Mathematics for Computer Science
Linear Algebra

Lecture 4: LU-decomposition

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Matrix decompositions

- A matrix decomposition (or factorisation) is a representation of a matrix as a product of two or more matrices with certain special properties.
- Decomposing matrices is a thread going through much of Linear Algebra
- Decompositions are widely used in applications (in CS and elsewhere)
- Matrix decompositions that we will see in this course:
  - LU-decomposition
  - PLU-decomposition
  - QR-decomposition
  - Eigenvalue decomposition
  - Singular value decomposition

Contents for today's lecture

LU-decomposition of matrices:

- What it is and what it's good for
- When it exists
- How to find it
- Improved version

LU-decomposition: definition

Definition

An LU-decomposition (or LU-factorisation) of a square matrix A is a (product) representation $A = LU$ where L is lower triangular and U is upper triangular.

Example: one can check that

$$A = \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = LU$$

Not every matrix has an LU-decomposition.

Exercise: Show that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has no LU-decomposition.

Exercise: A matrix can have more than one LU-decomposition.

Application: an algorithm for solving linear systems

Assume that we know an LU-decomposition $A = LU$.

Consider the following algorithm for solving the linear system $A\mathbf{x} = \mathbf{b}$:

1. re-write $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$,
2. denote $U\mathbf{x} = \mathbf{y}$ and substitute it in $LU\mathbf{x} = \mathbf{b}$ to obtain $L\mathbf{y} = \mathbf{b}$,
3. solve the triangular linear system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$,
4. now we know $\mathbf{y}$ and solve the triangular linear system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

Example

Solve the following linear system $A\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

We know an LU-decomposition for A (see previous slides):

$$A = \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = LU$$

Step 1. Rewrite the system as

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Example cont'd

$$LU\mathbf{x} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \mathbf{b}$$

Step 2. Denote $U\mathbf{x} = \mathbf{y}$ and substitute into the above equation to get

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

In equations, this is

$$\begin{array}{rrrl} 2y_1 & & & = 2 \\ -3y_1 & +y_2 & & = 2 \\ 4y_1 & -3y_2 & +7y_3 & = 3 \end{array}$$

Step 3. Solve the above (lower triangular) system by *forward substitution*:

Find $y_1 = 1$ from the 1st equation, then substitute it into the 2nd equation and find $y_2 = 5$, then substitute both values into the 3rd equation and find $y_3 = 2$.

Example cont'd

Step 4. Now we know $\mathbf{y}$, solve the linear system $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

In equations, this is

$$\begin{array}{rrrl} x_1 & +3x_2 & +x_3 & = 1 \\ & x_2 & +3x_3 & = 5 \\ & & x_3 & = 2 \end{array}$$

This is an upper triangular system, can solve it by *backward substitution*:

Find $x_3 = 2$ from the 3rd equation, then substitute it into the 2nd equation and find $x_2 = -1$, then substitute both values into the 1st equation and find $x_1 = 2$.

Discussion

LU method: Reduce solving a linear system to solving two triangular systems.

This method is widely used in Scientific Computing to solve huge linear systems.

Question: How is this better then solving $A\mathbf{x} = \mathbf{b}$ by Gauss-Jordan elimination or, if A is invertible, by finding A^{-1} and computing $\mathbf{x} = A^{-1}\mathbf{b}$?

Answer: Solving triangular linear systems is easy and fast.

Next question(s): OK, but we need to know an LU-decomposition before we start. When / how / how quickly can we find one?

Answer: Let's find out ...

LU-decomposition: sufficient condition for existence

Let A be a square matrix and let U be its (non-reduced) row echelon form, obtained by Gaussian elimination. Note that U is always upper triangular.

Theorem

If A and U are as above and no row exchanges were performed while obtaining U from A, then A can be factored $A = LU$, where L is lower triangular.

We exchange rows while computing U only when we get 0 in a pivot position.

The condition "no row exchanges" means that we never get this situation.

Before proving this theorem, we need to make some observations.

Useful observations

- Consider an elementary row operation $+(a \times R_i) \to R_j$ with $i < j$, for example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 5 & 3 \end{pmatrix} \quad + (-2 \times R1) \to R3$$

- Since $i < j$, the corresponding elementary matrix E is lower triangular.
- The inverse matrix E^{-1} corresponds to $+(-a \times R_i) \to R_j$, so also lower triangular.

- The matrix E corresponding to a row operation $a \times R_i$ is diagonal
- Since $a \neq 0$, E^{-1} is also diagonal (and so also lower triangular).

LU-decomposition: proof

Let A be a square matrix and let U be its (non-reduced) row echelon form, obtained by Gaussian elimination. Note that U is always upper triangular.

Theorem

If A and U are as above and no row exchanges were performed while obtaining U from A, then A can be factored $A = LU$, where L is lower triangular.

Proof.

We know that U is obtained from A by a sequence of elementary row operations of the form $a \times R_i$ and $+(a \times R_i) \to R_j$ (with $a \neq 0$ and $i < j$). Hence, we have

$$E_k \cdots E_1 A = U,$$

where each E_i is elementary and lower triangular (see previous slide). Then

$$A = E_1^{-1} \cdots E_k^{-1} U.$$

Now set $L = E_1^{-1} \cdots E_k^{-1}$. Each E_i^{-1} is lower triangular (see previous slide), and product of lower triangular matrices is also lower triangular (see lecture 1). $\square$

LU-decomposition: how to find it

The above proof gives a simple way to find it:

1. Keep track of row operations used to compute U by Gaussian elimination
2. Let $E_1, \ldots, E_k$ be the corresponding elementary matrices (E_1 corresponding to the first row operation and E_k to the last)
3. Then the inverse (elementary) matrices $E_1^{-1}, \ldots, E_k^{-1}$ are easy to find
4. Compute $L = E_1^{-1} \cdots E_k^{-1}$, e.g. by applying the corresponding row operations (starting from E_k^{-1}) to the identity matrix I.

Example

$$A = \begin{pmatrix} 2 & -4 \\ 3 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = U$$

$$E_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1/4 \end{pmatrix}$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$$

Observation: entries in L can in fact be computed in parallel to computing U – in the same order as we create 1s and 0s in U. (This is a general rule.)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} = L$$

Permutation matrices

Q: When does the LU method fail? A: When row exchanges must be used

Q: What can be done about this? A: Permute rows/equations in advance

Definition

A permutation matrix is a square matrix P obtained from I by permuting its rows.

Example:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Facts:

- If P has size $n \times n$, then, for any $n \times m$ matrix A, the product PA is the matrix obtained from A by permuting its rows in the same way (as P from I).
- P is invertible and $P^{-1} = P^T$ (which is also a permutation matrix)

PLU-decomposition

Definition

A PLU-decomposition of a square matrix A is a representation $A = PLU$ where P is a permutation matrix, L is lower triangular and U is upper triangular.

Note: $A = PLU$ is equivalent to $P^T A = LU$ (because $P^{-1} = P^T$).

Theorem

Every square matrix has a PLU-decomposition. *(Proof omitted).*

How to use it:

- Since P^T is invertible, $A\mathbf{x} = \mathbf{b}$ has the same solutions as $P^T A\mathbf{x} = P^T\mathbf{b}$
- Compute $\mathbf{b}' = P^T\mathbf{b}$, write the above system as $LU\mathbf{x} = \mathbf{b}'$ and solve as before

Beyond our scope:

- Algorithms for finding a PLU-decomposition

What we learnt today

LU-decomposition of matrices:

- What it is
- How to use it to solve linear systems
- Conditions for existence
- How to find it
- Improved version: PLU

Next time:

- Comparison of algorithms for solving linear systems