Section A Linear Algebra
(Prof. Andrei Krokhin)

Question 1

(a) Recall that, for an $m \times n$ matrix A, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map such that $T_A(\mathbf{x}) = A\mathbf{x}$. For the following matrices A, decide whether there exist distinct non-zero vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $T_A(\mathbf{x}_1) = T_A(\mathbf{x}_2)$.

i.
$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 2 \\ 1 & 0 & 3 & 8 & 5 \end{pmatrix}$$

ii.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 5 \\ 7 & 8 & 7 \end{pmatrix}$$

Justify your answers. **[5 Marks]**

(b) Show that the polynomials $\mathbf{p}_1 = 1 + x - 3x^2$, $\mathbf{p}_2 = 3 - x + 12x^2$, $\mathbf{p}_3 = x + 3x^2$ form a basis of the vector space P_2 of polynomials of degree at most 2 and find the coordinates of $1 + x$ in this basis. Show your working. **[6 Marks]**

(c) Find all real numbers x such that the vectors

$$(x, 1, 1, 1), (1, x, 1, 1), (1, 1, x, 1), (1, 1, 1, x)$$

do <u>not</u> form a basis in $\mathbb{R}^4$. For each of the values that you find, determine the dimension of the subspace of $\mathbb{R}^4$ that they span. Show your working. **[8 Marks]**

(d) Let U and W be two 2-dimensional subspaces in a 5-dimensional vector space V. Let $U + W$ denote the set of all vectors in V of the form $\mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Prove that $U + W$ is a subspace of V. Describe all dimensions that this subspace can possibly have. **[6 Marks]**

Question 2

(a) Find an orthonormal basis of the vector space P_3 of polynomials of degree at most 3

i. with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^2 f(x)g(x)\,dx$$

[7 Marks]

ii. with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1).$$

[7 Marks]

Show your working.

(b) Let A and B be $n \times n$ matrices such that $AB = 0$ is the zero matrix. Prove that if 0 is not an eigenvalue of B then every eigenvector of B is also an eigenvector of A. **[5 Marks]**

(c) Consider the set W of all 2×2 matrices A such that both $(1, 1)$ and $(1, -1)$ is an eigenvector of A. Prove that W is a subspace of the space of all 2×2 matrices and find the dimension of W. **[6 Marks]**

End of Section A

Section B Calculus
(Prof. Magnus Bordewich)

Question 3

(a) Give either a proof or a counterexample to each of the following assertions:

i. If $\{n^2 u_n\} \to 0$ as $n \to \infty$ then $\sum_{n=1}^{\infty} u_n$ converges.

ii. If $\{n u_n\} \to 0$ as $n \to \infty$ then $\sum_{n=1}^{\infty} u_n$ converges.

iii. If $\sum_{n=1}^{\infty} u_n$ converges then $\sum_{n=1}^{\infty} u_n^2$ converges.

iv. If $\sum_{n=1}^{\infty} u_n$ converges absolutely then $\sum_{n=1}^{\infty} u_n^2$ converges.

v. If $\sum_{n=1}^{\infty} u_n$ converges absolutely then $|u_n| < 1/n$ for all sufficiently large n. **[14 Marks]**

(b) Find the radius of convergence in each of the following power series. Show your working.

i. $\sum_{n=1}^{\infty} n^{2020} z^n$.

ii. $\sum_{n=1}^{\infty} \frac{z^n}{2^n n^4}$.

iii. $\sum_{n=1}^{\infty} \frac{(nz)^n}{n!}$. **[9 Marks]**

Question 4

Let $e(x_1, x_2) = x_1^3 - 2x_2 x_1 + \min\{x_1, x_2\}$.

(a) Draw a computation graph for e, identifying intermediate variables. List the intermediate variables and the operation by which each is obtained from its inputs. **[5 Marks]**

(b) Compute the partial derivatives of each intermediate variable with respect to its inputs. **[5 Marks]**

(c) Use a forward pass of your computation graph to compute $e(5, 1)$. Show your working. **[3 Marks]**

(d) Use forward mode Automatic Differentiation (AD) to compute the directional derivative of e at point $(5, 1)$ in the direction $(1, 2)$. Show your working. **[7 Marks]**

(e) Use reverse mode AD to compute a vector pointing in the direction of greatest increase in e from point $(5, 1)$. Show your working. **[7 Marks]**

END OF PAPER