The theory of Angular Momentum:

Translation:

Define operator J, such that:

$$|z\rangle \rightarrow |z+a\rangle = J(a)\,|z\rangle$$

$\uparrow$ translation operator

Then

$$J(a)|\psi\rangle = J(a)\int dz\; |z\rangle\langle z|\psi\rangle$$

$$\overset{!}{=} \int dz\; J(a)\,|z\rangle \underbrace{\langle z|\psi\rangle}_{\psi(z)}$$

$$\overset{!}{=} \int dz\; |z+a\rangle\langle z|\psi\rangle$$

$$J(a)|\psi\rangle \overset{!}{=} \int dz\; |z\rangle \underbrace{\langle z-a|\psi\rangle}_{\psi(z-a)}$$

so $\quad J(a)\,\psi(z) = \psi(z-a)$

Do taylor:

$$\psi(z-a) = \psi(z) - a\,\partial_z\psi(z) + \tfrac{1}{2}a^2\,\partial_z^2\psi(z) + \cdots$$

$$\overset{!}{=} \left(1 - a\,\partial_z + \tfrac{1}{2}a^2\partial_z^2 + \cdots\right)\psi(z)$$

$$\psi(z-a) \overset{!}{=} e^{-a\partial_z}\,\psi(z)$$

$\therefore \quad J(a)\psi(z) = e^{-a\partial_z}\psi(z)$

since $\quad P_z = -i\hbar\partial_z$

then $\quad \boxed{J(a) = e^{-\frac{i}{\hbar}aP_z}}$ ← Translator Operator.

Now let $\quad a \to \varepsilon$

then $\quad J(\varepsilon) = 1 - \frac{i}{\hbar}\varepsilon P_z$

then $\quad \psi' = J(\varepsilon)\psi = (1 - \frac{i}{\hbar}\varepsilon P_z)\psi$

then $\quad \delta\psi = \psi' - \psi = -\frac{i}{\hbar}\varepsilon P_z\psi$

P_z : Hermitian ← algebra

$J(a)$: unitary. ← group.

3D Generalization: P_x, P_y, P_z

$[P_i, P_j] = 0$: commutative (abelion) group.

translation commutes. → $[J(\vec{a}), J(\vec{b})] = 0$: commutative (abelion) group.

Translational Invariance:

$$H\left(J(a)|\psi\rangle\right) = E\left(J(a)|\psi\rangle\right)$$

↳ $H|\psi\rangle = E|\psi\rangle$ Hamiltonian doesn't change upon translation.

↳ $= J(a)E|\psi\rangle$

$\overset{!}{=} J(a)H|\psi\rangle$

↳ so $HJ(a) = J(a)H$

or $\boxed{[H, J(a)] = 0}$

or $\boxed{[H, P_z] = 0}$ = translational invariance.

P_z is infinitesimal translation of $J(a)$

with Heisenberg picture: $\frac{\partial P_z}{\partial t} = \frac{1}{i\hbar}\underbrace{[H, P_z]}_{=0} = 0$

↳ P_z is constant.

Time Evolution:

$$i\hbar \partial_t |\psi\rangle = H|\psi\rangle$$

If H is time independent:

$$|\psi\rangle(t) = \underbrace{e^{-\frac{i}{\hbar}Ht}}_{\text{finite time evolution, unitary operator, analogous to } J(a)}|\psi\rangle(t=0)$$

finite time evolution, unitary operator, analogous to $J(a)$,

Infinitesimal Time evolution: $J(dt) = 1 - \frac{i}{\hbar} H\, dt$

Hamiltonian is generator for time evolution. (Hermitian)

Rotation in 3D

$$v' = Rv$$

↳ $v'^T v' = v^T \underbrace{R^T R}_{=1} v = v^T v$

Therefore $R^T R = RR^T = 1$

So R is a 3x3 orthogonal Matrices. $R \in SO(3)$

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotate φ around z-axis.

$$R_z(\epsilon) \simeq \begin{pmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 + \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\equiv 1 - \frac{i}{\hbar}\epsilon L_z$$

angular momentum along axis-z.
generator of infinitesimal rotation.

Similarly:

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix} = 1 - \frac{i}{\hbar}\epsilon L_x$$

$$\boxed{\therefore \quad L_x = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_z = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

We see that $R_z(\varphi) R_x(\theta) \neq R_x(\theta) R_z(\varphi)$

↳ Not commutative (Non-abelian)

We can also observe that: after computation.

$$[L_x, L_y] = i\hbar L_z$$

or $[L_i, L_j] = i\hbar \epsilon^{ijk} L_k$

Define * $\boxed{R(\hat{n}, \varphi) = e^{-\frac{i}{\hbar}\varphi \hat{n}\cdot\hat{L}}}$ Finite Rotation operator

ex: $e^{-\frac{i}{\hbar}\varphi(\hat{z}\cdot\vec{L})} = e^{-\frac{i}{\hbar}\varphi L_z} = e^{\varphi\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

$\hookrightarrow = e^{-i\varphi\begin{pmatrix} \sigma_y & 0 \\ 0 & 0 \end{pmatrix}}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\hookrightarrow \overset{!}{=} \cos\left(\varphi\begin{pmatrix} \sigma_y & 0 \\ 0 & 0 \end{pmatrix}\right) - i\sin\left(\varphi\begin{pmatrix} \sigma_y & 0 \\ 0 & 0 \end{pmatrix}\right)$

$\overset{!}{=} \begin{pmatrix} \cos\varphi \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -i\sigma_y \sin\varphi & 0 \\ 0 & 0 \end{pmatrix}$

Note identity: $\vec{a}\cdot\vec{\sigma} = \vec{a}^2$, $e^{i\varphi(\vec{a}\cdot\vec{\sigma})} = \sum_{n=0}^{\infty} \frac{(i\varphi(\vec{a}\cdot\vec{\sigma}))^n}{n!}$

In Hilber Space

$$|\psi\rangle_R = \mathcal{D}(R)\,|\psi\rangle$$

Rotated

$\mathcal{D}(R)$: unitary operator of rotation; $N\times N$ matrix

$$\boxed{\begin{aligned} &\mathcal{D}(R)\big|_{\text{infinitesimal}} = 1 - \tfrac{i}{\hbar}\,\delta\phi\;\hat{n}\cdot\vec{J} \\ &[J_i, J_j] = i\hbar\,\epsilon^{ijk} J_k \\ &\mathcal{D}(R) = e^{-\frac{i}{\hbar}\varphi\,\hat{n}\cdot\vec{J}} \end{aligned}}$$

$\vec{J}$: generator of infinitesimal rotation or angular momentum.

$\mathcal{D}(R) = e^{-\frac{i}{\hbar}\varphi\,\hat{n}\cdot\vec{J}}$: unitary operator for finite operation

Analogous to $U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}$: finite time evo

$F(\Delta x'\hat{x}) = \exp\left(-\frac{i}{\hbar}P_x\,\Delta x'\right)$: translation

unitary, but not Hermitian

Example of spin $\frac{1}{2}$:

$|+\rangle, |-\rangle$ basis:

$$[S_i, S_j] = i\hbar\,\epsilon^{ijk} S_k$$

$$\left.\begin{aligned} S_x &= \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \\ S_y &= \frac{\hbar}{2}\left(-i\,|+\rangle\langle-| + i\,|-\rangle\langle+|\right) \\ S_z &= \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \end{aligned}\right\}$$

Representation of rotational algebra:

$$\mathcal{D}_z(\varphi) = e^{-\frac{i}{\hbar}\varphi S_z}$$

$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then $|\alpha\rangle = |+\rangle\underbrace{\langle+|\alpha\rangle} + |-\rangle\underbrace{\langle-|\alpha\rangle}$

$$|\alpha\rangle = \begin{pmatrix} \langle+|\alpha\rangle \\ \langle-|\alpha\rangle \end{pmatrix} \leftarrow \text{Arbitrary 2-component spinor}$$

$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

with Pauli-Matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\boxed{J_i = \frac{\hbar}{2}\sigma_i}$$

Properties:

① $[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k$.

② $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

③ $\sigma_i^\dagger = \sigma_i$ Hermitian

④ $Tr(\sigma_i) = 0$

⑤ $Det(\sigma_i) = 1$

⑥ $\vec{\sigma}\cdot\vec{a} = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}$

⑦ $(\vec{\sigma}\cdot\vec{a})^2 = a^2$

⑧ $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbb{1} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

⑨ $D(\varphi, \hat{n}) = e^{-\frac{i}{\hbar}\varphi \hat{n}\cdot\vec{S}} = e^{-\frac{i}{2}\varphi\hat{n}\cdot\vec{\sigma}}$

$= \cos\frac{\varphi}{2} - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\varphi}{2}$

⑩ $D(\varphi+2\pi, \hat{n}) = -D(\varphi, \hat{n})$

$D(2\pi, \hat{n}) = -D(0, \hat{n}) = -1$ ← Not itself. but with minus

still same state in QM:

<u>Transformation of operators under R</u>

$\langle\psi|A|\psi\rangle \to \langle\psi|e^{\frac{i}{2}\varphi\hat{n}\vec{\sigma}} A e^{-\frac{i}{2}\varphi\hat{n}\vec{\sigma}}|\psi\rangle$

Infinitesimal $\delta\varphi$:

$$(1+\frac{i}{2}\delta\varphi\,\hat{n}\cdot\vec{\sigma})A(1-\frac{i}{2}\delta\varphi\,\hat{n}\cdot\vec{\sigma}) = A \underbrace{-\frac{i}{2}\delta\varphi n^a[\sigma^a, A]}_{\delta A}$$

Consider $A \to \sigma^a$

$$\delta\sigma^b = \frac{i}{2}\delta\varphi\, n^a[\sigma^a \sigma^b] = \frac{i}{2}\delta\varphi\, n^a 2i\epsilon^{abc}\sigma^c = \delta\varphi[\hat{n}\times\vec{\sigma}]^b$$

If $\hat{n} = \hat{z}$

$$\left.\begin{aligned} \delta\sigma^x &= -\delta\varphi\,\sigma^y \\ \delta\sigma^y &= \delta\varphi\,\sigma^x \\ \delta\sigma^z &= 0 \end{aligned}\right\} \text{transform } \vec{\sigma} \text{ as vectors}$$

Groups, Algebra, Representations

Group: set $G = \{a, b, c, \dots\}$ element of group.

operation: $a \cdot b$ group multiplication.

Axioms: 1) if $a \in G$, $b \in G$, then $a \cdot b \in G$
2) $(ab)c = a(bc)$ association
3) $e \in G$, $a \cdot e = e \cdot a = a$ $\forall a$
4) Invertibility $a \in G$ $\exists$ $a^{-1} \in G$, $a \cdot a^{-1} = a^{-1} \cdot a = e$

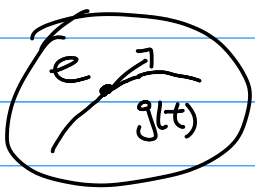
Lie Group: A group which is also differential manifold.

Lie group: Group operations are compatible with smooth structure

u $G \times G \to G$ - smooth
$u(x, y) = x \cdot y$ multiplication

v $G \to G$
$v(x) = x^{-1}$ - smooth inversion.

Differential calculus can be used.



$g(t)$: $g(0) = e$

$\left.\frac{dg}{dt}\right|_e$

Lie Algebra: (study of operations)

Vector Space $\mathfrak{g}$ (over some field F, i.e. $\mathbb{C}$ or $\mathbb{R}$)

with binary operation (Lie bracket)

$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

Axioms: 1) Lilinearity: $[ax+by, z] = a[x,z] + [y,z]$ #

2) Anticommutative: $[x,y] = -[y,x]$

3) Jacobi-Identity: $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$

Examples: J_i

1) $[J_i, J_j] = i\epsilon^{ijk} J_k$ satisfies all 3 properties.

2) From Associative algebra A, we can build Lie algebra $L(A)$:

$$[a,b] = a \cdot b - b \cdot a$$

Representation Theory: studies groups, algebra, etc. by representing their elements as linear operations of Hilbert space.

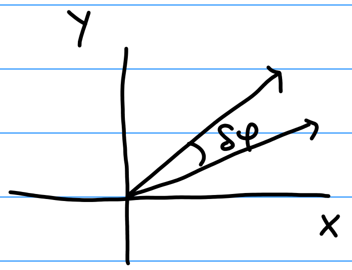
Ex: Orthogonality group: Pauli-Matrices acting on 2D Hilbert space (group of rotation)

Orbital Angular Momentum:

$\psi(x,y,z)$: wave function of 1 particle in 3D (no internal freedom)

$|x,y,z\rangle$ - basis

arbitrary state → $$|\psi\rangle = \int dx\,dy\,dz \; \underbrace{\psi(x,y,z)}_{\langle x,y,z|\psi\rangle} \, |x,y,z\rangle$$



$$\left.\begin{aligned} x &\to x - y\,\delta\varphi \\ y &\to y + x\,\delta\varphi \\ z &\to z \end{aligned}\right\} \begin{aligned} &x\cos\varphi - y\sin\varphi \\ &x\sin\varphi + y\cos\varphi \\ &z \end{aligned}$$

$$|x,y,z\rangle \to |x - y\delta\varphi,\, y + x\delta\varphi,\, z\rangle$$

$$= \left[1 - y\,\delta\varphi\left(\tfrac{-i}{\hbar}P_x\right) + x\,\delta\varphi\left(\tfrac{-i}{\hbar}P_y\right)\right]|x,y,z\rangle$$

Note $$|x+\epsilon, y, z\rangle = (1 + \epsilon\,\partial_x)\,|x,y,z\rangle$$

$$\uparrow \quad \partial_x = \tfrac{i}{\hbar}P_x$$

$$|x,y,z\rangle \to \left[1 - \tfrac{i}{\hbar}\,\delta\varphi\,(xP_y - yP_x)\right]|x,y,z\rangle$$

Postulate $$|\psi\rangle \to \left(1 - \tfrac{i}{\hbar}\,\delta\varphi\, J_z\right)|\psi\rangle$$

$J_z = L_z = xP_y - P_y x$ Note $\vec{J} = \vec{L}$ with no internal degrees of freedom.

$\boxed{\vec{L} = \vec{r} \times \vec{p}}$ ← generator of infinitesimal rotation.

Want to note that

L_z is the generator of rotation around z in coordinate representation.

$$L_z = xP_y - yP_x = -i\hbar \underbrace{(x\partial_y - y\partial_x)}_{\partial\varphi}$$

$$L_z \stackrel{!}{=} -i\hbar\,\partial_\varphi$$

$$\partial_x = \cos\varphi\,\partial_r - \frac{\sin\varphi}{r}\partial_\varphi$$
$$\partial_y = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\partial_\varphi$$

$$\psi_{rot}(\varphi) = \psi(\varphi - \delta\varphi) = (1 - \delta\varphi\,\partial_\varphi)\psi$$
$$\stackrel{!}{=} \left(1 - \delta\varphi\,\frac{i}{\hbar} L_z\right)\psi$$

$$\psi_{rot} = e^{-\frac{i}{\hbar}\Delta\varphi L_z}\psi = \psi(\varphi - \Delta\varphi)$$

Commutation Relation:

$$[L_x, L_y] \stackrel{!}{=} [yP_z - zP_y,\ zP_x - xP_z]$$
$$= [yP_z, zP_x] + [zP_y, xP_z]$$
$$\stackrel{!}{=} yP_x \underbrace{[P_z, z]}_{-i\hbar} + P_y x \underbrace{[z, P_z]}_{i\hbar}$$
$$= i\hbar(xP_y - yP_x)$$
$$\stackrel{!}{=} i\hbar L_z$$

In general: $[L_i, L_j] = i\hbar\,\epsilon^{ijk} L_k$
with $\vec{L} = \vec{r} \times \vec{p}$

Since $\hbar$ has same unit as L,

define dimensionless $\ell = \frac{L}{\hbar}$

then $[\ell_i, \ell_j] = i\epsilon^{ijk}\ell_k$

$$e^{-i\vec{\varphi}\cdot\vec{\ell}} = e^{-i\varphi\hat{n}\cdot\vec{\ell}} = e^{-\frac{i}{\hbar}\varphi\hat{n}\cdot\vec{L}} \quad \text{finite rotation}$$

$$[\ell_x, y] = [-i(y\partial_z - z\partial_y), y] = iz$$

$$[\ell_i, x_j] = [-i\underbrace{\epsilon^{imn} x_m \partial_n}_{\vec{x}\times\vec{\partial}}, x_j]$$

Recap: $\vec{L} = \vec{r} \times \vec{p}$, $\vec{p} = -i\hbar \vec{\nabla}$

$$\boxed{[L_i, L_j] = i\hbar \epsilon^{ijk} L_k}$$

$\vec{l} = \frac{\vec{L}}{\hbar}$ dimensionless $\vec{L}$

then $[l_i, l_j] = i \epsilon^{ijk} l_k$

$e^{-i\varphi \hat{n}\cdot\vec{l}}$ ← finite rotation.

$[l_i, x_j] = i \epsilon^{ijk} x_k$

$[l_i, p_j] = i \epsilon^{ijk} p_k$

$[l_i, a_j] = i\epsilon^{ijk} a_k$ ← a transforms as a vector rotation.

$[l_i, \underbrace{a_j b_j}_{\text{transform as scalar under rotation.}}] = 0$

Eigenvalues and Eigenstates of angular momentum:

$[J_i, J_j] = i\hbar \epsilon^{ijk} J_k$

build representation: $J_z|b\rangle = b|b\rangle$ ← eigenbasis of J_z but not $J_{x,y}$ since they don't commute

1) $[J_i, J^2] = 0$

J_z and J^2 diagonalize simultaneously.

$$J_z |a,b\rangle = b|a,b\rangle$$

$$J^2 |a,b\rangle = a|a,b\rangle \qquad , a \geq 0$$

2) Define

$$\boxed{\begin{aligned} J_\pm &= J_x \pm i J_y \\ \text{then} \quad [J_z, J_\pm] &= \pm\hbar J_\pm \\ [J_+, J_-] &= 2\hbar J_z \\ [J^2, J_{\pm,z}] &= 0 \end{aligned}}$$

$\rightarrow$ $J_z(J_\pm|a,b\rangle) = (\underbrace{[J_z, J_\pm]}_{\pm\hbar J_z} \pm J_\pm J_z)|a,b\rangle$

$= (b\pm\hbar) J_z |a,b\rangle$

$\rightarrow$ $J^2(J_z|a,b\rangle) = a(J_\pm|a,b\rangle)$

$\rightarrow$ $\boxed{J_\pm|a,b\rangle = C_\pm |a, b\pm\hbar\rangle}$

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

$$= \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$$

$$= J_- J_+ + J_z^2 + \hbar J_z$$

$$\underbrace{\langle a,b | \vec{J}^2 - J_z^2 | a,b \rangle}_{= a - b^2 \geq 0} = \frac{1}{2} \langle a,b | J_- J_+ + J_+ J_- | a,b \rangle$$

$$= \frac{1}{2} \langle a,b | J_+^\dagger J_+ + J_+ J_+^\dagger | a,b \rangle$$

or $a \geq b^2$

$$= \frac{1}{2} | J_+ | a,b \rangle |^2 + \frac{1}{2} | J_+^\dagger | a,b \rangle |^2$$

$$J_+ | a,b \rangle = c_+ | a, b+\hbar \rangle$$

$$(J_+)^n | a,b \rangle = \# | a, b+n\hbar \rangle$$

At some n, we have $(b+n\hbar)^2 > a$, which is not allowed

so there is a b_{max}.

then require $J_+ | a, b_{max} \rangle = 0$

$$J_- J_+ | a, b_{max} \rangle = 0$$

$$\hookrightarrow (J^2 - J_z^2 - \hbar J_z) | a, b_{max} \rangle = 0$$

$$\hookrightarrow (a - b_{max}^2 - \hbar b_{max}) | a, b_{max} \rangle = 0 \quad \text{or} \quad \boxed{a = b_{max}(b_{max}+1)}$$

Likewise, there is a b_{min} such that:

$$\begin{cases} \boxed{a = b_{min}(b_{min} - \hbar)} \\ a = b_{max}(b_{max} + \hbar) \end{cases}$$

then there are two solutions, one of them is

$$b_{min} = -b_{max}.$$

$$|a, b_{max}\rangle = \# \, J_+^n \underbrace{|a, b_{max}\rangle}_{|a, \underbrace{-b_{max}}_{b_{min}} + n\hbar\rangle}$$

$$b_{max} = -b_{max} + n\hbar$$

$\hookrightarrow$ $b_{max} = \dfrac{n\hbar}{2}$ $\rightarrow$ n : integer

let $\frac{n}{2} = j_{max}$ = integer or half-integer.

$$a = \hbar^2 j(j+1) \quad , \quad -\hbar j \le b \le \hbar j$$

$$b = -\hbar j, -\hbar j + \hbar \, \ldots\ldots$$

$|a, b\rangle \rightarrow |j, m\rangle$ Irreducible representation.

$$\boxed{\begin{array}{lll} \text{let} \quad \vec{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle & & j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots \\ \quad\quad J_z |j, m\rangle = \hbar m \, |j, m\rangle & & -j \le m \le j \\ & 2j+1 \text{ dim} \longrightarrow & m \in \{-j, -j+1 \, \ldots \, j-1, j\} \end{array}}$$

ex:

$j=0 \quad |0,0\rangle \rightarrow J_z|0,0\rangle = 0 \quad J^2|0,0\rangle = 0$

$j=\frac{1}{2} \quad J_i = S_i = \frac{1}{2}\hbar\sigma_i$

$$J_i^2 = S_i^2 = \frac{1}{4}\hbar^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{3}{4}\hbar^2 = \hbar^2 \underset{j=\frac{1}{2}}{j}(j+1)$$

$j=1 \quad L_x = i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = i\hbar\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$L_z = i\hbar\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = 2\hbar^2 = \hbar^2 \underset{j=1}{j}(j+1)$$

Matrix Elements of J_i:

$$\langle j',m'|\vec{J}^2|j\,m\rangle = \hbar^2 j(j+1)\,\delta_{j,j'}\,\delta_{m,m'}$$

$$\langle j',m'|J_z|j,m\rangle = \hbar m\,\delta_{j,j'}\,\delta_{m,m'}$$

$$J_- J_+ = \vec{J}^2 - \vec{J}_z^2 - \hbar J_z$$

$$\underbrace{\langle j,m|J_+^{\dagger} J_m|j,m\rangle}_{\langle j,m|J_+^{\dagger}|j',m'\rangle\langle j',m'|J_m|j,m\rangle} = \langle j,m|J^2 - J_z^2 - \hbar J_z|j,m\rangle$$

$$= \hbar^2[j(j+1) - m^2 - m]$$

$$= \hbar^2(j-m)(j+m+1)$$

Since $\langle j', m' | \underbrace{J_+ | j, m}_{\# |j, m+1\rangle} \rangle = \delta_{j,j'} \delta_{m', m+1}$

$\langle j', m' | \# | j, m+1 \rangle$

So $\langle j,m | J_+^\dagger | j', m' \rangle \langle j', m' | J_m | j, m \rangle = | \langle j, m+1 | \underbrace{J_+ | j, m}_{\# |j, m+1\rangle} \rangle |^2$

☆

$$\boxed{\begin{aligned} J_+ |j, m\rangle &= \hbar \sqrt{(j-m)(j+m+1)}\ |j, m+1\rangle \\ J_- |j, m\rangle &= \hbar \sqrt{(j+m)(j-m+1)}\ |j, m-1\rangle \end{aligned}}$$

so

$J_\pm = J_x \pm i J_y \rightarrow J_x = \dfrac{J_+ + J_-}{2} \quad J_y = \dfrac{J_+ - J_-}{2i}$

Representation of Rotation operator:

$D(R) = e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{J}}$ ← unitary operator. $\varphi = [-\pi, \pi]$

Question: what is $D(R)$ in j-notation:

$$\boxed{D^{(j)}_{m', m}(R) = \langle j, m' | e^{-\frac{i}{\hbar} \varphi \hat{n} \cdot \vec{J}} | j, m \rangle}$$

Wigner Function. linear function. Matrix of rotation operator.

$J^2 (D(R) |j, m\rangle) = \hbar (j+1) j (D(R) |j, m\rangle)$

Note that $D(R) |j, m\rangle$ is still an eigenfunction of J^2

Rotated state:

$$\mathcal{D}(R)\,|j,m\rangle = \sum_{m'} |j,m'\rangle \underbrace{\langle j,m'|\mathcal{D}(R)|j,m\rangle}_{\mathcal{D}^{(j)}_{m'm}(R)}$$

coefficient

Euler Angles:

$$\mathcal{D}(R) = e^{-\frac{i}{\hbar}J_z\alpha}\, e^{-\frac{i}{\hbar}J_y\beta}\, e^{-\frac{i}{\hbar}J_z\gamma}$$

$\alpha \rightarrow [0,2\pi]$, $\beta \rightarrow [0,\pi]$, $\gamma \rightarrow [0,2\pi]$

$$\mathcal{D}(R)^{j}_{m'm}(\alpha,\beta\,\gamma) = \langle j,m'|e^{-\frac{i}{\hbar}J_z\alpha}\, e^{-\frac{i}{\hbar}J_y\beta}\, e^{-\frac{i}{\hbar}J_z\gamma}|j,m\rangle$$

$$= \underbrace{e^{-im\gamma - im'\alpha}}_{\text{does not depend on } j}\ \underbrace{\langle j,m'|e^{-\frac{i}{\hbar}J_y\beta}|j,m\rangle}_{d^{(j)}_{m'm}(\beta)}$$

Ex:
$j = \frac{1}{2}$

$$e^{-\frac{i}{\hbar} J_y \beta} = e^{-\frac{i}{\hbar} S_y \beta} = \cos\left(\frac{\beta}{2}\right) - i\sigma_y \sin\left(\frac{\beta}{2}\right)$$

$$d^{(1/2)} = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

Then

$$D = e^{-im\gamma - im'\alpha} \, d^{1/2}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\left(\frac{\beta}{2}\right) e^{-i\frac{\gamma}{2}} & -\sin\frac{\beta}{2} \, e^{-i\frac{\alpha}{2}} e^{i\frac{\gamma}{2}} \\ e^{i\frac{\alpha}{2}} \sin\left(\frac{\beta}{2}\right) e^{-i\frac{\gamma}{2}} & \cos\frac{\beta}{2} \, e^{i\frac{\alpha}{2}} e^{i\frac{\gamma}{2}} \end{pmatrix}$$

Ex 2: $j = 1$.

$$J_y^{(j+1)} = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}$$

$$e^{-\frac{i}{\hbar} J_y \beta} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \beta^n \left(\frac{J_y^{(1)}}{2\hbar}\right)^n$$

Noting: $\left(\frac{J_y}{2\hbar}\right)^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $\qquad \left(\frac{J_y}{2\hbar}\right)^3 = \frac{J_y}{2\hbar}$

End result: $d^{(1)} = \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-1}{\sqrt{2}} \sin\beta & \frac{1-\cos\beta}{2} \\ -\frac{1}{\sqrt{2}} \sin\beta & \cos\beta & -\frac{1}{\sqrt{2}} \sin\beta \\ \frac{1-\cos\beta}{2} & \frac{1}{\sqrt{2}} \sin\beta & \frac{1+\cos\beta}{2} \end{pmatrix}$

Orbital Momentum and Spherical Harmonics:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[L_i, L_j] = i\hbar \epsilon^{ijk} L_k$$

$$[x_i, p_j] = \delta_{ij}(i\hbar)$$

$$L_z = -i\hbar \partial_\phi$$

$$\langle \vec{x} | L_z | \alpha \rangle = \langle \vec{x} | L_z | \vec{x} \rangle \langle \vec{x} | \alpha \rangle = -i\hbar \partial_\phi \langle \vec{x} | \alpha \rangle$$

$$L_z \varphi_\alpha(\vec{x}) = -i\hbar \partial_\phi \psi_\alpha(\vec{x})$$

let $|\vec{x}\rangle = |r, \varphi, \theta\rangle$ spherical coordinate.

$$L_z = -i\hbar (x\partial_y - y\partial_x) = -i\hbar \partial_\varphi$$

$$L_x = -i\hbar(y\partial_z - z\partial_y) = -i\hbar(-\sin\varphi\, \partial_\theta - \cot\theta \cos\varphi\, \partial_\varphi)$$

$$L_y = -i\hbar(z\partial_x - x\partial_z) = -i\hbar(\cos\varphi\, \partial_\theta - \cot\theta \sin\varphi\, \partial_\varphi)$$

$$L_\pm = L_x \pm iL_y = -i\hbar e^{\pm i\varphi}(\pm i\partial_\theta - \cot\theta\, \partial_\phi)$$

$$L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+) = -\hbar^2\left[\frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, \partial_\theta)\right]$$

$$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \underbrace{\frac{1}{r^2}\left[\frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, \partial_\theta)\right]}_{-\frac{1}{\hbar^2 r^2}L^2}$$

$$\boxed{\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{1}{\hbar^2 r^2}L^2}$$

$$\langle \vec{x} | n, l, m \rangle = R_{nl}(r)\underbrace{Y_l^m(\theta, \phi)}_{\langle \theta\phi | l, m \rangle}$$

First assume rotational symmetry $\Rightarrow [H, \vec{L}] = 0$

$|l,m\rangle \Rightarrow \quad L_z|l,m\rangle = \hbar m|l,m\rangle$

$$L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$$

$$H|n,l,m\rangle = E_{n,l,\cancel{m}}|n,l,m\rangle$$

* $\boxed{Y_l^m(\theta,\phi) = \langle\theta,\phi|l,m\rangle}$

$$\langle\vec{x}|n,l,m\rangle = R_{nl}(r) + Y_l^m(\theta,\phi)$$

$L_z|l,m\rangle = \hbar m|l,m\rangle$

$$-i\hbar\,\partial_\phi Y_l^m(\theta,\phi) = \hbar m\, Y_l^m(\theta,\phi)$$

$$Y_l^m(\theta,\phi) \propto e^{im\phi}$$

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\underbrace{\partial_\varphi^2}_{-m^2} + l(l+1)\right] Y_l^m(\theta,\phi) = 0$$

Know $\langle l',m'|l,m\rangle = \delta_{ll'}\delta_{mm'}$

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \;\underbrace{\langle l',m'|\hat{n}\rangle}_{Y_{l'}^{*m'}(\hat{n})}\underbrace{\langle\hat{n}|l,m\rangle}_{Y_l^m(\hat{n})} = \delta_{ll'}\delta_{mm'}$$

orthonormality.

Look at $|l, m=l\rangle$

then $L_+ |l, m=l\rangle = 0$ since already at max.

$$-i\hbar e^{i\phi}(i\partial_\theta - \cot\theta\,\partial_\phi) Y_l^l = 0$$

assume $Y_l^l \propto e^{il\phi}$ so $\partial_\phi Y_l^l \propto il Y_l^l$

$$\hookrightarrow -i\hbar e^{i\phi}(i\partial_\theta - il\cot\theta) Y_l^l = 0$$

$$Y_l^l(\theta,\phi) = c_l e^{il\phi} \sin^l\theta$$

with normalization: $c_l = \left[\frac{(-1)^l}{2^l l!}\right]\sqrt{\frac{(2l+1)(2l)!}{4\pi}}$

$$Y_l^{m-1} = \langle\hat{n}|l, m-1\rangle = \langle\hat{n}|L_-|l,m\rangle \frac{1}{\hbar\sqrt{(l+m)(l-m+1)}}$$

$$\overset{!}{=} \frac{-i\hbar}{\hbar\sqrt{(l+m)(l-m+1)}} e^{-i\phi}(-i\partial_\theta - \cot\theta\,\partial_\phi) Y_l^m$$

$$\overset{!}{=} \frac{1}{\sqrt{(l+m)(l-m+1)}} e^{-i\phi}(-\partial_\theta + i\cot\theta\,\partial_\phi) Y_l^m$$

$$(-\partial_\theta - m\cot\theta) Y_l^m$$

$L^2 Y_l^m = l(l+1) Y_l^m$ $\qquad \hookrightarrow = \sin\theta\,(-\partial_\theta) \sin^m\theta\, Y_l^m$

$$\hookrightarrow \boxed{\begin{gathered} Y_l^m(\theta,\phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}\, e^{im\phi} \frac{1}{\sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} \underbrace{(\sin\theta)^{2l}}_{(1-\cos^2\theta)^l} \\ Y_l^{-m} = (\theta,\phi) = (-1)^m \left(Y_l^m(\theta,\phi)\right)^* \end{gathered}}$$

$$\hookrightarrow Y_l^{m=0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \qquad \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, Y_{l'}^{*m'} Y_l^m = \delta_{ll'}\delta_{mm'}$$

$$e^{-\frac{i}{\hbar}(2\pi)L_z} = e^{-2\pi\partial\phi}$$

$$e^{-2\pi\partial\phi}\psi(\phi) = \psi(\phi-2\pi) = \psi(\phi)$$

but $\quad e^{-\frac{i}{\hbar}2\pi L_z}|l,m\rangle = e^{-i2\pi m}|l,m\rangle$

then $m \in$ integer, not half integer. for orbital angular momentum.

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}\, e^{im\phi}\, P_l^m(\cos\theta)$$

↖ associated Legendre polynomial.

let $\xi = \cos\theta$

$$\left[(1-\xi^2)P'\right]' + l(l+1)P = 0$$

$P_l \rightarrow$ legendre polynomial.

<u>Rodrigues Formula:</u> for Legendre. Poly

$$P_l(\xi) = \frac{1}{2^l l!}\frac{d^l}{d\xi^l}(\xi^2-1)^l \quad , \quad l = 0,1,2\ldots$$

Rodrigues Formula for associated Legendre Poly:

$$\left[(1-\xi^2)P'\right]' + \left(l(l+1) - \frac{m^2}{1-\xi^2}\right)P = 0$$

For associated Legendre Polynomial.

$l=0$: $Y_0^0 = \frac{1}{\sqrt{4\pi}}$ s-wave

$$\left.\begin{aligned} Y_1^1 &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \sim \frac{x+iy}{r} \\ l=1: \quad Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta \sim z/r \\ Y_1^{-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \sim \frac{x-iy}{r} \end{aligned}\right\} \text{p-wave.}$$

$$\left.\begin{aligned} l=2 \quad Y_2^2 \quad &\sim \frac{(x+iy)^2}{r^2} \\ &\sim \frac{z(x+iy)}{r^2} \\ &\sim \frac{3z^2-r^2}{r^2} \end{aligned}\right\} \text{d-wave.}$$

Addition of Angular Momentum:

Ex: 2 spin ½ particles. (no orbital dof)

H_1 : 2d Hilbert space for particle 1. $\vec{S}_1$
H_2 : 〃 particle 2. $\vec{S}_2$

Algebra
↳ $[S_{1,i}, S_{1,j}] = i\hbar S_{1,k}\, \epsilon^{ijk}$ (with subscripts 1 or 2) $\quad |\psi\rangle_{1,2} \in H_{1,2}$ with basis $|\pm\rangle_{1,2}$

Composite System:

$H = H_1 \otimes H_2$ ← tensor product $\quad \vec{S}_1 \to \vec{S}_1 \otimes 1 \quad \vec{S}_2 \to 1 \otimes \vec{S}_2$

↳ $\vec{S} = \vec{S}_1 \otimes 1 + 1 \otimes \vec{S}_2$: total spin.

$= \vec{S}_1 + \vec{S}_2$

$[S_i, S_j] = i\hbar S_k\, \epsilon^{ijk}$ ← same algebra

$S|+\rangle$

$\Rightarrow$ 4-D representation of $SU(2)$

$2j+1 = 4$ ← 4 dimension.

$\Rightarrow j = 3/2$ if irreducible.

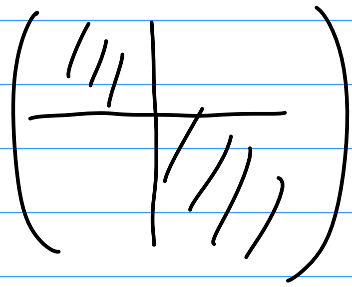
since $|+\rangle|+\rangle \sim \hbar$ ← Max value is just $\hbar$.

$|+\rangle|-\rangle \sim 0$ so can't be $j=3/2$.

$|-\rangle|+\rangle \sim 0$

$|-\rangle|-\rangle \sim -\hbar$

instead look for forms:



Consider S_+:

$$S_+|++\rangle = 0$$

$$S_+|+-\rangle = \cancel{S_+ \otimes 1}\,|+-\rangle + \underbrace{1 \otimes S_+\,|+-\rangle}_{=\hbar|++\rangle}$$

$$S_+|-+\rangle = \hbar|++\rangle$$

$$S_+|--\rangle = \hbar\left(|+-\rangle + |-+\rangle\right)$$

$$\boxed{S_{\pm} = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = 0 \quad \Leftarrow S=0 \text{ representation.}}$$

Call $\boxed{|--\rangle = |-1\rangle}$

$S_z |-1\rangle = -\hbar |-1\rangle$

$S_+ |-1\rangle = \sqrt{2}\hbar \boxed{\underbrace{\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)}_{|0\rangle}} = \sqrt{2}\hbar|0\rangle$

$S_+ |0\rangle = \sqrt{2}\hbar |++\rangle = \sqrt{2}\hbar |1\rangle$

$\boxed{|++\rangle = |1\rangle}$

since $S^2 | \rangle = 2\hbar^2 | \rangle$

$(j+1)j = 2$

$j = 1$

Clebsch - Gordan Coeffs: coefficient for linear combinations when forming eigenstate.

$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

$j=\frac{1}{2}$ $j=\frac{1}{2}$ $j=1$ $j=0$

Ex: spin $\frac{1}{2}$ with orbital distribution:

$|\pm\rangle$: spin states $\qquad$ $|\vec{r}\rangle$: orbital states.

$|\vec{r}, \pm\rangle = |\vec{r}\rangle \otimes |\pm\rangle$ $\qquad$ basis of H.

$$\vec{J} = \vec{L} \otimes 1 + 1 \otimes \vec{S} = \vec{L} + \vec{S}$$

$$D(R) = e^{-\frac{i}{\hbar}\varphi \hat{n}\cdot\vec{L}} \, e^{-\frac{i}{\hbar}\varphi \hat{n}\cdot\vec{S}} = e^{-\frac{i}{\hbar}\varphi \hat{n}\cdot(\vec{L}+\vec{S})}$$

$$\langle \vec{r}, \pm | \alpha \rangle = \psi_\pm(\vec{r}) \to \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

$\vec{L}^2 = \hbar^2 l(l+1)$ $\qquad$ $S^2 = \frac{3}{4}\hbar^2$

$L_z = \hbar m$ $\qquad$ $S_z = \pm\frac{\hbar}{2}$

$$|l, m; s=\tfrac{1}{2}, S_z = \pm\tfrac{\hbar}{2}\rangle \to |\underset{\vec{J}^2}{j}, \underset{J_z}{j_z}; \underset{L^2}{l}, \underset{S^2}{s}\rangle$$

fix l, then we have $\underbrace{(2l+1)}_{m} \times \underset{\text{spin}}{2}$ = # of states.

$$\boxed{l \otimes \tfrac{1}{2} = (l+\tfrac{1}{2}) \oplus (l-\tfrac{1}{2})}$$

Formal Theory:

$\vec{J}_1, \vec{J}_2$: $[J_{\alpha i}, J_{\alpha j}] = i\hbar \epsilon^{ijk} J_{\alpha k}$, $\alpha = 1, 2$

$$\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$$
$$\stackrel{!}{=} \vec{J}_1 + \vec{J}_2$$

For infinitesimal rotations:

$$\left(1 - \frac{i}{\hbar}\delta\phi \hat{n} \vec{J}_1\right) \otimes \left(1 - \frac{i}{\hbar}\delta\phi \hat{n} J_2\right) = 1 - \frac{i\overbrace{(\vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2)}^{\vec{J}}}{\hbar}\hat{n} \cdot \delta\phi$$

Or for finite rotation:

$$D_1(R) \otimes D_2(R) = e^{-\frac{i}{\hbar}\phi \hat{n} \cdot \vec{J}_1} \otimes e^{-\frac{i}{\hbar}\phi \hat{n} \cdot \vec{J}_2}$$

↳ $[J_i, J_j] = i\hbar \epsilon^{ijk} J_k$ → all results from SU(2) algebra hold for total angular momentum, $\vec{J}$

Consider 2 different basis:

Basis A: $(\vec{J}_1^2, \vec{J}_2, J_{1z}, J_{2z}) \rightarrow |j_1, j_2, m_1, m_2\rangle$

Since $-l < m < l$, so there are $(2j_1+1)(2j_2+1)$ states

ex: $J_1^2 |j_1, j_2, m_1, m_2\rangle = \hbar^2 (j_1+1) j_1 |j_1, j_2, m_1, m_2\rangle$

Basis B: $(\vec{J}_1^2, \vec{J}_2^2, J^2, J_z) \rightarrow |j_1, j_2, j, m\rangle$

Dim: $\sum_j (2j+1)$: sum over different j's.

Note that $[J_{1z}, J^2] \neq 0$.

Transforming Basis

Introduce projection: $\sum_{m_1}\sum_{m_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2| = 1$

then

$$|j_1\ j_2;\ j, m\rangle = \sum_{m_1, m_2} |j_1, j_2; m_1, m_2\rangle \underbrace{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordon Coefficients.}}$$

they form a unitary matrix of transformation from A to B.

Properties of CG Coefficients:

$$(J_z - J_{1z} - J_{2z})|j_1, j_2; j, m\rangle = 0$$

$$\hookrightarrow (m - m_1 - m_2)\underbrace{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j\, m\rangle}_{} = 0$$

$$\boxed{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j m\rangle \sim \delta_{m, m_1 + m_2}}$$

What values can j take ?

Suppose we start with the highest state: only one vector

$$|j_1, j_2 ; m_1 = j_1 , m_2 = j_2 \rangle \leftarrow J_z = j_1 + j_2 .$$

Now apply $J_- = J_{1-} + J_{2-}$ to obtain $J_z = j_1 + j_2 - 1$.

But there are two such vectors:

$$m_1 = j_1 , \quad m_2 = j_2 - 1 \qquad \text{or} \qquad m_1 = j_1 - 1 , \quad m_2 = j_2$$

Therefore: for $j_1 \otimes j_2 = |j_1 - j_2| \oplus |j_1 - j_2| \oplus \cdots\cdots \oplus j_1 + j_2$

$$\boxed{(2j_1 + 1)(2j_2 + 1) = \sum_{j=|j_1 - j_2|}^{j_1 + j_2} (2j + 1)}$$

Since $\langle j_1, j_2 ; j m | j_1, j_2, m_1, m_2 \rangle$ is unitary matrix elements, they can be chosen to be real.

$$\sum_{j,m} \langle j_1 j_2 ; m_1 m_2 | j_1 j_2\, j m \rangle \langle j_1, j_2 ; m_1' m_2' | j_1 j_2 ; j m \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\hookrightarrow \sum_{m_1 m_2} | \langle j_1 j_2 ; m_1 m_2 | j_1 j_2\, j m \rangle |^2 = 1$$

$$\boxed{\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle = (-1)^{j_1 - j_2 + m} \sqrt{2j+1} \underbrace{\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}}_{\text{Wigner's 3j symbol}}}$$

Recursion Relation for CG Coefficients:

Via $J_\pm |j_1 j_2 j m\rangle = \sum C (J_{1\pm} + J_{2\pm}) |j_1 j_2, m_1, m_2\rangle$

Ex: let $j_1 = l$ $\quad m_1 = m_l$
$\quad j_2 = s = \frac{1}{2}$ $\quad m_2 = \pm\frac{1}{2}$ $\Big\}$ $j = l \pm \frac{1}{2}$, $l > 0$

$$\langle l, \tfrac{1}{2}; \overset{l\pm\frac{1}{2}}{j}, m \,|\, l, \tfrac{1}{2}, m_1, \overset{\pm\frac{1}{2}}{m_2} \rangle = \begin{pmatrix} \sqrt{\dfrac{l+m+\frac{1}{2}}{2l+1}} & \sqrt{\dfrac{l-m+\frac{1}{2}}{2l+1}} \\ -\sqrt{\dfrac{l-m+\frac{1}{2}}{2l+1}} & \sqrt{\dfrac{l+m+\frac{1}{2}}{2l+1}} \end{pmatrix}$$

Spin-Angular Functions: $\langle \vec{r}, \pm 1, l, \frac{1}{2}, j, m \rangle$

$$\hookrightarrow \mathcal{Y}_l^{j=l\pm\frac{1}{2}, m} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm\sqrt{l \pm m + \frac{1}{2}} & Y_l^{m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{l \mp m + \frac{1}{2}} & Y_l^{m+\frac{1}{2}}(\theta,\phi) \end{pmatrix}$$

$|l, \frac{1}{2}; j, m\rangle$: Eigen functions of L^2, S^2, J^2, and J_z.

use $\quad J^2 = (\vec{L} + \vec{S})^2 = L^2 + \underbrace{\vec{L}\cdot\vec{S} + \vec{S}\cdot\vec{L}}_{\text{commute.}} + S^2$

$$\hookrightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2)$$

then $\quad \vec{L}\cdot\vec{S}\, |l, \frac{1}{2}; j, m\rangle = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - \frac{3}{4}\right]$

$\uparrow$

spin-orbit interaction $\quad = \begin{cases} \dfrac{l\hbar^2}{2} & j = l + \frac{1}{2} \\ -\dfrac{(l+1)\hbar^2}{2} & j = l - \frac{1}{2}. \end{cases}$

Tensor Operators:

$|\alpha\rangle \to \mathcal{D}(R)|\alpha\rangle$: Transformation of Hilbert space under rotation of real space.

How about transformation of operators?

$$\langle\alpha|O|\alpha\rangle \to \langle\alpha|\mathcal{D}^\dagger(R)\, O\, \mathcal{D}(R)|\alpha\rangle = \langle\alpha|\tilde{O}|\alpha\rangle$$

$$* \quad \boxed{\hookrightarrow O \to \mathcal{D}^\dagger(R)\, O\, \mathcal{D}(R) \ : \text{Transformation of linear operator.}}$$

Cases: 1) Scalar:

say $O = \vec{r}^{\,2} = r^2$ $\qquad$ $r^2 \to \mathcal{D}^\dagger(R)\, r^2\, \mathcal{D}(R) = r^2$

scalar operator is not affected by transformation.

Infinitesimal rotation. $\mathcal{D}(R) = 1 - \frac{i}{\hbar}\delta\phi\, \hat{n}\cdot\vec{J}$

$$\tilde{O}_R = \left(1 + \frac{i}{\hbar}\delta\phi\, \hat{n}\cdot\vec{J}\right) O \left(1 - \frac{i}{\hbar}\delta\phi\, \hat{n}\cdot\vec{J}\right) = O + \underbrace{\frac{i}{\hbar}\delta\phi\, \hat{n}[\vec{J}, O]}_{\delta O}$$

In general $\boxed{\delta O = \frac{i}{\hbar}\delta\phi\, \hat{n}[\vec{J}, O]}$

$$\boxed{\begin{array}{l} \text{Define Scalar when } \delta O = 0 \to [\vec{J}, O] = 0 \\ \qquad\qquad\qquad\qquad \text{or } \tilde{O}_R = \mathcal{D}^\dagger(R)\, O\, \mathcal{D}(R) = O \end{array}}$$

Ex 2: Vector:

$X_i \to \mathcal{D}^\dagger(R)\, X_i\, \mathcal{D}(R)$

Infinitesimal: $\delta X_i = \frac{i}{\hbar}\delta\phi\, \hat{n}_j [\vec{L}, X_i] = \frac{i}{\hbar}\delta\phi\, \hat{n}_j\, i\hbar\, \varepsilon^{jik} X_k$

$$\delta X_i \overset{!}{=} \delta\phi\, \varepsilon^{ijk}\, \hat{n}_j X_k$$

So $\boxed{[L_i, X_j] = i\hbar\, \varepsilon^{ijk} X_k}$

In general:

If $\langle\alpha|V_i|\alpha\rangle = \langle\alpha|\mathcal{D}^\dagger(R)\, V_i\, \mathcal{D}(R)|\alpha\rangle = R_{ij}\langle\alpha|V_j|\alpha\rangle$

then $\vec{V}$ is a vector operator,

i.e. $\boxed{[L_i, V_j] = i\hbar\, \varepsilon^{ijk} V_k}$ For vectors, V_j

ex:

$$[L_i, L_j] = i\hbar\, \varepsilon^{ijk} L_k$$

rotation generator (L_i); vector (L_j); transformed vector (L_k)

Ex 3: Dyadic:

Consider $T_{ij} = U_i V_j$ where U_i, V_j are vector operators

If $T_{ij} \to R_{ii'} U_{i'} R_{jj'} V_{j'} = R_{ii'} R_{jj'} T_{i'j'}$

then $U_i V_j$ is a rank 2 Cartesian Tensor.

$$\boxed{\begin{array}{l} \text{Generally:} \quad T_{i,j,k\cdots} \to \sum_{i',j',k'\cdots} R_{ii'} R_{jj'} R_{kk'} \cdots T_{i',j',k'\cdots} \\ \underbrace{T_{i,j,k,\cdots}}_{n} \ - \text{ Cartesian tensor of rank } n. \end{array}}$$

Note: Classification of scalar, vector and general tensor is for general linear transformation. However corresponding representation are <u>reducible</u> under rotation group.

$$\text{Ex: } \underbrace{U_i V_j}_{9\text{-component}} = \underbrace{\frac{U_i V_i}{3}\delta_{ij}}_{\substack{1\text{-component} \\ (\text{scalar})}} + \underbrace{\frac{U_i V_j - U_j V_i}{2}}_{\substack{\frac{1}{2}\varepsilon^{ijk} U_i V_j \\ 3\text{-component} \\ (\text{vector})}} + \underbrace{\left(\frac{U_i V_j + U_j V_i}{2} - \frac{U_i V_i}{3}\delta_{ij}\right)}_{\substack{\text{Symmetric traceless tensor} \\ 5\text{-component}}}$$

$$\begin{matrix} 1&2&3 \\ 4&5&6 \\ 7&8&9 \end{matrix} \qquad \begin{matrix} 1&2&3 \\ 4&5&6 \\ 7&8&9 \end{matrix} \ -1 \ \text{traceless}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$l=0,\ 1D \qquad l=1,\ 3D \qquad l=2,\ 5D$$

$$9 = 1 + 3 + 5$$

$$\text{or} \quad 1 \otimes 1 = 0 \oplus 1 \oplus 2$$

Spherical Tensor: $Y_l^m(\hat{n})$ under rotation of $\hat{n}$.

$$D^\dagger(R)\, Y_l^m(\hat{n})\, D(R) = \sum_{m'=-l}^{l} Y_l^{m'} \underbrace{D_{mm'}^{(R)*}(R)}_{\langle\, | e^{-\frac{i}{\hbar}\varphi_m \hat{n}\cdot\vec{J}} |\, \rangle}$$

Def: if $T_q^{(k)}$ $q = -k, \ldots\ldots +k$

transform as $D^\dagger(R)\, T_q^{(k)}\, D(R) = \sum_{q'=-m}^{m} D_{qq'}^{(k)*}\, T_{q'}^{(k)}$

then $T_q^{(k)}$ is called spherical tensor of rank k.

Infinitesimal version:

$$[J_z, T_q^{(k)}] = \hbar q\, T_q^{(k)}$$

$$[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\; T_{q\pm1}^{(k)}$$

Example:

instead of x, y, z

$$\begin{cases} U_{\pm1} = \dfrac{\mp U_x - iU_y}{\sqrt{2}} = Y_1^{\pm1}(\vec{U})\sqrt{\dfrac{4\pi}{3}} \\[2ex] U_0 = U_z = Y_1^0(\vec{U})\sqrt{\dfrac{4\pi}{3}} \end{cases}$$

(x, y written above U_x, U_y)

$U_q \rightarrow$ spherical tensor of rank 1.
$q = 0, \pm1$

so $U_q = T_q^{(1)} \approx Y_{l\sim k}^{m\sim q}$.

Ex:

Dyadic: $U_i V_j$

$$T_0^{(0)} = -\frac{\vec{U}\cdot\vec{V}}{3} = \frac{U_{+1}V_{+1} + U_{-1}V_{-1} - U_0 V_0}{3} \qquad \text{rank } 0$$

$$T_q^{(1)} = \frac{(\vec{U}\times\vec{V})_q}{i\sqrt{2}}, \quad q = 0, \pm 1 \qquad \text{rank } 1.$$

$$\left.\begin{aligned} T_{\pm 2}^{(2)} &= U_{\pm 1} V_{\pm 1} \\ T_{\pm 1}^{(2)} &= \frac{U_{\pm 1} V_0 + U_0 V_{\pm 1}}{\sqrt{2}} \\ T_0^{(2)} &= \frac{U_{+1}V_{-1} + 2U_0V_0 + U_{-1}V_{+1}}{\sqrt{6}} \end{aligned}\right\} \qquad \text{rank } 2.$$

General: spherical tensors.

$$X_{(q_1)}^{(k_1)}, \quad Z_{q_2}^{(k_2)}$$

then

$$T_q^{(k)} = \sum_{q_1}\sum_{q_2} \langle \overset{l_1}{k_1}, \overset{l_2}{k_2}; \overset{m_1}{q_1}, \overset{m_2}{q_2} \,|\, \overset{l_1}{k_1} \overset{l_2}{k_2}; \overset{l}{k}, \overset{m}{q} \rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

$$q_1 + q_2 = q \quad \sim \quad m_1 + m_2 = m$$

Matrix Elements of tensors:

Example: $\boxed{\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle \neq 0 \text{ only if } m' = q + m}$

m-selection rule: $\boxed{m' = q + m}$

Proof: $0 = \langle \alpha, j' m' | \underbrace{[J_z, T_q^{(k)}] - \hbar q T_q^{(k)}}_{=0} | \alpha, j, m \rangle$, with $[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$

$$0 = (\hbar(m' - m) - \hbar q) \underbrace{\langle \alpha', j' m' | T_q^{k} | \alpha, j m \rangle}_{\neq 0 \text{ unless}}$$

$$\hbar \underbrace{(m' - m - q)} = 0$$

$\boxed{m' = q + m}$ ← m-selection rule.

$$D(T_q^{(k)} | \alpha, j m \rangle) = \underbrace{(D T_q^{(k)} D^{+})}_{k\text{-repr}} D \underbrace{| \alpha, j m \rangle}_{j\text{-repr}}$$

Wigner-Eckart Theorem:

$$\boxed{\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle = \underbrace{\langle j k; m q | j k, j' m' \rangle}_{\text{Coeff.}} \frac{\langle \alpha', j' \| T_q^{(k)} \| \alpha, j \rangle}{\sqrt{2j' + 1}}}$$

$\boxed{\text{Triangular relation: } |j - k| \leq j' \leq j + k}$

$\boxed{\text{m-selection rule: } m' = q + m}$

ex: Scalar

$T_0^{(0)} = S$

$$\langle \alpha' j' m' | S | \alpha j m \rangle = \delta_{jj'} \delta_{mm'} \frac{\langle \alpha' j' || S || \alpha j \rangle}{\sqrt{2j+1}}$$

If $H = S$, not invariant: then $2j+1$ degeneracy.

ex 2: vector: $V_{q=0,\pm1}$

$$\langle \alpha', j' m' | V_q^{(1)} | \alpha, j m \rangle = \langle j, 1; m q | j 1; j' m' \rangle \frac{\langle \alpha j' || V_q | \alpha j \rangle}{\sqrt{2j+1}}$$

$$\boxed{\begin{array}{l} \text{then} \quad m' - m = \underbrace{\pm 1, 0}_{q} \\ \text{and} \quad j' - j = \begin{cases} \pm 1 \\ 0 \end{cases} \end{array}}$$

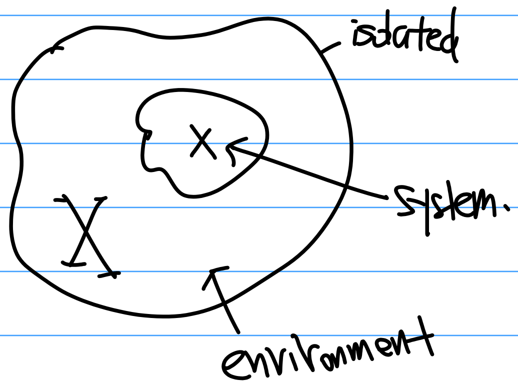
← Rotational Selection Rule for vector.

used triangular relation

Electric Field: $V(x) = -Ex \cos \omega t$.

Density Operators:

isolated quantum state:



$\psi(x, X)$

A - operator acting on system

$$\langle A \rangle = \int dx\, dx'\, dX\; \psi^*(x, X) A(x, x') \psi(x', X)$$
$$= \int dx\, dx'\, A(x, x') \underbrace{\int dX\; \psi^*(x', X) \psi(x, X)}_{\rho(x', x)}$$

$*$

$$\boxed{\langle A \rangle = \int dx\, dx'\, A(x, x')\, \rho(x', x) = \mathrm{Tr}_{sys}\{A \rho\}}$$

where $\rho(x', x) = \int dX\; \psi^*(x', X) \psi(x, X)$

density matrix, or density operator.

Important properties:

i) ρ is Hermitian, $\rho = \rho^\dagger$

ii) $\mathrm{Tr}\{\rho\} = 1$.

Diagonalization of ρ: $\rho = \sum_i \lambda_i |v_i\rangle\langle v_i|$

eigenbasis of ρ.

Since $\psi(x', X) = \langle x', X | \psi \rangle$

$\psi^*(x, X) = \langle \psi | x, X \rangle$

then $\rho(x', x) = \int dX \, \langle x', X | \psi \rangle \langle \psi | x, X \rangle$

$$\boxed{\rho(x', x) \overset{!}{=} \langle x' | \underbrace{\mathrm{Tr}_X \left(|\psi\rangle\langle\psi| \right)}_{\hat{\rho}} | x \rangle}$$

↳ $\rho(x', x) = \langle x' | \hat{\rho} | x \rangle$

↳ $\boxed{\hat{\rho} = \mathrm{Tr}_X \left(|\psi\rangle\langle\psi| \right)}$

over environment.

→ $\langle A \rangle = \mathrm{Tr}_X \left(\hat{\rho} A \right)$

→ $\mathrm{Tr}_x \hat{\rho} = \mathrm{Tr}_x \mathrm{Tr}_X \left(|\psi\rangle\langle\psi| \right)$

$\overset{!}{=} \mathrm{Tr} \left(|\psi\rangle\langle\psi| \right)$

$\overset{!}{=} 1$

so $\boxed{\mathrm{Tr}(\hat{\rho}) = 1}$

Ex: 2-spin half:

$$|s\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle \mp |-+\rangle\right)$$

$$\hat{\rho}_1 = \mathrm{Tr}_2\, |s\rangle\langle s|$$

of the first spin

$$\stackrel{!}{=} {}_2\langle +|s\rangle\langle s|+\rangle + {}_2\langle -|s\rangle\langle s|-\rangle$$

$$\stackrel{!}{=} \left(\mp\frac{1}{\sqrt{2}}|-\rangle_1\right)\left(\mp\frac{1}{\sqrt{2}}\, {}_1\langle -|\right) + \left(\frac{1}{\sqrt{2}}|+\rangle_1\right)\left(\frac{1}{\sqrt{2}}\, {}_1\langle +|\right)$$

$$\hat{\rho}_1 \stackrel{!}{=} \frac{1}{2}|-\rangle_1\, {}_1\langle -| + \frac{1}{2}|+\rangle_1\, {}_1\langle +| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$\mathrm{Tr}(\rho_1) = 1$

$\mathrm{Tr}(\rho_1^2) = 1/2$

$\langle \sigma_z\rangle = \mathrm{Tr}\left(\frac{1}{2}\cdot \mathbb{I}\ \sigma_z\right) = 0$

$\langle \sigma_{x,y}\rangle = \mathrm{Tr}\left(\frac{1}{2}\cdot \mathbb{I}\ \sigma_{x,y}\right) = 0$

Ex 2:

$$|s\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 + |-\rangle_1\right)|+\rangle_2$$

$$\stackrel{!}{=} \frac{1}{\sqrt{2}}\left(|++\rangle + |-+\rangle\right)$$

$$\hat{\rho}_1 = \mathrm{tr}_2\left(|s\rangle\langle s|\right)$$

$$\stackrel{!}{=} {}_2\langle +|s\rangle\langle s|+\rangle_2 + \cancel{{}_2\langle -|s\rangle\langle s|-\rangle_2}$$

$$\stackrel{!}{=} \frac{1}{2}\left(|+\rangle_1 + |-\rangle_1\right)\left(\langle +|_1 + \langle -|_1\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(1 + \sigma_x)$$

$\mathrm{Tr}\,\hat{\rho}_1 = 1$

$\mathrm{Tr}(\hat{\rho}_1^2) = 1$

$\langle \sigma_x\rangle = \mathrm{Tr}\left(\frac{1}{2}(1+\sigma_x)\sigma_x\right) = 1$

$\langle \sigma_{y,z}\rangle = 0$

<u>Ensemble:</u> $|\alpha^{(i)}\rangle$, w_i : probability at $|\alpha^{(i)}\rangle$

Not necessary orthogonal

$\sum w_i = 1$

$$[A] = \sum_i w_i \underbrace{\langle\alpha^{(i)}|A|\alpha^{(i)}\rangle}_{\text{Quantum Average.}}$$

Ensemble Average

$$= \sum_i w_i \sum_{b_1, b_2} \langle\alpha^{(i)}|b_1\rangle\langle b_1|A|b_2\rangle\langle b_2|\alpha^{(i)}\rangle$$

Complete basis

$$= \sum_{b_1, b_2} \underbrace{\left(\sum_i w_i \langle b_2|\alpha^{(i)}\rangle\langle\alpha^{(i)}|b_1\rangle\right)}_{\rho(b_1, b_2)} \underbrace{\langle b_1|A|b_2\rangle}_{A}$$

$$= \mathrm{Tr}(\rho A)$$

$$\boxed{\hat{\rho} = \sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|}$$ ← density operator for ensemble.

↳ $\mathrm{Tr}\,\rho = 1$ ↳ $\rho^\dagger = \rho$

$$\boxed{\begin{array}{l}\underline{\text{Pure Ensemble:}} \\ \text{If } w_i = 0 \text{ except one of them is } 1; \\ \text{then } \quad \rho = |\psi\rangle\langle\psi| \leftarrow \text{pure Ensemble.} \\ \text{then } \quad [A] = Tr(\rho A) = Tr(|\psi\rangle\langle\psi|A\rangle) = Tr(\langle\psi|A|\psi\rangle) \\ \qquad\qquad [A] = \langle A\rangle \\ \text{pure state: } \boxed{\rho^2 = \rho}, \ \boxed{Tr\,\rho^2 = 1}\end{array}}$$

In general $Tr\,\rho^2 = \lambda_1^2 + \lambda_2^2 + \cdots \leq \lambda_1 + \lambda_2 + \lambda_3 \cdots = 1$

Pure Ensemble: ex: $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\rho^2 = \rho \qquad \rho^2 = \rho.$

Diag $\rho_{pure} = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$ $\leftarrow$ 1 somewhere in diagonal, 0 else where.

<u>Completely Random</u>

Diag $\rho = \begin{pmatrix} \frac{1}{N} & & & 0 \\ & \frac{1}{N} & & \\ & & \ddots & \\ 0 & & & \frac{1}{N} \end{pmatrix} = \frac{1}{N} \cdot \mathbb{1}$

$Tr(\rho^2) = \frac{1}{N}^2 N = \frac{1}{N}$ $\leftarrow$ Minimum possible proof:

$$\sum w_i^2 - \lambda\left(\sum w_i - 1\right) = 0$$

$$\partial_{\rho_i} - \sum_i \rho_i \ln \rho_i - \lambda \sum_i \rho_i$$

$$\hookrightarrow -\ln \rho_i - 1 - \lambda = 0$$

$$2w_i - \lambda = 0 \qquad \sum w_i = 1 \qquad \rho_i = e^{-\lambda - 1}$$

$$\hookrightarrow w_i = \frac{\lambda}{2} = \text{const} \Rightarrow \text{so } w_i = \frac{1}{N}$$

Mixed Ensemble: $\boxed{\rho^2 \neq \rho, \quad Tr\rho^2 < 1}$

$|S_+ +\rangle$ with $w = 75\%$, $|S_- -\rangle$, with $w = 25\%$

$$\rho = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 7/8 & 1/8 \\ 1/8 & 1/8 \end{pmatrix}$$

$$Tr\,\rho = 1 \qquad Tr\,\rho^2 = \frac{13}{16} < 1$$

Evolution of ρ:

$$\rho(t_0) = \sum_i w_i \, |\alpha^{(i)}\rangle_{t_0} \, {}_{t_0}\langle\alpha^{(i)}|$$

$$\rho(t) = \sum_i w_i \, U_{t,t_0} |\alpha^{(i)}\rangle_{t_0} \, {}_{t_0}\langle\alpha^{(i)}| U^\dagger$$

$$* \quad \boxed{\rho(t) \overset{!}{=} U_{t,t_0} \, \rho(t_0) \, U^\dagger_{t,t_0}} \quad , \; U = e^{\frac{-i}{\hbar} H(t-t_0)}$$

$$i\hbar \partial_t \rho(t) = \sum_i w_i \left[\left(i\hbar \partial_t |\alpha^{(i)}\rangle \right) \langle\alpha^{(i)}| + |\alpha^{(i)}\rangle \left(i\hbar \partial_t \langle\alpha^{(i)}| \right) \right]$$

$$\overset{!}{=} \sum_i w_i \left[H |\alpha^{(i)}\rangle\langle\alpha^{(i)}| - |\alpha^{(i)}\rangle\langle\alpha^{(i)}| H \right]$$

$$* \quad \boxed{i\hbar \partial_t \rho(t) \overset{!}{=} [H, \rho] = -[\rho, H]}$$

vs. classical: $\frac{d}{dt}\rho_{classical} = \{\rho_{classical}, H\}_{PB} + \frac{\partial}{\partial t}\rho_{classical}$

$$\frac{\partial \rho_{classical}}{\partial t} = -\{\rho_{classical}, H\}_{PB}$$

vs. Heisenberg: $i\hbar \, d_t O = [O, H] + \cancel{i\hbar \partial_t O}$ No ~~explicite dependence~~ in t.

Entropy: $\sigma = -\mathrm{Tr}(\rho \ln \rho)$

$$\overset{!}{=} -\sum_i \rho_i \ln \rho_i$$

Pure: $\rho_i = 0$ except $\rho_1 = 1$
so $\sigma = 0$ for pure.

Random: $\rho_i = \frac{1}{w}$, $\sigma = -\sum_i \frac{1}{w} \ln \frac{1}{w} = \ln w \leftarrow$ Max entropy.

Two Independent solution of ρ_1 and ρ_2

$$\rho = \rho_1 \rho_2$$

$$S_{12} = -\mathrm{tr}\, \rho_1 \rho_2 \ln \rho_1 \rho_2$$

$$= -\mathrm{tr}\, \rho_2 (\rho_1 \ln \rho_1) - \mathrm{tr}\, \rho_1 \rho_2 \ln \rho_2$$

$$= -\underbrace{\mathrm{tr}_2 \rho_2}_{=1} \mathrm{tr}_1 (\rho_1 \ln \rho_1) - \underbrace{\mathrm{tr}_1 \rho_1}_{=1} \mathrm{tr}_2 \rho_2 \ln \rho_2$$

$$\boxed{S_{12} = S_1 + S_2}$$

Statistical Mechanics:

In equilibrium: $\frac{\partial \rho}{\partial t} = -[\rho, H] = 0$

then ρ and H diagonalize simultaneously.

$\mathrm{Tr}\,\rho = 1 \qquad \mathrm{Tr}\,\rho H = E$

want to maximize $S = -\mathrm{tr}\,\rho \ln \rho$

$\hookrightarrow -\mathrm{tr}\,\rho\ln\rho - \beta\,\mathrm{tr}\,\rho H - \gamma\,\mathrm{tr}\,\rho$

$\hookrightarrow -\sum_i \delta\rho_i (\ln\rho_i + 1) - \sum_i \beta\,\delta\rho_i E_i - \sum_i \gamma\,\delta\rho_i = 0$

$\hookrightarrow -\ln\rho_i - 1 - \beta E_i - \gamma = 0$

$$\rho_i = e^{-\beta E_i - \gamma - 1} = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$$

$$\boxed{\rho_i = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})} \qquad \langle A \rangle = \mathrm{tr}(\rho A)}$$

Entanglement

System A and B in pure state $|\psi\rangle_{AB}$

$$\rho_A = \mathrm{Tr}_B \, |\psi\rangle_{AB} \, {}_{AB}\langle\psi|$$

suppose $$|\psi\rangle_{AB} = \sum_i \overset{\geq 0}{\lambda_i} \, |\alpha^{(i)}\rangle_A \, |\beta^{(i)}\rangle_B$$

$$\rho_A = \mathrm{Tr}_B \sum_i \lambda_i^2 \cdot |\alpha^{(i)}\rangle_A \, |\beta^{(i)}\rangle_B \, {}_B\langle\beta^{(i)}| \, {}_A\langle\alpha^{(i)}|$$

$$\boxed{\begin{aligned} \rho_A &\overset{!}{=} \sum_i \underbrace{\lambda_i^2}_{w_i} \, |\alpha^{(i)}\rangle_A \, {}_A\langle\alpha^{(i)}| \\ \rho_A &\overset{!}{=} \sum_i w_i \, |\alpha^{(i)}\rangle\langle\alpha^{(i)}| \end{aligned}}$$

Schmidt Decomposition:

If $|\psi\rangle_{AB}$ is a pure state of A, B, then there exists orthonormal states $|\alpha^{(i)}\rangle_A$ of A and $|\beta^{(i)}\rangle_B$ of B such that

$$|\psi\rangle_{AB} = \sum_i \lambda_i \, |\alpha^{(i)}\rangle_A \, |\beta^{(i)}\rangle_B$$

and $\lambda_i \geq 0$ and $\sum \lambda_i^2 = 1$

$\uparrow$ schmid coefficients.

Corollary of Schmid decomposition:

Eigenvalues of ρ_A and ρ_B are the sum.

$$\rho_A = \sum \lambda_i^2 \, |\alpha^{(i)}\rangle_A \, {}_A\langle\alpha^{(i)}|$$

$$\rho_B = \sum \lambda_i^2 \, |\beta^{(i)}\rangle_B \, {}_B\langle\beta^{(i)}|$$

$$\sigma = -\mathrm{tr}(\rho^A \ln \rho^A) = -\mathrm{tr}(\rho^B \ln \rho^B)$$

entanglement entropy.

If $\sigma = 0$, then A and B are completely independent.

$$|\psi\rangle_{AB} = |\alpha\rangle_A \, |\beta\rangle_B$$

Purification: given ρ^A, find $|AB\rangle$, $\mathcal{H}^A \otimes \mathcal{H}^B$

then $\rho^A = \mathrm{tr}_B \, |A,B\rangle\langle A,B|$

Diagonalize $\rho^A = \sum_i p_i \, |\alpha^{(i)}\rangle_A \, {}_A\langle\alpha^{(i)}|$

consider $|AB\rangle = \sum_i \sqrt{p_i} \, |\alpha^{(i)}\rangle_A \, |\beta^{(i)}\rangle_B$

Given $|\psi\rangle_{AB}$

$\hookrightarrow \rho^A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$

then $S = -\mathrm{tr}\, \rho^A \ln \rho^A$ ← Entropy, measure of entanglement between A, B in $|\psi\rangle$

Density Matrices and measurements:

$$\rho \xrightarrow{\text{Time evolution}} U\rho U^\dagger$$

measure $\underset{\alpha\,|\alpha\rangle\langle\alpha|}{A \shortparallel}$ in state ρ, for $|\psi\rangle$ pure state

$$|\psi\rangle \xrightarrow{\text{measure } A} |\alpha\rangle \quad \text{with probability: } |\langle\alpha|\psi\rangle|^2$$

$$\rho = |\psi\rangle\langle\psi| \xrightarrow{\text{measure } A} |\alpha\rangle\langle\alpha| \text{ with probability } |\langle\alpha|\psi\rangle|^2$$

$$\rho' = \sum_\alpha |\langle\alpha|\psi\rangle|^2 |\alpha\rangle\langle\alpha|$$

$$\stackrel{!}{=} \sum_\alpha |\alpha\rangle\langle\alpha| \rho |\alpha\rangle\langle\alpha|$$

$$\boxed{\begin{gathered}\rho \rightarrow \rho' = \sum_\alpha M_\alpha \rho M_\alpha^\dagger \\ \sum_\alpha M_\alpha^\dagger M_\alpha = 1\end{gathered}}$$

← Time evolution of ρ.

general time evolution of ρ

M_α = Kraus operator.

$M_\alpha = |\alpha\rangle\langle\alpha|$ ← repeated measurements

$M_\alpha = U$

Weak Measurement:

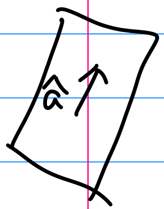
$$\rho \xrightarrow{\text{extent}} \hat{\rho} = |1\rangle\langle 1| \otimes \rho \xrightarrow[U]{\text{evolution}} \rho'$$

$$\rho' = U\hat{\rho}U^{+} = \sum_{\alpha,\alpha'} |\alpha\rangle\langle\alpha'| \otimes M_{\alpha}\rho M_{\alpha'}^{+}$$

$\big\downarrow$ trace out $|\alpha\rangle$

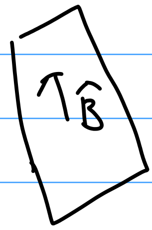
$$\rho \rightarrow \rho' = \sum_{\alpha} M_{\alpha}\rho M_{\alpha}^{+}$$

Bell Inequality:



$$e^- \leftarrow \pi^0 \rightarrow e^+$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) = |0\rangle$$



± 1 $\qquad\qquad$ ± 1

$P(\hat{a},\hat{b})$

average product of results of $\hat{a},\hat{b}$

$$P(\hat{a},\hat{b}) = \langle 0 | (\vec{\sigma}_1 \cdot \hat{a}) \; (\vec{\sigma}_2 \cdot \hat{b}) | 0 \rangle$$

$$\vec{\sigma}_1 \cdot \hat{a} = \begin{pmatrix} a_3 & a_1 - i a_2 \\ a_1 + i a_2 & -a_3 \end{pmatrix}$$

$$\hookrightarrow \frac{1}{2} \left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow| \right) (\vec{\sigma}_1 \cdot \hat{a})(\vec{\sigma}_2 \cdot \hat{b}) \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

$$\hookrightarrow \frac{1}{2}\left[(\vec{\sigma}_1 \cdot \hat{a})_{11} (\vec{\sigma}_2 \cdot \hat{b})_{22} + (\vec{\sigma}_1 \hat{a})_{22} (\vec{\sigma}_2 \hat{b})_{11} - (\vec{\sigma}_1 \hat{a})_{12} (\vec{\sigma}_2 \cdot \hat{b})_{21} - (\vec{\sigma}_1 \cdot \hat{a})_{21} (\vec{\sigma}_2 \cdot \hat{b})_{12} \right]$$

1 means $\uparrow$ $\qquad$ 2 means $\downarrow$.

$$\hookrightarrow = \frac{1}{2}\left[-a_3 b_3 - a_3 b_3 - (a_1 - i a_2)(b + i b_2) - (a_1 + i a_2)(b_1 - i b_2) \right]$$

$$= -\hat{a} \cdot \hat{b}$$

$$\boxed{P(\hat{a},\hat{b}) = -\hat{a} \cdot \hat{b}}$$

If $\hat{b} = \hat{a}$ $\rightarrow$ $\qquad P(\hat{a},\hat{b}) = -1$

Now assume local hidden variable λ:

Result $A(\hat{a}, \lambda) = \pm 1$

result. of electron

$$B(\hat{b}, \lambda) = \pm 1$$

$$A(\hat{a}, \lambda) = -B(\hat{b}, \lambda)$$

$$\rho(\lambda) \geq 0 \quad , \quad \int \rho(\lambda)\, d\lambda = 1$$

density probability of λ

$$\hat{P}(\hat{a}, \hat{b}) = \int d\lambda\, \rho(\lambda)\, A(\hat{a}, \lambda)\, B(\hat{b}, \lambda)$$

$$\overset{!}{=} -\int d\lambda\, \rho(\lambda)\, A(\hat{a}, \lambda)\, A(\hat{b}, \lambda)$$

$$\hat{P}(\hat{a}, \hat{b}) - \hat{P}(\hat{a}, \hat{c}) = -\int d\lambda\, \rho(\lambda) \left[A(\hat{a}, \lambda) A(\hat{b}, \lambda) - A(\hat{a}, \lambda) A(\hat{c}, \lambda) \right]$$

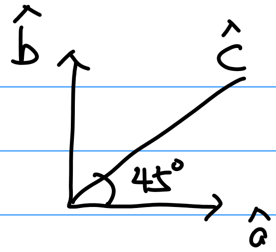
$$\overset{!}{=} -\int d\lambda\, \rho(\lambda) \left[1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda) \right] A(\hat{a}, \lambda) A(\hat{b}, \lambda)$$

$$\left| \hat{P}(\hat{a}, \hat{b}) - \hat{P}(\hat{a}, \hat{c}) \right| \overset{!}{\leq} \int d\lambda\, \rho(\lambda) \left| 1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda) \right| \underbrace{\left| A(\hat{a}, \lambda) A(\hat{b}, \lambda) \right|}_{=1}$$

$$\hookrightarrow \boxed{\left| \hat{P}(\hat{a}, \hat{b}) - \hat{P}(\hat{a}, \hat{c}) \right| \leq 1 + P(\hat{b}, \hat{c})}$$

Inequality result.

Now suppose.



$$\left.\begin{array}{ll} P(\hat{a},\hat{b}) = 0 & P(\hat{b},\hat{c}) = -\frac{\sqrt{2}}{2} \\ P(\hat{a},\hat{c}) = -\frac{\sqrt{2}}{2} & \end{array}\right\} \text{QM results.}$$

↳ Then: $\left|P(\hat{a},\hat{b}) - P(\hat{a},\hat{c})\right| \leq 1 + P(\hat{b},\hat{c})$

$$\left|0 - \left(-\frac{\sqrt{2}}{2}\right)\right| \leq 1 + -\frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2} \nleq 1 - \frac{\sqrt{2}}{2}$$

→ it doesn't violate causality because we can't demand the result on one end.