Zhi Chen HW#12

41) Lattice Gas

a) Show if $n_i = \frac{(S_i+1)}{2}$ with $S_i = \pm 1$,

$$\Xi = C \underbrace{\sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1} \exp\{\beta H' \sum_i S_i + \beta J \sum_{ij} S_i S_j\}}_{Q_{IM}(N,T,H')}$$

Determine C, H', J in terms of ε, u for a lattice that has q - nearest neighbors.

$$\Xi = \sum_\nu e^{-\beta(E_\nu - uN_\nu)}$$

$$= \sum_{n_1=0,1}\sum_{n_2=0,1}\cdots\sum_{n_N=0,1} \exp\{\beta u \sum_{i=1}^N n_i + \beta\varepsilon \sum_{(ij)} n_i n_j\}$$

$$= \sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1} \exp\{\beta u \sum_{i=1}^N \left(\frac{S_i+1}{2}\right) + \beta\varepsilon \sum_{(ij)} \left(\frac{S_i+1}{2}\right)\left(\frac{S_j+1}{2}\right)\}$$

$$= \sum_{S_1=\pm1}\cdots\sum_{S_N=\pm1} \exp\{\tfrac{1}{2}\beta u\left(\sum_i S_i + N\right) + \tfrac{1}{4}\beta\varepsilon \sum_{(ij)} (S_i S_j + S_i + S_j + 1)\}$$

If there are q nearest neighbors for S_i, then

$$\sum_{(ij)} (S_i + S_j + 1) = \left(\sum_{(ij)} S_i + \sum_{(ij)} S_j + \sum_{ij} 1\right)$$

$$= \frac{1}{2}\left(2q \sum_i^N S_i + qN\right)$$

to avoid double counting.

$$\hookrightarrow = \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \exp\left\{\frac{1}{2}\beta u\left(\sum_i^N S_i + N\right) + \frac{1}{4}\beta\varepsilon\left[\sum_{(ij)} S_i S_j + q\sum_i^N S_i + \frac{qN}{2}\right]\right\}$$

$$= \exp\left\{\frac{1}{2}\beta u N + \frac{q}{8}\beta\varepsilon N\right\} \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \exp\left\{\beta\left(\frac{u}{2} + \frac{q\varepsilon}{4}\right)\sum_i^N S_i + \frac{1}{4}\beta\varepsilon\sum_{(ij)} S_i S_j\right\}$$

Recognizing:

$$C = \exp\left\{\frac{1}{2}\beta u N + \frac{q}{8}\beta\varepsilon N\right\}$$

$$J = \frac{\varepsilon}{4}$$

$$H' = \frac{u}{2} + \frac{q\varepsilon}{4}$$

b) Show that in mean-field theory:

$$k_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T = \frac{1-\rho}{\rho k_B T - q\varepsilon\rho^2(1-\rho)}$$

where $\rho = \langle n_i \rangle$, $p = \left(\frac{k_B T}{N}\right)\ln \Xi_{LG}$

$\Rightarrow$ From part a, we see lattice gas is essentially Ising Model with const factors

$$\Xi_{LG} = \sum_{n_1=0,1} \cdots \sum_{n_N=0,1} \exp\left\{\beta u \sum_{i=1}^N n_i + \beta\varepsilon\sum_{(ij)} n_i n_j\right\}$$

$$\stackrel{!}{=} \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \exp\left\{\beta\left(\frac{u}{2} + \frac{q\varepsilon}{4}\right)\sum_i S_i + \frac{1}{4}\beta\varepsilon\sum_{(ij)} S_i S_j + \frac{1}{2}\beta u N + \frac{q}{8}\beta\varepsilon N\right\}$$

then $$\underbrace{\underbrace{\left(\frac{u}{2} + \frac{q\varepsilon}{4}\right)}_{H'}\sum_i^N S_i + \underbrace{\frac{1}{4}\varepsilon}_{J}\sum_{(ij)} S_i S_j}_{-\mathcal{H}_{IM}} + \frac{1}{2}uN + \frac{q}{8}\varepsilon N = \underbrace{u\sum_{i=1}^N n_i}_{uN} + \underbrace{\varepsilon\sum_{(ij)} n_i n_j}_{-\mathcal{H}_{LG}}$$

Then: $\mathcal{H}_{LG} - uN = \mathcal{H}_{IM} - \frac{1}{2}uN - \frac{q}{8}\varepsilon N$

So $\left(\mathcal{H}_{LG} - uN\right)_{MF} = \left(\mathcal{H}_{IM} - \frac{1}{2}uN - \frac{q}{8}\varepsilon N\right)_{MF}$

then $\Xi_{LG} = \exp\left\{\left(\frac{1}{2}u + \frac{q}{8}\varepsilon\right)\beta N\right\} Q_{IM}$

$\hookrightarrow \ln \Xi_{LG}^{MF} = \ln Q_{IM}^{MF} + \left(\frac{1}{2}uN + \frac{q}{8}\varepsilon N\right)\beta$

Derive $\mathcal{H}_{IM}^{MF}$ and Q_{IM}^{MF}:

$$\mathcal{H}_{IM} = -H'\sum_i^N s_i - J\sum_{(ij)} s_i s_j = -\sum_i H_i s_i$$

Approximate: $s_i s_j = \left(\langle s_i\rangle + \delta s_i\right)\left(\langle s_j\rangle + \delta s_j\right)$

$$\stackrel{!}{=} \langle s_i\rangle\langle s_j\rangle + \langle s_i\rangle \delta s_j + \langle s_j\rangle \delta s_i + \mathcal{O}(\delta^2)$$

$$= \langle s_i\rangle\langle s_j\rangle + \langle s_i\rangle\left(s_j - \langle s_j\rangle\right) + \langle s_j\rangle\left(s_i - \delta s_i\right)$$

$$s_i s_j \stackrel{!}{=} \langle s_i\rangle s_j + \langle s_j\rangle s_i - \langle s_i\rangle\langle s_j\rangle$$

In class we derived: $\langle s_i\rangle = \langle s_j\rangle = m = \tanh\left(\beta H' + \beta J q m\right)$

$$\mathcal{H}_{IM}^{MF} = -H'\sum s_i - J\sum_{(ij)} m s_j + m s_i - m^2$$

$$\mathcal{H}_{IM}^{MF} \stackrel{!}{=} -H'\sum s_i - J\sum_{(ij)}\left(2m s_i - m^2\right)$$

$$\stackrel{!}{=} -H'\sum s_i - J\frac{q}{2}\sum_i^N\left(2m s_i - m^2\right)$$

$$\mathcal{H}_{IM}^{MF} \stackrel{!}{=} \frac{JqNm^2}{2} - \left(H' + Jqm\right)\sum_i^N s_i$$

then $$Q_{IM} = \sum \exp\{-\beta H_{IM}^{MF}\}$$

$$\overset{|}{=} \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \exp\{\beta(H' + Jqm)\sum_i S_i\} \exp\{-\beta \frac{NqJm^2}{2}\}$$

Since $$\langle n_i \rangle = \rho = \frac{\langle S_i \rangle}{2} + \frac{1}{2} = \frac{m+1}{2}$$

or $$m = (2\rho - 1)$$

$$Q_{IM}^{MF} = \exp\{\beta \frac{NqJm^2}{2}\} \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \exp\{\beta(H' + Jq(2\rho-1)) \sum_i^N S_i\}$$

$$\overset{|}{=} \exp\{-\beta \frac{NqJ}{2}(2\rho-1)^2\} \left(\sum_{S_i=\pm1} \exp\{\beta(H' + Jq(2\rho-1)) S_i\} \right)^N$$

$$= \exp\{-\beta \frac{NqJ}{2}(2\rho-1)^2\} \left(\exp\{\beta(H' + Jq(2\rho-1))\} + \exp\{-\beta(H' + Jq(2\rho-1))\} \right)^N$$

$$= \exp\{-\beta \frac{NqJ}{2}(2\rho-1)^2\} \left(2\cosh[\beta(H' + Jq(2\rho-1))] \right)^N$$

Then $$\ln Q_{IM}^{MF} = -\beta \frac{NqJ}{2}(2\rho-1)^2 + N\ln[2\cosh(\beta(H' + Jq(2\rho-1)))]$$

$$\Rightarrow \ln \Xi_{LG}^{MF} = \ln Q_{IM} + \frac{1}{2}uN + \frac{q\varepsilon N}{8}$$

$$= -\beta \frac{Nq\varepsilon}{8}(2\rho-1)^2 + N\ln\left(2\cosh[\beta(H' + Jq(2\rho-1))]\right) + \left(\frac{1}{2}uN + \frac{q\varepsilon N}{8}\right)\beta$$

$$= -\beta \frac{Nq\varepsilon}{8}(2\rho-1)^2 + N\ln\left(2\cosh\left[\beta\left(\frac{u}{2} + \frac{q\varepsilon}{4} + \frac{q\varepsilon}{4}(2\rho-1)\right)\right]\right) + \left(\frac{1}{2}uN + \frac{q\varepsilon N}{8}\right)\beta$$

Since $m = \tanh(\beta H' + \beta J q m)$

$$2\rho - 1 = \tanh\left(\beta\left(\frac{u}{2} + \frac{q\varepsilon}{4} + \frac{q\varepsilon}{4}(2\rho-1)\right)\right)$$

$$2\rho - 1 \overset{!}{=} \tanh\left(\beta\left(\frac{u}{2} + \frac{q\varepsilon}{2}\rho\right)\right)$$

↳ $$u = \frac{2}{\beta}\tanh^{-1}(2\rho-1) - q\varepsilon\rho$$

with $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$$u = \frac{1}{\beta}\ln\left(\frac{\rho}{1-\rho}\right) - q\varepsilon\rho$$

↳ $$\ln \Xi_{LG}^{MF} = -\beta\frac{Nq\varepsilon}{8}(2\rho-1)^2 + N\ln\left[2\cosh\left(\frac{1}{2}\ln\left(\frac{\rho}{1-\rho}\right) - \cancel{\frac{\beta}{2}q\varepsilon\rho + \frac{\beta}{2}q\varepsilon\rho}\right)\right] + \frac{\beta N}{2}\left(\frac{1}{\beta}\ln\left(\frac{\rho}{1-\rho}\right) - q\varepsilon\rho\right) + \frac{q\varepsilon\beta N}{8}$$

↳ $$= -\beta\frac{Nq\varepsilon}{8}(2\rho-1)^2 + N\ln\left(e^{\frac{1}{2}\ln\left(\frac{\rho}{1-\rho}\right)} + e^{-\frac{1}{2}\ln\left(\frac{\rho}{1-\rho}\right)}\right) + \frac{N}{2}\ln\left(\frac{\rho}{1-\rho}\right) + \frac{\beta N}{2}q\varepsilon\left(\frac{1}{4} - \rho\right)$$

$$= -\beta\frac{Nq\varepsilon}{8}(2\rho-1)^2 + N\ln\left(\sqrt{\frac{\rho}{1-\rho}} + \sqrt{\frac{1-\rho}{\rho}}\right) + \frac{N}{2}\ln\left(\frac{\rho}{1-\rho}\right) + \frac{\beta Nq\varepsilon}{2}\left(\frac{1}{4} - \rho\right)$$

$$= -\beta\frac{Nq\varepsilon}{8}(2\rho-1)^2 + N\ln\left(\frac{1}{\sqrt{(1-\rho)\rho}}\sqrt{\frac{\rho}{1-\rho}}\right) + \frac{\beta Nq\varepsilon}{2}\left(\frac{1}{4} - \rho\right)$$

$$= -N\ln(1-\rho) - \beta\frac{Nq\varepsilon}{8}(4\rho^2 - \cancel{4\rho} + 1) + \cancel{\frac{\beta Nq\varepsilon}{2}\left(\frac{1}{4} - \rho\right)}$$

$$\ln \Xi_{LG}^{MF} \overset{!}{=} -N\left(\ln(1-\rho) + \frac{\beta q\varepsilon}{2}\rho^2\right)$$

Using hint given in question:

$$p = \frac{k_B T}{N} \ln \Xi$$

$$p = -k_B T \ln(1-\rho) - \frac{q\varepsilon}{2}\rho^2$$

$$\kappa_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T = \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_T^{-1}$$

$$\stackrel{!}{=} \frac{1}{\rho}\left[\frac{k_B T}{1-\rho} - q\varepsilon\rho\right]^{-1}$$

$$\stackrel{!}{=} \frac{1}{\rho}\left[\frac{k_B T - q\varepsilon\rho(1-\rho)}{1-\rho}\right]^{-1}$$

$$\boxed{\kappa_T \stackrel{!}{=} \frac{1-\rho}{\rho k_B T - q\varepsilon\rho^2(1-\rho)}}$$

c) Show near critical point, $K_T \to \frac{1}{k_B}(T-T_c)^{-1}$

At critical point, $\left(\frac{\partial p}{\partial \rho}\right)_T = \left(\frac{\partial^2 p}{\partial \rho^2}\right)_T = 0$

① $\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{k_B T}{1-\rho} - q\varepsilon\rho = 0$

$$\frac{k_B T}{1-\rho} = q\varepsilon\rho$$

$$\hookrightarrow \quad k_B T_c = (1-\rho_c) q\varepsilon\rho_c$$

② $\left(\frac{\partial^2 p}{\partial \rho^2}\right)_T = \frac{k_B T}{(1-\rho)^2} - q\varepsilon = 0$

$$\hookrightarrow \quad k_B T_c = (1-\rho_c)^2 q\varepsilon$$

Combine ① and ②:

$$\cancel{(1-\rho_c)}\cancel{q\varepsilon}\rho_c = (1-\rho_c)^{\cancel{2}}\cancel{q\varepsilon}$$

then $\boxed{\rho_c = 1/2}$

plug $\rho_c = 1/2$ to ①:

$$\boxed{k_B T_c = \left(1-\tfrac{1}{2}\right) q\varepsilon \tfrac{1}{2} = \frac{q\varepsilon}{4}}$$

Then

$$\kappa_T = \frac{1-\rho}{\rho} \frac{1}{k_B T - q\varepsilon\rho(1-\rho)}$$

plug $\rho = \rho_c = 1/2$

$$\kappa_T(\rho=\rho_c, T) = \frac{1-\frac{1}{2}}{\frac{1}{2}} \frac{1}{k_B T - q\varepsilon \frac{1}{2}(1-\frac{1}{2})}$$

$$= \frac{1}{k_B T - q\varepsilon \, 1/4}$$

but $k_B T_c = \frac{q\varepsilon}{4}$

$$\boxed{= \frac{1}{k_B}(T-T_c)^{-1}}$$

$\boxed{\text{since } \kappa_T \propto (T-T_c)^{-\gamma} \text{ we see } \gamma = 1}$

42) Monte-Carlo:

$$E = \sum_{i}^{N} \nu_i \hbar \omega_i$$

a) Calculate Q, determine T such $\langle E \rangle = 3678\,cm^{-1}$, also find $\langle (\delta E)^2 \rangle$

partition function for particle i.

$$Q_i = \sum_{\nu_i=0}^{\infty} \exp\{-\beta\hbar\omega_i \nu_i\}$$

$$= \sum_{\nu_i=0}^{\infty} \exp\{-\beta\hbar\omega_i\}^{\nu_i}$$

with $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

$$\stackrel{!}{=} \frac{1}{1-\exp\{-\beta\hbar\omega_i\}}$$

Then $$Q = \prod_{i}^{N=21} Q_i$$

$$\stackrel{!}{=} \prod_{i=1}^{N=21} \frac{1}{1-\exp\{-\beta\hbar\omega_i\}}$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta}(\ln Q)_{N,V} = 3678\,cm^{-1}$$

$$\stackrel{!}{=} \frac{-\partial}{\partial\beta}\left(\ln\left[\prod_{i=1}^{21} \frac{1}{1-\exp\{-\beta\hbar\omega_i\}}\right]\right)_{N,V}$$

$$\stackrel{!}{=} \sum_{i=1}^{21} \frac{-\partial}{\partial\beta} \ln\left(\frac{1}{1-\exp\{-\beta\hbar\omega_i\}}\right)$$

$$\stackrel{!}{=} \sum_{i=1}^{21} \frac{\partial}{\partial\beta} \ln\left(1-\exp\{-\beta\hbar\omega_i\}\right)$$

$$\langle E \rangle \stackrel{!}{=} \sum_{i=1}^{21} \hbar\omega_i \frac{\exp\{-\beta\hbar\omega_i\}}{1-\exp\{-\beta\hbar\omega_i\}} = 3678\,cm^{-1}$$

$\boxed{\text{Solve using code, and find } \underline{T = 870.6\,K}}$

$$\langle (\delta E)^2 \rangle = \langle (E - \langle E \rangle)^2 \rangle = \frac{\partial^2 \ln Q}{\partial \beta^2}$$

$$\hookrightarrow = \frac{\partial}{\partial \beta}\underbrace{\left(\frac{\partial \ln Q}{\partial \beta}\right)}_{-\langle E \rangle}$$

$$\overset{!}{=} -\frac{\partial}{\partial \beta} \sum_{i=1}^{21} \hbar\omega_i \frac{\exp\{-\beta\hbar\omega_i\}}{1-\exp\{-\beta\hbar\omega_i\}}$$

$$\overset{!}{=} \sum_{i=1}^{21} \hbar\omega_i \left(\frac{\hbar\omega_i \exp\{-\beta\hbar\omega_i\}(1-\exp\{-\beta\hbar\omega_i\}) + \hbar\omega_i \exp\{-\beta\hbar\omega_i\}^2}{(1-\exp\{-\beta\hbar\omega_i\})^2} \right)$$

$$\langle (\delta E)^2 \rangle \overset{!}{=} \sum_{i=1}^{21} (\hbar\omega_i)^2 \frac{\exp\{-\beta\hbar\omega_i\}}{(1-\exp\{-\beta\hbar\omega_i\})^2}$$

After setting $T = 870.6K$, then plug in #'s,

$$\boxed{\langle (\delta E)^2 \rangle \cong 4.47 \times 10^6 \text{ cm}^{-1}}$$

b) See code, average and variance labelled on plots

c) See code.

43) Masters Equation:

$$\frac{dP_i}{dt} = \sum_j \left[W_{i \leftarrow j} P_j - W_{j \leftarrow i} P_i \right]$$

$\Rightarrow$ P_i : State occupation probabilities

$\Rightarrow$ W_{ij} : transition matrix elements, rate from j-state to i-state.

$\Rightarrow$ assume $W_{ij} = W_{ji}$, symmetric.

a) Show $\sum_i P_i = 1$ is conserved by time evolution.

$$\frac{dP_i}{dt} = \sum_j W_{ij} P_j - W_{ji} P_i$$

$$\frac{d}{dt}\left(\sum_i P_i\right) = \sum_i \frac{d}{dt} P_i$$

$$= \sum_i \sum_j \left(W_{ij} P_j - W_{ji} P_i \right)$$

$$= \sum_i \sum_j W_{ij} P_j - \sum_i \sum_j W_{ji} P_i$$

$$= \sum_i \sum_j W_{ij} P_j - \sum_j \sum_i W_{ji} P_i$$

← same as first term, but with $i \rightleftarrows j$, so they're equal

$$= \sum_i (WP)_i - \sum_j (WP)_j$$

$$= 0$$

$$\boxed{\begin{array}{l} \text{or } \frac{d}{dt}\left(\sum_i P_i\right) = 0 \text{ at all times, so } \sum_i P_i = \text{constant} \\ \therefore \text{ If } \sum_i P_i \text{ is normalized to } 1 \text{ initially, then} \\ \qquad \sum_i P_i = 1 \text{ always} \end{array}}$$

b) show $\frac{dP_i}{dt} = \sum_j [W_{ij} P_j - W_{ji} P_i]$ is consistent with second law.

$$S = -k_B \sum_i P_i \ln P_i$$

$$\frac{dS}{dt} = -k_B \sum_i \left(\frac{dP_i}{dt} \ln P_i + \frac{dP_i}{dt} \right)$$

$$\overset{!}{=} -k_B \left[\sum_i \frac{dP_i}{dt} \ln P_i + \cancel{\sum_i \frac{dP_i}{dt}}^{\,0} \right]$$

$$\overset{!}{=} -k_B \sum_i \left(\sum_j (W_{ij} P_j - W_{ji} P_i) \ln P_i \right)$$

$$\overset{!}{=} -k_B \sum_i \left(\sum_j \left[W_{ij} \underbrace{(P_j - P_i)} \right] \ln P_i \right)$$

get zero if $i=j$,

then only contributing terms are off-diagonal,

let, $W_{ij} = \begin{pmatrix} W_{11} & W_{12} & W_{13} & \cdots \\ W_{21} & W_{22} & & \\ \vdots & & \ddots & \end{pmatrix}$ then we can disregard W_{ii} and only sum over off-diagonal.

$\Rightarrow$ Now if we observe two adjacent pair, $(i=1, j=2)$ and $(i=2, j=1)$ We will sum them up first:

1) $\left.\frac{dS}{dt}\right|_{i=1, j=2} = -k_B W_{12} (P_2 - P_1) \ln P_1$

2) $\left.\frac{dS}{dt}\right|_{i=2, j=1} = -k_B W_{21} (P_1 - P_2) \ln P_2 = -k_B W_{21} (P_2 - P_1)(-\ln P_2)$

but if $W_{21} = W_{12}$, then

$$\left.\frac{dS}{dt}\right|_{1,2} + \left.\frac{dS}{dt}\right|_{2,1} = -k_B W_{12}(P_2 - P_1)(\ln P_1 - \ln P_2)$$

case 1: if $\underline{P_2 \geq P_1}$:

$$\left.\begin{aligned} &\text{then} \quad P_2 - P_1 \geq 0 \\ &\text{and} \quad \ln P_1 - \ln P_2 \leq 0 \end{aligned}\right\} \text{ so } (P_2 - P_1)(\ln P_1 - \ln P_2) \leq 0$$

Case 2: if $\underline{P_2 \leq P_1}$:

$$\left.\begin{aligned} &\text{then} \quad P_2 - P_1 \leq 0 \\ &\text{and} \quad \ln P_1 - \ln P_2 \geq 0 \end{aligned}\right\} \text{ we still have } (P_2 - P_1)(\ln P_1 - \ln P_2) \leq 0$$

$\Rightarrow$ and since $W_{ij} > 0$ and extra negative sign in front, then $\left.\frac{dS}{dt}\right|_{1,2} + \left.\frac{dS}{dt}\right|_{2,1} \geq 0$ always

$\Rightarrow$ Similarly, this procedure can be done to all other pairs, such as $(1,3) + (3,1)$, $(2,3) + (3,2)$, and etc. but they all give positive $\frac{dS}{dt}$.

∴ After summing all pairs, we have

$$\boxed{\frac{dS}{dt} \geq 0},$$

which is consistent with second law, which says entropy is increasing monotonically.

c) consider two-level system, $P_1(t_0) = 1$, $P_2(t_0) = 0$
$W_{12} = W_{21} = \frac{1}{\tau}$, find $P_1(t)$ and $P_2(t)$

$$\frac{dP_1}{dt} = \frac{1}{\tau}(P_2 - P_1)$$

$$\frac{dP_2}{dt} = \frac{1}{\tau}(P_1 - P_2)$$

$$\frac{d}{dt}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{1}{\tau}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

let $P_i = q_i e^{\lambda t}$

$$\frac{d}{dt}P_i = \lambda q_i e^{\lambda t}$$

$$\hookrightarrow \left| \frac{1}{\tau}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \frac{1}{\tau}\begin{pmatrix} -(1+\lambda) & 1 \\ 1 & -(1+\lambda) \end{pmatrix} \right| = 0$$

$$\hookrightarrow \left(\frac{1}{\tau}\right)^2 \left[(1+\lambda)^2 - 1 \right] = \left(\frac{1}{\tau}\right)^2 \left[\lambda^2 + 2\lambda \right]$$

$$= \left(\frac{1}{\tau}\right)^2 \lambda(\lambda+2) = 0$$

then $\lambda = 0$ or $\lambda = -2/\tau$

For $\underline{\lambda = 0}$:

$$\left(\frac{1}{\tau}\right)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_1^+ \\ q_2^+ \end{pmatrix} = 0.$$

then $q_1^+ - q_2^+ = 0$ or $q^+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = -2/\tau$:

$$\frac{1}{\tau}\begin{pmatrix} -(1-2) & 1 \\ 1 & -(1-2) \end{pmatrix} = \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\hookrightarrow \quad q^- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then: $P_i = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{2}{\tau}t}$

① $P_1(t = t_0) = \underline{1} = A + Be^{\frac{-2}{\tau}t_0}$
② $P_2(t = t_0) = 0 = A - Be^{\frac{-2}{\tau}t_0}$

with ① + ② : $1 = 2A \Rightarrow A = \frac{1}{2}$

with ① − ② : $1 = 2Be^{\frac{-2}{\tau}t_0} \Rightarrow B = \frac{1}{2}e^{\frac{2}{\tau}t_0}$

then $$\boxed{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\frac{-2}{\tau}(t-t_0)}}$$