2 Elimination with Matrices

Example

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned}$$

$$A x = b$$

Elimination

first pivot

$$\underbrace{\begin{matrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix}}_{\text{this is the matrix that matters.}} \begin{matrix} 2 \\ 12 \\ 2 \end{matrix} \xrightarrow{(2,1)} \begin{matrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{matrix} \xrightarrow{(3,2)} \begin{matrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -1 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{matrix}$$

done here!

U c

let's call this U for upper triangular matrix

- pivots cannot be zero

how could I fail to come up with 3 pivots?

do row exchange

back substitution

$$\begin{aligned} x + 2y + z &= 2 \\ 2y - z &= 6 \\ 5z &= -10 \end{aligned}$$

$$\begin{aligned} z &= -2 \\ 2y - (-2) &= 6 \\ 2y + 4 &= 6 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x + 2(1) + (-2) &= 2 \\ x + 2 - 2 &= 2 \\ x &= 2 \end{aligned}$$

$$\begin{cases} x = 2 \\ y = 1 \\ z = -2 \end{cases}$$

Some things that made Gaussian elimination clearer to me

- all matrices can be transformed into row echelon form.
- even the zero matrix is in its row echelon form

☆ How to perform elimination with matrices?

hint:

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix} = \underbrace{1 * \text{row } 1 + 2 * \text{row } 2 + 7 * \text{row } 3}_{\text{linear comb. of the rows}}$$

$1 \times 3 \qquad 3 \times 3$

Step 1:

first step:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

E_{21} the matrix that make (2,1) zero

Step 2: subtract 2 * row 2 from row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

E_{32}

Summary

$$E_{32}(E_{21} A) = U$$

$$(E_{32} E_{21}) A = U \qquad \text{(associative law)}$$

☆ Permutation matrices

- row exchange $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$
- column exchange $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

☆ How to get from U back to A? (does this work for singular matrices?)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(by visual inspection)