Zhi Chen HW#11

38) Clouds and Cloud Chambers:

a) Show with inclusion of surface tension,

$$\mu_{gas} = \mu_{liq} + \frac{2\gamma}{\rho r}$$

use Gibbs free energy:

$$dG = \mu_{liq}\, dN_{liq} + \mu_{gas}\, dN_{gas} + \gamma\, d\sigma$$

due to conservation of particles,

$$dN_{liq} + dN_{gas} = 0$$

then $dN_{liq} = dN = -dN_{gas}$

so $dG = (\mu_{liq} - \mu_{gas})\, dN + \gamma\, d\sigma$

Assume water droplet is perfect sphere,

$$\sigma = 4\pi r^2 \Rightarrow d\sigma = 8\pi r\, dr$$

$$dN = d(\rho V) = d(\rho \tfrac{4}{3}\pi r^3) = \rho 4\pi r^2 dr$$

After substitution

$$dG = (\mu_{liq} - \mu_{gas})\, dN + \gamma\, d\sigma$$

$$= [(\mu_{liq} - \mu_{gas})\rho 4\pi r^2 + \gamma 8\pi r]\, dr$$

For phase transition, G is at minimum, so $dG = 0$

or $$(\mu_{liq} - \mu_{gas})\rho 4\pi r^{\cancel{2}} + \gamma 8\pi \cancel{r} = 0$$

↳ $$\mu_{liq} - \mu_{gas} = \frac{-2\gamma}{\rho r}$$

↳ $$\boxed{\mu_{gas} = \mu_{liq} + \frac{2\gamma}{\rho r}}$$

b) Show $u_{gas} = u_{liq} + \frac{2\gamma}{\rho r}$ implies $p(r) = p(\infty)\exp\left\{\frac{2\beta\gamma}{\rho r}\right\}$

Along phase-Transition boundary, take differential of Eq 38.5

$$du_{gas} = du_{liq} - \frac{2\gamma}{\rho r^2}\,dr$$

$$\frac{\partial u_{liq}}{\partial p}\,dp_{liq} - \frac{2\gamma}{\rho r^2}\,dr = \frac{\partial u_{gas}}{\partial p}\,dp_{gas}$$

Due to Gibbs-Duhem: $du = -\frac{S}{N}dT + \frac{V}{N}dp$

then $$\frac{\partial u_{liq}}{\partial p}\,dp - \frac{2\gamma}{\rho r^2}\,dr = \frac{\partial u_{gas}}{\partial p}\,dp$$

$$\frac{V}{N}\,dp - \frac{2\gamma}{\rho r^2}\,dr = \frac{V_{gas}}{N_{gas}}\,dp$$

with ideal gas law: $pV = Nk_BT$, then $\frac{V_{gas}}{N_{gas}} = \frac{k_BT}{p}$

$$-\frac{2\gamma}{\rho r^2}\,dr = \frac{k_BT}{p_{gas}}\,dp_{gas} - \cancel{\frac{V}{N}\,dp_{liq}}$$

$$-\int_r^{\infty} \beta\frac{2\gamma}{\rho r^2}\,dr = \int_{p(r)}^{p(r=\infty)} \frac{dp}{p}$$

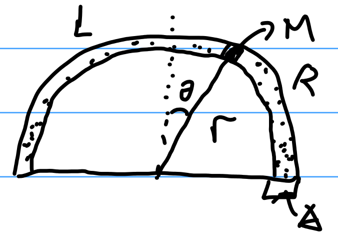
ignore liquid pressure since it is negligible compared to gas pressure.

$$\frac{2\beta\gamma}{\rho}\left(\frac{1}{r}\right)\Big|_r^{\infty} = \ln\left(\frac{p(r=\infty)}{p(r)}\right)$$

$$\hookrightarrow\quad -\frac{2\beta\gamma}{\rho r} = \ln\left(\frac{p(r=\infty)}{p(r)}\right)$$

$$\boxed{\therefore\quad p(r) = p(r=\infty)\exp\left\{\frac{2\beta\gamma}{\rho r}\right\}}$$

39) Continuous Phase Transitions.



$z = r\cos\theta$ $\downarrow g$

L: R: } $T_L = T_R$, $N_L = N_R$

a) Show for small θ, $A(\theta) = A_0 + A_2\theta^2 + A_4\theta^4$

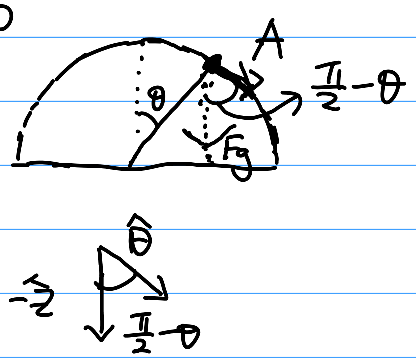
$$dA = -SdT - pdV + \mu dN \qquad A(\theta) \propto \cos\theta$$

Piston, M slides, need to find equilibrium position of the piston.

Since $T_L = T_R = T$ $\quad N = N_L = N_R$

→ since No friction, then $dQ = 0$, then $dT = 0$

→ since piston prevent exchange of particles $dN = 0$



$$\Rightarrow \quad dA = -pdV$$

$$dA_{piston} = \frac{-F}{d\sigma} dV = -\frac{\vec{F}}{\cancel{A}} \cancel{A} r d\vec{\theta} = -r\vec{F} \cdot d\vec{\theta}$$

$$\stackrel{!}{=} -(-Mg\hat{z}) \cdot r d\theta \hat{\theta}$$

$$= -Mg r d\theta (-\hat{z} \cdot \hat{\theta})$$

$$= -Mgr\cos\left(\frac{\pi}{2} - \theta\right)$$

$$\stackrel{!}{=} -Mg \sin\theta \, r d\theta$$

$$A_{piston.} \stackrel{!}{=} Mgr\cos\theta$$

For ideal gas: $q_1 = \frac{V}{\lambda_{th}^3}$

$$q_{1,R} = \frac{V}{\lambda_{th}^3} = \frac{\int \overset{\text{area}}{A} r\,d\theta}{\lambda_{th}^3} = \frac{Ar\left(\frac{\pi}{2}-\theta\right)}{\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}}$$

$$q_{1,L} = \frac{Ar\left(\frac{\pi}{2}+\theta\right)}{\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}}$$

then $A_R = -k_B T \ln\left[(q_{1,R})^N\right]$

$$\overset{!}{=} -Nk_BT\ln\left(\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} Ar\left(\frac{\pi}{2}-\theta\right)\right)$$

$$\overset{!}{=} -Nk_BT\frac{3}{2}\ln\left(\frac{2\pi m k_B T}{h^2}\right) - Nk_BT\ln(Ar) - Nk_BT\ln\left(\frac{\pi}{2}-\theta\right)$$

Similarly:

$$A_L = -Nk_BT\frac{3}{2}\ln\left(\frac{2\pi m k_B T}{h^2}\right) - Nk_BT\ln(Ar) - Nk_BT\ln\left(\frac{\pi}{2}+\theta\right)$$

$$A_L + A_R = -3Nk_BT\ln\left(\frac{2\pi m k_B T}{h^2}\right) - 2Nk_BT\ln(Ar) - Nk_BT\ln\left[\left(\frac{\pi}{2}+\theta\right)\left(\frac{\pi}{2}-\theta\right)\right]$$

$$A_L + A_R \overset{!}{=} -3Nk_BT\ln\left(\frac{2\pi m k_B T}{h^2}\right) - 2Nk_BT\ln(Ar) - Nk_BT\ln\left(\frac{\pi^2}{2}-\theta^2\right)$$

let $\theta^2 = x$, and taylor expand $\ln\left(\frac{\pi^2}{4}-x\right)$ for $x \ll \frac{\pi^2}{4}$

$$\ln\left(\frac{\pi^2}{4}-x\right) \cong \ln\left(\frac{\pi^2}{4}\right) - \left.\frac{1}{\frac{\pi^2}{4}-x}\right|_{x=0} x - \frac{1}{2}\left.\frac{1}{\left(\frac{\pi^2}{4}-x\right)^2}\right|_{x=0} x^2$$

$$\overset{!}{\cong} \ln\left(\frac{\pi^2}{4}\right) - \frac{4}{\pi^2}x - \frac{1}{2}\frac{16}{\pi^4}x^2$$

then with $x = \theta^2$

$$A_L + A_R = -3Nk_BT\ln\left(\frac{2\pi m k_BT}{h^2}\right) - 2Nk_BT\ln(Ar)$$
$$- Nk_BT\left(\ln\left(\frac{\pi^2}{4}\right) - \frac{4}{\pi^2}\theta^2 - \frac{8}{\pi^4}\theta^4\right)$$

Now $A_{tot} = A_L + A_R + A_{piston}$

$$A_{piston} = Mgr\cos\theta$$

for $\theta \ll 1$ $\overset{\text{def}}{=}$ $Mgr\left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4\right)$

$$\boxed{\begin{aligned} A_{tot}(\theta) = &-3Nk_BT\ln\left(\frac{2\pi m k_BT}{h^2}\right) - 2Nk_BT\ln(Ar) \\ &- Nk_BT\ln\left(\frac{\pi^2}{4}\right) + Mgr \\ &+ \underbrace{\left(Nk_BT\frac{4}{\pi^2} - \frac{1}{2}Mgr\right)}_{A_2}\theta^2 + \underbrace{\left(Nk_BT\frac{8}{\pi^4} + \frac{Mgr}{24}\right)}_{A_4}\theta^4 \end{aligned}}$$

b) $A_2 = 0 = Nk_BT\frac{4}{\pi^2} - \frac{Mgr}{2}$

then $T = T_c = \frac{Mgr}{2}\frac{\pi^2}{4}\frac{1}{Nk_B}$

$$\boxed{T_c = \frac{Mgr\pi^2}{8}\frac{1}{Nk_B}}$$

$$\frac{\partial A}{\partial \theta} = \frac{\partial}{\partial \theta}\left(A_0 + A_2\theta^2 + A_4\theta^4\right) = 0$$

$$= 2A_2\theta + 4A_4\theta^3 = 0$$

$\hookrightarrow 2\theta\left(A_2 + 2A_4\theta^2\right) = 0$

$\Rightarrow$ Need $\underline{\theta = 0}$ or $A_2 + 2A_4\theta^2 = 0$

$A_2 + 2A_4\theta^2 = 0$

$\hookrightarrow \theta_1 = \pm\sqrt{\frac{-A_2}{2A_4}}$

$$\pm\theta_1 = \pm\sqrt{\frac{-\left(Nk_BT\frac{4}{\pi^2} - \frac{Mgr}{2}\right)}{2\left(\frac{Mgr}{24} + Nk_BT\frac{8}{\pi^4}\right)}\left(\frac{\frac{2}{Mgr}}{\frac{2}{Mgr}}\right)}$$

$$\boxed{\pm\theta_1 = \pm\sqrt{\frac{1 - T/T_c}{4\left(\frac{1}{24} + \left(\frac{T}{T_c}\right)\frac{1}{\pi^2}\right)}}}$$

If $T > T_c$, or $\frac{T}{T_c} > 1$ then $1 - \frac{T}{T_c} < 0$

and since denominator is always positive, we see that the square root give imaginary value.

So, if $T > T_c$, we only have 1 solution, $\theta = 0$.

If $T < T_c$, $\frac{T}{T_c} < 1$, then $1 - \frac{T}{T_c} > 0$.

In this case, square root gives real answer for $\pm\theta$,

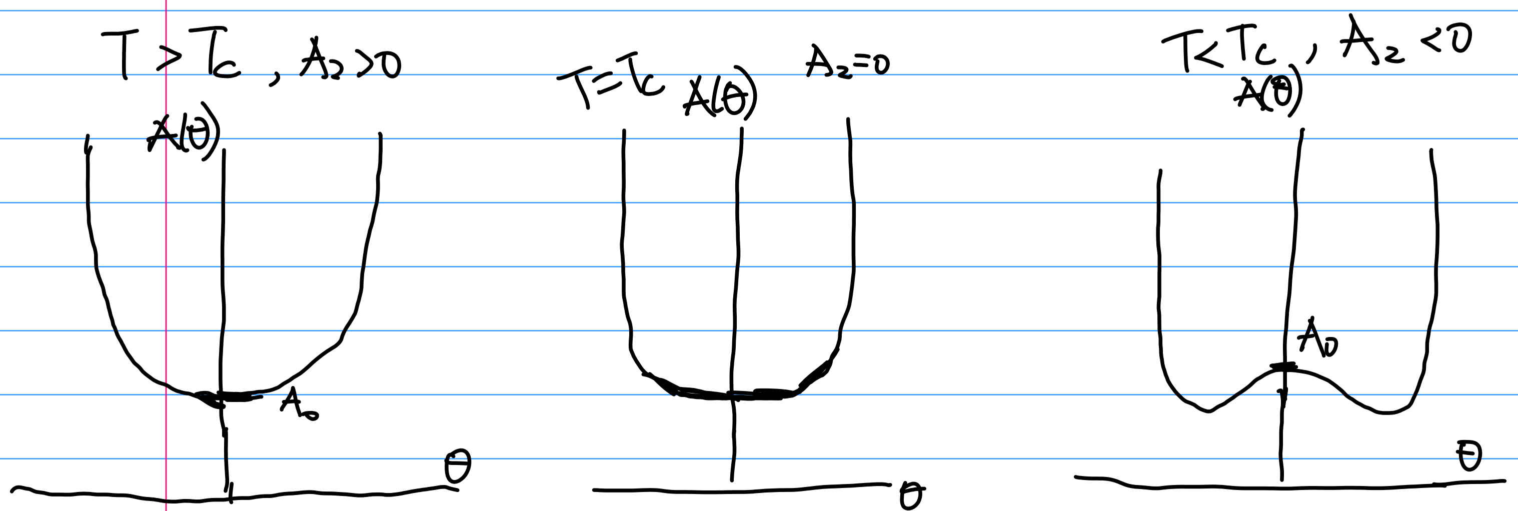
θ defined above

$$A(\theta) = \underbrace{A_0}_{\text{const}} + A_2\theta^2 + \underbrace{A_4}_{\text{always positive}}\theta^4$$

$$A_2 = Nk_BT\frac{4}{\pi^2} - \frac{Mgr}{2} = \frac{Mgr}{2}\left(\frac{T}{T_c} - 1\right)$$

positive for $T > T_c$

Negative for $T < T_c$



40) Three Ising Model:

$$\mathcal{H} = -J(s_1 s_2 + s_1 s_3 + s_2 s_3) - \mu H(s_1 + s_2 + s_3)$$

where $s_i = \pm 1$, $J > 0$, H is magnetic field

a) Enumerate the states of the system (energies and degeneracies) and calculate the partition function. What is the magnetization for $H=0$

$$Q = \sum_{s_1=\pm1} \sum_{s_2=\pm1} \sum_{s_3=\pm1} \exp\{\beta J(s_1 s_2 + s_1 s_3 + s_2 s_3) + \beta\mu H(s_1 + s_2 + s_3)\}$$

$$= \sum_{s_1=\pm1} \sum_{s_2=\pm1} \exp\{\beta J(s_1 s_2 + s_1 + s_2) + \beta\mu H(s_1 + s_2 + 1)\} \quad \uparrow$$

$$+ \exp\{\beta J(s_1 s_2 - s_1 - s_2) + \beta\mu H(s_1 + s_2 - 1)\} \quad \downarrow$$

$$= \sum_{s_1=\pm1} \exp\{\beta J(s_1 + s_1 + 1) + \beta\mu H(s_1 + 2)\} \quad \uparrow\uparrow$$

$$+ \exp\{\beta J(\cancel{s_1} - s_1 - 1) + \beta\mu H(s_1)\} \quad \uparrow\downarrow$$

$$+ \exp\{\beta J(\cancel{-s_1 + } s_1 - 1) + \beta\mu H(s_1)\} \quad \downarrow\uparrow$$

$$+ \exp\{\beta J(-s_1 - s_1 + 1) + \beta\mu H(s_1 - 2)\} \quad \downarrow\downarrow$$

$$Q = \sum \exp\{\beta J(3) + \beta\mu H(3)\} \overset{\uparrow\uparrow\uparrow}{} + \exp\{\beta J(-1) + \beta\mu H\} \overset{\uparrow\uparrow\downarrow}{}$$

$$+\exp\{\beta J(-1) + \beta\mu H\} \underset{\uparrow\downarrow\uparrow}{} + \exp\{\beta J(-1) + \beta\mu H(-1)\} \underset{\uparrow\downarrow\downarrow}{}$$

$$+\exp\{\beta J(-1) + \beta\mu H\} \overset{\downarrow\uparrow\uparrow}{} + \exp\{\beta J(-1) + \beta\mu H(-1)\} \overset{\downarrow\uparrow\downarrow}{}$$

$$+\exp\{\beta J(-1) + \beta\mu H(-1)\} \underset{\downarrow\downarrow\uparrow}{} + \exp\{\beta J(3) + \beta\mu H(-3)\} \underset{\downarrow\downarrow\downarrow}{}$$

$$H \begin{cases} = -(J + \mu H) & \text{degeneracy: } 1 \quad \uparrow\uparrow\uparrow \\ = -(\mu H - J) & \text{degeneracy: } 3 \quad \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \downarrow\uparrow\uparrow \\ = -(-\mu H - J) & \text{degeneracy: } 3 \quad \downarrow\downarrow\uparrow, \downarrow\uparrow\downarrow, \uparrow\downarrow\downarrow \\ = -(J - \mu H) & \text{degeneracy: } 1 \quad \downarrow\downarrow\downarrow \end{cases}$$

$$Q = \exp\{3(\beta J + \beta\mu H)\} + 3\exp\{\beta\mu H - \beta J\}$$

$$+3\exp\{-\beta\mu H - \beta J\} + \exp\{3(\beta J - \beta\mu H)\}$$

$$= 2\exp\{3\beta J\}\cosh(3\beta\mu H) + 6\exp\{-\beta J\}\cosh(\beta\mu H)$$

then $A = -k_B T \ln Q$

$$= -k_B T \ln\left[2\exp\{3\beta J\}\cosh(3\beta\mu H) + 6\exp\{-\beta J\}\cosh(\beta\mu H)\right]$$

$$dA = -SdT - MdH$$

$$M = \left(\frac{-\partial A}{\partial H}\right)_T = \frac{-\partial}{\partial H}\left(-k_B T \ln Q\right)$$

$$= k_B T \frac{1}{Q}\left(\frac{\partial}{\partial H} Q\right)_T$$

$$M = \frac{(\cancel{k_B T})\left[2\exp\{3\beta J\} 3\cancel{\beta} u \sinh(3\beta u H) + 6\exp\{-\beta J\} \cancel{\beta} u \sinh(\beta u H)\right]}{2\exp\{3\beta J\}\cosh(3\beta u H) + 6\exp\{-\beta J\}\cosh(\beta u H)}$$

$$= \frac{3u\left[\exp\{4\beta J\}\sinh(3\beta u H) + \sinh(\beta u H)\right]}{\exp\{4\beta J\}\cosh(3\beta u H) + 3\cosh(\beta u H)}$$

clearly as $\lim_{H\to 0} M(H,T)$, $\sinh \to 0$, $\cosh \to 1$

so $M(H=0,T) = 0$

b) Magnetic Susceptibility:

$$\chi = \left(\frac{\partial M}{\partial H}\right)_T = -\left(\frac{\partial^2 A}{\partial H^2}\right)_T$$

$$\chi = \frac{\partial}{\partial H}\left(\frac{3\mu\left[\exp\{4\beta J\}\sinh(3\beta\mu H) + \sinh(\beta\mu H)\right]}{\exp\{4\beta J\}\cosh(3\beta\mu H) + \cosh(\beta\mu H)}\right)_T$$

Find derivative using Sympy (see code)

$$\chi = \frac{3\beta\mu^2\left[3\exp\{8\beta J\} - 6\exp\{4\beta J\}\sinh(\beta\mu H)\sinh(3\beta\mu H) + 10\exp\{4\beta J\}\cosh(\beta\mu H)\cosh(3\beta\mu H) + 3\right]}{\left[\exp\{4\beta J\}\cosh(3\beta\mu H) + 3\cosh(\beta\mu H)\right]^2}$$

as $T \to$ large, $\beta\mu H, \beta J \ll 1$

Since $\chi \sim \beta\mu^2$, it is already first order, so only keep zero order when Taylor:

$\sinh(x) \sim x \sim 0 \qquad \exp(x) \sim 1 + x \sim 1$

$\cosh(x) \sim 1$

$$\boxed{\chi \sim 3\beta\mu^2 \frac{(3 + 10 + 3)}{(1+3)^2} \sim 3\beta\mu^2}$$

$$C(H,T) = \frac{\partial E}{\partial T} = \frac{\partial}{\partial T}\left(-\frac{\partial}{\partial \beta}\ln Q\right)$$

$$\hookrightarrow = \frac{\partial \beta}{\partial T}\frac{\partial}{\partial \beta}\left(-\frac{\partial}{\partial \beta}\ln Q\right)$$

$$= \frac{1}{k_B T^2}\frac{\partial^2}{\partial \beta^2}\ln Q$$

First find $\frac{\partial}{\partial \beta}\ln Q$

$$E = -\frac{\partial}{\partial \beta}\ln Q = \frac{-\partial}{\partial \beta}\ln\left(2\exp\{3\beta J\}\cosh(3\beta \mu H) + 6\exp\{-\beta J\}\cosh(\beta \mu H)\right)$$

$$= \frac{-6J\exp\{3\beta J\}\cosh(3\beta\mu H) - 6\mu H\exp\{3\beta J\}\sinh(3\beta\mu H) + 6J\exp\{-\beta J\}\cosh(\beta\mu H) - 6\mu H\exp\{-\beta J\}\sinh(\beta\mu H)}{2\exp\{3\beta J\}\cosh(3\beta\mu H) + 6\exp\{-\beta J\}\cosh(\beta\mu H)}$$

$$E = \frac{-3J\exp\{4\beta J\}\cosh(3\beta\mu H) - 3\mu H\exp\{4\beta J\}\sinh(3\beta\mu H) + 3J\cosh(\beta\mu H) - 3\mu H\sinh(\beta\mu H)}{\exp\{4\beta J\}\cosh(3\beta\mu H) + 3\cosh(\beta\mu H)}$$

$$C = \frac{\partial E}{\partial T} = \frac{\partial \beta}{\partial T}\frac{\partial}{\partial \beta}E$$

$$= -\frac{1}{k_B T^2}\frac{\partial}{\partial \beta}E$$

use simpl

$$\rightarrow = \frac{1}{k_B T^2} \frac{3}{\left[\exp\{4\beta J\}\cosh(3\beta\mu H) + 3\cosh(\beta\mu H)\right]^2} \Big[3\mu^2 H^2 \exp\{8\beta J\}$$

$$-6\mu^2 H^2 \exp\{4\beta J\}\sinh(\beta\mu H)\sinh(3\beta\mu H) + 10\mu^2 H^2 \exp\{4\beta J\}\cosh(\beta\mu H)\cosh(3\beta\mu H)$$

$$+ 3\mu^2 H^2 - 8 J\mu H \exp\{4\beta J\}\sinh(\beta\mu H)\cosh(3\beta\mu H)$$

$$+24 J\mu H \exp\{4\beta J\}\sinh(3\beta\mu H)\cosh(\beta\mu H)$$

$$+ 16 J^2 \exp\{4\beta J\}\cosh(\beta\mu H)\cosh(3\beta\mu H) \Big]$$

$$\rightarrow \simeq k_B \beta^2 \frac{3}{(1+3)^2}\left[3\mu^2 H^2 + 10\mu^2 H^2 + 3\mu^2 H^2 + 16 J^2\right]$$

$$\simeq k_B \beta^2 \frac{3}{16} 16 \left[\mu^2 H^2 + J^2\right]$$

$$\boxed{C \simeq 3k_B \left[\beta^2 \mu^2 H^2 + \beta^2 J^2\right]}$$