec{x}'] = d\vec{x}'$$

$$[X_i, 1 - iK_j\,dx_j] = dx_i'$$

$$-i\,dx_j\,[X_i, K_j] = dx_i'$$

$$[X_i, K_j] = i\delta_{ij}\,\mathbb{1}$$

<u>Classical Mechanics:</u>

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}$$

$$\frac{dA}{dt} = \{H, A\}$$

Correspond : $P = \hbar K$ ← generator, has unit $\frac{1}{m}$

$\hbar$ has unit $\frac{kg\,m^2}{s} = [E][t] = $ [angular momentum]

$$\boxed{\begin{aligned} &\text{Now: } \mathcal{F}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}' = 1 - \tfrac{i}{\hbar}\vec{P}\cdot d\vec{x}' \\ &\text{since } [X_i, K_j] = i\delta_{ij} \\ &\qquad [X_i, P_j] = i\hbar\delta_{ij} \end{aligned}}$$

$$\langle (\Delta X)^2 \rangle \langle (\Delta P_x)^2 \rangle \geq \frac{1}{4} \left| \langle [X, P_x] \rangle \right|^2 = \frac{\hbar^2}{4}$$

or $\delta x \, \delta P_x \geq \frac{\hbar}{2}$

Finite Translation:

$$\mathcal{F}(\Delta x') = \mathcal{F}\left[N \frac{\Delta x'}{N}\right] = \mathcal{F}\left[\frac{\Delta x'}{N}\right]^N$$

take $\lim\limits_{N\to\infty} \mathcal{F}\left[\frac{\Delta x'}{N}\right]^N = \lim\limits_{N\to\infty} \left(1 - i\vec{k} \frac{\Delta x'}{N}\right)^N$

$$\overset{!}{=} e^{-i\vec{k}\cdot\vec{\Delta x'}}$$

$$\mathcal{F}(\Delta x') \overset{!}{=} e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{\Delta x'}}$$

operator

P_x, P_y, P_z are compatible:

$|\vec{p}'\rangle$ such that $\hat{p}\,|\vec{p}'\rangle = \vec{p}'\,|\vec{p}'\rangle$

$$\mathcal{F}(\Delta x')\,|\vec{p}'\rangle = e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{\Delta x'}}\,|\vec{p}'\rangle$$

eigenvalues of unitary operators have magnitude = 1

Classical	Quantum
$\{x_i, P_j\} = \delta_{ij}$ $\longrightarrow$	$[x_i, P_j] = i\hbar\,\delta_{ij}$
canonical quantization.	
however, there's ordering problem e.g. $P_x^2 X$ $\longrightarrow$	$P_x^2 X$? $P_x X P_x$?

Poisson Bracket Identity:

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0$$

similarly:

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$$

Position space wave function:

In 1D: $X|x'\rangle = x'|x'\rangle \qquad \langle x''|x'\rangle = \delta(x''-x')$

$$|\alpha\rangle = \int dx'\, |x'\rangle\langle x'|\alpha\rangle = \int dx'\, |x'\rangle\, \psi_\alpha(x')$$

then $|\langle x'|\alpha\rangle|^2 dx' = |\psi_\alpha(x')|^2 dx' \Rightarrow$ probability of finding particle in $[x', x'+dx']$

$\psi_\alpha(x') = \langle x'|\alpha\rangle$ wave function in coordinate representation.

$$\langle\alpha|\alpha\rangle = \int dx'\, |\psi_\alpha(x')|^2 = 1$$

$$\langle\beta|\alpha\rangle = \int dx'\, \psi_\beta^* \psi_\alpha$$

overlap with inner product

$$|\alpha\rangle = \sum_{a'} |a'\rangle \underbrace{\langle a'|\alpha\rangle}_{C_{a'}}$$

$$\langle x'|\alpha\rangle = \sum \langle x'|a'\rangle\, C_{a'}$$

$$\overset{!}{=} \sum_{a'} \underbrace{U_{a'}(x')}_{\text{basis wave function}}\, C_{a'}$$

$$\langle\beta|A|\alpha\rangle = \int dx' dx'' \underbrace{\langle\beta|x'\rangle}_{\psi^*_\beta(x')} \underbrace{\langle x'|A|x''\rangle}_{A(x',x'')} \underbrace{\langle x''|\alpha\rangle}_{\psi_\alpha(x'')}$$

kernal of operator A.

Ex: $A = x^2$

$$\hookrightarrow \langle x'|x^2|x''\rangle = x''^2 \langle x'|x''\rangle = x''^2\,\delta(x'-x'')$$

For a general:

$$\langle\beta|x^2|\alpha\rangle = \langle\beta|x'\rangle\langle x'|x^2|x''\rangle\langle x''|\alpha\rangle$$

$$\stackrel{!}{=} \iint \psi^*_\beta(x')\, x''^2\, \delta(x'-x'')\psi_\alpha(x'')\, dx' dx''$$

$$\stackrel{!}{=} \int dx'\, \psi^*_\beta(x')\, x'^2 \psi_\alpha(x')$$

Momentum Operator: p:

$$\langle x'|\mathcal{F}(s')|x''\rangle = \langle x'|x''+ds'\rangle$$

$$\stackrel{!}{=} \delta(x'-x''-ds') = \delta(x'-x'') - ds' \frac{\partial}{\partial x'}(\delta(x'-x''))$$

$$\hookrightarrow \langle x'|1 - \tfrac{i}{\hbar}\vec{p}\cdot d\vec{s}'|x''\rangle = \delta(x'-x'') - \tfrac{i}{\hbar}ds'\langle x'|p|x''\rangle$$

By comparison we see that:

$$\langle x'|p|x''\rangle = -i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')$$

$$\langle\beta|P_x|\alpha\rangle = \int dx'\,dx'' \quad \langle\beta|x'\rangle\underbrace{\langle x'|P_x|x''\rangle}_{-i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')}\langle x''|\alpha\rangle$$

$$\overset{!}{=} \int dx'\,dx'' \quad \psi_\beta^*(x')\, -i\hbar\frac{\partial}{\partial x'}\,\delta(x'-x'')\,\psi_\alpha(x'')$$

$$\overset{!}{=} \int dx' \quad \psi_\beta^*(x')\left(-i\hbar\frac{\partial}{\partial x'} \quad dx''\delta(x'-x'')\psi_\alpha(x'')\right)$$

$$\overset{!}{=} \int dx' \quad \psi_\beta^*(x')\left(-i\hbar\frac{\partial}{\partial x'}\right)\psi_\alpha(x')$$

$$[P_x, x] = -i\hbar\frac{\partial}{\partial x}\left(x\phi(x)\right) - x\left(-i\hbar\frac{\partial}{\partial x}\right)\phi(x) = -i\hbar$$

then $[P_x, f(x)] = -i\hbar f'(x)$

$$[A, BC] = B[A,C] + [A,B]C$$

Momentum Representation:

$$P|p'\rangle = p'|p'\rangle \qquad \text{with} \quad \langle p'|p''\rangle = \delta(p'-p'')$$

$$|\alpha\rangle = \int dp' \; |p'\rangle\langle p'|\alpha\rangle$$

$$\overset{!}{=} \int dp' \quad \psi_\alpha(p')\;|p'\rangle$$

$$\langle\alpha|\alpha\rangle = \int dp' \; \underbrace{|\psi_\alpha(p')|^2}_{\text{probability density.}} = \int dp'\,\langle\alpha|p'\rangle\langle p'|\alpha\rangle$$

what is the relation: $\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$

$$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

$$\overset{!}{=} p'\langle x'|p'\rangle$$

Since $-i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle = p'\langle x'|p'\rangle$

$$\langle x'|p'\rangle = N e^{\frac{i}{\hbar}p'x'}$$

$$\langle x'|x''\rangle = \int dp' \langle x'|p'\rangle\langle p''|x''\rangle$$

$$\delta(x'-x'') \overset{!}{=} \int dp' \left(N e^{\frac{i}{\hbar}p'x'}\right)\left(N e^{\frac{-i}{\hbar}p'x''}\right)$$

$$\delta(x'-x'') \overset{!}{=} \int dp' |N|^2 e^{\frac{i}{\hbar}p'(x'-x'')}$$

$$\overset{!}{=} 2\pi\hbar |N|^2 \delta(x'-x'')$$

then $N = \frac{1}{\sqrt{2\pi\hbar}}$

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p'x'} = \psi_p(x')$$

$$\langle p'|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-i}{\hbar}p'x'} = \phi_{x'}(p')$$

$$\psi_\alpha(x') = \langle x'|\alpha\rangle = \int dp' \underbrace{\langle x'|p'\rangle}_{\phi_p(x)}\underbrace{\langle p'|\alpha\rangle}_{\phi_\alpha(p')}$$

$$\overset{!}{=} \int dp' \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p'x'} \phi_\alpha(p')$$

$$\phi_\alpha(p') = \int \frac{dx'}{\sqrt{2\pi\hbar}} e^{\frac{-i}{\hbar}p'x'} \psi_\alpha(x')$$

In summary:

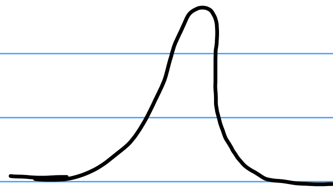
$$\psi_\alpha(x') = \int \frac{dp'}{2\pi\hbar} e^{\frac{i}{\hbar}p'x'} \phi_\alpha(p')$$

$$\phi_\alpha(p') = \int dx' \, e^{\frac{-i}{\hbar}p'x'} \psi_\alpha(x')$$

Gaussian Wave Packets:

$$\psi_\alpha(x') = \langle x' | \alpha \rangle = \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx' - \frac{x'^2}{2d^2}}$$

$$|\langle x' | x' \rangle|^2 = \frac{1}{\sqrt{\pi}\, d} e^{\frac{-x'^2}{d^2}}$$



$$\langle x \rangle = \int_{-\infty}^{\infty} dx' \, x' |\psi(x')|^2 = 0$$

$$\langle x^2 \rangle = \frac{d^2}{2} \quad \text{so} \quad \delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{d}{\sqrt{2}}$$

$$\langle p \rangle = \hbar k$$

$$\langle p^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

$$\langle (\Delta p)^2 \rangle = \frac{\hbar^2}{2d^2} \Rightarrow \delta p = \frac{\hbar}{\sqrt{2}\, d}$$

so its the best quantum approximation

$$\text{so} \quad \delta x \, \delta p = \frac{\hbar}{2} \quad \text{noting } \delta x \, \delta p \geq \frac{\hbar}{2}.$$

3d - generalization:

$$\vec{X}|\vec{x}'\rangle = \vec{x}'|\vec{x}'\rangle$$
$$\vec{p}|\vec{p}'\rangle = \vec{p}'|\vec{p}'\rangle$$

$$\langle\vec{x}'|\vec{x}''\rangle = \delta^3(x'-x'') = \delta(x'-x'')\delta(y'-y'')\delta(z'-z'')$$

$$\langle\vec{p}'|\vec{p}''\rangle = \delta^3(p'-p'')$$

Completeness: $\int d^3x' \, |\vec{x}'\rangle\langle\vec{x}'| = 1$

$$\int d^3p' \, |p'\rangle\langle p'| = 1$$

$$\psi_\alpha(\vec{x}') = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p' \, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \, \phi_\alpha(p')$$

With $|\alpha\rangle$ at $t=t_0$ find $|\alpha(t)\rangle$

with Schrodinger Eq: $i\hbar \frac{d}{dt}|\alpha\rangle = H|\alpha\rangle$

$H = H^\dagger$ Hermitian

$$i\hbar\,\partial_t|\alpha\rangle = H|\alpha\rangle \Rightarrow -i\hbar\,\partial_t\langle\alpha|^\dagger = \langle\alpha|H^\dagger$$

$$i\hbar\,\partial_t(\langle\alpha|\alpha\rangle) = \langle\alpha|\left(i\hbar\,\partial_t|\alpha\rangle\right) - \left(i\hbar\,\partial_t\langle\alpha|\right)|\alpha\rangle$$

$$\overset{!}{=} \langle\alpha|H|\alpha\rangle - \langle\alpha|H^\dagger|\alpha\rangle$$

$$\overset{!}{=} \langle\alpha|H-H^\dagger|\alpha\rangle = 0$$

$$\Rightarrow i\hbar\,\partial_t\langle\beta|\alpha\rangle = 0$$

$$|\alpha(t_0)\rangle \longrightarrow |\alpha(t)\rangle$$

with $|\alpha(t)\rangle = U(t,t_0)|\alpha(t_0)\rangle$

$\downarrow$ unitary operator

$U^\dagger U = UU^\dagger = 1$

Solving Time-Independent H:

$$i\hbar \partial_t |\alpha\rangle = H |\alpha\rangle \Rightarrow \frac{d|\alpha\rangle}{|\alpha\rangle} = dt \frac{-i}{\hbar} H \Rightarrow |\alpha\rangle = |\alpha_0\rangle e^{\frac{-i}{\hbar}H(t-t_0)}$$

1) Find eigenvalue: $H|\psi_j\rangle = E_j|\psi_j\rangle$

2) If $|\alpha\rangle \underset{t=0}{=} |\psi_j\rangle \Rightarrow |\alpha(t)\rangle = e^{\frac{-i}{\hbar}E_j(t-t_0)} |\psi_j\rangle$

3) $|\alpha(t_0)\rangle = \sum_j |\psi_j\rangle\langle\psi_j|\alpha(t_0)\rangle$

$|\alpha(t_0)\rangle \to e^{\frac{-i}{\hbar}E_j(t-t_0)} |\alpha(t_0)\rangle$

$$|\alpha(t_0)\rangle \to \underset{\text{solution}}{|\alpha(t)\rangle} = \sum_j |\psi_j\rangle e^{\frac{-i}{\hbar}E_j(t-t_0)} \langle\psi_j|\alpha(t_0)\rangle$$

$$|\alpha(t)\rangle = \sum_j e^{\frac{-i}{\hbar}H(t-t_0)} |\psi_j\rangle\langle\psi_j|\alpha(t_0)\rangle$$

$$\downarrow\ e^{\frac{-i}{\hbar}H(t-t_0)} \sum_j \underbrace{|\psi_j\rangle\langle\psi_j|}_{=1} \alpha(t_0)\rangle$$

$$|\alpha(t)\rangle \overset{!}{=} |\alpha(t_0)\rangle \underbrace{e^{\frac{-i}{\hbar}H(t-t_0)}}_{U(t,t_0)}$$

then $U^{+}(t,t_0) = e^{\frac{i}{\hbar}H(t-t_0)} = U^{-1}(t,t_0)$

For Time-dependent Hamiltonian, $H(t)$

$$|\alpha(t)\rangle = e^{\frac{-i}{\hbar}\int_{t_0}^{t} H(t')dt'}$$

, note $[H(t'), H(t'')] \neq 0$ Hamiltonian at different times do not commute in general.

Only works when $[H(t'), H(t'')] = 0$.

Conservation of Probability:

$$H = \frac{p^2}{2m} + U(x)$$

Where $p = -i\hbar \vec{\nabla}_x$

$$i\hbar \partial_t \psi = \frac{(-i\hbar\vec{\nabla})^2}{2m}\psi + U(\vec{x})\psi$$

take c.c.: $-i\hbar \partial_t \bar{\psi} = -\frac{\hbar^2 \nabla^2}{2m}\bar{\psi} + U(x)\bar{\psi}$

$$\bar{\psi}\left[i\hbar \partial_t \psi = -\frac{\hbar^2\nabla^2}{2m}\psi + U\psi\right]$$

$$\psi\left[-i\hbar \partial_t \bar{\psi} = \frac{-\hbar^2\nabla^2}{2m}\bar{\psi} + U\bar{\psi}\right]$$

subtract

$$\bar{\psi} i\hbar \partial_t \psi + \psi i\hbar \partial_t \bar{\psi} = \frac{-\hbar^2}{2m}\left[\bar{\psi}\nabla^2\psi - \psi\nabla^2\bar{\psi}\right]$$

$$\hookrightarrow i\hbar \partial_t |\psi|^2 = \frac{-\hbar^2}{2m}\vec{\nabla}\left[\bar{\psi}\vec{\nabla}\psi - \psi\vec{\nabla}\bar{\psi}\right]$$

$$\boxed{\begin{gathered}\partial_t \underbrace{|\psi|^2}_{\rho} + \vec{\nabla}\Big[\underbrace{\tfrac{\hbar}{2mi}\left(\bar{\psi}\vec{\nabla}\psi - \psi\vec{\nabla}\bar{\psi}\right)}_{j}\Big] = 0 \\ \partial_t \rho + \vec{\nabla}\cdot j = 0 \quad \leftarrow \text{continuity.}\end{gathered}}$$

$$\partial_t \int \rho \, d^3x = -\int \vec{\nabla}\cdot j \, d^3x = -\int_{\partial D}^{D} j\cdot n \, dS$$

let $\psi = \sqrt{\rho}\, e^{i\theta}$ where $\rho = |\psi|^2$

$$\vec{j} = \frac{\hbar}{2mi}\rho^{2i\nabla\theta} = \rho\left(\frac{\hbar}{m}\vec{\nabla}\theta\right)$$

Sectionally constant potential:

$$H = \frac{p^2}{2m} + U(x) \qquad p = -i\hbar\,\partial_x$$

$$\left[\frac{-\hbar^2}{2m}\partial_x^2 + U(x)\right]\psi = E\psi \qquad \text{given } \psi(0) \text{ and } \psi'(0)$$

Ex: Free particle in 1D: $U(x) = 0$

$$-\frac{\hbar^2}{2m}\partial_x^2\psi = E\psi \qquad \bar{j} = \frac{1}{2\pi\hbar}\frac{p'}{m}$$

$$\psi_{p'}(x) = C e^{\frac{i}{\hbar}p'x} \qquad \text{and} \quad E = \frac{p'^2}{2m}$$

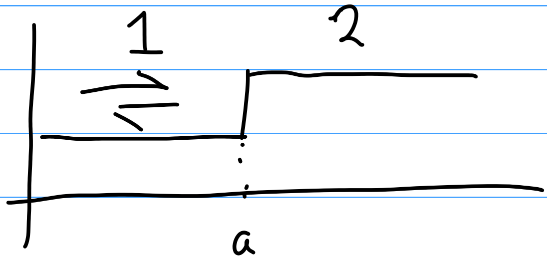
Normalize: $\int dp'\, \psi^*_{p'}(x')\psi_{p'}(x'') = \int dp'\, |C|^2\, 2\pi\hbar\, \delta(x''-x')$

So $|C| = \frac{1}{\sqrt{2\pi\hbar}}$

For $E > 0$:

$$p' = \pm\sqrt{2mE}$$

$$\psi_E = Ae^{\frac{i}{\hbar}\sqrt{2mE}\,x'} + Be^{\frac{-i}{\hbar}\sqrt{2mE}\,x'}$$



Multi-region, solve for each state

$$\left.\begin{array}{l}\phi(a-0) = \phi(a+0)\\ \phi'(a-0) = \phi'(a+0)\\ \text{Normalization}\end{array}\right\} \text{ leaves 1 constant left}$$

Ex: $H = w(\sigma_x + \sigma_y) = w\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$

Find eigenvalues:

$$\det\begin{vmatrix} -\lambda & w(1-i) \\ w(1+i) & -\lambda \end{vmatrix} = \lambda^2 - w^2(1-i)(1+i)$$

$$\stackrel{!}{=} \lambda^2 - 2w^2$$

$$\lambda = \pm\sqrt{2}\, w$$

$$H|\xi_1\rangle = \lambda_1 |\xi_1\rangle$$

$$(H - \lambda_1 \mathbb{1}) = \begin{pmatrix} -\sqrt{2}w & w(1-i) \\ w(1+i) & -\sqrt{2}\, w \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1-i \\ \sqrt{2} \end{pmatrix}$$

and to normalize

$$\sqrt{(1-i)^2 + (\sqrt{2})^2} = \sqrt{2+2} = 2$$

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1-i \\ \sqrt{2} \end{pmatrix} / 2$$

similarly: $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1-i \\ -\sqrt{2} \end{pmatrix} / 2$

$t=0$, $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find probability

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

let $|\psi\rangle = c_1(t)|\xi_1\rangle + c_2(t)|\xi_2\rangle$

$$i\hbar\left(\dot{c}_1|\xi_1\rangle + \dot{c}_2|\xi_2\rangle\right) = \lambda_1 c_1|\xi_1\rangle + \lambda_2 c_2|\xi_2\rangle$$

then $\quad i\hbar \dot{C}_1 = \lambda_1 C_1 \quad$ and $\quad i\hbar \dot{C}_2 = \lambda_2 C_2$

$$C_1 = C_1(0) e^{\frac{-i}{\hbar}\lambda_1 t} \qquad C_2 = C_2(0) e^{\frac{-i}{\hbar}\lambda_2 t}$$

$$\langle e_1 | \psi(0)\rangle = \frac{1}{2}\begin{pmatrix} 1+i & \sqrt{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1+i}{2} = C_2(0)$$

$$\langle e_2 | \psi(0)\rangle = \frac{1}{2}(1-i)$$

Then the probability is $P_t = |\langle + | \psi(t)\rangle|^2$

Ex: given $[p,x] = -i$

$$[p, x^3] = [p,x]x^2 + x[p,x]x + x^2[p,x]$$
$$= -3ix^2$$

b) $[p^3, x^3] = -[x^3, p] = -\{[x^3,p]p^2 + p[x^3,p]p + p^2[x^3,p]\}$

$$= -3ix^2p^2 - 3ipx^2p - 3ip^2x^2$$

c) $[p, e^{\alpha x}]$ $\quad$ knowing $[p, x^n] = -inx^{n-1} = -i\frac{\partial}{\partial x}x^n$

then $[p, f(x)] = -i\frac{\partial}{\partial x}f(x)$

then $[p, e^{\alpha x}] = -i\alpha e^{\alpha x}$

d) $[(x+ip)^3, (x-ip)^3] = 8[a^3, a^{+3}]$

$$a = \frac{x+ip}{\sqrt{2}} \qquad a^{+} = \frac{x-ip}{\sqrt{2}} \quad \Rightarrow \quad [a, a^{+}] = 1$$

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Time evolution of Gaussian wave packets:

$$\psi(t=0,x') = \frac{1}{\pi^{1/4}\sqrt{d}}\, e^{ik(x'-X) - \frac{(x'-X)^2}{2d^2}}$$

$$\xrightarrow{t} \stackrel{!}{=} \frac{1}{\sqrt{2\pi\hbar}} \int dp'\, e^{\frac{i}{\hbar}p'x' - \frac{i}{\hbar}\frac{p'^2}{2m}t} \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{-\frac{i}{\hbar}p'X - \frac{(p'-\hbar k)^2}{2\hbar^2}d^2}$$

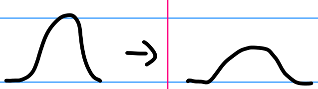
$$\psi(t,x) = \frac{\sqrt{d}}{\pi^{1/4}\Delta}\, e^{ik(x-X-\frac{\hbar k}{m}t) - \frac{(x-X-\frac{\hbar k}{m}t)^2}{2\Delta^2}}\, e^{\frac{i}{2\hbar}\frac{\hbar^2k^2}{2m}t}$$

with $\Delta^2 \stackrel{!}{=} d^2 + i\frac{\hbar}{m}t$

$$\left.\begin{aligned} \langle x\rangle &= X + \frac{\hbar k}{m}t \\ \langle p\rangle &= \hbar k \end{aligned}\right\}$$ Ehrenfest Theorem, only average x changes

$$|\psi|^2 = \frac{d}{\sqrt{\pi}|\Delta|^2}\, e^{-\frac{1}{2}(x-X-\frac{\hbar k}{m}t)^2\left(\frac{1}{\Delta^2}+\frac{1}{\bar{\Delta}^2}\right)}$$

$$\stackrel{!}{=} \frac{d}{\sqrt{\pi}\sqrt{d^4+(\frac{\hbar t}{m})^2}}\, e^{-\frac{1}{2}(x'-X-\frac{\hbar k}{m}t)^2 \frac{1}{d^2\left[1+\left(\frac{\hbar t}{md^2}\right)^2\right]}}$$

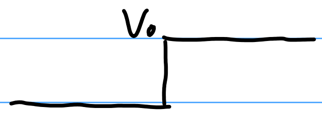


$\delta x = \frac{d}{\sqrt{2}}\sqrt{1+\left(\frac{\hbar t}{md^2}\right)^2}$ growing linearly in time.

$\delta p = \frac{\hbar}{2\sqrt{d}}$

Sectional constant Potential:

Step - Function Potential:

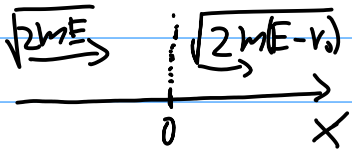


$$V(x) = V_0\,\Theta(x)$$

$$p = \begin{cases} \sqrt{2mE} & x < 0 \\ \sqrt{2m(E-V_0)} & x > 0 \end{cases}$$

classically

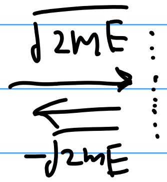
a) If $E > V_0$



$P = -\sqrt{2mE}$; $P = -\sqrt{2m(E-V_0)}$

0

b) $E < V_0$



Quantum:

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi = E\psi$$

$$-\psi'' = \psi \times \begin{cases} \frac{2m}{\hbar^2}(E - V_0) & x > 0 \\ \frac{2m}{\hbar^2}E & x < 0 \end{cases}$$

$$\psi = e^{ikx} \quad \text{where} \quad k^2 = \begin{cases} \frac{2m}{\hbar^2}(E - V_0) & x>0 \\ \frac{2m}{\hbar^2}E & x<0 \end{cases}$$

with $E>0$: let $\frac{2mE}{\hbar^2} = k^2$ $\quad \frac{2m}{\hbar^2}V_0 = q^2$

$$-\psi'' = \psi \begin{cases} k^2 - q^2 & x>0 \\ k^2 & x<0 \end{cases}$$

for $x<0$: $\psi = Ae^{ikx} + Be^{-ikx}$

$x>0$: $\psi = Ce^{i\sqrt{k^2-q^2}x} + De^{-i\sqrt{k^2-q^2}x}$

Continuity at $x=0$ $\quad \psi(x=0^+) = \psi(x=0^-)$

$\hookrightarrow A + B = C + D$

True for regular $\rightarrow$ $\psi'|_{x=0^+} = \psi'|_{x=0^-}$

$$ik(A-B) = i\sqrt{k^2-q^2}(C-D) = is(C-D)$$

1) let $D=0$ $\Rightarrow$ $A+B = C$

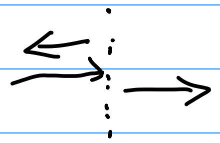
$k(A-B) = s(A+B)$

$C = \frac{2k}{k+s}A$ with $A=1$

$B = \frac{k-s}{k+s}A$

then $\psi_1 = e^{ikx} + \frac{k-s}{k+s} e^{-ikx}$ $\quad x<0$

$\psi_1 = \frac{2k}{k+s} e^{isx}$ $\quad x>0$



part of it is reflected
some of it is transmitted.

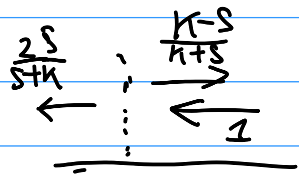
reflection coefficient: $R = \left|\frac{k-s}{k+s}\right|^2$

Transmission Coefficient: $T = \left|\frac{2k}{k+s}\right|^2 \frac{s}{k} = \frac{4ks}{(k+s)^2}$

Second solution: $A=0, \quad D=1$

$\psi_2 = \frac{2s}{s+k} e^{-ikx}$ $\quad x<0$

$\psi_2 = e^{-ikx} + \frac{k-s}{k+s} e^{isx}$ $\quad x>0$



General solution: when $E > V_0$

$\psi = C_1\psi_1 + C_2\psi_2$ with $E_1 = E_2 = \frac{\hbar^2 k^2}{2m}$ two-fold degenerate.

Now consider $0 < E < V_0$

$$0 < k^2 < q^2$$

define $\kappa^2 = q^2 - k^2 > 0$

$$\stackrel{!}{=} \frac{2m(V_0 - E)}{\hbar^2}$$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ \cancel{Ce^{\kappa x}} + De^{-\kappa x} & x > 0 \end{cases}$$

doesn't work
since goes ∞
as $x \to \infty$

With boundary condition: $A + B = D$

$$ik(A-B) = -\kappa D$$

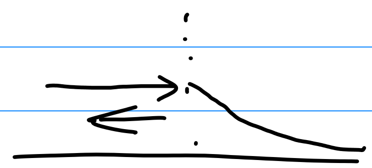
$$\hookrightarrow ik(A-B) = -\kappa(A+B)$$

then $B = -\dfrac{\kappa + ik}{\kappa - ik} A$

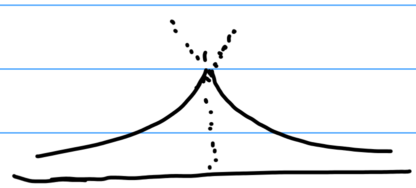
$$D = -A \frac{2i\kappa}{\kappa - ik}$$

for $0 < E < V_0$

$$\psi = \begin{cases} e^{ikx} - \dfrac{\kappa + ik}{\kappa - ik} e^{-ikx} \\ -\dfrac{2ik}{\kappa - ik} e^{-\kappa x} \end{cases}$$



For $E < 0$,



No solution
since wouldn't be
normalizable.

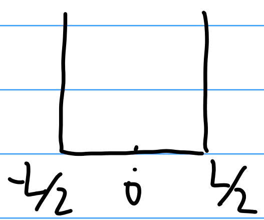
Infinite Wall: $V_0 = \infty$

$$\psi = \begin{cases} e^{ikx} - \dfrac{K+ik}{K-ik} e^{-ikx} & x < 0 \\ -\dfrac{2ik}{K+ik} e^{-Kx} & x > 0 \end{cases}$$

for $V_0 \to \infty$.

$$\psi = \begin{cases} e^{ikx} - e^{-ikx} & x > 0 \\ 0 & x < 0. \end{cases}$$

Particle in the box:



$$V(x) = \begin{cases} 0 & |x| < \frac{L}{2} \\ \infty & |x| > L/2 \end{cases}$$

$$\psi'' = -\frac{2m(E-V)}{\hbar^2}\psi = -k^2\psi$$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{for } |x| < L/2 \\ 0 & \text{for } |x| > L/2 \end{cases}$$

Boundary Condition. $\psi(x = \pm L/2) = 0$

$$Ae^{ik\frac{L}{2}} + Be^{-ik\frac{L}{2}} = Ae^{-ik\frac{L}{2}} + Be^{ik\frac{L}{2}} = 0$$

$$\begin{pmatrix} e^{ik\frac{L}{2}} & e^{-ik\frac{L}{2}} \\ e^{-ik\frac{L}{2}} & e^{ik\frac{L}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\underbrace{e^{ikL} - e^{-ikL}} = 0$$

$$\sin(kL) = 0$$

$$k = \frac{n\pi}{L}$$

since $Ae^{ik\frac{L}{2}} + Be^{-ik\frac{L}{2}} = 0$

$$A = -Be^{-ikL}$$

$$\overset{!}{=} -Be^{-i\pi n} = (-1)^{n+1} B$$

then $\psi = B((-1)^{n+1} e^{ikx} + e^{-ikx})$

with $E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2$ $\overset{!}{=} 2B\cos(\frac{\pi x}{L} n)$ for $n = 1, 3, 5 \cdots$

$2Bi\sin(\frac{\pi x}{L} n)$ for $n = 2, 4, 6 \cdots$

Symmetry:

$$\frac{-\hbar^2}{2m} \psi'' + V(x)\psi = E\psi$$

$x \to -x$:

$$\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(-x) + \underbrace{V(-x)}_{\substack{=V(x) \\ \text{if symmetric}}} \psi(-x) = E\psi(-x)$$

then $\psi(x)$ produce same sol as $\psi(-x)$

Parity Operator:

$$[P, H] = 0$$

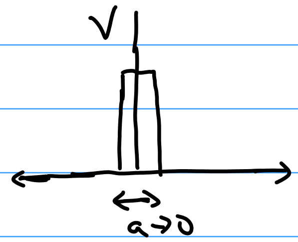
$$P\psi(x) = \psi(-x)$$

$$P^2\psi(x) = \psi(x) \Rightarrow P = \pm 1$$

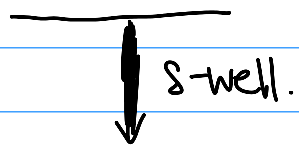
eigenvalue.

then $P\psi(x) = \psi(-x) \begin{cases} \psi(x) & : \text{even} \\ -\psi(x) & : \text{odd} \end{cases}$

Delta-Function Potential:



$$V(x) = W\delta(x)$$



Boundary Condition: $\psi(0^+) = \psi(0^-)$

$$\int_{-\epsilon}^{+\epsilon} dx \; \frac{-\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\psi\Big|_{-\epsilon}^{+\epsilon} + \int_{-\epsilon}^{\epsilon} dx\, V(x)\psi = \int_{-\epsilon}^{\epsilon} dx\, E\psi$$

If $V(x)$ is not singular, $\psi\Big|_{-\epsilon}^{+\epsilon} = 0 \Rightarrow \psi'(0^+) = \psi'(0^-)$

If $V(x) = W\delta(x)$: $\frac{-\hbar^2}{2m}\psi'\Big|_{-\epsilon}^{+\epsilon} + W\psi(0) = 0$

$$\boxed{\begin{array}{l}\text{Then} \quad \psi'(0^+) - \psi'(0^-) = \frac{2m}{\hbar^2} W \psi(0) \\ \text{with} \quad \psi'(0^+) = \psi'(0^-)\end{array}}$$ ← for delta function potential

Consider δ- well: $V(x) = -W\delta(x)$

$e^{\pm Kx}$

$$\psi(x) = \begin{cases} A e^{-Kx} & x > 0 \\ B e^{Kx} & x < 0 \end{cases}$$ for $x \to \infty$ $\psi(x) = 0$

Since $\psi(0^+) = \psi(0^-)$ $\Rightarrow A = B$

$$\psi(x) = A e^{-K|x|}$$

Apply derivative boundary condition:

$$\psi'(0^+) - \psi'(0^-) = -\frac{2m}{\hbar^2} W \psi(0)$$

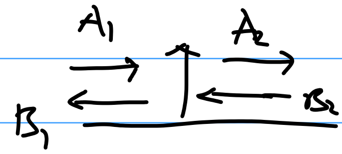
$$-2K = -\frac{2m}{\hbar^2} W$$

$$K = \frac{mW}{\hbar^2}$$

$$\psi(x) = A e^{-\frac{mW}{\hbar^2}|x|}$$

with normalization $\stackrel{!}{=} \sqrt{K}\, e^{-K|x|}$ with $E_n = \frac{-\hbar^2 K^2}{2m} = \frac{-m}{2\hbar^2} W^2$

Scattering by δ-barrier.



$$V(x) = W\,\delta(x) \qquad E = \frac{\hbar^2 w^2}{2m}$$

$$\psi(x) = \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} & x<0 \\ A_2 e^{ikx} + B_2 e^{-ikx} & x>0 \end{cases}$$

B.C. $\quad \psi(0^+) = \psi(0^-)$

$\hookrightarrow A_1 + B_1 = A_2 + B_2$

$$\psi'(0^+) - \psi'(0^-) = \frac{2mW}{\hbar^2}\psi(0)$$

$$ik(A_2 - B_2) - ik(A_1 - B_1) = \frac{2mW}{\hbar^2}(A_1 + B_1)$$

$$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = \hat{S} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}$$

outgoing $\qquad$ scattering matrix

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = T \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

$\hookrightarrow$ transfer matrix.

Let $\dfrac{mW}{\hbar^2 k} = \alpha$

$$\hat{S} = \frac{1}{1+i\alpha}\begin{pmatrix} -i\alpha & 1 \\ 1 & -i\alpha \end{pmatrix} \qquad T = \begin{pmatrix} 1-i\alpha & -i\alpha \\ i\alpha & 1+i\alpha \end{pmatrix}$$

$$\stackrel{!}{=} \begin{pmatrix} r_{LL} & t_{RL} \\ t_{LR} & r_{RR} \end{pmatrix}$$

$$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = S \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}$$

$A_1 = 1$, $B_2 = 0$ $$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = S \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix}$$

$A_1 = 0$, $B_2 = 1$ $$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}$$

$$K\left(\underbrace{|A_1|^2 + |B_2|^2}_{\text{Incoming}}\right) = K\left(\underbrace{|A_2|^2 + |B_1|^2}_{\text{outgoing}}\right)$$

$$\hookrightarrow \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}^\dagger \begin{pmatrix} A_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ A_2 \end{pmatrix}^\dagger \begin{pmatrix} B_1 \\ A_2 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}^\dagger \underbrace{S^\dagger S} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}$$

$S^\dagger S = 1$ ← S is Hermitian.

$$\begin{pmatrix} \bar{r}_{LL} & \bar{t}_{LR} \\ \bar{t}_{RL} & \bar{r}_{RR} \end{pmatrix} \begin{pmatrix} r_{LL} & t_{RL} \\ t_{LR} & r_{RR} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|r_{LL}|^2 + |t_{LR}|^2 = 1$$

$$|r_{RR}|^2 + |t_{RL}|^2 = 1$$

$$\bar{r}_{LL} t_{RL} + \bar{t}_{LR} r_{RR} = 0$$

$$\psi \to \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} \\ A_2 e^{ikx} + B_2 e^{-ikx} \end{cases}$$

$$\bar{\psi} \to \begin{cases} \bar{A}_1 e^{-ikx} + \bar{B}_1 e^{ikx} \\ \bar{A}_2 e^{-ikx} + \bar{B}_2 e^{ikx} \end{cases}$$

$\bar{A}_1 = B_1 \qquad \bar{B}_1 = A_1$

$\bar{A}_2 = B_2 \qquad \bar{B}_2 = A_2$

$$\begin{pmatrix} B_1' \\ A_1' \end{pmatrix} = S \begin{pmatrix} A_1' \\ B_2' \end{pmatrix}$$

$$\begin{pmatrix} \bar{A}_1 \\ \bar{B}_2 \end{pmatrix} = S \begin{pmatrix} \bar{B}_1 \\ \bar{A}_2 \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ B_2 \end{pmatrix} = S \begin{pmatrix} B_1 \\ A_2 \end{pmatrix}$$

$$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = S^{-1} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix} = S \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}$$

$$\hookrightarrow \boxed{S^{-1} = S = S^{+}}$$

since $S = S^{+} = \begin{pmatrix} r_{LL} & t_{RL} \\ t_{LR} & r_{RR} \end{pmatrix}$

so $t_{RL} = t_{LR} = t$. $\quad = \begin{pmatrix} r_{LL} & t \\ t & r_{RR} \end{pmatrix}$

let $\quad r_{LL} = r \qquad r_{RR} = -\bar{r}\dfrac{t}{\bar{t}}$

$$S = \begin{pmatrix} r & t \\ t & -\bar{r}\dfrac{t}{\bar{t}} \end{pmatrix}$$

With δ - barrier and α:

$$S_\alpha = \frac{1}{1+i\alpha}\begin{pmatrix} -i\alpha & 1 \\ 1 & -i\alpha \end{pmatrix} = \begin{pmatrix} r & t \\ t & \bar{r}\frac{t}{\bar{t}} \end{pmatrix}$$

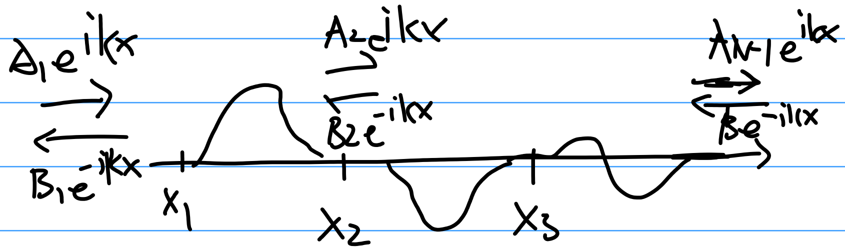
Then $T = |t|^2 = \dfrac{1}{1+\alpha^2}$

$R = |r|^2 = \dfrac{\alpha^2}{1+\alpha^2}$

transfer matrix

$$T = \begin{pmatrix} \frac{1}{\bar{t}} & -\frac{\bar{r}}{\bar{t}} \\ \frac{-r}{t} & \frac{1}{t} \end{pmatrix} = \begin{pmatrix} 1+i\alpha & -i\alpha \\ i\alpha & 1+i\alpha \end{pmatrix}$$

Transfer Matrices and Multiple Scattering



$$\begin{pmatrix} A_{N+1} \\ B_{N+1} \end{pmatrix} = T \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \qquad T = T_N T_{N-1} \cdots T_1$$

let $\psi_k = A_k e^{ik(x-x_k)} + B_k e^{-ik(x-x_k)}$

If $x_1 = x_2$ and no scattering.

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = T_0 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \Rightarrow T_0 = \mathbb{1}.$$

If $x_1 \neq x_2$, no scattering

$\psi_1 = A_1 e^{ik(x-x_1)} + B_1 e^{-ik(x-x_1)}$

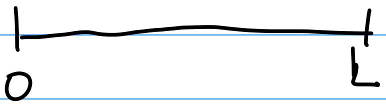
$\psi_2 = A_2 e^{ik(x-x_2)} + B_2 e^{-ik(x-x_2)}$

$\psi_2 = \psi_1$

So $\left.\begin{aligned} A_2 &= A_1 e^{ik(x_2-x_1)} \\ B_2 &= B_1 e^{-ik(x_2-x_1)} \end{aligned}\right\} \underset{a = x_2 - x_1}{T_a} = \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix}$

then $\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = T \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$

Density of States:



$$\psi(L) = \psi(0)$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad \psi = e^{\pm ikx}$$

$$\hookrightarrow e^{\pm ikL} = e^{0} = 1.$$

then $k = \frac{2\pi}{L} n$, $n = 0, \pm 1, \pm 2 + \cdots$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 n^2$$

$$\Delta n = \nu(E)\,\Delta E$$

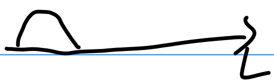
so $\frac{dn}{dE} = \nu(E)$

consider $\frac{dn_+}{dk} + \frac{dn_-}{dk} = \frac{L}{\pi}$

$$\frac{dn}{dk}\frac{dk}{dE} = \frac{dn}{dk}\left(\frac{\hbar^2 k}{m}\right)^{-1}$$

$$= \left(\frac{L}{\pi}\right)\left(\frac{m}{\hbar^2 k}\right) \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$= L\,\frac{1}{\hbar\pi}\sqrt{\left(\frac{m}{2E}\right)}$$

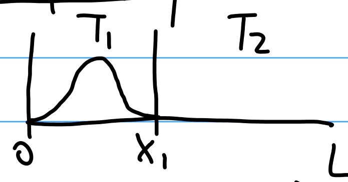
For 

$$T = T_{free\,propagation}\, T_{potential} = \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$$T = \begin{pmatrix} e^{ikL} T_{11} & e^{ikL} T_{12} \\ e^{-ikL} T_{21} & e^{-ikL} T_{22} \end{pmatrix}$$

$$\psi(L) = T\psi(0) = \psi(0) \quad \leftarrow \text{B.C.}$$

Impurity Effect on density of states.



With periodic B.C. $\phi(0) = \phi(L)$

$(L-x) \sim L$

$$T = T_L T_W$$

$$T = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with $\det(T) = 1$

Since we have B.C. $\psi(0) = \phi(L)$

$$T\phi(0) = \phi(L) = \phi(0)$$

$$\det(T - 1) = 0$$

$$\hookrightarrow = (e^{ikL} T_{11} - 1)(e^{-ikL} T_{22} - 1) - T_{12} T_{21} = 0$$

$$\stackrel{!}{=} \underbrace{\det(T_W)}_{=1} - e^{ikL} \underbrace{T_{11}}_{=\frac{1}{t}} - e^{-ikL} \underbrace{T_{22}}_{=1/t} + 1 = 0$$

$$\stackrel{!}{=} 2 - \frac{1}{t} e^{ikL} - \frac{1}{t} e^{-ikL} = 0$$

Solve for k to get $E = \frac{\hbar^2 k^2}{2m}$:

let $t = e^{i\delta}|t|$

then $2|t| - 2\cos(kL + \delta(k)) = 0$

$$kL + \delta(k) = \pm\cos^{-1}(|t|) + 2\pi n \quad , n = 0, 1, 2, \dots$$

$$\frac{d}{dk} \downarrow \quad L + \delta'(k) = \mp \frac{1}{\sqrt{1-|t|^2}} \frac{d|t|}{dk} + 2\pi \frac{dn}{dk}$$

density of state!

$$\nu(E) = \frac{m}{\hbar^2 k} \frac{dn_\pm}{dk} = \frac{m}{\hbar^2 k}\left(\frac{L}{2\pi}\left\{1 + \frac{1}{L}\delta'(k)\right\} \pm \frac{1}{2\pi}\frac{1}{\sqrt{1-|t|^2}}\frac{d|t|}{dk}\right)$$

$$\doteq \underbrace{\frac{m}{\hbar^2 k}\frac{L}{\pi}\left\{1 +\right.}_{\nu_0(E)} \underbrace{\left.\frac{1}{L}\delta'(k)\right\}}_{\delta\nu(E)}\Big)$$

$$\Rightarrow \quad \frac{\Delta\nu(E)}{\nu_0(E)} = \frac{1}{L}\delta'(k)$$

Ex: For $U = W\delta(x)$

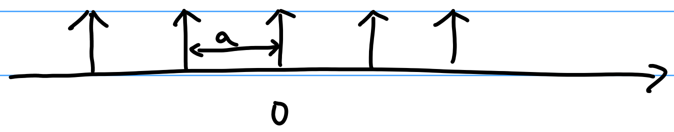
$$t = \frac{1}{1+i\alpha} = e^{i\delta}|t| \quad , \text{ then } \delta = -\tan^{-1}(\alpha)$$

then
$$\delta'(k) = -\frac{1}{1+\alpha^2}\underbrace{\frac{d\alpha}{dk}}_{-\frac{mW}{\hbar^2 k^2}} = \frac{\frac{mW}{\hbar^2}}{k^2 + \left(\frac{mW}{\hbar^2}\right)^2} = \frac{k_0}{k^2 + k_0^2}$$

then
$$\frac{\Delta\nu}{\nu_0} = \frac{1}{L}\frac{k_0}{k^2+k_0^2}$$

Infinitely Many Dirac-Delta: δ-comb:

$$U(x) = W \sum_{n=-\infty}^{\infty} \delta(x - na)$$



take $\phi(0) = \phi(Na)$

$$T = \left(T_a \, T_\alpha\right)^N$$

free propagation. $\uparrow$ (T_a) $\qquad$ $\uparrow$ δ-func (T_α)

To use periodic Boundary Condition:

$$T\,\phi(0) = \phi(Na) = \phi(0)$$

$$\hookrightarrow \det(T - \lambda) = 0$$

let $T = (T_a T_\alpha)^N = T_1{}^N$ with $\det(T_1) = 1$

To get convergence: $|\lambda_i| = 1$

$$\lambda_i = e^{i\tilde{k}a} \quad , \tilde{k} \text{ is real}$$

$$T_1 \psi = e^{\pm i\tilde{k}a} \psi \qquad \text{Bloch Theorem.}$$

use $\det(T_1 - \lambda) = 0$

$$\rightarrow \underbrace{\det(T_1)}_{1} + \lambda^2 - \lambda\left(\frac{1}{t} + \frac{1}{\bar{t}}\right) = 0$$

$$\Rightarrow \quad 1 - \lambda\left(\frac{1}{t} + \frac{1}{\bar{t}}\right) + \lambda^2 = 0$$

$$\lambda_{\pm} = \frac{1}{2}\left(\frac{1}{t} + \frac{1}{\bar{t}}\right) \pm \sqrt{\frac{1}{4}\left(\frac{1}{t} + \frac{1}{\bar{t}}\right)^2 - 1}$$

$$\lambda_+ + \lambda_- = \frac{1}{t} + \frac{1}{\bar{t}}$$
$$\lambda_+ \lambda_- = 1$$

require $|\lambda_{\pm}| = 1$, which means

$$\frac{1}{4}\left(\frac{1}{t} + \frac{1}{\bar{t}}\right)^2 - 1 < 0$$

$$\left(\frac{1}{t} + \frac{1}{\bar{t}}\right)^2 < 4$$

$$|t + \bar{t}|^2 < 4|t|^4$$

$$-2|t|^2 < \underbrace{t + \bar{t}}_{2|t|\cos\delta} < 2|t|^2$$

$$\boxed{\cos\delta < |t|}$$

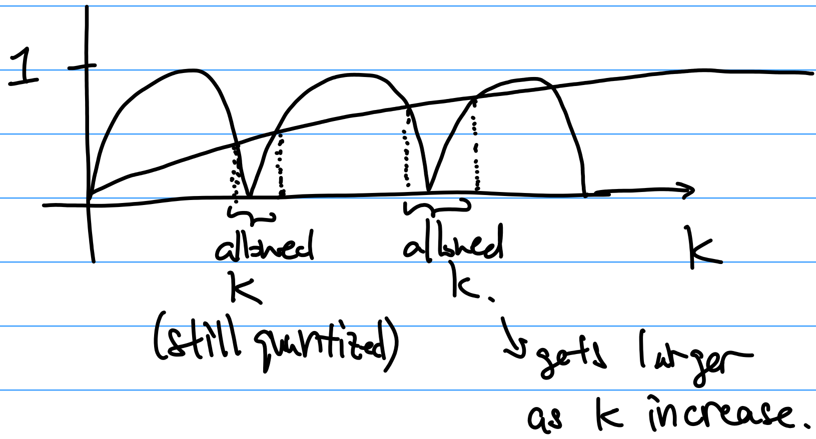
With δ - potential:

$$t = \frac{1}{1+i\alpha} e^{ika}$$

$$|t|^2 = \frac{1}{1+\alpha^2}$$

$$\delta = -\tan^{-1}(\alpha) + ka$$

then ↳ $\left|\cos\left(ka - \tan^{-1}(\alpha)\right)\right| < \frac{1}{\sqrt{1+\alpha^2}}$



Scattering Matrix and discrete spectrum:

$$S = \begin{pmatrix} r & t \\ t & -\bar{r}\frac{t}{\bar{t}} \end{pmatrix} \qquad T = \begin{pmatrix} \frac{1}{\bar{t}} & -\frac{\bar{r}}{\bar{t}} \\ -\frac{r}{t} & \frac{1}{t} \end{pmatrix}$$

with $U = -W\delta(x)$

$$E_0 = \frac{-\hbar^2 K^2}{2m} \qquad \psi = \sqrt{K}\, e^{-K|x|} \quad \text{and} \quad K = -\sqrt{\frac{2mE_0}{\hbar^2}}$$

Continuous spectrum: $E = \frac{\hbar^2 k^2}{2m}$ $\quad \alpha = \frac{mW}{\hbar^2 k}$ $\quad k = \sqrt{\frac{2mE}{\hbar^2}}$

$$t = \frac{1}{1+i\alpha} \qquad r = \frac{i\alpha}{1+i\alpha} \qquad \psi = Ae^{ikx} + Be^{-ikx}$$

$$t(E) = \frac{1}{\underbrace{1 + i\frac{mW}{\hbar^2}\frac{1}{\sqrt{\frac{2mE}{\hbar^2}}}}_{=0 \text{ when } E=-E_0}} \qquad \text{for } E>0 \quad |t| < 1$$

Harmonic Oscillator:

$$\hat{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \quad \text{for } p = -i\hbar\partial_x$$

Brute Force Approach:

$$\frac{1}{\hbar\omega}\left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2\right]\varphi(x) = E\varphi(x)\frac{1}{\hbar\omega}$$

let $\lambda = \sqrt{\frac{\hbar}{m\omega}}$ (oscillator length)

$$\left(-\frac{\lambda^2}{2}\partial_x^2 + \frac{1}{2\lambda^2}x^2\right)\varphi(x) = \frac{E}{\hbar\omega}\varphi(x)$$

let $\xi = \frac{x}{\lambda}$ and $E = \hbar\omega\left(n + \frac{1}{2}\right)$

$$\left(-\lambda^2\partial_x^2 + \xi^2\right)\varphi(x) = (2n+1)\varphi(x)$$

$$\varphi''_{\xi\xi} + (2n+1-\xi^2)\varphi(\xi) = 0$$

let $\varphi = e^{\frac{-\xi^2}{2}}f(\xi)$

$$f'' - 2\xi f' + 2nf = 0$$

Suppose $f(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$

$$\sum_{j=0}^{\infty}\left\{(j)(j-1)a_j\xi^{j-2} - 2ja_j\xi^j + 2na_j\xi^j\right\}$$

$$\hookrightarrow \sum_{j=-2}^{\infty}\left\{(j+2)(j+1)a_{j+2}\xi^j\right\} + \sum_{j=0}^{\infty}\left(-2ja_j\xi^j + 2na_j\xi^j\right)$$

$$\hookrightarrow \sum_{j=0}^{\infty}\left[(j+2)(j+1)a_{j+2} - 2ja_j + 2na_j\right]\xi^j = 0$$

then $$a_{j+2} = \frac{2(j-n)}{(j+2)(j+1)} a_j$$

as $j \to \infty$:

$$a_{j+2} = \frac{2a_j}{j}$$

consider even: $a_{2j} \sim \frac{1}{j!}$

and $\sum_j \frac{\xi^{2j}}{j!} \sim e^{\xi^2}$ so it diverges.

To have convergence:

require $a_{j+2} = 0$ at some point.

so let $j = n$. where j is integer $0, 1, 2, 3 \ldots$

For

$n=0$ $\quad f = a_0 \quad \varphi = a_0 e^{-\frac{\xi^2}{2}}$

$n=1$ $\quad f = a_1 \xi \quad \varphi = a_1 \xi e^{-\frac{\xi^2}{2}}$

$n=2$ $\quad a_2 = \frac{2(0-2)}{(0+2)(0+1)} a_0 = -2a_0$

$$\varphi = a_0 (1 - 2\xi^2) e^{-\frac{\xi^2}{2}}$$

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2/2}$$

$$\psi_n(\xi) = C_n H_n(\xi) e^{-\xi^2/2}$$

Normalization:

$$\lambda \int_{-\infty}^{\infty} \psi_m^*(\xi)\, \psi_n(\xi)\, d\xi \qquad m \leq n$$

$$\hookrightarrow = c_m^* c_n \int (-1)^n \left(e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \right) H_m(\xi) e^{-\xi^2} d\xi$$

$$\stackrel{!}{=} 0 \qquad m < n$$

$$\stackrel{!}{=} \lambda \delta_{mn} |c_n|^2 \underbrace{\int d\xi \; e^{-\xi^2}}_{\sqrt{\pi}} \underbrace{\frac{d^n}{d\xi^n} H_n(\xi)}_{2^n n!}$$

$$= \delta_{mn} \sqrt{\pi}\, 2^n n!\, |c_n|^2 \lambda$$

then $|c_n| = \dfrac{1}{\sqrt{2^n \sqrt{\pi}\, n!\, \lambda}}$

then $\psi_n = \dfrac{1}{\pi^{1/4} \sqrt{2^n n! \lambda}} H_n\left(\dfrac{x}{\lambda}\right) e^{-\frac{x^2}{2\lambda^2}}$

where $H_n(\xi) = (-1)^n e^{\xi^2} \dfrac{d^n}{d\xi^n} e^{-\xi^2}$

let $y = \zeta^2$

then $y f_{yy}'' + \left(\frac{1}{2} - y\right) f_y' + \frac{n}{2} f = 0$

$$f = \phi\left(\frac{-n}{2}, \frac{1}{2}; y\right)$$

with general form: $y f'' + (b-y) f' - af = 0$

$$f = \phi(a, b; y) = 1 + \sum_{k=1}^{\infty} \frac{a_k}{b_k} \frac{y^k}{k!}$$

Hypergeometric function.

$$f_{general} = C_1 \phi(a, b; y) + C_2 y^{1-b} \phi(a-b+1, 2-b; y)$$

general solution

$$= C_1 \phi\left(-\frac{n}{2}, \frac{1}{2}; \zeta^2\right) + C_2 \zeta \phi\left(\frac{1-n}{2}, \frac{3}{2}, \zeta^2\right)$$

$$\left.\begin{array}{lll} \text{If } n = \text{even}, & & C_2 = 0 \\ \phantom{\text{If }} n = \text{odd} & & C_1 = 0 \end{array}\right\} \text{so we converge.}$$

Hermite Polynomial:

$$H_n'' - 2\xi H_n' + 2n H_n = 0$$

∂_ξ ↳

$$H_n''' - 2H_n' - 2\xi H_n'' + 2n H_n' = 0$$

↳ $(H_n')'' - 2\xi (H_n')' + 2(n-1) H_n' = 0$

↳ Recognize $\boxed{H_n' = 2n H_{n-1}}$

$$(\xi H_n - n H_{n-1})'' - 2\xi (\xi H_n - n H_{n-1})' + 2(n+1)(\xi H_n - n H_{n-1}) = 0$$

↳ Find $\boxed{\xi H_n - n H_{n-1} = \frac{1}{2} H_{n+1}}$

Ladder operators:

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega} \right)$$

$$\overset{!}{=} \frac{1}{\sqrt{2}} \left(\xi + \partial_\xi \right) \qquad \text{for } \xi = \frac{x}{\lambda}, \lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$[a, a^\dagger] = 1$$

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \frac{m\omega^2}{4} \left[x + \frac{ip}{m\omega^2} \right] \left[x - \frac{ip}{m\omega^2} \right]$$

$$\overset{!}{=} \frac{1}{2} \hbar\omega \left[a^\dagger a + \underbrace{a a^\dagger}_{= a^\dagger a + 1} \right]$$

$$\overset{!}{=} \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

Consider $\hat{n} = a^\dagger a$

$$\hat{n} |n\rangle = n |n\rangle$$

$$n = \langle n | a^\dagger a | n \rangle = \left| a |n\rangle \right|^2 \geq 0$$

$$\boxed{\text{then } n \geq 0}$$

$$\hat{n} a^\dagger |n\rangle = a^\dagger a a^\dagger |n\rangle$$

$$\overset{!}{=} a^\dagger (\hat{n} + 1) |n\rangle$$

$$\overset{!}{=} (n+1) a^\dagger |n\rangle$$

then $a^+|n\rangle = C_n|n+1\rangle$

then a^+ is the raising operator.

same logic:

$$a|n\rangle = d_n|n-1\rangle$$

a is the lowering operator.

Take $a|n\rangle = d_l\,|n-1\rangle$

$$a^2|n\rangle = d_l\, d_{l-1}\,|n-2\rangle$$

$$a^l|n\rangle = d_l\, d_{l-1}\cdots d_{l-l+1}\,|n-l\rangle$$

$$n \in 0, 1, 2 \cdots$$

$a|0\rangle = 0$, so

$E_0 = \hbar w\left(\frac{1}{2}\right) =$ ground state.

$|0\rangle \quad a|0\rangle \quad \Rightarrow \quad \langle 0|0\rangle = 1.$

$a^+|0\rangle = C_1|1\rangle$:

$$|C_1|^2 = |C_1|^2\langle 1|1\rangle = \langle 0|a a^+|0\rangle = \langle 0|a^+a+1|0\rangle = 1$$

then $C_1 = 1$.

Generalize:

$$a^{+}|n\rangle = C_{n+1}|n+1\rangle$$

$$|C_{n+1}|^2 = \langle n+1| C_{n+1}^{*} C_{n+1} |n+1\rangle = \langle n|a^{+}a+1|n\rangle$$

$$\boxed{\begin{aligned} a^{+}|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a^{-}|n\rangle &= \sqrt{n}\,|n-1\rangle \end{aligned}}$$

$$|n\rangle = \frac{1}{\sqrt{n}} a^{+}|n-1\rangle$$

$$\stackrel{!}{=} \frac{1}{\sqrt{n}} a^{+} \frac{1}{\sqrt{n-1}} a^{+} |n-2\rangle$$

$$\vdots$$

$$\stackrel{!}{=} \frac{1}{\sqrt{n!}} (a^{+})^{n} |0\rangle$$

Matrix forms: $a^{+} = a^{+} |n\rangle\langle n|$

$$\stackrel{!}{=} \sqrt{n+1}\,|n+1\rangle\langle n|$$

$$\stackrel{!}{=} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot \\ \sqrt{1} & 0 & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ & & & & & \ddots \end{pmatrix}$$

Similarly $a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdot \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix}$

Momentum Operator in terms of a:

$$p = \frac{\hbar}{\lambda} \frac{a^\dagger - a}{\sqrt{2}\, i}$$

$$x = \lambda \frac{a^\dagger + a}{\sqrt{2}}$$

$$x|n\rangle = \frac{\lambda}{\sqrt{2}} \left(\sqrt{n+1}\, |n+1\rangle + \sqrt{n}\, |n-1\rangle \right)$$

$$p|n\rangle = \frac{\hbar}{\sqrt{2}\lambda i} \left(\sqrt{n+1}\, |n+1\rangle - \sqrt{n}\, |n-1\rangle \right)$$

$$x = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \end{pmatrix}$$

Connection back to coordinate representation:

$$a = \frac{1}{\sqrt{2}} \left(\xi + \partial_\xi \right)$$

$$a|0\rangle = 0$$

$$\frac{1}{\sqrt{2}} \left(\xi + \partial_\xi \right) \phi_0(\xi) = 0$$

$$\frac{1}{\sqrt{2}} \left(\xi \phi_0 + \phi_0' \right) = 0$$

$$\phi_0 = \frac{1}{\pi^{1/4} \sqrt{\lambda}} e^{-\frac{\xi^2}{2}}$$

$$\phi_n = \frac{1}{\sqrt{n!}} (a^+)^n \phi_0 = \frac{1}{\sqrt{n!}} \frac{1}{\pi^{1/4}\sqrt{\lambda}} \left(\frac{1}{\sqrt{2}}\right)^n (\xi - \partial_\xi)^n e^{-\frac{\xi^2}{2}}$$

$$e^{-\frac{\xi^2}{2}} = e^{\frac{\xi^2}{2}} e^{-\xi^2}$$

since $(\xi - \partial_\xi)^n e^{\frac{\xi^2}{2}} = e^{\frac{\xi^2}{2}} (-\partial_\xi)^n$

$$\phi_n = \frac{1}{\pi^{1/4}\sqrt{n!\, 2^n \lambda}}\, e^{-\frac{\xi^2}{2}} \underbrace{\left(e^{\xi^2} (-\partial_\xi)^n e^{-\xi^2} \right)}_{H_n(\xi)}$$

Supersymmetric QM:

$$B = \frac{ip + W(x)}{\sqrt{2}} \qquad B^{+} = \frac{-ip + W(x)}{\sqrt{2}} \quad , [p, x] = -i$$

$$[B, B^{+}] = W'(x)$$

$$\{B, B^{+}\} = p^2 + W^2(x)$$

$$BB^{+} = \frac{1}{2}(p^2 + W^2 + W')$$

$$B^{+}B = \frac{1}{2}(p^2 + W^2 - W')$$

Introduce $Q_1 = \begin{pmatrix} 0 & B \\ B^{+} & 0 \end{pmatrix}$ $\qquad Q_2 = \begin{pmatrix} 0 & -iB \\ iB^{+} & 0 \end{pmatrix}$

$$Q_1^2 = Q_2^2 = \begin{pmatrix} BB^{+} & 0 \\ 0 & B^{+}B \end{pmatrix} = H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

$$[H, Q_{1,2}] = 0$$

$$\{Q_i, Q_j\} = 2\delta_{ij} H$$

anti-commutator.

Think about Q-operator as $\sqrt{H}$

$W(x)$: super-potential

$$Q\psi_1 = \overset{\text{if } q\neq 0}{q}\psi_1 \Rightarrow \text{define } \psi_2 = Q_2\psi_1 \neq 0$$
$$H\psi_1 = q^2\psi_1$$

$$\hookrightarrow Q_1\psi_2 = Q_1Q_2\psi_1$$
$$= -Q_2Q_1\psi_1$$
$$= -Q_2\, q\psi_1$$
$$Q_1\psi_2 = -q\psi_2$$

$$\text{then } H\psi_2 = q^2\psi_2$$

$$H_+\psi_+ = q^2\psi_+ \qquad H\begin{pmatrix}\psi_+\\0\end{pmatrix} = q^2\begin{pmatrix}\psi_+\\0\end{pmatrix}$$

$$Q_1\begin{pmatrix}\psi_+\\0\end{pmatrix} = \begin{pmatrix}0\\B^\dagger\psi_+\end{pmatrix} \qquad H\begin{pmatrix}0\\B^\dagger\phi_+\end{pmatrix} = q^2\begin{pmatrix}0\\B^\dagger\phi_+\end{pmatrix}$$

$$H_-(B^\dagger\phi_+) = q^2 B^\dagger\phi_+$$

<u>Then:</u> $H_+ = BB^\dagger = B^\dagger B + 1 = H_- + 1$ ← <u>Relation</u>

$$H_-|0\rangle = 0 \Rightarrow H_+|0\rangle = (H_- + 1)|0\rangle = 0$$

$$\hookrightarrow H_-(B^\dagger|0\rangle) = B^\dagger|0\rangle \Rightarrow H_+(B^\dagger|0\rangle) = (H+1)(B^\dagger|0\rangle) = 2B^\dagger|0\rangle$$

$$\hookrightarrow H_-(B^{\dagger 2}|0\rangle) =$$

Example: $W(\alpha, x) = \alpha \tanh(x)$

$$W^2 = \alpha^2 \tanh^2 x = \alpha^2 \left(1 - \frac{1}{\cosh^2 x}\right)$$

$$W' = \frac{\alpha}{\cosh x}$$

$$W^2 \pm W' = \alpha^2 - \frac{\alpha^2 \mp \alpha}{\cosh^2 \alpha}$$

$$H_{\pm}(\alpha) = \frac{1}{2}\left[p^2 + \alpha^2 - \frac{\alpha(\alpha \mp 1)}{\cosh^2 x}\right]$$

$$H_+(\alpha) - H_-(\alpha-1) = \frac{1}{2}\left[\alpha^2 - (\alpha-1)^2\right]$$

Relation: $H_+(\alpha+1) = H_-(\alpha) + \frac{(\alpha+1)^2 - \alpha^2}{2}$

$$H_+ \psi_+ = q^2 \psi_+ \rightarrow H_-(B^+ \psi_+) = q^2 (B^+ \psi_+)$$

$$H_-(\alpha)\psi_\alpha = E_\alpha \psi_\alpha \xrightarrow{\text{relation}} H_+(\alpha+1)\psi_\alpha = \left[E_\alpha + \frac{(\alpha+1)^2 - \alpha^2}{2}\right]\psi_\alpha$$

shift $\alpha \to \alpha - 1$ $\hookrightarrow H_+(\alpha)\psi_{\alpha-1} = \left[E_{\alpha-1} + \frac{\alpha^2 - (\alpha-1)^2}{2}\right]\psi_{\alpha-1}$

use supersymmetry.

$$\hookrightarrow \quad H_-(\alpha)(B^+_\alpha \psi_{\alpha-1}) = \left[E_{\alpha-1} + \frac{\alpha^2 - (\alpha-1)^2}{2}\right](B^+ \psi_{\alpha-1})$$

$B_\alpha \psi_\alpha = 0 \quad \Rightarrow \psi_\alpha = \frac{c}{(\cosh)^\alpha} \quad \frac{1}{\sqrt{2}}(\partial_x + \alpha \tanh x)\psi_\alpha = 0.$

$$\left.\begin{aligned}
\psi_\alpha &\rightarrow E^0_\alpha = 0 \\
B^+_\alpha \psi_{\alpha-1} &\rightarrow E^1_\alpha = \frac{\alpha^2 - (\alpha-1)^2}{2} = \alpha - \frac{1}{2} \\
B^+_\alpha B_{\alpha-1} \psi_{\alpha-2} &\rightarrow E^2_\alpha = \frac{\alpha^2 - (\alpha-2)^2}{2} = 2\alpha - 2
\end{aligned}\right\} E_n = \frac{-(\alpha-n)^2}{2}$$

for $n = 0, 1, 2 \cdots \alpha - 1$

Another example: $\alpha = 1$, $U(x) = -\frac{1}{\cosh^2(x)}$

$$\psi_0(x) = \frac{1}{\cosh x}$$

$$H_- - \frac{1}{2} = \frac{p^2}{2} - \frac{1}{\cosh^2 x} \Rightarrow H_-(B^+ e^{ikx}) = \frac{k^2}{2} B^+ e^{ikx}$$

$$H_+ - \frac{1}{2} = \frac{p^2}{2} \Rightarrow H_+ e^{ikx} = \frac{k^2}{2} e^{ikx}$$

$$\psi_\alpha^- = B^+ e^{ikx} = \frac{1}{\sqrt{2}}\left(-\partial_x + \tanh x\right) e^{ikx} = \frac{1}{\sqrt{2}}(-ik + \tanh x) e^{ikx}$$

$$\psi_\alpha^-(x) \begin{cases} \xrightarrow{x \text{ from } +\infty} \dfrac{-ik-1}{\sqrt{2}} e^{ikx} & \text{with } A_1 = \frac{1}{\sqrt{2}}(-ik-1) \\ \xrightarrow{x \text{ from } -\infty} \dfrac{-ik+1}{\sqrt{2}} e^{ikx} & \text{with } A_2 = \frac{1}{\sqrt{2}}(-ik+1) \end{cases} \left\} |t|^2 = \left|\frac{A_2}{A_1}\right|^2 = 1 \right.$$

The Fundamental properties of Schrodinger Eq:

$$H = \frac{p^2}{2m} + U(x,y,z) \quad , \quad H\psi = E\psi$$

1) ψ : single-valued, continuous.
It is continuous even if U is not continuous

2) $\vec{\nabla}\psi$: continuous except when $U \to \infty$

3) In region $U \to \infty$, $\psi = 0$, and $\vec{\nabla}\psi$ is discontinuous

4) If U is finite everywhere, ψ is finite everywhere.

5) If $U_{min} = \min\{U(x,y,z)\}$, then $E_n \geq U_{min}$.

since $E_n = \langle \psi | \frac{p^2}{2m} + U | \psi \rangle \geq \langle \psi | U_{min} | \psi \rangle$

6) If $U(x,y,z) \to 0$, as $r \to \infty$,

$E < 0$: Discrete [Bound State] $\int |\psi|^2 d\vec{r} \leq \infty$

$E > 0$: continuous $\int |\psi|^2 d\vec{r} = \infty$

7) If $E_n < U$ somewhere, (classically forbidden region)

then $|\psi|$ decays, but $|\psi| \neq 0$.

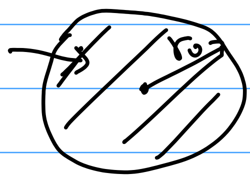
8) Suppose $U(x,y,z) > 0$ everywhere and $U \to 0$ at ∞

then there is no discrete spectrum
or no bound state.

9) If $U \to -\infty$ at somewhere, suppose origin:

$$U = -\alpha r^{-s} \quad (\alpha > 0)$$

finite ψ



$\psi_{out} = 0$

$$\delta r = r_0 \qquad \delta p = \frac{\hbar}{r_0}$$

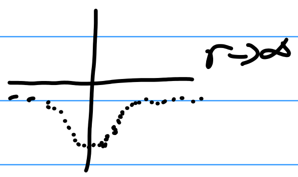
$$\left.\begin{array}{l} T \sim \dfrac{\hbar^2}{2mr_0^2} \\ U \sim \dfrac{-\alpha}{r_0^s} \end{array}\right\} E \sim \frac{\hbar^2}{2mr_0^2} - \frac{\alpha}{r_0^s}$$

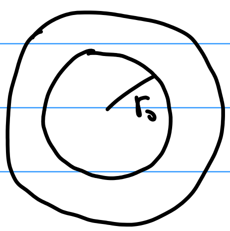
If $s < 2$, it doesn't fall to the center and spectrum is bounded from below.

If $s > 2$, then particle falls to center.

If $s = 2$, then depends on $\left(\frac{\hbar^2}{2m} - \alpha\right)$

10)





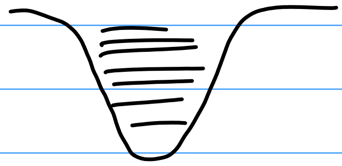
$$U = -\alpha r^{-s} \quad \alpha > 0$$

$$T = \frac{\hbar^2}{m \Delta r^2}$$

$$E \sim \frac{\hbar^2}{\beta^2 m r_0^2} - \frac{\alpha}{r_0^s}$$

$s < 2$: $\psi \sim e^{-\sqrt{-E}\, r}$, there are ∞ bound states.

$E \to 0$



$s > 2$: No bound state ?

For 1D Motion: Properties

1) None of the energy levels a
a discrete spectrum (bound state) is degenerate.

proof: If $\frac{\psi_1''}{\psi_1} = \frac{\psi_2''}{\psi_2} = \frac{2m}{\hbar^2}(U-E)$

$$0 = \psi_1''\psi_2 - \psi_2''\psi_1 = \big(\underbrace{\psi_1'\psi_2 - \psi_2'\psi_1}_{=\text{const.}}\big)'$$

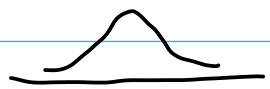
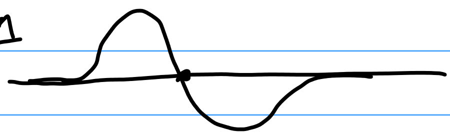
If they're bound state, $\psi_1 = \psi_2 = 0$ as $x \to \infty$
then const $= 0$.

or $\psi_1'\psi_2 = \psi_2'\psi_1$

$\frac{\psi_1'}{\psi_1} = \frac{\psi_2'}{\psi_2} \Rightarrow \psi_1 = C\psi_2$

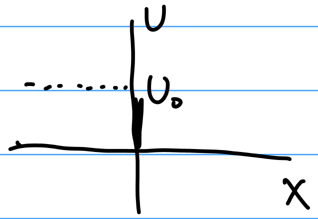
2) Oscillation Theorem:

$\psi_n(x)$ = Bound state with E_n, $E_i < E_{i+1} < E_{i+2}$

ψ_0  ψ_1 

so $\psi_n(x)$ vanish or oscillate n-times

3) Generalized step function:



$$U(x) = \begin{cases} U_0 > 0 & \text{as } x \to -\infty \\ 0 & \text{as } x \to +\infty \end{cases}$$

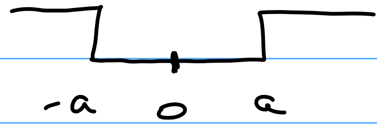
There are three cases.

$e^{-\kappa x}$ 1) Discrete spectrum: $U_{min} < E < 0$ ($U_{min} < 0$), non-degenerate, bound state (finite motion)

$e^{-\kappa x} + e^{ikx}$ 2) $0 < E < U_0$, continuous spectrum, non-degenerate, semi-finite motion.

$e^{ikx} + e^{-ikx}$ 3) $U_0 < E$, continuous 2-fold degenerate (infinite motion).

4) Shallow Potential Well:



$$-\int_{-\infty}^{\infty} U(x)\,dx > 0, \quad \text{so} \quad U(x) < 0.$$

With Schrodinger: $\frac{\psi''}{\psi} + \frac{2m}{\hbar^2}(E-U) = 0$

Since $\frac{\psi''}{\psi} = \left(\frac{\psi'}{\psi}\right)' + \left(\frac{\psi'}{\psi}\right)^2$

$$\int_{-a}^{a} dx\left[\left(\frac{\psi'}{\psi}\right)' + \left(\frac{\psi'}{\psi}\right)^2\right] + \frac{2m}{\hbar^2}\int_{-a}^{a} E\,dx = \frac{2m}{\hbar^2}\int_{-a}^{a} U(x)\,dx$$

$\psi = e^{-\kappa|x|}$

assume $\kappa a \ll 1$

$$\underbrace{\frac{\psi'}{\psi}\Big|_{-a}^{a}}_{-2\kappa} + \underbrace{\int_{-a}^{a}\left(\frac{\psi'}{\psi}\right)^2 dx}_{\sim \kappa^2 a} + \frac{2m}{\hbar^2}(E)(2a) = \frac{2m}{\hbar^2}\int_{-a}^{a} U(x)\,dx$$

$$\frac{2m}{\hbar^2}2a\left(-\frac{\hbar^2}{2m}\kappa^2\right) = \frac{2m}{\hbar^2}\int_{-\infty}^{\infty} U(x)\,dx.$$

ignore since $\propto \kappa^2$

$$\kappa = -\frac{m}{\hbar^2}\int_{-\infty}^{\infty} U(x)\,dx$$

The variational Principle:

constraint $\int |\psi|^2 d^3r = 1$

$$I(\psi(x), \psi^*(x)) = \int \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + U(r)|\psi|^2 - E|\psi|^2 \right) d^3r$$

Lagrange Multiplier, E

$$\delta I = I(\psi(x) + \delta\psi, \psi^* + \delta\psi^*) - I(\psi(x), \psi^*(x))$$

$$\overset{!}{=} \int \frac{\hbar^2}{2m} (\nabla \delta\psi^* \nabla\psi + \nabla\psi^* \nabla\delta\psi) + (U-E)(\delta\psi^*\psi + \psi^*\delta\psi) = 0$$

$$\overset{!}{=} \int \nabla\left(\frac{\hbar^2}{2m}(\delta\psi^*\nabla\psi + \nabla\psi^*\delta\psi)\right) + \delta\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2\psi + (U-E)\psi\right) + c.c.$$

If $\frac{\delta I}{\delta \psi^*} = 0$ then $\left[-\frac{\hbar^2}{2m}\nabla^2 + U\right]\psi = E\psi$

Alternative: $\delta\langle H \rangle = \dfrac{\delta\left(\int \psi^* H \psi \, d^3r\right)}{\delta\left(\int \psi\psi^* d^3r\right)}$

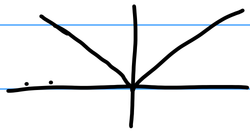
Examples:

Choose trial wave function, $\psi(\vec{r}, \underbrace{\alpha, \beta, \gamma \cdots}_{\text{variational parameters.}})$

$$\langle H \rangle = E(\alpha, \beta, \gamma \cdots)$$

minimize E over $\alpha, \beta, \gamma, \cdots$ find E_{min}.

Consider:



$U(x) = g|x|$

Choose $\psi_\alpha(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2}$

$$\langle H \rangle = \int_{-\infty}^{\infty} \psi^* \left(\frac{-\hbar^2}{2m} \partial_x^2 + g|x| \right) \psi \, dx$$

$$E = \frac{\hbar^2 \alpha}{2m} + \frac{g}{\sqrt{2\pi\alpha}}$$

$$\frac{\partial E}{\partial \alpha} = \frac{\hbar^2}{2m} - \frac{1}{2} \frac{g}{\sqrt{2\pi}\, \alpha^{3/2}} = 0$$

$$\alpha_0 = \left(\frac{gm}{\sqrt{2\pi}\, \hbar^2} \right)^{2/3}$$

$$E(\alpha_0) = \frac{1}{\pi^{1/3}} \left(\frac{1}{2^{4/3}} + \frac{1}{2^{1/3}} \right) \left(\frac{g^2 \hbar^2}{m} \right)^{1/3}$$

$$\doteq 0.813 \left(\frac{g^2 \hbar^2}{m} \right)^{1/3}$$

Excited States: use orthognolity to find excited state.

$m = 0, 1, \dots n$ $\quad \int \psi_i^* \psi_m = 0$

$$\psi(r) = \sum_{i=1}^{N} C_i \chi_i(r)$$ Rasleigh - Ritz Method.

Sudden Perturbation: (Time-dependent Schrodinger Eq)

$$H = H(t) = \begin{cases} H_0 & t<0 \\ H_0 + V & t>0 \end{cases}$$

$$H(t) = H_0 + \theta(t)\, V$$

$$\int_{-\tau}^{\tau} dt \; i\hbar\, \partial_t \psi = H(t)\psi$$

$$i\hbar\, \psi \Big|_{-\tau}^{\tau} = \int_{-\tau}^{\tau} dt \; H(t)\,\psi(t)$$

$$i\hbar\left[\psi(\tau) - \psi(-\tau)\right] \cong 0 \qquad \text{as } \tau \to 0$$

$$i\hbar\; \psi(0^+) - \psi(0^-) = 0$$

which implies ψ is continuous

Apply new-H is applied, wavefunction stays same, but evolves slowly

Shrodinger vs. Heinsenberg picture:

$$|\alpha\rangle_t = U(t,t_0)\,|\alpha\rangle_{t_0}$$

evolution operator, if H time-independent $U = e^{-\frac{i}{\hbar}H(t-t_0)}$

$$_t\langle\beta|X|\alpha\rangle_t = {}_{t_0}\langle\beta|U^{\dagger}XU|\alpha\rangle_{t_0}$$

Schrodinger picture

state is function of time, observable doesn't change.

$|\alpha\rangle_t = U|\alpha\rangle_{t_0}$

Heinsenberg Picture

observable function of time

$X(t) = U^{\dagger}XU$

For $t_0 = 0$, time-independent $U(t,t_0) = U(t) = e^{-\frac{i}{\hbar}Ht}$

Heinsenberg operator:

$$A^{(H)}(t) = U^{\dagger}(t)\,A^{(S)}\,U(t)$$

assume no time-dependent.

$$A^{(H)}(t=0) = A^{(S)}$$

$$|\alpha, t_0=0; t\rangle_H = |\alpha, t_0=0\rangle$$ ← states don't evolve.

$$|\alpha; t_0=0; t\rangle_S = U(t)\,|\alpha, t_0=0\rangle$$

$$_S\langle \alpha, t_0=0; t | A^{(S)} | \alpha, t_0=0; t\rangle_S = \langle \alpha, t_0=0 | \underbrace{U^\dagger A^{(S)}}_{A^{(H)}} U | \alpha, t_0=0\rangle$$

Heinsenberg Equation of Motion:

Assume $A^{(S)}$ doesn't dependent on t.

$$\frac{d}{dt}\left(A^{(H)}\right) = \frac{d}{dt}\left(U^\dagger A^{(S)} U\right)$$

$$\overset{!}{=} \frac{d}{dt}U^\dagger A^{(S)} U + U^\dagger \cancel{\frac{dA^{(S)}}{dt}} U + U^\dagger A^{(S)} \frac{dU}{dt}$$

If time-independent: $U = e^{-\frac{i}{\hbar}Ht} \Rightarrow \frac{\partial U}{\partial t} = \frac{1}{i\hbar} H U$

$$\frac{\partial U^\dagger}{\partial t} = \frac{-1}{i\hbar} H U^\dagger$$

$$\hookrightarrow = \frac{-1}{i\hbar} U^\dagger H (U U^\dagger) A^{(S)} U + U^\dagger A^{(S)} (U U^\dagger) \frac{1}{i\hbar} H U$$

$$\overset{!}{=} \frac{1}{i\hbar}\left[U^\dagger A^{(S)} U, U^\dagger H U\right]$$

$$\overset{!}{=} \frac{1}{i\hbar}\left[A^{(H)}, H\right]$$

then $$\boxed{\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}\left[A^{(H)}, H\right]}$$ Heinsenberg EOM

vs. classical: $\frac{dA}{dt} = \{A, H\}$

Ex: Free particle.

Note. $[x, F(\vec{p})] = i\hbar \frac{\partial F}{\partial p_i}$ $\qquad [p, G(\vec{x})] = -i\hbar \frac{\partial G}{\partial x}$

with $H = \frac{p^2}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$

$$\frac{dP}{dt} = \frac{1}{i\hbar}[P, H] = 0$$

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x, H] = \frac{1}{i\hbar} i\hbar \frac{p_i}{m} = \frac{p_i}{m} = \frac{p_i(t=0)}{m}$$

$$x(t) = x_i(0) + \frac{p_i(0)}{m} t$$

Remark: $\quad [x_i(0), x_j(0)] = 0$

$$[x_i(t), x_j(0)] = [x_i(0) + \frac{p_i(0)}{m} t, x_j(0)]$$

$$\overset{!}{=} -i\hbar \frac{t}{m} \delta_{ij}$$

Ex 2: $H = \frac{p^2}{2m} + V(x)$

$$\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, V(x)] = -\frac{\partial}{\partial x_i} V(x).$$

$$\frac{dx_i}{dt} = \frac{p_i}{m}$$

$$\underbrace{\frac{d^2 x_i}{dt}}_{a} = \frac{1}{m}\frac{dp_i}{dt} = \frac{1}{m} \underbrace{-\frac{\partial}{\partial x} V(x)}_{F}$$

$$\boxed{m \frac{d^2}{dt^2}\langle \vec{x} \rangle = \frac{d}{dt}\langle p \rangle = -\langle \vec{\nabla} V \rangle \quad \leftarrow \text{Eh renfest.}}$$

Transition Amplitude:

prepare system $|a\rangle$ at $t=0$, what is probability it goes to $|b\rangle$ at t ?

$U(t)|a\rangle$: state at t:

$\langle b|U|a\rangle$: Transition amplitude for $|a\rangle \to |b\rangle$

$|\langle b|U|a\rangle|^2$: transition probability.

Transition Amplitude (applied to a single particle)

$$\langle \vec{x}''|U(t,t_0)|\vec{x}'\rangle = \underset{\text{propagator}}{\underset{\uparrow}{K(\vec{x}''t;\vec{x}',t_0)}} \qquad t>t_0$$

$$= 0 \qquad = t<t_0$$

$|\psi\rangle_t = U|\psi\rangle_{t_0}$

$$\langle\vec{x}''|\psi\rangle_t = \int d\vec{x}' \langle x''|U(t,t_0)|x'\rangle\langle x'|\psi_0\rangle_{t_0}.$$

$$\psi(\vec{x}'',t) = \int d\vec{x}' \, K(\vec{x}'',t;\vec{x}',t_0)\,\psi_0(\vec{x}';t_0)$$

$$K(\vec{x}'',t;\vec{x}',t_0) = \langle\vec{x}''|e^{-\frac{i}{\hbar}H(t-t_0)}|\vec{x}'\rangle$$

6PM - S-240:

propagator $\rightarrow$ $K(\vec{x}'', t, \vec{x}', t_0) = \langle \vec{x}'' | e^{-\frac{i}{\hbar}H(t-t_0)} | \vec{x}' \rangle \theta(t-t_0)$

$i\hbar \frac{\partial}{\partial t} K = \langle \vec{x}'' | H \, e^{-\frac{i}{\hbar}H(t-t_0)} | \vec{x}' \rangle$

$\langle \vec{x}'' | H | \vec{x}' \rangle = \delta(\vec{x}'' - \vec{x}') \left[-\frac{\hbar^2}{2m} \nabla_x^2 + V(x) \right]$

$|\vec{x}''\rangle\langle\vec{x}'|$

$= \int dx \, \langle x'' | H | x \rangle \langle x | e^{-\frac{i}{\hbar}H(t-t_0)} | x'' \rangle$

$\stackrel{!}{=} \int dx \, \delta(x''-x) \left(-\frac{\hbar^2}{2m} \nabla_x^2 + V(x) \right) K(x'', t, x', t_0)$

$\stackrel{!}{=} \left[i\hbar \frac{\partial}{\partial t} - \left(-\frac{\hbar^2}{2m} \nabla_x^2 + V(x'') \right) \right] K(x'', t, x', t_0)$

$K(x'', t, x', t_0) = \langle \vec{x}'' | e^{-\frac{i}{\hbar}H(t-t_0)} | x' \rangle$

$\stackrel{!}{=} \delta(x'' - x')$

$\therefore \left[i\hbar \frac{\partial}{\partial t} - \left(-\frac{\hbar^2}{2m} \nabla_x^2 + V(x) \right) \right] K(x'', t, x', t_0) = \delta(x''-x') \delta(t-t_0)$

also Green's Function

$\hookrightarrow \psi(x'', t) = \int dx' \, K(x'', t, x', t_0) \psi_0(x', t_0)$

Calculation of $K = \sum \langle x'' | e^{\frac{-i}{\hbar} H(t-t_0)} | x' \rangle$

__Ex__ : Free Particle: $H = \frac{p^2}{2m}$

$$K = \sum_{\substack{a' \\ \text{eigenstate.}}} \langle x'' | e^{\frac{-i}{\hbar} H(t-t_0)} | x' \rangle$$

$$\stackrel{!}{=} \sum \underbrace{\langle x'' | a' \rangle}_{\phi_{a'}(x'')} \langle a' | e^{\frac{-i}{\hbar} H(t-t_0)} | a' \rangle \underbrace{\langle a' | x' \rangle}_{\psi_{a'}(x')}$$

$$H | p' \rangle = \frac{p^2}{2m} | p' \rangle = \frac{p'^2}{2m} | p' \rangle$$

$$\langle x' | p' \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p' x'}$$

$$K = \frac{1}{2\pi\hbar} \int dp' \, e^{\frac{i}{\hbar} p'(x''-x')} \, e^{\frac{-i}{\hbar} \frac{p'^2}{2m}(t-t_0)}$$

$$\stackrel{!}{=} \sqrt{\frac{m}{2\pi i \hbar (t-t_0)}} \, e^{\frac{i}{\hbar} \frac{m}{2} \frac{(x''-x')^2}{t-t_0}}$$

Ex 2: Harmonic Potential: $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$

$$\frac{E_n}{\hbar} = \omega(n+\tfrac{1}{2})$$

$$\xi = \frac{x}{\lambda}, \quad \lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$u_n(x) = \frac{1}{2^{n/2}\sqrt{n!}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{\frac{-m\omega^2 x'}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

$$K = \sum_{n=0}^{\infty} \frac{1}{2^n\, n!\sqrt{\pi}\,\lambda}\, e^{-\xi^2/2}\, e^{\frac{\xi'^2}{2}} H_n(\xi')H_n(\xi)\, e^{-i(n+\frac{1}{2})\omega(t-t_0)}$$

Identity: $$e^{-(\xi^2+\eta^2)}\sum_{n=0}^{\infty}\frac{s^n}{n!\,2^n} H_n(\xi)H_m(\xi) = \frac{1}{\sqrt{1-s^2}}\, e^{-\frac{\xi^2+\eta^2-2s\eta\xi}{1-s^2}}$$

then $s = e^{-i\omega(t-t_0)}$

Result:

$$K(x'', t; x', t_0) = \frac{1}{\lambda}\sqrt{\frac{1}{2\pi i \sin(\omega(t-t_0))}}\; e^{\frac{i}{\hbar}\frac{(\xi^2+\xi'^2)\cos\omega(t-t_0) - 2\xi\xi'}{\sin\omega(t-t_0)}}$$

Composition Properties of K:

$$\psi(x_0,t_0) \rightarrow \psi(x_1,t_1) \longrightarrow \psi(x_2,t_2)$$

$$K(x_2,t_2;x_0,t_0)=\int dx_1\, K(x_2,t_2;x_1,t_1)\, K(x_1,t_1;x_0,t_0)$$

$$\langle x''', t''' | x', t' \rangle = \int d^3x'' \langle x''' t''' | x'' t'' \rangle \langle x'' t'' | x', t' \rangle$$

$$K(x,t;x_0,t_0)=\int d^3x_1 \cdots d^3x_N\, K(x,t;x_N,t_N)\cdots K(x_1,t_1;x_0,t_0)$$

Multi-Dimensional wave:

$$H = \frac{-\hbar^2}{2m}\nabla^2 + V(\vec{x})$$

1) Free Particle: $H = \frac{p^2}{2m}$

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\vec{r}} \quad \text{with} \quad E_k = \frac{\hbar^2 \vec{k}^2}{2m} \qquad \vec{p} = \hbar\vec{k}$$

2) Particle in a box:

w/ periodic B.C.: $\psi(0, y, z) = \psi(L_x, y, z)\, e^{i\alpha_x}$

$$\frac{-\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\psi = E\psi$$

separation of variable:

let $\psi(x, y, z) = \psi_1(x)\,\psi_2(y)\,\psi_3(z)$

$$\frac{-\hbar^2}{2m}\left(\frac{\psi_1''}{\psi_1} + \frac{\psi_2''}{\psi_2} + \frac{\psi_3''}{\psi_3}\right) = E$$

$$\sum_i^3 \frac{-\hbar^2}{2m}\psi_i'' = E_i\psi_i$$

$$\psi = e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z} = e^{i\vec{k}\cdot\vec{r}}$$

$$E = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2 + k_z^2\right)$$

n_x = any integer, $-1, 0, 1 \ldots$

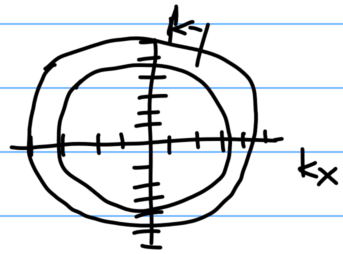
$$k_x = \frac{2\pi}{L_x} n_x \quad , \quad k_y = \frac{2\pi}{L_y} n_y \quad , \quad k_z = \frac{2\pi}{L_z} n_z$$

With Normalization:,

$$\int |\psi|^2 \, dx\,dy\,dz = |c|^2 L_x L_y L_z = 1$$

$$|c|^2 = \frac{1}{V}$$

$$\psi = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}}$$



$$dN = dn_x \, dn_y \, dn_z$$

$$\stackrel{!}{=} \frac{dk_x}{\frac{2\pi}{L_x}} \frac{dk_y}{\frac{2\pi}{L_y}} \frac{dk_z}{\frac{2\pi}{L_z}}$$

$$DoS = \frac{dN}{dE}$$

3D $$dN \stackrel{!}{=} \frac{V}{(2\pi)^3} d^3k = \frac{V}{(2\pi)^3} 4\pi k^2 dk = V \frac{m}{2\pi^2 \hbar^2} \sqrt{\frac{2mE}{\hbar}} \, dE$$

2D $$dN = \frac{A}{(2\pi)^2} 2\pi k \, dk = \frac{A}{(2\pi)^2} 2\pi \cancel{k} \frac{\hbar^2}{m} \frac{E}{\cancel{k}}$$

1D $$dN = \frac{L}{2\pi} dk \; (2) = \frac{L}{2\pi} 2 \frac{m}{\hbar} \frac{dE}{\sqrt{2mE}}$$

↑ extra factor of 2 to go from $0 \to \infty$

$$E = \frac{\hbar^2 k^2}{2m} \rightarrow k = \frac{\sqrt{2mE}}{\hbar}$$

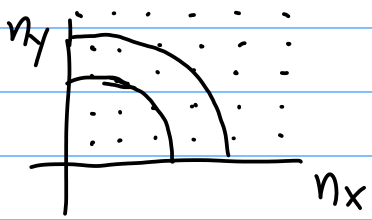
or $$dE = \frac{\hbar^2}{m} k \, dk \qquad dk = \frac{1}{2} \frac{\sqrt{2m}}{\hbar} \frac{1}{\sqrt{E}}$$

b) Now consider <u>hard B.C.</u>:

$$\frac{-\hbar^2}{2m}\psi_1'' = E_1\psi_1$$

↳ with solution: $\psi_1 = \sin(k_x x)$, $k_x = \frac{\pi}{L_x} n_x$, $n_x = 1, 2, 3 \dots$

then



→ <u>2D Harmonic Oscillator:</u>

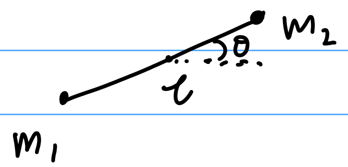
$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 x^2$$

$$\stackrel{!}{=} H_x + H_y$$

$$\psi(x,y) = \psi(x)\psi(y)$$

Ex: Plane Rotator:

→ 2 particles connected by rod:



$$\mathcal{L} = \frac{I\dot{\varphi}^2}{2} \qquad P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I\dot{\varphi}$$

$$H = P_\varphi \dot{\varphi} - \mathcal{L}$$

$$\stackrel{!}{=} \frac{P_\varphi^2}{I} - \frac{I}{2}\left(\frac{P_\varphi}{I}\right)^2 = \frac{P_\varphi^2}{2I}$$

$$\frac{\partial H}{\partial P_\varphi} = \dot{\varphi} = \frac{P_\varphi}{I} = \text{const} \quad \dot{P_\varphi} = \frac{-\partial H}{\partial \varphi} = 0.$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial P_\varphi}\right) = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow \frac{P_\varphi}{I} = \text{const.}$$

Now work with Quantum: $H = \frac{P_\varphi^2}{2I}$

$$\frac{-\hbar^2}{2I}\partial_\varphi^2 \psi = E\psi$$

then $\psi = e^{im\varphi}$ $\qquad E_n = \frac{\hbar^2 m^2}{2I}$, $m^2 = \frac{2IE_n}{\hbar^2}$

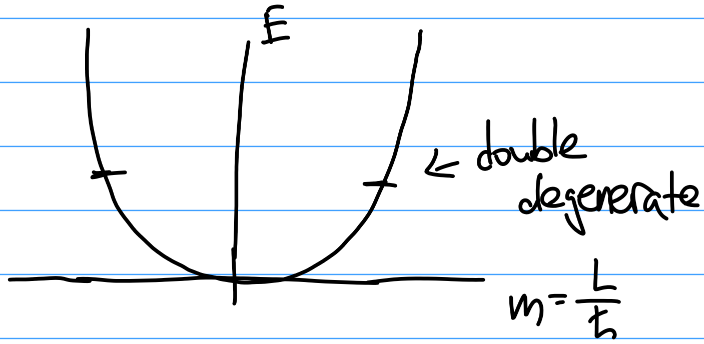
with BC. $\psi(2\pi) = \psi(0)$

$$e^{i2\pi m} = 1 \quad \Rightarrow m = 0, \pm 1, \pm 2 \dots$$

$$H = \frac{L^2}{2I} \Rightarrow H = \frac{-\hbar^2}{2I}\partial_t^2$$

$$\psi_n = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \qquad [L, \varphi] = -i\hbar$$

$\hookrightarrow L\psi_m = \hbar m \psi_m$



Since we can add arbitrary phase.

$$\psi(2\pi) = e^{i\alpha}\psi(0)$$

then $E_n = \frac{\hbar^2 m^2}{2I} = \frac{\hbar^2\left(n + \frac{\alpha}{2\pi}\right)^2}{2I}$

Central Potential in 2D:

$$H = \frac{\vec{p}^2}{2m} + V(r)$$

$$\left(\frac{-\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2\right) + V(r)\right)\psi = E\psi$$

should use r, θ.

with Polar Coordinates:

$$x = r\cos\varphi \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\varphi \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

then $$\partial_x = \cos\varphi\,\partial_r - \frac{\sin\varphi}{r}\partial_\varphi$$

$$\partial_y = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\partial_\varphi$$

let $$\Delta = \partial_x{}' + \partial_y{}' = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2\varphi$$
$$\stackrel{!}{=} \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial^2\varphi$$

Then: $$\left[\frac{-\hbar^2}{2m}\left(\frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\varphi{}^2\right) + V\right]\psi = E\psi$$

let $\psi(r) = R(r)\,\phi(\varphi)$

then $$\frac{-\hbar^2}{2m}\left[\frac{1}{R}\frac{1}{r}\partial_r(r\partial_r R) + V\right] - \frac{\hbar^2}{2mr^2}\frac{\partial_\varphi^2\phi}{\phi} = E'$$

Since $\frac{1}{\Phi}\partial_\varphi^2\Phi = -M^2$, then $\Phi = e^{iM\varphi}$

$$-\frac{\hbar^2}{2m}\frac{1}{r}\partial_r(r\partial_r R) + VR + \frac{\hbar^2 M^2}{2mr^2}R = ER$$

let $R = \frac{1}{\sqrt{r}}\tilde{R}(r)$

$$\frac{1}{r}\partial_r(r\partial_r R) = \frac{1}{\sqrt{r}}\left(\tilde{R}'' + \frac{1}{4r^2}\tilde{R}\right)$$

$$-\frac{\hbar^2}{2m}\tilde{R}'' + \left[V(r) + \frac{\hbar^2(m^2-\frac{1}{4})}{2mr^2}\right]\tilde{R} = E\tilde{R}$$

let $R_{M,E}$ be the solution:

$$\psi(r,\varphi) = C e^{im\varphi} R_{M,E}(r)$$

Consider free-particle: $V=0$.

$$\frac{-\hbar^2}{2m}\frac{1}{r}\partial_r(r\partial_r R)+\frac{\hbar^2 M^2}{2mr^2}R=ER \quad \text{with } E=\frac{\hbar^2k^2}{2m}$$

$$r^2R''+rR'+(k^2r^2-M^2)R=0$$

let $kr=\xi$

then $\xi^2R''+\xi R'+(\xi^2-M^2)R=0$

Bessel Equation

With solution: $R(r)=J_M(\xi)=J_M(kr)$ ← Bessel Function

$$J_M(\xi)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+|M|+1)}\left(\frac{\xi}{2}\right)^{m+|M|}$$

↑ Euler Gamma function.

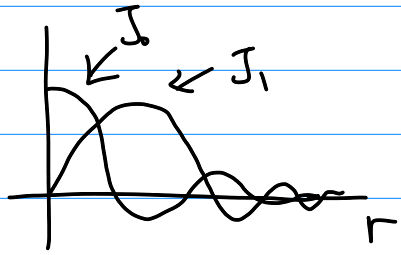
$$\Gamma(k+1)=k!$$

$$J_{-M}(\xi)=(-1)^M J_M(\xi)$$

All together: $\psi(r,\varphi)=J_{|M|}(kr)\,e^{iM\varphi}$

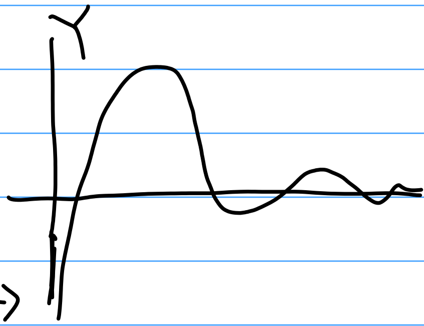
Bessel Function:

First kind →



second kind

Divergent at $r=0$ →



$$J_M(\alpha_{iM}) = 0$$

↑ zero point of J_M.

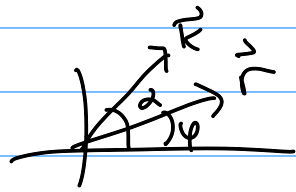
$$\alpha_{iM} = k R_0$$

then $R_0 = \dfrac{\alpha_{iM}}{k}$

Alternative Derivation:

$$\frac{-\hbar^2}{2m}\left(\partial_x^2+\partial_y^2\right)\psi = E\psi$$

$$\psi = e^{i\vec{k}\vec{r}} , \quad E_k = \frac{\hbar^2 k^2}{2m} , \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$



$$\psi_E = \sum A_{\vec{k}}\, e^{i\vec{k}\vec{r}}$$

$$\stackrel{!}{=} \int \frac{d\alpha}{2\pi} A_\alpha\, e^{ikr\cos(\alpha-\varphi)}$$

let $$A_\alpha = \sum_{M=-\infty}^{M=\infty} \tilde{A}_M\, e^{iM\alpha}$$

$$\psi_E = \sum_\alpha A_\alpha\, e^{i\vec{k}\cdot\vec{r}}$$

$$\psi_E \stackrel{!}{=} \sum_M \tilde{A}_M \int \frac{d\alpha}{2\pi}\, e^{ikr\cos(\alpha-\varphi)}\, e^{iM\alpha}$$

$$\stackrel{!}{=} \sum_M \tilde{A}_M\, e^{iM\varphi} \underbrace{\int \frac{d\alpha}{2\pi}\, e^{ikr\cos\alpha}\, e^{iM\alpha}}_{J_M(k\vec{r})}$$

Bessel's Equation:

$$z^2 w'' + z w' + (z^2 - \nu^2) w = 0$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^m$$

If ν = fraction:

then J_ν and $J_{-\nu}$ form fundamental set.

If ν = integer:

then $J_{-m}(z) = (-1)^m J_m(z)$, & they're not independent

then $$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)}$$

J_ν and Y_ν are always fundamental sets.

as $z \to \infty$:

$$J \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)$$

$$Y \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)$$

Modified Bessel Function: $z \to iz$

$$z^2 w'' + z w' - (z^2 + \nu^2) = 0$$

$$I_\nu(z) = e^{-\frac{\pi \nu}{2} i} J_\nu(iz)$$

$$K_\nu(z) = \frac{\pi i}{2} e^{\frac{i\pi\nu}{2}} \left[J_\nu(iz) + i Y_\nu(iz) \right]$$

$$L_z = -i\hbar\partial_\varphi$$

$$H = \frac{p^2}{2m} + V(r)$$

$$\frac{p^2}{2m} = \frac{-\hbar^2}{2m}\frac{1}{r}\partial_r r\partial_r + \frac{L_z^2}{2mr^2}$$

$$\left[L_z, \frac{p^2}{2m}\right] = 0 \qquad [L_z, V(r)] = 0$$

$$\underbrace{e^{i\alpha\frac{L_z}{\hbar}}}_{\text{generator of rotation.}}\psi(r,\varphi) = e^{\alpha\partial_\varphi}\psi(r,\varphi)$$

$$\stackrel{!}{=} \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\partial_\varphi^n\,\psi(r,\varphi)$$

$$\stackrel{!}{=} \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\psi^{(n)}_{(\varphi)}(r,\varphi)$$

$$\stackrel{!}{=} \psi(r,\varphi) + \alpha\psi'(r,\varphi) + \cdots$$

$$\stackrel{!}{=} \varphi(r,\varphi+\alpha)$$

$\Rightarrow$ Charged particle in e-m field:

$$H = \frac{(P - \frac{q}{c}A)^2}{2m} + q\phi \quad , \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad , \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$\dot{\vec{r}} = \frac{1}{i\hbar}[\vec{r}, H] \qquad [P_i, x_j] = -i\hbar\delta_{ij}$$

$$\boxed{= \frac{1}{m}(\vec{p} - \frac{q}{c}\vec{A}) \quad = \vec{v} \leftarrow \text{velocity operator.}}$$

with $\vec{A} = \vec{A}(\vec{r}, t)$ $\quad \phi = \phi(\vec{r}, t)$

$\vec{p}$: canonical momentum
$\vec{v}$: velocity operator.

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{1}{i\hbar}[\vec{v}, H]$$

$$= \frac{-q}{mc}\frac{\partial A}{\partial t} + \underbrace{\frac{1}{i\hbar}\left[\vec{v}^2, \frac{mv^2}{2} + q\phi\right]}_{\frac{1}{i\hbar}\frac{m}{2}(v_j[v_i, v_j] + [v_i, v_j]v_j) + \frac{1}{i\hbar}\frac{q}{m}[P_i, \varphi]}$$

$$[v_i, v_j] = \frac{1}{m^2}[P_i - \frac{q}{c}A_i, P_j - \frac{q}{c}A_j] = \frac{-q}{m^2c}\left[P_i, A_j\right] + [A_i, P_j]$$

$$\hookrightarrow = \frac{-q}{m^2c}\left(-i\hbar\frac{\partial A_j}{\partial x_i} \sim\!\!\sim\right)$$

$$= \frac{i\hbar q}{m^2c}\underbrace{(\partial_i A_j - \partial_j A_i)}_{F_{ij}}$$

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So $[v_i, v_j] = i\frac{\hbar q}{m^2 c} F_{ij}$

$\vec{\nabla} \times \vec{v} = i\hbar \frac{q}{m^2 c} \vec{B}$

$$\frac{d\vec{v}}{dt} = \frac{-q}{mc}\frac{\partial \vec{A}}{\partial t} - \frac{q}{m}\phi + \frac{m}{2i\hbar} i\hbar \frac{q}{m^2 c}\left(v_i \underset{\epsilon^{ijk}B_k}{\underset{\parallel}{F_{ij}}} + F_{ij} v_j\right)$$

$v_i \epsilon^{ijk} B_k = (v \times B)_j$

$$\boxed{\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{E} + \frac{q}{m}\frac{1}{c}\frac{1}{2}\left((\vec{v}\times\vec{B}) - (\vec{B}\times\vec{v})\right)}$$

Summarize: $\dot{\vec{r}} = \vec{v} = \frac{1}{m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)$

Heinsenberg. -> $\dot{\vec{v}} = \frac{q}{m}\left(\vec{E} + \frac{1}{2c}\left[(\vec{v}\times\vec{B}) - (\vec{B}\times\vec{v})\right]\right)$

Schrodinger picture:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}A\right)^2 + q\varphi\right]\psi$$

Gauge Invariant:

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}f$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \phi \to \phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

but $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \underbrace{\vec{\nabla}\times\vec{\nabla}f}_{=0} = \vec{\nabla}\times\vec{A} = B$

So this is gauge invariant of gauge transformation.

$\Rightarrow$ only gauge invariant quantities are measurable.

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{1}{2m}\left[-i\hbar\vec{\nabla} - \frac{q}{c}\left(\vec{A}' - \vec{\nabla}f\right)\right]^2 + q\left(\phi' + \frac{1}{c}\frac{\partial f}{\partial t}\right)\Big]\psi$$

$$\psi = e^{-i\frac{q}{\hbar c}f}\psi'$$

$\hookrightarrow \psi' = e^{i\frac{q}{\hbar c}f}\psi$

then $\vec{\nabla}\left(e^{-i\frac{q}{\hbar c}f}\psi'\right) = e^{-i\frac{q}{\hbar c}f}\left(\vec{\nabla} - \frac{iq}{\hbar c}\nabla f\right)\psi'$

$$\frac{\partial}{\partial t}\left(e^{-i\frac{q}{\hbar c}f}\psi'\right) = e^{-i\frac{q}{\hbar c}f}\left(\frac{\partial}{\partial t} - \frac{iq}{\hbar c}\frac{\partial}{\partial t}f\right)\psi'$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla}-\frac{q}{c}A\right)^2 + q\phi - \frac{q}{c}\frac{\partial f}{\partial t}\right]\psi'$$

$$i\hbar\left(\partial_t - \frac{iq}{\hbar c}\partial_t f\right)\psi'$$

$$i\hbar\partial_t\psi = H\psi$$

so it is gauge invariant

$$i\hbar\partial_t\psi' = H(\phi', A')\psi'$$

if $\psi \to \psi e^{\frac{iq}{\hbar c}f}$

let $U = e^{i\frac{q}{\hbar c}f}$, $U^\dagger = e^{-i\frac{q}{\hbar c}f}$

then $A' = UAU^\dagger + U i\frac{\hbar c}{q}\vec{\nabla}U^\dagger = \vec{A} + \vec{\nabla}f$

$O' = UOU^\dagger$

$\varphi' = U\varphi U^\dagger - U i\frac{\hbar}{q}\partial_t U^\dagger = \varphi - \frac{1}{c}\frac{\partial f}{\partial t}.$

$\vec{r}' = UrU^\dagger$

$H' = UHU^\dagger + \underbrace{i\hbar\frac{\partial U}{\partial t}U^\dagger}_{-\frac{q}{c}\frac{\partial f}{\partial t}}$

$Q' = UQU^\dagger$

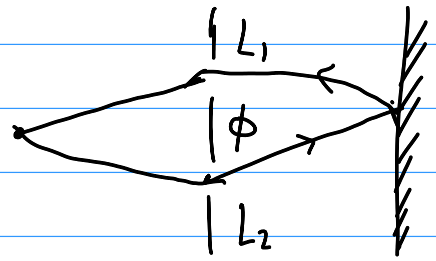
Ex: $\rho = \psi^*\psi$

$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t} = -\vec{\nabla}\cdot j$,

$j = \frac{\hbar}{2m}\Big[\psi^*\underbrace{\left(\vec{\nabla} - \frac{iq}{\hbar c}\right)}_{\frac{i}{\hbar}mv}\psi - \left(\vec{\nabla} + \frac{iq}{\hbar c}\vec{A}\right)\psi^*\Big]$

$= \frac{1}{2}\left[\psi^*\vec{v}\psi + (\vec{v}\psi)^*\psi\right]$

Aharonov-Bohr Effect:



$B \times \text{Area} = \phi$

$$H = \frac{(\vec{p} - \frac{q}{c}\vec{A})^2}{2m}$$

L_1: choose $A = 0$ $\quad -i\partial_x \psi = k\psi, \quad \psi = e^{ikx}$

L_2: choose $\vec{A} = A = \text{const.}$ $\quad -i(\partial_x - i\frac{q}{\hbar c}A)\psi = k\psi, \quad \psi = e^{i(k+\frac{q}{\hbar c})x}$

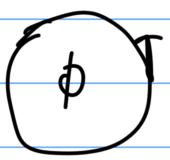
$$e^{ikl_1} + e^{i(k+\frac{q}{\hbar c}A)l_2} = e^{ikl_1}\left(1 + e^{ik(l_2 - l_1) + i\frac{q}{\hbar c}Al_2}\right)$$

$$\oint A\, d\vec{x} = Al_2$$

$$\int B \cdot dA = \phi$$

$$= e^{ikl_1}\left(1 + e^{ik(l_2 - l_1) + i\frac{q}{\hbar c}\phi}\right)$$

Particle in a ring with solenoid



Singular gauge transform:

$$\psi(\varphi) = \psi'(\varphi)\, e^{i \frac{\phi}{\phi_0} \varphi} \qquad \phi_0 = \frac{\hbar c}{e}$$