Zhi Chen HW#13 QM

1) a) write down eigenstates of S_x in $|1/2, \pm 1/2\rangle$, eigenstates of S_z

From previous HW, we know: $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

we see that S_z has eigenvalues $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$ since it is only non-zero in the diagonal.

$$\begin{pmatrix} \frac{\hbar}{2}-\lambda & 0 \\ 0 & -\frac{\hbar}{2}-\lambda \end{pmatrix} = 0$$ using $\lambda_\pm$, we get eigenvectors:

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now find eigenvalues of $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\left|\begin{pmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{pmatrix}\right| = \left(\lambda^2 - \frac{\hbar^2}{4}\right) = 0 \quad , \text{ or } \lambda = \pm\frac{\hbar}{2}$$

For $\boxed{\lambda = \frac{\hbar}{2}}$:

$$\frac{\hbar}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} -a_+ + b_+ \\ a_+ - b_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $a_+ = b_+ \rightarrow v_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)$

For $\boxed{\lambda = -\frac{\hbar}{2}}$:

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_- \\ b_- \end{pmatrix} = \begin{pmatrix} a_- + b_- \\ a_- + b_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenstate of S_x in S_z eigenbasis

or $a_- = -b_- \rightarrow v_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)$

Summary:

$$v_+^{(x)} = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) \quad \text{for } \lambda = \frac{\hbar}{2}$$

$$v_-^{(x)} = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) \quad \text{for } \lambda = \frac{-\hbar}{2}$$

b) write down eigenstates of L_x in terms of $|1, m\rangle$, eigenstates of L_z.

From previous HW, we know:

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$m=1, m=0, m=-1$

Since L_z is a diagonal matrix, clearly $\vec{\lambda} = \{\lambda_+, \lambda_0, \lambda_-\} = \{\hbar, 0, -\hbar\}$

with eigen vectors $v_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $v_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $v_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Now find eigenvalues for L_x:

$$\begin{vmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} = (-\lambda)\left[\lambda^2 - \left(\frac{\hbar}{\sqrt{2}}\right)^2\right] - \frac{\hbar}{\sqrt{2}}\left[-\lambda \frac{\hbar}{\sqrt{2}}\right]$$

$$= -\lambda^3 + \lambda\left(\frac{\hbar}{\sqrt{2}}\right)^2 + \lambda\left(\frac{\hbar}{\sqrt{2}}\right)^2$$

$$= -\lambda^3 + 2\lambda\left(\frac{\hbar}{\sqrt{2}}\right)^2$$

$$= \lambda\left[2\left(\frac{\hbar}{\sqrt{2}}\right)^2 - \lambda^2\right]$$

$$\boxed{\lambda = 0, \pm\hbar}$$

For $\boxed{\lambda = \pm\hbar}$:

$$\hbar\begin{pmatrix} \mp 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \mp 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \mp 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} \mp a + \frac{1}{\sqrt{2}} b \\ \frac{a}{\sqrt{2}} \mp b + \frac{c}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \mp c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ $\pm a = \frac{b}{\sqrt{2}}$, $\pm c = \frac{b}{\sqrt{2}}$ $\Rightarrow$ let $b = \frac{1}{\sqrt{2}}$, the $a = c = \pm 1$

$$\hookrightarrow \ v_{\pm}^{(x)} = \begin{pmatrix} 1 \\ \pm\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} = |1,1\rangle \pm \frac{1}{\sqrt{2}} |1,0\rangle + |1,-1\rangle \quad \text{for } \lambda = \pm\hbar$$

For $\boxed{\lambda = 0}$:

$$\hbar\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} \frac{b}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(a+c) \\ \frac{b}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ $b = 0$, $\frac{1}{\sqrt{2}}(a+c) = 0$ or $a = -c$, let $a = \frac{1}{\sqrt{2}}$, $c = \frac{-1}{\sqrt{2}}$

$$\hookrightarrow \quad v_0^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right) \quad \text{for } \lambda = 0$$

Summary:

$$v_+^{(x)} = |1,1\rangle + \frac{1}{\sqrt{2}} |1,0\rangle + |1,-1\rangle \qquad \text{for } \lambda = \hbar$$

$$v_0^{(x)} = \frac{1}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right) \qquad \text{for } \lambda = 0$$

$$v_-^{(x)} = -|1,1\rangle + \frac{1}{\sqrt{2}} |1,0\rangle - |1,-1\rangle \qquad \text{for } \lambda = -\hbar$$

2) Spherical Harmonics: $\tilde{Y}(\tilde{\theta},\tilde{\phi}) = \langle \tilde{\hat{n}} | l, m \rangle$ in momentum space.

$$\tilde{Y}(\tilde{\theta}|\tilde{\phi}) = \underbrace{\langle \tilde{\hat{n}} | \hat{n} \rangle}\underbrace{\langle \hat{n} | l,m \rangle}$$

$$\stackrel{!}{=} \int \left(e^{-\frac{i\vec{p}}{\hbar}\cdot\vec{x}}\right)_{\text{angular}} Y_l^m(\theta,\phi)\, d\Omega$$

since $\left(e^{-i\frac{\vec{p}}{\hbar}\cdot\vec{x}}\right)_{\text{angular}} \sim Y_{l'}^{m'}(\tilde{\theta},\tilde{\phi})\, Y_{l'}^{m'}(\theta,\phi)$

$$\stackrel{!}{=} Y_{l'}^{m'}(\tilde{\theta},\tilde{\phi}) \underbrace{\int Y_{l'}^{m'}(\theta,\phi)\, Y_l^m(\theta,\phi)\, d\Omega}_{=\delta_{m'm}\delta_{l'l}}$$

$$\tilde{Y}_l^m(\tilde{\theta},\tilde{\phi}) = Y_l^m(\tilde{\theta},\tilde{\phi})$$

so $\tilde{Y}_l^m(\tilde{\theta},\tilde{\phi})$ is simply spherical Harmonics in position space but let $\theta \to \tilde{\theta}$ and $\phi \to \tilde{\phi}$

$\Rightarrow$ We can also see this by recognizing $\vec{L} = \vec{x} \times \vec{p} = \vec{x} \times -i\hbar\vec{\nabla}_x \stackrel{!}{=} -i\hbar(\vec{x} \times \vec{\nabla}_x)$ in position space.

$\Rightarrow$ In momentum space, $X_i = i\hbar \frac{\partial}{\partial p_i} = i\hbar\vec{\nabla}_p$,

so $\vec{L} = i\hbar\vec{\nabla}_p \times \vec{p} = -i\hbar(\vec{p} \times \vec{\nabla}_p)$

By comparing: $L = -i\hbar(\vec{p} \times \vec{\nabla}_p) = -i\hbar(\vec{x} \times \vec{\nabla}_x)$

which has the same form as position space but $\vec{x} \rightleftarrows \vec{p}$

Then we have $\vec{L}_x^2\, \psi(\vec{x}) \to \vec{L}_p^2\, \psi(\vec{p})$

Therefore, the solution to $\hat{L}^2$ in position space, $L^2\psi(\vec{x})$

i.e. $Y_l^m(\theta,\phi)$, must have the same form, except that we swap the angular part of $\vec{x} \to \vec{p}$.

i.e. $\theta \to \tilde{\theta}$, $\phi \to \tilde{\phi}$

So

$$\boxed{\begin{gathered}\tilde{Y}_l^m(\tilde{\theta},\tilde{\phi}) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}}\, e^{im\tilde{\phi}}\frac{1}{\sin^m\tilde{\theta}}\frac{d^{l-m}}{d(\cos\tilde{\theta})^{l-m}}(\sin\tilde{\theta})^{2l} \\ \text{and } \tilde{Y}_l^{-m}(\tilde{\theta},\tilde{\phi}) = (-1)^m\left[\tilde{Y}_l^m(\tilde{\theta},\tilde{\phi})\right]^*\end{gathered}}$$

3) $\langle Y_3^{2*}(\theta,\phi) \left(\frac{1}{2}(L_xL_y + L_yL_x)\right) Y_3^2(\theta,\phi)\rangle$

$L_x = \frac{L_+ + L_-}{2}$ $\qquad L_y = \frac{L_+ - L_-}{2i}$

$L_xL_y = \left(\frac{L_+ + L_-}{2}\right)\left(\frac{L_+ - L_-}{2i}\right) = \frac{L_+^2 - L_+L_- + L_-L_+ - L_-^2}{4i}$

$L_yL_x = \left(\frac{L_+ - L_-}{2i}\right)\left(\frac{L_+ + L_-}{2}\right) = \frac{L_+^2 + L_+L_- - L_-L_+ - L_-^2}{4i}$

$\langle Y_3^{2*} \frac{1}{8i}(L_+^2 \cancel{-L_+L_-} \cancel{+L_-L_+} - L_-^2 + L_+^2 \cancel{+L_+L_-} \cancel{-L_-L_+} - L_-^2) Y_3^2\rangle$

$= \frac{1}{4i}\langle Y_3^{2*}(L_+^2 - L_-^2) Y_3^2\rangle$

$= \frac{1}{4i} \langle Y_3^{2*} \underbrace{L_+^2 Y_3^2}_{=0}\rangle - \underbrace{\langle Y_3^{2*} \underbrace{L_-^2 Y_3^2}_{\neq Y_3^0}\rangle}_{=0}$

$\boxed{=0}$

4) $\psi(\theta,\phi) = \dfrac{3Y_1^1 + 4Y_7^3 + Y_7^1}{\sqrt{26}}$

$$L_z \psi = \frac{1}{\sqrt{26}}\left(\underset{\uparrow}{\hbar}\,(3Y_1^1) + \underset{\uparrow}{3\hbar}\,(4Y_7^3) + \underset{\uparrow}{\hbar}\; Y_7^1\right)$$

we will measure L_z with values

L_z:

$$\boxed{\begin{array}{l} ① \ \hbar \text{ with probability } \left|\frac{3}{\sqrt{26}}\right|^2 + \left|\frac{1}{\sqrt{26}}\right|^2 = \frac{10}{26} \\ ② \ 3\hbar \text{ with probability } \left|\frac{4}{\sqrt{26}}\right|^2 = \frac{16}{26} \end{array}}$$

Find $L = \sqrt{L^2}$

$$\sqrt{L^2}\,\psi = \hbar^2\left(\sqrt{1(1+1)}\,3Y_1^1 + \sqrt{7(7+1)}\,4Y_7^3 + \sqrt{7(7+1)}\,Y_7^1\right)$$

$$\overset{!}{=}\hbar^2\left(\sqrt{2}\,3Y_1^1 + \sqrt{56}\,4Y_7^3 + \sqrt{56}\,Y_7^1\right)\frac{1}{\sqrt{26}}$$

L has two possible values:

L:

$$\boxed{\begin{array}{l} ① \ \sqrt{2}\,\hbar^2 \text{ with probability : } \left|\frac{3}{\sqrt{26}}\right|^2 = \frac{9}{26} \\ ② \ \sqrt{56}\,\hbar^2 \text{ with probability : } \left|\frac{4}{\sqrt{26}}\right|^2 + \left|\frac{1}{\sqrt{26}}\right|^2 = \frac{17}{26} \end{array}}$$

5) $\psi(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \sin\phi \sin\theta$

By looking at table, $Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$

so $\psi(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \sin\theta \left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)$

$$= \frac{1}{\sqrt{2}i}\left(\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} - \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}\right)$$

$$= \frac{1}{\sqrt{2}i}\left(-Y_1^1 - Y_1^{-1}\right)$$

$$\psi(\theta,\phi) = \frac{i}{\sqrt{2}}\left(Y_1^1 + Y_1^{-1}\right)$$

a) $L_z \psi = \frac{i}{\sqrt{2}}\left((\hbar) Y_1^1 + (-\hbar) Y_1^{-1}\right)$

It will have measurement $\hbar$ and $-\hbar$ with probability $\frac{1}{2}$ for both measurement.

b) $$\langle L_x \rangle = \left\langle \frac{L_+ + L_-}{2} \right\rangle = \psi^*(\theta,\phi)\left\{\frac{1}{2}\frac{i}{\sqrt{2}}\left(0 + \sqrt{(1-(-1))(1-1+1)}\, Y_1^0 + \sqrt{(1+1)(1-1+1)}\, Y_1^0 + 0\right)\right\}$$

$$= \psi^*(\theta,\phi)\, \frac{1}{2}\frac{i}{\sqrt{2}}\left(2\sqrt{2}\, Y_1^0\right)$$

$$= \frac{-i}{\sqrt{2}}\left(Y_1^1 + Y_1^{-1}\right)^* i\, Y_1^0$$

$$\boxed{\langle L_x \rangle = 0}$$

c) $\langle L^2 \rangle = \psi^* L^2 \psi = \psi^* \frac{i}{\sqrt{2}} \hbar^2 \left(\sqrt{(1)(1+1)}\, Y_1^1 + \sqrt{(1)(1+1)}\, Y_1^{-1} \right)$

$$\stackrel{!}{=} \psi^* \frac{i}{\sqrt{2}} \hbar^2 \sqrt{2} \left(Y_1^1 + Y_1^{-1} \right)$$

$$\stackrel{!}{=} \sqrt{2}\hbar^2 \psi^* \psi$$

$$\boxed{\langle L^2 \rangle \stackrel{!}{=} \sqrt{2}\hbar^2}$$