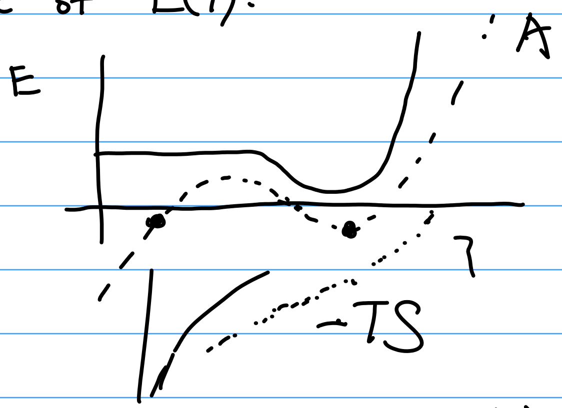
Minimize Free energy:

$$A = E - TS$$

consider an order parameter, ξ

$$A = E(\xi) - TS(\xi)$$

An example of $E(\xi)$:



two ξ giving the same minimum A.

Review: Classical gas, non interacting

$$Q = \frac{1}{N!} q_1^N = \frac{1}{N!}\left(\frac{V}{\lambda_{th}^3} q_{int}\right)^N$$

$$A = -k_BT\ln Q = Nk_BT\ln\left(\frac{\rho\lambda_{th}^3}{q_{int}}\right) - Nk_BT$$

For phase transition, chemical potential stays same.

Use Gibbs-Free energy:

$$G = \mu N = A + pV$$

$$= A + N k_B T$$

$$G = N k_B T \ln\left(\frac{\rho \lambda_{th}^3}{q_{int}}\right)$$

Now do mean-field theory:

$$A(N, T, V) \rightarrow A(N, T, \underset{\substack{\uparrow \\ \text{excluded} \\ \text{volume.}}}{V - Nb}) + \underset{\substack{\uparrow \\ \text{Mean-Field.}}}{E_{MF}}$$

$$E_{MF} \propto \rho^2 V = N\rho$$

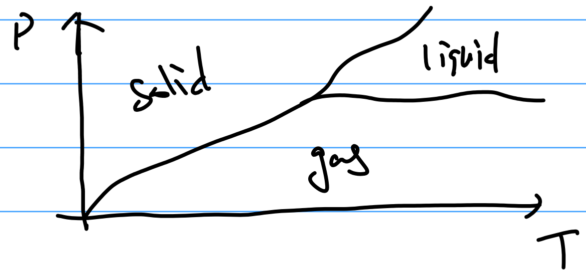
$$\boxed{E_{MF} = -N\rho a}$$

$$\boxed{\begin{aligned} A &= N\left[k_B T \ln\left(\frac{\rho \lambda_{th}^3}{q_{int}}\right) - k_B T - \rho a\right] \\ p &= -\rho^2 a + \rho \frac{k_B T}{1 - \rho b} \end{aligned}}$$

For van-der-Waals gas

$$\boxed{\mu = \frac{G}{N} = \frac{A + pV}{N} \Rightarrow \mu = k_B T \ln\left(\frac{\rho \lambda_{th}^3}{1 - \rho b}\right) - 2\rho a + k_B T \frac{\rho b}{1 - \rho b}}$$

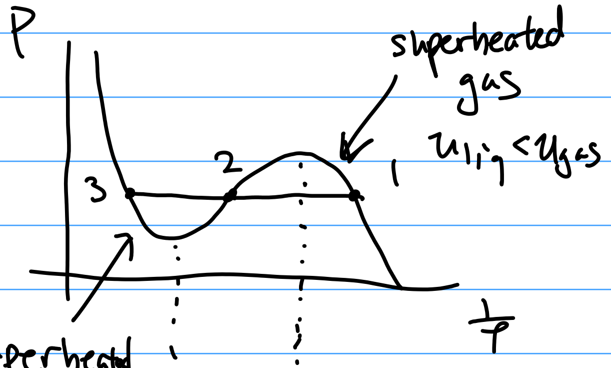
This phase transition has a latent heat:



Heating but stays at constant temp.

To find the latent heat:

$$\frac{dP_0}{dT} = \frac{S_2 - S_1}{V_2 - V_1} = \frac{L}{T\Delta V}$$



$u_{liq} < u_{gas}$

Superheated liquid

$u_{liq} > u_{gas}$

metastable

unstable, So it goes through line rather than curve.

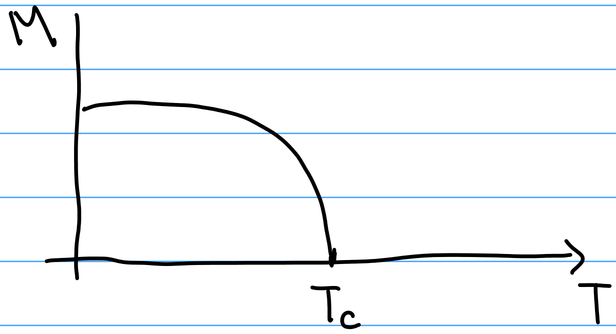
$$\int_1^3 dG = 0$$

$$\int_1^3 \cancel{-SdT} + Vdp + \cancel{\mu dN} = 0$$

$$\int_1^3 dV\left(p - p_0(T)\right) = 0$$

Continuous Phase Transition: No latent heat.

1st order Phase Transition: Has Latent heat



$$\partial A = -SdT - mdH$$

$$\langle M \rangle = -\frac{\partial A}{\partial H} = \text{continuous}$$

↑ applied field

$$\chi = \frac{\partial \langle m \rangle}{\partial H} = -\left(\frac{\partial^2 A}{\partial H^2}\right)_{N,T} \xrightarrow{T=T_c} \infty$$

Near critical point

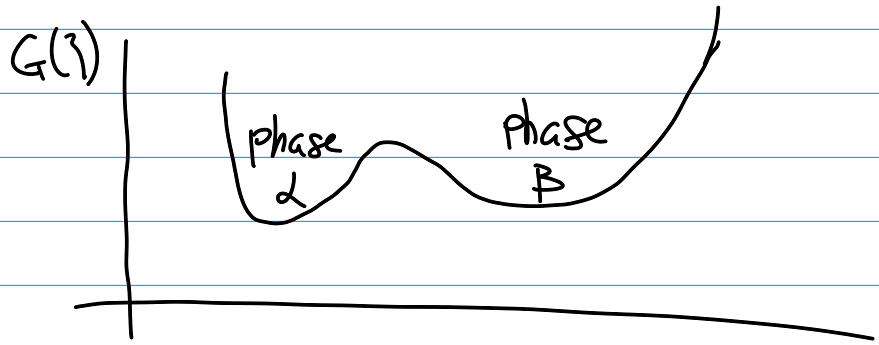
$$\frac{\partial M}{\partial H} = \chi \propto \frac{1}{\left(1-\frac{T}{T_c}\right)^{\gamma}} = \frac{1}{\tau^{\gamma}} \qquad \gamma \cong 1.3$$

$$C \propto \frac{1}{\left(1-\frac{T}{T_c}\right)^{\alpha}} \qquad \alpha \cong \frac{1}{8}$$

$$\eta \propto \tau^{\beta} \qquad \beta \approx \frac{1}{3}$$

$$\alpha + 2\beta + \gamma = 2$$

Landau Theory:



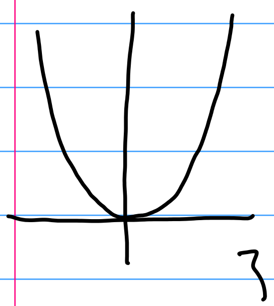
Near critical point:

$$G(\xi) = G_0 + G_2 \xi^2 + G_4 \xi^4$$

depend on T

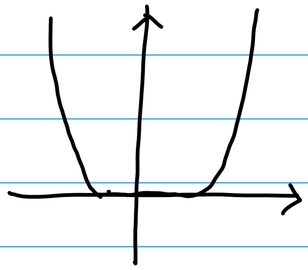
$T > T_c$

$G_2, G_4 > 0$



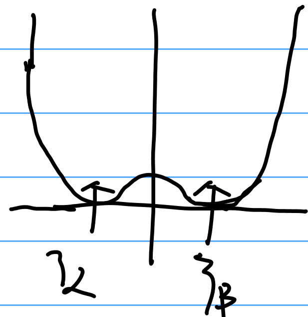
$T = T_c$

$G_2 = 0$



$T < T_c$

$G_2 < 0, \quad G_4 > 0$



symmetry breaking

External Field: $G \to G - uH\langle m \rangle$

applied field

$$g = \frac{G}{N} \Rightarrow \Delta g = \frac{G - G_0}{N} = -a\tau\xi^2 + \frac{1}{2} b \xi^4 - ph\xi + f(|\vec{\nabla}\xi|^2)$$

$\tau = 1 - \frac{T}{T_c}$

$ph \simeq uH$

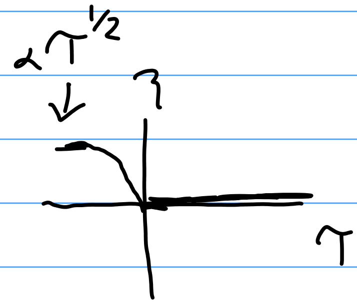
magnetic field

kinetic

Take $h=0$, $f=0$

$$\frac{\partial g}{\partial \eta} = 0 = -2a\tau\eta + 2b\eta^3$$

For $T < T_c$: $\eta = \left(\frac{a\tau}{b}\right)^{1/2}$



Since $\tau = 0$ for $T > T_c$ $\quad$ $T > T_c$: $\eta = 0$

Since $\eta \propto \tau^\beta$ $\quad$ $\boxed{\text{Find } \beta = \frac{1}{2}}$

<u>If $h \neq 0$</u>:

ignore higher order.

$$\frac{\partial g}{\partial \eta} = 0 = -2a\tau\eta + \cancel{2b\eta^3} - fh$$

$$\eta = \frac{-fh}{2a\tau}$$

Since $\chi \propto \frac{1}{\tau^\gamma}$ $\quad$ Find $\gamma = 1$

Ising Model:

⇒ Each particle is fixed on a lattice. (No Motion)

⇒ Each lattice can only be in one of two states. ↑↓

⇒ Each spin only interacts (parwise) with nearest neighbor.

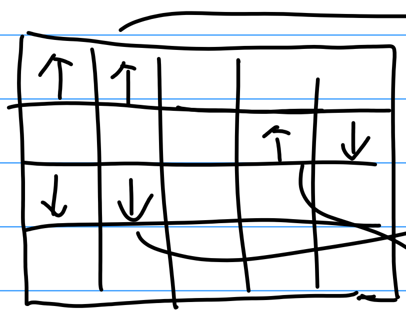
$$\mathcal{H} = -J \sum_{\substack{(ij) \\ \text{pair}}} S_i S_j - \sum_{i=1}^{N} H \mu S_i$$

$S_{ij} = \pm 1$, classical spin

J = Interaction Strength

H = external field

μ = Magnetic moment



$-J(1)(1) = -J$

$-J(-1)(-1) = -J$

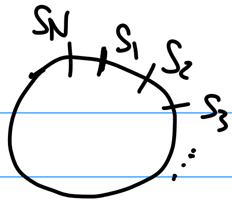
$\Big\}$ lower energy

$-J(-1)(1) = +J$ → Higher Energy

$$Q(N, \beta, H) = \sum_{S_1=\pm 1} \cdots\cdots \sum_{S_N=\pm 1} \exp\{\underbrace{\beta \mu H}_{\equiv h} \sum S_i + \underbrace{\beta J}_{\equiv K} \sum_{\substack{ij \\ \text{pair}}} S_i S_j\}$$

For 1D: $\displaystyle\sum_{ij} S_i S_j \Rightarrow \sum_i S_i S_{i+1}$

Consider 1D model:



$$Q = \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \exp\left[K S_i S_{i+1} + \tfrac{1}{2} h (S_i + S_{i+1})\right]$$

↓ 4 possibilities in the end.

If $S_i = 1$, $S_{i+1} = 1$, $Q \propto e^{K+h}$

If $S_i = -1$, $S_{i+1} = 1$, $Q \propto e^{-K}$

If $S_i = 1$, $S_{i+1} = -1$, $Q \propto e^{K}$

If $S_i = -1$, $S_{i+1} = -1$, $Q \propto e^{K-h}$

Transfer Matrix:

$$P_{i,j} = \begin{array}{c} \begin{array}{cc} (1,1) & (-1,1) \end{array} \\ \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \\ \begin{array}{cc} (-1,1) & (-1,-1) \end{array} \end{array}$$

$$Q = \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \underbrace{\langle S_1 | \hat{P} | S_2 \rangle \langle S_2 | \hat{P} | S_3 \rangle \cdots \langle S_{N-1} | \hat{P} | S_N \rangle \langle S_N | \hat{P} | S_1 \rangle}_{\text{Diagonal}}$$

$$\mathrm{Tr}\left([P]^N\right) = \lambda_1^N + \lambda_2^N$$

Get : $\lambda = e^K \cosh(h) \pm \left[e^{-2K} + e^{2K} \sinh^2(h)\right]^{1/2}$

For $N \gg 1$,

then just consider the larger λ, for λ^N:

If $\lambda_+ \gg \lambda_-$, then

$$Q \approx \lambda_1^{\,N}$$

$$\frac{1}{N}\ln Q = \ln \lambda_1$$

$$= \ln\left[e^{K}\cosh(h) + (e^{-2K} + e^{2K}\sinh^2(h))^{1/2}\right]$$

$$A = -NJ - Nk_BT\ln\left[\cosh(h) + (e^{-4K} + \sinh^2(h))^{1/2}\right]$$

<u>For $h=0$:</u>

$$Q = (2\cosh(K))^N$$

$$A = -Nk_BT\ln(2\cosh(K))$$

$$\langle M\rangle = -\left(\frac{\partial A}{\partial H}\right)_T \qquad dE = TdS - MdH$$

$$dA = -SdT - MdH$$

then

$$\langle M\rangle = \frac{N\mu\sinh(h)}{\left[e^{-4K} + \sinh^2(h)\right]^{1/2}}$$

Spontaneous Magnetization

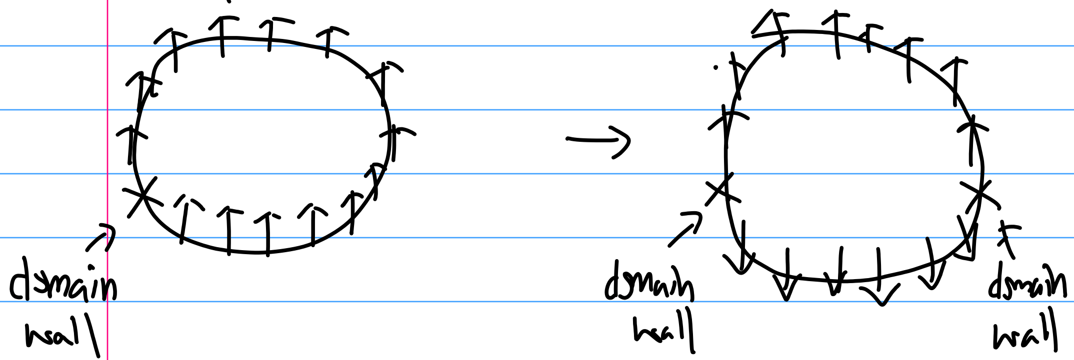
<u>$M(H=0, T)$</u>

as $\lim_{h\to 0} \dfrac{N\mu\sinh(h)}{e^{-2K}} \to 0$ unless $T \to 0$,

so no phase transition at finite T.

If $J=0$: $\langle M \rangle = N\mu \tanh(\beta\mu H)$

No phase transition, why?:



$$\Delta E = 2J + 2J$$

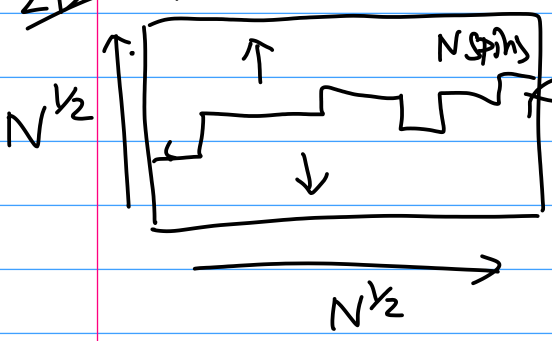
choice of putting wall

$$\Delta S = k_B \ln\big(N(N-1) \big)$$

$$\Delta A = \Delta E - T\Delta S$$
$$= 4J - 2k_BT \ln N$$

For $\Delta A > 0$, $T_C = \dfrac{2J}{k_B \ln N} \xrightarrow{\text{as } N\to\infty} 0$

2D: peierl's Argument:



domain wall has strength (Length) $L \sim N^{1/2}$

$\Delta E = 2JL$, 2 spin neighboring wall.

$2N^{1/2}$ starting point, 3 choice after.

$\Omega \sim 2N^{1/2} 3^L$

$L \propto N^{1/2}$

$$\Delta A = 2JL - k_B T \ln (2N^{1/2} 3^L)$$

$$\simeq \underbrace{2JL}_{N^{1/2}} - \underbrace{k_B T L \ln(3)}_{N^{1/2}} - \cancel{\frac{1}{2} k_B T \ln(N)} - \cancel{k_B T \ln 2}$$

For $N \gg 1$

$\underline{\Delta A = 0}$

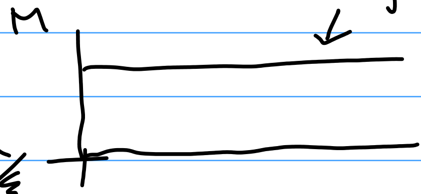
$$T_c \approx \frac{2J}{k_B \ln 3} = \frac{J}{k_B}(1.82)$$

exact: $T_c = 2.269\, J/k_B$

going above: disorder
going below: ordered

all spin up or down.

long range.



Correlation Function:

ensemble average

$$g(r) = \langle S_k S_{k+r} \rangle \sim e^{-r/\xi}$$

correlation length.

$$\uparrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow$$

$k \qquad\qquad k+r$

→ Higher Temp → more disorder → shorter correlation length.

→ lower Temp → less disorder → longer correlation length

For $r=1$: Nearest Neighbor Interaction:

$H=0$:

trick $J \to J_i$

$$Q = \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \exp\{\beta J_i S_i S_{i+1}\}$$

$$= \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \prod_i \exp\{\beta J_i S_i S_{i+1}\}$$

$$= \prod_i \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \exp\{\beta J_i S_i S_{i+1}\}$$

At end-point:

$$\sum_{S_N=\pm 1} \exp\{\beta J_{N-1} S_{N-1} S_N\} = e^{\beta J_{N-1} S_{N-1}} + e^{-\beta J_{N-1} S_{N-1}}$$

$$= 2\cosh(\beta J_{N-1} S_{N-1})$$

Since $S_{N-1} = \pm 1$ and cosh is even, we can ignore.

Then $$Q = \prod_{i=1}^{N-1} 2\cosh(\beta J_i)$$

$$= 2^N \prod_{i=1}^{N-1} \cosh(\beta J_i)$$

$$\langle S_i S_{i+1} \rangle = \frac{\sum S_i S_{i+1} \, e^{\beta[\sum_i J_i S_i S_{i+1}]}}{Q}$$

$$= \frac{1}{Q} \underbrace{\left(\frac{1}{\beta} \frac{\partial}{\partial J_k} Q \right)}_{S_i S_{i+1} Q}$$

$$= \frac{1}{\beta} \frac{\partial}{\partial J_k} \ln Q$$

$$\ln Q = N \ln 2 + \sum_{i=1}^{N-1} \ln(\cosh(\beta J_i))$$

$$\langle S_k , S_{k+1} \rangle = \frac{1}{\cancel{\beta} \cosh(\beta J_k)} \sinh(\beta J_k) \cancel{\beta}$$

$$= \tanh(\beta J_k)$$

$$\langle S_k S_{k+r} \rangle = \langle S_k \underbrace{(S_{k+1} S_{k+1})}_{(\pm 1)(\pm 1)=1} (S_{k+2} S_{k+2}) \cdots (S_{k+r-1})(S_{k+r-1}) S_{k+r} \rangle$$

$$= \langle (S_k S_{k+1})(S_{k+1} S_{k+2}) \cdots (S_{k+r-1} S_{k+r}) \rangle$$

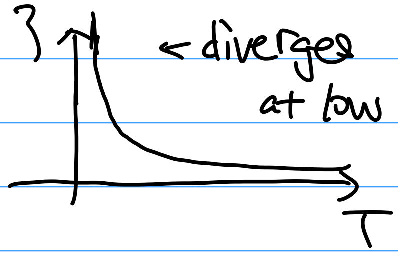
$$= \langle S_k S_{k+1} \rangle \langle S_{k+1} S_{k+2} \rangle \cdots \langle S_{k+r-1} S_{k+r} \rangle$$

$$= \langle S_k S_{k+1} \rangle^r$$

$$\boxed{\begin{array}{l} \text{then } g(r) = \langle S_k S_{k+r} \rangle = \langle S_k S_{k+1} \rangle^r = \tanh(\beta J)^r \\ \text{since } g(r) \propto e^{-r/\xi} \quad \text{then } \xi = \dfrac{1}{\ln\left(\dfrac{1}{\tanh(\beta J)}\right)} \end{array}}$$

For $\beta J \gg 1$, $k_B T \ll J$

taylor: $\xi \to \frac{1}{2} e^{2\beta J} \gg 1$



< diverges at low T.

Magnetic Susceptibility:

$$\chi = \frac{\partial \langle M \rangle}{\partial H} = -\left(\frac{\partial^2 A}{\partial H^2}\right)_T = \beta\Big(\underbrace{\langle M^2 \rangle - \langle M \rangle^2}_{(\delta M)^2}\Big)$$

$$\hookrightarrow = \mu^2 \beta \sum_i \sum_j \langle S_i S_j \rangle - \langle S \rangle^2$$

$$\stackrel{!}{=} N \mu^2 \beta \underbrace{\sum g(r)}_{\approx \int d^3 r \, g(r)}$$

$$g(r) \propto e^{-r/\xi}$$

at $\sim T_c$, little H gives large $\langle M \rangle$, or $\chi \sim \infty$

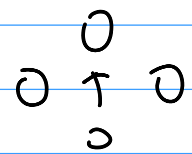
Mean-Field Theory for Ising Model:

$$\mathcal{H} = -\mu H \underbrace{\sum_i S_i}_{\substack{N\langle S_i\rangle \\ Nm}} - J \sum_{\substack{ij \\ \text{pairs}}} S_i S_j$$

↳ $$\mathcal{H}_{MF} = -\mu \sum_i H_i(m)\, S_i$$

↑ effective magnetic field

$$H_i = -\frac{1}{\mu}\frac{\partial \mathcal{H}}{\partial S_i}$$



$\Rightarrow q=4$ for 2D.

$$\stackrel{!}{=} H + \frac{J}{\mu}\underbrace{\sum_{j=1}^{q} S_j}_{\text{Mean-Field: } q\langle S_i\rangle}$$

Then $H_i = \langle H_i\rangle = H + \frac{J}{\mu} q m$ ← Mean-Field approximation

$$m = \langle S_i\rangle = \sum_{S_i=\pm 1} \frac{1}{Q_1} e^{-\beta(-\mu H_i S_i)} S_i$$

↳ $$Q_1 = \sum_{S_i=\pm 1} e^{-\beta(-\mu H_i S_i)}$$

Self-consistent Eq: $m = \langle S_i \rangle = \tanh(\beta\mu H + \beta J q m)$

For $H=0$, is $m(H=0,T) \neq 0$?

↳ $m = \tanh(\beta J q m)$, with critical point $q\beta J = 1$

$T_c = q\frac{J}{k_B}$

Phase transition when $q\beta J > 1$.

Dimension	T_c exact (Ising)	MFT
1	0	$2\ J/k_B$
2	$2.269\ J/k_B$	$4\ J/k_B$
3	$4.513\ J/k_B$	$6\ J/k_B$

In Mean-Field Theory:

$$\frac{A}{N} = \frac{k_BT}{2}\ln\left(\frac{1-m^2}{4}\right) + \frac{m^2 J q}{2}$$

as $T > T_c$: $\frac{A}{N} = \frac{k_BT}{2}\ln\left(\frac{1}{4}\right) = -k_BT\ln(2)$

$S = Nk_B\ln(2) \Rightarrow C \overset{?}{=} 0$ which is wrong.