Durham University

Maths for Computer Science
Calculus

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Series

Series vs Sequences

A **sequence** is an ordered countably infinite set of numbers $\{a_n\}$.

E.g. $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}, \ldots$

A **series** is the sum of members of a sequence:

E.g. $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2 = 1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$.

This is a **finite series.**

E.g. $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

This in an **infinite series.**

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Series value

Any **finite series** has a well defined value:

E.g. $\sum_{n=1}^{5} n^2 = 1 + 4 + 9 + 16 + 25 = 55.$

But an **infinite series** may not:

E.g. $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. This is "convergent"

But $\sum_{n=0}^{\infty} n^2 \to \infty$ is "divergent"

And $\sum_{n=0}^{\infty} (-1)^n \to \infty$ is "divergent"

Series value

For any infinite series $\sum_{n=0}^{\infty} a_n$, we define the partial sum

$$S_n = \sum_{r=0}^{n} a_r = a_0 + a_1 + \cdots + a_n$$

And the remainder

$$R_n = \sum_{r=n+1}^{\infty} a_r = a_{r+1} + a_{r+2} + \cdots$$

A **series** $\sum_{n=0}^{\infty} a_n$ is convergent to a finite value S if

$$\lim_{n\to\infty} S_n = S$$

Equivalently: if

$$\lim_{n\to\infty} R_n = 0 \quad \left(= \lim_{n\to\infty} (S - S_n)\right).$$

Example $\sum_{n=0}^{\infty} \frac{1}{3^n}$

$\sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$ is a geometric series with initial value 1 and ratio 1/3.

Hence $S_n = \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$.

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right) = \frac{3}{2}$.

So this series is convergent to 3/2.

Also $R_n = \frac{3}{2} - S_n = \frac{3}{2}\left(\frac{1}{3}\right)^n = \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}$. Clearly $\lim_{n\to\infty} R_n = 0$.

Given $\epsilon = 0.01$ there is some N_ϵ such that $R_n < \epsilon$ for all $n > N_\epsilon$.

Here we can take $N_\epsilon = 5$ since

$$R_5 = \frac{1}{2}\left(\frac{1}{3}\right)^4 = \frac{1}{162} < 0.01.$$

Note: if we are using a series to approximate some complex function, analysis of R_n can tell us how far we need to go to get sufficient accuracy.

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Sum and difference of convergent series

If the series $\sum_{n=0}^{\infty} a_n$ and the series $\sum_{n=0}^{\infty} b_n$ are convergent, with respective sums α and β, then

$$\sum_{n=0}^{\infty}(a_n + b_n) = \alpha + \beta$$

And

$$\sum_{n=0}^{\infty}(a_n - b_n) = \alpha - \beta$$

First test for divergence

Suppose $\sum_{n=0}^{\infty} a_n$ converges. Then given arbitrary $\epsilon > 0$ there exists some N_ϵ such that $S_n < \epsilon$ for all $n > N_\epsilon$.

I.e. $|S - S_n| < \epsilon$.

But also $|S - S_{n+1}| < \epsilon$, so combining these*:

$$2\epsilon > |S - S_n| + |S - S_{n+1}| = |S_n - S| + |S - S_{n+1}| \geq |S_n - S_{n+1}| = |a_{n+1}|$$

But ϵ was arbitrary, so we have shown for any $\epsilon' > 0$ there exists an $N_{\epsilon'}$ such that $|a_n| < \epsilon$' for all $n > N_\epsilon$.

I.e. $\lim_{n\to\infty} |a_n| = 0$, and so $\lim_{n\to\infty} a_n = 0$.

The contrapositive:

If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.

*Using triangle inequality: $|a| + |b| \geq |a + b|$.

Example

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^2+1}{4n+2}$ diverges.

Since $\lim_{n\to\infty} \frac{2n^2+1}{4n+2} = \lim_{n\to\infty} \frac{2n+\frac{1}{n}}{4+\frac{2}{n}} \to \infty$.

Equally, $\lim_{n\to\infty} |a_n| = 0$ is a necessary condition for convergence, but not a sufficient condition:

E.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Note: $\lim_{n\to\infty} |a_n| = 0$ does not imply convergence, but $\lim_{n\to\infty} R_n = 0$ does.

Absolute convergence implies convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof:

Let $S_n = \sum_{r=1}^{n} a_r$ and $T_n = \sum_{r=1}^{n} |a_r|$.

For each r we have $0 \leq a_r + |a_r| < 2|a_r|$, so $0 \leq S_n + T_n \leq 2T_n$.

But we know $\lim_{n\to\infty} T_n = T$ exists, so

$$0 \leq \lim_{n\to\infty} (S_n + T_n) \leq 2T$$

And so the series $\sum_{n=1}^{\infty} a_n + |a_n|$ converges too, and taking the difference $\sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$ converges and is equal to $\sum_{n=1}^{\infty} a_n$.

Comparison test

Convergence:

Let $\sum_{n=1}^{\infty} b_n$ be a convergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and N be an integer such that

$$|a_n| \leq b_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

Divergence:

Let $\sum_{n=1}^{\infty} b_n$ be a divergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and N be an integer such that

$$0 \leq b_n \leq a_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

Examples

$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$?

Converges: $a_n \leq \frac{3}{2^n}$, and as $\sum_{n=1}^{\infty} \frac{3}{2^n} = 3\sum_{n=1}^{\infty} \frac{1}{2^n} = 3$ converges, we can apply the comparison test with $b_n = \frac{3}{2^n}$.

$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$?

Diverges: $a_n = \frac{n+1}{n^2} = \frac{1}{n}\frac{n+1}{n} > \frac{1}{n}$, and as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we can apply the comparison test with $b_n = \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series and we shall prove later that it diverges.

The Ratio test

Convergence:

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

Example:

$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$?

Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n}$,

so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 1/\left(1 + \frac{1}{n}\right)^n = \frac{1}{e}$. As $\frac{1}{e} < 1$, the series absolutely converges.

The Ratio test

Convergence:

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

Proof a):

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then we can pick r: $L < r < 1$ and N_r such that for all $n > N_r$

$\left|\frac{a_{n+1}}{a_n}\right| < r$ i.e. $|a_{n+1}| < r|a_n|$.

Then $|a_{n+2}| < r|a_{n+1}| < r^2|a_n|, |a_{n+3}| < r|a_{n+2}| < r^3|a_n|$, etc.

So $R_{n+m} = a_{n+m} + a_{n+m+1} + \cdots < |a_n| r^{m-1}(1 + r + r^2 + r^3 + \cdots) = a_n \frac{r^{m-1}}{1-r} \to 0$ as $m \to \infty$. Hence the series converges.

The n^{th} root test

Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{nk}{3n+1}\right)^n$, where k is a constant.

First observe that $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{nk}{3n+1}\right) = \frac{k}{3}$.

If $k < 3$, then there is an r: $\frac{k}{3} < r < 1$ and N_r so that for all $n > N_r$, $\sqrt[n]{a_n} < r$.

Thus $R_n = a_n + a_{n+1} + \cdots < r^n + r^{n+1} + \cdots = r^n \frac{1}{1-r} \to 0$ as $n \to \infty$.

So the series is convergent.

If $k > 3$, then there is an r: $1 < r < \frac{k}{3}$ and N_r so that for all $n > N_r$, $\sqrt[n]{a_n} > r$.

Thus $R_n = a_n + a_{n+1} + \cdots > r^n + r^{n+1} + \cdots = r^n \frac{1}{1-r} \to \infty$ as $n \to \infty$.

So the series is divergent.

The n^{th} root test

Consider the series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

Alternating series test

The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

- $a_n > 0$,
- $a_{n+1} \le a_n$ for all n, and
- $\lim_{n\to\infty} a_n = 0$.

Proof:

Consider the partial sum

$$S_{2r} = a_1 - a_2 + a_3 - a_4 + a_5 \dots - a_{2r-2} + a_{2r-1} - a_{2r}.$$

First note that

$$S_{2r} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2r-1} - a_{2r}) \ge 0$$

Also

$$S_{2r} = a_1 - (a_2 - a_3) - (a_4 - a_5) \dots - (a_{2r-2} - a_{2r-1}) - a_{2r} < a_1$$

So $\{S_{2r}\}$ is bounded and monotonically increasing, hence converges to S.

But $S_{2r+1} = S_{2r} + a_{2r+1}$, and so $\lim_{r\to\infty} S_{2r+1} = \lim_{r\to\infty} S_{2r} + \lim_{r\to\infty} a_{2r+1} = S + 0 = S$.

And as $\{S_{2r}\}$ and $\{S_{2r+1}\}$ both tend to S, $\{S_n\}$ does also.

Grouping and rearrangement of series

For any finite sum we can group and rearrange terms as much as we like.

$$1 + 3 - 5 - 6 = (1 - 5) - (6 - 3)$$

For an infinite sum we need to be careful!

Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

By the alternating series test is is convergent, to some value S.

So

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Rearranging

$$S = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} \dots + \left(\frac{1}{2k+1} - \frac{1}{4k+2}\right) - \frac{1}{4k+4} + \cdots$$

$$S = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{1}{2} S$$

So as $S = \frac{1}{2} S$ we must have $S = 0$, but actually $S > \frac{1}{2}$. (Exercise).

Grouping and rearrangement of series

If $\sum_{n=1}^{\infty} a_n$ converges, then we can insert brackets/groupings without altering the sum.

Proof:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

Consider bracketing $(a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + \cdots$ etc.

Define b_n to be the n^{th} bracketed term.

Now $\sum_{n=1}^{\infty} b_n = (a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + \cdots$ etc.

But the partial sums T_n of this series are a subsequence of the partial sums S_n of $\sum_{n=1}^{\infty} a_n$.

Since $\lim\limits_{n\to\infty} S_n = S$, it follows that $\lim\limits_{n\to\infty} T_n = S$ and so $\sum_{n=1}^{\infty} b_n$ converges.

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Grouping and rearrangement of series

If $\sum_{n=1}^{\infty} a_n$ absolutely converges, then we can reorder the terms without altering the sum.

Proof:

Let $\sum_{n=1}^{\infty} b_n$ be a reordering of $\sum_{n=1}^{\infty} a_n$.

Since $\sum_{n=1}^{\infty} |a_n|$ converges, say to S, and the partial sums $T_n = \sum_{r=1}^{n} |b_r|$ are monotonic increasing and bounded above by S, $\sum_{r=1}^{n} |b_r|$ also converges.

Since $\sum_{r=1}^{n} |b_r|$ converges, $\sum_{r=1}^{n} b_r$ also converges.

Let $\sum_{r=1}^{n} a_r = S_n'$ and $\sum_{n=1}^{\infty} a_n = S'$. Let $\sum_{r=1}^{n} b_r = T_n'$ and $\sum_{n=1}^{\infty} b_n = T'$

Then given $\epsilon > 0, \exists\, N$: $|S_n' - S'| < \epsilon$.

Then for large enough m, $T_m' = S_n' + a_{i_1} + a_{i_2} + \cdots + a_{i_k}$ for some a_{i_j}.

Then $|T_m' - S'| \leq |S_n' - S'| + |a_{i_1}| + |a_{i_2}| + \cdots + |a_{i_k}| < 2\epsilon$. So $T_m' \to S'$ as $n \to \infty$.