Mathematics for Computer Science Linear Algebra

Lecture 13: Diagonalisation

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Reminder from the last lecture

- For an $n \times n$ matrix, a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of A if $A\mathbf{x} = \lambda\mathbf{x}$.
- In this case, λ is called an eigenvalue of A, and $\mathbf{x}$ is an eigenvector corresponding to λ.
- The polynomial $det(\lambda I - A)$ is called the characteristic polynomial of A and the equation $det(\lambda I - A) = 0$ the characteristic equation of A.
- The eigenvalues of A are the solutions of $det(\lambda I - A) = 0$.
- For an eigenvalue λ_0 of A, the null space of matrix $\lambda_0 I - A$ is the eigenspace of A corresponding to λ_0. The non-zero vectors in this subspace are the eigenvectors of A corresponding to λ_0.
- For every eigenvalue of A, its algebraic multiplicity is <u>greater than</u> or equal to its geometric multiplicity.

Contents for today's lecture

- Similarity of matrices;
- Diagonalisation and how to find it
- Eigendecomposition

Similarity of matrices

Definition

Square matrices A and B are called similar if $A = P^{-1}BP$ for some invertible P.

Note that if $A = P^{-1}BP$ then $B = Q^{-1}AQ$ where $Q = P^{-1}$.

Similar matrices have many features in common, including determinant, trace, rank, nullity, characteristic polynomial, eigenvalues, dimensions of corresponding eigenspaces etc.

Lemma

If A and B are similar then $det(A) = det(B)$.

Proof.

$det(A) = det(P^{-1}BP) = det(P^{-1})det(B)det(P) = \frac{1}{det(P)}det(B)det(P) = det(B)$. $\square$

Similarity and linear maps

- Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ a basis in $\mathbb{R}^n$.
- Let $[T]_S$ be the $n \times n$ matrix

$$[T]_S = [(T(\mathbf{v}_1))_S | (T(\mathbf{v}_2))_S | \dots | (T(\mathbf{v}_n))_S]$$

  whose columns are the coordinate vectors of vectors $T(\mathbf{v}_i)$ in basis S. This matrix is called the matrix of T in S.
- We also say that the matrix $[T]_S$ represents T in basis S.
- If S is the standard basis of $\mathbb{R}^n$ then $[T]_S$ is the standard matrix of T.

Theorem

Matrices A and B are similar iff they represent the same linear operator $f : \mathbb{R}^n \to \mathbb{R}^n$, possibly in different bases. *(Proof omitted.)*

That is, matrices A and B are similar iff there is a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and two bases S, S' of $\mathbb{R}^n$ such that $A = [T]_S$ and $B = [T]_{S'}$.

Diagonalisable matrices

Definition

A matrix A is called diagonalisable if it is similar to a diagonal matrix –
in other words, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.
Then P is said to diagonalise A.

Note that A is diagonalisable if it decomposes as $A = PDP^{-1}$ where P is invertible and $D = diag(\lambda_1, \ldots, \lambda_n)$ is diagonal

Diagonalisation is useful for many things, e.g. computing powers of matrices.

If we know that $A = PDP^{-1}$ where $D = diag(\lambda_1, \ldots, \lambda_n)$ then

$$A^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})(PDP^{-1}) = PD^kP^{-1},$$

where $D^k = diag(\lambda_1^k, \ldots, \lambda_n^k)$.

A characterisation

Theorem

An $n \times n$ matrix is diagonalisable iff it has n linearly independent eigenvectors.

Proof.

($\Rightarrow$). Assume that there is an invertible matrix P and
a diagonal matrix $D = diag(\lambda_1, \ldots \lambda_n)$ such that $D = P^{-1}AP$, or $AP = PD$.

Denote the column vectors of P by $\mathbf{p}_1, \ldots, \mathbf{p}_n$, so $P = [\mathbf{p}_1 | \ldots | \mathbf{p}_n]$. Then

$$AP = A[\mathbf{p}_1 | \ldots | \mathbf{p}_n] = [A\mathbf{p}_1 | \ldots | A\mathbf{p}_n].$$

On the other hand,

$$PD = [\lambda_1 \mathbf{p}_1 | \ldots | \lambda_n \mathbf{p}_n].$$

Since $AP = PD$, we conclude that $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$ for all $1 \leq i \leq n$.

Since P is invertible, its rank is n and so the vectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are linearly independent. Then none of $\mathbf{p}_1, \ldots, \mathbf{p}_n$ is $\mathbf{0}$, so each of them is an eigenvector. $\square$

Proof cont'd

Theorem

An $n \times n$ matrix is diagonalisable iff it has n linearly independent eigenvectors.

Proof.

($\Leftarrow$). Assume that A has n linearly independent eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues (not necessarily distinct). Define

$$P = [\mathbf{p}_1 | \ldots | \mathbf{p}_n] \quad \text{and} \quad D = diag(\lambda_1, \ldots, \lambda_n).$$

Then

$$AP = A[\mathbf{p}_1 | \ldots | \mathbf{p}_n] = [A\mathbf{p}_1 | \ldots | A\mathbf{p}_n] = [\lambda_1 \mathbf{p}_1 | \ldots | \lambda_n \mathbf{p}_n] = PD.$$

The columns of P are linearly independent, so its rank is n and it is invertible. Finally $AP = PD$ is equilvalent to $D = P^{-1}AP$. $\square$

An algorithm for diagonalisation

Given a matrix A, this algorithm diagonalises it (or reports that this is impossible):

1. Find the eigenvalues of A (e.g. by solving its characterstic equation).
2. Find a basis in each eigenspace of A and merge these bases into one set S.
3. If S has fewer than n vectors then A is not diagonalisable.
4. Else, form the matrix $P = [\mathbf{p}_1 | \dots | \mathbf{p}_n]$ where $S = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$.

   \# The set S contains n vectors and is linearly independent (will prove this),
   \# so S is a basis for $\mathbb{R}^n$. Hence, P is invertible.
5. The matrix $D = P^{-1}AP$ is diagonal, $D = diag(\lambda_1, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue corresponding to $\mathbf{p}_i$.

Remark:

- If an $n \times n$ matrix has n distinct eigenvalues then it is diagonalisable.

Example 1

For $k \neq 0$, is the following matrix (corresponding to shear) diagonalisable?

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Solution. First compute the characteristic polynomial $det(\lambda I - A)$:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -k \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2.$$

The characteristic equation is $(\lambda - 1)^2 = 0$, so A has only one eigenvalue $\lambda_1 = 1$.

The corresponding eigenspace is the nullspace of $\lambda_1 I - A = I - A = \begin{pmatrix} 0 & -k \\ 0 & 0 \end{pmatrix}$.

Does it have two linearly independent vectors? (Is $nullity(I - A) = 2$?)

Since $rank(I - A) + nullity(I - A) = 2$ (by the Dimension Theorem for matrices) and $rank(I - A) = 1$ (obviously), we conclude that $nullity(I - A) = 1$, and therefore A is **not** diagonalisable.

Example 2

Diagonalise the following matrix, if possible,

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Solution. We have

$$det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, can factor it (as in last lecture): $(\lambda - 1)(\lambda - 2)^2 = 0$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

The corresponding eigenspaces are the nullspaces of $I - A$ and $2I - A$, resp. Using the algorithms for finding a basis in a nullspace, get the following:

$$\lambda_1 = 1 : \mathbf{p}_1 = (-2, 1, 1); \quad \lambda_1 = 2 : \mathbf{p}_2 = (-1, 0, 1) \text{ and } \mathbf{p}_3 = (0, 1, 0).$$

Example 2 cont'd

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$, and the bases of nullspaces are

$$\lambda_1 = 1 : \mathbf{p}_1 = (-2, 1, 1); \quad \lambda_1 = 2 : \mathbf{p}_2 = (-1, 0, 1) \text{ and } \mathbf{p}_3 = (0, 1, 0).$$

Hence, A is diagonalisable, and one possible matrix that diagonalises it is

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

One can check that $P^{-1}AP$ is

$$\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Remark: The order of the $\mathbf{p}_i$'s in P can be changed arbitrarily, this will result in changing the order of the λ_i's in D accordingly.

Eigendecomposition

- Let A be a diagonalisable matrix.
- From the proof of characterisation, we have

$$AP = PD$$

where P is invertible and its columns $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are eigenvectors of A, and $D = diag(\lambda_1, \ldots, \lambda_n)$ and each λ_i is the eigenvalue of A corresponding to $\mathbf{p}_i$

- Then we have a decomposition

$$A = PDP^{-1}.$$

Since the factors in this decomposition are made of eigenvectors and eigenvalues, it is called an eigendecomposition of A.

(Can use the diagonalisation algorithm to find it)

Linear independence of eigenvectors

Theorem

If vectors $\mathbf{v}_1 \ldots, \mathbf{v}_k$ *are eigenvectors of a matrix* A *corresponding to (pairwise) distinct eigenvalues* $\lambda_1, \ldots, \lambda_k$ *then the set* $\{\mathbf{v}_1 \ldots, \mathbf{v}_k\}$ *is linearly independent.*

The theorem and the proof immediately extend to the case when we take several (i.e. possibly more than one) linearly independent vectors for each λ_i.

Proof.

Let $r \leq k$ be the largest number such that $\{\mathbf{v}_1 \ldots, \mathbf{v}_r\}$ is linearly independent. Assume for contradiction that $r < k$, so $\{\mathbf{v}_1 \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}\}$ is linearly dependent:

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

where not all $c_1, \ldots, c_r, c_{r+1}$ are 0.

Since $\{\mathbf{v}_1 \ldots, \mathbf{v}_r\}$ is linearly independent, we conclude that $c_{r+1} \neq 0$.

Since $\mathbf{v}_{r+1}$ is an eigenvector, we conclude that $c_i \neq 0$ for some $i \leq r$.

Continued on next slide ... □

Proof continued

Proof.

We assumed that $\{\mathbf{v}_1 \ldots, \mathbf{v}_r\}$ is linearly independent, but $\{\mathbf{v}_1 \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}\}$ is not:

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} = \mathbf{0} \tag{1}$$

We derived that $c_{r+1} \neq 0$ and $c_i \neq 0$ for some $i \leq r$.

Multiply both sides of equation (1) by A from the right and use $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$:

$$c_1\lambda_1\mathbf{v}_1 + \ldots + c_r\lambda_r\mathbf{v}_r + c_{r+1}\lambda_{r+1}\mathbf{v}_{r+1} = \mathbf{0}. \tag{2}$$

Now multiply both sides of (1) by λ_{r+1} and subtract that from (2) to obtain

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + \ldots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r = \mathbf{0}. \tag{3}$$

So $c_i(\lambda_i - \lambda_{r+1}) = 0$, and hence $c_i = 0$, for all $i \leq r$, a contradiction. $\square$

What we learnt today

- Similarity of matrices
- Diagonalisable matrices (= similar to a diagonal one)
- A characterisation of diagonalisable matrices
- An algorithm for diagonalisation
- Eigendecomposition of matrices

Next time:

- Complex vector spaces.

! If you haven't met complex numbers before, read parts 1-3 and 5 of this Wikipedia page and/or watch this lockdown lecture by 3Blue1Brown in advance of the lecture.