Lecture 5: Breadth-First Search

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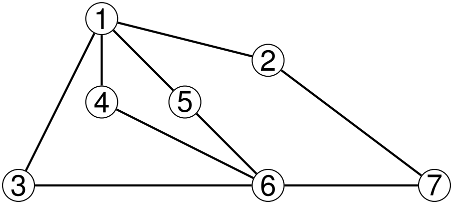
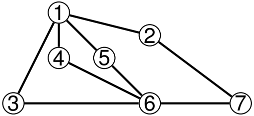
Graphs

- A graph $G = (V, E)$ has a pair of sets: vertices V and edges E.
- To give an adjacency list representation of a graph, for each vertex v list all of the vertices adjacent to v.
- To give an adjacency matrix representation of a graph create a square matrix A and label the rows and columns with the vertices: the entry in row i column j is 1 if vertex j is adjacent to vertex i and 0 if it is not.
- Can also represent a graph by an array of its edges.



Graphs: Representations

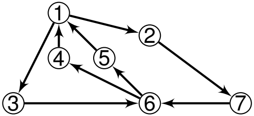
Undirected



1 : 2, 3, 4, 5
2 : 1, 7
3 : 1, 6
4 : 1, 6
5 : 1, 6
6 : 3, 4, 5, 7
7 : 2, 6

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Directed



1 : 2, 3
2 : 7
3 : 6
4 : 1
5 : 1
6 : 4, 5
7 : 6

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Graphs: Representations

For each representation

- How much space does we need to store it?
- How long does it take to initialize an empty graph?
- How long does it take to make a copy?
- How long does it take to insert an edge?
- How long does it take to list the vertices adjacent to a vertex u?
- How long does it take to find out if the edge (u, v) belongs to G?

Breadth-First Search (Graph Traversal)

- Input: a graph $G = (V, E)$ and a source vertex s.
- Aim: to find the distance from s to each of the other vertices in the graph.

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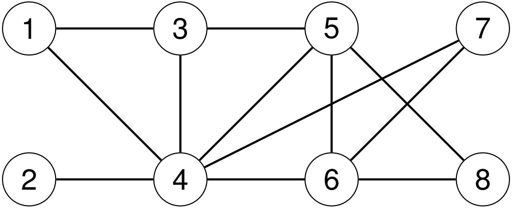
- Idea: send out a wave from s.
  - The wave first hits vertices at distance 1
  - Then the wave hits vertices at distance 2
  - and so on

Breadth-First Search

- BFS maintains a queue that contains vertices that have been discovered but are waiting to be processed.
- BFS colours the vertices:
  - White indicates that a vertex is undiscovered
  - Grey indicates that a vertex is discovered but unprocessed
  - Black indicates that a vertex has been processed.
- The algorithm maintains an array d (distance)
  - $d[s] = 0$, where s is the source vertex;
  - if we discover a new vertex v while processing u, we set $d[v] = d[u] + 1$.

Example

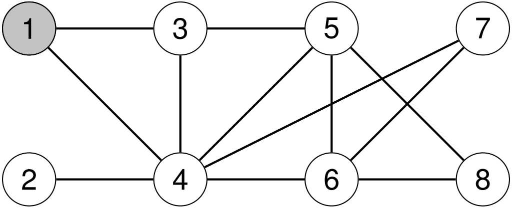


- Initialization: source vertex grey, others are white; distance to source is 0; add source to the queue
- while the queue is not empty
  - remove the first vertex v from the queue (why the first one?)
  - add white neighbours of v to queue and colour them grey; distance is 1 greater than to v
  - colour v black

Example

$Q = 1$

	1	2	3	4	5	6	7	8
d	0							



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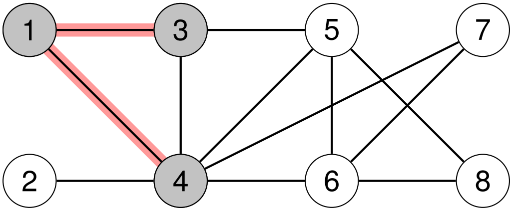
Example

$Q = 3, 4$

	1	2	3	4	5	6	7	8
d	0		1	1				



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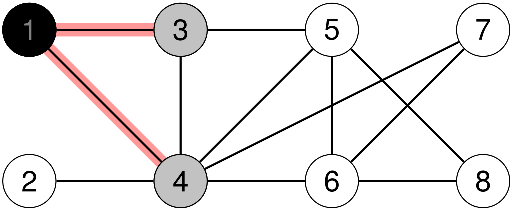

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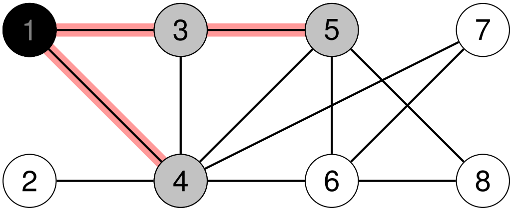
Example

$Q = 4, 5$

	1	2	3	4	5	6	7	8
d	0		1	1	2			



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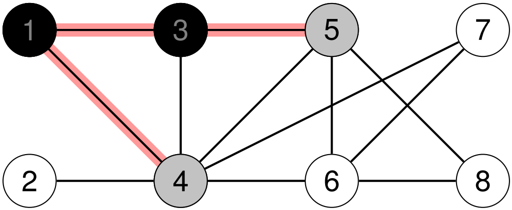

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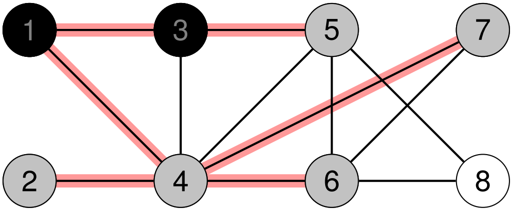
Example

$Q = 5, 2, 6, 7$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	



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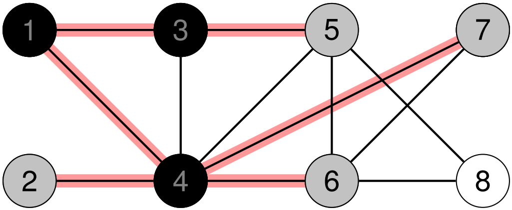

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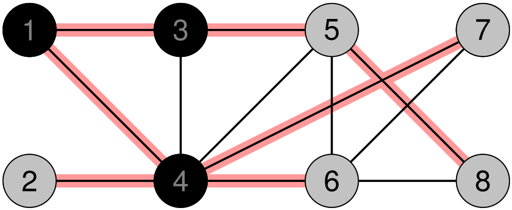
Example

$Q = 2, 6, 7, 8$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	3



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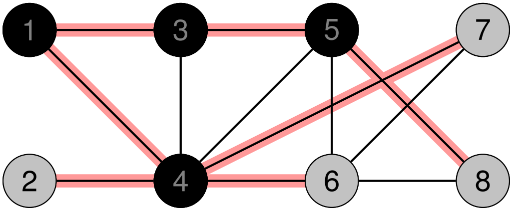
Example

$Q = 2, 6, 7, 8$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	3



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- while the queue is not empty
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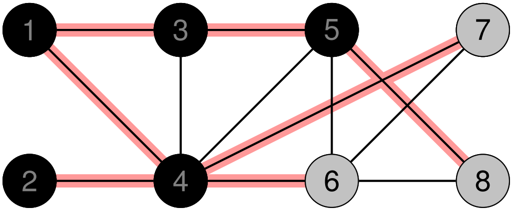
  - colour v black

Example

$Q = 6, 7, 8$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	3



- Initialization: source vertex grey, others are white; distance to source is 0; add source to the queue
- while the queue is not empty
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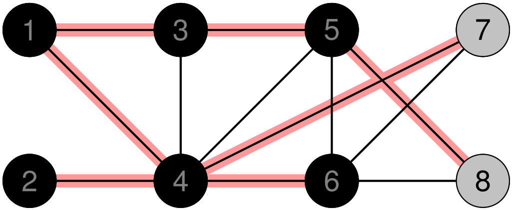
Example

$Q = 7, 8$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	3



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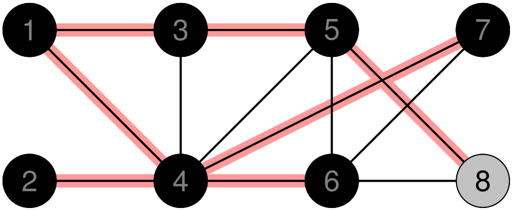
Example

$Q = 8$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	3



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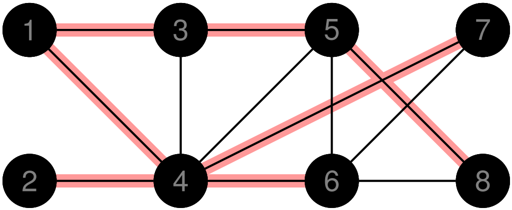
Example

$Q =$

	1	2	3	4	5	6	7	8
d	0	2	1	1	2	2	2	3



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Breadth-first search

```
BFS (G, s)
for each vertex u ∈ V[G] − {s} do
    colour[u] ← WHITE
        d[u] ← ∞
            π[u] ← NIL
colour[s] ← GREY
d[s] ← 0
π[s] ← NIL
Q ← ∅
ENQUEUE(Q, s)
while Q ≠ ∅ do
    u ← DEQUEUE(Q)
        for each v ∈ Adj[u] do
            if colour[v] = WHITE
                then colour[v] = GREY
                    d[v] ← d[u] + 1
                    π[v] ← u
                    ENQUEUE(Q, v)
        colour[u] ← BLACK
```

Analysis of running time

- We want an upper bound on the worst-case running time.
- Assume that it takes constant time for each operation such as to test and update colours, to make changes to distance (and predecessor) and to enqueue and dequeue.
- Initialization takes time $O(V)$.
- Each vertex enters (and leaves) the queue exactly once. So queuing operations take $O(V)$.
- In the loop the adjacency lists of each vertex are scanned. Each list is read once, and the combined lengths of the lists is $O(E)$.
- Thus the total running time is $O(V + E)$.

More than distances

- What if as well as finding the distance to each vertex, we want to be able to find a shortest-possible path from the source to each vertex?
- What should we add to the algorithm to achieve this?

Breadth-first search

- Note that the algorithm runs on both directed and undirected graphs.
- Notice that the highlighted edges (the ones used to discover new vertices) form a tree: we call this the Breadth-first tree. A path from s to another vertex v through the tree is the shortest path between s and v
- The predecessor of a vertex is the one from which it was discovered (i.e its parent in the Breadth-first tree). We can record predecessors in an array Π when we run the algorithm and then use this array to construct the Breadth-first tree.