Maths for Computer Science
Calculus

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Sequences and limits

Sequences

Sequences of events are common in the world around us:

1. Your birthdays occurring each year
2. The sequence of events leading up to the first world war
3. The barcode of items going through a till…

What they have in common is an order of a set of things.
Mathematically we define a sequence to be exactly this:

A sequence is the ordered values of some function $\boldsymbol{f}: \mathbb{N} \mapsto \boldsymbol{S}$ given by

$$\boldsymbol{f(0), f(1), f(2), f(3), \ldots}$$

E.g. The sequence $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$
Is given by $f(x) = \frac{1}{2^x}$.
It could also be written $\left\{\frac{1}{2^n}\right\}$, or $u_0, u_1, \ldots,$ where $u_n = \frac{1}{2^n}$.

A subsequence can be written $\left\{\frac{1}{2^n}\right\}_{n=3}^{5} = \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}$

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Sequences

A sequence $\{u_n\}$ may be:

• Monotonic	• Either
• Increasing, or	• $u_{i+1} \geq u_i$ for all i.
• Decreasing	• $u_{i+1} \leq u_i$ for all i.
• Strictly monotonic	• $u_{i+1} > u_i$ for all i or $u_{i+1} < u_i$ for all i.
• Bounded above	• $u_i \leq M$ for some $M \in \mathbb{R}$.
• Bounded below	• $u_i \geq m$ for some $m \in \mathbb{R}$.
• Bounded: bounded above and below	• $m \leq u_i \leq M$ for some $\mathrm{m}, M \in \mathbb{R}$.

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Sequences: examples

1. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is a bounded, strictly monotonic, decreasing sequence. Upper bound is 1, and is attained, the lower bound is 0 and is never attained.
2. $\{(-2)^n\}$ is an oscillating, unbounded sequence.
3. $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ is an oscillating, bounded sequence. Lower bound -1 is attained at $n = 1$, upper bound $1/2$ is attained at $n = 2$.
4. $\left\{1 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ is not monotonic, but values appear to be clustering closer and closer to 1.

Limits

A point $\boldsymbol{u^*}$ is the limit of sequence $\{\boldsymbol{u_n}\}$ if for every $\boldsymbol{\epsilon > 0}$, there is a number $\boldsymbol{N_\epsilon}$ such that for all $\boldsymbol{n > N_\epsilon}$, $\boldsymbol{|u_n - u^*| < \epsilon}$.

This is written $\lim_{n\to\infty} u_n = u^*$.

Examples:

1. The limit of $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is 0, or $\lim_{n\to\infty} \frac{1}{n} = 0$.

Pick any small $\epsilon > 0$, e.g. $\epsilon = 0.01$. Then for any $n > 100$ we have $\frac{1}{n} < 0.01$, so we could use $N_{0.01} = 100$.

But we need to be able to do this for every $\epsilon > 0$.

In this case it is OK: take $N_\epsilon = \frac{1}{\epsilon}$.

Limits

A point u^* is the limit of sequence $\{u_n\}$ if for every $\epsilon > 0$, there is a number N_ϵ such that for all $n > N_\epsilon$, $|u_n - u^*| < \epsilon$.

Examples:

1. What is the limit of $\left\{\frac{(-1)^n}{n^{2+(-1)^n}}\right\}_{n=1}^{\infty}$?

$\frac{-1}{1}, \frac{1}{2^3}, \frac{-1}{3}, \frac{1}{4^3}, \dots$

It doesn't matter that this is oscillating or that it is not monotonic, if we take $N_\epsilon = \frac{1}{\epsilon}$, then $|u_n - 0| < \epsilon$ for all $n > N_\epsilon$ still. So the limit is still 0.

Limit examples

What is $\lim_{n\to\infty}\left(\frac{5^{n+1}+7^{n+1}}{5^n-7^n}\right)$?

The general term is $\left(\frac{5^{n+1}+7^{n+1}}{5^n-7^n}\right) = \left(\frac{5\left(\frac{5}{7}\right)^n+7}{\left(\frac{5}{7}\right)^n-1}\right)$

Since $\frac{5}{7} < 1$, as n gets large $\left(\frac{5}{7}\right)^n$ goes to 0, so the bracket above tends to $\frac{7}{-1} = -7$.

Hence $\lim_{n\to\infty}\left(\frac{5^{n+1}+7^{n+1}}{5^n-7^n}\right) = -7$.

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Limit examples

What is $\lim_{n\to\infty}\left(\frac{1^2+2^2+\cdots+n^2}{n^2}\right)$?

The general term u_n is $\left(\frac{1^2+2^2+\cdots+n^2}{n^2}\right) = \left(\frac{\frac{n(n+1)(2n+1)}{6}}{n^2}\right) = \frac{n}{3}+\frac{1}{2}+\frac{1}{6n}$

As n increases without bound, so will u_n. The sequence diverges, and we write

$$\lim_{n\to\infty}\left(\frac{1^2+2^2+\cdots+n^2}{n^2}\right) \to \infty.$$

Note: not =, as infinity is not a number.

Limits: arithmetic

Let $\{u_n\}$ and $\{v_n\}$ be sequences such that $\lim_{n\to\infty} u_n = L$ and $\lim_{n\to\infty} v_n = M$.

Then

- $\{u_n + v_n\}$ is a sequence such that $\lim_{n\to\infty}(u_n + v_n) = L + M$.
- $\{u_n v_n\}$ is a sequence such that $\lim_{n\to\infty}(u_n v_n) = LM$.
- $\left\{\frac{u_n}{v_n}\right\}$ is a sequence such that, provided $M \neq 0$, $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{L}{M}$.

Indeterminate form

A limit of the form $\{u_n v_n\}$ or $\left\{\frac{u_n}{v_n}\right\}$ where the above does not apply.

I.e. if $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} v_n \to \infty$, we cannot say if $\lim_{n\to\infty}(u_n v_n)$ exists.

Or if $L = M = 0$, we cannot say if $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ exists.

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Fundamental Theorem for Sequences

Theorem: Every increasing sequence that is bounded above tends to a limit. Conversely, every decreasing sequence that is bounded below tends to a limit.

Proof: Let $\{u_n\}$ be an increasing sequence that is bounded above. Then there must be a least upper bound L such that L is an upper bound and no number less than L is an upper bound.

Since L is the least upper bound, if we take any smaller number $L - \epsilon$, then there is some u_N such that $u_N > L - \epsilon$. But since the sequence is increasing, for all $n > N$ we have $L - \epsilon < u_n \leq L$, where the second inequality is because L is an upper bound. Therefore L is the limit of $\{u_n\}$.

The proof for a decreasing sequence is similar.

Algorithmic consequences: roots

Consider the sequence defined by:

$$u_n = \frac{1}{2}\left(u_{n-1} + \frac{a}{u_{n-1}}\right)$$

for some positive number a and $u_0 > 0$.

Let $u_i = k\sqrt{a}$ for some $k > 0$ then

$$u_{i+1} - \sqrt{a} = \frac{1}{2}\left(k\sqrt{a} + \frac{1}{k}\sqrt{a}\right) - \sqrt{a} = \sqrt{a}\left(\frac{k}{2} + \frac{1}{2k} - 1\right)$$

$$= \frac{\sqrt{a}}{2k}(k^2 + 1 - 2k) = \frac{\sqrt{a}}{2k}(k-1)^2 > 0.$$

So $u_i > \sqrt{a}$ for all $i \geq 1$.

Observe that

$$u_i - u_{i+1} = u_i - \frac{1}{2}\left(u_i + \frac{a}{u_i}\right) = \frac{1}{2u_i}\left(u_i^2 - a\right) > 0,$$

so the sequence is decreasing for all $i > 1$.

Hence $\{u_n\}_1^\infty$ is decreasing and bounded below, therefore converges to a limit.

Algorithmic consequences: roots

Consider the sequence defined by:

$$u_n = \frac{1}{2}\left(u_{n-1} + \frac{a}{u_{n-1}}\right)$$

for some positive number a and $u_0 > 0$.

Hence $\{u_n\}_1^\infty$ is decreasing and bounded below, therefore converges to a limit L.

L must satisfy $L = \frac{1}{2}\left(L + \frac{a}{L}\right)$, i.e. $L^2 = \frac{L^2}{2} + \frac{a}{2}$ whence $L^2 = a$.

So we can **algorithmically compute** a square root for a using the recurrence relation above and the process will converge.

Euler's number e

Euler's number e and the related exponential function $f(x) = e^x$ (sometimes written $\exp(x)$) have great significance in mathematics and calculus.

We will define e to be the limit: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

How do we even know the limit exists?

Consider u_n. Expanding out the bracket $u_n = 1 + n.\frac{1}{n} + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \cdots + \binom{n}{n}\left(\frac{1}{n}\right)^n$

$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$

$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \ldots + \frac{1}{2^{n-1}} < 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 3.$

where the penultimate inequality comes from the geometric progression formula.

The sequence is increasing (extra term in u_n and coefficients increase), and bounded above by 3, hence converges to a limit we call e.

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