4 LU and LDU factorization

Inverse of AB ✓

$$A A^{-1} = I = A^{-1} A$$

$$\underbrace{(AB)(B^{-1}A^{-1})}_{(AB)^{-1}} = I = \underbrace{(B^{-1}A^{-1})}(AB)$$

Inverse of A^T ✓

$$A A^{-1} = I$$

transpose both sides, need to switch order

$$\underbrace{(A^{-1})^T}_{\text{this is } (A^T)^{-1}!} A^T = I$$

Elimination as $A = LU$ (instead of $EA = U$)

lower triangular (L), upper triangular (U)

ex. 2x2

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}$$

ex
3x3
(this example is meant to show that $A = LU$ has a big advantage over $EA = U$ for bigger matrices)

$E_{32}E_{31}E_{21}A = U$ (no row exchanges) assuming

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L}U$$

this product turns out to be better than that one

To see this, suppose $E_{31} = I$ for simplicity

$$\underset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}}\underset{E_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}} = \begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}$$

don't like this

10 is here bc we first removed 2 * row 1 from row 2, then 5 times row 2 from row 3

Now as L

$$\underset{E_{21}^{-1}}{\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}}\underset{E_{32}^{-1}}{\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}} = \begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}$$

no huge values

now 2 & 5 don't interfere

Summary

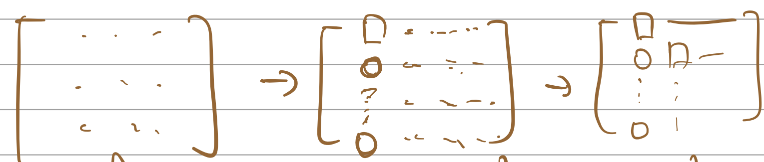
$$\boxed{A = LU}$$

If no row exchanges, multipliers go directly into L.

(* & −)

How many operations on an $n \times n$ matrix? for elimination to split into L & U.

$n = 100$ one step sec. step



A about 100^2 99^2

$$n^2 + (n-1)^2 + \dots + 1^2 \approx \frac{1}{3} n^3$$

cost of b is n^2 ← then we can process different b's at lower costs

What if we allow row exchanges?

Transpose & permutations

Next lecture