Zhi Chen QM 2:

1) Suppose $g(t)$ is some matrix smoothly dependent on t.

a) prove $\partial_t(g^{-1}) = -g^{-1}(\partial_t g)g^{-1}$

$$\partial_t(g g^{-1}) = \partial_t(\mathbb{1}) = 0$$

$$0 \overset{!}{=} \partial_t(g)\, g^{-1} + g\, \partial_t(g^{-1})$$

$$-g\, \partial_t g^{-1} = \partial_t(g)\, g^{-1}$$

Multiply $-g^{-1}$ to both sides

$$\hookrightarrow \underbrace{g^{-1} g}_{\mathbb{1}}\, \partial_t g^{-1} = -g^{-1}\, \partial_t(g)\, g^{-1}$$

$$\hookrightarrow \boxed{\partial_t g^{-1} = -g^{-1}\, \partial_t(g)\, g^{-1}}$$

b) Assume $g(t)$ is orthogonal matrix $\rightarrow$ $g^T = g^{-1}$

Prove: $(g^{-1}\partial_t g)^T = -g^{-1}\partial_t g$

$$(g^{-1}\partial_t g)^T = (\partial_t g)^T (g^{-1})^T$$

$$\overset{!}{=} \partial_t(g^T)\,(g^T)^{-1}$$

For orthongonal Matrix, $g^T = g^{-1}$

$$\overset{!}{=} \partial_t(g^{-1})\,\underbrace{(g^{-1})^{-1}}_{=g}$$

but $\partial_t(g^{-1}g) = \partial_t(\mathbb{1}) = 0$

$$\hookrightarrow 0 = \partial_t(g^{-1})g + g^{-1}\partial_t g$$

$$\boxed{\Omega^T = (g^{-1}\partial_t g)^T \overset{!}{=} -g^{-1}\partial_t g = -\Omega}$$

c) If $g(t)$ is unitary matrix, $g^{-1} = g^{+}$

$$\left(g^{-1}\partial_t g\right)^{+} = \left(\partial_t g\right)^{+}\left(g^{-1}\right)^{+}$$

$$\overset{!}{=} \underbrace{\partial_t(g^{+})}_{\partial_t(g^{-1})}\ \underbrace{(g^{+})^{-1}}_{(g^{-1})^{-1} = g}$$

$$\overset{!}{=} \partial_t(g^{-1})\, g$$

again:

$$0 = \partial_t(g^{-1})g + g^{-1}\partial_t g \qquad \overset{!}{=} -g^{-1}\partial_t g$$

$$\boxed{\left(g^{-1}\partial_t g\right)^{+} = -g^{-1}\partial_t g}$$

2) a) Calculate $\mathrm{Tr}\{(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma})\}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{a}\cdot\vec{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$$

$$\overset{!}{=} \begin{pmatrix} 0 & a_x \\ a_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ia_y \\ ia_y & 0 \end{pmatrix} + \begin{pmatrix} a_z & 0 \\ 0 & -a_z \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}$$

so $\mathrm{Tr}\{(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma})\}$

$$\overset{!}{=} \mathrm{Tr}\left\{\begin{pmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{pmatrix}\begin{pmatrix} b_z & b_x - i b_y \\ b_x + i b_y & -b_z \end{pmatrix}\begin{pmatrix} c_z & c_x - i c_y \\ c_x + i c_y & -c_z \end{pmatrix}\right\}$$

$$\mathrm{Tr}\left\{\begin{pmatrix} a_z b_z + (a_x - i a_y)(b_x + i b_y) & a_z(b_x - i b_y) - b_z(a_x - i a_y) \\ b_z(a_x + i a_y) - a_z(b_x + i b_y) & (a_x + i a_y)(b_x - i b_y) + a_z b_z \end{pmatrix}\begin{pmatrix} c_z & c_x - i c_y \\ c_x + i c_y & -c_z \end{pmatrix}\right\}$$

$$= \left[\cancel{a_z b_z} + (a_x - i a_y)(b_x + i b_y)\right] c_z + \left[a_z(b_x - i b_y) - b_z(a_x - i a_y)\right](c_x + i c_y)$$

$$+ \left[b_z(a_x + i a_y) - a_z(b_x + i b_y)\right](c_x - i c_y) - c_z\left[(a_x + i a_y)(b_x - i b_y) + \cancel{a_z b_z}\right]$$

$$= (\cancel{a_x b_x} - i a_y b_x + i a_x b_y + \cancel{a_y b_y}) c_z - c_z(\cancel{a_x b_x} - i a_x b_y + i a_y b_x + \cancel{a_y b_y})$$

$$(a_z b_x - i a_z b_y - b_z a_x + i b_z a_y)(c_x + i c_y) + (b_z a_x + i b_z a_y - a_z b_x - i a_z b_y)(c_x - i c_y)$$

$$\overset{!}{=} c_z(2i a_x b_y - 2i a_y b_x)$$

$$+ c_x(\cancel{a_z b_x} - i a_z b_y - \cancel{b_z a_x} + i b_z a_y + \cancel{b_z a_x} + i b_z a_y - \cancel{a_z b_x} - i a_z b_y)$$

$$+ i c_y(a_z b_x - \cancel{i a_z b_y} - b_z a_x + \cancel{i b_z a_y} - b_z a_x - \cancel{i b_z a_y} + a_z b_x + \cancel{i a_z b_y})$$

$$= 2i(a_x b_y - a_y b_x) c_z + 2i(a_y b_z - a_z b_y) c_x + 2i(a_z b_x - a_x b_z) c_y$$

$$\boxed{\mathrm{Tr}\{(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma})\} = 2i\,\varepsilon^{ijk} a_i b_j c_k}$$

b) Calculate $\exp\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ using Pauli - Matrices.

From part c, we know any 2×2 matrix can be decomposed into linear combinations of Pauli-Matrices.

$$\vec{A} \rightarrow \vec{a} \cdot \vec{\sigma}$$

we know via Taylor:

$$e^{A} = \sum_{n=0}^{\infty} \frac{A^n}{n!} \rightarrow \sum_{n=0}^{\infty} \frac{\vec{a}\cdot\vec{\sigma}}{n!}$$

split into even and odd terms.

$$= \sum_{p=0}^{\infty} \frac{(\vec{a}\cdot\vec{\sigma})^{2p}}{(2p)!} + \sum_{q=0}^{\infty} \frac{(\vec{a}\cdot\vec{\sigma})^{2q+1}}{(2q+1)!}$$

consider

$$(\vec{a}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) = (a_i \sigma_i)(a_j \sigma_j)$$

$$\stackrel{!}{=} \frac{a_i a_j}{2} \Big(\underbrace{\sigma_i \sigma_j - \sigma_j \sigma_i}_{[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k} + \underbrace{\sigma_i \sigma_j + \sigma_j \sigma_i}_{\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}} \Big)$$

$$\stackrel{!}{=} a_i a_j \left[i\varepsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1} \right]$$

$$\stackrel{!}{=} a_i a_i + i \underbrace{a_i a_j \varepsilon_{ijk}}_{\vec{a}\times\vec{a}=0} \sigma_k$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) \stackrel{!}{=} \vec{a}\cdot\vec{a}$$

Therefore:

$$= \sum_{p=0}^{\infty} \frac{(\vec{a}\cdot\vec{\sigma})^{2p}}{(2p)!} + \sum_{q=0}^{\infty} \frac{(\vec{a}\cdot\vec{\sigma})^{2q+1}}{(2q+1)!}$$

$$\overset{!}{=} \sum_{p=0}^{\infty} \frac{(\vec{a}\cdot\vec{a})^{p}}{(2p)!} + (\vec{a}\cdot\vec{\sigma}) \sum_{q=0}^{\infty} \frac{(\vec{a}\cdot\vec{a})^{q}}{(2q+1)!}$$

$$\overset{!}{=} \underbrace{\sum_{p=0}^{\infty} \frac{(\sqrt{\vec{a}\cdot\vec{a}})^{2p}}{(2p)!}}_{\cosh(\sqrt{\vec{a}\cdot\vec{a}})} + (\vec{a}\cdot\vec{\sigma})\frac{1}{\sqrt{\vec{a}\cdot\vec{a}}} \underbrace{\sum_{q=0}^{\infty} \frac{(\sqrt{\vec{a}\cdot\vec{a}})^{2q+1}}{(2q+1)!}}_{\sinh(\sqrt{\vec{a}\cdot\vec{a}})}$$

$$\boxed{e^{\vec{a}\cdot\vec{\sigma}} \overset{!}{=} \mathbb{1}\cosh(\sqrt{\vec{a}\cdot\vec{a}}) + \frac{(\vec{a}\cdot\vec{\sigma})}{\sqrt{\vec{a}\cdot\vec{a}}}\sinh(\sqrt{\vec{a}\cdot\vec{a}})}$$

using results from part C: we can decompose $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ into:

$$\hookrightarrow \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hookrightarrow \exp\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} = \exp\{2\mathbb{1} + 3\sigma_x + i\sigma_y + \sigma_z\}$$

since $\mathbb{1}$ commutes with everything.

$$\overset{!}{=} \exp\{2\mathbb{1}\}\exp\{3\sigma_x + i\sigma_y + \sigma_z\}$$

$$\overset{!}{=} \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \exp\{3\sigma_x + i\sigma_y + \sigma_z\}$$

let $\vec{a} = (3, i, 1) \rightarrow \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{3^2 - 1 + 1} = 3$

$$e^{\vec{a}\cdot\vec{\sigma}} = \cosh(\sqrt{\vec{a}\cdot\vec{a}})\mathbb{I} + \frac{\vec{a}\cdot\vec{\sigma}}{\sqrt{\vec{a}\cdot\vec{a}}}\sinh(\sqrt{\vec{a}\cdot\vec{a}})$$

$$\hookrightarrow = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix}\left\{\begin{pmatrix} \cosh 3 & 0 \\ 0 & \cosh 3 \end{pmatrix} + \sinh 3\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & -1/3 \end{pmatrix}\right]\right\}$$

$$\boxed{\exp\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} e^2\left(\cosh 3 + \frac{\sinh 3}{3}\right) & e^2 \frac{4}{3}\sinh 3 \\ e^2 \frac{2}{3}\sinh 3 & e^2\left(\cosh 3 - \frac{\sinh 3}{3}\right) \end{pmatrix}}$$

c) express $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ using pauli - Matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{1}} + Q \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{!}{=} \begin{pmatrix} \alpha+\beta & Q-i\gamma \\ Q+i\gamma & \alpha-\beta \end{pmatrix}$$

so

$$\left.\begin{aligned} \alpha+\beta &= a \\ d &= \alpha-\beta \end{aligned}\right\} \quad \beta+d+\beta = a \quad \rightarrow \quad \boxed{\beta = \tfrac{1}{2}(a-d)}$$

and $\alpha = \beta + d = \frac{1}{2}(a-d) + d = \boxed{\frac{1}{2}(a+d) = \alpha}$

also

$$\left.\begin{aligned} Q - i\gamma &= b \\ Q + i\gamma &= c \end{aligned}\right\} \quad b + i\gamma + i\gamma = c \quad \rightarrow \quad \boxed{\gamma = \tfrac{-i}{2}(c-b)}$$

$Q = b + i\gamma = b + \frac{1}{2}(c-b) = \boxed{\frac{1}{2}(c+b) = Q}$

then

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a+d)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(a-d)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}(c+b)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{i}{2}(c-b)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

3) $\psi = N e^{-\alpha r^2}(x+y)z \quad , r^2 = x^2+y^2+z^2$

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz \; \psi^* \psi$

$\hookrightarrow = \iiint dxdydz \; |N|^2 e^{-2\alpha(x^2+y^2+z^2)} (x+y)^2 z^2$

$= |N|^2 \iiint e^{-2\alpha x^2} e^{-2\alpha y^2} e^{-2\alpha z^2} (x^2+2xy+y^2) z^2$

$\overset{!}{=} |N|^2 \int dydz \, e^{-2\alpha y^2} e^{-2\alpha z^2} z^2 \left(\frac{\sqrt{\pi}}{2^{5/2}\alpha^{3/2}} + \sqrt{\frac{\pi}{2a}} y^2 \right) z^2$

$\overset{!}{=} |N|^2 \int dz \, e^{-2\alpha z^2} z^2 \underbrace{\left(\frac{\sqrt{\pi}}{2^{5/2}\alpha^{3/2}} \sqrt{\frac{\pi}{2\alpha}} \; 2 \right)}_{\frac{\pi}{(2\alpha)^2}}$

$= |N|^2 \frac{\sqrt{\pi}}{2^{5/2}\alpha^{3/2}} \frac{\pi}{(2\alpha)^2}$

$1 \overset{!}{=} |N|^2 \pi^{3/2} \frac{1}{2^{9/2}\alpha^{7/2}}$

$\hookrightarrow \boxed{|N| = \sqrt{\frac{2^{9/2}\alpha^{7/2}}{\pi^{3/2}}} = \frac{2^{9/4}\alpha^{7/4}}{\pi^{3/4}}}$

b) $\vec{L} = \vec{r} \times \vec{p} = -i\hbar \varepsilon^{ijk} r_i \partial_j \hat{e}_k$

$$L_x = -i\hbar (y\partial_z - z\partial_y)$$
$$L_y = -i\hbar (z\partial_x - x\partial_z)$$
$$L_z = -i\hbar (x\partial_y - y\partial_x)$$

Calculate derivatives: $\psi = Ne^{-\alpha(x^2+y^2+z^2)}(x+y)z$

$$\partial_z \psi = N(x+y) e^{-\alpha(x^2+y^2)} \left[(-2\alpha z) e^{-\alpha z^2} z + e^{-\alpha z^2}\right]$$
$$= Ne^{-\alpha(x^2+y^2+z^2)} \left[-2\alpha z^2 + 1\right](x+y)$$

$$\partial_x \psi = Nz\, e^{-\alpha(y^2+z^2)} \left[-2\alpha x e^{-\alpha x^2}(x+y) + e^{-\alpha x^2}\right]$$
$$= Ne^{-\alpha(x^2+y^2+z^2)} \left[-2\alpha x(x+y) + 1\right] z$$

$$\partial_y \psi = Nz\, e^{-\alpha(y^2+z^2)} \left[-2\alpha y\, e^{-\alpha y^2}(x+y) + e^{-\alpha y^2}\right]$$
$$= Ne^{-\alpha(x^2+y^2+z^2)} \left[-2\alpha y(x+y) + 1\right] z$$

$$\langle L_x \rangle = \iiint dx\,dy\,dz\ \psi^* (-i\hbar)(y\partial_z - z\partial_y)\psi$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,dz\,(-i\hbar)\,N^2 e^{-2\alpha(x^2+y^2+z^2)}(x+y)z$$

$$\left[y(x+y)(-2\alpha z^2+1) - z^2\left(-2\alpha y(x+y)+1\right)\right]$$

$$= \iiint dx\,dy\,dz\,(-i\hbar)N^2 e^{-2\alpha(x^2+y^2+z^2)}(xz+yz)$$

$$\left[-2\alpha(\cancel{xyz^2} + y^2z^2) + xy + y^2 + 2\alpha(\cancel{xyz^2} + y^2z^2) - z^2\right]$$

$$= \iiint dx\,dy\,dz\,(-i\hbar)N^2 \underbrace{e^{-2\alpha(x^2+y^2+z^2)}}_{\text{even term}} \underbrace{\left(x^2yz + xy^2z - xz^3 + xy^2z + y^3z - yz^3\right)}_{\text{Each term has at one least one odd term in either } x,y,\text{ or } z}$$

$= 0$ since entire integrand is odd and we integrate over $(-\infty, +\infty)$.

$$\langle L_y \rangle = \iiint dx\,dy\,dz\ \psi^* (-i\hbar)(z\partial_x - x\partial_z)\psi$$

$$= \iiint dx\,dy\,dz\,(-i\hbar)\,N^2 e^{-2\alpha(x^2+y^2+z^2)}(x+y)z$$

$$\left[z^2(-2\alpha x(x+y)+1) - x(x+y)(-2\alpha z^2+1)\right]$$

$$= \iiint dx\,dy\,dz\,(-i\hbar)N^2 e^{-2\alpha(x^2+y^2+z^2)}(xz+yz)$$

$$\left(-2\alpha(\cancel{x^2z^2} + xyz^2) + z^2 + 2\alpha(\cancel{x^2z^2} + xyz^2) - x^2 - xy\right)$$

$= 0$ odd integrand just like L_x

$$\langle L_z \rangle = \iiint dx\,dy\,dz\ \psi^*(-i\hbar)(x\partial_y - y\partial_x)\psi$$

$$\overset{!}{=} \iiint dx\,dy\,dz\,(-i\hbar)N^2 e^{-2\alpha(x^2+y^2+z^2)}(x+y)z$$

$$[xz(-2\alpha y(x+y)+1) - yz(-2\alpha x(x+y)+1)]$$

$$\overset{!}{=} \iiint dx\,dy\,dz\,(-i\hbar)N^2 e^{-2\alpha(x^2+y^2+z^2)}(xz+yz)$$

$$(-2\alpha(\cancel{x^2yz}+xy^2z)+xz+2\alpha(\cancel{x^2yz}+xy^2z)-yz)$$

$$\overset{!}{=} \iiint dx\,dy\,dz\,(-i\hbar)N^2 e^{-2\alpha(x^2+y^2+z^2)}\underbrace{\{(xz)^2-(yz)^2\}}_{=z^2\underbrace{(x^2-y^2)}}$$

since there is a symmetry in x and y, x^2-y^2 leads to 0 expectation value.

$$= 0$$

Therefore $\boxed{\langle \vec{L} \rangle = \langle L_x \rangle + \langle L_y \rangle + \langle L_z \rangle = 0}$

Find $\langle L^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle$

$$L^2 = \underbrace{(-i\hbar)^2}_{-\hbar^2} \left\{ (y\partial_z - z\partial_y)^2 + (z\partial_x - x\partial_z)^2 + (x\partial_y - y\partial_x)^2 \right\}$$

$$L_x^2 \psi = L_x \, L_x \psi$$

$$= (-i\hbar)^2 (y\partial_z - z\partial_y) N e^{-\alpha(x^2+y^2+z^2)} (y^2 + xy - z^2)$$

$$= (-i\hbar)^2 N e^{-\alpha(x^2+y^2+z^2)} \{ y[(-2\alpha z)(y^2+xy-z^2) - 2z]$$

$$- z[(-2\alpha y)(y^2+xy-z^2) + 2y + x]$$

$$L_x^2 \psi = \hbar^2 N e^{-\alpha(x^2+y^2+z^2)} (x+4y)z$$

$$L_y^2 \psi = (-i\hbar)^2 L_y \, L_y \psi$$

$$= -\hbar^2 (z\partial_x - x\partial_z)(N e^{-\alpha(x^2+y^2+z^2)} (-x^2 - xy + z^2))$$

$$= -\hbar^2 N e^{-\alpha(x^2+y^2+z^2)} \{ z[-2\alpha x(-x^2-xy+z^2) - 2x - y]$$

$$- x[-2\alpha z(-x^2-xy+z^2) + 2z]$$

$$L_y^2 \psi = \hbar^2 N e^{-\alpha(x^2+y^2+z^2)} (4x+y)z$$

$$L_z^2\psi = -\hbar^2 (x\partial_y - y\partial_x) Ne^{-\alpha(x^2+y^2+z^2)}(x-y)z$$

$$\overset{!}{=} -\hbar^2 Ne^{-\alpha(x^2+y^2+z^2)} z(-x-y)$$

$$\overset{!}{=} \hbar^2 Ne^{-\alpha(x^2+y^2+z^2)}(x+y)z$$

$$L^2\psi = L_x^2\psi + L_y^2\psi + L_z^2\psi$$

$$\overset{!}{=} \hbar^2 Ne^{-\alpha(x^2+y^2+z^2)} z(4x+y+4y+x+x+y)$$

$$\overset{!}{=} 6\hbar^2 \underbrace{Ne^{-\alpha(x^2+y^2+z^2)}(x+y)z}_{=\psi}$$

$$\overset{!}{=} 6\hbar^2\psi$$

$$\langle L^2\rangle = \iiint dxdydz\ \psi^* \underbrace{L^2\psi}_{6\hbar^2\psi}$$

$$\overset{!}{=} 6\hbar^2 \int \psi^*\psi$$

$$\boxed{\langle L^2\rangle \overset{!}{=} 6\hbar^2}$$

c) variance of L, $\Delta^2(L) = \langle L^2\rangle - \cancel{\langle L\rangle^2}$

$$\boxed{\Delta^2(L) \overset{!}{=} 6\hbar^2}$$

$$\boxed{\Delta^2(L^2) = \langle L^4\rangle - \langle L^2\rangle^2 = 0}$$

since L^2 commutes with L^4

$[L^4, L^2] = [L^2L^2, L^2] = 0$

4) work out J- Matrices for $j=\frac{1}{2}$, $j=1$, $j=\frac{3}{2}$

$$J_z|j,m\rangle = \hbar m |j,m\rangle$$

$$J_+|j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\ |j,m+1\rangle \qquad J_x = \frac{J_+ + J_-}{2}$$

$$J_-|j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\ |j,m-1\rangle \qquad J_y = \frac{J_+ - J_-}{2i}$$

$\underline{\underline{j=\frac{1}{2}}}$, $m = \frac{1}{2}, -\frac{1}{2}$

$$J_z^{\frac{1}{2}} = \hbar\begin{pmatrix} \langle \frac{1}{2},\frac{1}{2}|J_z|\frac{1}{2},\frac{1}{2}\rangle & \langle \frac{1}{2},\frac{1}{2}|J_z|\frac{1}{2},-\frac{1}{2}\rangle \\ \langle \frac{1}{2},-\frac{1}{2}|J_z|\frac{1}{2},\frac{1}{2}\rangle & \langle \frac{1}{2},-\frac{1}{2}|J_z|\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix}$$

$$J_z^{\frac{1}{2}} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_+^{\frac{1}{2}} = \begin{pmatrix} \langle \frac{1}{2},\frac{1}{2}|J_+|\frac{1}{2},\frac{1}{2}\rangle & \langle \frac{1}{2},\frac{1}{2}|J_+|\frac{1}{2},-\frac{1}{2}\rangle \\ \langle \frac{1}{2},-\frac{1}{2}|J_+|\frac{1}{2},\frac{1}{2}\rangle & \langle \frac{1}{2},-\frac{1}{2}|J_+|\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix}$$

$$= \hbar\begin{pmatrix} 0 & \sqrt{\left(\frac{1}{2}-\left(-\frac{1}{2}\right)\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} = 1 \\ 0 & 0 \end{pmatrix}$$

$$J_-^{\frac{1}{2}} = \begin{pmatrix} \langle \frac{1}{2},\frac{1}{2}|J_-|\frac{1}{2},\frac{1}{2}\rangle & \langle \frac{1}{2},\frac{1}{2}|J_-|\frac{1}{2},-\frac{1}{2}\rangle \\ \langle \frac{1}{2},-\frac{1}{2}|J_-|\frac{1}{2},\frac{1}{2}\rangle & \langle \frac{1}{2},-\frac{1}{2}|J_-|\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix}$$

$$= \hbar\begin{pmatrix} 0 & 0 \\ \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} = 1 & 0 \end{pmatrix}$$

$$J_x^{\frac{1}{2}} = \frac{J_+ + J_-}{2} = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y^{\frac{1}{2}} = \frac{J_+ - J_-}{2i} = \frac{\hbar}{2i}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Summary: $j = \frac{1}{2}$:

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$j = 1$: $m = -1, 0, 1$

$$J_z^{1} = \hbar\begin{pmatrix} \langle 1|J_z|1\rangle & \langle 1|J_z|0\rangle & \langle 1|J_z|-1\rangle \\ \langle 0|J_z|1\rangle & \langle 0|J_z|0\rangle & \langle 0|J_z|-1\rangle \\ \langle -1|J_z|1\rangle & \langle -1|J_z|0\rangle & \langle -1|J_z|-1\rangle \end{pmatrix}$$

$$= \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$J_+ = \hbar \begin{pmatrix} \langle 1|J_+|1\rangle & \langle 1|J_+|0\rangle & \langle 1|J_+|-1\rangle \\ \langle 0|J_+|1\rangle & \langle 0|J_+|0\rangle & \langle 0|J_+|-1\rangle \\ \langle -1|J_+|1\rangle & \langle -1|J_+|0\rangle & \langle -1|J_+|-1\rangle \end{pmatrix}$$

$$= \hbar \begin{pmatrix} 0 & \sqrt{(1-0)(1+0+1)} = \sqrt{2} & 0 \\ 0 & 0 & \sqrt{(1-(-1))(1+(-1)+1)} = \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_-^1 = \hbar \begin{pmatrix} \langle 1|J_-|1\rangle & \langle 1|J_-|0\rangle & \langle 1|J_-|-1\rangle \\ \langle 0|J_-|1\rangle & \langle 0|J_-|0\rangle & \langle 0|J_-|-1\rangle \\ \langle -1|J_-|1\rangle & \langle -1|J_-|0\rangle & \langle -1|J_-|-1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{(1+1)(1-1+1)} = \sqrt{2} & 0 & 0 \\ 0 & \sqrt{(1+0)(1-0+1)} = \sqrt{2} & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} J_x^1 &= \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ J_y^1 &= \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{\sqrt{2}\, i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ \text{with } J_z^1 &= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}}$$

$j = \frac{3}{2}$, $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$$J_z^{3/2} = \begin{pmatrix} \langle \frac{3}{2} | J_z | \frac{3}{2} \rangle & \langle \frac{3}{2} | J_z | \frac{1}{2} \rangle & \langle \frac{3}{2} | J_z | -\frac{1}{2} \rangle & \langle \frac{3}{2} | J_z | -\frac{3}{2} \rangle \\ \langle \frac{1}{2} | J_z | \frac{3}{2} \rangle & \langle \frac{1}{2} | J_z | \frac{1}{2} \rangle & \langle \frac{1}{2} | J_z | -\frac{1}{2} \rangle & \langle \frac{1}{2} | J_z | -\frac{3}{2} \rangle \\ \langle -\frac{1}{2} | J_z | \frac{3}{2} \rangle & \langle -\frac{1}{2} | J_z | \frac{1}{2} \rangle & \langle -\frac{1}{2} | J_z | -\frac{1}{2} \rangle & \langle -\frac{1}{2} | J_z | -\frac{3}{2} \rangle \\ \langle -\frac{3}{2} | J_z | \frac{3}{2} \rangle & \langle -\frac{3}{2} | J_z | \frac{1}{2} \rangle & \langle -\frac{3}{2} | J_z | -\frac{1}{2} \rangle & \langle -\frac{3}{2} | J_z | -\frac{3}{2} \rangle \end{pmatrix}$$

$$J_z^{3/2} = \hbar \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

$$J_+^{3/2} = \hbar \begin{pmatrix} \langle \frac{3}{2} | J_+ | \frac{3}{2} \rangle & \langle \frac{3}{2} | J_+ | \frac{1}{2} \rangle & \langle \frac{3}{2} | J_+ | -\frac{1}{2} \rangle & \langle \frac{3}{2} | J_+ | -\frac{3}{2} \rangle \\ \langle \frac{1}{2} | J_+ | \frac{3}{2} \rangle & \langle \frac{1}{2} | J_+ | \frac{1}{2} \rangle & \langle \frac{1}{2} | J_+ | -\frac{1}{2} \rangle & \langle \frac{1}{2} | J_+ | -\frac{3}{2} \rangle \\ \langle -\frac{1}{2} | J_+ | \frac{3}{2} \rangle & \langle -\frac{1}{2} | J_+ | \frac{1}{2} \rangle & \langle -\frac{1}{2} | J_+ | -\frac{1}{2} \rangle & \langle -\frac{1}{2} | J_+ | -\frac{3}{2} \rangle \\ \langle -\frac{3}{2} | J_+ | \frac{3}{2} \rangle & \langle -\frac{3}{2} | J_+ | \frac{1}{2} \rangle & \langle -\frac{3}{2} | J_+ | -\frac{1}{2} \rangle & \langle -\frac{3}{2} | J_+ | -\frac{3}{2} \rangle \end{pmatrix}$$

$$J_+^{3/2} = \hbar \begin{pmatrix} 0 & \sqrt{\left(\frac{3}{2}-\frac{1}{2}\right)\left(3/2+\frac{1}{2}+1\right)} = \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{\left(\frac{3}{2}-\left(-\frac{1}{2}\right)\right)\left(\frac{3}{2}-\frac{1}{2}+1\right)} = 2 & 0 \\ 0 & 0 & 0 & \sqrt{\left(\frac{3}{2}-\left(-\frac{3}{2}\right)\right)\left(\frac{3}{2}-\frac{3}{2}+1\right)} = \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_-^{3/2} = \hbar \begin{pmatrix} \langle \frac{3}{2} | J_- | \frac{3}{2} \rangle & \langle \frac{3}{2} | J_- | \frac{1}{2} \rangle & \langle \frac{3}{2} | J_- | -\frac{1}{2} \rangle & \langle \frac{3}{2} | J_- | -\frac{3}{2} \rangle \\ \langle \frac{1}{2} | J_- | \frac{3}{2} \rangle & \langle \frac{1}{2} | J_- | \frac{1}{2} \rangle & \langle \frac{1}{2} | J_- | -\frac{1}{2} \rangle & \langle \frac{1}{2} | J_- | -\frac{3}{2} \rangle \\ \langle -\frac{1}{2} | J_- | \frac{3}{2} \rangle & \langle -\frac{1}{2} | J_- | \frac{1}{2} \rangle & \langle -\frac{1}{2} | J_- | -\frac{1}{2} \rangle & \langle -\frac{1}{2} | J_- | -\frac{3}{2} \rangle \\ \langle -\frac{3}{2} | J_- | \frac{3}{2} \rangle & \langle -\frac{3}{2} | J_- | \frac{1}{2} \rangle & \langle -\frac{3}{2} | J_- | -\frac{1}{2} \rangle & \langle -\frac{3}{2} | J_- | -\frac{3}{2} \rangle \end{pmatrix}$$

$$J_-^{3/2} = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{3}{2}+1)} = \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{(\frac{3}{2}+\frac{1}{2})(\frac{3}{2}-\frac{1}{2}+1)} = 2 & 0 & 0 \\ 0 & 0 & \sqrt{(\frac{3}{2}-\frac{1}{2})(\frac{3}{2}+\frac{1}{2}+1)} = \sqrt{3} & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} J_x^{3/2} &= \frac{1}{2}\left(J_+^{3/2} + J_-^{3/2}\right) = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\ J_y^{3/2} &= \frac{1}{2i}\left(J_+^{3/2} - J_-^{3/2}\right) = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} \\ \text{with} \quad J_z^{3/2} &= \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \end{aligned}}$$

5) show that in a state with definite value of L_z, $\langle L_{x,y} \rangle = 0$

$\langle L_x \rangle \& \langle L_y \rangle = 0$

If in eigenstate with L_z, then $L_z|m\rangle = m|m\rangle$

or $\langle m|L_z = \langle m|m$

know

$$[L_z, L_x] = i\hbar L_y$$

$$\hookrightarrow \quad \frac{1}{i\hbar}(L_zL_x - L_xL_z) = L_y$$

$$\frac{1}{i\hbar}\langle m|L_zL_x|m\rangle - \langle m|L_xL_z|m\rangle = \langle m|L_y|m\rangle$$

$$\hookrightarrow \frac{1}{i\hbar}\langle m|mL_x|m\rangle - \langle m|L_x m|m\rangle$$

$$\hookrightarrow \frac{1}{i\hbar} m\left(\underbrace{\langle m|L_x|m\rangle - \langle m|L_x|m\rangle}_{=0}\right) = \langle L_y\rangle$$

$$\hookrightarrow \boxed{\langle L_y\rangle = 0}$$

similarly, we know $[L_y, L_z] = i\hbar L_x$

then $\frac{1}{i\hbar}\langle m|L_yL_z - L_zL_y|m\rangle = \langle L_x\rangle$

$$\hookrightarrow \frac{1}{i\hbar}\underbrace{\langle m|L_y m - mL_y|m\rangle}_{=0} = \langle L_x\rangle$$

$$\boxed{\langle L_x\rangle = 0}$$