Computational Thinking

Logic

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Lecture 7

More on Resolution for Propositional Logic

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Recall the basic rule of Resolution

- Recall the rule of inference known as resolution

$$\frac{p \vee q \qquad\qquad \neg p \vee r}{q \vee r}$$

  – forms the basis of the proof system for propositional logic known as Resolution.

- However, the basic rule of Resolution is a more general one than that above

$$\frac{p_1 \vee \ldots \vee p_{i-1} \vee \boxed{x \vee} p_{i+1} \vee \ldots \vee p_m \qquad q_1 \vee \ldots \vee q_{i-1} \vee \boxed{\neg x} \vee q_{i+1} \vee \ldots \vee q_n}{p_1 \vee \ldots \vee p_{i-1} \vee p_{i+1} \vee \ldots \vee p_m \vee q_1 \vee \ldots \vee q_{i-1} \vee q_{i+1} \vee \ldots \vee q_n}$$

  – the p's and the q's are literals
    - that is, variables or negated variables (not necessarily distinct)
  – this is the *only* rule of Resolution.

Example 1

Let φ be the formula $\neg((p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q))$.

Is φ a theorem?

In order to prove this using Resolution we negate φ and put it in conjunctive normal form if necessary.

So $\neg\varphi$ is the formula $\underline{(p \vee \cancel{q}) \wedge (\neg p \vee q)} \wedge \underline{(p \vee \neg q) \wedge (\neg p \vee \neg q)}$ which is in conjunctive normal form already.

$(q \vee q) \wedge (\neg q \vee \neg q) \mid q \wedge \neg q = \emptyset$

We now try to apply Resolution on $\neg\varphi$ until

- either we infer the empty clause, which means that $\neg\varphi$ is a contradiction, and hence φ is a theorem, or
- we do not infer the empty clause but at some point we do not find any new clauses either; in that case we can find a truth assignment that makes $\neg\varphi$ true, and hence, φ false, which means that φ is not a theorem.

$\neg q \vee r \vdash \neg\varphi$

$\neg\varphi$ is True

$q = F$, $r = T$

φ is false,

φ is not a theorem

Example 2

- Use Resolution to prove that if
  - "It is not raining or I have my umbrella"
  - "I do not have my umbrella or I do not get wet"
  - "It is raining or I do not get wet"

  then
  - "I do not get wet"

$\neg R \vee U$ (1), $\neg U \vee \neg W$ (2), $R \vee \neg W$ (3), W (4)

4: $U \vee \neg W$ 1, 3

5: $\neg W \vee \neg W$ 4, 2

6: $\neg W$ 5

7: W premise

8: $\emptyset$

- A formula $\varphi \Rightarrow \psi$ is logically equivalent to $\neg\varphi \vee \psi$
  - so the negation of our formula is $\varphi \wedge \neg\psi$
    - that is

$$(\neg R \vee U) \wedge (\neg U \vee \neg W) \wedge (R \vee \neg W) \wedge W$$

$\varphi \wedge \neg\psi$ is a contradiction

$\varphi \Rightarrow \psi$ is a theorem.

So we must apply resolution on clauses $\neg R \vee U,\ \neg U \vee \neg W,\ R \vee \neg W,\ W$

Example 3

Apply Resolution to the following set of clauses:

$a \vee b \vee c \qquad a \vee \neg c \vee d \qquad \neg a \vee e \vee f \qquad c \vee \neg e \vee f \qquad c \vee \neg d \vee \neg f$

Resolving a gives us: $b \vee c \vee e \vee f$ and $\neg c \vee d \vee e \vee f$.

From now on we ignore the three clauses $a \vee b \vee c$, $a \vee \neg c \vee d$, and $\neg a \vee e \vee f$.

Resolving c on $c \vee \neg e \vee f$ and $\neg c \vee d \vee e \vee f$;

on $c \vee \neg d \vee \neg f$ and $\neg c \vee d \vee e \vee f$; and

on $b \vee c \vee e \vee f$ and $\neg c \vee d \vee e \vee f$

$f \vee d \vee f$

e

$b \vee e \vee f$

~~True ∧ True~~ $\wedge$ $\boxed{b \vee e \vee f \vee d \vee e \vee f}$

gives us: $\neg e \vee f \vee d \vee e \vee f$ and $\wedge$ $\neg d \vee \neg f \vee d \vee e \vee f$ and $\wedge$ $b \vee e \vee f \vee d \vee e \vee f$, resp.

We resolved c so we ignore the clauses with c or $\neg c$ in it.

The clauses $\boxed{\neg e} \vee f \vee d \vee \boxed{e} \vee f$ and $\neg d \vee \neg f \vee d \vee e \vee f$ are tautologies and won't be useful for us anymore (check this).

→ True   → True

The last clause $b \vee e \vee f \vee d \vee e \vee f$ simplifies to $b \vee d \vee e \vee f$. This clause is the only "remaining" clause and we cannot resolve any other variables. We can satisfy this clause via the truth assignment that makes b, d, e, f true. Making c true as well (and making a either true or false) we then find a satisfying truth assignment for the initial set of clauses:

$a \vee b \vee c \qquad a \vee \neg c \vee d \qquad \neg a \vee e \vee f \qquad c \vee \neg e \vee f \qquad c \vee \neg d \vee \neg f$

Hence the initial set is **not** a contradiction. If we applied Resolution formally we would have to create many more clauses, but in the end the process would have stopped in the case where no new clauses can be generated without finding the empty clause (check this).

The above is an example where we systematically resolved variables one by one until in the end we found a satisfying truth assignment. Formally we needed to apply Resolution as long as possible, which we did not do: By resolving variable after variable we found a satisfying truth assignment, and to either find a satisfying truth assignment or the empty clause was our goal, so we can stop.