$$|\alpha\rangle = \sum_i \sum_j c_i^{(j)} |a_i^{(j)}\rangle$$

$$|\alpha\rangle \xrightarrow{A} P_a |\alpha\rangle$$

probability: $P_i = \sum |c_i|^2 = \langle\alpha| P_a |\alpha\rangle$

$$\langle A\rangle = \sum P_i a_i = \langle\alpha| \sum a_i P_i |\alpha\rangle = \langle\alpha|A|\alpha\rangle$$

Measurement in 2-level system

$$|\alpha\rangle = c_+ \underset{\binom{1}{0}}{|+\rangle} + c_- \underset{\binom{0}{1}}{|-\rangle}$$ eigenvector of S_z

so $\langle\alpha|\alpha\rangle = |c_+|^2 + |c_-|^2$

eigenvalue of S_z

$$\langle\alpha| S_z |\alpha\rangle = \frac{\hbar}{2}|c_+|^2 + \left(\frac{-\hbar}{2}\right)|c_-|^2$$

$$\stackrel{!}{=} \frac{\hbar}{2}\left(|c_+|^2 - |c_-|^2\right)$$

$$\left.\begin{array}{l} |S_{x}+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ |S_x -\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \end{array}\right\} \begin{array}{l} |+\rangle = \frac{1}{\sqrt{2}}(|S_x+\rangle + |S_x-\rangle) \\ |-\rangle = \frac{1}{\sqrt{2}}(|S_x+\rangle - |S_x-\rangle) \end{array}$$

$$|S_y \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle) \qquad S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} =$$

Compatible Observables:

Observable A, B are compatible if $[A,B]=0$
incompatible if $[A,B] \neq 0$

If they're compatible: , they can be diagonalized together.

$$A|a_i, b_i\rangle = a_i |a_i, b_i\rangle$$
$$B|a_i, b_i\rangle = b_i |a_i, b_i\rangle$$

So the eigenvector works for both A and B.

For a set of compatible observables:

$|a_i, b_i, c_i, d_i \cdots \rangle = 0$
if these labeling are unique, then they're unique.

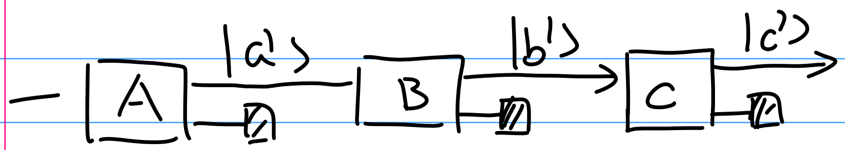
assume compatible.

$$|\alpha\rangle \xrightarrow{A} |a'\rangle \xrightarrow{B} |a', b'\rangle \xrightarrow{C} |a', b', c'\rangle \cdots$$

ultimately: $|a', b', c', \ldots\rangle$

Now if we apply $A|a', b', c' \ldots\rangle = a'|a', b', c' \ldots\rangle$

Now consider incompatible:



1) $P_{a' \to c'} = \sum_{b'} |\langle a'|b'\rangle|^2 |\langle b'|c'\rangle|^2$ from $a \to b \to c$

2) $\tilde{P}_{a' \to c'} = |\langle a'|c'\rangle|^2$

$$P_{a' \to c'} = \sum_{b'} \langle a'|b'\rangle\langle b'|c'\rangle\langle c'|b'\rangle\langle b'|a'\rangle$$

due to sum, cannot use complete relation

Not Equal

$$\tilde{P}_{a' \to c'} = |\langle a'|b'\rangle\langle b'|a'\rangle| = \sum_{b',b''} \langle a'|b'\rangle\langle b'|c'\rangle\langle c'|b''\rangle\langle b''|a'\rangle$$

Uncertainty Principle:

Observable A: with state $|\ \rangle$

$\langle A\rangle = \langle\ |A|\ \rangle$ ← Expectation value.

$\Delta A = A - \langle A\rangle \mathbb{1}$

and $\langle \Delta A\rangle = \langle A\rangle - \langle A\rangle = 0$

With variance (dispersion of A in $|\rangle$)

$$\langle (\Delta A)^2 \rangle = \langle (A - \langle A \rangle \mathbb{1})^2 \rangle$$
$$\stackrel{!}{=} \langle A^2 - 2A\langle A \rangle + \langle A \rangle^2 \rangle$$
$$\stackrel{!}{=} \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2$$
$$\langle (\Delta A)^2 \rangle \stackrel{!}{=} \langle A^2 \rangle - \langle A \rangle^2$$

Ex: $\langle (\Delta S_z)^2 \rangle = \langle S_z^2 \rangle - \langle S_z \rangle^2$
$$\stackrel{!}{=} \left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2$$
$$\stackrel{!}{=} \left(\frac{\hbar}{2}\right)^2 - 0 = \frac{\hbar^2}{4}$$

Theorem: If A and B are Hermitian (observable)

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

let $|\alpha\rangle = \Delta A |\rangle$ and $|\beta\rangle = \Delta B |\rangle$

then $\langle (\Delta A)^2 \rangle = \langle \alpha | \alpha \rangle$
$\langle (\Delta B)^2 \rangle = \langle \beta | \beta \rangle$

With Schwarz-inequality:

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2 = |\langle \Delta A \, \Delta B \rangle|^2$$

let $|\langle \Delta A\, \Delta B\rangle|^2 = |\langle \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}\rangle|^2$

$$\overset{!}{=} \frac{1}{4} |\langle[\Delta A, \Delta B]\rangle + \langle\{\Delta A, \Delta B\}\rangle|^2$$

$$\geq \frac{1}{4} |\langle[\Delta A, \Delta B]\rangle|^2$$

Tensor product:

Hilbert space of composite system is a tensor product of the Hilbert space.

$H^{(1)}, H^{(2)} \Rightarrow H = H^{(1)} \times H^{(2)}$

then $H \rightarrow |\varphi_{ij}\rangle = |\varphi_i\rangle \otimes |\varphi_j\rangle$

$$\langle \varphi_{ij} | \varphi_{kl}\rangle = \langle \varphi_i^{(1)} | \varphi_k^{(1)}\rangle \langle \varphi_j^{(2)} | \varphi_l^{(2)}\rangle$$

$$\overset{!}{=} \delta_{ik}\, \delta_{jl}$$

$$|\alpha\rangle = \sum c_i |\varphi_i^{(1)}\rangle \in H^{(1)}$$
$$|\beta\rangle = \sum d_j |\varphi_j^{(2)}\rangle \in H^{(2)}$$

$$|\alpha\rangle \otimes |\beta\rangle = \sum_{ij} c_i d_j |\varphi_{ij}\rangle \in H$$

If $\hat{A}$ act on $H^{(1)}$
$\hat{B}$ act on $H^{(2)}$

Then: $A \otimes B |\varphi_{ij}\rangle = A|\varphi_i^{(1)}\rangle \otimes B|\varphi_j^{(2)}\rangle$

If A, B are observables, then $A \otimes B$ also observable.

Ex: $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes B$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

For spin $\frac{1}{2}$ particle $H^{(1)}$

for two $\frac{1}{2}$ spin particle: $H^{(1)} \otimes H^{(2)} = H$

Basis of H: $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

Complete set of operators:

$$\otimes \begin{cases} \mathbb{1}^{(1)}, S_x^{(1)}, S_y^{(1)}, S_z^{(1)} \\ \mathbb{1}^{(2)}, S_x^{(2)}, S_y^{(2)}, S_z^{(2)} \end{cases}$$

$\hookrightarrow$ 16 combination.

Complete set of compatible operators:

for $H^{(1)}$: $\mathbb{1}, S_z$

so 4 total for $H = H^{(1)} \otimes H^{(2)}$

$$S_z \otimes \mathbb{1} = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 1 & & \\ \hline & & -1 & \\ 0 & & & -1 \end{array}\right) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\mathbb{1} \otimes S_z = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & -1 & & \\ \hline & & 1 & \\ 0 & & & -1 \end{array}\right) = \begin{pmatrix} S_z & 0 \\ 0 & S_z \end{pmatrix}$$

General Procedure : for time independent H:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

← time - independent.

① $\hat{H}|E\rangle = E_i|E_i\rangle$

② $|\psi(t=0)\rangle = \sum c_i|E_i\rangle$

3) $|\psi(t)\rangle = \sum c_i e^{-\frac{i}{\hbar}E_i t}|E_i\rangle$