Mathematics for Computer Science
Linear Algebra

Lecture 9: Linear independence, basis and dimension

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Reminder from the previous lecture

- A vector space is a set V equipped with operations of "addition" and "multiplication by scalars" satisfying certain axioms
  - Examples: $\mathbb{R}^n$ (n-tuples of reals), $\mathbb{M}_{mn}$ (matrices of size $m \times n$).
- A subset W of V is called a subspace of V if W is closed under the operations of V, i.e. if $\mathbf{u}, \mathbf{v} \in W$ and $k \in \mathbb{R}$ then $\mathbf{u} + \mathbf{v} \in W$ and $k\mathbf{u} \in W$.
- A vector $\mathbf{w} \in V$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ if $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$ for some scalars $k_1, \ldots, k_r$.
- Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a non-empty subset of a vector space V. The set $span(S)$ of all linear combinations of the vectors in S is a subspace of V.
- For any $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, $span(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbb{R}^n$ iff $det([\mathbf{v}_1|\ldots|\mathbf{v}_n]) \neq 0$.

Contents for today's lecture

- Linear (in)dependence in vector spaces
- Basis of a vector space
- Dimension of a vector space

Linear (in)dependence

Definition

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are called linearly independent if

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0} \;\Rightarrow\; k_1 = k_2 = \ldots = k_r = 0.$$

Otherwise, they are linearly dependent.

Example: Standard unit vectors in $\mathbb{R}^n$ are linearly independent. Indeed, if $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \ldots + k_n\mathbf{e}_n = (k_1, k_2, \ldots, k_n) = \mathbf{0}$ then $k_1 = k_2 = \ldots = k_r = 0$.

Theorem

A set S of two or more vectors is linearly dependent iff at least one of the vectors is expressible as a linear combination of the other vectors in S.

Proof.

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. Let $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$, and $k_i \neq 0$ for some i.
Let k_s be the first non-zero coefficient. Then $\mathbf{v}_s = -\frac{k_{s+1}}{k_s}\mathbf{v}_{s+1} - \ldots - \frac{k_r}{k_s}\mathbf{v}_r$.
The other direction is equally easy and left as an exercise. $\square$

Determining linear (in)dependence in $\mathbb{R}^n$

How do we determine whether a given set of vectors in $\mathbb{R}^n$ is linearly independent?

We show how to do this on an example: take vectors $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$, and $\mathbf{v}_3 = (3, 2, 1)$ in $\mathbb{R}^3$.

Equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ can be written as a homogenous linear system:

$$\begin{array}{rrrl} k_1 & +5k_2 & +3k_3 & = 0 \\ -2k_1 & +6k_2 & +2k_3 & = 0 \\ 3k_1 & -k_2 & +k_3 & = 0 \end{array}$$

Then the question is: does this linear system have a <u>non-trivial</u> solution?
Recall: non-trivial solution exists $\Leftrightarrow$ the vectors are linearly dependent.

- Algorithm: given $\mathbf{v}_1, \ldots, \mathbf{v}_r$, transform $A = [\mathbf{v}_1 | \ldots | \mathbf{v}_r]$ to row echelon form. $A\mathbf{x} = \mathbf{0}$ has a non-triv solution iff number of variables $>$ number of leading 1s
- Another option for square matrices: By the theorem about invertible matrices, $A\mathbf{x} = \mathbf{0}$ has only trivial solution iff A is invertible iff $det(A) \neq 0$.

For the above vectors, $det([\mathbf{v}_1 | \mathbf{v}_1 | \mathbf{v}_3]) = 0$, so the vectors are linearly dependent.

Determining linear (in)dependence in $\mathbb{R}^n$

We now have a general method for determining linear independence in $\mathbb{R}^n$.

Moreover, n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$ are linearly independent iff the matrix

$$A = [\mathbf{v}_1 | \mathbf{v}_2 | \ldots | \mathbf{v}_n]$$

whose columns are our vectors is non-singular, i.e. iff $det(A) \neq 0$.

What if we have more vectors than n?

Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a subset of $\mathbb{R}^n$. If $r > n$ then S is linearly dependent.

Proof.

Assume that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$ and write this as a linear system. This is a homogeneous linear system with more variables (r) than equations (n). As proved in lecture 2, it has a non-trivial solution, so S is linearly dependent. $\square$

Basis

Definition

If V is a vector space and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a set of vectors in V then B is a basis for V if (1) B is linearly independent, and (2) B spans V.

- The standard unit vectors form a basis for $\mathbb{R}^n$, called the standard basis.
- Generally, we know that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis in $\mathbb{R}^n$ iff $det([\mathbf{v}_1| \ldots |\mathbf{v}_n]) \neq 0$.
- The $m \times n$ matrices M_{ij} whose entries are all 0 except $a_{ij} = 1$ form the standard basis for the space $\mathbb{M}_{mn}$ of all $m \times n$ matrices.

  Consider the case $m = n = 2$ (other cases are similar). Then

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  - Span: It is clear that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = aM_{11} + bM_{12} + cM_{21} + dM_{22}$.
  - Linear independence: If $aM_{11} + bM_{12} + cM_{21} + dM_{22}$ is the zero matrix then $a = b = c = d = 0$.

Basis representation is unique

Theorem

If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V then each vector $\mathbf{v} \in V$ can be expressed as $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$ in exactly one way.

Proof.

B spans V, hence each vector can be represented as above in at least one way. Assume some vector $\mathbf{v}$ has two different representations:

$$\begin{aligned} \mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n \\ \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n. \end{aligned}$$

Subtracting one from the other, one gets

$$\mathbf{0} = (k_1 - c_1)\mathbf{v}_1 + (k_2 - c_2)\mathbf{v}_2 + \ldots + (k_n - c_n)\mathbf{v}_n.$$

Since the two representations of $\mathbf{v}$ are different, we have $k_i \neq c_i$ for some i. Then the last equality contradicts the fact that $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent. $\square$

Coordinates

Definition

If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V then the coordinates of a vector $\mathbf{v} \in V$ relative to the basis B are the (unique) numbers $k_1, k_2, \ldots, k_n$ such that $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$.
The vector $(\mathbf{v})_B = (k_1, k_2, \ldots, k_n) \in \mathbb{R}^n$ is the coordinate vector of $\mathbf{v}$ relative to B.

Example: If $V = \mathbb{R}^n$ and E is the standard basis then $\mathbf{v}$ and $(\mathbf{v})_E$ are the same.

For any B as above, $\mathbf{v} \leftrightarrow (\mathbf{v})_B$ is a one-to-one correspondence between V and $\mathbb{R}^n$.

How do we find the coordinates of a given vector relative to a given basis?

Example: Let $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, $\mathbf{v}_3 = (3, 3, 4)$. As $det([\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]) = -1$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis in $\mathbb{R}^3$. Find the coordinates of $\mathbf{v} = (5, -1, 9)$ in this basis.

We need to find numbers k_1, k_2, k_3 such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{v}$.

This equation can be re-written as a linear system in the usual way. Solving the system, get $k_1 = 1, k_2 = -1, k_3 = 2$, these are the coordinates of $\mathbf{v}$ in this basis.

Dimension

A vector space V is finite-dimensional if it can be spanned by a finite set of vectors. Otherwise, V is infinite-dimensional.

Theorem

Let V be a finite-dimensional vector space and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be any basis in V.

1. *Any subset of V with more than n vectors is linearly dependent.*
2. *Any subset of V with fewer than n vectors does not span V.*

Corollary

All bases of a finite-dimensional vector space have the same number of vectors.

Definition

The dimension of a finite-dimensional vector space V, denoted by $dim(V)$, is the number of vectors in any of its bases. By convention, $dim(\{\mathbf{0}\}) = 0$.

Examples:

- $dim(\mathbb{R}^n) = n$, the standard basis has n vectors.
- $dim(\mathbb{M}_{mn}) = mn$, the standard basis has mn vectors.

Basis and dimension of a solution set of a linear system

For any $m \times n$ matrix A, the solution set of the linear system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$. How do we find a basis for and the dimension of this subspace?

We'll show the general method on an example:

$$\begin{array}{rrrrrrr}
x_1 & +3x_2 & -2x_3 & & +2x_5 & & = 0 \\
2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = 0 \\
 & & 5x_3 & +10x_4 & & +15x_6 & = 0 \\
2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = 0
\end{array}$$

We can find the general solution (e.g. by applying Gauss-Jordan elimination):

$$x_2, x_4, x_5 \text{ are free} \quad \text{and} \quad x_1 = -3x_2 - 4x_4 - 2x_5,\ x_3 = -2x_4,\ x_6 = 0.$$

To find a basis, draw a table, with as many vectors as the free variables: put the identity matrix for the free variables and find the rest from the general solution.

	x_1	x_2	x_3	x_4	x_5	x_6
$\mathbf{v}_1$	-3	1	0	0	0	0
$\mathbf{v}_2$	-4	0	-2	1	0	0
$\mathbf{v}_3$	-2	0	0	0	1	0

Basis and dimension of a solution set of a linear system

	x_1	x_2	x_3	x_4	x_5	x_6
$\mathbf{v}_1$	-3	1	0	0	0	0
$\mathbf{v}_2$	-4	0	-2	1	0	0
$\mathbf{v}_3$	-2	0	0	0	1	0

We claim that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of the solution space.

- Why is this set linearly independent? For any k_1, k_2, k_3, we have

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = (*, k_1, *, k_2, k_3, *)$$

  Hence, if $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ then $k_1 = k_2 = k_3 = 0$.
- Why does it span all solutions?
  - All solutions to the system are obtained by arbitrarily choosing values for the free variables (i.e. k_1, k_2, k_3) and computing the rest from the general solution.
  - For any choice of k_1, k_2, k_3, such a vector is in $span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Dimension of the solution set = the number of free variables in general solution.

Plus/Minus Theorem

Theorem (Plus/Minus Theorem)

Let S be a non-empty set of vectors in a vector space V.

1. *If S is linearly independent and $\mathbf{v} \in V$ is not in $span(S)$ then $S \cup \{\mathbf{v}\}$ is also linearly independent.*
2. *If some $\mathbf{v} \in S$ can be expressed as a linear combination of other vectors in S then $span(S) = span(S \setminus \{\mathbf{v}\})$.*

Proof.

We prove only (1). Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be linearly independent and $\mathbf{v} \notin span(S)$.
Assume that $k_0\mathbf{v} + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$ for some scalars $k_0, k_1, \ldots, k_r$.
If $k_0 \neq 0$ then $\mathbf{v} \in span(S)$ which contradicts the choice of $\mathbf{v}$.
So $k_0 = 0$, and $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$. This implies $k_1 = \ldots = k_r = 0$ because S is linearly independent. Hence $S \cup \{\mathbf{v}\}$ is also linearly independent. $\square$

Corollary

Let V be an n-dimensional vector space and let S be a subset of V with exactly n vectors. If S is linearly independent or S spans V then S is a basis for V.

Dimension of a subspace

Theorem

Let W be a subspace of a finite-dimensional vector space V. Then

1. *W is finite-dimensional and $dim(W) \leq dim(V)$,*
2. *$W = V$ if and only if $dim(W) = dim(V)$.*

Proof.

- The case $W = \{\mathbf{0}\}$ is obvious, so let's assume $W \neq \{\mathbf{0}\}$.
- Take any non-0 vector in W, it obviously forms a linearly independent set S.
- By part (1) of Plus/Minus Theorem (applied to W), we can add vectors from W to S, one by one, so that S remains linearly independent, until it spans W.
- If $dim(V) = n$, S cannot contain $> n$ vectors, so the process will stop.
- The final S will be a basis for W and it cannot contain more than n vectors, so W is finite-dimensional and $dim(W) \leq dim(V)$.
- Moreover, if $dim(W) = dim(V)$ then any basis for W is a linearly independent set of n vectors in V, which forms a basis for V. (Why?)
  – By Corollary from the previous slide.

□

Change of basis

Fix two bases $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \dots, \mathbf{v}'_n\}$ in a vector space V.
For a vector $\mathbf{u} \in V$, how are the coordinate vectors $(\mathbf{u})_B$ and $(\mathbf{u})_{B'}$ related?

Let $[\mathbf{u}]_B$ denote $(\mathbf{u})_B$ written as the column, i.e. as an $n \times 1$ matrix.

Let $P_{B' \to B}$ be the matrix $[\ [\mathbf{v}'_1]_B \mid \dots \mid [\mathbf{v}'_n]_B\ ]$.

The columns of $P_{B' \to B}$ are the coordinate vectors of the new basis in the old basis.

Then the following holds (as can be checked by direct computation):

$$[\mathbf{u}]_B = P_{B' \to B} [\mathbf{u}]_{B'}.$$

What we learnt today

- Linear (in)dependence
- Basis and coordinates
- Dimension of vector space
- Basis and dimension for solution space of a linear system

Next time:

- The four fundamental spaces of a matrix