$$E = E_1(\{x_1\}) + E_2(\{x_2\}) + \cdots\cdots$$

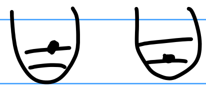
$\uparrow$ quantum #'s of system 1 $\qquad$ $\uparrow$ quantum #'s of system 2

$$e^{-\beta E} = e^{-\beta(E_1+E_2+\cdots\cdots)} = e^{-\beta E_1} e^{-\beta E_2} \cdots$$

$$Q = \sum_{\nu} e^{-\beta E_\nu} = \left(\sum_{\{x_1\}} e^{-\beta E_1}\right)\left(\sum_{\{x_2\}} e^{-\beta E_2}\right) \cdots\cdots$$

If particles are distinguishable:

$|v_1 = 0, v_2 = 1\rangle$



$|v_1 = 1, v_2 = 0\rangle$

However with

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{1s}(1)\rangle |\psi_{2s}(2)\rangle \pm |\psi_{2s}(1)\rangle |\psi_{1s}(2)\rangle \right.$$

$\hat{P}_{12} |\psi\rangle \Rightarrow$ same observable.

$\hat{P}_{12} \rightarrow$ interchange 2 identical particles

$$\langle o \rangle = \left(\langle\phi| e^{-i\phi} \hat{o} e^{i\phi} |\phi\rangle \right)$$

So $\hat{P}_{12} |\psi\rangle = e^{i\phi} |\psi\rangle$

if interchange twice, get same thing.

but $\hat{P}_{12}^{2} |\psi\rangle = |\psi\rangle$

so $\hat{P}_{12} = e^{i2\phi} = 1$

so $\phi = 0, \pi.$

Boson Fermion.

Fermions:

anti-symmetric under change.

$$\Psi(1,2,3,\cdots,N) = -\Psi(2,1\cdots N)$$

If 2, 1 are the same system, suppose x, x

$$\Psi(x,x\cdots) = -\Psi(x,x\cdots)$$

So $\Psi = 0$

or $\Psi = \frac{1}{\sqrt{2}}\left(|\psi_a(1)\rangle |\psi_b(2)\rangle - |\psi_b(1)\rangle |\psi_a(2)\rangle\right) \overset{\text{if } a=b}{=} 0$

Pauli-Exclusion Principle: No two fermions can be in the same state.

Bosons:

symmetric under change.

$$\Psi(1,2,3\cdots,N) = +\Psi(2,1,\cdots N)$$

⇒ Occupation number: n_j, # of particles with a corresponding wave function ϕ_j.

⇒ A energy eigenstate ν can be described by a set of occupation #s.

$$\nu = (n_1, n_2 \cdots)$$

Bosons

occupation numbers

orbitals = single particle state.

$$\Psi_N = \sqrt{\frac{n_1!\, n_2! \cdots}{N!}} \sum_{P_{ij}} \hat{P}_{ij}\, \phi_1(1) \cdots\cdots \phi_N(N)$$

all permutations

Fermions

$$\Psi_N = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(1) & \phi_1(2) & \cdots & \phi_1(N) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \phi_N(1) & \cdots & \cdots & \phi_N(N) \end{vmatrix}$$

Slater determinant

$$E_\nu = n_1 \varepsilon_1 + n_2 \varepsilon_2 + \cdots$$

State occupancy (# of particles occupying n_1)

single particle energy

Basis of fock state: $|n_1, n_2, n_3, n_4 \cdots \rangle$

$$Q(N,V,T) = \sum_{\{n_j\}} e^{-\beta \sum_j n_j \varepsilon_j} \quad \text{such that } \sum n_j = N$$

Difficult to do so instead:

$$\boxed{\Xi(\mu, V, T) = \sum_{n_1} e^{-\beta(n_1 \varepsilon_1 - \mu n_1)} \sum_{n_2} e^{-\beta(n_2 \varepsilon_2 - \mu n_2)} \sum_{n_3} \cdots}$$

then find $\langle N \rangle$ and dial μ such that

$\langle N \rangle = N \qquad \langle N \rangle = z\left(\frac{\partial}{\partial z} \ln \Xi\right)_{\beta, V} \qquad z = e^{\beta\mu}$

fugacity.

If $\hat{H} = \hat{H}_1 + \hat{H}_2 + \cdots$

e.g. $\frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \cdots$

This solves TISE:

$$\hat{H}\Psi_N = \left(\hat{H}_1 + \hat{H}_2 \cdots + \hat{H}_N\right)\phi_1(1)\cdots\phi_N(N)$$

$$\overset{!}{=} [\phi_2(2)\cdots\phi_N(N)]\underbrace{\hat{H}_1\phi_1(1)}_{\varepsilon_1\phi_1} + [\phi_1(1)\cdots\phi_N(N)]\underbrace{\hat{H}_2\phi_2(2)}_{\varepsilon_2\phi_2}$$

$$\hat{H}\Psi_N = (\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_N)(\phi_1(1)\phi_2(2)\cdots\phi_N(N))$$

$$\overset{!}{=} (\varepsilon_1 + \varepsilon_2 \cdots \varepsilon_N)\Psi_N$$

But it does not satisfy non-commital requirement:

$$[\hat{H}, \hat{P}_{ij}] = 0$$

permutation operator

$$P_{ij}\Psi(\ldots i, \ldots, j \ldots) = \Psi(\ldots, j, \ldots, i \ldots)$$

Last time:

since $P_{ij}^2\Psi_N = \Psi_N$

$$\hat{P}_{ij}\Psi_N = \lambda\Psi_N$$

$e^{i\theta}$

$\lambda^2 = 1 \Rightarrow \theta = 0, \pi$

$\lambda = +1$ bosons $\theta = 0$

$\lambda = -1$ fermions $\theta = \pi$

$$Q = \sum_{|\Psi\rangle} \langle \Psi_N | e^{-\beta \hat{H}} | \Psi_N \rangle = \mathrm{Tr}(e^{-\beta \hat{H}})$$

Fock state: $|\Psi_N\rangle \to |n_1 \dots n_N\rangle$

$\uparrow$ occupation #:

Fermions: $n_i = 0, 1$

Bosons: $n_i = 0 \dots N$

Grand Canonical:

# of particles having energy ε_j

$$\Xi = \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \sum_{n_3}^{\infty} \cdots e^{-\beta(\underbrace{\sum_j n_j \varepsilon_j}_{E_\nu} - \mu \underbrace{\sum_j n_j}_{N_\nu})} = \sum_{\nu} e^{-\beta(E_\nu - \mu N_\nu)}$$

$\nwarrow$ some configuration of N-particles

$$\stackrel{!}{=} \sum_{n_1=0}^{\infty} e^{-\beta(n_1 \varepsilon_1 - \mu n_1)} \sum_{n_2=0}^{\infty} e^{-\beta(n_2 \varepsilon_2 - \mu n_2)} \sum_{n_3=0}^{\infty} \cdots\cdots$$

$$\stackrel{!}{=} \prod_j \left[\sum_{n_j=0}^{\infty} e^{-\beta(n_j \varepsilon_j - \mu n_j)} \right]$$

$$\ln \Xi = \sum_{\substack{\text{single} \\ \text{particle} \\ \text{states}, j}} \ln\left(\sum_{n_j} e^{-\beta(\varepsilon - \mu) n_j} \right)$$

$\{n_j\} = 0, 1$ for fermion

$= 0, 1, 2, 3 \dots$ for boson

$\nwarrow$ sum over all occupation # for a single particle state.

$\nwarrow$ sum over

For Bosons: $n: 0 \to \infty$

$$\sum_{n_j=0}^{\infty} e^{-\beta(\varepsilon - \mu) n_j} = \frac{1}{1 - e^{-\beta(\varepsilon_j - \mu)}}$$

$$\ln \Xi = -\sum_{\substack{j \\ \text{single} \\ \text{particle} \\ \text{states}}} \ln(1 - e^{-\beta(\varepsilon_j - \mu)})$$

$$\sum_{\nu} e^{-\beta(E_\nu - \mu N_\nu)}$$ $\leftarrow$ ν configuration of N-particle system.

split to

$$\hookrightarrow \sum_{\nu} e^{\sum_{j}^{\infty}\left(-\beta(\varepsilon_{j,\nu} n_{j,\nu} - \mu n_{j,\nu})\right)}$$

$$\hookrightarrow \sum_{\nu} \prod_{j(\nu)}^{\infty} e^{-\beta(\varepsilon_{j,\nu} n_{j,\nu} - \mu n_{j,\nu})}$$

since # of particles is not fixed

$$\hookrightarrow \prod_{j}^{\infty} \sum_{\nu}$$

For Fermions:

$$\sum_{n_j=0}^{1} e^{-\beta(\varepsilon_j - \mu)n_j} = 1 + e^{-\beta(\varepsilon_j-\mu)}$$

$$\ln \Xi = \sum_j \ln\left(1 + e^{-\beta(\varepsilon_j-\mu)}\right)$$

In the end:

$$\ln \Xi = \mp \sum_{j} \ln\left(1 \mp e^{-\beta(\varepsilon_j - \mu)}\right)$$

single particle states

$-$ Bosons

$+$ Fermions

$$\langle n_j \rangle = \sum_\nu n_j \frac{e^{-\beta(E_\nu - \mu N_\nu)}}{\Xi}$$

$$= \sum_{n_j} n_j \, e^{-\beta(\varepsilon_j - \mu)n_j} \frac{1}{\Xi} \sum_{\substack{\text{all}\\ \text{other}\\ \text{states}}}$$

$$\langle n_j \rangle = \frac{-\partial}{\partial(\beta\varepsilon_j)} \ln \Xi$$

$$\langle n_j \rangle = \frac{1}{e^{\beta(\varepsilon_j-\mu)} \mp 1}$$

$-$ for bosons

$+$ for fermions

$N \to \langle N \rangle = \sum_j \langle n_j \rangle$ dial μ

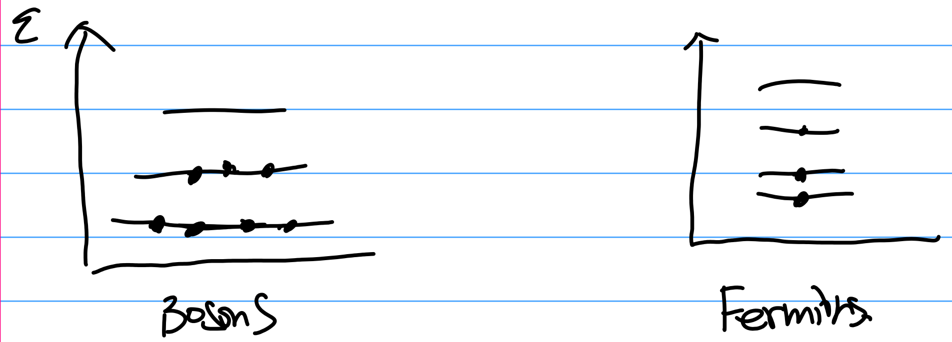
Bosons:

$$\Xi = \prod_j \sum_{n_j=0}^{\infty} \underbrace{\left(e^{-\beta(\varepsilon_j - \mu)}\right)}^{n_j}$$

$r < 1$ for geometric series to converge.
$\hookrightarrow \varepsilon - \mu > 0$

So $\mu < \varepsilon_j$ for all ε.

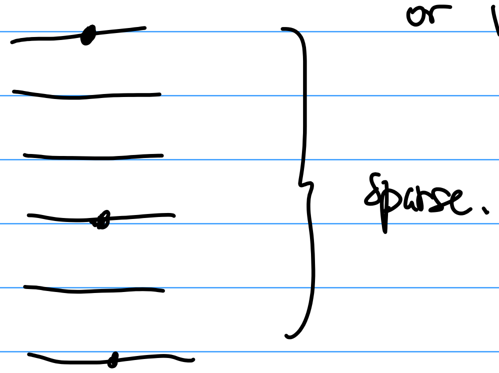
So for $\varepsilon_{min} = 0$, μ must be negative.



in low T, high ρ in low T, high ρ

Classical limit: $\rightarrow$ $\langle n_j \rangle \ll 1$
or when $\mu \ll 0$

$\mu = \frac{H}{N} - \frac{TS}{N}$



At classical limit $\mu \ll \varepsilon_j$ (0)

$$\langle n \rangle = \frac{1}{e^{-\beta\mu} e^{\beta\varepsilon_j} \mp 1} \rightarrow e^{\beta\mu} \underbrace{e^{-\beta\varepsilon_j}}_{\propto \text{ Boltzmann factor.}}$$

Dial μ:

$$N = \langle N \rangle = \sum_j \langle n_j \rangle$$

$$\stackrel{!}{=} e^{\beta\mu} \underbrace{\sum_j e^{-\beta\varepsilon_j}}_{q_1 \text{ : \# of thermally accessible states}}$$

$$e^{\beta\mu} = \frac{N}{q_1}$$

$$\hookrightarrow \mu = k_B T \ln\left(\frac{N}{q_1}\right)$$

$\hookrightarrow$ require $\frac{N}{q_1} \ll 1$

so that μ is very negative

μ : large negative #.

$$= -k_B T \ln\left(\frac{q_1}{N}\right)$$

classical limit : $q_1 \gg N \Rightarrow \mu$ large and negative.

Find the partition function: Q:

$$G = \mu\langle N\rangle = A + pV$$

$$\mu = k_BT\ln\frac{N}{q_1} \qquad A = -k_BT\ln Q \qquad pV = k_BT\ln\Xi:$$

$$\ln\Xi \doteq \mp\sum_j \ln\left(1\mp e^{-\beta(\epsilon_j-\mu)}\right)$$

$$= \mp\sum_j \ln\left(1\mp e^{-\beta\epsilon_j}e^{\beta\mu}\right)$$

Taylor expand

$$\simeq \sum_j e^{-\beta\epsilon_j}e^{\beta\mu}$$

$$\simeq \langle N\rangle$$

↳ $$Nk_BT\left(\ln N-\ln q_1\right) = -k_BT\ln Q + k_BTN$$

↳ $$\ln Q = \ln q_1^{\,N} - (N\ln N - N)$$

$$\boxed{Q = \frac{1}{N!}q_1^{\,N}}$$

identical particles.

$$Q \overset{0}{\doteq} \sum_{\substack{\text{state}\\\text{of particle 1}}}\sum\sum e^{-\beta(\epsilon_1+\epsilon_2+\cdots)}$$ ← gets overcount by $N!$

Density of States:

$$q_1 = \sum_j e^{-\beta \varepsilon_j} \to \int d\varepsilon\, g(\varepsilon) e^{-\beta \varepsilon}$$