Charged Particle in EM Field:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

with $\vec{B} = \vec{\nabla} \times \vec{A}$, $E = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$\vec{p} = -i\hbar\vec{\nabla} \qquad [P_i, X_j] = -i\hbar\delta_{ij}$$

velocity operator: $\dot{\vec{r}} = \cancelto{0}{\frac{\partial \vec{r}}{\partial t}} + \frac{1}{i\hbar}[\vec{r}, H] = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right) = \vec{v}$

$$\dot{\vec{v}} = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{-\frac{e}{mc}\frac{\partial A}{\partial t}} + \frac{1}{i\hbar}[\vec{v}, H] =$$

Heisenberg Picture.

$$\hookrightarrow \boxed{m\frac{d\vec{v}}{dt} = \frac{q}{2c}\left(\vec{v} \times \vec{B} - \vec{B} \times \vec{v}\right) + q\vec{E}}$$

Schrodinger Picture:

$$i\hbar\frac{\partial \psi}{\partial t} = H\psi$$

If
$$\left.\begin{aligned} A' &= A + \vec{\nabla} f(\vec{x}, t) \\ \phi' &= \phi - \frac{1}{c}\partial_t f(\vec{x}, t) \\ \psi' &= \psi e^{i\frac{e}{\hbar c}f} \end{aligned}\right\} \text{Gauge Transformation.}$$

Then we have gauge invariant:

$E' = E$, $B' = B$ and $i\hbar\frac{\partial \psi'}{\partial t} = H'\psi'$

Problem: Particle in const $\vec{B}$ field. (2D)

$$H = \frac{(P_x - \frac{e}{c}A_x)^2}{2m} + \frac{(P_y - \frac{e}{c}A_y)^2}{2m}$$

with assumption $\partial_x A_y - \partial_y A_x = B =$ constant.

Choose Gauge:

Landau Gauge: $A_x = -By$, $A_y = 0$

With Landau Gauge:

$$H = \frac{(-i\hbar\partial_x + \frac{eB}{c}y)^2}{2m} + \frac{(-i\hbar\partial_y)^2}{2m}$$

$$= -\frac{\hbar^2}{2m}\left(\partial_x + i\frac{y}{l^2}\right)^2 - \frac{\hbar^2}{2m}\partial_y^2$$

magnetic length

where $l^2 = \frac{\hbar c}{eB}$

Define cyclotron frequency $\omega_B = \frac{eB}{mc}$

Energy scale $\frac{\hbar^2}{ml^2} = \hbar\omega_B$

Note H doesn't depend on x, so it has translational symmetry in x.

$$\psi(x,y) \to e^{-ikx}\,\psi_k(y)$$

look for: $H_k \psi_k = E_k \psi_k$

then $H_k = -\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2} m \omega_B^2 (y - k\ell^2)^2$ ← Harmonic Oscillator with displaced center.

We know solutions to HO:

$$\psi_{n,k}(y) = e^{-\frac{(y-k\ell)^2}{2\ell^2}} H_n\left(\frac{y-k\ell^2}{\ell}\right)$$

$$E_{n,k} = \hbar\omega_B\left(n + \tfrac{1}{2}\right)$$

↑ Doesn't depend on k, so degeneracy in k.

$$\psi_{n,k}(x,y) = \frac{1}{\sqrt{L_x}} e^{-ikx} \frac{1}{\pi^{1/4}\sqrt{2^n n!\,\ell}} e^{-\frac{(y-k\ell^2)^2}{2\ell^2}} H_n\left(\frac{y-k\ell^2}{\ell}\right)$$

with $k = \frac{2\pi}{L_x} m$, $m = 0, \pm 1, \pm 2 \cdots$

with constraint $0 < k\ell^2 < L_y$

Since $k\ell^2$ is the distance displaced in y direction. So don't want the wave function outside the box. L_x, L_y.

$$0 < m < \frac{L_x L_y}{2\pi \ell^2}$$

Then energy is not infinitely degenerate by k but limited by $\frac{\text{Area}}{2\pi \ell^2}$

So:

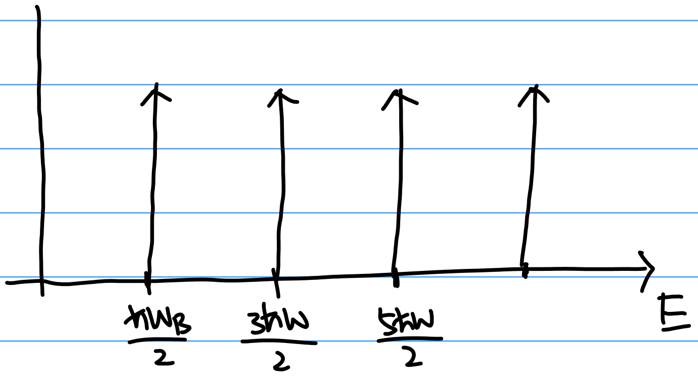
$$E_{n,k} = \hbar\omega_B\left(n+\frac{1}{2}\right)\left(\frac{\text{Area}}{2\pi \ell^2}\right)$$

# of degeneracy

$$N = \frac{\text{Area}}{2\pi \ell^2} = \frac{\text{Area}}{2\pi \frac{\hbar c}{eB}} = \frac{B \cdot \text{Area}}{\frac{\hbar c}{e}} = \frac{\Phi}{\Phi_0}$$

$\frac{\nu(E)}{\text{Area}}$

Density of state.



DoS: $$\frac{\nu(E)}{\text{Area}} = \sum_{n=0}^{\infty} \frac{1}{2\pi \ell^2}\,\delta\!\left(E - \hbar\omega_B\left(n+\frac{1}{2}\right)\right)$$

Focus on the lowest level $n=0$:

$$\psi_k(x,y) = \frac{1}{\sqrt{L_x}} e^{ikx} \frac{1}{\pi^{1/4}\sqrt{l}} e^{-\frac{(y-kl^2)^2}{2l^2}} \quad , k = \frac{2\pi}{L_x} m, \; m = 0, \pm 1 \cdots$$

$$\psi(x,y) = \int \frac{dk}{2\pi} \psi_k(x,y) A_k$$

1) example: $A_k = e^{-\frac{1}{2}k^2 l^2}$

then $\psi(x,y) \sim e^{-\frac{(x^2+y^2)}{4l^2}} e^{-\frac{i}{2l^2} xy}$

2) Example: $A_k = e^{-\frac{1}{2}k^2 l^2 + (y_0 + i x_0) k}$

$$\psi(x,y) = \frac{1}{\sqrt{2\pi l^2}} e^{-\frac{(\vec{r}-\vec{r}_0)^2}{4l^2} - \frac{i}{2l^2}(x-x_0)(y-y_0)}$$

3D problems: choose z-axis to point along magnetic field

$$H = \frac{\left(P_x - \frac{e}{c}A_x\right)^2}{2m} + \frac{\left(P_y - \frac{e}{c}A_y\right)^2}{2m} + \frac{P_z^2}{2m}$$

Look for $e^{ikz}\,\psi_q(x,y)$

$$H_q \psi_q = E_q \psi_q$$

$$H_q = \frac{\left(P_x - \frac{e}{c}A_x\right)^2}{2m} + \frac{\left(P_y - \frac{e}{c}A_y\right)^2}{2m} + \frac{\hbar^2 q^2}{2m}$$

$$E_{n,q} = \hbar\omega_B\left(n + \tfrac{1}{2}\right) + \frac{\hbar^2 q^2}{2m}$$

next week,

OH: Thurs 4-5, Mon: 4-5, HW Feb 5.

Radial Gauge:

$A_x = -\frac{1}{2}By \qquad A_y = \frac{1}{2}Bx \quad , \quad \vec{A} = \frac{1}{2}(\vec{B}\times\vec{r})$

then $\quad H = -\frac{\hbar^2}{2m}\left(\partial_x + i\frac{eB}{2\hbar c}y\right)^2 - \frac{\hbar^2}{2m}\left(\partial_y - i\frac{eB}{2\hbar c}x\right)^2$

$$= -\frac{\hbar^2}{2m}\underbrace{(\partial_x^2 + \partial_y^2)}_{\nabla^2} + i\frac{eB}{2mc}\hbar\underbrace{(x\partial_y - y\partial_x)}_{\partial\phi} + \frac{\hbar^2}{2m}\left(\frac{eB}{2\hbar c}\right)^2(x^2+y^2)$$

with $\omega_B = \frac{eB}{mc}$

$l^2 = \frac{\hbar c}{eB}$

$$= -\frac{\hbar^2}{2m}\left(\frac{1}{r}\partial_r r\partial_r + \frac{1}{r^2}\partial\phi^2\right) + i\frac{eB\hbar}{2mc}\partial\phi + \frac{\hbar^2}{2m}\left(\frac{eB}{2\hbar c}\right)^2 r^2$$

$$= -\frac{\hbar\omega_B}{2}\left(\frac{1}{r}\partial_r r\partial_r + \frac{1}{r^2}\underbrace{\partial\phi^2}_{m^2}\right) + \frac{\hbar\omega_B}{2}\underbrace{i\partial\phi}_{im} + \frac{\hbar\omega_B}{8}\frac{r^2}{l^2}$$

then let $\psi = e^{im\phi}\psi_m(r)$, $H_m\psi_m = E_m\psi_m$

then we will get answer depend by generalized Laguerre polynomial.

$\psi_0 \propto e^{-\frac{r^2}{4l^2}}$

Consider Lowest Landau Level (LLL) in complex coordinates.

$z = x + iy$, let $\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$

$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$

then $f(x,y) = f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = f(z,\bar{z})$

$\rightarrow$ $H = -\frac{\hbar\omega_B}{2}\left[\left(\partial_x + \frac{i}{2\ell^2}y\right)^2 + \left(\partial_y - \frac{i}{2\ell^2}x\right)^2\right]$

define $x' = \frac{x}{\ell}$, $y' = y/\ell$ $H' = \frac{H}{\hbar\omega_B}$.

then $H' = -\frac{1}{2}\left[\left(\partial_{x'} + \frac{i}{2}y'\right)^2 - \frac{1}{2}\left(\partial_{y'} - \frac{i}{2}x'\right)^2\right]$

Now we will drop $'$:

$$H = \underbrace{\frac{1}{\sqrt{2}}\left(-i\partial_x - \partial_y + \frac{i}{2}(x - iy)\right)}_{a^+}\underbrace{\frac{1}{\sqrt{2}}\left(-i\partial_x + \partial_y - \frac{i}{2}(x+iy)\right)}_{a^-} + \frac{1}{2}$$

$H \overset{!}{=} a^+a^- + \frac{1}{2}$

Show $[a^+, a] = 1$, then $a^+a = n$

then $E = n + \frac{1}{2}$

$H = \frac{1}{2}\left(-2i\partial + \frac{i}{2}\bar{z}\right)\left(-2i\bar{\partial} - \frac{i}{2}z\right) + \frac{1}{2}$

let $\psi = e^{-\frac{|z|^2}{4}} \chi = e^{-\frac{z\bar{z}}{4}} \chi$

$$\left(-2i\bar{\partial} - \frac{i}{2} z\right) e^{-\frac{z\bar{z}}{4}} \chi$$

$$\hookrightarrow = e^{-\frac{z\bar{z}}{4}} \left(-2i\bar{\partial} - \frac{i}{2} z \cancel{- 2i\left(-\frac{z}{4}\right)}\right) \chi$$

then

$$(-2i\bar{\partial})\chi = e^{-\frac{z\bar{z}}{4}} E \chi$$

$$\cancel{e^{-\frac{z\bar{z}}{4}}} \frac{1}{2}(-2i\partial + i\bar{z}) \underbrace{(-2i\bar{\partial})}_{\frac{\partial}{\partial\bar{z}}} \chi = \underbrace{\left(E - \frac{1}{2}\right)}_{E=\frac{1}{2} \text{ in LLL so RHS}=0} \chi \cancel{e^{-\frac{z\bar{z}}{4}}}$$

let $\chi = f(z)$

$\hookrightarrow$ analytic, not function of $\bar{z}$

then LHS $=0$. let $\chi = z^m$, $m=0,1,2\ldots$

so it is ∞ degenerate.

then $z^m = (x+iy)^m = (re^{i\phi})^m = r^m e^{im\phi}$

then $\psi(x,y)_{n=0} = e^{-\frac{|z|^2}{4}} z^m$

Magnetic Monopole:

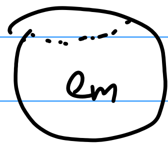
$$\vec{B} = \frac{e_m}{r^2}\hat{r} \qquad \vec{\nabla}\cdot\vec{B} = 4\pi \rho_m$$

$$\vec{E} = \frac{e}{r^2}\hat{r} \qquad \vec{\nabla}\cdot\vec{E} = 4\pi \rho$$

$\vec{B} = \vec{\nabla}\times\vec{A}$ but $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ if $\vec{A}$ is not singular

Therefore, let $\vec{A}$ be singular.

$$\underbrace{\oiint \vec{B}\cdot d\vec{S}}_{4\pi e_m} \xrightarrow{\text{Stokes.}} = \int_{\partial\Theta} \vec{A}\cdot d\vec{l} = 0$$

Suppose 

, magnetic monopole in sphere.

$$\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)$$

$$e^{i\frac{e}{\hbar c}\oint \vec{A}\cdot d\vec{l}} = e^{i\frac{e}{\hbar c}\int_S \vec{\nabla}\times\vec{A}\, d^2s} = e^{i\frac{e}{\hbar c}\Phi_S}$$

Flux going from top cap

$$\hookrightarrow = e^{i2\pi\left(\frac{e}{hc}\right)\Phi_S} = e^{i2\pi\frac{\Phi_S}{\Phi_0}}, \quad \Phi_0 = \frac{hc}{e}$$

$$\overset{!}{=} e^{-i2\pi\frac{\Phi_S'}{\Phi_0'}}$$

Flux going through rest of sphere.

$$\hookrightarrow e^{i2\pi\frac{\Phi_S+\Phi_S' = \Phi}{\Phi_0}} = 1$$

then $\Phi = N\Phi_0$

since $\phi = 4\pi e_m$ $\qquad \phi_0 = \frac{\hbar c}{e} 2\pi$

$$e_m e = N \frac{\hbar c}{2}$$

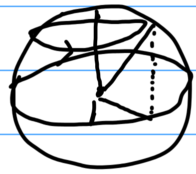
$$e_m = N\left(\frac{\hbar c}{2e^2}\right) e \qquad \text{and } \frac{\hbar c}{e^2} \sim 137$$

$$= N \frac{137}{2} e$$

or $\qquad e = N \frac{\hbar c}{2 e_m}$ $\leftarrow$ meaning that electric charge is quantized and depend on e_m if it exists.

Dirac String:

$$\vec{A} = \frac{e_m(1-\cos\theta)}{r\sin\theta}\hat{\varphi}$$

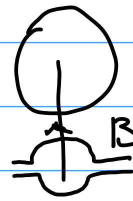


$$\vec{\nabla}\times\vec{A} = \frac{e_m}{r^2}\hat{r}$$

$$\oint \vec{A}\cdot d\vec{l} = \frac{e_m(1-\cos\theta)}{r\sin\theta}2\pi\sin\theta = \underbrace{\frac{e_m}{r^2}}_{\vec{\nabla}\times\vec{A}=B}\underbrace{2\pi(1-\cos\theta)r^2}_{A}$$

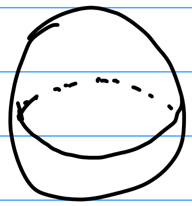
when $\theta=0$, it is fine, but as $\theta=\pi$, $\sin\theta=0$, then it is divergent.

When $\theta=\pi$ $\quad A = \frac{2e_m}{r(\pi-\theta)}2\pi r(\pi-\theta)$



$$\Phi = N\phi_0$$

Wu-Yang Monopole:



$$\vec{A}^{(I)} = \frac{e_m(1-\cos\theta)}{r\sin\theta}\hat{\varphi} \qquad\qquad \vec{A}^{(II)} = -\frac{e_m(1+\cos\theta)}{r\sin\theta}\hat{\varphi}$$

$$0\le\theta<\pi-\varepsilon \qquad\qquad \varepsilon<\theta\le\pi$$

$$A^{II}-A^{I} = \vec{\nabla}\Lambda, \quad \Lambda=-2e_m\varphi$$

$$\underset{\text{single-valued}}{\psi^{II}} = \psi^{I}e^{i\frac{e}{\hbar c}\Lambda} = \underset{\text{single-valued}}{\psi^{I}}\,e^{-i\frac{2e e_m}{\hbar c}\varphi} \quad \text{so } N\varphi = 2\pi.$$

then this should be single-valued