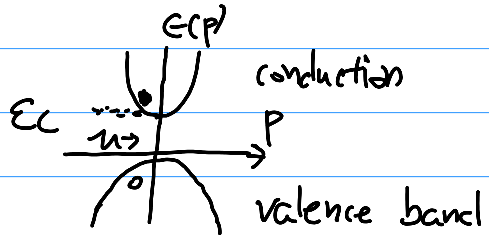
Zhi Chen HW#9



31) Electron and Holes:

consider a system with a single valence band and a single conduction band!

$$\varepsilon_V = -\frac{p^2}{2m_V} \qquad \varepsilon_c = \frac{p^2}{2m_c} + \Delta$$

consider $\mu \gg k_B T$, $(\Delta - \mu) \gg k_B T$

a) Assume electron gas is ideal (non-interacting)
find $\rho_e = N_e^{cond}/V$ in conduction band.
and # of holes $\rho_h = N_h^{val}/V = (N - N_e^{val})/V$
as a function of T and μ. N is the total # of e^-.

From HW#5, we know $g(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon$

then $N_e = \int d\varepsilon \, g(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}+1} 2$ ← spin

$$= \int_{\varepsilon_c}^{\infty} d\varepsilon \, \underbrace{2\frac{V}{4\pi^2}\left(\frac{2m_c}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon-\varepsilon_c}}_{g(\varepsilon)} \frac{1}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$$

Since $\Delta - \mu \gg k_B T$

$$\langle n \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \approx e^{-\beta(\varepsilon-\mu)}$$

let $\varepsilon' = \varepsilon - \varepsilon_c$ or $\varepsilon = \varepsilon' + \varepsilon_c$ or $d\varepsilon = d\varepsilon'$

then $\Rightarrow \displaystyle\int_{\varepsilon_c}^{\infty} d\varepsilon' \frac{V}{4\pi^2}\left(\frac{2m_c}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon'} \quad e^{-\beta(\varepsilon' + \varepsilon_c - \mu)} \; 2$

$$\hookrightarrow = \frac{V}{4\pi^2}\left(\frac{2m_c}{\hbar^2}\right)^{3/2}\frac{\sqrt{\pi}}{2}(k_B T)^{3/2} e^{-\beta(\varepsilon_c - \mu)} \; 2$$

$$N_e \overset{!}{=} 2V\left(\frac{m_c k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-\beta(\varepsilon_c - \mu)}$$

then $\boxed{\rho_e = \frac{N_e}{V} = 2\left(\frac{m_c k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-\beta(\varepsilon_c - \mu)}}$

Since $N_h^{val} = N - N_e^{val} \Rightarrow$ then probability to # of holes is $1 - \langle n\rangle$ i.e. probability not occupied by electron

so $$N_h^{val} = \int_{-\infty}^{\varepsilon_h} d\varepsilon \, 2g(\varepsilon)(1 - \langle n\rangle) \, d\varepsilon$$

$$\overset{!}{=} \int_{-\infty}^{\varepsilon_h} d\varepsilon \, 2g(\varepsilon)\left(1 - \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right) d\varepsilon$$

Since for valence band, ε is negative, then $\varepsilon - \mu$ is very negative, or $e^{\beta(\varepsilon-\mu)} \ll 1$

by applying Taylor: $\dfrac{1}{e^{\beta(\varepsilon-\mu)}+1} \approx 1 - e^{\beta(\varepsilon-\mu)}$

then $$N_h = \int_{-\infty}^{\varepsilon_h} d\varepsilon \; 2 g(\varepsilon) \left(1 - \langle n \rangle\right) d\varepsilon$$

$$= \int_{-\infty}^{\varepsilon_h} d\varepsilon \; 2 \underbrace{\frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon_h - \varepsilon}}_{g(\varepsilon)} \quad \left[1 - \left(1 - e^{\beta(\varepsilon - \mu)}\right)\right]$$

$$\stackrel{!}{=} \int_{-\infty}^{\varepsilon_h} d\varepsilon \; 2 \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon_h - \varepsilon} \quad e^{\beta(\varepsilon - \mu)}$$

let $\varepsilon' = \varepsilon_h - \varepsilon$ or $\varepsilon = \varepsilon_h - \varepsilon'$ then $d\varepsilon' = -d\varepsilon$

$$\hookrightarrow \int_{\infty}^{0} - d\varepsilon' \; 2 \frac{V}{4\pi^2} \left(\frac{2m_v}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon'} \quad e^{\beta(\varepsilon_h - \varepsilon' - \mu)}$$

$$\hookrightarrow \int_{0}^{\infty} d\varepsilon' \, 2 \frac{V}{4\pi^2} \left(\frac{2m_v}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon'} \; e^{-\beta\varepsilon'} \; e^{\beta(\varepsilon_h - \mu)}$$

$$N_h^{val} = 2 \frac{V}{4\pi^2} \left(\frac{2m_v}{\hbar^2}\right)^{3/2} \frac{\sqrt{\pi}}{2} e^{\beta(\varepsilon_h - \mu)} \quad (k_B T)^{3/2}$$

then $$\boxed{p_h = \frac{N_h^{val}}{V} = 2 \left(\frac{m_v k_B T}{2\pi\hbar^2}\right)^{3/2} e^{\beta(\varepsilon_h - \mu)}}$$

b) Find u is chemical equilibrium:

$$\rho_h = \rho_e$$

$$2\left(\frac{m_v k_B T}{2\pi\hbar^2}\right)^{3/2} e^{\beta(\varepsilon_h - u)} = 2\left(\frac{m_c k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-\beta(\varepsilon_c - u)}$$

$$\hookrightarrow \quad m_v^{3/2} e^{\beta(\varepsilon_h - u)} = m_c^{3/2} e^{-\beta(\varepsilon_c - u)}$$

$$\left(\frac{m_v}{m_c}\right)^{3/2} = e^{-\beta(\varepsilon_c + \varepsilon_h - 2u)}$$

$$\frac{1}{\beta}\frac{3}{2}\ln\left(\frac{m_v}{m_c}\right) + (\varepsilon_c + \varepsilon_h) = 2u$$

$$\hookrightarrow u = \frac{3}{4} k_B T \ln\left(\frac{m_v}{m_c}\right) + \frac{1}{2}\underbrace{(\varepsilon_c + \varepsilon_h)}_{= \frac{p^2}{2m_c} + \Delta - \frac{p^2}{2m_v} \approx \Delta}$$

$$\boxed{u = \frac{3}{4} k_B T \ln\left(\frac{m_v}{m_c}\right) + \frac{\Delta}{2}}$$

32) Consider a box of volume V and N_0 of $\bar{e}$ at $T=0$

a) Calculate # of positrons in the temperature in range $\varepsilon_F \ll k_B T \ll m_e c^2$

$$\Rightarrow \varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N_0}{V}\right)^{2/3}$$

$\Rightarrow$ Assume N_0 is large in this Temp range, so the # of positrons $\ll N_0$. Discuss under what condition this assumption is valid.

assume $\varepsilon_F \ll k_B T \ll mc^2$ (nonrelativistic)

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon)\left(\exp\left\{\frac{(\varepsilon-\mu)}{k_B T}\right\}+1\right)^{-1}$$

For $k_B T \ll mc^2$, $\varepsilon = \sqrt{m^2c^4 + p^2c^2} \simeq mc^2 + \frac{p^2}{2m} = \frac{p^2}{2m}$

ignore mc^2, or change zero point energy.

For nonrelativistic: $g(\varepsilon) = 2\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$

$$= \int d\varepsilon\, 2\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\exp\left\{\frac{\varepsilon-\mu}{k_B T}\right\}+1\right]^{-1}\varepsilon^{1/2}$$

since $k_B T \gg \varepsilon_F$

$$\frac{1}{\exp\left\{\frac{\varepsilon-\mu}{k_B T}\right\}+1} \approx e^{\frac{-(\varepsilon-\mu)}{k_B T}}$$

$$\hookrightarrow = \int_0^\infty d\varepsilon \, 2 \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} e^{-\beta(\varepsilon-\mu)} \varepsilon^{1/2}$$

$$N_{\pm} = \cancel{2}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\sqrt{\pi}}{\cancel{2}} (k_B T)^{3/2} e^{\beta\mu_\pm}$$

<u>Particle Equilibrium:</u>

$$N_{e^-} - N_{e^+} = N_0$$

$$\hookrightarrow \quad \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\pi} \, (k_B T)^{3/2} \left(e^{\beta\mu_-} - e^{\beta\mu_+}\right) = N_0$$

For chemical equilibrium:

$$\mu_{e^-} + \mu_{e^+} = 0 \quad \text{or} \quad \mu_{e^-} = \mu_{e^+}$$

$$\hookrightarrow \quad \underbrace{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\pi} \, (k_B T)^{3/2}}_{\equiv \alpha} \underbrace{\left(e^{\beta\mu} - e^{-\beta\mu_-}\right)}_{2\sinh(\beta\mu_-)} = N_0$$

$$\hookrightarrow \quad 2\alpha \sinh(\beta\mu_-) = N_0$$

$$\hookrightarrow \mu_- = \frac{1}{\beta}\sinh^{-1}\left(\frac{N_0}{2\alpha}\right)$$

use identity: $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$$\mu_- = \frac{1}{\beta}\ln\left(\frac{N_0}{2\alpha} + \sqrt{\left(\frac{N_0}{2\alpha}\right)^2 + 1}\right)$$

then $N_{e^+} = \underbrace{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\pi}\,(k_BT)^{3/2}}_{\equiv\alpha} e^{-\beta u_-}$

$$= \alpha e^{-\beta u}$$

plug in u_- results

$$= \alpha \exp\left\{-\ln\left(\frac{N_0}{2\alpha} + \sqrt{\left(\frac{N_0}{2\alpha}\right)^2 + 1}\right)\right\}$$

$$\boxed{N_+ = \frac{\alpha}{\frac{N_0}{2\alpha} + \sqrt{\left(\frac{N_0}{2\alpha}\right)^2 + 1}} \quad \text{where} \quad \alpha = \frac{V}{4\pi^2}\sqrt{\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_BT)^{3/2}}$$

The condition that N_0 is large such that positron is much less than N_0 is valid when the electron gas is dilute inside the box, so that there is not terribly many collisions of making pair productions.

b) For $k_B T \gg mc^2$, relativistic: $\varepsilon = \sqrt{m^2c^4 + p^2c^2} \approx pc = \hbar c k$

Know $g(\varepsilon) = \frac{V}{2\pi^2}\frac{\varepsilon^2}{(\hbar c)^3}$ for relativistic from HW#5

$$N_\pm = \int_0^\infty d\varepsilon\; \underset{\text{extra factor for spin}}{2}\; \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\,\varepsilon^2\, e^{-\beta(\varepsilon - \mu_\pm)}$$

$$\overset{!}{=} \cancel{2}\frac{V}{\cancel{2}\pi^2}\frac{1}{(\hbar c)^3}\, 2(k_B T)^3\, e^{\beta\mu_\pm}$$

with $N_{e^-} - N_{e^+} = N_0$ and $\mu_- = -\mu_+$

$$\underbrace{2\frac{V}{\pi^2}\frac{1}{(\hbar c)^3}(k_B T)^3}_{\equiv \alpha}\left(\underbrace{e^{\beta\mu_-} - e^{-\beta\mu_-}}_{2\sinh(\beta\mu_-)}\right) = N_0$$

$$\Rightarrow = 2\alpha\sinh(\beta\mu_-) = N_0$$

we see it's same as part a, but different α

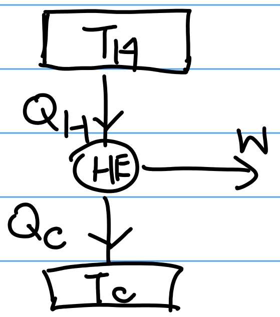
$$\mu_- = \frac{1}{\beta}\ln\left(\frac{N_0}{2\alpha} + \sqrt{\left(\frac{N_0}{2\alpha}\right)^2 + 1}\right)$$

$$\boxed{N_+ = 2\frac{V}{\pi^2}\frac{1}{(\hbar c)^3}(k_B T)^3 e^{-\beta\mu_-} \overset{!}{=} \frac{\alpha}{\frac{N_0}{2\alpha} + \sqrt{\left(\frac{N_0}{2\alpha}\right)^2 + 1}} \quad \text{for } \alpha = 2\frac{V}{\pi^2}\frac{1}{(\hbar c)^3}(k_B T)^3}$$

33) Landsberg Limit:

a) Show the second law, $\Delta S \geq 0$ implies the maximum efficiency for a heat engine between two thermal energy reservoirs with T_H and T_C is given by:

$$\eta \leq 1 - \frac{T_C}{T_H}$$



$$\eta = \frac{W}{Q_H}$$

$$dE = dQ + dW = 0.$$

$$Q_H - Q_C - W = 0$$

or $$W = Q_H - Q_C.$$

$$\eta = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$$

$$\Delta S_{tot} = \Delta S_H + \Delta S_C \geq 0$$

$TdS = dQ$

$$\stackrel{!}{=} \left(\frac{dS}{dQ}\right)_H dQ_H + \left(\frac{\partial S}{\partial Q}\right) dQ_C$$

$$\stackrel{!}{=} \frac{1}{T_H} dQ_H + \frac{1}{T_C} dQ_C$$

$dQ_H = Q_H$

$dQ_C = -Q_C$

$$= \frac{Q_H}{T_H} - \frac{Q_C}{T_C} \geq 0$$

We get max work with reversible, i.e. $\Delta S = 0$.

rearrange: $$\frac{Q_C}{Q_H} \geq \frac{T_C}{T_H}$$

i.e. $$\boxed{\eta = 1 - \frac{Q_C}{Q_H} \leq 1 - \frac{T_C}{T_H}}$$

b) Consider Blackbody radiation:

energy density: $\frac{E}{V} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T_H^4$

Derive Entropy density:

$$dA = -SdT - pdV + \mu dN.$$

$$S = -\frac{\partial A}{\partial T}\Big)_{V,N}$$

$$A = -k_B T \ln Q$$

For Bosons: $\sum_{n_j=0}^{\infty} e^{-\beta(\varepsilon-\mu)n_j} = \frac{1}{1-e^{-\beta(\varepsilon_j-\mu)}}$

for photons, $\mu=0$ $\qquad \stackrel{!}{=} \frac{1}{1-e^{-\beta\varepsilon_j}}$

$$\ln Q = \ln \Xi = -\sum_j \ln(1-e^{-\beta\varepsilon_j})$$

$\varepsilon_j = \hbar c k_j$ for photons.

$$\ln Q = -\sum_j \ln(1-e^{-\beta\hbar c k_j})$$

$$\stackrel{!}{=} -\int dk\, g(k) \ln(1-e^{-\beta\hbar c k})$$

$$\stackrel{!}{=} -\int_0^\infty dk\, 2\frac{V}{2\pi^2} k^2 \ln(1-e^{-\beta\hbar c k})$$

$$\ln Q \stackrel{!}{=} \frac{V}{\pi^2}\frac{\pi^4}{45}\left(\frac{k_B T}{\hbar c}\right)^3$$

$$A = -k_B T \ln Q$$

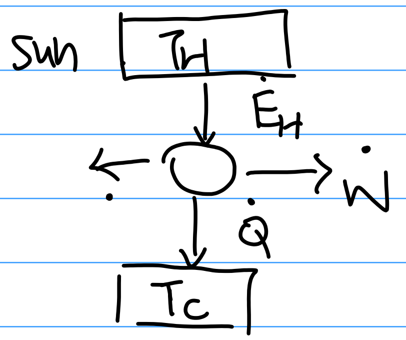
$$A = -V \frac{\pi^2}{45} \frac{(k_B T)^4}{(\hbar c)^3}$$

$$\Rightarrow \quad S = -\left(\frac{\partial A}{\partial T}\right)_{N,V}$$

$$= \frac{4\pi^2}{45} V \frac{k_B^4}{(\hbar c)^3} T^3$$

or $$\boxed{\frac{S}{V} = \frac{4\pi^2}{45} \frac{k_B^4}{(\hbar c)^3} T_H^3}$$

c) Consider that the solar cells emit blackbody radiation, derive an expression for the maximum efficiency turning solar energy to work. Evaluate $T_H = 5800K$, $T_c = 300K$.



$$\dot{E} = \dot{E}_H - \dot{E}_{emit} - \dot{Q} - \dot{W} = 0$$

$$\dot{W} = \dot{E}_H - \dot{E}_{emit} - \dot{Q}$$

$$\eta \leq \frac{\dot{W}}{\dot{E}_H} = \frac{\dot{E}_H - \dot{E}_{emit} - \dot{Q}}{\dot{E}_H}$$

From part b:

we know $S = \frac{4\pi^2}{45} \frac{k_B^4}{(\hbar c)^3} V T^3$

also $E = \frac{\pi^2}{15} \frac{1}{(\hbar c)^3} (k_B T)^4$

so $\frac{\dot{E}}{\dot{S}} = \frac{3}{4} T$

some constant

Then $\dot{E}_H = \frac{3}{4} \dot{S}_H T_H = \frac{3}{4} \alpha T_H^4$

$\dot{E}_{emit} = \frac{3}{4} \dot{S}_{emit} T_C = \frac{3}{4} \alpha T_C^4$

Find $\dot{Q}$ Since $\dot{S}_{tot} = 0$

entropy generated from solar cell = 0 ideally

$\dot{S} = \dot{S}_H - \dot{S}_C - \dot{S}_{emit} - \cancel{\dot{S}_{cell}}^{\,0} = 0$

or $\dot{S}_C = \dot{S}_H - \dot{S}_{emit}$

$\dot{S}_C \stackrel{!}{=} \alpha (T_H^3 - T_C^3)$

Then $\dot{Q} = T_C \dot{S}_C$

$\stackrel{!}{=} \alpha T_C (T_H^3 - T_C^3)$

$$\eta \leq \frac{\dot{E}_H - \dot{E}_{emit} - \dot{Q}}{\dot{E}_H}$$

$$\leq 1 - \frac{\dot{E}_{emit}}{\dot{E}_H} - \frac{\dot{Q}}{\dot{E}_H}$$

since $\frac{\dot{E}_{emit}}{\dot{E}_H} = \frac{\dot{S}_{emit} T_c}{\dot{S}_H T_H} = \frac{T_c{}^4}{T_H{}^4}$

and since $\frac{\dot{Q}}{\dot{E}_H} = \frac{\alpha T_c (T_H^3 - T_c^3)}{\frac{3}{4}\alpha T_H{}^4} = \frac{4}{3}\frac{T_c}{T_H} - \frac{4}{3}\frac{T_c^4}{T_H^4}$

Altogether:

$$\eta \leq 1 - \left(\frac{T_c}{T_H}\right)^4 - \frac{4}{3}\frac{T_c}{T_H} + \frac{4}{3}\left(\frac{T_c}{T_H}\right)^4$$

$$\hookrightarrow \eta \leq 1 - \frac{4}{3}\frac{T_c}{T_H} + \frac{1}{3}\left(\frac{T_c}{T_H}\right)^4$$

for $T_c = 300\,K$ $T_H = 5800\,K$

$$\eta_{SQ} = 1 - \frac{4(300K)}{3(5800K)} + \frac{1}{3}\left(\frac{300}{5800}\right)^4 = 0.931$$

compare to Carnot

$$\eta_{carnot} = 1 - \frac{300}{5800} = 0.94827$$

we see that $\eta_{SQ} < \eta_{carnot}$.

d) Consider heat transport via a gas of optical phonons at temperature T_H. With $\omega(\vec{k}) = \omega_0 = \text{const.}$

Calculate $\dot{E}/\dot{S}$

Partion function: $$Q = \prod_{k=1}^{3N} q_k$$

$$\ln Q = \sum_{k=1}^{3N} \ln q_k$$

Each mode is modelled by SHO:

$$q_k = \sum_{n=0}^{\infty} e^{-\beta E} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega_0}$$

$$q_k = \frac{e^{\frac{-\beta\hbar\omega_0}{2}}}{1 - e^{-\beta\hbar\omega_0}}$$

$$\ln Q = \sum_{k=1}^{3N} \ln q_k = \ln(q_k^{3N}) = 3N \ln q_k$$

$$\ln Q = 3N\left(\frac{-\beta\hbar\omega_0}{2} - \ln(1 - e^{-\beta\hbar\omega_0})\right)$$

$$E = -\frac{\partial}{\partial\beta}\ln Q = \frac{3N}{2}\hbar\omega_0 + 3N\frac{\hbar\omega_0 e^{-\beta\hbar\omega_0}}{1 - e^{-\beta\hbar\omega_0}}$$

$$E = \frac{3N}{2}\hbar\omega_0 + \frac{3N\hbar\omega_0}{e^{\beta\hbar\omega_0} - 1}$$

$$S = -\frac{\partial}{\partial T}\Big(\underbrace{-\frac{1}{\beta}\ln Q}_{A}\Big)_{V,N} = -\frac{\partial}{\partial \beta}\frac{\partial \beta}{\partial T}\Big(-\frac{1}{\beta}\ln Q\Big)_{V,N}$$

$$= \frac{-3N}{k_B T^2}\frac{\partial}{\partial \beta}\left(\frac{\cancel{\beta}}{\cancel{\beta}}\frac{1}{2}\hbar w_0 - \frac{\ln(1-e^{-\beta\hbar w_0})}{\beta}\right)_{V,N}$$

$$= \frac{-3N}{k_B T^2}\left(\frac{1}{\beta^2}\ln(1-e^{-\beta\hbar w_0}) - \frac{\hbar w_0 e^{-\beta\hbar w_0}}{\beta(1-e^{-\beta\hbar w_0})}\right)$$

$$= -3Nk_B\beta\left(\frac{1}{\beta}\ln(1-e^{-\beta\hbar w_0}) - \frac{\hbar w_0}{e^{\beta\hbar w_0}-1}\right)$$

$$= 3Nk_B\left(\frac{\beta\hbar w_0}{e^{\beta\hbar w_0}-1} - \ln(1-e^{-\beta\hbar w_0})\right)$$

$$S = \frac{3N\hbar w_0}{T}\left(\frac{1}{e^{\beta\hbar w_0}-1} - \frac{\ln(1-e^{-\beta\hbar w_0})}{\beta\hbar w_0}\right)$$

ignore zero-point energy

$$\Rightarrow \frac{\dot{E}}{\dot{S}} = \frac{E}{S} = \frac{3N\hbar w_0\left(\cancel{\frac{1}{2}} + \frac{1}{e^{\beta\hbar w_0}-1}\right)}{\frac{3N\hbar w_0}{T}\left(\frac{1}{e^{\beta\hbar w_0}-1} - \frac{\ln(1-e^{-\beta\hbar w_0})}{\beta\hbar w_0}\right)}$$

$$\boxed{T_f = \frac{E}{S} = T\frac{1}{1-\frac{\ln(1-e^{-\beta\hbar w_0})(e^{\beta\hbar w_0}-1)}{\beta\hbar w_0}}}$$

we see that T_f is maximized when

$$\frac{\ln(1-e^{-\beta\hbar w_0})(e^{\beta\hbar w_0}-1)}{\beta\hbar w_0}$$ is minmized.

$$\frac{\ln\left(1-e^{-\beta\hbar W_0}\right)\left(e^{\beta\hbar W_0}-1\right)}{\beta\hbar W_0} = 0$$

$$\boxed{\begin{array}{l} \text{when} \quad e^{\beta\hbar W_0} - 1 = 0 \\ \qquad \text{or} \quad W_0 = 0 \quad \text{for max } T_f \end{array}}$$

$$\lim_{W_0 \to 0} \frac{\ln\left(1-e^{-\beta\hbar W_0}\right)\left(e^{\beta\hbar W_0}-1\right)}{\beta\hbar W_0} \approx \frac{-e^{-\beta\hbar W_0}\left(\cancel{1}+\beta\hbar W_0 \cancel{-1}\right)}{\beta\hbar W_0}$$

$$\approx -e^{-\beta\hbar W_0}$$

Then $$\boxed{T_{f,max} = \frac{1}{1+e^{-\beta\hbar W_0}} T_H}$$