Zhi Chen HW#20

1) $H = H_0 + H_{LS} + H_B$, $H_0 = \frac{p^2}{2m} + V_c(r)$

a) $$\boxed{H_{LS} = \vec{\mu}\cdot\vec{B}_{eff} = \frac{1}{2m_e^2c^2}\frac{1}{r}\frac{dV_c}{dr}(\vec{L}\cdot\vec{S})}$$

$$\boxed{H_B = \frac{-eB}{2m_ec}(L_z + 2S_z)}$$

b) $$H_{LS} = \underbrace{\frac{1}{2m_e^2c^2}}_{\sim\frac{1}{m_e^2c^2}}\underbrace{\frac{1}{r}}_{\sim\frac{1}{a_0}}\underbrace{\frac{dV_c}{dr}}_{\sim\frac{e^2}{a_0^2}}\underbrace{(\vec{L}\cdot\vec{S})}_{\sim\hbar^2}$$

$$\sim \frac{\hbar^2}{m_e^2c^2}\frac{e^2}{a_0^3}$$

with $\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$ and $a_0 = \frac{\hbar^2}{me^2}$

$$H_{LS} \sim \frac{\hbar^2}{m_e^2c^2}\left(\frac{m_e e^2}{\hbar^2}\right)^2\frac{e^2}{a_0} \sim \frac{e^2}{\underbrace{\hbar^2c^2}_{\alpha^2}}\frac{e^2}{a_0} \sim \boxed{\alpha^2\frac{e^2}{a_0}}$$

$$H_B = -\frac{eB}{2m_ec}\underbrace{(L_z + 2S_z)}_{\sim\hbar} \sim -\frac{\hbar e}{m_ec}B$$

$$\hookrightarrow \sim \frac{\hbar e}{\left(\frac{\hbar^2}{e^2a_0}\right)\frac{e^2}{\alpha\hbar}}B \sim \boxed{\alpha e a_0 B \sim H_B}$$

By setting strong magnetic limit when $H_B \gg H_{LS}$.

$$\cancel{\alpha}\cancel{e} a_0 B \gtrsim \alpha^{\cancel{2}} \frac{e^{\cancel{2}}}{a_0}$$

strong magnetic when $\rightarrow B \gtrsim \alpha \frac{e}{a_0^2} \sim B_{crit}$

$$\underbrace{B_{crit}} \sim \alpha \frac{e}{a_0^2} \sim \frac{1}{137}(4.803\times10^{-10}\text{statcoul})\frac{1}{(5.3\times10^{-9}\text{ cm})^2}$$

$$\sim 1.25\times10^6 \text{ Gauss}$$

10^4 Gauss $= 1$T

$$\boxed{B_{crit} \sim 12.5 \text{ Tesla}}$$

$\begin{cases} \text{If } B \gg B_{crit} \cong 12.5\text{ T} & \text{then strong magnetic field} \\ \text{If } B \ll B_{crit} \cong 12.5\text{ T} & \text{then weak magnetic field.} \end{cases}$

c) Consider $\underline{B \gg B_{crit}}$ (strong case): so H_B and H_0 dominate.

with $H_0 = \frac{P^2}{2m} + V_c$, $H_B = \frac{-eB}{2m_e c}(L_z + 2S_z)$

$\Rightarrow$ After diagonalizing $H_0 + H_B$, we should use $|n, l, s, m_l, m_s\rangle$

we see $H_B \sim (L_z + 2S_z)$, so we should use quantum #, m_l, m_s which diagonalizes L_z and S_z.

We cannot use j for J^2 because now there is a preferred direction in $\hat{z}$ which breaks spherical symmetry, leaving only rotational symmetry around z.

By using $|n, l, s=\frac{1}{2}, m_l, m_s\rangle$:

$$E_B = \langle n, l, \tfrac{1}{2}, m_l, m_s | \frac{-eB}{2m_e c}(L_z + 2S_z) | n, l, \tfrac{1}{2}, m_l, m_s\rangle$$

using $L_z|m_l, m_s\rangle = \hbar m_l |m_l, m_s\rangle$ and $S_z|m_l, m_s\rangle = \hbar m_s |m_l, m_s\rangle$

$$\boxed{E_B = \frac{-eB}{2m_e c}(\hbar m_l + 2\hbar m_s)}$$

d) Now do perturbation theory on H_{LS}, since $H_{LS} \ll H_B$

we have unperturbed states $|n, l, s=\frac{1}{2}, m_l, m_s\rangle$

If there is just H_0, then we have degeneracy $= (2s+1)(2l+1)$

for $s = 1/2$, # of degeneracy $= 2(2l+1)$

Now with $\langle H_B\rangle = \frac{-eB\hbar}{2m_e c}(m_l + 2m_s)$, we get degeneracy, i.e. same energy when

$$m_l' + 2m_s' = m_l + 2m_s$$

where $m_s = \pm\frac{1}{2}$ so $2m_s = \pm 1$

$\hookrightarrow$ $m_l' \pm 1 = m_l \pm 1$

$\hookrightarrow$ $m_l' = m_l \pm 1 \mp 1$

$$\hookrightarrow m_l' = \begin{cases} m_l & \text{when } m_s = m_s' \\ m_l + 2 & \text{when } m_s = \frac{1}{2},\ m_s' = -\frac{1}{2} \\ m_l - 2 & \text{when } m_s = -\frac{1}{2},\ m_s' = \frac{1}{2} \end{cases}$$

d) Now do perturbation theory on H_{LS}, since $H_{LS} \ll H_B$

we have unperturbed states $|n, l, s=\frac{1}{2}, m_l, m_s\rangle$

First order perturbation:

$$E_{LS}^{(1)} = \langle n,l,s=\tfrac{1}{2}, m_l, m_s \Big| \frac{1}{2m_e^2c^2} \frac{1}{r} \frac{dV_c}{dr} (\vec{L}\cdot\vec{S}) \Big| n,l,s=\tfrac{1}{2}, m_l, m_s\rangle$$

$$\stackrel{!}{=} \frac{1}{2m_e^2c^2} \langle \frac{1}{r}\frac{dV_c}{dr} \rangle \langle \vec{L}\cdot\vec{S}\rangle$$

Now $\vec{L}\cdot\vec{S} = L_zS_z + L_xS_x + L_yS_y$

use $S_x = \frac{S_+ + S_-}{2}$ $\qquad S_y = \frac{S_+ - S_-}{2i}$

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then $\vec{L}\cdot\vec{S} = L_zS_z + \frac{1}{4}\left(\cancel{S_+L_+} + L_+S_- + L_-S_+ + \cancel{S_-L_-} - \cancel{S_+L_+} + L_+S_- + L_-S_+ - \cancel{S_-L_-}\right)$

$$\stackrel{!}{=} L_zS_z + \frac{1}{2}(L_+S_- + L_-S_+)$$

$$\langle \vec{L}\cdot\vec{S}\rangle = \langle m_l, m_s | L_zS_z | m_l, m_s\rangle + \frac{1}{2}\cancel{\langle m_l, m_s | L_+S_- | m_l, m_s\rangle}^{0} + \frac{1}{2}\cancel{\langle m_l, m_s | L_-S_+ | m_l, m_s\rangle}^{0}$$

$$\stackrel{!}{=} \hbar^2 m_l m_s$$

so $\boxed{E_{LS}^{(1)} = \frac{1}{2m_e^2c^2}\langle \frac{1}{r}\frac{dV_c}{dr}\rangle \langle \vec{L}\cdot\vec{S}\rangle = \frac{\hbar^2 m_l m_s}{2m_e^2c^2}\langle \frac{1}{r}\frac{dV_c}{dr}\rangle}$

2) spin-$\frac{1}{2}$ particle with magnetic moment μ.

$$\vec{B}(t) = B_0 \cos\omega_0 t\, \hat{x} + B_0 \sin\omega_0 t\, \hat{y} + B_1 \hat{z}$$

At $t=0$ $S_z = 1/2$, purely in S_z. Find $P(t)$

The Hamiltonian of spin-$\frac{1}{2}$ system subject to external field is:

$$H = -\vec{\mu}\cdot\vec{B} = -\frac{e}{m_e c}\vec{S}\cdot\vec{B}$$

$$\hookrightarrow H = -\frac{e}{m_e c}\left[S_z B_1 + B_0\left(S_x \cos\omega_0 t + S_y \sin\omega_0 t\right)\right]$$

using $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Separate H into H_0 (time-independent) and $V(t)$ (time-dependent)

$$\hookrightarrow H = \underbrace{-\frac{e\hbar B_1}{2m_e c}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{H_0} \underbrace{- \frac{e\hbar B_0}{2m_e c}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos\omega_0 t + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\omega_0 t\right]}_{V(t)}$$

so $$H_0 = \frac{-e\hbar B_1}{2m_e c}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$$

and $$V(t) = \frac{-e\hbar B_0}{2m_e c}\Big\{\left(|-\rangle\langle+| + |+\rangle\langle-|\right)\cos\omega_0 t$$
$$+ \left(-i|+\rangle\langle-| + i|-\rangle\langle+|\right)\sin\omega_0 t\Big\}$$

use state $|s=\frac{1}{2}, m_s = \pm\frac{1}{2}\rangle$ to represent ~~state~~, i.e.

$$|\alpha\rangle = C_+(t)\,|s=\tfrac{1}{2}, m_s=+\tfrac{1}{2}\rangle + C_-(t)\,|s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\rangle$$

$$= C_+(t)\,|+\rangle + C_-(t)\,|-\rangle$$

then we need to solve for

$$i\hbar \frac{d}{dt} C_n(t) = \sum_m V_{nm}\, e^{i\omega_{nm}t}\, C_m(t)$$

If $e<0$, then:

let $|-\rangle \to |1\rangle$, lower energy state.
$|+\rangle \to |2\rangle$, higher energy state

$$V_{nm} = -\frac{e\hbar B_0}{2m_e c}\begin{pmatrix} 0 & \cos\omega t - i\sin\omega t \\ \cos\omega t + i\sin\omega t & 0 \end{pmatrix}$$

know $E_+ = E_2 = \langle H_0\rangle_2 = \dfrac{-e\hbar B_1}{2m_e c}$

and $E_- = E_1 = \langle H_0\rangle_1 = \dfrac{e\hbar B_1}{2m_e c}$

with $\omega_{nm} = \dfrac{E_n - E_m}{\hbar} \rightarrow \boxed{\omega_{21} = \dfrac{-eB_1}{m_e c} = \dfrac{|e|B_1}{m_e c}}$ with $e<0$

then

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{-e\hbar B_0}{2m_e c}\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{+i\omega t} & 0 \end{pmatrix} e^{i\omega_{21}t}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

The above equation has the same exact form as the equation of the 2-state system that we did in class with

$$\boxed{\frac{-e\hbar B_0}{2m_e c} = \gamma}$$

so we get

$$|C_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + \frac{(\omega_0-\omega_{21})^2}{4}} \sin^2\left\{\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega_0-\omega_{21})^2}{4}}\; t\right\}$$

with $\Omega = \sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega_0-\omega_{21})^2}{4}}$, then $|C_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\Omega^2}\sin^2(\Omega t)$

$$|C_1(t)|^2 = 1 - |C_2(t)|^2$$

Now find $P(t)$, absorption power:

$|C_2(t)|^2$ = probability of going to higher level (absorption)

then $P(t) = \frac{d}{dt}\left(|C_2(t)|^2 \Delta E_{21}\right)$ where $\Delta E_{21} = E_2 - E_1$

energy difference between 2 states.

$$= \frac{d}{dt}\left(|C_2|^2 \hbar\omega_{21}\right)$$

$$= \hbar\omega_{21} \frac{\gamma^2/\hbar^2}{\Omega^2} 2\Omega \sin\Omega t \cos\Omega t$$

$$\boxed{P(t) = \hbar\omega_{21} \frac{2\gamma^2/\hbar^2}{\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega_0-\omega_{21})^2}{4}}} \sin\Omega t \cos\Omega t}$$

If $B_0 \ll B_1$

then $\frac{\gamma}{\hbar} = -\frac{eB_0}{2m_e c} = \frac{|e|B_0}{2m_e c}$

$\omega_{21} = \frac{-eB_1}{m_e c} = \frac{|e|B_1}{m_e c}$

so $\frac{\gamma}{\hbar} \ll \omega_{21}$.

So when $\omega_0 \simeq \omega_{21} \rightarrow \omega_0 - \omega_{21} \sim 0$, so $\Omega \sim \frac{\gamma}{\hbar}$

and $\Omega = \sqrt{\left(\frac{\gamma}{\hbar}\right)^2 + \frac{(\omega-\omega_{21})^2}{4}} \sim \frac{\gamma}{\hbar}$, which is very small.

so the oscillation behavior of absorption power will be slow.

3) Show $W = e^{A+\lambda B} = e^{A}\left[1+\lambda\int_0^1 d\tau_1 \underbrace{e^{-\tau_1 A} B e^{\tau_1 A}}_{B_I} + \dots\right]$

$$\frac{d}{d\tau}U(\tau) = \frac{d}{d\tau}\left[e^{-\tau A} e^{\tau(A+\lambda B)}\right]$$

Note: τ is just a number, so $[\tau A, A] = 0$

$$\frac{d}{d\tau} e^{\tau A} = A e^{\tau A} = e^{\tau A} A$$

$$= -A e^{-\tau A} e^{\tau(A+\lambda B)} + e^{-\tau A}(A+\lambda B) e^{\tau(A+\lambda B)}$$

$$= e^{-\tau A}(-A) e^{\tau(A+\lambda B)} + e^{-\tau A}(A+\lambda B) e^{\tau(A+\lambda B)}$$

$$\Rightarrow \frac{d}{d\tau}U(\tau) = e^{-\tau A} \lambda B e^{\tau(A+\lambda B)}$$

Now integrate above expression:

$$U(\tau) - U(\tau=0) = \int_0^{\tau} e^{-\tau_1 A} \lambda B e^{\tau_1(A+\lambda B)} d\tau_1$$

Note $U(\tau) = e^{-\tau A} e^{\tau(A+\lambda B)}$ and $U(\tau=0) = 1$

$$e^{-\tau A} e^{\tau(A+\lambda B)} - 1 = \lambda \int_0^{\tau} e^{-\tau_1 A} B e^{\tau_1(A+\lambda B)} d\tau_1$$

$$e^{\tau(A+\lambda B)} = e^{\tau A}\left[1 + \lambda \int_0^{\tau} e^{-\tau_1 A} B e^{\tau_1(A+\lambda B)} d\tau_1\right]$$

Use iteration:

notice on RHS, we have $e^{\tau_1(A+\lambda B)}$, which is LHS evaluated at $\tau = \tau_1$

then

$$e^{\tau(A+\lambda B)} = e^{\tau A}\left[1 + \lambda \int_0^{\tau} e^{-\tau_1 A} B e^{\tau_1 (A+\lambda B)} d\tau_1\right]$$

$$= e^{\tau A}\left[1 + \lambda \int_0^{\tau} d\tau_1 e^{-\tau_1 A} B \left\{ e^{\tau_1 A}\left[1 + \lambda \int_0^{\tau_1} d\tau_2 e^{-\tau_2 A} B e^{\tau_2 (A+\lambda B)}\right\}\right.$$

$$= e^{\tau A}\left[1 + \lambda \int_0^{\tau} d\tau_1 e^{-\tau_1 A} B e^{\tau_1 A} + \lambda^2 \int_0^{\tau} d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-\tau_1 A} B e^{\tau_1 A} e^{-\tau_2 A} B e^{\tau_2 (A+\lambda B)}\right]$$

Do iteration again :

$$= e^{\tau A}\left[1 + \lambda \int_0^{\tau} d\tau_1 e^{-\tau_1 A} B e^{\tau_1 A} + \lambda^2 \int_0^{\tau} d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-\tau_1 A} B e^{\tau_1 A} e^{-\tau_2 A} B \left\{ e^{\tau_2 A} + \mathcal{O}(\lambda) \right\}\right.$$

ignore terms with λ, since we're already λ^2

$$= e^{\tau A}\left[1 + \lambda \int_0^{\tau} d\tau_1 e^{-\tau_1 A} B e^{\tau_1 A} + \lambda^2 \int_0^{\tau} d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-\tau_1 A} B e^{\tau_1 A} e^{-\tau_2 A} B e^{\tau_2 A} + \mathcal{O}(\lambda^3)\right]$$

Lastly, set $\boxed{\tau = 1}$, then LHS, $e^{\tau(A+\lambda B)} = e^{A+\lambda B} = W$

$$\boxed{e^{A+\lambda B} = e^{A}\left[1 + \lambda \int_0^{1} d\tau_1 e^{-\tau_1 A} B e^{\tau_1 A} + \underbrace{\lambda^2 \int_0^{1} d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-\tau_1 A} B e^{\tau_1 A} e^{-\tau_2 A} B e^{\tau_2 A}}_{\text{second order term.}} + \mathcal{O}(\lambda^3)\right]}$$

4) Sudden approximation.

$t<0$: $H_{old} = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2$

$t>0$: $H_{new} = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 - q\varepsilon x$

$$\overset{!}{=} \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 \left(x^2 - \frac{2q\varepsilon}{m\omega_0^2} x\right)$$

complete the square

$$\overset{!}{=} \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 \left[x^2 - \frac{2q\varepsilon}{m\omega_0^2} x + \left(\frac{q\varepsilon}{m\omega_0^2}\right)^2 - \left(\frac{q\varepsilon}{m\omega_0^2}\right)^2\right]$$

$$H \overset{!}{=} \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 \left(x - \frac{q\varepsilon}{m\omega_0^2}\right)^2 - \frac{1}{2} \frac{q^2\varepsilon^2}{m\omega_0^2}$$

so new Hamiltonian is just old H shifted by $\frac{q\varepsilon}{m\omega_0^2}$

with eigenenergy. $E_\alpha = \hbar\omega_0 \left(\alpha + \frac{1}{2}\right) - \frac{1}{2} \frac{q^2\varepsilon^2}{m\omega_0^2}$

→ We can also define a_{new}, to be the annihilation operator in H_{new} by shifting the annihilation operator, a, in H_{old}:

For H_{old}: $a = \sqrt{\frac{m\omega_0}{2\hbar}} \left(x + i \frac{p}{m\omega_0}\right)$

since $H_{new} = H_{old}\left(x - \frac{q\varepsilon}{m\omega_0^2}\right) - \frac{1}{2} \frac{q^2\varepsilon^2}{m\omega_0^2}$

by analogy define annihilation operator for H_{new} as:

$$a_{new} = \sqrt{\frac{m\omega_0}{2\hbar}} \left(\left[x - \frac{q\varepsilon}{m\omega_0^2}\right] + i \frac{p}{m\omega_0}\right)$$

$$\overset{!}{=} \sqrt{\frac{m\omega_0}{2\hbar}} \left(x + i \frac{p}{m\omega_0}\right) - \sqrt{\frac{m\omega_0}{2\hbar}} \frac{q\varepsilon}{m\omega_0^2}$$

$$a_{new} \overset{!}{=} a - \sqrt{\frac{m\omega_0}{2\hbar}} \frac{q\varepsilon}{m\omega_0^2}$$

$$a_{new} \overset{!}{=} a - \lambda$$

$$\boxed{\lambda = \sqrt{\frac{m\omega_0}{2\hbar}} \frac{q\varepsilon}{m\omega_0^2}}$$

Now define a new state $|\alpha=0\rangle$ to denote the ground state of H_{new}.

Now since a_{new} is annihilation operator of H_{new}, we should have:

$$a_{new} |\alpha=0\rangle = 0$$

but $a_{new} = a - \lambda$

$$(a-\lambda) |\alpha=0\rangle = 0$$

$$a |\alpha=0\rangle = \lambda |\alpha=0\rangle$$

By the above expression, we observe that the eigenstate $|\alpha=0\rangle$ for H_{new} is also eigenstate for the annihilation operator for H_{old} and the eigenvalue is the difference between a_{new} and a.

$$\hookrightarrow \boxed{a |\alpha=0\rangle = \lambda |\alpha=0\rangle}$$

now lets define $|\alpha\rangle$ as the general eigenstate for H_{new}

so expanding $|n=0\rangle$ in $|\alpha\rangle$

$$|n=0\rangle = \sum_{\alpha=0}^{\infty} |\alpha\rangle \underbrace{\langle\alpha|n=0\rangle}_{C_\alpha}$$

so $P_\alpha = \left| \langle \alpha' | n=0 \rangle \right|^2$

$P_\alpha \overset{!}{=} |C_\alpha|^2$ ← so need to find C_α

Now determine C_α:

$$C_\alpha = \langle \alpha | n=0 \rangle$$

note $|\alpha\rangle = \frac{1}{\sqrt{\alpha!}} (a^+_{new})^\alpha |\alpha=0\rangle$

$\hookrightarrow \langle \alpha | = \frac{1}{\sqrt{\alpha!}} \langle \alpha=0 | (a_{new})^\alpha$

$$\overset{!}{=} \langle \alpha | \frac{1}{\sqrt{\alpha!}} (a_{new})^\alpha | n=0 \rangle$$

$$\overset{!}{=} \langle \alpha=0 | \frac{1}{\sqrt{\alpha!}} (a-\lambda)^\alpha | n=0 \rangle$$

note

$(a-\lambda)|n=0\rangle = \cancel{a|n=0\rangle} - \lambda |n=0\rangle$

$\hookrightarrow (a-\lambda)^\alpha |n=0\rangle = (-\lambda)^\alpha |n=0\rangle$

$$C_\alpha \overset{!}{=} \frac{1}{\sqrt{\alpha!}} (-\lambda)^\alpha \underbrace{\langle \alpha=0 | n=0 \rangle}_{C_{\alpha=0}}$$

Now find $C_{\alpha=0}$ via normalization:

$$\langle n=0 | n=0 \rangle = 1$$

$$\hookrightarrow \sum_{\alpha=0}^{\infty} |C_\alpha|^2 = 1$$

$$\hookrightarrow |C_{\alpha=0}|^2 \underbrace{\sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \lambda^{2\alpha}}_{} = 1$$

Taylor expansion of $\exp\{|\lambda|^2\}$

$$\hookrightarrow C_{\alpha=0} = \exp\left\{ -\frac{|\lambda|^2}{2} \right\}$$

so $\boxed{C_\alpha = \frac{1}{\sqrt{\alpha!}} (-\lambda)^\alpha e^{\frac{-|\lambda|^2}{2}} \quad \text{with} \quad \lambda = \sqrt{\frac{m\omega}{2\hbar}} \frac{q\varepsilon}{m\omega^2}}$

Therefore:

$$\boxed{P_\alpha = |C_\alpha|^2 = \frac{1}{\alpha!}\left[\frac{m\omega_0}{2\hbar}\left(\frac{q\varepsilon}{m\omega_0^2}\right)^2\right]^\alpha \exp\left\{-\frac{m\omega_0}{2\hbar}\left(\frac{q\varepsilon}{m\omega_0^2}\right)^2\right\}}$$

probability of going to the α-th excited state in the new Hamiltonian from the ground state in old Hamiltonian

5) $$H_{new} = \underbrace{\frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2}_{H_0} \underbrace{- q\varepsilon_0 x \exp\{-t^2/\tau^2\}}_{V(t)}$$

a) $t_0 = -\infty$, $\quad |m, t_0=-\infty, t=t_0\rangle = |n\rangle$

$$P_{n\to m} = |c_m^{(1)}(t)|^2$$

$$|n, t_0=-\infty, t\rangle_I = U_I(t, t_0=-\infty)\,|n, t_0=-\infty; t=t_0\rangle$$

$$\stackrel{!}{=} \sum_m |m\rangle \underbrace{\langle m| U_I(t, t_0=-\infty)|n, t_0=-\infty; t=t_0\rangle}_{c_m}$$

From first order perturbation theory:

$$c_m^{(1)}(t) = \frac{-i}{\hbar}\int_{t_0=-\infty}^{t} \langle m| V_I |n\rangle dt'$$

$$\stackrel{!}{=} \frac{-i}{\hbar}\int_{-\infty}^{t} \langle m| e^{\frac{iH_0 t}{\hbar}} V e^{\frac{-iH_0 t}{\hbar}} |n\rangle dt'$$

for Harmonic oscillator $H_0|n\rangle = E_n|n\rangle$ for $E_n = \hbar\omega_0(n+\frac{1}{2})$

$$= \lim_{t\to\infty} \frac{-i}{\hbar}\int_{-\infty}^{t=\infty} e^{\frac{i}{\hbar}(E_m-E_n)t'} \langle m|(-q\varepsilon_0 x) e^{\frac{-t'^2}{\tau^2}}|n\rangle dt'$$

$$\stackrel{!}{=} \frac{i}{\hbar} q\varepsilon_0 \int_{-\infty}^{\infty} e^{i\omega_0(m-n)t'} \langle m|x|n\rangle e^{\frac{-t'^2}{\tau^2}} dt'$$

$$\stackrel{!}{=} \frac{i}{\hbar} q\varepsilon_0 \langle m|x|n\rangle \underbrace{\int_{-\infty}^{\infty} e^{i\omega_0(m-n)t'} e^{\frac{-t'^2}{\tau^2}} dt'}_{}$$

$$= \sqrt{\pi}\,\tau \exp\left\{\frac{-\tau^2\omega_0^2(m-n)^2}{4}\right\}$$

$$= \frac{i\sqrt{\pi}}{\hbar} q\varepsilon_0 \tau \exp\left\{\frac{-\tau^2\omega_0^2(m-n)^2}{4}\right\} \sqrt{\frac{\hbar}{2m\omega_0}} \langle m|a^\dagger + a|n\rangle$$

$$= \frac{i\sqrt{\pi}}{\hbar} q\varepsilon_0 \tau \exp\left\{\frac{-\tau^2\omega_0^2(m-n)^2}{4}\right\} \sqrt{\frac{\hbar}{2m\omega_0}} \left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right)$$

then $P_m^{(1)}(t=\infty) = \left|C_m^{(1)}(t=\infty)\right|^2$

$$= \frac{\pi q^2\varepsilon_0^2\tau^2}{2\hbar m\omega_0} \exp\left\{\frac{-\tau^2\omega_0^2(m-n)^2}{2}\right\}\left((n+1)\delta_{m,n+1} + n\,\delta_{m,n-1}\right)$$

So

$$P_m^{(1)}(t=\infty) = \begin{cases} = (n+1)\dfrac{\pi q^2\varepsilon_0^2\tau^2}{2\hbar m\omega_0}\exp\left\{\dfrac{-\tau^2\omega_0^2}{2}\right\} & \text{if } m = n+1 \\[2ex] = n\dfrac{\pi q^2\varepsilon_0^2\tau^2}{2\hbar m\omega_0}\exp\left\{\dfrac{-\tau^2\omega_0^2}{2}\right\} & \text{if } m = n-1 \\[2ex] = 0 & \text{otherwise.} \end{cases}$$

b) The condition of application when $\left|C_n^{(1)}(t=\infty)\right| \ll 1$

$$\left|C_n^{(1)}(t=\infty)\right|^2 \approx \frac{(q\varepsilon_0\tau)^2}{\hbar m\omega_0}\exp\left\{-\frac{\tau^2\omega_0^2}{2}\right\} \ll 1$$

So $$\underbrace{\hbar\omega_0}_{\text{energy difference between levels.}} \gg \frac{(q\varepsilon_0\tau)^2}{m}\underbrace{\exp\left\{-\frac{\tau^2\omega_0^2}{2}\right\}}_{\text{observe for this exponent to be small}}$$

we just want $\tau\omega_0 \gg 1$

or $\boxed{\tau \gg \frac{1}{\omega_0}}$