Mathematics for Computer Science Linear Algebra

Lecture 7: Euclidean vector spaces

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November 18, 2020

Contents for today's lecture

- Euclidean vector spaces $\mathbb{R}^n$
- Norm, dot product, and orthogonality in $\mathbb{R}^n$
- Useful properties of these notions (with proofs ...)
- Dot product and linear systems

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

- You are familiar with vectors in two and three dimensions
- Such a vector can be identified with an ordered tuple of real numbers: (a_1, a_2) or (a_1, a_2, a_3), respectively.
- The numbers in the tuple are the components of the vector.
- The sets of all 2D and 3D vectors are denoted by $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

- Two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are equal iff all corresponding coordinates are equal.
- Main operations on vectors: addition and scaling (multiplication by a scalar).
  - If $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ are vectors in $\mathbb{R}^2$ then $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$.
  - If k is a scalar (i.e. a real number) and $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ then $k\mathbf{a} = (ka_1, ka_2)$.
- For example, if $\mathbf{a} = (-1, 3)$ and $\mathbf{b} = (2, 1)$ then $2\mathbf{a} - 5\mathbf{b} = (-12, 1)$.

Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^n$

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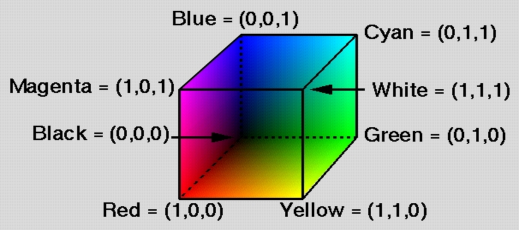
  - If k is a scalar (i.e. a real number) and $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ then $k\mathbf{a} = (ka_1, ka_2)$.

- All the above can be generalised to n-tuples of real numbers, for any fixed n.
- Notation: $\mathbb{R}^n = \{(a_1, \ldots, a_n) \mid \text{ all } a_i \in \mathbb{R}\}$.
- Euclidean vector spaces $= \mathbb{R}^n$ equipped with the two above operations
- Note that one can view a vector in $\mathbb{R}^n$ as a $1 \times n$ (or $n \times 1$) matrix.

The RGB colour model

- Colours in computer monitors are commonly based on the RGB colour model.
- Colours are created by adding percentages of primary colours: red (R), green (G) and blue (B).
- We can identify primary colours with vectors, as follows:
  $\mathbf{r} = (1, 0, 0)$ (pure red)
  $\mathbf{g} = (0, 1, 0)$ (pure green)
  $\mathbf{b} = (0, 0, 1)$ (pure blue)
- Each colour vector $\mathbf{c}$ can be expressed as a sum (aka linear combination) $\mathbf{c} = k_1\mathbf{r} + k_2\mathbf{g} + k_3\mathbf{b} = (k_1, k_2, k_3)$, where $0 \leq k_i \leq 1$ for $i = 1, 2, 3$.
- This can be visualised as the RGB colour cube.

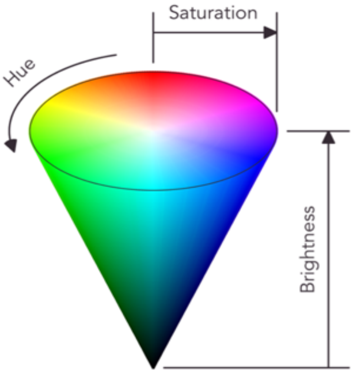


An alternative colour model - HSB (aka HSV)

An image on a computer screen is a set of pixels.

Each pixel is a vector $(x, y, h, s, b) \in \mathbb{R}^5$
where (x, y) are the pixel's coordinates,
(h, s, b) describe the colour of the pixel:

- h is hue = "colour of the rainbow"
  it's a number between 0 and 360
  0 is red, 120 is green, 240 is blue
- s is saturation = "richness"
  "how injected with colour it is"
  it's a number between 0 and 100
- b is brightness (aka value)
  "how much lightbulb is turned on"
  it's a number between 0 and 100
  brightness 0 is always black



Norm in $\mathbb{R}^n$

- The length (aka norm) of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is defined by formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

- It holds that
  - $||\mathbf{v}|| \geq 0$, and $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$,
  - $||k\mathbf{v}|| = |k| \cdot ||\mathbf{v}||$.
- A vector of length 1 is called a unit vector.
- For any vector $\mathbf{v}$, the vector $\frac{1}{||\mathbf{v}||}\mathbf{v}$ is a unit vector in the same direction as $\mathbf{v}$. It is obtained by normalizing $\mathbf{v}$.

Example: To normalize vector $\mathbf{v} = (3, -2, -4, 4, 2)$, compute its length

$$||\mathbf{v}|| = \sqrt{3^2 + (-2)^2 + (-4)^2 + 4^2 + 2^2} = \sqrt{49} = 7.$$

So the unit vector obtained by normaizing $\mathbf{v}$ is $\frac{1}{||\mathbf{v}||}\mathbf{v} = (\frac{3}{7}, -\frac{2}{7}, -\frac{4}{7}, \frac{4}{7}, \frac{2}{7})$.

Dot product in $\mathbb{R}^n$

- The dot product (aka Euclidean inner product) of vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

  Notice similarity to matrix product: If you think of $\mathbf{u}$ and $\mathbf{v}$ as column matrices (of size $n \times 1$) then $\mathbf{u} \cdot \mathbf{v}$ is the same as the (matrix) product $\mathbf{u}^T \mathbf{v}$.
- Note that $||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$.
- For example, if $\mathbf{u} = (-1, 3, 5, 7)$ and $\mathbf{v} = (2, -1, 3, -5) \in \mathbb{R}^4$ then

$$\mathbf{u} \cdot \mathbf{v} = (-1) \cdot 2 + 3 \cdot (-1) + 5 \cdot 3 + 7 \cdot (-5) = -25.$$

If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then the following properties (obviously) hold:

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Symmetry, aka Commutativity)
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Distributivity)
- $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ (Homogeneity)
- $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ iff $\mathbf{v} = \mathbf{0}$ (Positivity)

Dot product and ISBN numbers

- Each book has a unique code: ISBN (International Standard Book Number).
- It used to be 10 digits (ISBN-10) before 2007, now it's 13 digits (ISBN-13).
- So it can thought of as a vector $\mathbf{d} = (d_1, d_2, \ldots, d_{12}, d_{13}) \in \mathbb{R}^{13}$
- 10th edition of "Elementary Linear Algebra" has ISBN 978-0-470-56157-7

- The last digit is the check digit. For each valid ISBN $\mathbf{d}$ the following holds:
  - Let $\mathbf{d}' = (d_1, d_2, \ldots, d_{12}) \in \mathbb{R}^{12}$, i.e. it is $\mathbf{d}$ without the last component.
  - Let $\mathbf{a} = (1, 3, 1, 3, \ldots, 1, 3) \in \mathbb{R}^{12}$.
  - Compute the dot product $\mathbf{a} \cdot \mathbf{d}'$ and let x be the last digit of this number.
  - Then we must have

$$d_{13} = \begin{cases} 0 & \text{if } x = 0 \\ 10 - x & \text{if } x \neq 0 \end{cases}$$

- For (the 10th edition of) our textbook, $\mathbf{a} \cdot \mathbf{d}' =$

$$1 \cdot 9 + 3 \cdot 7 + 1 \cdot 8 + 3 \cdot 0 + 1 \cdot 4 + 3 \cdot 7 + 1 \cdot 0 + 3 \cdot 5 + 1 \cdot 6 + 3 \cdot 1 + 1 \cdot 5 + 3 \cdot 7 = 113$$

The last digit is 3, so we should have $d_{13} = 10 - 3 = 7$, and we do!

The Cauchy-Schwarz and triangle inequalities

Theorem (Cauchy-Schwarz inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^n$ *then* $\mathbf{u} \cdot \mathbf{v} \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$.

The proof will appear later today.

Corollary (Triangle inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^n$ *then* $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

Proof.

$$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = ||\mathbf{u}||^2 + 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2 \leq$$

(use Cauchy-Schwarz)

$$\leq ||\mathbf{u}||^2 + 2||\mathbf{u}|| \cdot ||\mathbf{v}|| + ||\mathbf{v}||^2 = (||\mathbf{u}|| + ||\mathbf{v}||)^2.$$

Now have $||\mathbf{u} + \mathbf{v}||^2 \leq (||\mathbf{u}|| + ||\mathbf{v}||)^2$, take square roots. □

Orthogonality in $\mathbb{R}^n$

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal (or perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$.
- Example: vectors $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ in $\mathbb{R}^4$ are orthogonal because
$$\mathbf{u} \cdot \mathbf{v} = (-2) \cdot 1 + 3 \cdot 2 + 1 \cdot 0 + 4 \cdot (-1) = 0.$$

Theorem (Pythogoras' theorem in $\mathbb{R}^n$)

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$ then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Proof.

Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we have $\mathbf{u} \cdot \mathbf{v} = 0$, hence
$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = ||\mathbf{u}||^2 + 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$. $\square$

Projection theorem

Theorem (Projection theorem)

If $\mathbf{u}$ *and* $\mathbf{v} \neq \mathbf{0}$ *are vectors in* $\mathbb{R}^n$ *then* $\mathbf{u}$ *can be uniquely expressed as* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ *where* $\mathbf{w}_1 = k\mathbf{v}$*, and* $\mathbf{v}$ *and* $\mathbf{w_2}$ *are orthogonal.*

The vector $\mathbf{w}_1$ is called the orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$.

Proof.

First, find out what k must be if we were to have $\mathbf{u} = k\mathbf{v} + \mathbf{w}_2$ and $\mathbf{v} \cdot \mathbf{w_2} = 0$.

Take the dot product of the two sides of equality $\mathbf{u} = k\mathbf{v} + \mathbf{w}_2$ with $\mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} = (k\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v} = k(\mathbf{v} \cdot \mathbf{v}) + \mathbf{w}_2 \cdot \mathbf{v} = k(\mathbf{v} \cdot \mathbf{v}) = k||\mathbf{v}||^2.$$

So, we must have $k = (\mathbf{u} \cdot \mathbf{v})/||\mathbf{v}||^2$. Then $\mathbf{w_2} = \mathbf{u} - k\mathbf{v}$ is uniquely determined.

It remains to check that, with this k, the vectors $\mathbf{v}$ and $\mathbf{w_2}$ are indeed orthogonal:

$$\mathbf{v} \cdot (\mathbf{u} - k\mathbf{v}) = \mathbf{v} \cdot \mathbf{u} - k(\mathbf{v} \cdot \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} - k||\mathbf{v}||^2 = \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}||\mathbf{v}||^2 = 0$$

$\square$

Proof of Cauchy-Schwarz

Theorem (Cauchy-Schwarz inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^n$ *then* $\mathbf{u} \cdot \mathbf{v} \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$.

Proof.

The inequality clearly holds if $\mathbf{v} = \mathbf{0}$. Assume now $\mathbf{v} \neq \mathbf{0}$.

Apply Projection theorem to $\mathbf{u}$ and $\mathbf{v}$: we get $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w_1} = k\mathbf{v}$ and $\mathbf{v} \cdot \mathbf{w}_2 = 0$. Moreover, we have $k = (\mathbf{u} \cdot \mathbf{v})/||\mathbf{v}||^2$.

Let's prove that $||\mathbf{u}||^2 \geq k^2||\mathbf{v}||^2$ by applying Pythagoras' theorem to $\mathbf{w}_1$ and $\mathbf{w}_2$

$$||\mathbf{u}||^2 = ||\mathbf{w}_1 + \mathbf{w}_2||^2 = ||\mathbf{w}_1||^2 + ||\mathbf{w}_2||^2 \geq ||\mathbf{w}_1||^2 = ||k\mathbf{v}||^2 = (k\mathbf{v}) \cdot (k\mathbf{v}) = k^2||\mathbf{v}||^2.$$

Substituting $k = (\mathbf{u} \cdot \mathbf{v})/||\mathbf{v}||^2$ into $||\mathbf{u}||^2 \geq k^2||\mathbf{v}||^2$, we get

$$||\mathbf{u}||^2 \geq k^2||\mathbf{v}||^2 = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{(||\mathbf{v}||^2)^2}||\mathbf{v}||^2 = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{||\mathbf{v}||^2}.$$

Re-arrange to get $(\mathbf{u} \cdot \mathbf{v})^2 \leq ||\mathbf{u}||^2 \cdot ||\mathbf{v}||^2$ and take square roots. $\square$

Dot product and linear systems

A linear equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ can be re-written as follows:

Letting $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the equation is

$$\mathbf{a} \cdot \mathbf{x} = b.$$

Similarly, a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ can be re-written as a system

$$\begin{gathered} \mathbf{r}_1 \cdot \mathbf{x} = 0 \\ \mathbf{r}_2 \cdot \mathbf{x} = 0 \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} = 0 \end{gathered}$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the row vectors of A.

Fact

For any $m \times n$ matrix A, the set of solutions of the linear system $A\mathbf{x} = \mathbf{0}$ is equal to the set of all vectors in $\mathbb{R}^n$ which are orthogonal to every row vector of A.

What we learnt today

Euclidean vector spaces $\mathbb{R}^n$:

- What they are
- Norm, dot product, and orthogonality in $\mathbb{R}^n$
- Useful properties: Cauchy-Schwarz, triangle, Pythagoras
- Dot product and linear systems

Next time:

- General vector spaces