Mathematics for Computer Science
Linear Algebra

Lecture 10: The fundamental spaces of a matrix

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Reminder from the previous lecture

- Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent if $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$ implies that $k_1 = \ldots = k_r = 0$. Otherwise, they are linearly dependent.
- A set B of vectors in a vector space V is a basis for V if (1) B is linearly independent and (2) B spans V.
- We have $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis in $\mathbb{R}^n$ iff $det([\mathbf{v}_1|\ldots|\mathbf{v}_n]) \neq 0$.
- All bases of a finite-dimensional vector space V have the same number of vectors. This number is called the dimension of V and denoted by $dim(V)$.

Contents for today's lecture

- Row, column, and null spaces of a matrix
- How to find bases in these spaces
- The rank of a matrix

Row space, column space, and null space of a matrix

Definition

Let A be an $m \times n$ matrix.
The row space of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
The column space of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.
The null space of A is the subspace of $\mathbb{R}^n$ equal to the solution space of $A\mathbf{x} = \mathbf{0}$.

Example: let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{pmatrix}$$

The row vectors of A are

$$\mathbf{r}_1 = (2, 1, 0) \text{ and } \mathbf{r}_2 = (3, -1, 4).$$

The column vectors of A are

$$\mathbf{c}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{c}_3 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Elementary row operations and the row space

Lemma

Elementary row operations do not change the row space of a matrix.

Proof.

For any (row) vectors $\mathbf{r}_i$ and $\mathbf{r}_j$ from a matrix A, and any scalar $k \neq 0$, we have

$$span(\mathbf{r}_i, \mathbf{r}_j) = span(\mathbf{r}_j, \mathbf{r}_i) = span(k\mathbf{r}_i, \mathbf{r}_j) = span(\mathbf{r}_i, \mathbf{r}_j + k\mathbf{r}_i).$$

Clearly, the equalities hold if we add the remaining row vectors of A to each span:

$$span(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_1, \ldots) = \ldots = span(k\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_1, \ldots) = span(\mathbf{r}_i, \mathbf{r}_j + k\mathbf{r}_i, \mathbf{r}_1, \ldots).$$

So, any elementary row operation does not change the row space of A. $\square$

Elementary row operations and the row space

Lemma

If R is a matrix in row echelon form, then its non-zero rows form a basis for its row space.

These rows obviously span the row space, need to check linear independence.

The idea of a proof on an example: let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be the first three rows of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & -2 & 9 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = \mathbf{0}$. Since $k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = (k_1, *, *, \ldots, *)$, have $k_1 = 0$

Then $k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = \mathbf{0}$. Since $k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = (0, 0, k_2, *, \ldots, *)$, have $k_2 = 0$.

Then $k_3\mathbf{r}_3 = \mathbf{0}$. Since $k_3\mathbf{r}_3 = (0, 0, 0, 0, 0, 0, k_3)$, have $k_3 = 0$.

Finding basis for the row space and the null space

In order to find a basis for the row space of a matrix A, do

- transform A (by elementary row operations) to row echelon form R;
- the rows in R with the leading 1s form a basis for the row space of A.

Bonus: In order to find a basis for $span(S)$ for a finite set S of vectors, do

- form a matrix whose row vectors are the vectors in S and then do as above.

From the last lecture, we know:

In order to find a basis for the null space of A, do

- find the general solution to the system $A\mathbf{x} = \mathbf{0}$;
- for each free variable x, take the solution (vector $\mathbf{v}_x$) in which $x = 1$ and the other free variables are set to 0;
- these vectors $\mathbf{v}_x$ together form a basis for the null space.

Elementary row operations and the column space

Elementary row operations can change the column space: for example, let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}.$$

B is obtained from A by an elementary row operation: adding $(-2) \cdot r_1$ to r_2.

The column space of A is $span(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}) = \{k \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid k \in \mathbb{R}\}$.

The column space of B is $span(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}) = \{k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid k \in \mathbb{R}\}$.

Clearly, these are different spaces.

Generally, how can we find a basis for the column space of a matrix?

Elementary row operations and the column space

Elementary row operations do not change dependencies between column vectors of a matrix: if, for example, $\mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_5, \mathbf{c}_7$ are column vectors of a matrix such that $k_2\mathbf{c}_2 + k_3\mathbf{c}_3 + k_5\mathbf{c}_5 + k_7\mathbf{c}_7 = \mathbf{0}$ holds before an operation then it also holds after.

Exercise: Check this (the proof is easy).

Theorem

Let B be a matrix obtained from matrix A by an elementary row operation.

- *A subset S of column vectors in A is linearly independent iff the corresponding set of column vectors in B is such.*
- *A subset S of column vectors in A is a basis for the column space of A iff the corresponding set of column vectors in B is a basis for the column space of B.*

Theorem

If a matrix R is in row echelon form then the column vectors with the leading 1s form a basis for the column space of R.

Proofs of both theorems are straightforward, but a bit cumbersome - omitted.

Finding basis for the column space

Column space of a matrix $A =$ row space of A^T. They have the same bases.

To find a basis for the column space that is a subset of the columns :

- transform A (by elementary row operations) to row echelon form R;
- select all columns <u>in R</u> that have leading 1s;
- the corresponding columns <u>in A</u> form the required basis.

Example: columns 1, 3, and 5 form a basis of the column space of this matrix

$$\begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Btw, how do we find a basis for the row space that is a subset of the rows?

Rank of a matrix

Theorem

The row space and the column space of a matrix have the same dimension.

Proof.

Dimension of each space is equal to the number of leading 1s in row echelon form of the matrix. □

Definition

The rank of a matrix A, denoted by $rank(A)$, is the dimension of its row space.

Properties of rank

Definition

The rank of a matrix A, denoted by $rank(A)$, is the dimension of its row space.

Some properties of rank:

- $rank(A) = rank(A^T)$ — because row space of A = column space of A^T
- If A has size $m \times n$ then $rank(A) \leq \min(m, n)$ — this is because $rank(A)$ is the dimension of a subspace in $\mathbb{R}^m$ and of a subspace in $\mathbb{R}^n$.
- If A has size $n \times n$, then $rank(A) = n$ iff $det(A) \neq 0$ iff A is invertible — because $rank(A) = n$ means that the rows of A span (and form a basis in) $\mathbb{R}^n$.
- If A has size $m \times n$ then $rank(A) = 1$ iff $A = CR$ where C is an $m \times 1$ matrix (i.e. a single column) and R is an $1 \times n$ matrix (i.e a single row).
  This is a special case of rank decomposition (we might prove it later.)

How to compute $rank(A)$? Many ways.
One is to transform to row echelon form and count leading 1s. Additional trick: can use elementary column operations too (because $rank(A) = rank(A^T)$).

Rank and nullity

Definition

The nullity of A, denoted by *nullity*(A), is the dimension of the null space of A.

Lemma

For any $m \times n$ matrix A, $rank(A)$ and $nullity(A)$ are the numbers of leading and free variables, respectively, in the general solution to $A\mathbf{x} = \mathbf{0}$.

Theorem (Dimension Theorem for Matrices)

For any matrix A with n columns, $rank(A) + nullity(A) = n$.

Proof.

The system $A\mathbf{x} = \mathbf{0}$ has n variables. Now use the previous lemma. $\square$

To rephrase: $rank(A)$ determines the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Orthogonal complement

Definition

If W is a subspace of $\mathbb{R}^n$ then the orthogonal complement of W, denoted by $W^\perp$, is defined as

$$W^\perp = \{\mathbf{v} \in V \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for every } \mathbf{w} \in W\}.$$

Lemma

If W is a subspace of $\mathbb{R}^n$ then

1. *$W^\perp$ is a subspace of $\mathbb{R}^n$;*
2. *$W \cap W^\perp = \{\mathbf{0}\}$;*
3. *$(W^\perp)^\perp = W$.*

Theorem

If A is an $m \times n$ matrix then the null space of A and the row space of A are orthogonal complements in $\mathbb{R}^n$.

Exercise: Prove the above lemma and theorem (all proofs are one-line).

An application of rank in CS: low-rank approximation

- Fact: data is often stored in matrix form.
- Popular idea: approximate a matrix by a matrix of a low rank.
- Intuitively, rank measures complexity of a matrix. The low rank constraint is related to a constraint on the complexity of a model that fits the data. Go from high-complexity data to a low-rank model to make computation feasible.
- Some CS applications of low-rank approximation:
  - Data compression (via SVD - singular value decomposition for matrices)
  - Machine learning
  - Recommender systems
  - Natural language processing

What we learnt today

- The four fundamental spaces of an $m \times n$ matrix A:

subspaces of $\mathbb{R}^n$	subspaces of $\mathbb{R}^m$
row space of A	column space of A
null space of A	null space of A^T

  The subspaces in each column are orthogonal complements of each other.

- How to find bases of these spaces
- Rank and nullity of a matrix, and how they are related

Next time:

- Matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$