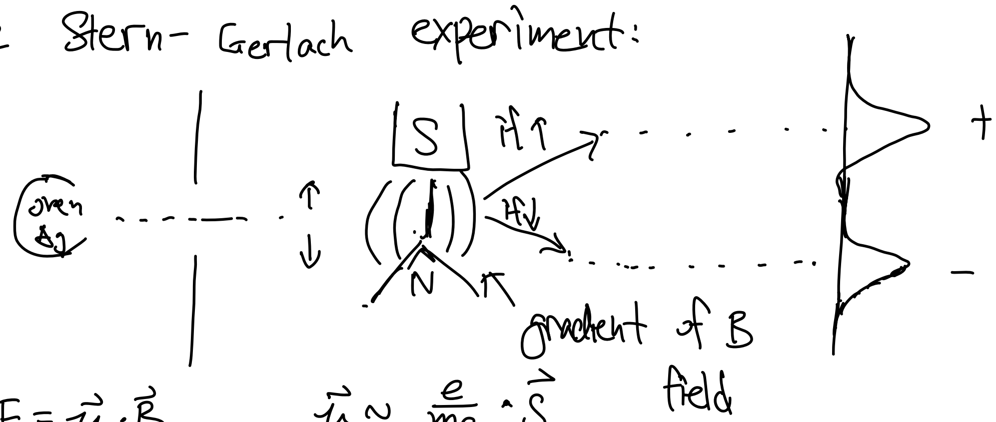
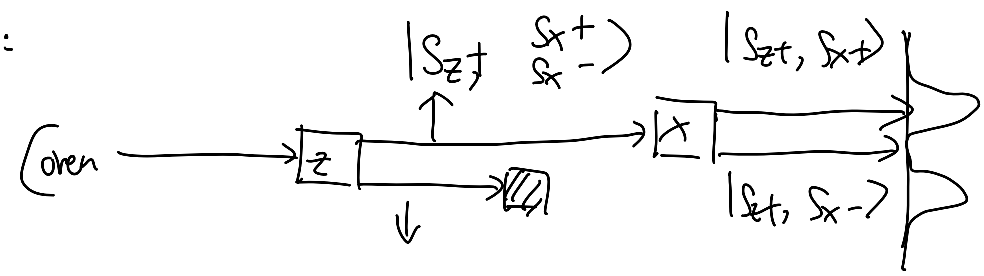
The Stern-Gerlach experiment:

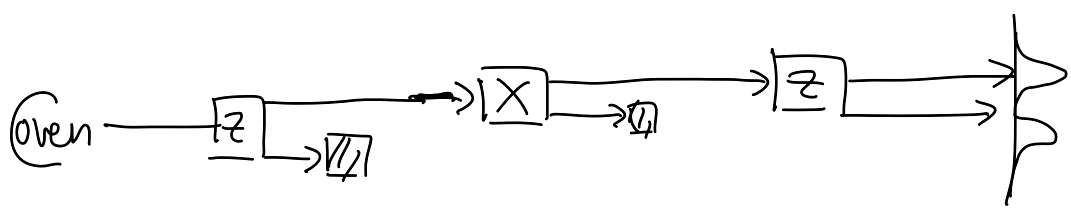


$E = \vec{u} \cdot \vec{B}$, $\vec{u} \sim \frac{e}{mc} \cdot \vec{S}$

$F = -\frac{\partial E}{\partial z} = u \frac{\partial B}{\partial z}$

If:





let $\vec{E} = E_0 \hat{x} \cos(kx - wt)$
$\vec{E} = E_0 \hat{z} \cos(kx - wt)$

	Classical Mechanics	Quantum Mechanics
States	point in phase-space. (x, p) $\{x_i, p_j\} = \delta_{ij}$	vector in a complete Hilbert space. $\|\psi\rangle \in \mathcal{H}$

	poisson brackets.	
observable	$O(x,p)$	operator act on $\mathcal{H}$ $\hat{O}\vert\psi\rangle = \vert\hat{O}\psi\rangle$ for observables, $\hat{O}^\dagger = \hat{O}$
Dynamics	$H(x,p)$: Hamiltonian $\dot{x}_i = \frac{\partial H}{\partial p_i}$ $\quad \dot{p}_i = \frac{-\partial H}{\partial x_i}$	$H^\dagger = H$ $\hat{H}\vert\psi\rangle = i\hbar \frac{d}{dt}\vert\psi\rangle$
Measurement	(x,p) , $O(x,p)$ State is unchanged under measurement.	$\hat{A}\vert\lambda_i\rangle = \lambda\vert\lambda_i\rangle$ $\uparrow$ eigenstate $\vert\psi\rangle = \sum_i \alpha_i \vert\lambda_i\rangle$ produces λ_i with probability $\vert\alpha_i\vert^2$ After measurement $\vert\psi\rangle$ collapses $\vert\psi\rangle \to \vert\lambda_i\rangle$

Math Preliminaries:

- The state of QM system at time t is given by a ray $|\psi\rangle$ in a complete Hilbert space.

- Hilbert Space: a vector space with inner product which is complete.

- Vector Space: V is a set of vectors, $|\alpha\rangle \in V$ having the following properties:

$\Longrightarrow$ V is a commutative group w.r.t. $+$

① $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$

two vectors give another vector.

② $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

③ $\left(|\alpha\rangle + |\beta\rangle\right) + |\gamma\rangle = |\alpha\rangle + \left(|\beta\rangle + |\gamma\rangle\right)$

④ $\exists\, |0\rangle$ such that

$$|\alpha\rangle + |0\rangle = |\alpha\rangle \quad \forall\, |\alpha\rangle \in V$$

⑤ $\forall\, |\alpha\rangle \in V \quad \exists\, |-\alpha\rangle \in V : |\alpha\rangle + |-\alpha\rangle = |0\rangle$

$\Longrightarrow$ For some field F ($\mathbb{R}$ or $\mathbb{C}$): multiplication by number from F:

$$\underset{\in F}{c}\,\underset{\in V}{|\alpha\rangle} \in V$$

① $c\left(d|\alpha\rangle\right) = (cd)|\alpha\rangle$

② $1|\alpha\rangle = |\alpha\rangle$

③ $c(|\alpha\rangle + |\beta\rangle) = c|\alpha\rangle + c|\beta\rangle$

④ $(c+d)|\alpha\rangle = c|\alpha\rangle + d|\alpha\rangle$

If $F = \mathbb{R}$, then we have real vector space.

If $F = \mathbb{C}$, then we have complex vector space.

<u>Examples</u>:

1) Euclidean d-dimensional space $\mathbb{R}^d$
   ↳ real-vector space.

2) 2D complex vector space:

$$|\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad \text{where } \alpha_{1,2} \in \mathbb{C}$$

$$\stackrel{!}{=} \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

3) space of complex valued functions $f(x) \in \mathbb{C}$
   for $x \in \mathbb{R}$

Linear Independence and bases:

$|\alpha_1\rangle, |\alpha_2\rangle \cdots |\alpha_i\rangle$

$$c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + c_3|\alpha_3\rangle + \cdots + c_n|\alpha_n\rangle = |0\rangle$$

If $c_i = 0$ is the only solution, then $\{|\alpha_i\rangle\}$ are linearly independent.

V: $|\alpha_1\rangle \cdots |\alpha_n\rangle$ linearly independent vectors are called basis of V.

but any $|\beta_1\rangle \cdots |\beta_n\rangle$ are linearly dependent.

then we have n-dimension

Theorem:

V: $|\alpha_i\rangle$ $i = 1, \cdots n$, basis of V, then $\forall\ |\beta\rangle \in V$

$$|\beta\rangle = \sum_{i=1}^{n} c_i |\alpha_i\rangle$$ and c_i are defined uniquely.

when n is finite, then we have n-dimension.

when n is ∞, we can have:

1. Countable, separable basis: $|\alpha_1\rangle, |\alpha_2\rangle, \cdots |\alpha_n\rangle$
2. uncountable $|\alpha_x\rangle$ for $x \in \mathbb{R}$.

Dual Spaces:

- Complex Vector V, $|\alpha\rangle \in V$
- then V^* - the dual space: space of all linear complex-valued function on V.

$$\beta(|\alpha\rangle) = \text{some complex number.}$$

here $\beta = \langle\beta|$

For example: V: $|e_1\rangle, |e_2\rangle \cdots$

let $|\alpha\rangle = \sum_i \alpha_i |e_i\rangle$

then $\langle e_i|\alpha\rangle = \alpha_i$

Inner product space:

V is "inner product space" (unitary space)

if $\forall\ |\alpha\rangle, |\beta\rangle \in V$ then

$$\underbrace{\text{inner product}(|\alpha\rangle, |\beta\rangle)}_{\langle\alpha|\beta\rangle} = \underset{\uparrow\text{some complex \#.}}{C} \Rightarrow V \otimes V \to \mathbb{C}$$

they obey the following properties:

① $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$

② $\langle\alpha|\left(|\beta\rangle + |\gamma\rangle\right) = \langle\alpha|\beta\rangle + \langle\alpha|\gamma\rangle$

③ $\langle\alpha|c|\beta\rangle = c\langle\alpha|\beta\rangle$

④ $\langle\alpha|\alpha\rangle \geq 0$

if $\langle\alpha|\alpha\rangle = 0$ then $|\alpha\rangle = |0\rangle$

Ex: a) $V = \mathbb{C}^N$ or $|z\rangle = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$ where $z_i \in \mathbb{C}$

$$\langle z|w\rangle = z^\dagger w = \sum_{i=1}^{N} z_i^* w_i$$

b) $V = \{f : [a,b] \to \mathbb{C}\}$

$$\langle f|g\rangle = \int_a^b f^*(x)\, g(x)\, dx$$

Norm of α: $\sqrt{\langle\alpha|\alpha\rangle}$

If $\langle\alpha|\beta\rangle = 0$ then $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal.

Isomorphism $V \to V^*$

$\forall \, |\beta\rangle \in V$, define $\langle\beta| \in V^*$ then

$$\beta(|\alpha\rangle) = \langle\beta|\alpha\rangle$$

Hilbert Space:

V - inner product complex vector space.

then we have a norm $\| \, |\alpha\rangle \, \|$,

then define Cauchy Sequence:

$\{ |\alpha_n\rangle \}$ $\quad \forall \, \epsilon > 0 \quad \exists \, N > 0$

so that $\| \, |\alpha_n\rangle - |\alpha_m\rangle \, \| < \epsilon \quad \forall (n, m) > N$

If $\forall \, |\alpha_n\rangle$ - Cauchy sequence

and $\lim_{n \to \infty} |\alpha_n\rangle - |\alpha\rangle = 0$

then V is complete.

Example of incomplete space:

$V : \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$ then $\begin{array}{ccc} |\alpha_1\rangle & |\alpha_2\rangle & |\alpha_3\rangle \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} & \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \end{array}$

$$\begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1/3 \\ \vdots \\ 1 \end{pmatrix}$$

Here V has Cauchy sequence, but not complete.

Since. $\lim\limits_{n\to\infty} |\alpha_n\rangle - |\alpha\rangle \neq 0$.

Dimensions:

a) n-dimension

b) ∞ dimension but countable ($|\alpha_i\rangle$, $|\alpha_1\rangle, |\alpha_2\rangle \cdots$)
or separable.

c) ∞ dimension but uncountable $|\alpha_x\rangle$ for $x \in \mathbb{R}$

Separable:

V - separable if $\exists$ countable set $D \subset V$ so that D is dense in V.

Orthonormal basis:

$\{|\phi_i\rangle\}$ orthonormal basis of V

then $\langle\phi_i, \phi_j\rangle = \delta_{i,j}$

Gram - Schmidt procedure: to get orthonormal basis.

$$|\alpha_n'\rangle = |\alpha_n\rangle - \sum_{i=1}^{n-1} \langle\phi_i | \alpha\rangle |\phi_i\rangle$$

$$|\alpha\rangle = \sum_i \langle \phi_i|\alpha\rangle\, |\phi_i\rangle$$

$$\overset{!}{=} \sum_i |\phi_i\rangle\langle\phi_i|\alpha\rangle$$

So $\boxed{\sum_i |\phi_i\rangle\langle\phi_i| = 1}$ ← completeness relation.

Schwarz Inequality:

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle^2| \quad \forall\ |\alpha\rangle, |\beta\rangle \in H$$

proof: $|\gamma\rangle = |\alpha\rangle + \lambda|\beta\rangle$

then $\langle\gamma|\gamma\rangle \geq 0$

↳ $\left(\langle\alpha| + \lambda^*\langle\beta|\right)\left(|\alpha\rangle + \lambda|\beta\rangle\right)$

↳ $\langle\alpha|\alpha\rangle + |\lambda|^2\langle\beta|\beta\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle \geq 0$

$$\lambda = \frac{-\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} \qquad \lambda\langle\beta|\beta\rangle + \langle\beta|\alpha\rangle = 0$$

↳ $\langle\alpha|\alpha\rangle + \dfrac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} - 2\dfrac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} \geq 0$

↳ $\langle\alpha|\alpha\rangle - \dfrac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle} \geq 0$

↳ $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\beta|\alpha\rangle|^2$

Operators

linear operator A: such that $V \to W$ or $W^* \to V^*$

$$A\left(a_1|\alpha\rangle + a_2|\beta\rangle\right) = a_1 A|\alpha\rangle + a_2 A|\beta\rangle$$

$$\left(\langle\beta|A\right)|\alpha\rangle \equiv \langle\beta|\left(A|\alpha\rangle\right) \equiv \langle\beta|A|\alpha\rangle$$

$|\alpha\rangle \in V$ and $|\beta\rangle \in W$

Outer products:

class of operators

$$A = |\beta\rangle\langle\alpha|$$

$$A|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle$$

Adjoint Operators

Isomorphism between V and V^*

$$\left(|\alpha\rangle\right)^\dagger = \langle\alpha|$$

$$\left(\langle\alpha|\right)^\dagger = |\alpha\rangle$$

For $A: V \to V$

$$A^\dagger|\alpha\rangle = (\langle\alpha|A)^\dagger$$

property: $(AB)^\dagger = B^\dagger A^\dagger$

<u>proof:</u> $(B^\dagger A^\dagger|\alpha\rangle, |\beta\rangle)_{\text{inner}} = (A^\dagger|\alpha\rangle, B|\beta\rangle)$

$$\stackrel{!}{=} (|\alpha\rangle, AB|\beta\rangle)$$

$$\stackrel{!}{=} ((AB)^\dagger|\alpha\rangle, |\beta\rangle)$$

$\langle\alpha|A|\beta\rangle$ = Inner product of $(A^\dagger|\alpha\rangle, |\beta\rangle)$

$\stackrel{!}{=}$ Inner product of $(|\beta\rangle, A^\dagger|\alpha\rangle)^*$

$$\stackrel{!}{=} (\langle\beta|A^\dagger|\alpha\rangle)^*$$

$$\hookrightarrow \boxed{\langle\alpha|A|\beta\rangle = (\langle\beta|A^\dagger|\alpha\rangle)^*}$$

let $A^\dagger = (|\alpha\rangle\langle\beta|)^\dagger = |\beta\rangle\langle\alpha|$

$$\langle\lambda|A^\dagger|u\rangle = \langle\lambda|\beta\rangle\langle\alpha|u\rangle$$

then $\langle u|A|u\rangle = \langle u|\alpha\rangle\langle\beta|\lambda\rangle$

$$\langle u|\alpha\rangle\langle\beta|\lambda\rangle = (\langle\lambda|\beta\rangle\langle\alpha|u\rangle)^*$$

$$\stackrel{!}{=} \langle u|\alpha\rangle\langle\beta|\lambda\rangle$$

Hermitian Operators:

$$\boxed{A \text{ is Hermitian operator if } A = A^{\dagger}\text{, or self-adjoint.}}$$

Operators form vector space themselves:

$$(A+B)|\alpha\rangle = A|\alpha\rangle + B|\alpha\rangle$$

$$(aA)|\alpha\rangle = a(A|\alpha\rangle)$$

$$(AB)|\alpha\rangle = A(B|\alpha\rangle)$$

$$A(BC)|r\rangle = (AB)C|r\rangle = ABC|r\rangle$$

Functions on Operators

$$f(x) = e^{x} = \sum_{i=0}^{\infty} C_n x^n$$

$$f(A) = \sum_{i=0}^{\infty} C_n A^n = \sum_{i=0}^{\infty} \frac{A^n}{n!} \quad \text{for } f(x) = e^{x}$$

For $f(x,y) = e^{x+y}$ ← for multi values, it is more tricky since order matters. (orderness prescription)

Unitary Operators:

If U is isometries (preserves inner product)

then $U^\dagger U = \mathbb{1}$ notice here we don't require $UU^\dagger = 1$

$$\boxed{\begin{array}{c}\text{Operator } U \text{ is an unitary operator if:}\\ U^\dagger U = UU^\dagger = 1\\ \text{or } U^\dagger = U^{-1}\end{array}}$$

So all unitary operators are isometries, but not all isometries operators are unitary.

For $S|n\rangle = |n+1\rangle$

$S = |n+1\rangle\langle n|$

$$S^\dagger S = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \, |m+1\rangle\langle m|$$

$$\overset{!}{=} \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$$\overset{!}{=} \mathbb{1}$$

$$SS^\dagger = |n+1\rangle\langle n| \, |m\rangle\langle m+1|$$

$$\overset{!}{=} \sum_{n=0}^{\infty} |n+1\rangle\langle n+1|$$

$= \mathbb{1} - |0\rangle\langle 0|$ ← since start with $|1\rangle\langle 1|$

$\boxed{\begin{array}{l}\text{Projection Operator:} \\ \quad A \text{ is projector if } A^2 = A.\end{array}}$

ex: $A = |\alpha\rangle\langle\alpha|$

$A^2 = |\alpha\rangle\langle\alpha|\alpha\rangle\langle\alpha| = |\alpha\rangle\langle\alpha|.$

or $A = |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$

Eigenvalues and Eigenvectors:

$A|\alpha\rangle = a|\alpha\rangle$, a is the eigenvalue of A.

$|\alpha\rangle$ is eigenvector of A

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_+ = +1$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda_- = -1$

Spectrum: a collection of eigenvalues for operator, (results of measurement)

$\mathrm{Spec}(A) = \{a\}$

ex: $\mathrm{Spec}(\sigma_x) = \{1, -1\}$

Theorem:

If operator is Hermitian, $\hat{A} = A^{\dagger}$, then all eigenvalues a_i are real, and all eigenstates associated with different eigenvalues are $\perp$.

$$A|\alpha\rangle = a|\alpha\rangle \qquad \text{if } a \neq b, \ \langle\beta|\alpha\rangle = 0$$
$$A|\beta\rangle = b|\beta\rangle$$

proof:

$$A|a\rangle = a|a\rangle$$
$$\langle a|A^{\dagger} = a^{*}\langle a|$$

$$0 = \langle\beta| A - A^{\dagger}|a\rangle = \left(a - \beta^{*}\right)\langle\beta|a\rangle = 0$$

if $|\beta\rangle = |a\rangle$:

then $(a - a^{*})\langle a|a\rangle = 0$

so $a = a^{*}$ or a is real.

if $|\beta\rangle \neq |a\rangle$:

then $(a - \beta)\underbrace{\langle\beta|a\rangle}_{=0} = 0$

$\underline{\text{Completeness Relation:}}$

If $|\varphi_i\rangle$ orthnormal basis of Hilber space.

$$|\alpha\rangle = \underbrace{\sum_i |\varphi_i\rangle\langle\varphi_i|}\,\alpha\rangle$$

$$\therefore \sum_i |\varphi_i\rangle\langle\varphi_i| = 1$$

Matrix and Vector Representations:

$$|\alpha\rangle = \sum_i |\varphi_i\rangle\langle\varphi_i|\alpha\rangle = \begin{pmatrix} \langle\varphi_1|\alpha\rangle \\ \langle\varphi_2|\alpha\rangle \\ \vdots \\ \vdots \end{pmatrix}$$

$$\langle\beta| = \sum_i \langle\beta|\varphi_i\rangle\langle\varphi_i| = \left(\langle\beta|\varphi_1\rangle, \langle\beta|\varphi_2\rangle, \langle\beta|\varphi_3\rangle, \cdots\right)$$

$$\stackrel{!}{=} \begin{pmatrix} \langle\varphi_1|\beta\rangle \\ \langle\varphi_2|\beta\rangle \\ \vdots \end{pmatrix}^{\dagger}$$

$$A = \sum_{i,j} |\varphi_i\rangle\langle\varphi_i|A|\varphi_j\rangle\langle\varphi_j| = \begin{pmatrix} \langle\varphi_1|A|\varphi_1\rangle & \langle\varphi_1|A|\varphi_2\rangle & \cdots \\ \langle\varphi_2|A|\varphi_1\rangle & \langle\varphi_2|A|\varphi_2\rangle & \cdots \\ \vdots & & \ddots \end{pmatrix}$$

$$A|a_i\rangle = a_i|a_i\rangle$$
$$\langle a_j|A|a_i\rangle = a_i \underbrace{\langle a_j|a_i\rangle}_{\delta_{ij}}$$

Pauli Matricesi

Consider 2-D Hilbert space:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1|1\rangle + \alpha_2|2\rangle$$

$$\langle 1| = (1, 0)$$
$$\langle 2| = (0, 1)$$

$$|i\rangle\langle j|$$

$$|1\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 2| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|2\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1, 0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|2\rangle\langle 2| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|i\rangle\langle j| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Hermitian Basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 = \sigma_x$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2 = \sigma_y \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 = \sigma_z$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$+ \frac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\frac{b-c}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$A = \frac{a+d}{2}\sigma_0 + \frac{a-d}{2}\sigma_3 + \frac{b+c}{2}\sigma_1 + i\frac{b-c}{2}\sigma_2$$

$$A^\dagger = \frac{a^*+d^*}{2}\sigma_0 + \frac{a^*-d^*}{2}\sigma_3 + \frac{b^*+c^*}{2}\sigma_1 - i\frac{b^*-c^*}{2}\sigma_2$$

If $A = A^{\dagger} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ or Hermitian

so $a^* = a$ and $d = d^*$, both a,d are real.

$b^* + c^* = b^* + b =$ real

$i(b^* - c^*) = i(b^* - b) =$ real.

So if A is Hermitian:

$$\underline{\underline{A = \sum_{i=0}^{3} A^i \sigma_i}}$$

Do eigenstate of A (for Hermitian, A.) form a complete Hilbert space?

→ Yes, if $\dim(\mathcal{H}) < \infty$, which is what we consider.

<u>Trace</u> : $\mathrm{Tr}\, A = \sum_i \langle \varphi_i | A | \varphi_i \rangle = \sum_i A_{ii}$

$\sum_i \langle a_i | A | a_i \rangle = \sum a_i$

Unitary Transformation:

$\{|a\rangle\}$ $\{|b\rangle\}$ orthonormal basis

Define $U|a_i\rangle = |b_i\rangle$

transforms to another orthonormal basis

$$\boxed{U = U\sum|a_i\rangle\langle a_i| = \sum|b_i\rangle\langle a_i|}$$

$$U^\dagger = |a_i\rangle\langle b_i|$$

$$UU^\dagger = \sum_{i,j}|b_i\rangle\underbrace{\langle a_i|a_j\rangle}_{\delta_{ij}}\langle b_j| = \sum|b_i\rangle\langle b_i|$$

and $U^\dagger U = UU^\dagger = 1$

so $U^{-1} = U^\dagger$ so U is unitary.

ex:

$$|\alpha\rangle = c_i|a_i\rangle$$
$$|\alpha\rangle = d_i|b_i\rangle \qquad U = |b_i\rangle\langle a_i|$$

$$|\alpha\rangle = \sum_j d_j \underbrace{U|a_j\rangle} = \sum_{i,j} d_j |a_i\rangle\langle a_i|U|a_j\rangle$$
$$\overset{!}{=} \sum_i \underbrace{\sum_j d_j \langle a_i|U|a_j\rangle}_{c_i} |a_i\rangle$$

$$c_i = U_{ij}d_j \qquad \text{for } U_{ij} = \langle a_i|U|a_j\rangle$$

For X operator:

$$X = |a_i\rangle X_{ij} \langle a_j|$$

$$\stackrel{!}{=} |b_k\rangle Y_{kl} \langle b_l|$$

$$\boxed{X_{ij} = U_{ik} Y_{kl} U^{\dagger}_{lj}}$$

Diagonalization of Hermitian Operators:

Theorem:

A Hermitian Matrix (Finite Dim)

$$H_{ij} = \langle \varphi_i | H | \varphi_j \rangle$$

can always be diagonalized by U

proof! $|\varphi_i\rangle$ orthonormal basis:

$$|h_i\rangle = U|\varphi_i\rangle$$

$$\langle h_i | H | h_j \rangle = \langle \varphi_i | U^{\dagger} H U | \varphi_j \rangle$$

Now the basis are orthonormal basis, so diagonalized.

Algorithm:

1) Solve determine $(H - \lambda \mathbb{1}) = 0$ find $\lambda_1, \lambda_2 \cdots \lambda_d$

2) $H|h_i\rangle = \lambda^{(i+1)}|h_i\rangle$

$$|h_i\rangle = \sum_i c_i|\varphi_i\rangle$$

$$H_{ij} c_j = \lambda c_j$$

solve for c_j

ex: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm 1$$

$$\sigma_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = +1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{or} \quad \alpha = \beta$$

let $\alpha = \frac{1}{\sqrt{2}} = \beta$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \quad \text{or} \quad \beta = -\alpha$$

let $\alpha = \frac{1}{\sqrt{2}} \quad \beta = \frac{-1}{\sqrt{2}}$

then $U = E^+ + E^-$

$$\overset{!}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Invariant:

$\Rightarrow$ $f(A)$ is invariant under $A \to U^+ A U$
then f is invariant.

Ex: $Tr(A)$: $\Rightarrow$ $Tr\, U^+ A U = Tr(\underbrace{U U^+}_{=1} A)$

then Trace is invariant.

Determinant (A) $\Rightarrow$ $Det(A) = Det(U^+ A U)$

$$\overset{!}{=} Det(U^+)\, Det(A)\, Det(U)$$

$$\overset{!}{=} Det(\underbrace{U^+ U}_{=1})\, Det(A)$$

For a general $f(A) = \sum c_m A^m$

Invariant if $f(A) = f(U^+ A U)$

$$\overset{!}{=} \sum c_m (U^+ A U)^m$$

$$\overset{!}{=} \sum c_m U^+ A^m U = f(diag(\lambda_1, \lambda_2 \dots \lambda_n))$$

Simultaneous Diagonalization:

A, B : diagonalized operator:

$U^{+} A U$ and $U^{+} B U$

is there U that diagonalizes both A and B.

It happens when $[A, B] = 0$ or $AB = BA$.

Proof:

1) If Diagonalizable: $A|\alpha_1\rangle = a|\alpha_1\rangle$
$B|\alpha_1\rangle = b|\alpha_1\rangle$

then they commute.

$$[AB - BA]\,|\alpha_1\rangle = (a_1 b_1 - b_1 a_1)|\alpha_1\rangle = 0$$

2) If they commute: $[A, B] = 0$

and

$$A|\alpha_1\rangle = a|\alpha_1\rangle$$

Show B is also diagonalizable.

$$0 = [A, B]\,|\alpha_1\rangle = (AB - BA)|\alpha_1\rangle$$

$$= AB|\alpha_1\rangle - a\,B|\alpha_1\rangle = 0$$

so $A B|\alpha_1\rangle = a B|\alpha_1\rangle$

$\boxed{\text{If } [A,B]=0 \text{ then } A, B \text{ are simultaneously diagonalizable}}$

or compatible observable