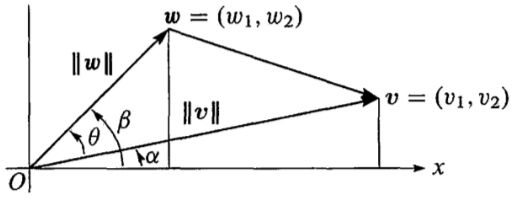
Problem Set 1

Sec 1.2 # 23

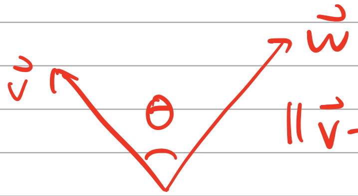


23 The figure shows that $\cos\alpha = v_1/\|v\|$ and $\sin\alpha = v_2/\|v\|$. Similarly $\cos\beta$ is _____ and $\sin\beta$ is _____. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos\beta\cos\alpha + \sin\beta\sin\alpha$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

$\cos\beta = \dfrac{w_1}{\|w\|} \qquad \sin\beta = \dfrac{w_2}{\|w\|}$

$$\cos(\theta) = \cos(\beta-\alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$
$$= \frac{w_1}{\|w\|}\frac{v_1}{\|v\|} + \frac{w_2}{\|w\|}\frac{v_2}{\|v\|}$$
$$= v\cdot w / (\|v\|\|w\|)$$

In general, for $\vec{v}, \vec{w} \in \mathbb{R}^n$,



$$\|\vec{v}-\vec{w}\|^2 = \underline{\|\vec{v}\|^2 + \|w\|^2 -}$$
$$\underline{2\|\vec{v}\|\|w\|\cos(\theta)}$$

LHS: $\|\vec{v} - \vec{w}\|^2$ //

$= (\vec{v} - \vec{w})\cdot(\vec{v} - \vec{w})$

$= \vec{v}\cdot\vec{v} - 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$

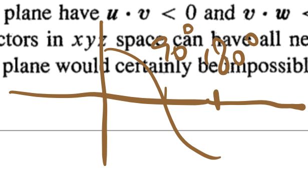
$= \underline{\|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2}$

$\vec{v}\cdot\vec{w} =$
$\|\vec{v}\|\|\vec{w}\|\cos\theta$

Sec 1.2 # 28

28 Can three vectors in the xy plane have $\boldsymbol{u}\cdot\boldsymbol{v} < 0$ and $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ and $\boldsymbol{u}\cdot\boldsymbol{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would certainly be impossible . . .).

90° 180°



$\vec{u}\cdot\vec{v} < 0 \Rightarrow \cos\theta < 0 \Rightarrow 90^\circ < \theta < 180^\circ$

at most 3

Sec 1.3 # 4

Find a combination $x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3$ that gives the zero vector:

$$\boldsymbol{w}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix} \quad \boldsymbol{w}_3 = \begin{bmatrix}7\\8\\9\end{bmatrix}.$$

Those vectors are (independent) (dependent). The three vectors lie in a plane. The matrix W with those columns is *not invertible*.

$$2w_2 - w_1 = \begin{bmatrix}8\\10\\12\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}7\\8\\9\end{bmatrix}$$

Sec 1.3 #13

The very last words say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $C\boldsymbol{x} = \boldsymbol{b}$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero? (The 5 columns lie on a "4-dimensional hyperplane" in 5-dimensional space.)

5 by 5 central diff. matrix

same plane

$$\begin{bmatrix}0&1&0&0&0\\-1&0&1&0&0\\0&-1&0&1&0\\0&0&-1&0&1\\0&0&0&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\\b_5\end{bmatrix}$$

1 0 1 0 1

for the system to be solvable, we need $b_1 + b_3 + b_5 = 0$

could be easier seen if equations were written out

$$\vec{x} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \vec{b} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -4 \end{bmatrix}$$

indeed add up to zero...

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$b_1 = \alpha$

$b_2 = 2\alpha$

$\cancel{b_1} \; b_2 = 2b_1$

$2b_1 - b_2 = 0,$

Sec 2.1 #29

Start with the vector $\boldsymbol{u}_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" $A = [.8\ .3;\ .2\ .7]$. The next three vectors are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$:

$$\boldsymbol{u}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = \_\_\_\_ \qquad \boldsymbol{u}_3 = A\boldsymbol{u}_2 = \_\_\_\_.$$

What property do you notice for all four vectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$?

$$u_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = [$$

skipped; too easy

Sec 2.1 #30

skipped; too easy

✓

Sec 2.2 #20

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a ____ of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

linear combination

$x + y + z = 5$

✓

Sec 2.2 # 32

Start with 100 equations $Ax = \mathbf{0}$ for 100 unknowns $\boldsymbol{x} = (x_1, \ldots, x_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".

(a) Elimination takes linear combinations of the rows. So this singular system has the singular property: Some linear combination of the 100 **rows** is a row of zeros

(b) Singular systems $Ax = \mathbf{0}$ have infinitely many solutions. This means that some linear combination of the 100 ***columns*** is zero

(c) Invent a 100 by 100 singular matrix with no zero entries. a matrix of ones

(d) For your matrix, describe in words the row picture and the column picture of $Ax = \mathbf{0}$. Not necessary to draw 100-dimensional space.

row pic: just overlapping lines

$$x_1 + \cdots + x_{100} = 0$$

col pic: all $\underbrace{(1, \cdots, 1)}_{100\text{ D.}}$ vectors

Sec 2.3 # 22

The entries of A and $\boldsymbol{x}$ are a_{ij} and x_j. So the first component of $A\boldsymbol{x}$ is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for

(a) the third component of $A\boldsymbol{x}$

(b) the $(2, 1)$ entry of $E_{21}A$

(c) the $(2, 1)$ entry of $E_{21}(E_{21}A)$

(d) the first component of $(E_{21}A)\boldsymbol{x}$.

a) $\sum a_{3j}x_j = a_{31}x_1 + \cdots + a_{3n}x_n$

b) $a_{21} - a_{11}$

c) $a_{21} - 2a_{11}$

d) bc $E_{21}A$ doesn't change the first row of A

Sec 2.3 #29

Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

Eliminate column 1 $\quad E\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{bmatrix}.$

Which matrix M (multiplying several E's) reduces Pascal all the way to I? Pascal's triangular matrix is exceptional, all of its multipliers are $\ell_{ij}=1$.

$$E=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$$

$$M=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$$

Sec 2.4 #32

(*Very important*) Suppose you solve $A\boldsymbol{x}=\boldsymbol{b}$ for three special right sides $\boldsymbol{b}$: b_1, b_2, b_3

$$A\boldsymbol{x}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}\quad\text{and}\quad A\boldsymbol{x}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}\quad\text{and}\quad A\boldsymbol{x}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

If the three solutions $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3$ are the columns of a matrix X, what is A times X?

$$A[x_1\ x_2\ x_3]=[b_1\ b_2\ b_3]$$
$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Sec 2.4 #36

Practical question Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The same answer comes from A times BC with $mnq + npq$ separate multiplications. Notice npq for BC.

$m=2$ $n=4$ $p=7$ $q=10$

(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?

(b) With N-component vectors, would you choose $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ or $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$?

(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

a)

$(AB)C$ cost: $mnp + mpq$
$= 2\times4\times7 + 2\times7\times10$
$= 196$ ← preferred

$A(BC)$ cost: $mnq + npq$
$= 2\times4\times10 + 4\times7\times10$
$= 360$

b) $(\underbrace{u^T v}_{n})\underbrace{w^T}_{n} = 2n$ ← (multi. count preferred.

$\underbrace{u^T(\underbrace{vw^T}_{n^2})}_{n^2} = 2n^2$

c) $mnp + mpq < mnq + npq$
$q^{-1} + n^{-1} < p^{-1} + m^{-1}$ (divide by mnpq)

Sec 2.5 # 7

(Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\boldsymbol{x} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{x} = \boldsymbol{b}$?

(c) What happens to row 3 in elimination?

Suppose $\vec{x}$ is such a solution.

a) $(\text{row } 1)\cdot\vec{x} = 1$

$(\text{row } 2)\cdot\vec{x} = 0$

$(\text{row } 3)\cdot\vec{x} = 0$

but

$(\text{row } 3)\cdot\vec{x} = (\text{row } 1 + \text{row } 2)\cdot\vec{x} = 1$

contradiction, so $\vec{x}$ cannot exist.

b) $b_1 + b_2 = b_3 \Rightarrow b_1 + b_2 - b_3 = 0$

c) It will become all zeros.