Fourier Transform of N-Dimensional Gaussian Distribution

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TL;DR

Let $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \Sigma)$, $\mathbf{x} \in R^n$, the multivariate gaussian distribution is defined as

$$f(\mathbf{x}) = \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}\}$$

Fourier transform of $f(\mathbf{x})$ is given by

$$F(\mathbf{s}) = \mathcal{F}(f) = \int_{R^n} f(\mathbf{x}) \exp\{-i2\pi \mathbf{s}^T \mathbf{x}\} d\mathbf{x}$$

where $\mathbf{s} \in R^n$ and $F(s)$ is a scalar, i.e. F maps from R^n to R.

The Fourier transform of $f(\mathbf{x})$ is

$$F(\mathbf{s}) = \mathcal{F}(\mathcal{N}(\mathbf{0}, \Sigma)) = \exp\{-\frac{1}{2}(2\pi\mathbf{s})^T \Sigma (2\pi\mathbf{s})\}$$

furthermore,

$$\mathcal{F}(\mathcal{N}(\mu, \Sigma)) = \exp\{-i2\pi\mathbf{s}^T \mu\} \cdot \exp\{-\frac{1}{2}(2\pi\mathbf{s})^T \Sigma (2\pi\mathbf{s})\}$$

Derivation Steps

Expanding $F(\mathbf{s})$

$$\begin{aligned}
F(\mathbf{s}) = \mathcal{F}(f) &= \int_{R^n} f(\mathbf{x}) \exp\{-i2\pi \mathbf{s}^T \mathbf{x}\} d\mathbf{x} \\
&= \int_{R^n} \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}\} \exp\{-i2\pi \mathbf{s}^T \mathbf{x}\} d\mathbf{x} \\
&= \int_{R^n} \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}[\mathbf{x}^T \Sigma^{-1} \mathbf{x} - 2(-i2\pi\mathbf{s})^T \mathbf{x}]\} d\mathbf{x} \\
&= \int_{R^n} \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}\Delta^2\} d\mathbf{x}
\end{aligned}$$

where

$$\Delta^2 = \mathbf{x}^T \Sigma^{-1} \mathbf{x} - 2(-i2\pi\mathbf{s})^T \mathbf{x}$$

Now we use a standard trick called complete the square

$$\begin{aligned}
&\mathbf{x}^T A \mathbf{x} - 2\mathbf{b}^T \mathbf{x} = (\mathbf{x} - \mathbf{u})^T A (\mathbf{x} - \mathbf{u}) - \mathbf{u}^T A \mathbf{u} \\
&\text{with } \mathbf{u} = A^{-1}\mathbf{b} \\
&\quad A \text{ is symmetric and invertible}
\end{aligned}$$

The convariance matrix Σ is symmetric and invertible, hence it satisfies the condition. Therefore, we can substitute as

$$\begin{aligned}
A &= \Sigma^{-1} \\
\mathbf{b} &= -i2\pi\mathbf{s} \\
\rightarrow & \\
\mathbf{u} &= (\Sigma^{-1})^{-1}(-i2\pi\mathbf{s}) \\
&= \Sigma(-i2\pi\mathbf{s}) \\
\mathbf{u}^T A\mathbf{u} &= -i(2\pi\mathbf{s})^T\Sigma^T\Sigma^{-1}\Sigma(-i2\pi\mathbf{s}) \\
&= -(2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})
\end{aligned}$$

Put $\mathbf{u}$ and $\mathbf{u}^T A\mathbf{u}$ back into Δ^2, we have

$$\begin{aligned}
\Delta^2 &= (\mathbf{x}-\mathbf{u})^T A(\mathbf{x}-\mathbf{u}) - \mathbf{u}^T A\mathbf{u} \\
&= (\mathbf{x}-\mathbf{u})^T\Sigma^{-1}(\mathbf{x}-\mathbf{u}) + (2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})
\end{aligned}$$

Put Δ^2 back into $F(\mathbf{s})$ we have

$$\begin{aligned}
F(\mathbf{s}) &= \int_{R^n} \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}[(\mathbf{x}-\mathbf{u})^T\Sigma^{-1}(\mathbf{x}-\mathbf{u}) + (2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})]\}d\mathbf{x} \\
&= \int_{R^n} \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mathbf{u})^T\Sigma^{-1}(\mathbf{x}-\mathbf{u})\} \exp\{-\frac{1}{2}(2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})\}d\mathbf{x} \\
&= \left(\int_{R^n} \frac{1}{\sqrt{det(\Sigma)}(2\pi)^{n/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mathbf{u})^T\Sigma^{-1}(\mathbf{x}-\mathbf{u})\}d\mathbf{x}\right) \cdot \left(\exp\{-\frac{1}{2}(2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})\}\right)
\end{aligned}$$

where the first term is simply the sum of probability, i.e. equals to 1, therefore

$$F(\mathbf{s}) = \exp\{-\frac{1}{2}(2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})\}$$

By applying the shift property of Fourier transform, we can easily obtain

$$\begin{aligned}
\mathcal{F}(\mathcal{N}(\mu,\Sigma)) &= \mathcal{F}(f(\mathbf{x}-\mu)) \\
&= \exp\{-i2\pi\mathbf{s}^T\mu\} \cdot F(\mathbf{s}) \\
&= \exp\{-i2\pi\mathbf{s}^T\mu\} \cdot \exp\{-\frac{1}{2}(2\pi\mathbf{s})^T\Sigma(2\pi\mathbf{s})\}
\end{aligned}$$

References

[1] Solving ODEs and Fourier transforms. `https://warwick.ac.uk/fac/sci/mathsys/courses/msc/ma934/resources/notes8.pdf`