Zhi Chen HW#21

1) $H_0 = \frac{p^2}{2m} - a\,\delta(x)$ for $t<0$

$H = H_0 - xF_0 \sin\omega_0 t$ for $t>0$

Initially at ground state of $U(x) = -a\,\delta(x)$

we know the bound state (ground state) of $U(x) = -a\,\delta(x)$ is

$\psi(x, t=0) = \frac{\sqrt{ma}}{\hbar} e^{-\frac{ma}{\hbar^2}|x|}$ with $E_0 = -\frac{ma^2}{2\hbar^2}$

For $E \gg E_0$, we can approximate with free particle, and at $t=0$, we have bound state wave function as initial state.

and free particle wavefunction is given by: $\psi(x) \sim e^{ikx}$

With Fermi's Golden Rule: $W_{0\to k} = \left(\frac{2\pi}{\hbar}\right)|V_{k0}|^2 \delta(E_k - E_0 \pm \hbar\omega_0)$

from ground state to kth state

then $\langle k|V|0\rangle = \int dx\, \psi_{free}^* (-xF_0 \sin\omega_0 t)\, \psi_{bound}$

$$= -F_0 \sin\omega_0 t \int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{\sqrt{ma}}{\hbar} e^{-\frac{ma}{\hbar^2}|x|}$$

$$= -\frac{F_0\sqrt{ma}}{\hbar} \sin\omega_0 t \underbrace{\int_{-\infty}^{\infty} dx\, e^{-ikx}\, x\, e^{-\frac{ma}{\hbar^2}|x|}}$$

Fourier Transform

$$= -\frac{F_0\sqrt{ma}}{\hbar} \sin\omega_0 t \left(\frac{-4i\left(\frac{ma}{\hbar^2}\right)k}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + k^2\right]^2} \right)$$

then $$W_{0\to k} = \frac{\hbar}{2\pi} \left|\langle k|V|0\rangle\right|^2 \delta(E_k - E_0 \pm \hbar\omega_0)$$

let $E_k = E$

choose $-\hbar\omega_0$ since we're already in ground state so want absorption

$$= \frac{\hbar}{2\pi}\left(\frac{F_0^2 ma}{\hbar^2} \frac{\left(4\left(\frac{ma}{\hbar^2}\right)k\right)^2}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + k^2\right]^4}\right) \sin^2\omega_0 t \;\; \delta(E - E_0 - \hbar\omega_0)$$

since $E = \frac{\hbar^2 k^2}{2m}$

$E_0 = \frac{-ma^2}{2\hbar^2}$

$$= \frac{\hbar}{2\pi}\left(\frac{F_0^2 ma}{\hbar^2} \frac{\left(4\left(\frac{ma}{\hbar^2}\right)k\right)^2}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + k^2\right]^4}\right) \sin^2\omega_0 t \;\; \delta\left(\frac{\hbar^2 k^2}{2m} + \frac{ma^2}{2\hbar^2} - \hbar\omega_0\right)$$

Since $W_{0\to k}$ is the transition rate of going from state 0 to k to get probability we simply integrate over t.

$$P_{0\to k} = \int dt \; W_{0\to k}(t)$$

$$= \frac{\hbar}{2\pi}\left(\frac{F_0^2 ma}{\hbar^2} \frac{\left(4\left(\frac{ma}{\hbar^2}\right)k\right)^2}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + k^2\right]^4}\right) \delta\left(\frac{\hbar^2 k^2}{2m} + \frac{ma^2}{2\hbar^2} - \hbar\omega_0\right) \int_0^t dt' \sin^2\omega_0 t'$$

$$P_{0\to k} = \frac{\hbar}{2\pi}\left(\frac{F_0^2 ma}{\hbar^2} \frac{\left(4\left(\frac{ma}{\hbar^2}\right)k\right)^2}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + k^2\right]^4}\right) \delta\left(\frac{\hbar^2 k^2}{2m} + \frac{ma^2}{2\hbar^2} - \hbar\omega_0\right) \left(\frac{t}{2} - \frac{\sin 2\omega_0 t}{4\omega_0}\right)$$

Since $P_{0\to k}$ is the probability of going from ground state to the k-state, we can sum up all probability going to all the k-state.

Then $$P_{0\to 0} = 1 - \int P_{0\to k}\, dk$$

but since $$P_{0\to k} \sim \delta\left(\frac{\hbar^2 k^2}{2m} + \frac{ma^2}{2\hbar^2} - \hbar\omega_0\right)$$

so $$\frac{\hbar^2 k^2}{2m} = -\frac{ma^2}{2\hbar^2} + \hbar\omega_0$$

but we know $\hbar\omega_0 \gg |E_0| = \frac{ma^2}{2\hbar^2}$

so $$\frac{\hbar^2 k^2}{2m} \approx \hbar\omega_0$$

or $$\boxed{k^2 \approx \frac{2m\omega_0}{\hbar}}$$

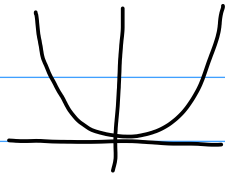
then plug this back to $P_{0\to k}$, then we get

$$P_{0\to 0} = 1 - \frac{\hbar}{2\pi}\left(\frac{F_0^2 ma}{\hbar^2}\right)\left(\frac{\left(4\left(\frac{ma}{\hbar^2}\right)k\right)^2}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + k^2\right]^4}\right)\sin^2\omega_0 t$$

$$\boxed{P_{0\to 0}(t) = 1 - \frac{F_0^2 ma}{2\pi\hbar}\, \frac{16\left(\frac{ma}{\hbar^2}\right)^2 \frac{2m\omega_0}{\hbar}}{\left[\left(\frac{ma}{\hbar^2}\right)^2 + \frac{2m\omega_0}{\hbar}\right]^4}\sin^2\omega_0 t}$$

probability to remain ground state.

2) $V(x) = \alpha |x|^{\nu}$



$$\frac{-\hbar^2}{2m} \partial_x^2 \psi = (E - V(x)) \psi$$

In class, we derived the Bohr-Sommerfeld formula for Bound state:

$$\int_b^a k(x)\, dx = \left(n + \tfrac{1}{2}\right)\pi \quad \text{for } n = 0, 1, 2 \dots$$

In this problem: $k(x) = \dfrac{\sqrt{2m(E - V(x))}}{\hbar}$

$$= \frac{\sqrt{2m(E - \alpha |x|^{\nu})}}{\hbar}$$

then $\displaystyle\int_b^a \frac{\sqrt{2m}}{\hbar} \sqrt{E - \alpha |x|^{\nu}}\, dx = \left(n + \tfrac{1}{2}\right)\pi$

Since the potential is symmetric, $b = -a$ or $\int_b^a \rightarrow 2\int_0^a$

and since we integrate from $[0, a]$, $|x| \rightarrow x$ since its always positive

$$\hookrightarrow 2\int_0^a \frac{\sqrt{2m}}{\hbar} \sqrt{E - \alpha x^{\nu}}\, dx = \left(n + \tfrac{1}{2}\right)\pi$$

$$\hookrightarrow \frac{2}{\hbar}\sqrt{2mE} \int_0^a \sqrt{1 - \frac{\alpha x^{\nu}}{E}}\, dx = \left(n + \tfrac{1}{2}\right)\pi$$

change variable let $z = \frac{\alpha x^{\nu}}{E}$ or $\left(\frac{zE}{\alpha}\right)^{\frac{1}{\nu}} = x$

then $dz = \frac{\alpha}{E} \nu x^{\nu-1} dx$

$$dz = \left(\frac{\alpha}{E}\right)\nu\left(\frac{zE}{\alpha}\right)^{1-\frac{1}{\nu}} dx$$

$$\hookrightarrow \quad dx = dz \frac{1}{\nu}\left(\frac{E}{\alpha}\right)^{\frac{1}{\nu}} z^{\frac{1}{\nu}-1}$$

we know at turning point, a, kinetic energy is zero, and all energy goes to potential.

so $\quad E = V(x=a) = \alpha a^{\nu}$

so $\quad x = \left(\frac{zE}{\alpha}\right)^{\frac{1}{\nu}} = \left(z\frac{\alpha a^{\nu}}{\alpha}\right)^{\frac{1}{\nu}} = z^{\frac{1}{\nu}} a$

so $\quad x = a = z^{\frac{1}{\nu}} a \Rightarrow z = 1$

then the upper bound of integral $\int_{x=0}^{x=a} \rightarrow \int_{z=0}^{z=1}$

then integral becomes:

$$\frac{2}{\hbar}\sqrt{2mE}\,\frac{1}{\nu}\left(\frac{E}{\alpha}\right)^{\frac{1}{\nu}} \underbrace{\int_{z=0}^{z=1} z^{\frac{1}{\nu}-1}\sqrt{1-z}\,dz}_{\text{Mathematica : } \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{\nu}\right)}{2\Gamma\left(\frac{3}{2}+\frac{1}{\nu}\right)}} = \left(n+\frac{1}{2}\right)\pi$$

$$\hookrightarrow \quad \frac{\cancel{2}}{\hbar}\sqrt{2mE}\,\frac{1}{\nu}\left(\frac{E}{\alpha}\right)^{\frac{1}{\nu}} \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{\nu}\right)}{\cancel{2}\Gamma\left(\frac{3}{2}+\frac{1}{\nu}\right)} = \left(n+\frac{1}{2}\right)\pi$$

Rearrange to get E:

then
$$E^{\frac{1}{\nu}+\frac{1}{2}} = (n+\tfrac{1}{2})\nu\hbar\sqrt{\frac{\pi}{2m}}\ \alpha^{\frac{1}{\nu}}\ \frac{\Gamma(\frac{3}{2}+\frac{1}{\nu})}{\Gamma(\frac{1}{\nu})}$$

$$\boxed{E_n = \left[(n+\tfrac{1}{2})\nu\hbar\sqrt{\frac{\pi}{2m}}\ \alpha^{\frac{1}{\nu}}\ \frac{\Gamma(\frac{3}{2}+\frac{1}{\nu})}{\Gamma(\frac{1}{\nu})}\right]^{\frac{2\nu}{2+\nu}}}$$

For Harmonic oscillator $\alpha|x|^\nu = \frac{1}{2}m\omega^2 x^2$

so $\alpha = \frac{1}{2}m\omega^2$ and $\nu = 2$

$$E_n = \left[(n+\tfrac{1}{2})\hbar\ \cancel{2}\sqrt{\frac{\pi}{\cancel{2m}}}\ \sqrt{\tfrac{1}{\cancel{2}}\cancel{m}\omega^2}\ \frac{\Gamma(\frac{3}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right]^{\frac{4}{2+2}}$$

$\Gamma(2) = 1$

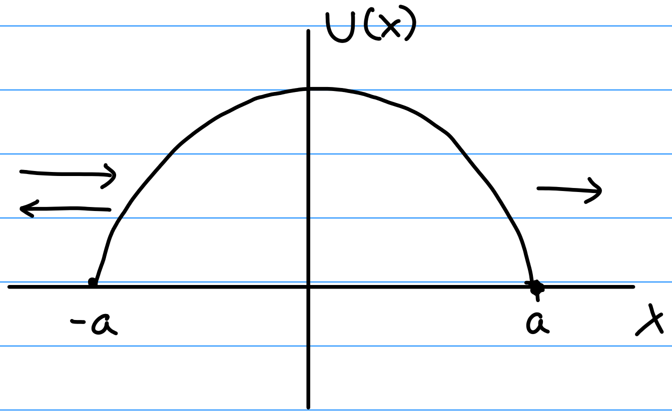
$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega}$$ for $n = 0, 1, 2, 3\cdots$

same answer as Harmonic oscillator.

3) Find transmission coefficient:

$$U(x) = U_0\left(1 - \frac{x^2}{a^2}\right) \qquad \text{for} \qquad |x| < a$$



From Landau-Lifshitz:

Transmission Coefficient: $D = \exp\left\{-\frac{2}{\hbar}\left|\int_{-a}^{a} p(x)dx\right|\right\}$

$$= \exp\left\{-\frac{2}{\hbar}\int_{-a}^{a}\sqrt{2m\,(U(x)-E)}\,dx\right\}$$

let's evaluate the integral:

$$\int_{-a}^{a}\sqrt{U_0\left(1-\left(\frac{x}{a}\right)^2\right)-E}\,dx$$

$$= \int_{-a}^{a}\sqrt{(U_0-E) - U_0\left(\frac{x}{a}\right)^2}\,dx$$

$$= \sqrt{(U_0-E)}\int_{-a}^{a}\sqrt{1-\left(\frac{U_0}{U_0-E}\right)\left(\frac{x}{a}\right)^2}\,dx$$

Since the potential is symmetric, so the integral is symmetric about $x=0$.

So

$$\sqrt{(U_0-E)}\int_{-a}^{a}\sqrt{1-\left(\frac{U_0}{U_0-E}\right)\left(\frac{x}{a}\right)^2}\,dx = 2\sqrt{U_0-E}\int_0^a\sqrt{1-\frac{U_0}{U_0-E}\left(\frac{x}{a}\right)^2}\,dx$$

let $x = az$ then $dx = a\,dz$

$$\rightarrow \quad 2a\sqrt{U_0-E}\underbrace{\int_0^1\sqrt{1-\frac{U_0}{U_0-E}z^2}\,dz}$$

using Mathematica : $\frac{1}{2}\left\{\sqrt{1-\frac{U_0}{U_0-E}}+\sqrt{\frac{U_0-E}{U_0}}\left(\pi+2\,\mathrm{arccot}\left[\frac{-1+\sqrt{1-\frac{U_0}{U_0-E}}}{\sqrt{\frac{U_0}{U_0-E}}}\right]\right)\right.$

So $D = \exp\left\{-\frac{2}{\hbar}\int_{-a}^{a}\sqrt{2m(U(x)-E)}\,dx\right\}$

$$\boxed{D = \exp\left\{\frac{-2a}{\hbar}\sqrt{2m(U_0-E)}\left[\sqrt{1-\frac{U_0}{U_0-E}}+\sqrt{\frac{U_0-E}{U_0}}\left(\pi+2\,\mathrm{arccot}\left[\frac{-1+\sqrt{1-\frac{U_0}{U_0-E}}}{\sqrt{\frac{U_0}{U_0-E}}}\right]\right)\right]\right\}}$$

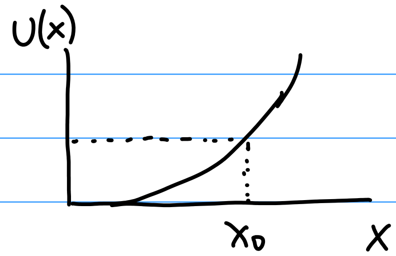
$\uparrow$ transmission coefficient.

In order for this approximation to hold, we require the transmission coefficient to be very small or very negative exponent.

so $$\frac{a\sqrt{m(U_0-E)}}{\hbar} \gg 1$$

or $$\boxed{E \ll U_0 - \frac{\hbar^2}{ma^2}}$$

4)



we observe that at $x=0$, wave function must go to 0.

i.e. $\psi(x=0)=0$

using the connection formula:

$$\frac{2A}{\sqrt{k}}\cos\left(\int_x^{x_0} k(x')dx' - \frac{\pi}{4}\right) - \frac{B}{\sqrt{k}}\sin\left(\int_x^{x_0} k(x)dx - \frac{\pi}{4}\right) \quad \text{for } x \ll x_0$$

$$= \frac{A}{\sqrt{\kappa}}\exp\left\{-\int_{x_0}^{x} \kappa(x')dx'\right\} + \frac{B}{\sqrt{\kappa}}\exp\left\{\int_{x_0}^{x} \kappa(x)dx\right\} \quad \text{for } x \gg x_0$$

We know as $x \gg x_0$, wavefunction should decay, so set the coefficient of growing solution for $x \gg x_0$,

i.e. $\boxed{B=0}$

This means for solution $x \ll x_0$, we have

$$\psi(x) = \frac{2A}{\sqrt{k}}\cos\left(\int_x^{x_0} k(x')dx' - \frac{\pi}{4}\right) \quad \text{for } x \ll x_0$$

using B.C. $\psi(x=0) = \frac{2A}{\sqrt{k}}\cos\left(\int_0^{x_0} k(x')dx' - \frac{\pi}{4}\right) = 0$

this means $\cos\left(\int_0^{x_0} k(x')dx' - \frac{\pi}{4}\right) = 0$

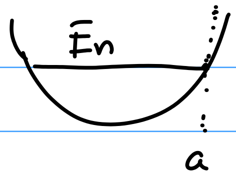
$$\hookrightarrow \int_0^{x_0} k(x')dx' - \frac{\pi}{4} = \left(n+\frac{1}{2}\right)\pi$$

$$\hookrightarrow \int_0^{x_0} k(x')dx' = \left(n+\frac{3}{4}\right)\pi$$

using $k(x') = \frac{1}{\hbar}\sqrt{2m(E-U(x))}$

$$\hookrightarrow \boxed{\int_0^{x_0}\sqrt{2m[E-U(x)]}\,dx = \pi\hbar\left(n+\frac{3}{4}\right)}$$

5) Bohr - Sommerfeld quantization:



$$\int_b^a p(x)\,dx = \int_b^a \sqrt{2m\left[E - U(x)\right]}\,dx = \hbar\pi\left(n+\tfrac{1}{2}\right)$$

consider small perturbation:

$$E = E^{(0)}(x) + \lambda\,\delta E$$
$$U(x) = U^{(0)}(x) + \lambda\,\delta U(x)$$

$\hookrightarrow$ order-tracking parameter, set to 1 in the end.

Since we have a small perturbation in the potential there should be a small perturbation around the turning point.

So $\quad a = a_0 + \lambda\,\delta a \qquad b = b_0 + \lambda\,\delta b$

So we have

$$\int_b^a p(x)\,dx = \int_b^a \sqrt{2m\left(E - U(x)\right)} = \hbar\pi\left(n+\tfrac{1}{2}\right)$$

$$\hookrightarrow \int_{b_0+\lambda\delta b}^{a_0+\lambda\delta a} \sqrt{2m\left\{E^0 - U^{(0)}(x) + \lambda(\delta E - \delta U(x)\right)}\,dx = \hbar\pi\left(n+\tfrac{1}{2}\right)$$

Now lets expand around center values i.e. $\lambda = 0$:

$$\int_{b(\lambda)}^{a(\lambda)} P(x,\lambda)dx = \left.\int_{b(\lambda)}^{a(\lambda)} P(x,\lambda)dx\right|_{\lambda=0} + \lambda \frac{d}{d\lambda}\left\{\int_{b(\lambda)}^{a(\lambda)} P(x,\lambda)dx\right\}_{\lambda=0}$$

use Leibniz rule for differentiation

$$= \int_{b_0}^{a_0} \underbrace{\sqrt{2m(E^{(0)} - U^{(0)}(x))}}_{=p^{(0)}(x)} \, dx$$

$$+ \lambda \left\{ \underbrace{\frac{da}{d\lambda}}_{=\delta a} p(\lambda, a(\lambda))\Big|_{\lambda=0} - \underbrace{\frac{db}{d\lambda}}_{=\delta b} p(\lambda, b(\lambda))\Big|_{\lambda=0} + \int_b^a \frac{d}{d\lambda} P(x,\lambda) dx \Big|_{\lambda=0} \right\}$$

$$= \int_{b_0}^{a_0} p^{(0)}(x)\, dx + \lambda \Big\{ \delta a \, \cancel{p^{(0)}(x=a_0)}^{=0} - \delta b \, \cancel{p^{(0)}(x=b_0)}^{=0}$$

$$\int_{b_0}^{a_0} \frac{1}{2}(\delta E - \delta U(x)) \frac{\sqrt{2m}}{\sqrt{E^{(0)} - U^{(0)}(x)}} dx \Big\}$$

we note that since a_0 and b_0 are the turning points of the unperturbed system, $p^{(0)}(x)$, this means $p^{(0)}(x = a_0, b_0) = 0$ since at turning point kinetic energy is zero.

Then by setting $\lambda = 1$

$$= \int_{b_0}^{a_0} p^{(0)}(x)dx + \int_{b_0}^{a_0} \frac{1}{2}\sqrt{2m} \frac{\delta E - \delta U(x)}{\sqrt{E^{(0)} - U^{(0)}(x)}} dx = \hbar\pi(n + \tfrac{1}{2})$$

but since the unperturbed system should also satisfy the quantization condition

$$\underbrace{\cancel{\int_{b_0}^{a_0} p^{(0)}(x)\,dx}}_{=\hbar\pi\left(n+\frac{1}{2}\right)} + \int_{b_0}^{a_0} \frac{1}{2}\sqrt{2m}\,\frac{\delta E - \delta U(x)}{\sqrt{E^{(0)} - U^{(0)}(x)}}\,dx = \cancel{\hbar\pi\left(n+\tfrac{1}{2}\right)}$$

$$\Rightarrow \int_{b_0}^{a_0} \frac{1}{2}\sqrt{2m}\,\frac{\delta E - \delta U(x)}{\sqrt{E^{(0)} - U^{(0)}(x)}}\,dx = 0$$

then rearrange:

$$\delta E \int_{b_0}^{a_0} \frac{dx}{\sqrt{E^{(0)} - U^{(0)}(x)}} = \int_{b_0}^{a_0} \frac{\delta U(x)}{\sqrt{E^{(0)} - U^{(0)}(x)}}\,dx$$

or $$\delta E = \frac{\int_{b_0}^{a_0} \frac{\delta U(x)}{\sqrt{E^{(0)} - U^{(0)}(x)}}\,dx}{\int_{b_0}^{a_0} \frac{dx}{\sqrt{E^{(0)} - U^{(0)}(x)}}\,dx}$$

recall $$E = \frac{p^2}{2m} + U = \frac{1}{2}mv^2 + U$$

so $$\sqrt{\frac{m}{2}}\,v = \sqrt{E^{(0)} - U^{(0)}(x)}$$

then $$\delta E = \frac{\int_{b_0}^{a_0} \frac{\delta U(x)}{v}\,dx}{\int_{b_0}^{a_0} \frac{dx}{v}}$$

since $v = \frac{dx}{dt}$

then $\int_{b_0}^{a_0} \frac{dx}{v} = \int_{t(x=b_0)}^{t(x=a_0)} dt = t(x=a_0) - t(x=b_0) = \frac{T}{2}$

we know since a_0 and b_0 are turning points, so the time difference is exactly half of a period.

So $$\boxed{\delta E = \int_{b_0}^{a_0} \frac{2}{T} \frac{1}{v(x)} \delta U(x)\, dx}$$

we observe that the quantity $\frac{2}{T}\frac{1}{v(x)}$ is analogous to the classical probability density of finding a particle between x and $x+dx$.

And therefore, the right-hand-size is like the average change of potential energy throughout $[a_0, b_0]$.

or $$\delta E \cong \langle \delta U(x) \rangle_{[a_0, b_0]}$$

This overall procedure also reminds me of the adiabatic invariance in classical mechanics where we slowly perturb the system and the adiabatic invariant $\oint p\,dq$ of the system stays invariant.

6) N-identical spin-½ in 1D Harmonic oscillator
Fermion

know $E_n = \hbar\omega(n+\frac{1}{2})$ for $n = 0, 1, 2, 3 \cdots$

due to pauli-exclusion principle, no two fermion can occupy the same state.

For spin-½, each state, n, can have two fermions, spin-up and spin down.

If N is even, then we will occupy from 0 and up to $\frac{N}{2}-1$ states.

then $E_{ground} = 2\hbar\omega(n+\frac{1}{2})$ for $n = 0, 1, 2 \cdots \frac{N}{2}-1$

$$= \hbar\omega\underbrace{[1+3+5+7+\cdots+N-3+N-1]}$$

using the arithmetic series summation $S_n = n\left(\frac{a_1+a_n}{2}\right)$

$$= \hbar\omega\frac{N}{2}\left(\frac{1+N-1}{2}\right)$$

$$\boxed{E_{ground} = \hbar\omega\frac{N^2}{4} \quad \text{for } N = \text{even, i.e. } N = 0, 2, 4\cdots}$$

If <u>N is odd:</u> then N-1 is even, then use formula we derived above but let $N \to N-1$. Then we still need to add one additional state which occupies the most energy at $n = \frac{N-1}{2}$

so

$$\boxed{\begin{aligned} E_{ground} &= \hbar\omega\frac{(N-1)^2}{4} + \hbar\omega\left(\frac{N-1}{2}+\frac{1}{2}\right) \\ &= \frac{\hbar\omega}{4}(N^2+1) \quad \text{for } N = \text{odd, i.e. } N = 1, 3, 5, 7\cdots \end{aligned}}$$

Since Fermi-energy is the energy of the highest populated state,

For N= even, the highest populated state is when

$$n = \frac{N}{2} - 1$$

then $E_F = E_{n=\frac{N}{2}-1} = \hbar\omega\left(\frac{N}{2} - 1 + \frac{1}{2}\right)$

$$\boxed{E_F = \frac{1}{2}\hbar\omega(N-1) \quad \text{for} \quad N = \text{even}}$$

For N= odd, state that occupies highest energy is $n = \frac{N-1}{2}$

so $\boxed{E_F = \hbar\omega\left(\frac{N-1}{2} + \frac{1}{2}\right) = \hbar\omega\frac{N}{2} \quad \text{for} \quad N = \text{odd}}$