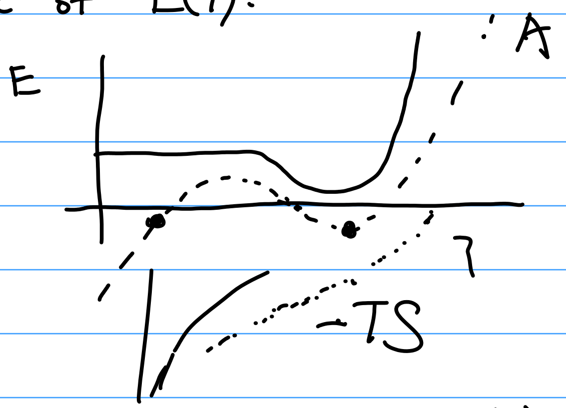
Minimize Free energy:

$$A = E - TS$$

consider an order parameter, ξ

$$A = E(\xi) - TS(\xi)$$

An example of $E(\xi)$:



two ξ giving the same minimum A.

Review: Classical gas, noninteracting

$$Q = \frac{1}{N!} q_1^N = \frac{1}{N!}\left(\frac{V}{\lambda_{th}^3} q_{int}\right)^N$$

$$A = -k_B T \ln Q = N k_B T \ln\left(\frac{\rho \lambda_{th}^3}{q_{int}}\right) - N k_B T$$

For phase transition, chemical potential stays same.

Use Gibbs - Free energy:

$$G = \mu N = A + pV$$

$$= A + N k_B T$$

$$G = N k_B T \ln\left(\frac{\rho \lambda_{th}^3}{q_{int}}\right)$$

Now do mean-field theory:

$$A(N, T, V) \to A(\underset{\substack{\uparrow \\ \text{excluded volume.}}}{N, T, V - Nb}) + \underset{\substack{\uparrow \\ \text{Mean-Field.}}}{E_{MF}}$$

$$E_{MF} \propto \rho^2 V = N\rho$$

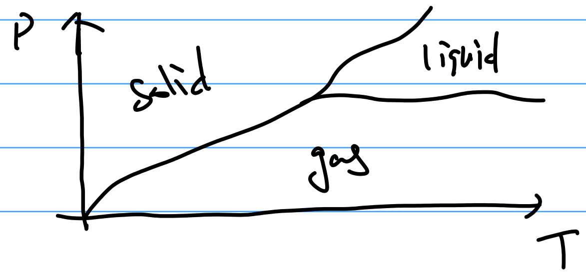
$$\boxed{E_{MF} = -N\rho a}$$

$$\boxed{\begin{aligned} A &= N\left[k_B T \ln\left(\frac{\rho \lambda_{th}^3}{q_{int}}\right) - k_B T - \rho a\right] \\ p &= -\rho^2 a + \rho \frac{k_B T}{1 - \rho b} \end{aligned}}$$

For van-der-Waals gas

$$\boxed{\mu = \frac{G}{N} = \frac{A + pV}{N} \Rightarrow \mu = k_B T \ln\left(\frac{\rho \lambda_{th}^3}{1 - \rho b}\right) - 2\rho a + k_B T \frac{\rho b}{1 - \rho b}}$$

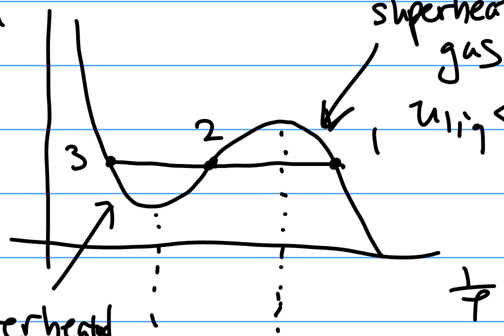
This phase transition has a latent heat:



Heating but stays at constant temp.

To find the latent heat:

$$\frac{dP_0}{dT} = \frac{S_2 - S_1}{V_2 - V_1} = \frac{L}{T\Delta V}$$



superheated gas

$u_{liq} < u_{gas}$

Superheated liquid

$u_{liq} > u_{gas}$

metastable

unstable, so it goes through line rather than curve.

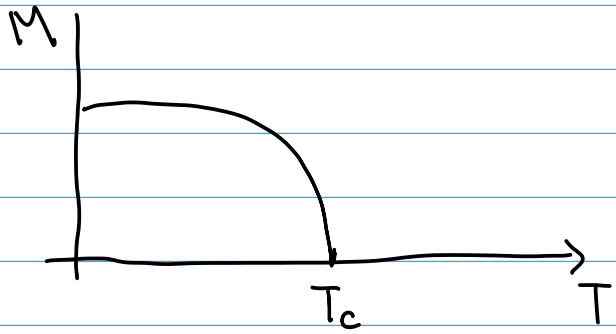
$$\int_1^3 dG = 0$$

$$\int_1^3 -\cancel{SdT} + Vdp + \cancel{\mu dN} = 0$$

$$\int_1^3 dV\left(p - p_0(T)\right) = 0$$

Continuous Phase Transition: No latent heat.

1st order Phase Transition: Has Latent heat



$$\partial A = -S\,dT - m\,dH$$

$$\langle M \rangle = -\frac{\partial A}{\partial H} = \text{continuous}$$

applied field

$$\chi = \frac{\partial \langle m \rangle}{\partial H} = -\left(\frac{\partial^2 A}{\partial H^2}\right)_{N,T} \xrightarrow{T=T_c} \infty$$

Near critical point

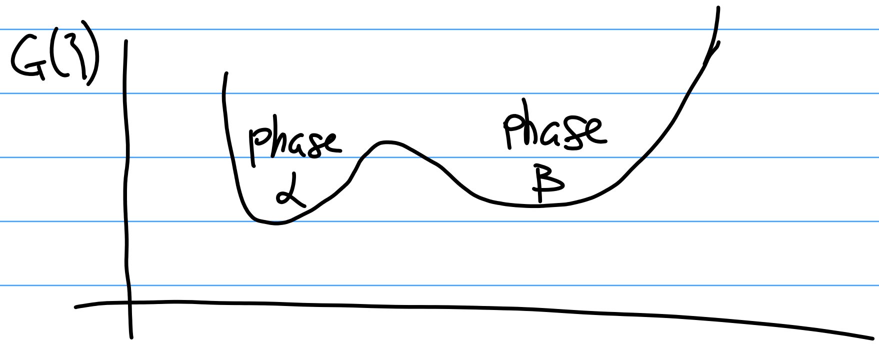
$$\chi \propto \frac{1}{\left(1-\frac{T}{T_c}\right)^{\gamma}} = \frac{1}{\tau^{\gamma}} \qquad \gamma \cong 1.3$$

$$C \propto \frac{1}{\left(1-\frac{T}{T_c}\right)^{\alpha}} \qquad \alpha \cong \frac{1}{8}$$

$$\zeta \propto \frac{1}{\tau^{\beta}} \qquad \beta \approx \frac{1}{3}$$

$$\alpha + 2\beta + \gamma = 2$$

Landau Theory:



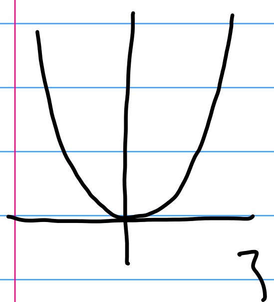
Near critical point:

$$G(\xi) = G_0 + G_2\xi^2 + G_4\xi^4$$

depend on T

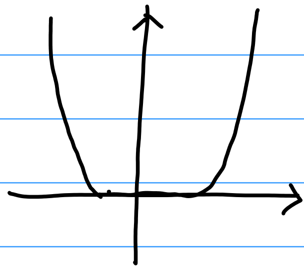
$T > T_c$

$G_2, G_4 > 0$



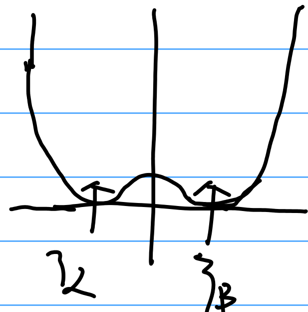
$T = T_c$

$G_2 = 0$



$T < T_c$

$G_2 < 0, \quad G_4 > 0$



symmetry breaking

External Field: $G \to G - \mu H\langle m \rangle$

applied field

$$g = \frac{G}{N} \Rightarrow \Delta g = \frac{G - G_0}{N} = -a\tau\xi^2 + \frac{1}{2}b\xi^4 - h\xi + f(|\vec{\nabla}\xi|^2)$$

$\tau = 1 - \frac{T}{T_c}$

$h \approx \mu H$ → magnetic field

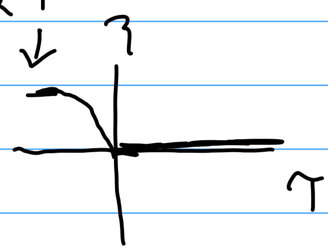
$f(|\vec{\nabla}\xi|^2)$ → kinetic

Take $h=0$, $f=0$

$$\frac{\partial g}{\partial \eta} = 0 = -2a\tau\eta + 2b\eta^3$$

For $T < T_c$: $\eta = \left(\frac{a\tau}{b}\right)^{1/2}$ $\propto \tau^{1/2}$

$T > T_c$: $\eta = 0$



Since $\eta \propto \frac{1}{\tau^\beta}$ $\boxed{\text{Find } \beta = \frac{1}{2}}$

<u>If $h \neq 0$</u>:

$$\frac{\partial g}{\partial \eta} = 0 = -2a\tau\eta + 2b\eta^3 - fh$$

$$\eta = \frac{-fh}{2a\tau}$$

Since $\chi \propto \frac{1}{\tau^\gamma}$ Find $\gamma = 1$

Ising Model:

⇒ Each particle is fixed on a lattice. (No Motion)

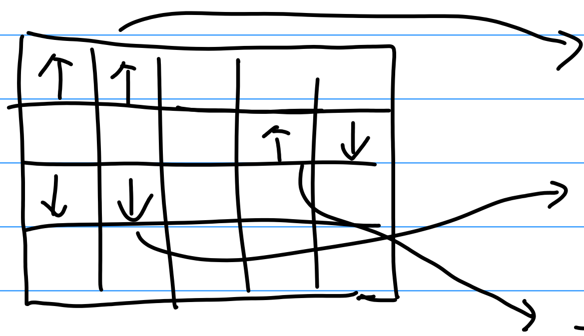
⇒ Each lattice can only be in one of two states. ↑↓

⇒ Each spin only interacts (parwise) with nearest neighbor.

$$\mathcal{H} = -J \sum_{\substack{(ij) \\ \text{pair}}} S_i S_j - \sum_{i=1}^{N} H u S_i$$

$S_{ij} = \pm 1$, classical spin

J = Interaction strength
H = external field
u = Magnetic moment

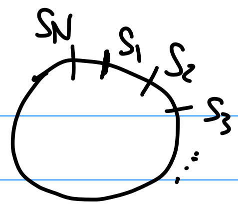


$-J(1)(1) = -J$
$-J(-1)(-1) = -J$
} lower energy

$-J(-1)(1) = +J$ ⇒ Higher Energy

$$Q(N, \beta, H) = \sum_{S_1 = \pm 1} \cdots\cdots \sum_{S_N = \pm 1} \exp\Big\{\underbrace{\beta u H}_{= h} \sum S_i + \underbrace{\beta J}_{= K} \sum_{\substack{ij \\ \text{pair}}} S_i S_j\Big\}$$

Consider 1D model:



$$Q = \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \exp\left[K S_i S_{i+1} + \tfrac{1}{2} h (S_i + S_{i+1})\right]$$

$\downarrow$ 4 possibilities in the end.

If $S_i = 1$, $S_{i+1} = 1$, $Q \propto e^{K+h}$

If $S_i = -1$, $S_{i+1} = 1$, $Q \propto e^{-K}$

If $S_i = 1$, $S_{i+1} = -1$, $Q \propto e^{K}$

If $S_i = -1$, $S_{i+1} = -1$, $Q \propto e^{K-h}$

Transfer Matrix:

$$P_{ij} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$

$$Q = \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \underbrace{\langle S_1 | \hat{P} | S_2 \rangle \langle S_2 | \hat{P} | S_3 \rangle \cdots \langle S_{N-1} | \hat{P} | S_N \rangle \langle S_N | \hat{P} | S_1 \rangle}_{\text{Diagonal}}$$

$$\mathrm{Tr}\left([P]^N\right) = \lambda_1^N + \lambda_2^N$$

Get: $\lambda = e^{K} \cosh(h) \pm \left[e^{-2K} + e^{2K} \sinh^2(h)\right]^{1/2}$

For $N \gg 1$,

then just consider the larger λ, for λ^N:

If $\lambda_+ \gg \lambda_-$, then

$$Q \approx \lambda_1^N$$

$$\frac{1}{N}\ln Q = \ln \lambda_1$$

$$= \ln\left[e^K \cosh(h) + (e^{-2K} + e^{2K}\sinh^2(h))^{1/2}\right]$$

$$A = -NJ - Nk_BT\ln\left[\cosh(h) + (e^{-4K} + \sinh^2(h))^{1/2}\right]$$

For $h=0$:

$$Q = (2\cosh(K))^N$$
$$A = -Nk_BT\ln(2\cosh(K))$$

$$\langle m\rangle = -\left(\frac{\partial A}{\partial H}\right)_T \qquad dE = TdS - MdH$$
$$dA = -SdT - MdH$$

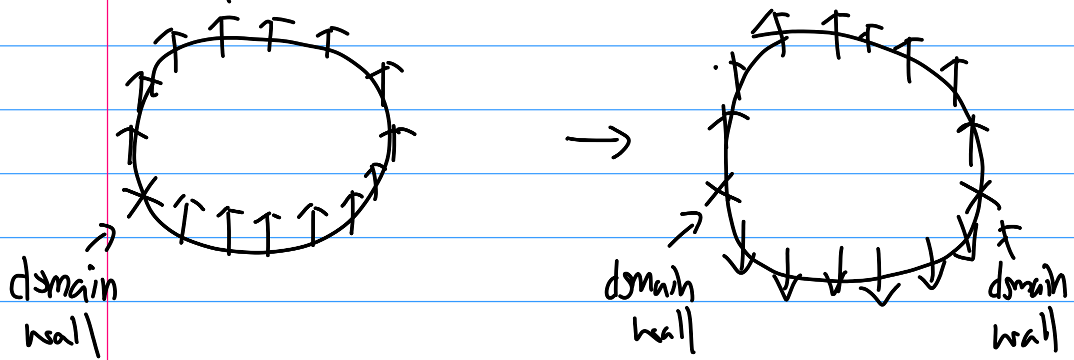
then

$$\langle M\rangle = \frac{N\mu\sinh(h)}{\left[e^{-4K} + \sinh^2(h)\right]^{1/2}}$$

as $$\lim_{h\to 0}\frac{N\mu\sinh(h)}{e^{-2K}} \to 0$$ unless $T \to 0$, so no phase transition at finite T.

If $J=0$: $\quad \langle M \rangle = N\mu \tanh(\beta\mu H)$

No phase transition, why?:



$$\Delta E = 2J + 2J$$

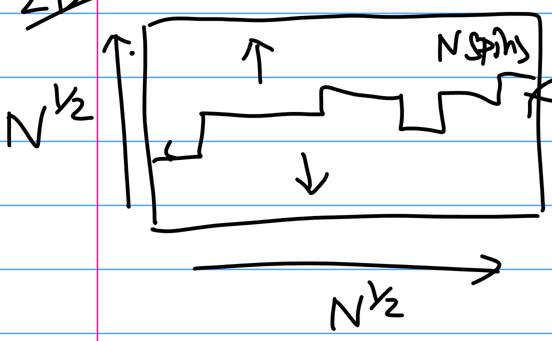
choice of putting wall

$$\Delta S = k_B \ln(N(N-1))$$

$$\Delta A = \Delta E - T\Delta S$$
$$= 4J - 2k_B T \ln N$$

For $\Delta A > 0$, $\quad T_c = \dfrac{2J}{k_B \ln N} \xrightarrow{\text{as } N\to\infty} 0$

2D:



domain wall has strength L

$\Delta E = 2JL$

$2N^{1/2}$ starting point, 3 choice after.

$\Omega \sim 2N^{1/2} 3^L$

$L \propto N$

$$\Delta A = 2JL - k_B T \ln(2N^{1/3}\, 3^L)$$

$\underline{\Delta A = 0}$

$$T_c \approx \frac{J}{k_B}(1.82)$$