

## The theory of Angular Momentum:

Translation:

Define operator  $J$ , such that:

$$|z\rangle \rightarrow |z+a\rangle = J(a) |z\rangle$$

$\hat{T}$  translation operator

Then

$$\begin{aligned} J(a)|\psi\rangle &= J(a) \int dz |z\rangle \langle z|\psi\rangle \\ &\stackrel{|}{=} \int dz J(a) |z\rangle \underbrace{\langle z|\psi\rangle}_{\psi(z)} \\ &\stackrel{|}{=} \int dz |z+a\rangle \langle z|\psi\rangle \\ J(a)|\psi\rangle &\stackrel{|}{=} \int dz |z\rangle \underbrace{\langle z-a|\psi\rangle}_{\psi(z-a)} \end{aligned}$$

$$\text{so } J(a)\psi(z) = \psi(z-a)$$

Do Taylor:

$$\psi(z-a) = \psi(z) - a \partial_z \psi(z) + \frac{1}{2} a^2 \partial_z^2 \psi(z) + \dots$$

$$\stackrel{|}{=} (1 - a \partial_z + \frac{1}{2} a^2 \partial_z^2 + \dots) \psi(z)$$

$$\psi(z-a) \stackrel{|}{=} e^{-a\partial_z} \psi(z)$$

$$\therefore J(a) \psi(z) = e^{-a\partial z} \psi(z)$$

Since  $P_z = -ik\partial z$

then

$$J(a) = e^{-\frac{i}{\hbar} a P_z}$$

$\leftarrow$  Translation Operator.

Now let  $a \rightarrow \epsilon$

$$\text{then } J(\epsilon) = 1 - \frac{i}{\hbar} \epsilon P_z$$

$$\text{then } \psi' = J(\epsilon) \psi = (1 - \frac{i}{\hbar} \epsilon P_z) \psi$$

$$\text{then } S\psi = \psi' - \psi = -\frac{i}{\hbar} \epsilon P_z \psi$$

$P_z$  : Hermitian  $\leftarrow$  algebra

$J(a)$  : Unitary.  $\leftarrow$  group.

3D Generalization:  $P_x, P_y, P_z$

$$[P_i, P_j] = 0 \quad : \text{Commutative (abelian) group.}$$

translation  $\rightarrow [J(\vec{a}), J(\vec{b})] = 0$  : commutative (abelian) group.  
commutes.

## Translational Invariance:

$$H(J(a)|\psi\rangle) = E(J(a)|\psi\rangle)$$

$\hookrightarrow H|\psi\rangle = E|\psi\rangle$  Hamiltonian doesn't change upon translation.

$$\hookrightarrow = J(a) E|\psi\rangle$$

$$\stackrel{!}{=} J(a) H|\psi\rangle$$

$$\hookrightarrow \text{so } H J(a) = J(a) H$$

or 
$$\boxed{[H, J(a)] = 0}$$

or 
$$\boxed{[H, P_z] = 0} = \text{translational invariance.}$$

$P_z$  is infinitesimal translation of  $J(a)$

With Heisenberg picture:  $\frac{\partial P_z}{\partial t} = \frac{1}{i\hbar} \underbrace{[H, P_z]}_{=0} = 0$

$\hookrightarrow P_z$  is constant.

## Time Evolution:

$$i\hbar \partial_t |\psi\rangle = H|\psi\rangle$$

If H is time independent:

$$|\psi\rangle(t) = \underbrace{e^{-\frac{i}{\hbar}Ht}}_{\text{finite time evolution, unitary}} |\psi\rangle(t=0)$$

operator, and gives to  $J(a)$ ,

Infinitesimal time evolution:  $J(dt) = | -\frac{i}{\hbar}H dt \rangle$

Hamiltonian is generator  
for time evolution. (Hermitian)

## Rotation in 3D

$$\mathbf{v}' = R\mathbf{v}$$

$$\hookrightarrow \mathbf{v}'^T \mathbf{v}' = \mathbf{v}^T \underbrace{R^T R}_{=I} \mathbf{v} = \mathbf{v}^T \mathbf{v}$$

Therefore  $R^T R = R R^T = I$

So  $R$  is a  $3 \times 3$  orthogonal Matrices.  $R \in SO(3)$

$$R_z(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{rotate } \varphi \text{ around } z\text{-axis.}$$

$$R_z(\epsilon) \approx \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1 - \frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 + \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

|

$$= 1 - \frac{i}{\hbar} \epsilon L_z$$

<sup>↑ angular momentum  
along axis -z.</sup>

generator of infinitesimal

Similarly:

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1 - \frac{\epsilon^2}{2} \end{pmatrix} = 1 - \frac{i}{\hbar} \epsilon L_x$$

<sup>rotation.</sup>

∴

$$L_x = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_z = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that  $R_z(\varphi) R_x(\theta) \neq R_x(\theta) R_z(\varphi)$

↳ Not commutative (Non-abelian)

We can also observe that: after computation.

$$[L_x, L_y] = i\hbar L_z$$

$$\text{or } [L_i, L_j] = i\hbar \epsilon^{ijk} L_k$$

\* Define  $R(\hat{n}, \varphi) = e^{\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{L}}$  Finite Rotation operator

ex:  $e^{\frac{-i}{\hbar} \varphi (\hat{z} \cdot \vec{L})} = e^{\frac{-i}{\hbar} \varphi L_z} = e^{\varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

$$\hat{L}_y = e^{-i\varphi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\hat{L}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{L}_y = \cos(\varphi \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) - i \sin(\varphi \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$$

$$= \begin{pmatrix} \cos \varphi & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -i \sin \varphi & 0 \\ 0 & 0 \end{pmatrix}$$

Note identity:  $\vec{a} \cdot \vec{a} = \vec{a}^2$ ,  $e^{i\varphi (\vec{a} \cdot \vec{a})} = \sum_{n=0}^{\infty} \frac{(i\varphi (\vec{a} \cdot \vec{a}))^n}{n!}$

In Hilber Space

$$|\psi\rangle_R = D(R) |\psi\rangle$$

Rotated  $\rightarrow$   $\downarrow$  unitary operator of rotation  
 $N \times N$  matrix

$$D(R)|_{\text{infinitesimal}} = 1 - \frac{i}{\hbar} \delta\phi \hat{n} \cdot \vec{J}$$

generator of infinitesimal rotation  
or angular momentum.

$$[J_i, J_j] = i\hbar \epsilon^{ijk} J_k$$

$$D(R) = e^{\frac{-i}{\hbar} \varphi \hat{n} \cdot \vec{J}}$$

unitary operator for finite operation

Analogous to  $U(t, t_0) = e^{-i\hbar H(t-t_0)}$ : finite time evolution

Example of spin  $\frac{1}{2}$ :

$F(\Delta x' \hat{x}) = \exp\left(-\frac{i}{\hbar} p_x \Delta x'\right)$ : translation

$|+\rangle, |-\rangle$  basis:

unitary, but not Hermitian

$$[S_i, S_j] = i\hbar \epsilon^{ijk} S_k$$

$$\left. \begin{aligned} S_x &= \frac{\hbar}{2} (|+\rangle\langle-| + |- \rangle\langle+|) \\ S_y &= \frac{\hbar}{2} (-i|+\rangle\langle-| + i|- \rangle\langle+|) \\ S_z &= \frac{\hbar}{2} (|+\rangle\langle+| - |- \rangle\langle-|) \end{aligned} \right\} \begin{aligned} &\text{Representation of} \\ &\text{rotational algebra:} \\ &D_z(\varphi) = e^{\frac{-i}{\hbar} \varphi S_z} \end{aligned}$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then  $|\alpha\rangle = |\langle +|\alpha\rangle \underbrace{|+\rangle}_{\langle +|} + |\langle -|\alpha\rangle \underbrace{|-\rangle}_{\langle -|}$

$$|\alpha\rangle = \begin{pmatrix} \langle +|\alpha\rangle \\ \langle -|\alpha\rangle \end{pmatrix} \leftarrow \text{Arbitrary 2-component spinor}$$

$$\chi = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

With Pauli-Matrices:  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$   $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\boxed{\sigma_i = \frac{i}{2} \delta_i}$$

Properties:

$$\textcircled{1} \quad [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k.$$

$$\textcircled{2} \quad \{\sigma_i, \sigma_j\} = 2 \delta_{ij}$$

$$\textcircled{3} \quad \sigma_i^\dagger = \sigma_i \quad \text{Hermitian}$$

$$\textcircled{4} \quad \text{Tr}(\sigma_i) = 0$$

$$\textcircled{5} \quad \text{Det}(\sigma_i) = 1$$

$$\textcircled{6} \quad \vec{\sigma} \cdot \vec{a} = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}$$

$$\textcircled{7} \quad (\vec{\sigma} \cdot \vec{a})^2 = a^2$$

$$\textcircled{8} \quad (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = (\vec{a} \cdot \vec{b}) \mathbb{1} + i \vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

$$\textcircled{9} \quad D(\varphi, \hat{n}) = e^{\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{\sigma}} = e^{\frac{-i}{2} \varphi \hat{n} \cdot \vec{\sigma}}$$

$$= \cos \frac{\varphi}{2} - i(\hat{n} \cdot \vec{\sigma}) \sin \frac{\varphi}{2}$$

$$\textcircled{10} \quad D(\varphi + 2\pi, \hat{n}) = -D(\varphi, \hat{n})$$

$$D(2\pi, \hat{n}) = -D(0, \hat{n}) = -1 \quad \leftarrow \text{Not itself, but with minus}\right.$$

↑  
still same state in QM:

Transformation of operators under R

$$\langle + | A | + \rangle \rightarrow \langle + | e^{\frac{i}{2} \varphi \hat{n} \cdot \vec{\sigma}} A e^{-\frac{i}{2} \varphi \hat{n} \cdot \vec{\sigma}} | + \rangle$$

Infinitesimal  $\delta\varphi$ :

$$(1 + \frac{i}{2} \delta\varphi \hat{n} \cdot \vec{\sigma}) A (1 - \frac{i}{2} \delta\varphi \hat{n} \cdot \vec{\sigma}) = A - \underbrace{\frac{i}{2} \delta\varphi n^a [\delta^a, A]}_{SA}$$

Consider  $A \rightarrow \delta^a$

$$\delta \delta^b = \frac{i}{2} \delta\varphi n^a [\delta^a \delta^b] = \frac{i}{2} \delta\varphi n^a 2i \epsilon^{abc} \delta^c = \delta\varphi [\hat{n} \times \vec{\sigma}]^b$$

If  $\hat{n} = \hat{z}$

$$\left. \begin{array}{l} \delta \delta^x = -\delta\varphi \delta^y \\ \delta \delta^y = \delta\varphi \delta^x \\ \delta \delta^z = 0 \end{array} \right\} \text{transform } \vec{\sigma} \text{ as vectors}$$

## Groups, Algebra, Representations

Group: set  $G = \{a, b, c, \dots\}$  ← element of group.

operation:  $a \cdot b$  group multiplication.

Axioms:

- 1) if  $a \in G, b \in G$ , then  $a \cdot b \in G$
- 2)  $(ab)c = a(bc)$  association
- 3)  $e \in G$ ,  $a \cdot e = e \cdot a = a \quad \forall a$
- 4) Invertibility  $a \in G \exists a^{-1} \in G, a \cdot a^{-1} = a^{-1} \cdot a = e$

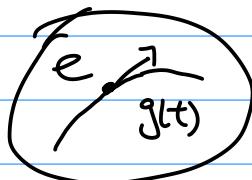
Lie Group: A group which is also differential manifold.

Lie group: Group operations are compatible with smooth structure

$\mu: G \times G \rightarrow G$  - smooth  
 $\mu(x, y) = x \cdot y$  multiplication

$\nu: G \rightarrow G$   
 $\nu(x) = x^{-1}$  - smooth Inversion.

Differential Calculus can be used.



$g(t): g(0) = e$

$\frac{dg}{dt}|_e$

Lie Algebra: (study of operations)

Vector Space  $\mathfrak{g}$  (over some field  $F$ , i.e.  $\mathbb{C}$  or  $\mathbb{R}$ )

with binary operation (Lie bracket)

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

- Axioms:
- 1) Linearity:  $[ax+by, z] = a[x, z] + [y, z]$   
↑      ↑  
#
  - 2) Anticommutative:  $[x, y] = -[y, x]$
  - 3) Jacobi-Identity:  $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

Example :  $J_i$

- 1)  $[J_i, J_j] = i \epsilon^{ijk} J_k$  ↑  
satisfies all 3 properties.
- 2) From Associative algebra  $A$ , we can build Lie algebra  $L(A)$ :

$$[a, b] = a \cdot b - b \cdot a$$

Representation Theory: studies groups, algebra, etc. by representing their elements as linear operations of Hilbert space.

Ex: Orthogonality group: Pauli-Matrices acting on 2D Hilbert space.  
(group of rotation)

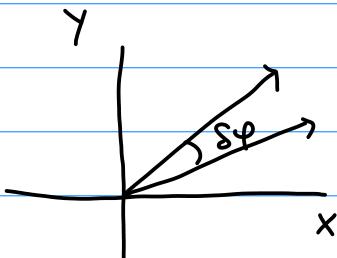
## Orbital Angular Momentum:

$\Psi(x, y, z)$ : wave function of 1 particle in 3D (no internal freedom)

$|x, y, z\rangle$  - basis

$$|\psi\rangle = \int dx dy dz \underbrace{\Psi(x, y, z)}_{\langle x, y, z | \psi \rangle} |x, y, z\rangle$$

arbitrary state



$$\begin{aligned} x &\rightarrow x - y \sin \varphi \\ y &\rightarrow y + x \sin \varphi \\ z &\rightarrow z \end{aligned} \quad \left. \begin{array}{l} x \cos \varphi - y \sin \varphi \\ x \sin \varphi + y \cos \varphi \\ z \end{array} \right\}$$

$$|x, y, z\rangle \rightarrow |x - y \sin \varphi, y + x \sin \varphi, z\rangle$$

$$= \left[ 1 - y \sin \varphi \left( \frac{-i}{\hbar} P_x \right) + x \sin \varphi \left( \frac{i}{\hbar} P_y \right) \right] |x, y, z\rangle$$

Note  $|x + \epsilon, y, z\rangle = (1 + \frac{\epsilon}{\hbar} \frac{\partial}{\partial x}) |x, y, z\rangle$

$$\frac{\partial}{\partial x} = \frac{i}{\hbar} P_x$$

$$|x, y, z\rangle \rightarrow \left[ 1 - \frac{i}{\hbar} \sin \varphi (x P_y - y P_x) \right] |x, y, z\rangle$$

Postulate  $|\psi\rangle \rightarrow (1 - \frac{i}{\hbar} \sin \varphi J_z) |\psi\rangle$

$$J_z = L_z = x P_y - P_y x$$

Note  $\vec{J} = \vec{L}$  with no internal degrees of freedom

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

< generator of infinitesimal rotation.

Want to note that

$L_z$  is the generator of rotation around  $z$  in coordinate representation.

$$L_z = xP_y - yP_x = -i\hbar \underbrace{(x\frac{\partial y}{\partial \varphi} - y\frac{\partial x}{\partial \varphi})}_{\partial \varphi}$$

$$L_z = -i\hbar \partial \varphi$$

$$\begin{aligned} \partial x &= \cos \varphi dr - \frac{\sin \varphi}{r} d\varphi \\ \partial y &= \sin \varphi dr + \frac{\cos \varphi}{r} d\varphi \end{aligned}$$

$$\begin{aligned} \Psi_{\text{rot}}(\varphi) &= \Psi(\varphi - \delta\varphi) = (1 - \delta\varphi \partial \varphi) \Psi \\ &\stackrel{|}{=} (1 - \delta\varphi \frac{i}{\hbar} L_z) \Psi \end{aligned}$$

$$\Psi_{\text{rot}} = e^{\frac{-i}{\hbar} \Delta\varphi L_z} \Psi = \Psi(\varphi - \Delta\varphi)$$

Commutation Relation:

$$\begin{aligned} [L_x, L_y] &= [yP_z - zP_y, zP_x - xP_z] \\ &= [yP_z, zP_x] + [zP_y, xP_z] \\ &= yP_x \underbrace{[P_z, z]}_{-i\hbar} + P_y \times \underbrace{[z, P_z]}_{i\hbar} \\ &= i\hbar (xP_y - yP_x) \\ &= i\hbar L_z \end{aligned}$$

In general:  $[L_i, L_j] = i\hbar e^{ijk} L_k$   
with  $\vec{L} = \vec{r} \times \vec{p}$

Since  $\hbar$  has same unit as  $L$ ,

define dimensionless  $\ell = \frac{L}{\hbar}$

$$\text{then } [\ell_i, \ell_j] = i\epsilon^{ijk} \ell_k$$

$$e^{-i\vec{\varphi} \cdot \vec{i}} = e^{-i\varphi \hat{n} \cdot \vec{i}} = e^{\frac{i}{\hbar}\varphi \hat{n} \cdot \vec{L}} \quad \text{finite rotation}$$

$$[\ell_x, Y] = [-i(\gamma \partial_z - z \partial_y), Y] = iz$$

$$[\ell_i, X_j] = \underbrace{[-i\epsilon^{imn} x_m \partial_n]}_{\vec{x} \times \vec{\partial}}, X_j]$$

Recap:  $\vec{L} = \vec{r} \times \vec{p}$ ,  $\vec{p} = -i\hbar \vec{\nabla}$

$$[L_i, L_j] = i\hbar \epsilon^{ijn} L_n$$

$$\vec{\epsilon} = \frac{\vec{L}}{\hbar} \text{ dimensionless } \vec{\epsilon}$$

then  $[l_i, l_j] = i \epsilon^{ijn} l_n$

$$e^{-i\varphi \vec{n} \cdot \vec{\epsilon}} \leftarrow \text{finite rotation.}$$

$$[l_i, x_j] = i \epsilon^{ijn} x_n$$

$$[l_i, p_j] = i \epsilon^{ijn} p_n$$

$$[l_i, a_j] = i \epsilon^{ijn} a_n \leftarrow a \text{ transforms as a vector rotation.}$$

$$[l_i, \underbrace{a_j b_j}_{}] = 0$$

transform as scalar under rotation.

Eigenvalues and Eigenstates of angular momentum:

$$[J_i, J_j] = i\hbar \epsilon^{ijk} J_k$$

build representation:  $J_z |b\rangle = b |b\rangle$  eigenbasis of  $J_z$   
but not  $J_x, y$ .  
since they don't commute

$$i) [J_i, J^2] = 0$$

$J_z$  and  $J^2$  diagonalize simultaneously.

$$J_z |a, b\rangle = b |a, b\rangle$$

$$J^2 |a, b\rangle = a |a, b\rangle, a \geq 0$$

2) Define  $J_{\pm} = J_x \pm i J_y$

then  $[J_z, J_{\pm}] = \pm \hbar J_{\pm}$

$[J_+, J_-] = 2\hbar J_z$

$[J^2, J_{\pm}] = 0$

$$\rightarrow J_z (J_{\pm} |a, b\rangle) = (\underbrace{[J_z, J_{\pm}]}_{\pm \hbar J_z} \pm J_z) |a, b\rangle$$

$$= (b \pm \hbar) J_z |a, b\rangle$$

$$\rightarrow J^2 (J_z |a, b\rangle) = a (J_z |a, b\rangle)$$

$$\rightarrow J_{\pm} |a, b\rangle = C_{\pm} |a, b \pm \hbar\rangle$$

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

$$= \frac{1}{2} (J_+ J_- + J_- J_+) + J_z^2$$

$$= J_- J_+ + J_z^2 + \hbar J_z$$

$$\langle a, b | \vec{J}^2 - J_z^2 | a, b \rangle = \frac{1}{2} \langle a, b | J_- J_+ + J_+ J_- | a, b \rangle$$

$$= a - b^2 \geq 0 = \frac{1}{2} \langle a, b | J_+^+ J_+ + J_+ J_+^+ | a, b \rangle$$

$$\text{or } a \geq b^2 = \frac{1}{2} |J_+|_{a,b}^2 + \frac{1}{2} |J_+^+|_{a,b}^2$$

$$J_+ |a, b\rangle = c |a, b + \hbar\rangle$$

$$(J_+)^n |a, b\rangle = \# |a, b + n\hbar\rangle$$

At some  $n$ , we have  $(b + n\hbar)^2 > a$ , which is not allowed

so there is a  $b_{\max}$ .

$$\text{then require } J_+ |a, b_{\max}\rangle = 0$$

$$J_- J_+ |a, b_{\max}\rangle = 0$$

$$\rightarrow \hookrightarrow (J^2 - J_z^2 - \hbar J_z) |a, b_{\max}\rangle = 0$$

$$\hookrightarrow (a - b_{\max}^2 - \hbar b_{\max}) |a, b_{\max}\rangle = 0 \quad \text{or}$$

$$a = b_{\max}(b_{\max} + 1)$$

Likewise, there is a  $b_{\min}$  such that:

$$\left. \begin{array}{l} a = b_{\min} (b_{\min} - \frac{\hbar}{2}) \\ a = b_{\max} (b_{\max} + \frac{\hbar}{2}) \end{array} \right\}$$

then there are two solutions, one of them is

$$b_{\min} = -b_{\max}$$

$$|a, b_{\max}\rangle = \# \mathbb{J}_+^n |a, \underbrace{b_{\max}}\rangle$$

$$|a, -b_{\max} + n\frac{\hbar}{2}\rangle$$

$\underbrace{b_{\min}}$

$$b_{\max} = -b_{\max} + n\frac{\hbar}{2}$$

$$\hookrightarrow b_{\max} = \frac{n\hbar}{2} \quad \rightarrow n: \text{integer}$$

let  $\frac{n}{2} = j = \text{integer or half-integer}$

$$a = \frac{\hbar^2}{2} j(j+1) \quad , \quad -\frac{\hbar}{2}j \leq b \leq \frac{\hbar}{2}j$$

$$b = -\frac{\hbar}{2}j, -\frac{\hbar}{2}j + \frac{\hbar}{2}, \dots$$

$$|a, b\rangle \rightarrow |j, m\rangle \quad \text{Irreducible representation.}$$

$$\boxed{\begin{array}{ll} \text{let } \hat{J}^2 |j, m\rangle = \frac{\hbar^2}{2} j(j+1) |j, m\rangle & j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \\ \hat{J}_z |j, m\rangle = \hbar m |j, m\rangle & -j \leq m \leq j \\ 2j+1 \dim \rightarrow m \in \{-j, -j+1, \dots, j-1, j\} \end{array}}$$

Ex:

$$j=0 \quad |0,0\rangle \rightarrow J_z|0,0\rangle = 0 \quad J^2|0,0\rangle = 0$$

$$j=\frac{1}{2} \quad J_i = S_i = \frac{1}{2}\hbar\sigma_i$$

$$J_i^2 = S_i^2 = \frac{1}{4}\hbar^2 (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{3}{4}\hbar^2 = \hbar^2 j(j+1)$$

$$j=1 \quad L_x = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$L_z = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = 2\hbar^2 = \hbar^2 j(j+1)$$

Matrix Elements of  $J_i$ :

$$\langle j', m' | \vec{J}^2 | j, m \rangle = \hbar^2 j(j+1) \delta_{j,j'} \delta_{m,m'}$$

$$\langle j', m' | J_z | j, m \rangle = \hbar m \delta_{j,j'} \delta_{m,m'}$$

$$J_+ J_- = \vec{J}^2 - \vec{J}_z^2 - \hbar J_z$$

$$\underbrace{\langle j, m | J_+^\dagger J_- | j, m \rangle}_{\langle j, m | J^2 - J_z^2 - \hbar J_z | j, m \rangle} = \langle j, m | J^2 - J_z^2 - \hbar J_z | j, m \rangle$$

$$\langle j, m | J_+^\dagger | j', m' \rangle \langle j', m' | J_- | j, m \rangle = \hbar^2 [ j(j+1) - m^2 - m ]$$

$$= \hbar^2 (j-m)(j+m+1)$$

Since  $\langle j', m' | \underbrace{J+1}_{\#} | j, m \rangle = \delta_{j,j'} \delta_{m,m+1}$

$$\langle j', m' | \# | j, m+1 \rangle$$

$\Rightarrow \langle j, m | J_+^+ | j', m' \rangle \langle j', m' | J_m | j, m \rangle = |\langle j, m+1 | \underbrace{J_+}_{\#} | j, m \rangle|^2$

\*

so  $J_+ | j, m \rangle = \hbar \sqrt{(j-m)(j+m+1)} | j, m+1 \rangle$

$J_- | j, m \rangle = \hbar \sqrt{(j+m)(j-m+1)} | j, m-1 \rangle$

$$J_{\pm} = J_x \pm i J_y \rightarrow J_x = \frac{J_+ + J_-}{2} \quad J_y = \frac{J_+ - J_-}{2i}$$

Representation of Rotation operator:

$$D(R) = e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{J}} \quad \leftarrow \text{unitary operator.} \quad \varphi = [-\pi, \pi]$$

Question: what is  $D(R)$  in  $j$ -notation:

Wigner Function.

$$D_{m',m}^{(j)}(R) = \langle j, m' | e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{J}} | j, m \rangle$$

linear function. Matrix of rotation operator.

$$J^2 (D(R) | j, m \rangle) = \hbar (j+1) j (D(R) | j, m \rangle)$$

Note that  $D(R) | j, m \rangle$  is still an eigenfunction of  $J^2$

Rotated state:

$$D(R) |j, m\rangle = \sum_{m'} |j, m'\rangle \underbrace{\langle j, m' | D(R) | j, m \rangle}_{D_{m'm}^{(ij)}(R)}$$

← coefficients

Euler Angles

$$D(R) = e^{-\frac{i}{\hbar} J_z \alpha} \downarrow e^{-\frac{i}{\hbar} J_y \beta} \downarrow e^{-\frac{i}{\hbar} J_z \gamma} \downarrow$$
$$[0, 2\pi] \quad [0, \pi] \quad [0, 2\pi]$$

$$D(R)_{m'm}^{(ij)}(\alpha, \beta, \gamma) = \langle j, m' | e^{-\frac{i}{\hbar} J_z \alpha} e^{-\frac{i}{\hbar} J_y \beta} e^{-\frac{i}{\hbar} J_z \gamma} | j, m \rangle$$

$$= \underbrace{e^{-im\gamma - im\alpha}}_{\text{does not depend on } j} \underbrace{\langle j, m' | e^{-\frac{i}{\hbar} J_y \beta} | j, m \rangle}_{d_{m'm}^{(ij)}(\beta)}$$

on  $j$

Ex:  
 $\hat{j} = \frac{1}{2}$

$$e^{\frac{i}{\hbar} J_y \beta} = e^{-\frac{i}{\hbar} S_y \beta} = \cos\left(\frac{\beta}{2}\right) - i \sin\left(\frac{\beta}{2}\right)$$

$$d^{(1/2)} = \begin{pmatrix} \cos \beta/2 & -\sin \beta/2 \\ \sin \beta/2 & \cos \beta/2 \end{pmatrix}$$

Then

$$\mathcal{D} = e^{-imY - im'Z} d^{1/2}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\left(\frac{\beta}{2}\right) e^{-i\frac{\gamma}{2}} & -\sin\frac{\beta}{2} e^{-i\frac{\alpha}{2}} e^{i\frac{\gamma}{2}} \\ e^{i\frac{\alpha}{2}} \sin\left(\frac{\beta}{2}\right) e^{-i\frac{\gamma}{2}} & \cos\frac{\beta}{2} e^{i\frac{\alpha}{2}} e^{i\frac{\gamma}{2}} \end{pmatrix}$$

Ex 2:  $j = 1$ .

$$J_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}$$

$$e^{-\frac{i}{\hbar} J_y \beta} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \beta^n \left( \frac{J_y^{(1)}}{2\hbar} \right)^n$$

Noting:

$$\left( \frac{J_y}{2\hbar} \right)^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\left( \frac{J_y}{2\hbar} \right)^3 = \frac{J_y}{2\hbar}$$

End result:

$$d^{(1)} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{1}{\sqrt{2}} \sin\beta & \frac{1-\cos\beta}{2} \\ -\frac{1}{\sqrt{2}} \sin\beta & \cos\beta & -\frac{1}{\sqrt{2}} \sin\beta \\ \frac{1-\cos\beta}{2} & \frac{1}{\sqrt{2}} \sin\beta & \frac{1+\cos\beta}{2} \end{pmatrix}$$

## Orbital Momentum and Spherical Harmonics:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[L_i, L_j] = i\hbar \epsilon^{ijk} L_k$$

$$[x_i, p_j] = \delta_{ij}(i\hbar)$$

$$L_z = -i\hbar \partial_\phi$$

$$\langle \vec{x} | L_z | \alpha \rangle = \langle \vec{x} | L_z | \vec{x} \rangle \langle \vec{x} | \alpha \rangle = -i\hbar \partial_\phi \langle \vec{x} | \alpha \rangle$$

$$L_z \psi_2(\vec{x}) = -i\hbar \partial_\phi \psi_2(\vec{x})$$

let  $|\vec{x}\rangle = |r, \varphi, \theta\rangle$  spherical coordinate.

$$L_z = -i\hbar (x\partial_y - y\partial_x) = -i\hbar \partial_\varphi$$

$$L_x = -i\hbar (-y\partial_z - z\partial_y) = -i\hbar (-\sin\varphi \partial_\theta - \cot\theta \cos\varphi \partial_\varphi)$$

$$L_y = -i\hbar (z\partial_x - x\partial_z) = -i\hbar (\cos\varphi \partial_\theta - \cot\theta \sin\varphi \partial_\varphi)$$

$$L_{\pm} = L_x \pm iL_y = -i\hbar e^{\pm i\varphi} (\pm i\partial_\theta - \cot\theta \partial_\varphi)$$

$$L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+) = -\hbar^2 \left[ \frac{1}{\sin^2\theta} \partial_\varphi^2 + \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) \right]$$

$$\nabla^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} \underbrace{\left[ \frac{1}{\sin^2\theta} \partial_\varphi^2 + \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) \right]}_{-\frac{1}{\hbar^2 r^2} L^2}$$

$$\boxed{\nabla^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) - \frac{1}{\hbar^2 r^2} L^2}$$

$$\langle \vec{x} | n, l, m \rangle = R_{nl}(r) \underbrace{Y_l^m(\theta, \phi)}_{\langle \theta, \phi | l, m \rangle}$$

First assume rotational symmetry  $\Rightarrow [H, L] = 0$

$$|l,m\rangle \Rightarrow L_z |l,m\rangle = \hbar m |l,m\rangle$$

$$L^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle$$

$$H |n,l,m\rangle = E_{n,l,m} |n,l,m\rangle$$

\*

$$\boxed{Y_l^m(\theta, \phi) = \langle \theta, \phi | l, m \rangle}$$

$$\langle \vec{X} | n, l, m \rangle = R_{nl}(r) + Y_l^m(\theta, \phi)$$

$$L_z |l,m\rangle = \hbar m |l,m\rangle$$

$$-i\hbar \partial_\phi Y_l^m(\theta, \phi) = \hbar m Y_l^{m'}(\theta, \phi)$$

$$Y_l^m(\theta, \phi) \propto e^{im\phi}$$

$$\left[ \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} \underbrace{\partial_\phi^2}_{-m^2} + l(l+1) \right] Y_l^m(\theta, \phi) = 0$$

$$\text{Know } \langle l', m' | l, m \rangle = \delta_{ll'} \delta_{mm'}$$

$$\int_0^{2\pi} d\phi \int_{-1}^1 d\cos\theta \underbrace{\langle l', m' | \hat{n} \rangle}_{Y_{l'}^{m'}(\hat{n})} \underbrace{\langle \hat{n} | l, m \rangle}_{Y_l^m(\hat{n})} = \delta_{ll'} \delta_{mm'}$$

↑  
orthonormality.

Look at  $|l, m=1\rangle$

then  $L_+ |l, m=1\rangle = 0$  since already at max.

$$-i\hbar e^{i\phi} (i\partial_\theta - \cot\theta \partial_\phi) Y_l^l = 0$$

$$\begin{aligned} & \text{assume } \\ & -l^2 e^{il\phi} Y_l^l \xrightarrow{\text{cancel}} -i\hbar e^{i\phi} (i\partial_\theta - i l \cot\theta) Y_l^l = 0 \\ & \text{so } \partial_\phi Y_l^l = l Y_l^l \end{aligned}$$

$$Y_l^l(\theta, \phi) = C_l e^{il\phi} \sin^l \theta$$

$$\text{with normalization: } C_l = \left[ \frac{(-1)^l}{2^l l!} \right] \sqrt{\frac{(2l+1)(2l)!}{4\pi}}$$

$$Y_l^{m-1} = \langle \hat{n} | l, m-1 \rangle = \langle \hat{n} | L - | l, m \rangle \frac{1}{\hbar \sqrt{(l+m)(l-m+1)}}$$

$$= \frac{-i\hbar}{\hbar \sqrt{(l+m)(l-m+1)}} e^{-i\phi} (-i\partial_\theta - \cot\theta \partial_\phi) Y_l^m$$

$$= \frac{1}{\sqrt{(l+m)(l-m+1)}} e^{-i\phi} (-\partial_\theta + i \cot\theta \partial_\phi) Y_l^m$$

$$(-\partial_\theta - m \cot\theta) Y_l^m$$

$$L^2 Y_l^m = l(l+1) Y_l^m \quad \hookrightarrow \sin\theta (-\partial_\theta) \sin^m \theta Y_l^m$$

$$\hookrightarrow Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \frac{(l+m)!}{(l-m)!} e^{im\phi} \frac{1}{\sin^m \theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} \frac{(\sin\theta)^{2l}}{(1-\cos^2\theta)^l}$$

$$Y_l^{-m}(\theta, \phi) = (-1)^m (Y_l^m(\theta, \phi))^*$$

$$\hookrightarrow Y_l^m = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \quad \int_0^{2\pi} d\phi \int_{-1}^1 d\cos\theta Y_l^{*m} Y_l^m = \delta_{l1} \delta_{mm}$$

$$e^{-\frac{i}{\hbar}(2\pi)L_z} = e^{-2\pi i \partial_\phi}$$

$$e^{-2\pi i \partial_\phi} \psi(\phi) = \psi(\phi - 2\pi) = \psi(\phi)$$

but  $e^{\frac{-i}{\hbar}2\pi L_z} |l,m\rangle = e^{-i2\pi m} |l,m\rangle$

then  $m \in \text{integer}$ , not half integer. for orbital angular momentum.

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}} e^{im\phi} P_l^m(\cos\theta)$$

associated Legendre polynomial.

$$\text{Let } \gamma = \cos\theta$$

$$[(1-\gamma^2)P']' + l(l+1)P = 0$$

$\downarrow$   
 $P_l \rightarrow \text{legendre polynomial.}$

Rodrigues Formula for Legendre-Poly

$$P_l(\gamma) = \frac{1}{2^l l!} \frac{d^l}{d\gamma^l} (1-\gamma^2)^l, \quad l=0,1,2\dots$$

Rodrigues Formula for associated Legendre Poly:

$$[(1-\gamma^2)P']' + (l(l+1) - \frac{m^2}{1-\gamma^2})P = 0$$

$\uparrow$   
For associated Legendre Polynomial.

$$l=0 : \quad Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad S\text{-wave}$$

$$l=1 : \quad \begin{aligned} Y_1^1 &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \sim \frac{x+iy}{r} \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos\theta \sim \frac{z}{r} \\ Y_1^{-1} &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \sim \frac{x-iy}{r} \end{aligned} \quad \left. \right\} P\text{-wave.}$$

$$l=2 \quad Y_2^2 \quad \sim \quad \left. \begin{aligned} &\sim \frac{(x+iy)^2}{r^2} \\ &\sim \frac{z(x+iy)}{r^2} \\ &\sim \frac{3z^2 - r^2}{r^2} \end{aligned} \right\} d\text{-wave.}$$

## Addition of Angular Momentum:

Ex: 2 spin  $\frac{1}{2}$  particles. (no orbital dof)

$H_1$  : 2d Hilbert space for particle 1.  $\vec{S}_1$ ,  
 $H_2$  : " particle 2.  $\vec{S}_2$

Algebra

$$\hookrightarrow [S_{1,i}, S_{1,j}] = i\hbar S_{1,k} \epsilon^{ijk} \quad |+\rangle_{1,2} \in H_{1,2} \text{ with basis } |+\rangle_{1,2}$$

Composite System:

$$H = H_1 \otimes H_2 \quad \vec{S}_1 \rightarrow \vec{S}_1 \otimes 1 \quad \vec{S}_2 \rightarrow 1 \otimes \vec{S}_2$$

↑  
tensor product

$$\hookrightarrow \vec{S} = \vec{S}_1 \otimes 1 + 1 \otimes \vec{S}_2 : \text{total spin.}$$
$$= \vec{S}_1 + \vec{S}_2$$

$$[S_i, S_j] = i\hbar S_k \epsilon^{ijk} \leftarrow \text{same algebra}$$

$S|+\rangle$

↑ 4-D representation of  $SU(2)$

$$2j+1 = 4 \leftarrow 4 \text{ dimension.}$$

↑  $j = \frac{3}{2}$  if irreducible.

Since  $|+\rangle|+\rangle \sim \hbar \leftarrow$  Max value is just  $\hbar$ .  
 $|+\rangle|-\rangle \sim 0$  so can't be  $j = \frac{3}{2}$ .  
 $|-\rangle|+\rangle \sim 0$   
 $|-\rangle|-\rangle \sim -\hbar$

instead look for forms:

$$\begin{pmatrix} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{pmatrix}$$

Consider  $S_+$ :

$$S_+ |++\rangle = 0$$

$$S_+ |+-\rangle = S_+ \cancel{\otimes} |+-\rangle + 1 \otimes S_+ |+-\rangle$$

$$= \hbar |++\rangle$$

$$S_+ |-+\rangle = \hbar |++\rangle$$

$$S_+ |--\rangle = \hbar (|+-\rangle + |-+\rangle)$$

$$S_{\pm} = \frac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) = 0 \quad \leftarrow S=0 \text{ representation.}$$

$$\text{Call } |\underline{\underline{-}}\rangle = |\underline{-}\rangle$$

$$S_z |\underline{-}\rangle = -\hbar |\underline{-}\rangle$$

$$S_t |\underline{-}\rangle = \sqrt{2} \hbar \left[ \frac{1}{\sqrt{2}} (|\underline{+-}\rangle + |\underline{-+}\rangle) \right] = \sqrt{2} \hbar |\underline{0}\rangle$$

$$S_t |\underline{0}\rangle = \sqrt{2} \hbar |\underline{++}\rangle = \sqrt{2} \hbar |\underline{..}\rangle$$

Since  $S^2 |\underline{\underline{1}}\rangle = \frac{2\hbar^2}{3} |\underline{1}\rangle$

$$\begin{aligned} (j+1) j &= 2 \\ j &= 1 \end{aligned}$$

$$|\underline{++}\rangle = |\underline{..}\rangle$$

Clebsch - Gordan Coeff's: coefficient for linear combinations when forming eigenstate.

$$\frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2}$$

$$\begin{array}{cccc} \nearrow & \uparrow & \nearrow & \uparrow \\ j=\frac{1}{2} & j=\frac{1}{2} & j=1 & j=0 \end{array}$$

Ex: spin  $\frac{1}{2}$  with orbital distribution:

$| \pm \rangle$ : spin states

$| \vec{r} \rangle$ : orbital states.

$$| \vec{r}, \pm \rangle = | \vec{r} \rangle \otimes | \pm \rangle \quad \text{basis of } H.$$

$$\vec{J} = \vec{L} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{S} = \vec{L} + \vec{S}$$

$$D(R) = e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{L}} e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{S}} = e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot (\vec{L} + \vec{S})}$$

$$\langle \vec{r}, \pm | \alpha \rangle = \psi_{\pm}(\vec{r}) \rightarrow \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

$$\vec{L}^2 = \hbar^2 l(l+1)$$

$$S^2 = \frac{3}{4}\hbar^2$$

$$L_z = \hbar m$$

$$S_z = \pm \frac{\hbar}{2}$$

$$| l, m; S=\frac{1}{2}, S_z=\pm \frac{\hbar}{2} \rangle \rightarrow | j, j_z; \frac{l}{j}, \frac{l_z}{j}; S, S_z \rangle$$

fix  $l$ , then we have  $\underbrace{(2l+1)}_m \times \underbrace{2}_{\text{spin}} = \# \text{ of states.}$

$$\boxed{l \otimes \frac{1}{2} = (l+\frac{1}{2}) \oplus (l-\frac{1}{2})}$$

## Formal Theory:

$$\vec{J}_1, \vec{J}_2 : [J_{\alpha i}, J_{\alpha j}] = i\hbar \epsilon^{ijk} J_{\alpha k}, \quad \alpha=1,2$$

$$\begin{aligned}\vec{J} &= \vec{J}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \vec{J}_2 \\ &\stackrel{!}{=} \vec{J}_1 + \vec{J}_2\end{aligned}$$

For infinitesimal rotations:

$$(1 - \frac{i}{\hbar} \delta\phi \hat{n} \cdot \vec{J}_1) \otimes (1 - \frac{i}{\hbar} \delta\phi \hat{n} \cdot \vec{J}_2) = 1 - \underbrace{\frac{i(\vec{J}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \vec{J}_2)}{\hbar}}_{\vec{J}} \hat{n} \cdot \delta\phi$$

Or for finite rotation:

$$D(R) \otimes D_2(k) = e^{-\frac{i}{\hbar} \phi \hat{n} \cdot \vec{J}_1} \otimes e^{\frac{-i}{\hbar} \phi \hat{n} \cdot \vec{J}_2}$$

↪  $[J_i, J_j] = i\hbar \epsilon^{ijk} J_k \rightarrow$  all results from  $SU(2)$  algebra hold for total angular momentum,  $\vec{J}$

Consider 2 different basis:

$$\text{Basis A: } (\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}) \rightarrow |j_1, j_2, m_1, m_2\rangle$$

Since  $-l < m < l$ , so there are  $(2j_1+1)(2j_2+1)$  states  
ex:  $J_1^2 |j_1, j_2, m_1, m_2\rangle = \hbar^2 (j_1+1) j_1 |j_1, j_2, m_1, m_2\rangle$

$$\text{Basis B: } (\vec{J}_1^2, \vec{J}_2^2, J^2, J_z) \rightarrow |j_1, j_2, j, m\rangle$$

Dim:  $\sum_j (2j+1)$  : sum over different  $j$ 's.

Note that  $[J_{1z}, J^2] \neq 0$ .

## Transforming Basis

Introduce projection:  $\sum_{m_1} \sum_{m_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2| = 1$

then

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} |j_1, j_2; m_1, m_2\rangle \underbrace{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan Coefficients}}$$

they form a unitary matrix  
of transformation from A to B.

Properties of CG Coefficients:

$$(J_z - J_{1z} - J_{2z}) |j_1, j_2; j, m\rangle = 0$$

$$\hookrightarrow \underbrace{(m - m_1 - m_2) \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle}_{} = 0$$

$$\boxed{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle \sim \delta_{m_1, m_1 + m_2}}$$

What values can  $j$  take?

Suppose we start with the highest state: only one vector

$$|j_1, j_2; m_1=j_1, m_2=j_2\rangle \leftarrow J_z = j_1 + j_2.$$

Now apply  $J_- = J_{1-} + J_{2-}$  to obtain  $J_z = j_1 + j_2 - 1$ .

But there are two such vectors:

$$m_1=j_1, m_2=j_2-1 \quad \text{or} \quad m_1=j_1-1, m_2=j_2$$

Therefore: for  $j_1 \otimes j_2 = |j_1-j_2|\oplus|j_1-j_2|\oplus\dots\oplus j_1+j_2$

★  $\hookrightarrow \boxed{(2j_1+1)(2j_2+1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1)}$

Since  $\langle j_1, j_2; \bar{j}m | j_1, j_2, m_1, m_2 \rangle$  is unitary matrix elements, they can be chosen to be real.

$$\sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2, \bar{j}m \rangle \langle j_1, j_2; m'_1, m'_2 | j_1, j_2, \bar{j}m \rangle = \delta_{m'm'_1} \delta_{m'm'_2}$$

$\hookrightarrow \sum_{m_1, m_2} |\langle j_1, j_2; m_1, m_2 | j_1, j_2, \bar{j}m \rangle|^2 = 1$

Wigner's 3j symbol

$\langle j_1, j_2; m_1, m_2 | j_1, j_2, \bar{j}m \rangle = (-1)^{\bar{j}_1 - \bar{j}_2 + m} \sqrt{2j+1} \begin{Bmatrix} j_1 & j_2 & \bar{j} \\ m_1 & m_2 & m \end{Bmatrix}$

## Recursion Relation for CG Coefficients:

$$\text{Via } J_{\pm} |j_1, j_2, j, m\rangle = \sum C(J_{1\pm} + J_{2\pm}) |j_1, j_2, m_1, m_2\rangle$$

Ex: let  $j_1 = l$        $m_1 = m_1$   
 $j_2 = S = \frac{1}{2}$        $m_2 = \pm \frac{1}{2}$        $\left\{ \begin{array}{l} j = l \pm \frac{1}{2}, l > 0 \\ \end{array} \right.$

$$\langle l, \frac{1}{2}; j, m | l, \frac{1}{2}, m_1, m_2 \rangle = \begin{pmatrix} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} & \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} \\ -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} & \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} \end{pmatrix}$$

## Spin-Angular Functions:      $\langle \vec{r}, \pm l, l, \frac{1}{2}, j, m \rangle$

$$\hookrightarrow Y_l^{\pm l \pm \frac{1}{2}, m} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm \sqrt{l \pm m + \frac{1}{2}} & Y_l^{m-\frac{1}{2}}(\theta, \phi) \\ \sqrt{l \mp m + \frac{1}{2}} & Y_l^{m+\frac{1}{2}}(\theta, \phi) \end{pmatrix}$$

$|l, \frac{1}{2}; j, m\rangle$  : Eigen functions of  $L^2, S^2, J^2$ , and  $J_z$ .

use  $J^2 = (\vec{L} + \vec{S})^2 = L^2 + \underbrace{\vec{L} \cdot \vec{S} + \vec{S} \cdot \vec{L}}_{\text{commute.}} + S^2$

$$\hookrightarrow \vec{L} \cdot \vec{S} = \frac{1}{2} (J^2 - L^2 - S^2)$$

then  $\vec{L} \cdot \vec{S} |l, \frac{1}{2}; j, m\rangle = \frac{\hbar^2}{2} [j(j+1) - l(l+1) - \frac{3}{4}]$

↑  
Spin-orbit interaction.      =  $\begin{cases} \frac{1}{2}\hbar^2 & j = l + \frac{1}{2} \\ -\frac{(l+1)\hbar^2}{2} & j = l - \frac{1}{2} \end{cases}$

## Tensor Operators:

$|\alpha\rangle \rightarrow D(R)|\alpha\rangle$  : Transformation of Hilbert space under rotation of real space.

How about transformation of operators?

$$\langle \alpha | O | \alpha \rangle \rightarrow \langle \alpha | D^+(R) O D(R) | \alpha \rangle = \langle \alpha | \tilde{O} | \alpha \rangle$$

$O \rightarrow D^+(R) O D(R)$  : Transformation of linear operator.

Cases: 1) Scalar:

Say  $O = \vec{r}^2 = r^2$        $r^2 \rightarrow D^+(R) r^2 D(R) = r^2$

Scalar operator is not affected by transformation.

Infinitesimal rotation.  $D(R) = 1 - \frac{i}{\hbar} \delta\phi \hat{n} \cdot \vec{j}$

$$\tilde{O}_R = (1 + \frac{i}{\hbar} \delta\phi \hat{n} \cdot \vec{j}) O (1 - \frac{i}{\hbar} \delta\phi \hat{n} \cdot \vec{j}) = O + \underbrace{\frac{i}{\hbar} \delta\phi \hat{n} [ \vec{j}, O ]}_{SO}$$

In general

$$SO = \frac{i}{\hbar} \delta\phi \hat{n} [ \vec{j}, O ]$$

Define Scalar when  $SO = O \rightarrow [ \vec{j}, O ] = O$   
 or  $\tilde{O}_R = D^+(R) O D(R) = O$

Ex 2: Vector:

$$x_i \rightarrow D^+(R) x; D(R)$$

$$\text{Infinitesimal: } \delta x_i = \frac{i}{\hbar} \delta \phi \hat{n}_j [L, x_i] = \frac{i}{\hbar} \delta \phi \hat{n}_j i \hbar \epsilon^{ijk} x_k.$$

$$\delta x_i = \delta \phi \epsilon^{ijk} \hat{n}_j x_k$$

So  $[L_i, x_j] = i \hbar \epsilon^{ijk} x_k$

In general:

$$\text{If } \langle \alpha | v_i | \alpha \rangle = \langle \alpha | D^+(R) v_i | \alpha \rangle = R_{ij} \langle \alpha | v_j | \alpha \rangle$$

then  $\vec{v}$  is a vector operator,

i.e.  $[L_i, v_j] = i \hbar \epsilon^{ijk} v_k$

ex:

$$[L_i, L_j] = i \hbar \epsilon^{ijk} L_k$$

rotation generator  $\nearrow$   $\uparrow$  vector  $\nwarrow$  transformed vector.

Ex 3: Dyadic:

Consider  $T_{ij} = U_i V_j$  where  $U_i, V_j$  are vector operators

$$\text{If } T_{ij} \rightarrow R_{ii} U_i, R_{jj} V_j = R_{ii} R_{jj} T_{ij}$$

then  $U_i V_j$  is a rank 2 Cartesian Tensor.

Generally:  $T_{i_1 i_2 \dots i_n} \rightarrow \sum_{j_1 j_2 \dots j_n} R_{i_1 j_1} R_{i_2 j_2} \dots R_{i_n j_n} T_{j_1 j_2 \dots j_n}$

$\underbrace{T_{i_1 i_2 \dots i_n}}_n$  - Cartesian tensor of rank n.

Note: Classification of scalar, vector and general tensor is for general linear transformation. However corresponding representation are reducible under rotation group.

$$\text{Ex: } \underbrace{U_i V_j}_{9\text{-component}} = \underbrace{\frac{U_i V_j}{3} \delta_{ij}}_{1\text{-component (scalar)}} + \underbrace{\frac{U_i V_j - V_j V_i}{2}}_{\frac{1}{2} \epsilon^{ijk} U_i V_j} + \underbrace{\left( \frac{U_i V_j + V_j V_i}{2} - \frac{U_i V_i}{3} \delta_{ij} \right)}_{\text{Symmetric traceless tensor}}$$

$\downarrow$                      $\downarrow$                      $\downarrow$

$t=0, 1D$              $t=1, 3D$              $t=2, 5D$

$$9 = 1 + 3 + 5$$

$$\text{or } 1 \otimes 1 = 0 \oplus 1 \oplus 2$$

Spherical Tensor:  $Y_i^m(\hat{n})$  under rotation of  $\hat{n}$ .

$$D^+(R) Y_i^m(\hat{n}) D(R) = \sum_{m'=-1}^1 Y_i^{m'} \underbrace{D_{mm'}^{(R)*}(R)}_{\langle |e^{\frac{i}{\hbar} \varphi_m \hat{n} \cdot \vec{r}}| \rangle}$$

Def: if  $T_q^{(k)}$   $q = -k, \dots, +k$

$$\text{transform as } D^+(R) T_q^{(k)} D(R) = \sum_{q'=m}^m D_{qq'}^{(k)*} T_{q'}^{(k)}$$

then  $T_q^{(k)}$  is called spherical tensor of rank  $k$ .

Infinitesimal version:

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$$

$$[J_{\pm}, T_q^{(k)}] = \hbar \sqrt{(k+q)(k+1)} T_{q \pm 1}^{(k)}$$

Example:

$$\left. \begin{array}{l} U_{\pm 1} = \frac{\mp U_x - i U_y}{\sqrt{2}} = Y_1^{\pm 1}(\vec{U}) \sqrt{\frac{4\pi}{3}} \\ U_0 = U_z = Y_1^0(\vec{U}) \sqrt{\frac{4\pi}{3}} \end{array} \right\} \text{instead of } x, y, z$$

$U_q \rightarrow$  spherical tensor of rank 1.  
 $\downarrow q=0, \pm 1$

$$\text{so } U_q = T_q^{(1)}$$

Ex:

Dyadic:  $\cup_i \nabla_j$

$$T_0^{(0)} = -\frac{\vec{U} \cdot \vec{V}}{2} = \frac{U_{+1}V_{+1} + U_{-1}V_{-1} - U_0V_0}{3} \quad \text{rank 0}$$

$$T_q^{(1)} = \frac{(\vec{U} \times \vec{V})_q}{i\sqrt{2}}, \quad q=0, \pm 1 \quad \text{rank 1.}$$

$$T_{\pm 2}^{(2)} = U_{\pm 1} V_{\mp 1}$$

$$T_{\pm 1}^{(2)} = \frac{U_{\pm 1}V_0 + U_0V_{\pm 1}}{\sqrt{2}}$$

$$T_0^{(2)} = \frac{U_{+1}V_{-1} + 2U_0V_0 + U_{-1}V_{+1}}{\sqrt{6}}$$

}

rank 2.

General: spherical tensors.

$$X_{(q_1)}^{(k_1)}, \quad Z_{q_2}^{(k_2)}$$

Then

$$T_q^{(k)} = \sum_{q_1} \sum_{q_2} \langle k_1, k_2; q_1, q_2 | k_1 k_2; k, q \rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

$$q_1 + q_2 = q.$$

### Matrix Elements of tensors:

Example:  $\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle \neq 0$  only if  $m' = q + m$

$\uparrow$

m-selection rule.  $\rightarrow$   $m' = q + m$

Proof:  $0 = \langle \alpha, j m' | [J_z, T_q^{(k)}] + \hbar q T_q^{(k)} | \alpha, j m \rangle$

$$= (\hbar(m' - m) - \hbar q) \underbrace{\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle}_{\neq 0 \text{ unless}} \neq 0$$

$\underbrace{\hbar(m' - m - q)}_{\text{or } m' = q + m} = 0$

$\leftarrow$  m-selection rule.

$$D(T_q^{(k)} | \alpha, j m \rangle) = \underbrace{(D T_q^{(k)} D^+)}_{k-\text{repr}} D | \alpha, j m \rangle$$

$\leftarrow$  j-repr

Wigner-Eckart Theorem:

$$\langle \alpha', j' m' | T_i^{(k)} | \alpha, j m \rangle = \underbrace{\langle j k; m q | j k', j' m' \rangle}_{\text{CG coeff}} \frac{\langle \alpha', j' | T^{(k)} | \alpha, j \rangle}{\sqrt{2j+1}}$$

Triangular relation:  $|j-k| \leq j' \leq j+k$

ex: scalar

$$T_0^{(0)} = S$$

$$\langle \alpha' j' m' | S | \alpha j m \rangle = \delta_{jj'} \delta_{mm'}, \frac{\langle \alpha' j' || S || \alpha j \rangle}{\sqrt{2j+1}}$$

If  $H = S$ , not invariant: then  $2j+1$  degeneracy.

ex 2: vector:  $V_{q=0, \pm 1}$

$$\langle \alpha' j' m' | V_i^{(1)} | \alpha j m \rangle = \langle j'-1; m_q | j | j'; m' \rangle \frac{\langle \alpha' j' || V_q || \alpha j \rangle}{\sqrt{2j+1}}$$

then  $m' - m = \begin{cases} \pm 1, 0 \\ q \end{cases}$

and  $j' - j = \begin{cases} \pm 1 \\ 0 \end{cases}$

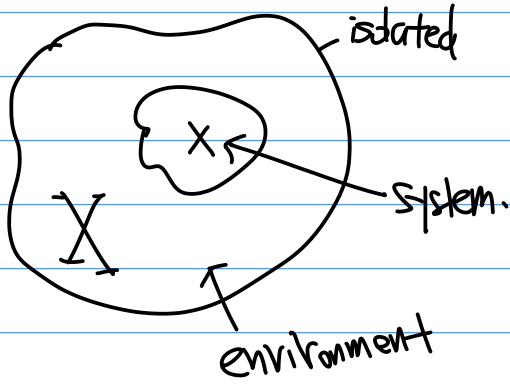
← Rotational Selection Rule  
for vector.

Used triangular relation

Electric Field:  $V(x) = -Ex \cos \alpha t$ .

## Density Operators:

isolated quantum state:



$$\psi(x, \bar{x})$$

$A$  - operator. acting on system

$$\begin{aligned} \langle A \rangle &= \int dx dx' d\bar{x} \psi^*(x, \bar{x}) A(x, x') \psi(x', \bar{x}) \\ &= \int dx dx' A(x, x') \underbrace{\int d\bar{x} \psi^*(x, \bar{x}) \psi(x, \bar{x})}_{\rho(x', x)} \end{aligned}$$

\*

$$\langle A \rangle = \int dx dx' A(x, x') \rho(x, x) = \text{Tr}_{\text{sys}} \{ A \rho \}$$

$$\text{where } \rho(x', x) = \int d\bar{x} \psi^*(x', \bar{x}) \psi(x, \bar{x})$$

density matrix, or density operator.

Important properties:

- i)  $\rho$  is Hermitian,  $\rho = \rho^\dagger$
- ii)  $\text{Tr}\{\rho\} = 1$ .

$$\text{Diagonalization of } \rho: \quad \rho = \sum_i \lambda_i |v_i\rangle \langle v_i|$$

↑  
eigenbasis of  $\rho$ .

$$\text{Since } \hat{\rho}(x', x) = \langle x', x | \hat{\rho} \rangle$$

$$\hat{\rho}^*(x, x') = \langle x' | \hat{\rho} | x \rangle$$

then  $\hat{\rho}(x', x) = \int d\bar{x} \langle x', \bar{x} | \hat{\rho} \rangle \langle \bar{x} | x, x' \rangle$

$$\hat{\rho}(x', x) = \langle x' | \underbrace{\text{Tr}_{\bar{x}} (\hat{\rho})}_{\hat{\rho}} | x \rangle$$

$$\hookrightarrow \hat{\rho}(x', x) = \langle x' | \hat{\rho} | x \rangle$$

$$\hookrightarrow \boxed{\hat{\rho} = \text{Tr}_{\bar{x}} (\hat{\rho})}$$

over environment.

$$\rightarrow \langle A \rangle = \text{Tr}_{\bar{x}} (\hat{\rho} A)$$

$$\rightarrow \text{Tr}_{\bar{x}} \hat{\rho} = \text{Tr}_{\bar{x}} \text{Tr}_{\bar{x}} (\hat{\rho})$$

$$\stackrel{!}{=} \text{Tr} (\hat{\rho})$$

$$\stackrel{!}{=} 1$$

so  $\boxed{\text{Tr} (\hat{\rho}) = 1}$

Ex: 2-spin half:

$$|S\rangle = \frac{1}{\sqrt{2}} (|+-\rangle \mp |-+\rangle)$$

$$\begin{aligned} \hat{\rho}_1 &= \text{Tr}_2 |S\rangle\langle S| \\ \text{of the} \\ \text{first spin} \quad &= \frac{1}{2} \langle + |S\rangle\langle S|+ \rangle + \langle - |S\rangle\langle S|-\rangle \end{aligned}$$

$$= \left( \mp \frac{1}{\sqrt{2}} |-\rangle_1 \right) \left( \mp \frac{1}{\sqrt{2}} \langle -1 \right) + \left( \frac{1}{\sqrt{2}} |+\rangle_1 \right) \left( \frac{1}{\sqrt{2}} \langle +1 \right)$$

$$\hat{\rho}_1 = \frac{1}{2} |-\rangle\langle -1| + \frac{1}{2} |+\rangle\langle +1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\text{Tr}(\hat{\rho}) = 1$$

$$\langle \sigma_z \rangle = \text{Tr} \left( \frac{1}{2} \cdot \mathbb{I} \cdot \sigma_z \right) = 0$$

$$\text{Tr}(\hat{\rho}^2) = \frac{1}{2}$$

$$\langle \sigma_{x,y} \rangle = \text{Tr} \left( \frac{1}{2} \cdot \mathbb{I} \cdot \sigma_{x,y} \right) = 0$$

Ex 2:

$$|S\rangle = \frac{1}{\sqrt{2}} (|+\rangle_1 + |-\rangle_1) |+\rangle_2$$

$$= \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle)$$

$$\hat{\rho}_1 = \text{Tr}_2 (|S\rangle\langle S|)$$

$$= \frac{1}{2} \langle + |S\rangle\langle S|+ \rangle_2 + \frac{1}{2} \cancel{\langle - |S\rangle\langle S|-\rangle_2}$$

$$= \frac{1}{2} (|+\rangle_1 + |-\rangle_1) (\langle +1, +\langle -1)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} (1 + \sigma_x)$$

$$\begin{aligned} \text{Tr} \hat{\rho}_1 &= 1 \\ \text{Tr} (\hat{\rho}_1^2) &= 1 \end{aligned}$$

$$\begin{aligned} \langle \sigma_x \rangle &= \text{Tr} \left( \frac{1}{2} (1 + \sigma_x) \sigma_x \right) = 1 \\ \langle \sigma_{y,z} \rangle &= 0 \end{aligned}$$

Ensemble:  $|\alpha^{(i)}\rangle$ ,  $w_i$ : probability at  $|\alpha^{(i)}\rangle$   
 Not necessarily  
 orthogonal

$$\sum w_i = 1$$

$$\begin{aligned}
 [A] &= \sum_i w_i \underbrace{\langle \alpha^{(i)} | A | \alpha^{(i)} \rangle}_{\text{Quantum Average.}} \\
 \xrightarrow{\substack{\text{Ensemble} \\ \text{Average}}} \quad &= \sum_i w_i \sum_{b_1, b_2} \langle \alpha^{(i)} | b_1 \rangle \langle b_1 | A | b_2 \rangle \langle b_2 | \alpha^{(i)} \rangle \\
 &\quad \left| \begin{array}{l} \uparrow \\ \text{(complete basis)} \end{array} \right. \\
 &= \sum_{b_1, b_2} \left( \sum_i w_i \langle b_2 | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b_1 \rangle \right) \underbrace{\langle b_1 | A | b_2 \rangle}_{f(b_1, b_2)} \\
 &= \text{Tr}(f A)
 \end{aligned}$$

$$\boxed{\hat{\rho} = \sum_i w_i |\alpha^{(i)}\rangle \langle \alpha^{(i)}|}$$

< density operator for ensemble

$$\hookrightarrow \text{Tr } \hat{\rho} = 1$$

$$\hookrightarrow \hat{\rho}^\dagger = \hat{\rho}$$

### Pure Ensemble:

If  $w_i = 0$  except one of them is 1;

then  $\rho = |4\rangle\langle 4| \leftarrow \text{pure Ensemble.}$

then  $[A] = \text{Tr}(\rho A) = \text{Tr}(|4\rangle\langle 4|A) = \text{Tr}(\langle 4|A|4\rangle)$

$$[A] = \langle A \rangle$$

Pure state:  $\boxed{\rho^2 = \rho}, \boxed{\text{Tr } \rho^2 = 1}$

In general  $\text{Tr } \rho^2 = \lambda_1^2 + \lambda_2^2 + \dots \leq \lambda_1 + \lambda_2 + \lambda_3 \dots = 1$

Pure Ensemble: ex:  $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$        $\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$   
 $\rho^2 = \rho$        $\rho^2 = \rho.$

Diag  $\rho_{\text{pure}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \leftarrow 1 \text{ somewhere in diagonal, } 0 \text{ elsewhere.}$

### Completely Random

$$\text{Diag } \rho = \begin{pmatrix} \frac{1}{N} & & & & 0 \\ & \frac{1}{N} & & & \\ & & \ddots & & \\ 0 & & & \ddots & \frac{1}{N} \end{pmatrix} = \frac{1}{N} \cdot \mathbb{I}$$

$$\text{Tr}(\rho^2) = \frac{1^2}{N} N = \frac{1}{N} \leftarrow \text{minimum possible}$$

proof:

$$\sum w_i^2 - \lambda(\sum w_i - 1) = 0$$
$$2w_i - \lambda = 0 \quad \sum w_i = 1$$
$$\hookrightarrow w_i = \frac{\lambda}{2} = \text{const} \Rightarrow s_0 \quad w_i = \frac{1}{N}$$
$$\begin{aligned} \partial p_i - \sum_i p_i \ln p_i - \lambda \sum_i p_i \\ \hookrightarrow -\ln p_i - 1 - \lambda = 0 \\ p_i = e^{-\lambda-1} \end{aligned}$$

Mixed Ensemble:  $\boxed{p^2 \neq p, \text{Tr } p^2 < 1.}$

$|S_+ + \rangle$  with  $w=75\%$ ,  $|S_- \rangle$ , with  $w=25\%$

$$p = \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

$$\text{Tr } p = 1 \quad \text{Tr } p^2 = \frac{13}{16} < 1$$

## Evolution of $\rho$ :

$$\rho(t_0) = \sum_i w_i | \alpha^{(i)} \rangle_{t_0} \langle \alpha^{(i)} |$$

$$\rho(t) = \sum_i w_i U_{t,t_0} | \alpha^{(i)} \rangle_{t_0} \langle \alpha^{(i)} | U^+$$

$$*\boxed{\rho(t) = U_{t,t_0} \rho(t_0) U_{t,t_0}^+}, U = e^{\frac{i}{\hbar} H(t-t_0)}$$

$$i\hbar \partial_t \rho(t) = \sum_i w_i [(i\hbar \partial_t | \alpha^{(i)} \rangle \langle \alpha^{(i)} | + | \alpha^{(i)} \rangle \langle i\hbar \partial_t | \alpha^{(i)} |)]$$

$$= \sum_i w_i [H | \alpha^{(i)} \rangle \langle \alpha^{(i)} | - | \alpha^{(i)} \rangle \langle \alpha^{(i)} | H]$$

$$*\boxed{i\hbar \partial_t \rho(t) = [H, \rho] = -[\rho, H]}$$

vs. Classical:  $\frac{d}{dt} \rho_{\text{classical}} = \{ \rho_{\text{classical}}, H \}_{\text{PB}} + \frac{\partial}{\partial t} \rho_{\text{classical}}$

$$\frac{\partial \rho_{\text{classical}}}{\partial t} = - \{ \rho_{\text{classical}}, H \}_{\text{PB}}$$

vs. Heisenberg:  $i\hbar \partial_t O = [O, H] + i\hbar \cancel{\partial_t O}$  No explicit dependence in t.

Entropy:  $\delta = -\text{Tr}(\rho \ln \rho)$

$$= - \sum_i \rho_i \ln \rho_i$$

Pure:  $\rho_i = 0$  except  $\rho_i = 1$   
 so  $\delta = 0$  for pure.

Random:  $\rho_i = \frac{1}{w}$ ,  $\delta = - \sum_i \frac{1}{w} \ln \frac{1}{w} = \ln w \leftarrow \underset{\text{entropy.}}{\text{Max}}$

Two Independent solution of  $\rho_1$  and  $\rho_2$

$$\rho = \rho_1 \rho_2$$

$$\begin{aligned} S_{12} &= -\text{tr} \rho_1 \rho_2 \ln \rho_1 \rho_2 \\ &\stackrel{!}{=} -\text{tr} \rho_2 (\rho_1 \ln \rho_1) - \text{tr} \rho_1 \rho_2 \ln \rho_2 \\ &\stackrel{!}{=} -\underbrace{\text{tr}_2 \rho_2}_{=1} \text{tr}_1 (\rho_1 \ln \rho_1) - \underbrace{\text{tr}_1 \rho_1}_{=1} \text{tr}_2 \rho_2 \ln \rho_2 \end{aligned}$$

$$\boxed{S_{12} = S_1 + S_2}$$

## Statistical Mechanics:

In equilibrium:  $\frac{\partial \rho}{\partial t} = -[\rho, H] = 0$

then  $\rho$  and  $H$  diagonalize simultaneously.

$$\text{Tr } \rho = 1 \quad \text{Tr } \rho H = E$$

want to maximize  $S = -\text{Tr } \rho \ln \rho$

$$\hookrightarrow -\text{Tr } \rho \ln \rho - \beta \text{Tr } \rho H - \gamma \text{Tr } \rho$$

$$\hookrightarrow -\sum_i \delta \rho_i (\ln \rho_i + 1) - \sum_i \beta \delta \rho_i E_i - \sum_i \gamma \delta \rho_i = 0$$

$$\hookrightarrow -\ln \rho_i - 1 - \beta E_i - \gamma = 0$$

$$\rho_i = e^{-\beta E_i - \gamma - 1} = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$$

$$\boxed{\rho_i = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}}$$
$$\langle A \rangle = \text{Tr}(\rho A)$$

## Entanglement

System A and B in pure state  $|\psi\rangle_{AB}$

$$\rho_A = \text{Tr}_B |\psi\rangle_{AB} \langle \psi|$$

Suppose  $|\psi\rangle_{AB} = \sum_i \lambda_i^{>0} |\alpha^{(i)}\rangle_A |\beta^{(i)}\rangle_B$

$$\rho_A = \text{Tr}_B \sum_i \lambda_i^2 |\alpha^{(i)}\rangle_A |\beta^{(i)}\rangle_B \langle \beta^{(i)}| \langle \alpha^{(i)}|$$

$$\boxed{\begin{aligned} \rho_A &= \sum_i \lambda_i^2 |\alpha^{(i)}\rangle_A \langle \alpha^{(i)}| \\ &\quad \downarrow \\ \rho_A &= \sum_i w_i |\alpha^{(i)}\rangle \langle \alpha^{(i)}| \end{aligned}}$$

## Schmidt Decomposition:

If  $|\psi\rangle_{AB}$  is a pure state of A, B, then there exists orthonormal states  $|\alpha^{(i)}\rangle_A$  of A and  $|\beta^{(i)}\rangle_B$  of B such that

$$|\psi\rangle_{AB} = \sum_i \lambda_i |\alpha^{(i)}\rangle_A |\beta^{(i)}\rangle_B$$

and  $\lambda_i \geq 0$  and  $\sum \lambda_i^2 = 1$

$\uparrow$   
Schmid coefficients.

Corollary of Schmidt decomposition:

Eigenvalues of  $\rho_A$  and  $\rho_B$  are the sum.

$$\rho_A = \sum \lambda_i^2 |\alpha^{(i)}\rangle_A \langle \alpha^{(i)}|$$

$$\rho_B = \sum \lambda_i^2 |\beta^{(i)}\rangle_B \langle \beta^{(i)}|$$

$$S = -\text{tr}(\rho^A \ln \rho^A) = -\text{tr}(\rho^B \ln \rho^B)$$

entanglement entropy.

If  $S=0$ , then A and B are completely independent.

$$|\Psi\rangle_{AB} = |\alpha\rangle_A |\beta\rangle_B$$

Purification: given  $\rho^A$ , find  $|\Psi_{AB}\rangle$ ,  $\mathcal{H}^A \otimes \mathcal{H}^B$

$$\text{then } \rho^A = \text{tr}_B |\Psi_{AB}\rangle \langle \Psi_{AB}|$$

$$\text{Diagonalize } \rho^A = \sum_i p_i |\alpha^{(i)}\rangle_A \langle \alpha^{(i)}|$$

$$\text{consider } |\Psi_{AB}\rangle = \sum_i \sqrt{p_i} |\alpha^{(i)}\rangle_A |\beta^{(i)}\rangle_B$$

Given  $|\psi\rangle_{AB}$

$$\hookrightarrow \rho^A = \text{Tr}_B |\psi\rangle\langle\psi|$$

then  $S = -\text{tr } \rho^A \ln \rho^A$  ← Entropy, measure of entanglement between A, B in  $|\psi\rangle$

Density Matrices and measurements:

$$\rho \xrightarrow[\text{Time evolving}]{U} \rho' = U \rho U^\dagger$$

measure A in state  $\rho$ , for  $|\psi\rangle$  pure state  
 $\alpha |1\rangle\langle 1|$

$$|\psi\rangle \xrightarrow{\text{measure A}} |\alpha\rangle \text{ with probability: } |\langle\alpha|\psi\rangle|^2$$

$$\rho = |\psi\rangle\langle\psi| \xrightarrow{\text{measure A}} |\alpha\rangle\langle\alpha| \text{ with probability } |\langle\alpha|\psi\rangle|^2$$

$$\rho' = \sum_{\alpha} |\langle\alpha|\psi\rangle|^2 |\alpha\rangle\langle\alpha|$$

$$= \sum_{\alpha} |\alpha\rangle\langle\alpha| \rho |\alpha\rangle\langle\alpha|$$

$$\boxed{\begin{aligned} \rho \rightarrow \rho' &= \sum_{\alpha} M_{\alpha} \rho M_{\alpha}^\dagger \\ \sum_{\alpha} M_{\alpha}^\dagger M_{\alpha} &= I \end{aligned}} \quad \begin{array}{l} \text{← Time evolution} \\ \text{of } \rho. \end{array}$$

↑  
general time evolution of  $\rho$

$M_{\alpha} = |\alpha\rangle\langle\alpha|$  Operator.

$M_{\alpha} = |\alpha\rangle\langle\alpha|$  ← repeated measurement

$$M_{\alpha} = U$$

### Weak Measurement:

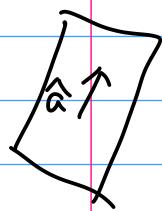
$$\rho \xrightarrow{\text{extent}} \hat{\rho} = |1\rangle\langle 1| \otimes \rho \xrightarrow[\cup]{\text{evolution}} \rho'$$

$$\rho' = \cup \hat{\rho} \cup^+ = \sum_{\alpha, \alpha'} |\alpha\rangle\langle\alpha'| \otimes M_\alpha \rho M_{\alpha'}^+$$

↓  
trace out  $|\alpha\rangle$

$$\rho \rightarrow \rho' = \sum_{\alpha} M_\alpha \rho M_\alpha^+$$

## Bell Inequality:



$\pm 1$

$$\frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle) = 10$$



$\pm 1$

$$P(\hat{a}, \hat{b})$$

average product of results of  $\hat{a}, \hat{b}$

$$P(\hat{a}, \hat{b}) = \langle 0 | (\vec{s}_1 \cdot \hat{a}) (\vec{s}_2 \cdot \hat{b}) | 0 \rangle$$

$$\vec{s}_1 \cdot \hat{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$$

$$\hookrightarrow \frac{1}{2} \langle \uparrow \downarrow | - \langle \downarrow \uparrow | (\vec{s}_1 \cdot \hat{a}) (\vec{s}_2 \cdot \hat{b}) | \uparrow \downarrow \rangle - \langle \downarrow \uparrow \rangle$$

$$\hookrightarrow \frac{1}{2} [ \underbrace{(\vec{s}_1 \cdot \hat{a})_{11} (\vec{s}_2 \cdot \hat{b})_{22}}_{1 \text{ means } \uparrow} + \underbrace{(\vec{s}_1 \cdot \hat{a})_{22} (\vec{s}_2 \cdot \hat{b})_{11}}_{2 \text{ means } \downarrow} - (\vec{s}_1 \cdot \hat{a})_{12} (\vec{s}_2 \cdot \hat{b})_{21} - (\vec{s}_1 \cdot \hat{a})_{21} (\vec{s}_2 \cdot \hat{b})_{12} ]$$

1 means  $\uparrow$  2 means  $\downarrow$ .

$$\hookrightarrow = \frac{1}{2} [-a_3 b_3 - a_3 b_3 - (a_1 - ia_2)(b_1 + ib_2) - (a_1 + ia_2)(b_1 - ib_2)]$$

$$= -\hat{a} \cdot \hat{b}$$

$P(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$

$$\text{If } \hat{b} = \hat{a} \rightarrow P(\hat{a}, \hat{b}) = -1$$

Now assume local hidden variable  $\lambda$ :

$$\text{Result } A(\hat{a}, \lambda) = \pm 1$$

$\uparrow$   
result. of electron

$$B(\hat{b}, \lambda) = \pm 1$$

$$A(\hat{a}, \lambda) = -B(\hat{b}, \lambda)$$

$$\stackrel{\text{density probability of } \lambda}{\overbrace{p(\lambda)}} \geq 0, \quad \int p(\lambda) d\lambda = 1$$

$$\begin{aligned} \hat{P}(\hat{a}, \hat{b}) &= \int d\lambda \ p(\lambda) \ A(\hat{a}, \lambda) \ B(\hat{b}, \lambda) \\ &= - \int d\lambda \ p(\lambda) \ A(\hat{a}, \lambda) \ A(\hat{b}, \lambda) \end{aligned}$$

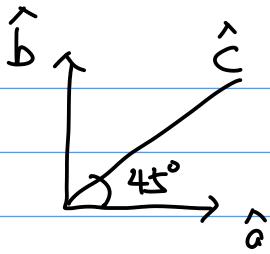
$$\begin{aligned} \hat{P}(\hat{a}, \hat{c}) - \hat{P}(\hat{a}, \hat{b}) &= - \int d\lambda \ p(\lambda) [A(\hat{a}, \lambda) A(\hat{b}, \lambda) - A(\hat{a}, \lambda) A(\hat{c}, \lambda)] \\ &= - \int d\lambda \ p(\lambda) [1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda)] A(\hat{a}, \lambda) \end{aligned}$$

$$\left| \hat{P}(\hat{a}, \hat{b}) - \hat{P}(\hat{a}, \hat{c}) \right| \leq \int d\lambda \ p(\lambda) |1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda)| \underbrace{|A(\hat{a}, \lambda) A(\hat{b}, \lambda)|}_{=1}$$

$$\hookrightarrow \boxed{|\hat{P}(\hat{a}, \hat{b}) - \hat{P}(\hat{a}, \hat{c})| \leq 1 + P(\hat{b}, \hat{c})}$$

$\downarrow$   
Inequality result.

Now suppose.



$$P(\hat{a}, \hat{b}) = 0$$
$$P(\hat{a}, \hat{c}) = -\frac{\sqrt{2}}{2}$$

$$P(\hat{b}, \hat{c}) = -\frac{\sqrt{2}}{2} \quad \left. \right\} \text{QM results.}$$

$$\hookrightarrow \text{Then: } |P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| \leq 1 + P(\hat{b}, \hat{c})$$
$$|0 - (-\frac{\sqrt{2}}{2})| \leq 1 + -\frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2} \neq 1 - \frac{\sqrt{2}}{2}$$

→ it doesn't violate causality because we can't demand the result on one end.