7 Solving $Ax=0$: pivot variables, special solutions

$$Ax=0$$
$$\downarrow$$
$$Ux=0$$
$$\downarrow$$
$$Rx=0$$

TOC:
- computing the nullspace ($Ax=0$)
- pivot variables, free variables
- special solutions, $\text{rref}(A)=R$

$$A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

row operations does not change solution to $Ax=0$

$$\hookrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

row-echelon form

rank = # of pivots = 2

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

free columns

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \\ 2x_3 + 4x_4 = 0 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 - 2x_3 - 2x_4 \\ x_3 = -2x_4 \end{cases}$$

Free to chose whatever value

$$\begin{cases} x_1 = -2x_2 - 2(-2x_4) - 2x_4 \\ x_3 = -2x_4 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 + 2x_4 \\ x_3 = -2x_4 \end{cases}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}$$

Values for pivot variables can be expressed as a linear equation with the free variables

to get every vector in the nullspace, we try to find its basis vectors

$$\begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ 0 \\ -2x_4 \\ x_4 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underbrace{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

Prof Strang calls them the "special solutions" → basis

Reduced row echelon form

zeros above and below pivots

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Rx = 0$$

$$\hookrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A)$$

in rref, pivot variables no longer depend on each other

$$\begin{cases} x_1 + 2x_2 + \cancel{0x_3} - 2x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases}$$

signs switched

$$\begin{cases} x_1 = -2x_2 + 2x_4 \\ x_3 = -2x_4 \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Remark: if we organize $\vec{x}$ as $\begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix}$

then the RREF becomes

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

identity

negate these to get sols. values for x_1 & x_3 in special

RREF more generally (building on this)

Suppose we organize the system s.t.

$$x = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \begin{array}{l} \} \text{ pivot variables} \\ \} \text{ free variables} \end{array}$$

Then the RREF of A would be

$$R = \begin{array}{c} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{array}{l} r \text{ pivot rows} \\ \end{array} \\ \begin{array}{cc} \text{pivot cols } (r) & \text{free cols } (n-r) \end{array} \end{array}$$

(block notation)

The special solutions would satisfy

$$\underbrace{R N}_{} = \underbrace{0}_{n \text{ by } n-r}$$

N: n by n−r
each col is a special solution, one for each free variable

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = 0$$

$$\Rightarrow IN_1 + FN_2 = 0$$

Since we pick N_2 to be I,

$$\Rightarrow N_1 = -F$$

$$\Rightarrow N = \begin{bmatrix} -F \\ I \end{bmatrix}$$