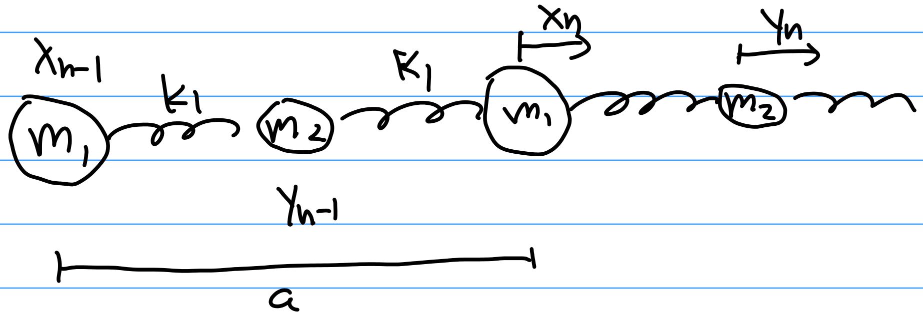
Zhi Chen HW# 10:

34) Optical and Acoustic Phonons:



$$x_n(t) = \tilde{A} e^{i(kna - wt)}$$

$$y_n(t) = \tilde{B} e^{i(kna - wt)}$$

} where $k = \frac{2\pi}{\lambda}$

a) Derive EOM for m_1 and m_2.

$$V = \frac{1}{2}k_1(y_{n-1} - x_n)^2 + \frac{1}{2}k_1(x_n - y_n)^2 + \frac{1}{2}k_1(y_n - x_{n+1})^2 + \text{other terms}$$

$$= \frac{1}{2}k_1\{ y_{n-1}^2 - 2y_{n-1}x_n + x_n^2 + x_n^2 + y_n^2 - 2x_n y_n + y_n^2 + x_{n+1}^2 - 2y_n x_{n+1} \}$$

$$= \frac{1}{2}k_1\{ y_{n-1}^2 - 2y_{n-1}x_n + 2x_n^2 + 2y_n^2 - 2x_n y_n + x_{n+1}^2 - 2y_n x_{n+1} \}$$

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m_1 \dot{x}_n^2 + \frac{1}{2} m_2 \dot{y}_n^2 + \text{other kinetic terms} - V$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_n} = \frac{\partial \mathcal{L}}{\partial x_n}$$

$$\hookrightarrow m_1 \ddot{x}_n = -\frac{\partial}{\partial x_n} V$$

$$= -k_1 \left(-y_{n-1} + 2x_n - y_n\right)$$

$$\hookrightarrow \boxed{\ddot{x}_n = \frac{-k_1}{m_1}\left(-y_{n-1} + 2x_n - y_n\right)}$$

$$\hookrightarrow m_2 \ddot{y}_n = -\frac{\partial}{\partial y_n} V$$

$$= -k_1 \left(2y_n - x_n - x_{n+1}\right)$$

$$\hookrightarrow \boxed{\ddot{y}_n = \frac{-k_1}{m_2}\left(-x_{n+1} + 2y_n - x_n\right)}$$

b) know $x_n(t) = \tilde{A} e^{i(kna - wt)}$

$$y_n(t) = \tilde{B} e^{i(kna - wt)}$$

$$\ddot{x}_n(t) = -w^2 \tilde{A} e^{i(kna - wt)} = -w^2 x_n$$

then $$\ddot{x}_n(t) = \frac{-k_1}{m_1}\left(2x_n - \tilde{B} e^{i(k(n-1)a - wt)} - y_n\right)$$

$$\overset{!}{=} -\frac{k_1}{m_1}\left(2x_n - \tilde{B} e^{i(kna - wt)} e^{-ika} - y_n\right)$$

$$-w^2 x_n \overset{!}{=} -\frac{k_1}{m_1}\left(2x_n - y_n\left(e^{-ika} + 1\right)\right)$$

$$\hookrightarrow \left(\frac{2k_1}{m_1} - w^2\right) x_n - \frac{k_1}{m_1}\left(e^{-ika} + 1\right) y_n = 0$$

$$\Rightarrow \left(\frac{2k_1}{m_1} - w^2\right)\tilde{A} - \frac{k_1}{m_1}\left(e^{-ika} + 1\right)\tilde{B} = 0 \qquad (1)$$

$$\ddot{y}_n(t) = -w^2 \tilde{B} e^{i(kna - wt)} = -w^2 y_n$$

$$\ddot{y}_n(t) = \frac{-k_1}{m_2}\left(-\tilde{A} e^{i(k(n+1)a - wt)} + 2y_n - x_n\right)$$

$$\overset{!}{=} \frac{-k_1}{m_2}\left(-\underbrace{\tilde{A} e^{i(kna - wt)}}_{x_n} e^{ika} - x_n + 2y_n\right)$$

$$-w^2 y_n = \frac{-k_1}{m_2}\left(2y_n - x_n\left(e^{ika} + 1\right)\right)$$

$$\hookrightarrow \left(2\frac{k_1}{m_2} - w^2\right) y_n - \frac{k_1}{m_2}\left(e^{ika} + 1\right) x_n = 0$$

$$\hookrightarrow \left(2\frac{k_1}{m_2} - w^2\right)\tilde{B} - \frac{k_1}{m_2}\left(e^{ika} + 1\right)\tilde{A} = 0 \qquad (2)$$

With $\ddot{x}_n$ and $\ddot{y}_n$, we have 2 EOMs giving:

$$① \quad \left(\frac{2k_1}{m_1} - w^2\right)\tilde{A} - \frac{k_1}{m_1}\left(e^{-ika}+1\right)\tilde{B} = 0$$

$$\frac{\tilde{A}}{\tilde{B}} = \frac{\frac{k_1}{m_1}\left(e^{-ika}+1\right)}{\frac{2k_1}{m_1} - w^2}$$

$$② \quad \left(2\frac{k_1}{m_2} - w^2\right)\tilde{B} - \frac{k_1}{m_2}\left(e^{ika}+1\right)\tilde{A} = 0$$

$$\frac{\tilde{A}}{\tilde{B}} = \frac{\frac{2k_1}{m_2} - w^2}{\frac{k_1}{m_2}\left(e^{ika}+1\right)}$$

Combine ① and ②:

$$\frac{\frac{k_1}{m_1}\left(e^{-ika}+1\right)}{\frac{2k_1}{m_1} - w^2} = \frac{\frac{2k_1}{m_2} - w^2}{\frac{k_1}{m_2}\left(e^{ika}+1\right)}$$

$$\hookrightarrow \frac{k_1^2}{m_1 m_2}\left(e^{-ika}+1\right)\left(e^{ika}+1\right) = \left(\frac{2k_1}{m_2} - w^2\right)\left(\frac{2k_1}{m_1} - w^2\right)$$

$$\hookrightarrow \frac{k_1^2}{m_1 m_2}\left(2 + \underbrace{e^{-ika} + e^{ika}}_{2\cos(ka)}\right) = \frac{4k_1^2}{m_1 m_2} - 2k_1\left(\frac{1}{m_2} + \frac{1}{m_1}\right)w^2 + w^4$$

$$\hookrightarrow \frac{2k_1^2}{m_1 m_2}\left(1 + \cos(ka)\right) = w^4 - 2k_1\left(\frac{m_1+m_2}{m_1 m_2}\right)w^2 + \frac{4k_1^2}{m_1 m_2}$$

$$\hookrightarrow \omega^4 - 2k_1\left(\frac{m_1+m_2}{m_1 m_2}\right)\omega^2 + \frac{2k_1^2}{m_1 m_2}\left(1-\cos ka\right) = 0$$

$$\omega^2 = \frac{\cancel{2}k_1\left(\frac{m_1+m_2}{m_1 m_2}\right) \pm \sqrt{4k_1^2\left(\frac{m_1+m_2}{m_1 m_2}\right)^2 - \frac{8k_1^2}{m_1 m_2}(1-\cos ka)}}{\cancel{2}}$$

$$\overset{!}{=} k_1\left(\frac{m_1+m_2}{m_1 m_2}\right) \pm k_1\left(\frac{m_1+m_2}{m_1 m_2}\right)\sqrt{1 - 2\frac{m_1 m_2}{(m_1+m_2)^2}(1-\cos ka)}$$

$$\omega^2 \overset{!}{=} k_1\left(\frac{m_1+m_2}{m_1 m_2}\right)\left\{1 \pm \sqrt{1-2\frac{m_1 m_2}{(m_1+m_2)^2}(1-\cos ka)}\right\}$$

know $1-\cos ka = 2\sin^2\left(\frac{ka}{2}\right)$

$$\boxed{\omega_\pm = \sqrt{k_1\left(\frac{m_1+m_2}{m_1 m_2}\right)\left\{1 \pm \sqrt{1-4\frac{m_1 m_2}{(m_1+m_2)^2}\sin^2\left(\frac{ka}{2}\right)}\right\}}}$$

only positive root here

Both +, − to give 2 solutions.

c) Plot from $ka = [-\pi, +\pi]$

use $m_1 = \frac{126}{6} m_2$.

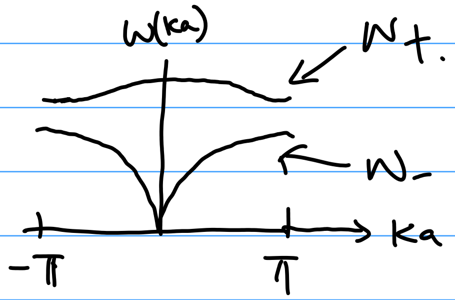
express y-axis in terms of $w_2 = \sqrt{\frac{2k_1}{m_2}}$

$$\frac{w_\pm}{\sqrt{\frac{2k_1}{m_2}}} = \sqrt{\frac{\frac{\left(\frac{126}{6}m_2 + m_2\right)\cancel{k_1}}{\frac{126}{6}m_2^{\cancel{2}}}}{\frac{2\cancel{k_1}}{\cancel{m_2}}} \left\{1 \pm \sqrt{1 - 4\frac{\frac{126}{6}\cancel{m_2^2}}{\left(\frac{126}{6}+1\right)^2\cancel{m_2^2}} \sin^2\left(\frac{ka}{2}\right)}\right\}}$$

$$\frac{w_\pm}{w_2} = \sqrt{\frac{1}{2}\left(\frac{132}{126}\right)\left\{1 \pm \sqrt{1 - 4\frac{\frac{126}{6}}{\left(\frac{132}{6}\right)^2} \sin^2\left(\frac{ka}{2}\right)}\right\}}$$

c) Interpret Result:

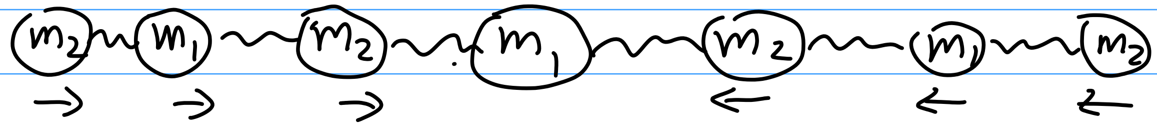
The plot looked like



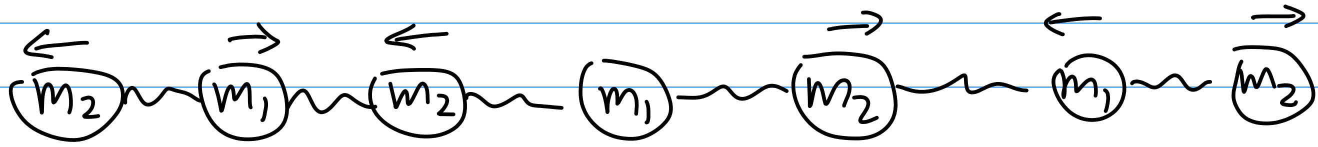
see actual plot in attached file.

we first observe that W- has a much larger bandwidth or larger range w(k). This means the difference between the max and min w(k) is significantly larger compared to W+.

we see that for W-, dispersion relation looks like what we got when considering a chain of identical atoms that we derived in class. This suggests that atoms are moving in-phase with neighboring atoms or



On the other hand, W+ goes out of phase between neighboring atoms



35) Heat capacity of a 1D atom-chain:

Find $c_L = \frac{C_L}{L}$ as a function of temperature: from $T=0$ to $T \gg \frac{\hbar\omega_2}{k_B} = \Theta_2$ or $\frac{T}{\Theta_2} = T' \gg 1$

Plot $\frac{c_L}{\frac{2k_B}{a}}$ vs. $\frac{T}{\Theta_2} = \frac{k_B T}{\hbar\omega_2} = T'$

$$\frac{c_L}{\frac{2k_B}{a}} = \left(\frac{dQ}{dT}\right)_L \frac{a}{2k_B L} = \left(\frac{dE}{dT}\right)_L \frac{a}{2k_B L}$$

$$T = \frac{\hbar\omega_2}{k_B} T' \Rightarrow dT = \frac{\hbar\omega_2}{k_B} dT'$$

$$\hookrightarrow \left(\frac{dE}{dT}\right)_L \frac{a}{2k_B L} = \left(\frac{dE}{dT'}\right)_L \frac{\cancel{k_B}}{\hbar\omega_2} \frac{a}{2\cancel{k_B} L}$$

$$\overset{!}{=} \left(\frac{dE}{dT'}\right)_L \frac{a}{2\hbar\omega_2 L}$$

$$\langle E \rangle = \sum_{k\text{-modes}} \langle n \rangle(k)\, \underbrace{\hbar\omega(k)}_{\equiv \varepsilon}$$

$$\overset{!}{=} \int g(k) \frac{1}{e^{\beta(\hbar\omega - \cancel{\mu})} - 1} \hbar\omega \, dk$$

$$\overset{!}{=} \int_{-\pi/a}^{\pi/a} \frac{L}{2\pi} \hbar\omega \left[\exp\left\{\frac{\cancel{\hbar}\omega}{\cancel{k_B}} \frac{\cancel{k_B}}{T' \cancel{\hbar}\omega_2}\right\} - 1\right]^{-1} dk$$

$$\overset{!}{=} \int_{-\pi/a}^{\pi/a} \frac{L}{2\pi} \hbar\omega \frac{1}{\exp\left\{\frac{\omega}{\omega_2} \frac{1}{T'}\right\} - 1} dk$$

then
$$\langle E' \rangle \equiv \frac{\langle E \rangle}{2\hbar \omega_2 L} a = \int_{-\pi/a}^{\pi/a} \frac{a}{4\pi}\left(\frac{\omega}{\omega_2}\right) \frac{1}{\exp\{\frac{\omega}{\omega_2}\frac{1}{T'}\} - 1} dk$$

define $k' \equiv ka$ then $dk = \frac{dk'}{a}$ define $\omega' \equiv \frac{\omega}{\omega_2}$

$$\langle E' \rangle \hookrightarrow = \int_{-\pi}^{\pi} \frac{1}{4\pi} \omega' \frac{1}{\exp\{\omega' \frac{1}{T'}\} - 1} dk'$$

Since the integrand is symmetric regardless of $\omega_\pm$ then

$$\langle E' \rangle = 2\int_0^{\pi} \frac{1}{4\pi} \omega' \frac{1}{\exp\{\omega' \frac{1}{T'}\} - 1} dk'$$

We see $\frac{C_L}{\frac{2k_B}{a}}$ indeed goes to 1 at $T \gg \Theta_2$

This is expected since at high T, we have equipartion theorem. Since each normal mode attribute to $k_B T$ due to both kinetic and potential energy, then we have $E = 2k_B T$ for 2 modes. So $\frac{dE}{dT} \sim 2k_B$.

36) Bose-Einstein Condensation with Tight Confinement:

a) Below Transition Temp, $T < T_c$, derive approximate expression for $z = e^{\beta\mu}$, in terms of N_0. Use result to estimate the magnitude of error in the approximation $g_{3/2}(z) \approx g_{3/2}(z)$ for $T < T_c$ in terms of N_0.

$$\Rightarrow \quad N_0 = \frac{z}{1-z} \qquad \text{as} \quad z \to 1$$

$$N_0(1-z) = z$$

$$N_0 = (1+N_0)z$$

$$\hookrightarrow \boxed{z = \frac{N_0}{1+N_0} = \frac{1}{1+\frac{1}{N_0}} \cong 1 - \frac{1}{N_0}}$$

$$\Rightarrow g_{3/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \left(1 - \frac{1}{N_0}\right)^n$$

$$\cong \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \left(1 - \frac{n}{N_0}\right)$$

$$= g_{3/2}(z=1) - \frac{1}{N_0} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\Rightarrow g_{3/2}(z) - g_{3/2}(1) = \frac{-1}{N_0} \sum_{n=1}^{\infty} \lim_{z \to 1} \frac{z}{n^{1/2}}$$

$$= \lim_{z \to 1} \frac{-1}{N_0} g_{1/2}(z)$$

since $z = e^{\beta\mu} = e^{-\alpha}$

$$= \lim_{\alpha \to 0} \frac{-1}{N_0} g_{1/2}(e^{-\alpha})$$

use Appendix D.8 from Pathria: $g_\nu(e^{-\alpha}) \approx \frac{\Gamma(1-\nu)}{\alpha^{1-\nu}}$ for $0 < \nu < 1$

$$= \frac{-1}{N_0} \frac{\Gamma(1/2)}{\sqrt{\alpha}}$$

$$= \frac{-1}{N_0}\sqrt{\pi}\,\frac{1}{\sqrt{-\beta u}}$$ since $z = e^{\beta u} = e^{-\alpha}$

$\alpha = -\beta u$

note $u < 0$ for boson

Since $N_0 = \dfrac{z}{1-z} = \dfrac{1}{\frac{1}{z}-1} = \dfrac{1}{e^{-\beta u}-1}$

as $z = e^{\beta u} \to 1$, βu must be small and negative.

then $N_0 = \dfrac{1}{e^{-\beta u}-1} \approx \dfrac{1}{1-\beta u-1} = \dfrac{1}{-\beta u}$

or $\sqrt{N_0} = \sqrt{\dfrac{1}{-\beta u}}$

$$= \frac{-1}{N_0}\sqrt{\pi}\sqrt{N_0}$$

$\boxed{\text{Error} \cong -\sqrt{\dfrac{\pi}{N_0}}}$ or $\text{Error} \propto \dfrac{1}{\sqrt{N_0}}$

b) Generally make small error with $N = \sum \langle n(\varepsilon_j) \rangle \cong \int_0^\infty d\varepsilon\, g(\varepsilon) \langle n(\varepsilon) \rangle$ if the occupation of any particular state is small compared to total

Show that for $T \lessapprox T_c$, $\frac{\langle n(\varepsilon_1) \rangle}{N_0} \propto N^{-1/3}$

such that it is negligible compared to N_0 and N'

$$N = \sum_\varepsilon \langle n(\varepsilon) \rangle = \sum_\varepsilon \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} = \sum_\varepsilon \frac{1}{z^{-1} e^{\beta\varepsilon} - 1}$$

$$N_1 = \langle n(\varepsilon_1) \rangle = \frac{1}{e^{\beta(\varepsilon_1 - \mu)} - 1} = \frac{z}{e^{\beta\varepsilon_1} - z}$$

For $T < T_c$, $z \sim 1$, assume $\beta\varepsilon_1$ is small

$$N_1 = \frac{1}{e^{\beta\varepsilon_1} - 1} \cong \frac{1}{1 + \beta\varepsilon_1 - 1} = \frac{1}{\beta\varepsilon_1} = \frac{k_B T}{\varepsilon_1}$$

For $T < T_c$, let $T \approx T_c \propto \left(\frac{N}{V}\right)^{2/3}$

So $N_1 \propto N^{2/3}$

For $T < T_c$, we know $N_0 = N\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)$

or $N_0 \propto N$

$$\boxed{\therefore \frac{N_1}{N_0} \propto \frac{N^{2/3}}{N} \propto N^{-1/3}}$$

← since $\propto N^{-1/3}$, it is negligible as N goes large.

c) Workout BEC for 2D:

$$N' = \langle N' \rangle = \sum_j \langle n(\varepsilon_j) \rangle \approx \int_0^\infty d\varepsilon\, g(\varepsilon) \langle n(\varepsilon) \rangle$$

$$\hookrightarrow = \int_0^\infty d\varepsilon \underbrace{\frac{mA}{2\pi\hbar^2}}_{2D\ g(\varepsilon)} \left(\frac{1}{z^{-1} e^{\beta\varepsilon} - 1} \right)$$

$$\stackrel{!}{=} \frac{mA}{2\pi\hbar^2} \frac{-\ln(1-z)}{\beta}$$

$$\stackrel{!}{=} \frac{mA}{2\pi\hbar^2} \left[-\ln(1 - e^{\beta\mu}) \right] k_B T$$

Since $\beta\mu$ is small, $\ln(1-e^{\beta\mu}) \approx \ln(1-1-\beta\mu)$

$$N' = \frac{mA}{2\pi\hbar^2} \left[-\ln(\beta\mu) \right] k_B T.$$

Define T_c when $N \sim N'$

$$T_c = \frac{2\pi\hbar^2}{m} \frac{1}{\ln\left(\frac{1}{\beta\mu}\right)} k_B \frac{N}{A}$$

as $\lim_{\mu \to 0}, \frac{1}{\ln\left(\frac{1}{\beta\mu}\right)} \to 0$

$$\boxed{\therefore \quad T_c \approx 0}$$

Indeed we see no BEC at nonzero temp, as BEC form when $T \lesssim T_c \cong 0$ for 2D.

Then $N = N_0 + N'$

$$\hookrightarrow \frac{N_0}{N} = 1 - \frac{N'}{N} = 1 - \left(\frac{T}{T_c} \right)^3$$

$\leftarrow$ as $T_c \to 0$, it becomes unbounded.

Consider a gas of N non-interacting spin less bosons confined in potential:

$$V(x,y,z) = \frac{1}{2} m \omega_z^2 z^2 + \frac{1}{2} m \omega_{xy}^2 (x^2+y^2)$$

d) Find T_c when $\hbar\omega_z \ll k_B T_c$ and $\hbar\omega_{xy} \ll k_B T_c$

$$H = \frac{p^2}{2m} + V$$

$$= \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + \frac{1}{2}m\left(\omega_z^2 z^2 + \omega_{xy}^2 x^2 + \omega_{xy}^2 y^2\right)$$

With separation of variables, theses are 3 1-D HO problem

With: $E_x = (n_x + \frac{1}{2})\hbar\omega_{xy}$

$E_y = (n_y + \frac{1}{2})\hbar\omega_{xy}$

$E_z = (n_z + \frac{1}{2})\hbar\omega_z$

$$E = E_x + E_y + E_z = \hbar\omega_{xy}(n_x + n_y + 1) + \hbar\omega_z(n_z + \frac{1}{2})$$

Ignore zero-point energy

$$E_{x,y,z} = \sum_{n_x,n_y,n_z=0}^{\infty} \hbar\omega_{xy} n_x + \hbar\omega_{xy} n_y + \hbar\omega_z n_z$$

$$N(E) = \iiint dn_x\, dn_y\, dn_z$$

$$= \iiint \frac{dE_x}{\hbar\omega_{xy}} \frac{dE_y}{\hbar\omega_{xy}} \frac{dE_z}{\hbar\omega_z}$$

$$= \frac{1}{(\hbar\omega_{xy})^2 \hbar\omega_z} \int_0^E \int_0^{E-E_x} \int_0^{E-E_x-E_y} dE_z\, dE_y\, dE_x.$$

$$= \frac{1}{(\hbar\omega_{xy})^2 \hbar\omega_z} \int_0^E \int_0^{E-E_x} (E - E_x - E_y)\, dE_y\, dE_x$$

$$= \frac{1}{(\hbar\omega_{xy})^2 (\hbar\omega_z)} \int_0^E \left[(E - E_x)(E - E_x) - \frac{(E-E_x)^2}{2} \right] dE_x$$

$$= \frac{1}{(\hbar\omega_{xy})^2 (\hbar\omega_z)} \int_0^E \frac{(E-E_x)^2}{2}\, dE_x$$

$$N(E) = \frac{1}{(\hbar\omega_{xy})^2 (\hbar\omega_z)} \frac{E^3}{6}$$

then $g(E) = \frac{dN}{dE} = \frac{E^2}{2(\hbar\omega_{xy})^2 \hbar\omega_z}$

change $E \to \varepsilon$:

Density of state.

$$g(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega_{xy})^2 \hbar\omega_z}$$

with

$$\langle N \rangle = \sum_{n_x, n_y, n_z} \langle n(\varepsilon) \rangle = \frac{1}{\exp\{\beta(\varepsilon - \mu)\} - 1}$$

$$\to = \sum \left[\exp\{\beta(\hbar\omega_{xy} n_x + \hbar\omega_{xy} n_y + \hbar\omega_z n_z)\} \exp\{-\beta\mu\} - 1 \right]^{-1}$$

$$N' = \int_0^\infty d\varepsilon \; g(\varepsilon) \frac{1}{\exp\{\beta(\varepsilon-u)\}-1}$$

$$= \int_0^\infty d\varepsilon \frac{\varepsilon^2}{2(\hbar\omega_{xy})^2(\hbar\omega_z)} \frac{1}{\exp\{\beta(\varepsilon-u)\}-1}$$

let $x=\beta\varepsilon$

$z=e^{\beta u}$

$$= \frac{1}{2(\hbar\omega_{xy})^2(\hbar\omega_z)} \int_0^\infty \frac{x^2\frac{1}{\beta^2}}{z^{-1}e^x-1} dx \frac{1}{\beta}$$

$$= \frac{(k_B T)^3}{2(\hbar\omega_{xy})^2(\hbar\omega_z)} \underbrace{\int_0^\infty \frac{x^2}{z^{-1}e^x-1} dx}_{\underbrace{\Gamma(3)}_{=2} g_3(z)}$$

$$N' = \frac{(k_B T)^3}{(\hbar\omega_{xy})^2(\hbar\omega_z)} g_3(z)$$

Define T_c when $N \sim N'$ as $z \to 1$

$$N = \frac{(k_B T_c)^3}{(\hbar\omega_{xy})^2 \hbar\omega_z} g(1)$$

$$g_3(z) = \sum_{n=1}^\infty \frac{z^n}{n^3} \xrightarrow{z=1} \sum_{n=1}^\infty \frac{1}{n^3} = \zeta(3) = 1.20206$$

then $$N = \frac{(k_B T_c)^2}{(\hbar\omega_{xy})^2} \frac{k_B T_c}{\hbar\omega_z} 1.20206$$

$$\boxed{k_B T_c = \left[\frac{N(\hbar\omega_{xy})^2\hbar\omega_z}{1.20206}\right]^{1/3}}$$

e) If $\hbar\omega_z \gg k_B T_c$, then we can approximate particles not having excited states in z, since $E = \hbar\omega_z n_z$, so approximate it is always in ground state, or $n_z = 0$

Then $E \cong \hbar\omega_{xy}(n_x + n_y)$, and solve 2D:

$$N = \int dn_x dn_y$$

$$= \frac{1}{(\hbar\omega_{xy})^2} \int_0^E \int_0^{E-E_y} dE_x dE_y$$

$$= \frac{1}{(\hbar\omega_{xy})^2} \int_0^E (E - E_y) dE_y$$

$$\stackrel{!}{=} \frac{1}{(\hbar\omega_{xy})^2} \left(E E_y \Big|_0^E - \frac{E_y^2}{2} \Big|_0^E \right)$$

$$N \stackrel{!}{=} \frac{E^2}{2(\hbar\omega_{xy})^2}$$

then $$\frac{dN}{dE} = g(E) = \frac{E}{(\hbar\omega_{xy})^2}$$

$$N' = \int_0^\infty d\varepsilon \; g(\varepsilon) \frac{1}{\exp\{\beta(\varepsilon - \mu)\} - 1}$$

let $x = \beta\varepsilon$

$$= \int_0^\infty \frac{\varepsilon}{(\hbar\omega_{xy})^2} \frac{1}{z^{-1}\exp\{\beta\varepsilon\} - 1} d\varepsilon$$

$$\stackrel{!}{=} \frac{(k_B T)^2}{(\hbar\omega_{xy})^2} \underbrace{\int_0^\infty \frac{x}{z^{-1}e^x - 1} dx}_{\underbrace{\Gamma(2)}_{=1} g_2(z)}$$

$$N' = \left(\frac{k_B T}{\hbar \omega_{xy}}\right)^2 g_2(z)$$

set $N = N'$, $z \to 1$, $T \to T_c$

$$N = \left(\frac{k_B T_c}{\hbar \omega_{xy}}\right)^2 \underbrace{g_2(1)}_{\frac{\pi^2}{6}}$$

then $$\boxed{k_B T_c = \sqrt{\frac{6}{\pi^2} N (\hbar \omega_{xy})^2}}$$

37) Bose-Einstein Condensate in a gas with weak attraction

$\psi(\vec{r})$ for ultra-cold atom clouds can be described by mean-field theory using the Gross-Pitaevski Equation:

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r}) + V_{ext}(\vec{r})\psi(\vec{r}) + \underbrace{\frac{Ng}{2}|\psi(r)|^2}_{\text{Mean-Field}}\psi(r) = \varepsilon_s\psi(\vec{r})$$

g: parameter characterizing interactions
For Li: $g < 0$.

a) $V_{ext}(r) = \frac{1}{2}m\omega^2 r^2$

use $\psi(r) = \frac{1}{a^{3/2}\pi^{3/4}}\exp\left\{\frac{-r^2}{2a^2}\right\}$ to estimate ground state energy: $E(a) = \langle\hat{H}\rangle(a)$

① $$\nabla^2\psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\frac{1}{a^{3/2}\pi^{3/4}}\exp\left\{\frac{-r^2}{2a^2}\right\}$$

$$= \frac{1}{a^{3/2}\pi^{3/4}r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{-r}{a^2}\right)\exp\left\{\frac{-r^2}{2a^2}\right\}\right)$$

$$= \frac{1}{a^{3/2}\pi^{3/4}r^2}\left(\frac{-1}{a^2}\right)\left(3r^2\exp\left\{\frac{-r^2}{2a^2}\right\} - \frac{r^4}{a^2}\exp\left\{\frac{-r^2}{2a^2}\right\}\right)$$

$$= \frac{1}{a^{3/2}\pi^{3/4}a^2}\left(\frac{r^2}{a^2} - 3\right)\exp\left\{\frac{-r^2}{2a^2}\right\}$$

$$\varepsilon_0 = \langle H \rangle = \int_0^\infty \left(\psi \nabla^2 \psi + \frac{1}{2} m \omega^2 r^2 |\psi|^2 + \frac{Ng}{2} |\psi|^2 |\psi|^2 \right) 4\pi r^2 dr$$

$$= \int_0^\infty \frac{-\hbar^2}{2m} \frac{1}{a^5 \pi^{3/2}} \left(\frac{r^2}{a^2} - 3 \right) \exp\left\{ \frac{-r^2}{a^2} \right\}$$

$$+ \frac{1}{2} m \omega^2 r^2 \frac{1}{a^3 \pi^{3/2}} \exp\left\{ \frac{-r^2}{a^2} \right\}$$

$$+ \frac{Ng}{2} \frac{1}{a^6 \pi^3} \exp\left\{ \frac{-2r^2}{a^2} \right\} \quad 4\pi r^2 dr$$

$$= 4\pi \left\{ \left(\frac{-\hbar^2}{2m} \right) \left(\frac{-3\sqrt{\pi}\, a^3}{8} \right) \frac{1}{a^5 \pi^{3/2}} + \frac{1}{2} m \omega^2 \frac{1}{a^3 \pi^{3/2}} \frac{3\sqrt{\pi}\, a^5}{8} \right.$$

$$\left. + \frac{Ng}{2} \frac{1}{a^6 \pi^3} \frac{\sqrt{\pi}\, a^3}{2^{7/2}} \right\}$$

$$\boxed{\varepsilon_0 = \frac{3}{4} \frac{\hbar^2}{m a^2} + \frac{3}{4} m \omega^2 a^2 + \frac{Ng}{a^3 \pi^{3/2} 2^{5/2}}}$$

b) Suppose $g>0$, neglect kinetic energy and find E_{min}

$$\frac{\partial \mathcal{E}_0}{\partial a} = \frac{\partial}{\partial a}\left(\frac{3}{4} m\omega^2 a^2 + \frac{Ng}{a^3 \pi^{3/2} 2^{5/2}} \right)$$

$$\overset{!}{=} \frac{3}{2} m\omega^2 a - \frac{3Ng}{\pi^{3/2} 2^{5/2}} \frac{1}{a^4} = 0$$

$$\frac{3}{2} m\omega^2 a = \frac{3Ng}{\pi^{3/2} 2^{5/2}} a^{-4}$$

$$a_{min}^5 = \frac{2}{3} \frac{1}{m\omega^2} \frac{3Ng}{\pi^{3/2} 2^{5/2}}$$

$$\boxed{a_{min} = \left(\frac{1}{m\omega^2} \frac{Ng}{\pi^{3/2} 2^{3/2}} \right)^{1/5}}$$

$\hookrightarrow$ we see $a_{min} \propto N^{1/5}$

$$\mathcal{E}_0(a=a_{min})\ H_{min} = \frac{3}{4} m\omega^2 \left(\frac{1}{m\omega^2} \frac{Ng}{\pi^{3/2} 2^{3/2}} \right)^{2/5} + \frac{Ng}{\pi^{3/2} 2^{5/2}} \left(\frac{m\omega^2 \pi^{3/2} 2^{3/2}}{Ng} \right)^{3/5}$$

$$H_{min} \overset{!}{=} (m\omega^2)^{3/5} (Ng)^{2/5} \left(\frac{3}{4} \frac{1}{\pi^{3/5} 2^{3/5}} + \frac{1}{\pi^{3/2} 2^{5/2}} \pi^{9/10} 2^{9/10} \right)$$

$$\overset{!}{=} (m\omega^2)^{3/5} (Ng)^{2/5} \left(\frac{3}{4} \frac{1}{\pi^{3/5} 2^{3/5}} + \frac{1}{\pi^{3/5} 2^{8/5}} \right)$$

$$\boxed{H_{min} = (m\omega^2)^{3/5} (Ng)^{2/5} \left(\frac{1}{\pi^{3/5} 2^{8/5}} \right) \left(\frac{5}{2} \right)}$$

$\therefore\ H_{min} \propto N^{3/5}$

c) For $g = \frac{4\pi\hbar^2}{m} l$ with $l = -1.5\,nm$

$$\mathcal{E}_0 = \frac{3}{4}\frac{\hbar^2}{ma^2} + \frac{3}{4} m\omega^2 a^2 + \frac{Ng}{a^3 \pi^{3/2} 2^{5/2}}$$

$$m_{Li7} = 7.01601 \times 1.66 \times 10^{-27}\,kg = 1.1646 \times 10^{-26}\,kg$$

$$\omega = 2\pi \times 145\,Hz$$

Vary N such that there is a local min for $\mathcal{E}(a)$
or choose N such that there is a_{min} such that $\frac{\partial \mathcal{E}}{\partial a}\Big|_{min} = 0$

$$\frac{\partial \mathcal{E}_0}{\partial a} = -\frac{3}{2}\frac{\hbar^2}{ma^3} + \frac{3}{2} m\omega^2 a - \frac{3Ng}{\pi^{3/2} 2^{5/2}} \frac{1}{a^4} = 0$$

$$= \frac{3}{2} m\omega^2 a^5 - \frac{3}{2}\frac{\hbar^2}{m} a + \frac{3N|g|}{\pi^{3/2} 2^{5/2}} = 0$$

$\Rightarrow$ see plots of $\frac{\partial \mathcal{E}_0}{\partial a}$ for different N

$\Rightarrow$ To find N_{max}, we keep increasing N, until we don't observe a divergent in $\frac{\partial \mathcal{E}_0}{\partial a}$ in a log scale, a divergent in log scale means $\frac{\partial \mathcal{E}_0}{\partial a} = 0$, or have a minimum in $\mathcal{E}_0(a)$.

$\Rightarrow$ This happens when $N_{max} = 1409$. Increasing further, we see finite numbers in $\frac{\partial \mathcal{E}_0}{\partial a}$, which is no longer minimum.