Zhi Chen HW# 16

1) $AB = -BA$

If Ψ is simultaneously eigenstate of A, B, then (Hermitian)

$$A|\Psi\rangle = a|\Psi\rangle$$
$$B|\Psi\rangle = b|\Psi\rangle$$

a, b are real since A, B are Hermitian.

$AB|\Psi\rangle = Ab\Psi\rangle \stackrel{!}{=} bA|\Psi\rangle \stackrel{!}{=} ab|\Psi\rangle$

$$AB|\Psi\rangle + BA|\Psi\rangle = 0$$

$$ab|\Psi\rangle + ab|\Psi\rangle = 0$$

$$2ab|\Psi\rangle = 0$$

For a nonzero $|\Psi\rangle$, we see either $\boxed{a=0 \text{ or } b=0}$

This can be illustrated using by $A = \pi$ and $B = p$

Since we know $\{\pi, \vec{p}\} = 0$ or $\pi\vec{p}\pi^{+} = -\vec{p}$

or $\pi\vec{p} = -\vec{p}\pi$

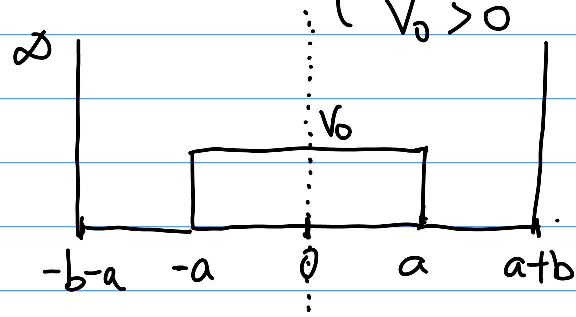
$$\pi\vec{p}|\Psi\rangle = -\vec{p}\pi|\Psi\rangle$$

We know π has two possible eigenvalues, ± 1, so can't be 0

$$(\pm 1)p'|\Psi\rangle = -p'(\pm 1)|\Psi\rangle$$

$2(\pm 1)p'|\Psi\rangle = 0$ ← we see for nonzero $|\Psi\rangle$, require $\boxed{p'=0}$ as the only eigenvalue for $\vec{p}$ in $|\Psi\rangle$ if $|\Psi\rangle$ is also eigenstate of π

2) Consider $V = \begin{cases} \infty & \text{for } |x| > a+b \\ 0 & \text{for } a < |x| < a+b \\ V_0 > 0 & \text{for } |x| < a \end{cases}$



Assume $V_0 \gg E_0$

Find ΔE between E_0 and E_1

Since $V(x) = V(-x)$, so Hamiltonian is even under parity $[H, \Pi] = 0$, which means they diagonalize simultaneously

For $|x| \geq b$, $V \Rightarrow \infty$ $\rightarrow$ $\psi(x) = 0$ for $|x| > b$

For $|x| < a$: $V = V_0$: $\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \psi = (E - V_0)\psi$

$$\frac{d^2}{dx^2}\psi = -\frac{2m(E-V_0)}{\hbar^2}\psi$$

Since assume $V_0 \gg E$, $V_0 - E \approx V_0 > 0$

define $\frac{2m(V_0-E)}{\hbar^2} \cong \frac{2mV_0}{\hbar^2} = K$

$$\frac{d^2}{dx^2}\psi = K^2\psi$$

or $\psi(x) = C\sinh Kx + D\cosh Kx$ for $|x| < a$

$\uparrow$ asymmetric sol $\qquad$ $\uparrow$ symmetric sol

Since solutions are subject to parity: $\psi_a(-x) = -\psi_a(x)$
$\psi_s(-x) = \psi_s(x)$

let's just consider $|x| > 0$:

so For $a<x<a+b$ $\quad V=0$: $\quad \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$

$$\hookrightarrow \frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi$$

define: $K^2 = \frac{2mE}{\hbar^2}$

$\hookrightarrow \frac{d^2}{dx^2}\psi = -K^2\psi$ subject to B.C. $\psi(x=a+b) = 0$

the solution that matches this B.C. is

$\psi = A\sin[K(x-a-b)]$ for $a<x<a+b$.

Now lets first consider <u>odd (asymetric) solutions:</u>

For $x<a$: the odd solution is $\psi(x) = C\sinh Kx$

For $a<x<a+b$, we choose $k=k_a$, where k_a makes $\sin(k_a(x-a-b))$ odd in the interval $a<x<a+b$.

so $\psi(x) = A\sin[k_a(x-a-b)]$

Now we need to match B.C. at $x=a$, $\psi(a)$ and $\psi'(a)$

$$\psi(a) = C\sinh Ka = A\sin(-k_a b)$$

$$\hookrightarrow \quad C\sinh Ka + A\sin k_a b = 0 \qquad ①$$

$$\psi'(a) = CK\cosh Ka = Ak_a \cos(-k_a b)$$

$$\hookrightarrow \quad CK\cosh Ka - Ak_a \cos(k_a b) = 0 \qquad ②$$

Put ① and ② into matrix:

$$\begin{pmatrix} \sin k_a b & \sinh Ka \\ -k_a \cos k_a b & K\cosh Ka \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = 0$$

Solve by setting determinant to zero.

$$\sin(k_a b) K\cosh(Ka) + \sinh(Ka) k_a \cos(k_a b) = 0$$

$$K\sin(k_a b)\cosh(Ka) = -k_a \sinh(Ka)\cos(k_a b)$$

$$\hookrightarrow \quad \boxed{\frac{1}{K}\tanh(Ka) + \frac{1}{k_a}\tan(k_a b) = 0} \qquad ①$$

similarly for even (symmetric solutions)

For $x<a$: the even solution is $\psi(x) = D\cosh Ka$

For $a<x<a+b$, we choose $k=k_s$ for symmetric solution.

so $\psi(x) = A\sin[k_s(x-a-b)]$

$\psi(x=a) = D\cosh Ka = A\sin(k_s(-b))$

$\hookrightarrow D\cosh Ka + A\sin k_s b = 0$

$\psi'(x=a) = KD\sinh Ka = k_s A\cos(k_s(-b))$

$\hookrightarrow KD\sinh Ka - k_s A\cos k_s b = 0$

$$\begin{pmatrix} \sin k_s b & \cosh Ka \\ -k_s\cos k_s b & K\sinh Ka \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix} = 0$$

$\hookrightarrow K\sin k_s b \sinh Ka + k_s\cos k_s b \cosh Ka = 0$

$\hookrightarrow \boxed{\frac{1}{k_s}\tan k_s b + \frac{1}{K}\coth Ka = 0}$ ②

Now we have the 2 equations that relate k, a function of E to K, a function of V_0.

Now suppose $V_0 \to \infty$, then we have solution to ∞ potential well which is

$$\psi(x) = A\sin kx \quad \text{for } k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{\pi n}{b} \quad \text{for } n = \pm 1, \pm 2 \cdots$$

$$\text{or} \quad kb = \pi n$$

at lowest energy level $\quad kb = \pi$

As we decrease V_0, we would only expect kb to deviate from π slightly, so $kb - \pi \ll 1$, then we can taylor expand tan kb around π

$$\hookrightarrow \tan kb = \tan\big((\underbrace{kb - \pi}_{\text{small}}) + \pi\big)$$

$$\hookrightarrow = \frac{\sin((kb-\pi)+\pi)}{\cos((kb-\pi)+\pi)} = \frac{\overset{0}{\cancel{\sin(\pi)}} + (kb-\pi)\cancel{\cos(\pi)}}{\cancel{\cos(\pi)} - (kb-\pi)\underset{0}{\cancel{\sin(\pi)}}}$$

$$\boxed{\tan kb \approx kb - \pi}$$

Now with asymmetric solution

$$\Rightarrow \frac{1}{\kappa}\tanh(\kappa a) + \frac{1}{k_a}\tan(k_a b) = 0$$

$$\hookrightarrow \cong \frac{1}{\kappa}\tanh(\kappa a) + \frac{1}{k_a}(k_a b - \pi) = 0$$

$$k_a \tanh \kappa a + \kappa k_a b - \kappa\pi = 0$$

or $$k_a = \frac{\kappa\pi}{\tanh \kappa a + \kappa b}$$

since $k_a^2 = \frac{2mE_a}{\hbar^2}$ $\rightarrow$ $$\boxed{E_a = \frac{k_a^2\hbar^2}{2m} = \frac{\hbar^2\pi^2}{2m}\left(\frac{\kappa}{\tanh \kappa a + \kappa b}\right)^2}$$

Similarly: for symmetric solution:

$$\Rightarrow \frac{1}{k_s}\tan k_s b + \frac{1}{\kappa}\coth \kappa a = 0$$

$$\hookrightarrow \frac{1}{k_s}(k_s b - \pi) + \frac{1}{\kappa}\coth \kappa a = 0$$

$$k_s \kappa b - \kappa\pi + k_s \coth \kappa a = a$$

$$\hookrightarrow k_s = \frac{\kappa\pi}{\coth \kappa a + \kappa b}$$

then $$\boxed{E_s = \frac{k_s^2\hbar^2}{2m} = \frac{\hbar^2\pi^2}{2m}\left(\frac{\kappa}{\coth \kappa a + \kappa b}\right)^2}$$

So energy-splitting:

$$E_a - E_s = \frac{\hbar^2\pi^2\kappa^2}{2m}\left(\frac{1}{(\tan \kappa a + \kappa b)^2} - \frac{1}{(\coth \kappa a + \kappa b)^2}\right)$$

for $\kappa \approx \sqrt{\frac{2mV_0}{\hbar^2}}$

3) Selection rule for $\langle \alpha'; j', m' | z | \alpha; j m \rangle$

By selection rule: $m' = q + m$

Here we note z is the z-component of the position vector.

For a vector, $V^{k=1}_{q=\pm1,0}$, X_z correspond to $V^{k=1}_{q=0}$

then $m' = \cancel{q}^{\,0} + m \rightarrow \boxed{m' = m}$

By triangle relation $|j-k| \le j' \le j+k$

and for vector, $k=1$, so $|j-1| \le j' \le j+1$

from this we can deduce that there are only 3 possibilities for $\Rightarrow \boxed{j'-j = \pm1, 0}$

Now by parity-selection rule:

$$\langle \alpha'; j' m' | z | \alpha; j m \rangle = \underbrace{\langle \alpha'; j', m' | \pi^\dagger}_{\varepsilon_{\alpha'} \langle \alpha'; j', m'|} \underbrace{\pi z \pi^\dagger}_{-z} \underbrace{\pi | \alpha; j m \rangle}_{\varepsilon_\alpha |\alpha; j m\rangle}$$

$$\langle \alpha'; j' m' | z | \alpha; j m \rangle = \underbrace{-\varepsilon_{\alpha'} \varepsilon_\alpha}_{1} \langle \alpha'; j' m' | z | \alpha; j m \rangle$$

require $-\varepsilon_{\alpha'} \varepsilon_\alpha = 1$ $\qquad \varepsilon_{\alpha'} = -\varepsilon_\alpha$

π parity-selection rule.

$\pi |\alpha; j m\rangle = (-1)^j |\alpha; j m\rangle$

$$= -(-1)^{j'} (-1)^j \langle \alpha'; j' m' | z | \alpha; j m \rangle$$

by parity require $-(-1)^{j'+j} = 1$, so $j'+j = \text{odd}$, then we eliminate $j'-j=0$ case. Then $\boxed{j'-j = \pm 1}$

4) $H = -W \sum_{n=1}^{N} \left(e^{i\theta} |n\rangle\langle n+1| + e^{-i\theta} |n+1\rangle\langle n| \right)$

real (W), real (θ)

$\rightarrow$ periodic B.C. : $|n\rangle \equiv |n+N\rangle$

$\rightarrow$ Translational invariant $n \rightarrow n+1$.

a) let current state $|\alpha\rangle = \sum_{n=1}^{N} |n\rangle \langle n|\alpha\rangle$

define operator $\hat{T}$ such that $\hat{T}|n\rangle = |n+1\rangle$

If $[H,T] = 0 \rightarrow HT - TH = 0.$

$$(HT-TH)|\alpha\rangle = \sum_{n=1}^{N} (H\hat{T} - \hat{T}H)|n\rangle\langle n|\alpha\rangle$$

$$= \sum_{n=1}^{N} \sum_{n'=1}^{N} -W \Big[\left(e^{i\theta}|n'\rangle\langle n'+1| + e^{-i\theta}|n'+1\rangle\langle n'| \right) \hat{T} |n\rangle\langle n|\alpha\rangle$$
$$- \hat{T} \left(e^{i\theta}|n'\rangle\langle n'+1| - e^{-i\theta}|n'+1\rangle\langle n'| \right) |n\rangle\langle n|\alpha\rangle \Big]$$

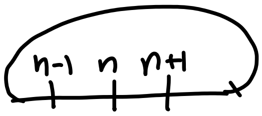
$$= -W \sum_{n=1}^{N} \sum_{n'=1}^{N} \Big[\left(e^{i\theta}|n'\rangle\langle n'+1| + e^{-i\theta}|n'+1\rangle\langle n'| \right) |n+1\rangle \langle n|\alpha\rangle$$
$$- \left(e^{i\theta}|n'+1\rangle\langle n'+1| - e^{-i\theta}|n'+2\rangle\langle n'| \right) |n\rangle\langle n|\alpha\rangle$$

$$= -W \sum_{n} \sum_{n'} \Big(e^{i\theta}|n'\rangle \delta_{n'+1,n+1} \langle n|\alpha\rangle + e^{-i\theta} |n'+1\rangle \delta_{n',n+1} \langle n|\alpha\rangle$$
$$- e^{i\theta} |n'+1\rangle \delta_{n'+1,n} \langle n|\alpha\rangle - e^{-i\theta} |n'+2\rangle \delta_{n,n'} \langle n|\alpha\rangle \Big)$$

$$= -W \sum_{n} \Big(e^{i\theta}|n\rangle\langle n|\alpha\rangle + e^{-i\theta}|n+2\rangle\langle n|\alpha\rangle$$
$$- e^{i\theta}|n\rangle\langle n|\alpha\rangle - e^{-i\theta}|n+2\rangle\langle n|\alpha\rangle \Big)$$

$$= \underline{\underline{0}}$$

commutator $[H,T] = 0$ so they commute.



b) Find spectrum of T, use $T^N = I$.

eigenvalue.

$$T|\alpha\rangle = T\sum_{n=1}^{N} |n\rangle\langle n|\alpha\rangle = t\sum_{n=1}^{N} |n\rangle\langle n|\alpha\rangle$$

$$\stackrel{!}{=} \sum_{n=1}^{N} |n+1\rangle\langle n|\alpha\rangle$$

$$\stackrel{!}{=} \sum_{n=1}^{N} |n\rangle\langle n-1|\alpha\rangle$$

$\hookrightarrow \quad t\sum_{n} |n\rangle\langle n|\alpha\rangle = \sum_{n=1}^{N} |n\rangle\langle n-1|\alpha\rangle$

$\Rightarrow$ or $t_n\langle n|\alpha\rangle = \langle n-1|\alpha\rangle$ for every n.

since $T^N|\alpha\rangle = \sum_{n=1}^{N} |n+N\rangle\langle n|\alpha\rangle = \sum_{n=1}^{N} |n\rangle\langle n-N|\alpha\rangle$

$\langle n-N| = \langle n|$ by periodic boundary

$\hookrightarrow \quad t^N|\alpha\rangle = 1\,|\alpha\rangle$

$t^N|\alpha\rangle \stackrel{|}{=} e^{2\pi m i}|\alpha\rangle$ for $m = \pm 1, \pm 2 \cdots \pm N$

$\hookrightarrow \quad t|\alpha\rangle = e^{\frac{2\pi m i}{N}}|\alpha\rangle$

then $\boxed{t = e^{\frac{2\pi m i}{N}} \quad \text{for } m = 1, 2, 3 \ldots N}$

Assume the eigenket $|\alpha\rangle$, follows plane wave, $|\alpha\rangle = \frac{1}{\sqrt{N}} e^{in\phi}|n\rangle$

then $\quad T|\alpha\rangle = \frac{1}{\sqrt{N}} e^{in\phi}|n+1\rangle = \frac{1}{\sqrt{N}} e^{in\phi} e^{-i\phi}|n\rangle = e^{-i\phi}|\alpha\rangle = e^{-\frac{2\pi}{N}im}|\alpha\rangle$

so we see $\underline{\underline{\phi = \frac{2\pi}{N}m}}$

So $|\alpha_m\rangle = \sum_n |n\rangle\langle n|\alpha_m\rangle = \sum_n \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}mn}|n\rangle$

c) $H = -W\sum_{n'}\left(e^{i\theta}|n'\rangle\langle n'+1| + e^{-i\theta}|n'+1\rangle\langle n'|\right)$

since $|n'+1\rangle = T|n'\rangle$
$\langle n'+1| = (|n'+1\rangle)^{+} = (T|n'\rangle)^{+} = \langle n'|T^{+}$

then $H = -W\sum_{n'}\left(e^{i\theta}|n'\rangle\langle n'|T^{+} + e^{-i\theta}T|n'\rangle\langle n'|\right)$

$\overset{!}{=} -W\sum_{n'}\left(e^{i\theta}|n'\rangle\langle n'|T^{+} + e^{-i\theta}T|n'\rangle\langle n'|\right)$

Then $H|\alpha\rangle_m = -W\sum_{n'}\left(e^{i\theta}|n'\rangle\langle n'|T^{+} + e^{-i\theta}T|n'\rangle\langle n'|\right)|\alpha_m\rangle$

$\overset{!}{=} -W\sum_{n'}\left(e^{i\theta}|n'\rangle\langle n'|t^{*}|\alpha\rangle_m + e^{-i\theta}T|n'\rangle\langle n'|\alpha\rangle_m\right)$

$\overset{!}{=} -W\sum_{n}\sum_{n'}\left[e^{i\theta}t^{*}|n'\rangle\underbrace{\langle n'|n\rangle}_{\delta_{n,n'}}\langle n|\alpha\rangle_m + e^{-i\theta}T|n'\rangle\underbrace{\langle n'|n\rangle}_{\delta_{n,n'}}\langle n|\alpha\rangle_m\right]$

$\overset{!}{=} -W\sum_{n}\left[e^{i\theta}t^{*}|n\rangle\langle n|\alpha\rangle_m + e^{-i\theta}T|n\rangle\langle n|\alpha\rangle_m\right]$

$\overset{!}{=} -W\sum_{n}\left[e^{i\theta}e^{\frac{2\pi i}{N}m}|n\rangle\langle n|\alpha\rangle_m + e^{-i\theta}e^{-\frac{2\pi i}{N}m}|n\rangle\langle n|\alpha\rangle_m\right]$

$H|\alpha\rangle_m \overset{!}{=} -W\left(e^{i(\theta+\frac{2\pi}{N}m)} + e^{-i(\theta+\frac{2\pi}{N}m)}\right)|\alpha\rangle_m$

$H|\alpha\rangle_m \overset{!}{=} \underbrace{-2W\cos\left(\theta+\frac{2\pi}{N}m\right)}|\alpha\rangle_m$

$\boxed{\text{then eigenvalue of } H \text{ is } E_m = -2W\cos\left(\theta+\frac{2\pi}{N}m\right) \text{ for } |\alpha_m\rangle}$

5) $H = -W \sum_{n=1}^{3} \left(e^{i\theta} |n\rangle\langle n+1| + e^{-i\theta} |n+1\rangle\langle n| \right)$

$$\rho(t=0) = \frac{1}{3}\left(2\,|1\rangle \otimes \langle 1| + |2\rangle \otimes \langle 2|\right)$$

$$\rho = \sum_i w_i\, |\alpha^{(i)}\rangle\langle\alpha^{(i)}| = \sum_{n=1}^{3} w_n\, |n\rangle\langle n|$$

$$\overset{!}{=} \sum_n w_n\, U(t,t_0)\,|n\rangle\langle n|\, U^{+}(t,t_0)$$

$$\rho(t) \overset{!}{=} U(t,t_0)\, \rho(t_0)\, U(t,t_0)^{+}$$

Since Hamiltonian is time-independent:

$$U(t,t_0=0) = \exp\left(\frac{-i}{\hbar}\hat{H}(t-t_0)\right)$$

Rewrite $|n\rangle$ in the basis of $|\alpha\rangle$, which is eigenstate of H.

$$|n\rangle = \sum_\alpha |\alpha\rangle\langle\alpha|n\rangle$$

we know $|\alpha_m\rangle = \sum_n |n\rangle\langle n|\alpha_m\rangle = \sum_{n=1}^{3} e^{\frac{i2\pi}{3}nm} \frac{1}{\sqrt{N}} |n\rangle$

$$\begin{pmatrix} \alpha_{m=1} \\ \alpha_{m=2} \\ \alpha_{m=3} \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} e^{\frac{i2\pi}{3}} & e^{\frac{4\pi i}{3}} & e^{\frac{6\pi i}{3}} \\ e^{\frac{4\pi i}{3}} & e^{\frac{8\pi i}{3}} & e^{\frac{12\pi i}{3}} \\ e^{\frac{i6\pi}{3}} & e^{\frac{12\pi i}{3}} & e^{\frac{18\pi i}{3}} \end{pmatrix} \begin{pmatrix} n=1 \\ n=2 \\ n=3 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{m=1} \\ \alpha_{m=2} \\ \alpha_{m=3} \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} e^{\frac{i2\pi}{3}} & e^{\frac{4\pi i}{3}} & 1 \\ e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n=1 \\ n=2 \\ n=3 \end{pmatrix}$$

then $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} & 1 \\ e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

By mathematica:

$$= \frac{1}{3} \begin{pmatrix} -\frac{1}{2}(1+i\sqrt{3}) & -\frac{1}{2}(1-i\sqrt{3}) & 1 \\ -\frac{1}{2}(1-i\sqrt{3}) & -\frac{1}{2}(1+i\sqrt{3}) & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

then $\rho(0) = \omega_n |n\rangle\langle n| = \frac{2}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2|$

$$2|1\rangle\langle 1| = 2\left(c_{11}|\alpha_1\rangle + c_{12}|\alpha_2\rangle + c_{13}|\alpha_3\rangle\right)\left(c_{11}^*\langle\alpha_1| + c_{12}^*\langle\alpha_2| + c_{13}^*\langle\alpha_3|\right)$$

$$= 2 \begin{pmatrix} |c_{11}|^2 & c_{11}c_{12}^* & c_{11}c_{13}^* \\ c_{12}c_{11}^* & |c_{12}|^2 & c_{12}c_{13}^* \\ c_{13}c_{11}^* & c_{13}c_{12}^* & |c_{13}|^2 \end{pmatrix} = \frac{2}{9} \begin{pmatrix} 1 & -\frac{1}{2}(1-i\sqrt{3}) & -\frac{1}{2}(1+i\sqrt{3}) \\ -\frac{1}{2}(1+i\sqrt{3}) & 1 & -\frac{1}{2}(1-i\sqrt{3}) \\ -\frac{1}{2}(1-i\sqrt{3}) & -\frac{1}{2}(1+i\sqrt{3}) & 1 \end{pmatrix}$$

$$|2\rangle\langle 2| = \left(c_{21}|\alpha_1\rangle + c_{22}|\alpha_2\rangle + c_{23}|\alpha_3\rangle\right)\left(c_{21}^*\langle\alpha_1| + c_{22}^*\langle\alpha_2| + c_{23}^*\langle\alpha_3|\right)$$

$$= \begin{pmatrix} |c_{21}|^2 & c_{21}c_{22}^* & c_{21}c_{23}^* \\ c_{22}c_{21}^* & |c_{22}|^2 & c_{22}c_{23}^* \\ c_{23}c_{21}^* & c_{23}c_{22}^* & |c_{23}|^2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & -\frac{1}{2}(1+i\sqrt{3}) & -\frac{1}{2}(1-i\sqrt{3}) \\ -\frac{1}{2}(1-i\sqrt{3}) & 1 & -\frac{1}{2}(1+i\sqrt{3}) \\ -\frac{1}{2}(1+i\sqrt{3}) & -\frac{1}{2}(1-i\sqrt{3}) & 1 \end{pmatrix}$$

then $\sum_n w_n |n\rangle\langle n| = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6}+\frac{i\sqrt{3}}{18} & -\frac{1}{6}-\frac{i\sqrt{3}}{18} \\ -\frac{1}{6}-\frac{i\sqrt{3}}{18} & \frac{1}{3} & -\frac{1}{6}+\frac{i\sqrt{3}}{18} \\ -\frac{1}{6}+\frac{i\sqrt{3}}{18} & -\frac{1}{6}-\frac{i\sqrt{3}}{18} & \frac{1}{3} \end{pmatrix}$

then $U\rho U^\dagger = w_n U |n\rangle\langle n| U^\dagger = U \sum_{\alpha\alpha'} w_n |\alpha\rangle\langle\alpha|n\rangle\langle n|\alpha'\rangle\langle\alpha'| \; U^\dagger$

Since $|\alpha_m\rangle$ is the eigenbasis for H, we get corresponding eigenenergy $E_m = -2W\cos(\theta + \frac{2\pi}{3}m)$

then

$$U\rho U^\dagger = \begin{pmatrix} 1 & \exp(\frac{-i}{\hbar}(E_1-E_2)t)(\frac{-1}{2}+\frac{i\sqrt{3}}{6}) & \exp(\frac{-i}{\hbar}(E_1-E_3)t)(\frac{-1}{2}-\frac{i\sqrt{3}}{6}) \\ \exp(\frac{-i}{\hbar}(E_2-E_1)t)(\frac{-1}{2}-\frac{i\sqrt{3}}{6}) & 1 & \exp(\frac{-i}{\hbar}(E_2-E_3)t)(\frac{-1}{2}+\frac{i\sqrt{3}}{6}) \\ \exp(\frac{-i}{\hbar}(E_3-E_1)t)(\frac{-1}{2}+\frac{i\sqrt{3}}{6}) & \exp(\frac{-i}{\hbar}(E_3-E_2)t)(\frac{-1}{2}-\frac{i\sqrt{3}}{6}) & 1 \end{pmatrix}$$

where $E_m = -2W\cos(\theta + \frac{2\pi}{3}m)$

6) $S^2 = 1$ $T^3 = 1$, $TST = S$, H is symmetric.

a) suppose $S|\alpha\rangle = \lambda_s|\alpha\rangle$

$S^2|\alpha\rangle = \lambda_s^2|\alpha\rangle = |\alpha\rangle$

So $\lambda_s^2 = 1$ or $\boxed{\lambda_s = \pm 1}$ for $\hat{S}$

suppose $T|\alpha\rangle = \lambda_T|\alpha\rangle$

$T^3|\alpha\rangle = \lambda_T^3|\alpha\rangle = |\alpha\rangle$

$\lambda_T^3 = 1$

$$\boxed{\begin{array}{l} \lambda_T = e^{\frac{2\pi n}{3}i} \quad \text{for } n = 0, 1, 2 \\ \text{so } \lambda_T = 1,\ e^{\frac{2\pi}{3}i},\ e^{\frac{4\pi}{3}i} \end{array}}$$

← for T

b) Show we get double degeneracy with symmetric H

Since Hamiltonian, H is symmetric, then we expect Hamiltonian to commute with the generator of symmetry i.e. S and T.

So $[H,S]=0$ and $[H,T]=0$

If $H|\alpha\rangle = E|\alpha\rangle$, so $|\alpha\rangle$ is eigenstate of H.

Now since $[H,S]=0$,

Case 1: if $|\alpha\rangle$ is simultaneously eigenket for S with eigenvalue λ_s
then $HS|\alpha\rangle = SH|\alpha\rangle = \lambda_s E|\alpha\rangle$.

Case 2: if $|\alpha\rangle$ is not eigenket of S, then $S|\alpha\rangle \neq |\alpha\rangle$ so
$HS|\alpha\rangle = SH|\alpha\rangle = ES|\alpha\rangle$, so $S|\alpha\rangle$ is a new eigenket of H.

And for $[H,T]=0$, we have the same cases:

Case 1: $|\alpha\rangle$ is also eigenket of T with eigenvalue λ_t, then
$HT|\alpha\rangle = TH|\alpha\rangle = \lambda_t E|\alpha\rangle$

Case 2: $|\alpha\rangle$ is not eigenket of T, then $T|\alpha\rangle \neq |\alpha\rangle$,
so $HT|\alpha\rangle = TH|\alpha\rangle = ET|\alpha\rangle$, so $T|\alpha\rangle$ is eigenket for H.

However, since $S = TST$, so $ST = TSTT$, but $TS = TTST$, and since ST, T, and T^2 are different element of the group, this means $ST \neq TS$, i.e. $[T, S] \neq 0$

Since $[T,S] \neq 0$, there's no eigenket between the two operator T and S.

→ So suppose $|\alpha\rangle$ is eigenket of S, then $|\alpha\rangle$ cannot be eigenket of T, then $T|\alpha\rangle$ is the other eigenket for H aside from $|\alpha\rangle$

→ Similarly if $|\alpha\rangle$ is eigenket of T, then $S|\alpha\rangle$ is the second eigenket for H aside from $|\alpha\rangle$

→ Therefore, we expect double degeneracy for H.

suppose the degenerate eigenkets are $|\alpha\rangle$ and $S|\alpha\rangle$, this means $|\alpha\rangle$ is eigenket of T

eigenvalue of $|\alpha\rangle$

then $T|\alpha\rangle = e^{\frac{2\pi}{3}ni}|\alpha\rangle$ using eigenvalue from part i)

using $S = TST$

$$TS|\alpha\rangle = T(TST)|\alpha\rangle$$

$$\overset{!}{=} TT(TST)T|\alpha\rangle$$

$$= \underbrace{TTT}_{=1}STT|\alpha\rangle$$

eigenvalue for $S|\alpha\rangle$

$$TS|\alpha\rangle \overset{!}{=} Se^{\frac{4\pi ni}{3}}|\alpha\rangle = e^{\frac{4\pi ni}{3}}S|\alpha\rangle$$

c) An example of this symmetry is a 3-atom 1D lattice with periodic boundary conditions, like problem 5.

In this example, S represents parity operator π and T is the translational operator that $|n\rangle \xrightarrow{T} |n+1\rangle$

we observe $1 \ 2 \ 3 \xrightarrow{\pi} 3 \ 2 \ 1$

and $1\ 2\ 3 \xrightarrow{T} 3\ 1\ 2 \xrightarrow{\pi} 2\ 1\ 3 \xrightarrow{T} 3\ 2\ 1$

which satisfies $TST = S$