Mathematics for Computer Science
Linear Algebra

Lecture 12: Eigenvalues and eigenvectors

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Reminder from the last lecture

- Let V and W be vector spaces. A function $T : V \to W$ is called a linear map, or a linear transformation, from V to W if, for all $\mathbf{u}, \mathbf{v} \in V, k \in \mathbb{R}$,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = k \cdot T(\mathbf{u}).$$

  If $V = W$ then T is called a linear operator.

- If A is an $m \times n$ matrix then the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$.

- Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and matrix transformations are the same things.

Contents for today's lecture

- Eigenvalues and eigenvectors of matrices and linear maps;
- Characteristic polynomial and characteristic equation of a matrix;

Eigenvalues and eigenvectors

Definition

Let A be an $n \times n$ matrix. A non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of A (or, equivalently, of the operator $T_A : \mathbb{R}^n \to \mathbb{R}^n$) if, for some scalar λ,

$$A\mathbf{x} = \lambda\mathbf{x} \quad \text{(or, equivalently, } T_A(\mathbf{x}) = \lambda\mathbf{x}.\text{)}$$

In this case, λ is called an eigenvalue of A (and of T_A), and $\mathbf{x}$ is an eigenvector corresponding to λ.

- The assumption $\mathbf{x} \neq \mathbf{0}$ is necessary to avoid the case $A\mathbf{0} = \lambda\mathbf{0}$ which always holds.
- The meaning of the notion is that T_A does not change the direction of $\mathbf{x}$ (up to reversal), it only scales $\mathbf{x}$ by λ.

Example: vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ corresponding to eigenvalue 3. Indeed,

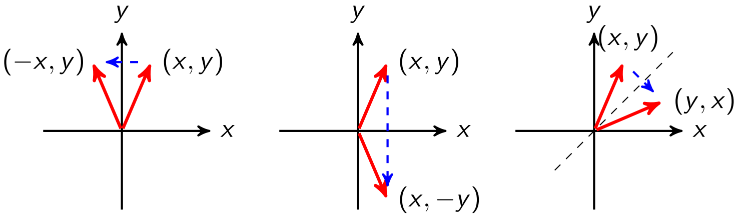
$$A\mathbf{x} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\mathbf{x}.$$

Example in $\mathbb{R}^2$

Consider linear operators T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

They correspond to reflections of $\mathbb{R}^2$ about y-axis, x-axis, and line $x = y$, resp.



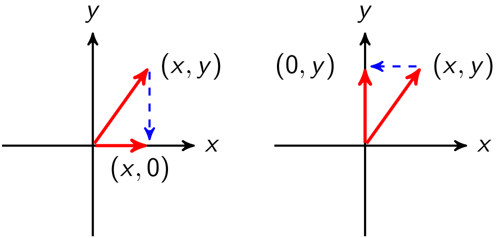
1. The eigenvectors are all non-zero vectors $(x, 0)$ and $(0, y)$, corresponding to eigenvalues -1 and 1, respectively.
2. The eigenvectors are all non-zero vectors $(x, 0)$ and $(0, y)$, corresponding to eigenvalues 1 and -1, respectively.
3. The eigenvectors are all non-zero vectors (x, x) and $(-x, x)$, corresponding to eigenvalues 1 and -1, respectively.

Example in $\mathbb{R}^2$

Consider linear operators T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

They correspond to orthogonal projections of $\mathbb{R}^2$ onto x-axis and y-axis, resp.



1. The eigenvectors are all non-zero vectors $(x, 0)$ and $(0, y)$, corresponding to eigenvalues 1 and 0, respectively.
2. The eigenvectors are all non-zero vectors $(x, 0)$ and $(0, y)$, corresponding to eigenvalues 0 and 1, respectively.

Example in $\mathbb{R}^2$

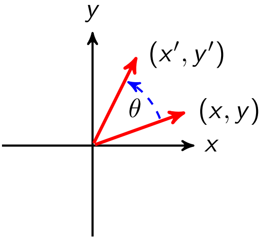
Consider the linear operator T_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The corresponding linear map T_A satisfies

$$T_A(x,y) = (x',y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

This corresponds to the rotation of $\mathbb{R}^2$ by angle θ counterclock-wise.



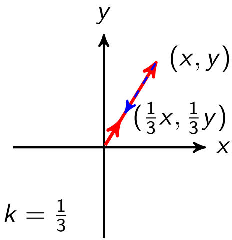
1. This linear map has no eigenvectors for any $0 < \theta < 180^\circ$.

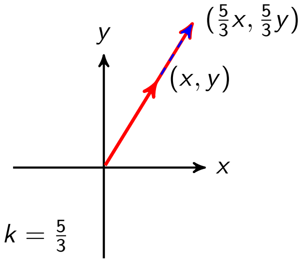
Example in $\mathbb{R}^2$

Consider linear operators T_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

This is contraction (if $k < 1$) or dilation (if $k > 1$) of $\mathbb{R}^2$.





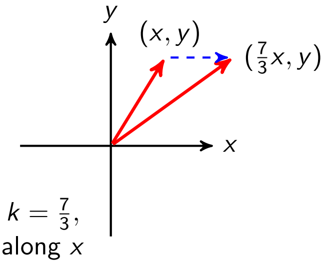
1. The eigenvectors are all non-zero vectors, corresponding to eigenvalue k.

Example in $\mathbb{R}^2$

Consider linear operators T_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

They correspond to compressions (if $k < 1$) and expansions (if $k > 1$) of $\mathbb{R}^2$ along x-axis.



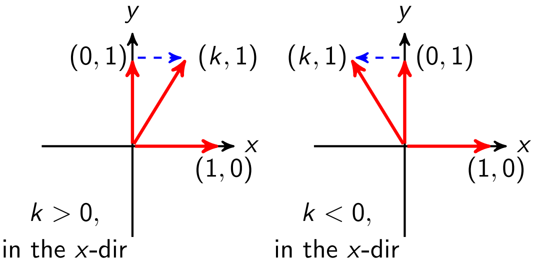
1. The eigenvectors are all non-zero vectors $(x, 0)$ and $(0, y)$, corresponding to eigenvalues k and 1, respectively.

Example in $\mathbb{R}^2$

Consider the transformation T_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

The transformation T_A satisfies $T_A(x, y) = (x + ky, y)$. For $k \neq 0$, it corresponds to shear of $\mathbb{R}^2$ in the x-direction with factor k.



1. The eigenvectors are all non-$\mathbf{0}$ vectors $(x, 0)$, corresponding to eigenvalue 1.

Characteristic equation of a matrix

Theorem

If A is an $n \times n$ matrix then λ is an eigenvalue of A iff it satisfies $det(\lambda I - A) = 0$.

The equation $det(\lambda I - A) = 0$ is called the characteristic equation of A.

Proof.

By definition, λ is an eigenvalue of A iff $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. We have

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad A\mathbf{x} = \lambda I\mathbf{x} \quad \Leftrightarrow \quad (\lambda I - A)\mathbf{x} = \mathbf{0}.$$

By theorem about invertible matrices, the last equation has a solution $\mathbf{x} \neq \mathbf{0}$ iff $det(\lambda I - A) = 0$. $\square$

Examples

Example: find eigenvalues of the matrix $A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$. We have

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 1 \\ -10 & \lambda + 9 \end{vmatrix} = (\lambda - 2) \cdot (\lambda + 9) - 1 \cdot (-10) = \lambda^2 + 7\lambda - 8.$$

So, the characteristic equation of A is $\lambda^2 + 7\lambda - 8 = 0$.

Its solutions $\lambda_1 = 1$ and $\lambda_2 = -8$ are the eigenvalues of A.

Example: find eigenvalues of the matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have

$$det(\lambda I - B) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1.$$

The characteristic equation of B is $\lambda^2 + 1 = 0$, so B has no (real) eigenvalues.

Characteristic polynomial of a matrix

- In general, the expression $det(\lambda I - A)$ is a polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n.$$

  where n is the order of A. It is called the characteristic polynomial of A.

- Solving the equation $p(\lambda) = 0$ is difficult in general — there is no closed formula or exact algorithm.
- There are numerical algorithms for computing eigenvalues approximately. (See e.g. Chapter "Numerical Methods" in the textbook).
- If all coefficients of $p(\lambda)$ are integers and the equation $p(\lambda) = 0$ has an integer solution $\lambda = k$ then $k|c_n$. This can be used to find some eigenvalues.

Example

Example: find eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$. We have

$$det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0.$$

If A has an integer eigenvalue then it is a divisor of -4, i.e., $\pm 1, \pm 2, \pm 4$. Checking these numbers in turn, we find that $\lambda = 4$ is a solution.

Divide $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ by $\lambda - 4$ to get

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = (\lambda - 4)(\lambda^2 - 4\lambda + 1).$$

Solving the equation $\lambda^2 - 4\lambda + 1 = 0$, we get that the eigenvalues of A are

$$\lambda_1 = 4, \lambda_2 = 2 + \sqrt{3}, \text{ and } \lambda_3 = 2 - \sqrt{3}.$$

Eigenspaces and their bases

- Let λ_0 be an eigenvalue of A and consider the equation $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$.
- The null space of $\lambda_0 I - A$ is called the eigenspace of A corresponding to λ_0.
- The non-$\mathbf{0}$ vectors in this space are the eigenvectors of A corresponding to λ_0.
- To find a basis in this subspace, use the algorithm for finding a basis in the null space of a matrix.

Find (a basis of) the eigenspace of $A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$ corresponding to $\lambda = -8$.

Solution. Form the equation $(-8I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -10 & 1 \\ -10 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} -10x_1 + x_2 & = & 0 \\ -10x_1 + x_2 & = & 0 \end{array}$$

The subspace consists of all vectors of the form $(x, 10x)$. One basis is $\{(1, 10)\}$.

Exercise: Find the eigenspace of A corresponding to eigenvalue $\lambda = 1$.

Multiplicities of an eigenvalue

Let λ_0 be an eigenvalue of a matrix A.

- The algebraic multiplicity of λ_0 is the power k with which $(\lambda - \lambda_0)$ appears as a factor of $det(\lambda I - A)$ - the characteristic polynomial of A.
  - E.g. if $det(\lambda I - A) = (\lambda - 2)^3 \cdot (\lambda + 5)^2 \cdots$, then it's 3 for 2 and 2 for -5
- The geometric multiplicity of λ_0 is the dimension of the eigenspace corresponding to λ_0.

Theorem

Let A be any square matrix. For every eigenvalue of A, its algebraic multiplicity is greater than or equal to its geometric multiplicity. (Proof omitted)

Exercise: Find an example of A and its eigenvalue where the inequality in the theorem is strict.

What we learnt today

- Eigenvalues and eigenvectors of matrices
- Examples in $\mathbb{R}^2$
- Characteristic equation of a matrix – how to find eigenvalues
- Eigenspaces and how to find their bases

Next time:

- Diagonalisation of matrices.