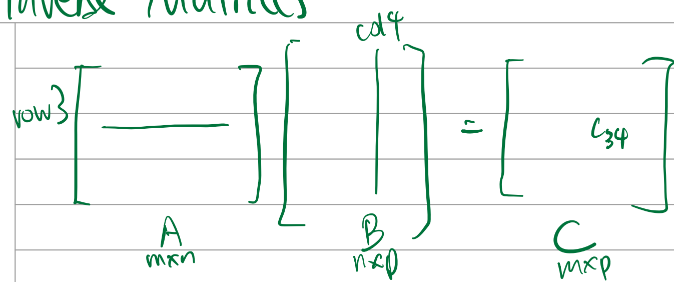
3 Multiplication & Inverse Matrices

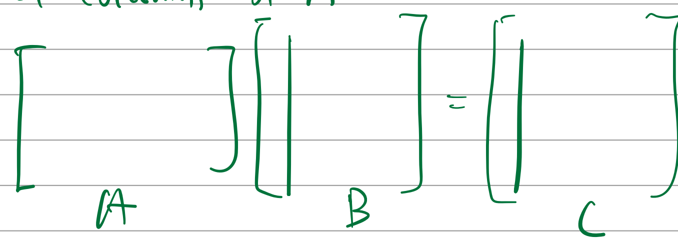
① matmul.
(standard rule)



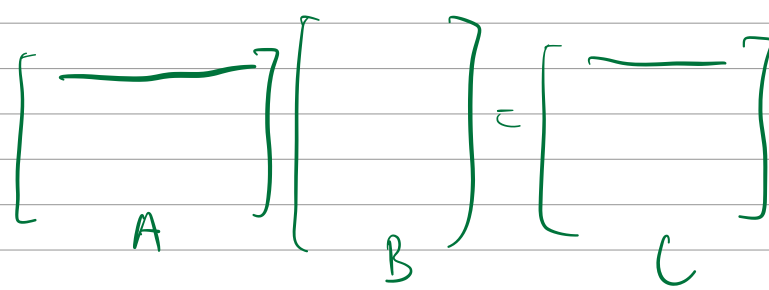
$$c_{34} = (\text{row } 3 \text{ of } A) \cdot (\text{col } 4 \text{ of } B)$$
$$= a_{31} b_{14} + a_{32} b_{24} + \cdots$$
$$= \sum_{k=1}^{n} a_{3k} b_{k4}$$

② matmul
(matrix - matrix multiplication as many matrix - vector products)

Columns of C are linear combinations of columns of A



③ matmul
(matrix - matrix multiplication as many vector - matrix products)



④ matmul as sum of all outer products

col of A $\times$ row of B $\Rightarrow$ $(m \times n)$
$(m \times 1)$ $(1 \times n)$

4 ways of looking at matmul!

$A_1 B_1 + A_2 B_3$

Block multiplication

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}\underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$$

(non-trivial but it's elegant)

Inverses (square matrix)

A^{-1}, if exist, $A^{-1}A = AA^{-1} = I$.

$\downarrow$

A is called invertible / nonsingular

ex. $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

can find $\vec{x} \neq 0$ such that $A\vec{x} = 0$

Proof: suppose A^{-1} exists. Then

$$A^{-1}(Ax) = A^{-1}0$$

$$x = 0$$

but $x \neq 0$, so A^{-1} doesn't exist.

ex. $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

A should be invertible b/c 2 cols not dependent
How should we find A^{-1}

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Finding the inverse is like solving 2 systems!

$A \times$ col j of A^{-1} = col j of I

Gauss-Jordan (solve 2 eqs at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Downarrow$

Jordan says keep going

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$A \quad I \qquad\qquad I \quad A^{-1}$

It turns out that if we do elim. for the left part, the right part will become the inverse.

why?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$\Rightarrow a = 7$
$c = -2$

$$E\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

$$E = A^{-1}$$