5. Transposes, permutations, $\mathbb{R}^n$

$\mathbb{R}^n$ — vector space / — subspace / — column space $C(A)$

Permutations : row exchanges

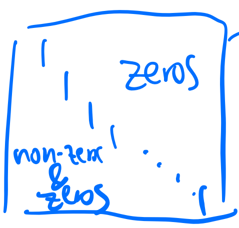
useful when, if a zero shows up in pivot position

MATLAB also avoids small pivots

What happens to $A = LU$ (~~[illegible]~~ represents elimination but assumes no row exchange)

$\boxed{PA = LU}$ (for any invertible A)

any square matrix



P: identity matrix with re-ordered rows (n! total orderings)

$\boxed{P^{-1} = P^T}$ $\quad P^T P = I$

Some PA = LU examples in JAX :

```
A = np.array([
    [1, 2, 3, 4],
    [1, 2, 3, 4],
    [3.2, 1.2, 3.7, 4.8],
    [-5, 1, 3, 9]
])
```

> the same — leading to

```
lu(A)
```

```
(Array([[0., 0., 0., 1.],
        [0., 1., 0., 0.],
        [0., 0., 1., 0.],
        [1., 0., 0., 0.]], dtype=float32),
 Array([[ 1.        ,  0.        ,  0.        ,  0.        ],
        [-0.2       ,  1.        ,  0.        ,  0.        ],
        [-0.64000005,  0.8363637 ,  1.        ,  0.        ],
        [-0.2       ,  1.        ,  0.        ,  1.        ]],       dtype=float32),
 Array([[-5.       ,  1.       ,  3.       ,  9.       ],
        [ 0.       ,  2.2      ,  3.6      ,  5.8      ],
        [ 0.       ,  0.       ,  2.6090913,  5.709091 ],
        [ 0.       ,  0.       ,  0.       ,  0.       ]], dtype=float32))
```

non-zero pivots show up in diagonal

```
A = np.array([
    [0, 0, 0, 0],
    [0, 0, 0, 0],
    [0, 0, 0, 0],
    [0, 0, 0, 0]
])
```

```
lu(A)
```

```
(Array([[1., 0., 0., 0.],
        [0., 1., 0., 0.],
        [0., 0., 1., 0.],
        [0., 0., 0., 1.]], dtype=float32),
 Array([[1., 0., 0., 0.],
        [0., 1., 0., 0.],
        [0., 0., 1., 0.],
        [0., 0., 0., 1.]], dtype=float32),
 Array([[0., 0., 0., 0.],
        [0., 0., 0., 0.],
        [0., 0., 0., 0.],
        [0., 0., 0., 0.]], dtype=float32))
```

Transpose

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

symmetric matrices : $A^T = A$

$R^T R$ is always symmetric why?

any real matrix

$$(R^T R)^T = R^T (R^T)^T = R^T R \,!$$

if $v, w \in$ space,
then $av + bw \in$ space
$(a, b \in \mathbb{R})$

together form linear combination

Vector spaces

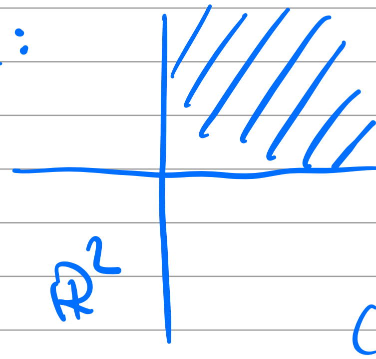
Set of vectors, that support vector operations
(add, scalar multiplication)

Example: $\mathbb{R}^2$, e.g. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\mathbb{R}^n$

only an intro, real lecture on column space & its relationship to $Ax = b$ starts on next lecture.

Not example:



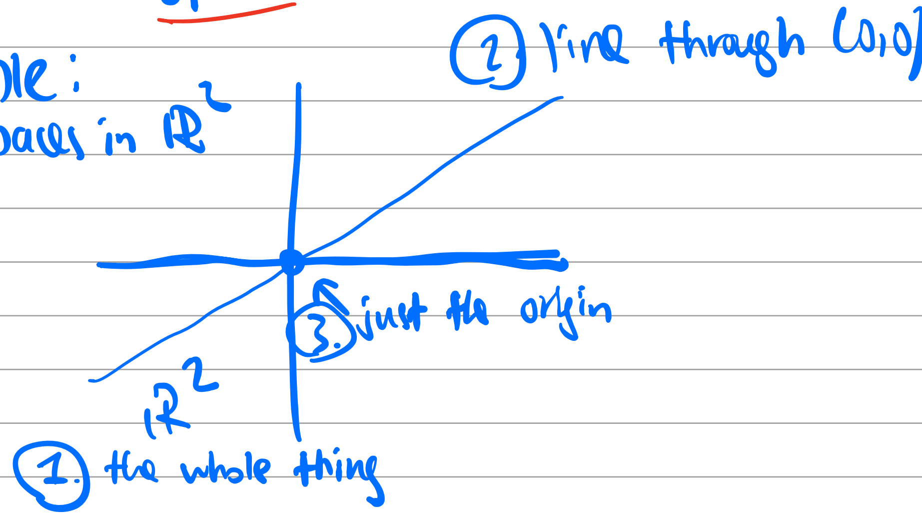
addition ✓

multiplying by scalar ✗

(not closed under scalar multiplying)

Subspaces: part of $\mathbb{R}^n$ but is still vector space

Example:
3 subspaces in $\mathbb{R}^2$



Example:
4 subspaces in $\mathbb{R}^3$

1. $\mathbb{R}^3$
2. planes thru 0,0,0
3. lines thru 0,0,0
4. origin

How do subspaces come out of matrices?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

columns in $\mathbb{R}^3$

all their combinations form a subspace,

called column space $C(A)$