Lecture 4: Trees, Rooted Trees, Graph Isomorphism

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Reminder from last lecture

- A tree is a connected acyclic graph.
- Each tree contains at least one leaf, i.e. a vertex of degree 1.
- A tree on n vertices has exactly $n-1$ edges.
- Conversely, a connected graph with n vertices and $n-1$ edges is a tree.
- A tree contains a unique path between any two vertices in it.
- A rooted tree has one distinguished vertex, the root, and all edges are directed away from the root.

Contents for today's lecture

- Exercises involving trees;
- Examples of proof techniques;
- Properties of rooted trees.

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- Step: Now suppose that every tree on $< n$ vertices has at least $n_3 + 2$ leaves (induction hypothesis), and consider a tree T on $n \geq 3$ vertices.

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- Then $T' = T - u$ is a tree on $n - 1$ vertices. By the induction hypothesis, we have $\ell(T') \geq n_3(T') + 2$.

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- The rest of the proof is on the next slide.

Proof continued

- We have: a leaf u in T, a tree $T' = T - u$, $\ell(T') \geq n_3(T') + 2$.
- Let v be the (unique) neighbour of u in T.
- T is connected and has at least 3 vertices, so v has at least 2 neighbors in T.

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This finishes the proof.

Note that induction on n_3 is also possible.

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- Choose any vertex v and put this vertex in the set V_1.
- For every vertex $u \neq v$, there is a unique path from v to u in T, consider the length of this path.
- If the length is odd, put u in V_2; otherwise put u in V_1.

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- If the length is odd, put u in V_2; otherwise put u in V_1.
- We have to show that this is a valid bipartition.
- V_1 and V_2 are disjoint and together make up $V(T)$. (Why?)
- Every edge has end vertices in both V_1 and V_2. (Why?)
- This completes the proof.

□

Graph isomorphism

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Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijective function $f : V \to V'$ such that for every $u, v \in V$ we have: $uv \in E$ if and only if $f(u)f(v) \in E'$. Then we write: $G \cong G'$.

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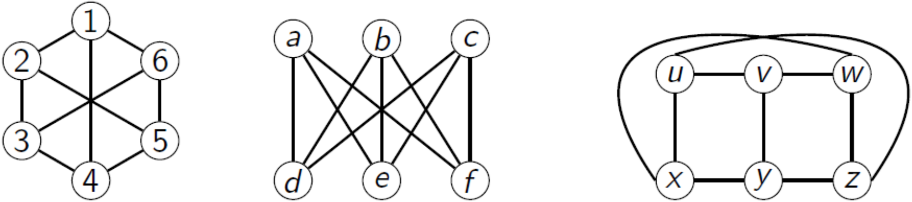
Example: which of these graphs are isomorphic?



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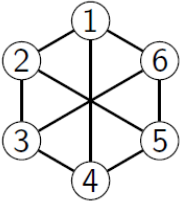

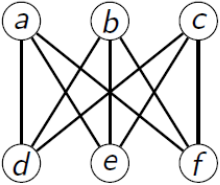

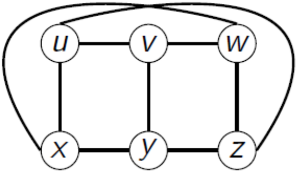

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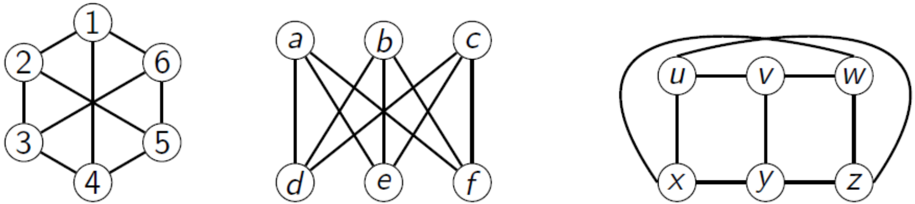

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bijective function f for the second and third graph:
$a \mapsto x,\ b \mapsto v,\ c \mapsto z,\ d \mapsto u,\ e \mapsto w,\ f \mapsto y$

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Can we quickly decide whether G and G' are isomorphic?

- one of the few most tantalising and tricky questions in Computer Science
- presumably not an "easy" problem (i.e. polynomial-time) and not a "hard" problem (i.e. NP-complete), but "somewhere in the middle"

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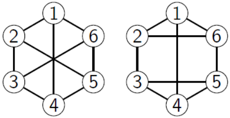

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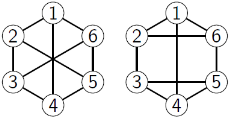

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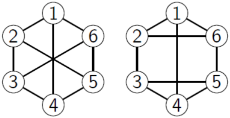

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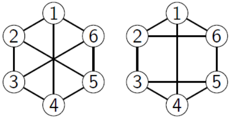

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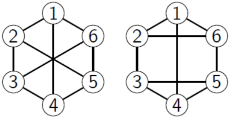

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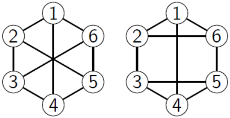

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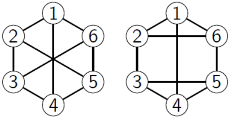

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All these (and all other characteristics):

- can only be used to show non-isomorphism

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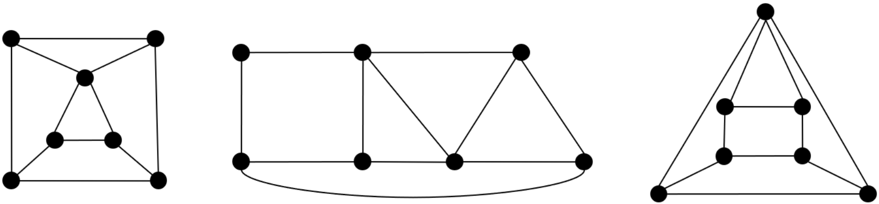
What about these graphs?



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Two rooted trees $T_1(V_1, E_1, r_1)$ and $T_2(V_2, E_2, r_2)$ are called isomorphic if there exists an isomorphism bijection $f : V_1 \mapsto V_2$ such that $f(r_1) = r_2$.

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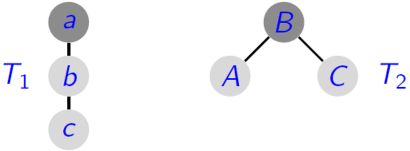
Example: T_1 and T_2 are isomorphic as graphs, but not as rooted trees !



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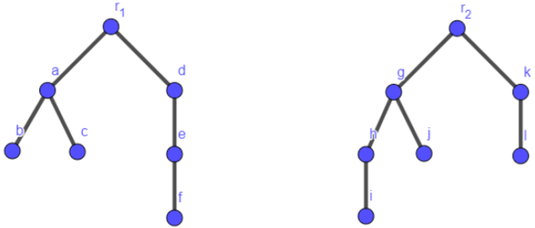

Graph isomorphism on rooted trees

An isomorphism algorithm for rooted trees (Algorithm 2):

Algorithm 1 LabelRootedTree(T, v)

```
if v is a leaf of T then
    label(v) ← "10"
else
    for every child w of v do
        ℓ(w) ← label(w)
        Sort the labels of the children of v decreasingly: ℓ_1 ≥ ℓ_2 ≥ ... ≥ ℓ_k
        label(v) ← "1" ℓ_1 ℓ_2 ... ℓ_k "0"
return label(v)
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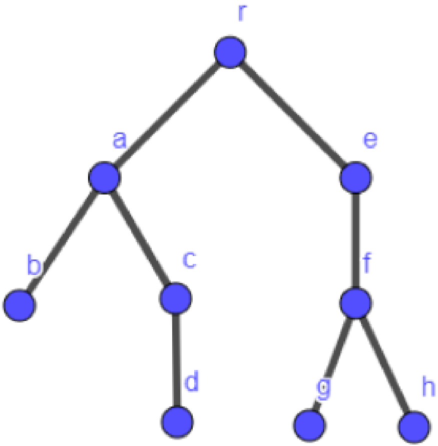
else
    for every child w of v **do**
        $\ell(w) \leftarrow label(w)$
        Sort the labels of the children of v decreasingly: $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_k$
        $label(v) \leftarrow$ "1" $\ell_1\ \ell_2 \ldots \ell_k$ "0"
return $label(v)$

Algorithm 2 $\textsc{Tree Isomorphism}((T_1, r_1), (T_2, r_2))$

$label(r_1) \leftarrow \textsc{LabelRootedTree}(T_1, r_1)$
$label(r_2) \leftarrow \textsc{LabelRootedTree}(T_2, r_2)$
if $label(r_1) = label(r_2)$ **then**
    return YES
else
    return NO

Graph isomorphism on rooted trees

Example:



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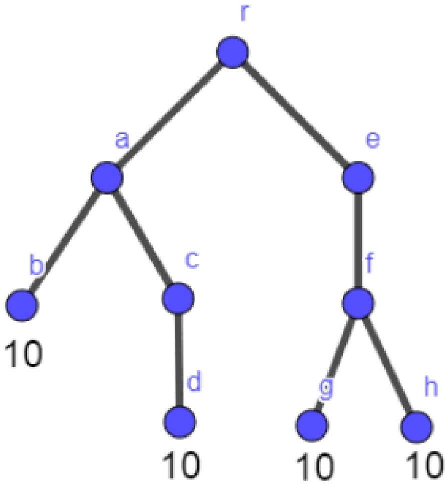

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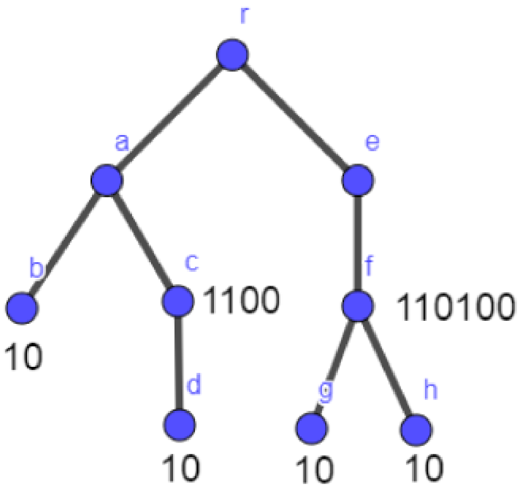

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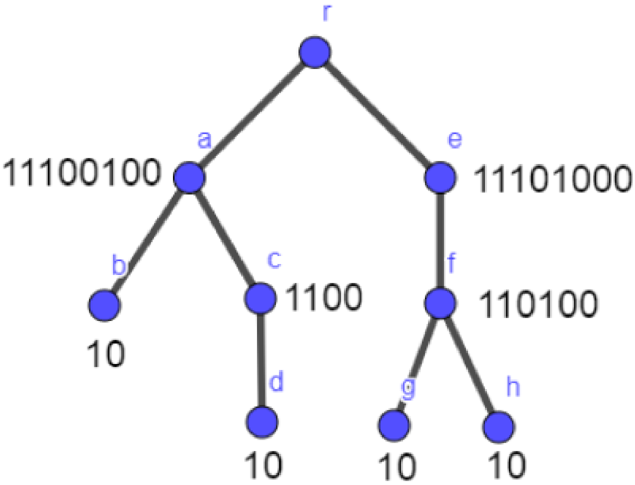

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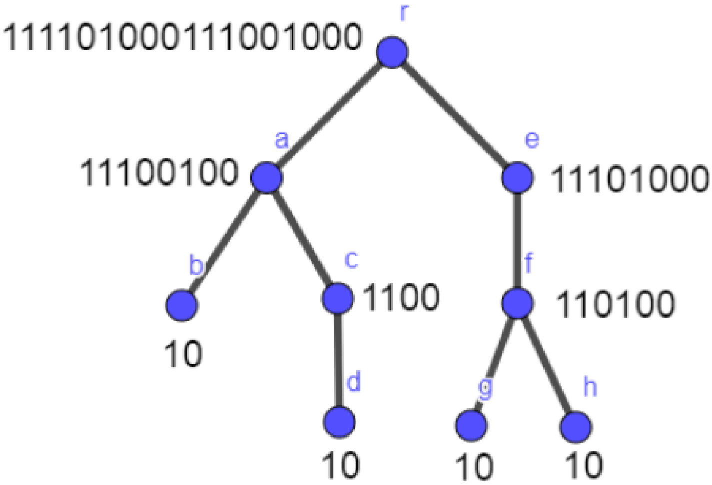

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Induction hypothesis: If two rooted trees (X_1, a_1) and (X_2, a_2) are isomorphic and have both k levels then the algorithm returns YES on input $((X_1, a_1), (X_2, a_2))$.

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Induction hypothesis: If two rooted trees (X_1, a_1) and (X_2, a_2) are isomorphic and have both k levels then the algorithm returns YES on input $((X_1, a_1), (X_2, a_2))$.

Induction step: Let (T_1, r_1) and (T_2, r_2) be two isomorphic rooted trees, each with $k + 1$ levels.

Let f be the bijection between T_1 and T_2 (from the definition). Then $f(r_1) = f(r_2)$ and, for every child a_1 of r_1 there exists a child a_2 of r_2 such that:

- the subtrees $(T_1(a_1), a_1)$ and $(T_2(a_2), a_2)$ are isomorphic, and
- $f(a_1) = f(a_2)$

Graph isomorphism on rooted trees

Proof (cont.):

Since every such pair of trees $(T_1(a_1), a_1)$ and $(T_2(a_2), a_2)$ is isomorphic, they have both k levels. Thus, by the induction hypothesis, Algorithm 1 returns the same labels for their roots, i.e. $label(a_1) = label(a_2)$.

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Algorithm 1 now computes $label(r_1)$ by decreasingly sorting the labels of the children of r_1, say $\ell_1 \geq \ldots \geq \ell_p$, and then it sets $label(r_1) =$ "1" $\ell_1 \ell_2 \ldots \ell_p$ "0".

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Since all these labels are the same (by the induction hypothesis): $label(r_1) = label(r_2)$, and thus Algorithm 2 returns YES. $\square$

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Does this theorem show that Algorithm 2 is a correct isomorphism algorithm for rooted trees?

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Does this theorem show that Algorithm 2 is a correct isomorphism algorithm for rooted trees?

- we also need the reverse direction: if two trees are not isomorphic, then the algorithm returns NO
- this can be also proved (a bit more tricky)

Additional slides

How to find and write down proofs?

Let's point out relevant stages in the process of proving statements by going in detail through one example.

Before doing so, let's point out the following questions you could ask yourself to help finding a possible proof approach.

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- How to approach/attack the question? Can I use induction; does a direct proof have any chance; or does it help to use contradiction?
- Is my solution valid and convincing? Write a draft first; check all the steps; critically examine the steps for errors or counterexamples; modify and revise the solution and write it down in a clear way.

The start: Write down what you see

We will consider the process of finding the proof on the following example:

Lemma

Let T be a tree on $n \geq 2$ vertices, and let $e \in E(T)$. Then $T - e$ is a forest consisting of precisely two trees.

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- Here it (probably) helps to draw a picture that roughly sketches the situation and concepts. I will not include it on the slides.

Example continued

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- The question is how to write it down (and check that the picture did not fool you).
- This requires certain skills and experience.

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- The question is how to write it down (and check that the picture did not fool you).
- This requires certain skills and experience.
- You can only learn this by doing it yourself.
- You do not learn it by reading (although this helps if you force yourself to understand every step you read).

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- A tree is a connected graph without cycles.
- A forest is a graph without cycles.
- Since a tree is a connected graph, between any two vertices there is a path in a tree.
- We even know from the previous lecture that this path is unique.

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How to use (some of) the above facts to prove that $T - e$ is a forest consisting of precisely two trees?

Let us consider the first part of the statement first. Can we prove that $T - e$ is a forest?

Example: the first part

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Let T be a tree on $n \geq 2$ vertices, and let $e \in E(T)$. Then $T - e$ is a forest.

This is an easy consequence of the definitions and the observation that removing edges from a tree, we cannot introduce cycles. So if T is a tree, then T contains no cycles and $T - e$ contains no cycles either, so $T - e$ is a forest. (This is a direct proof.)

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At least 2: you have to show that $T - e$ is not connected (not 1 tree). This is easy: if u and v are the end vertices of the edge e, then in $T - e$ there is no path between u and v. (Why not?) (This is also a direct proof.)

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The proof seems to be complete. Now you have to write it down and carefully check the details.

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Let ℓ be the number of leaves a full m-ary tree. Since $n = i + \ell$ and $n = m \cdot i + 1$, if we know any of n, i, ℓ then we can find all of them.

Exercise

- Suppose someone starts a chain letter. Each person who receives it is asked to send it on to four other people. Some people do this, some don't.
- It ended after there have been 100 people who have seen the letter, but did not send it out.
- How many people have seen this letter, including the first person, if no-one received it more than once?
- How many people sent out the letter?

Solution: model the situation as

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- Solving this, get $n = 133$ and $i = 33$.

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- Hence, T has at most $m \cdot m^{h-1} = m^h$ leaves.

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- **Exercise:**: how many are there on 7 vertices?