Mathematics for Computer Science
Linear Algebra

Lecture 14: Complex vector spaces

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Reminder from the last two lectures

Let A be an $n \times n$ matrix.

- A non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of A if $A\mathbf{x} = \lambda\mathbf{x}$.
- In this case, λ is called an eigenvalue of A, and $\mathbf{x}$ is an eigenvector corresponding to λ.
- The polynomial $det(\lambda I - A)$ is called the characteristic polynomial of A and the equation $det(\lambda I - A) = 0$ the characteristic equation of A.
- The eigenvalues of A are the solutions of $det(\lambda I - A) = 0$.
  In particular, A is singular (non-invertible) iff 0 is an eigenvalue of A
- A is called diagonalisable if there is invertible P such that $P^{-1}AP$ is diagonal.

  A is diagonalisable iff it has n linearly independent eigenvectors.

Contents for today's lecture

- Complex numbers
- Complex vector spaces
- Eigenvalues of symmetric real matrices

Complex numbers: motivation

Assume that we want to analyse the "eigen"-properties of the following matrix

$$A = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}.$$

Computing its characteristic equation, we get

$$det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = \lambda^2 + 1 = 0.$$

- This equation has no real roots, so we know that A has no real eigenvalues, but this is all we can say at the moment. Can we do more?
- It would (probably) be useful to work with some number set that extends $\mathbb{R}$ and where every polynomial can be factorised into linear polynomials, i.e.

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n),$$

where the λ_i's are not necessarily distinct.

Complex numbers: reminder

A complex number is a number of the form $z = a + bi$ where $a, b \in \mathbb{R}$ and

- i is the imaginary unit: the number such that $i^2 = -1$.

Then

- $Re(z) = a$ is the real part of z and $Im(z) = b$ is the imaginary part of z
- $|z| = \sqrt{a^2 + b^2}$ is the modulus (or absolute value) of z (note that $|z| \in \mathbb{R}$)
- The number $\overline{z} = a - bi$ is the complex conjugate of z (and $z\overline{z} = |z|^2$)

The set of all complex numbers is denoted by $\mathbb{C}$.

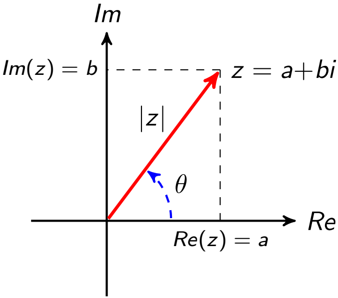
The arithmetic operations on $\mathbb{C}$ work as follows:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

It is easy to check that $\overline{\overline{z}} = z$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

Complex numbers geometrically: reminder

- Each complex number $z = a + bi$ can be viewed as a vector $(a, b) \in \mathbb{R}^2$
- Addition and multiplication by a real number are the same in $\mathbb{C}$ and in $\mathbb{R}^2$



- The angle $\theta = \arctan(b/a)$ in the diagram is called the argument of z.
- The expression $z = |z|(\cos\theta + i\sin\theta)$ is the polar form of z.
  - Example: $\sqrt{2} - \sqrt{2}i = 2(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$.

The fundamental theorem of algebra

Theorem

Each polynomial of degree $n \geq 1$ with complex coefficients has n complex roots (counting with multiplicities). That is, each such polynomail can be factored into linear polynomials,

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n),$$

where the λ_i's are not necessarily distinct.

(Proof omitted)

For example,

$$\lambda^2 + 1 = (\lambda - i)(\lambda + i) \quad \text{and} \quad \lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = (\lambda - i)^2(\lambda + i)^2.$$

All quadratic polynomials can now be factorised by using the standard formula for solving quadratic equations and the fact that, for $D < 0$, we have $\sqrt{D} = i\sqrt{|D|}$.

The vector space $\mathbb{C}^n$ and complex matrices

- Similarly to $\mathbb{R}^n$, the vector space $\mathbb{C}^n$ is defined to consist of all n-tuples $(v_1, \ldots, v_n)$, where each $z_i \in \mathbb{C}$.
- Each vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n$, where $v_i = a_i + b_i i$, can be represented as

$$\mathbf{v} = (v_1, \ldots, v_n) = (a_1 + b_1 i, \ldots, a_n + b_n i) = (a_1, \ldots, a_n) + i(b_1, \ldots, b_n) = \mathbf{a} + i\mathbf{b},$$

  where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then $\mathbf{a} = Re(\mathbf{v})$ and $\mathbf{b} = Im(\mathbf{v})$.
- Can extend the complex conjugate to $\mathbb{C}^n$: If $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ then $\bar{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$.
- Example: if $\mathbf{v} = (3 + i, -2i, 5)$ then

$$Re(\mathbf{v}) = (3, 0, 5), \quad Im(\mathbf{v}) = (1, -2, 0), \quad \bar{\mathbf{v}} = (3 - i, 2i, 5).$$

One can also consider complex matrices, i.e. matrices with complex entries.

All the above notions extend to complex matrices in a natural way.

We will call a matrix a real matrix to emphasize that all its entries are real.

Algebraic properties of the complex conjugate

The facts that $\overline{\overline{z}} = z$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ immediately imply

Theorem

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and a scalar $k \in \mathbb{C}$, the following holds:

- $\overline{\overline{\mathbf{u}}} = \mathbf{u}$
- $\overline{k\mathbf{u}} = \overline{k}\,\overline{\mathbf{u}}$
- $\overline{\mathbf{u} + \mathbf{v}} = \overline{\mathbf{u}} + \overline{\mathbf{v}}$
- $\overline{\mathbf{u} - \mathbf{v}} = \overline{\mathbf{u}} - \overline{\mathbf{v}}$

Theorem

If A is an $m \times k$ complex matrix and B is a $k \times n$ complex matrix, then

- $\overline{\overline{A}} = A$
- $\overline{(A^T)} = (\overline{A})^T$
- $\overline{AB} = \overline{A}\,\overline{B}$

Complex dot product

The complex dot product in $\mathbb{C}^n$ is defined as follows: if $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1\overline{v_1} + \ldots + u_n\overline{v_n}.$$

The Euclidean norm in $\mathbb{C}^n$ is then defined as follows:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{|v_1|^2 + \ldots + |v_n|^2}.$$

Example: let $\mathbf{u} = (1+i, i, 3-i)$ and $\mathbf{v} = (1+i, 2, 4i)$. Then

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (1+i)(1-i) + (i)(2) + (3-i)(-4i) = -2 - 10i \\
\mathbf{v} \cdot \mathbf{u} &= (1+i)(1-i) + 2(-i) + 4i(3+i) = -2 + 10i \\
||\mathbf{u}|| &= \sqrt{|1+i|^2 + |i|^2 + |3-i|^2} = \sqrt{2+1+10} = \sqrt{13}
\end{aligned}$$

Properties of complex dot product

For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, viewed as columns (i.e. $n \times 1$ matrices), we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u} \text{ and } ||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u} = \mathbf{u}^T \mathbf{u} = \mathbf{u}^T \mathbf{u}.$$

(The first product is the dot product and the other two are matrix products.)

For complex vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, this becomes

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \overline{\mathbf{v}} = \overline{\mathbf{v}}^T \mathbf{u} \text{ and } ||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u} = \mathbf{u}^T \overline{\mathbf{u}} = \overline{\mathbf{u}}^T \mathbf{u}.$$

Theorem

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ and a scalar $k \in \mathbb{C}$, the following holds:

- $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$ *and* $\mathbf{u} \cdot (k\mathbf{v}) = \overline{k}(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{v} \cdot \mathbf{v} \geq 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *iff* $\mathbf{v} = \mathbf{0}$

Complex eigenvalues and eigenvectors

If A is an $n \times n$ matrix with complex entries.

As in the real case, $\lambda \in \mathbb{C}$ is an eigenvalue of A if $A\mathbf{x} = \lambda\mathbf{x}$ for a non-zero $\mathbf{x} \in \mathbb{C}^n$. Then $\mathbf{x}$ is a complex eigenvector corresponding to λ.

As in the real case,

- the eigenvalues of A are the complex roots of $det(\lambda I - A) = 0$.
- the eigenspace of A corrresponding to an eigenvalue λ_0 is the solution space of the linear system $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$ (considered over $\mathbb{C}$).

Theorem

If λ is an eigenvalue of a real $n \times n$ matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{\mathbf{x}}$ is a corresponding eigenvector.

Proof.

Since A is real, i.e. $\overline{A} = A$, we have $A\overline{\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$. (And $\overline{\mathbf{x}} \neq \mathbf{0}$.) $\square$

Eigenvalues of real symmetric matrices

Theorem

If A is a real symmetric matrix then all (complex) eigenvalues of A are real.

Proof.

- Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $\mathbf{x} \in \mathbb{C}^n$ a corresponding eigenvector.
- Take the complex conjugate of both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$.
- We get $\overline{A}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$, and, since $A = \overline{A}$ (A is real), it follows that $A\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$.
- Then, using $A = A^T$, we compute the number $\overline{\mathbf{x}}^T A\mathbf{x}$ in two different ways:

$$\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T (A\mathbf{x}) = \overline{\mathbf{x}}^T (\lambda\mathbf{x}) = \lambda(\overline{\mathbf{x}}^T\mathbf{x}) = \lambda(\mathbf{x}\cdot\mathbf{x}) = \lambda||\mathbf{x}||^2,$$
$$\overline{\mathbf{x}}^T A\mathbf{x} = (A\overline{\mathbf{x}})^T\mathbf{x} = (\overline{\lambda}\,\overline{\mathbf{x}})^T\mathbf{x} = \overline{\lambda}(\overline{\mathbf{x}}^T\mathbf{x}) = \overline{\lambda}(\mathbf{x}\cdot\mathbf{x}) = \overline{\lambda}||\mathbf{x}||^2.$$

- Since $\mathbf{x} \neq \mathbf{0}$, have $||\mathbf{x}|| \neq 0$. So $\lambda(\overline{\mathbf{x}}^T\mathbf{x}) = \overline{\lambda}(\overline{\mathbf{x}}^T\mathbf{x})$ implies $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

□

Real 2×2 matrices with complex eigenvalues

Theorem

The complex eigenvalues of the real matrix $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *are* $\lambda = a \pm bi$.

If a, b *are not both zero, then* C *can be factored as*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ *is the argument of* $\lambda = a + bi$.

Geometrically, the operator T_C is equal to rotation by θ followed by scaling by $|\lambda|$.

Theorem

Let A *be a real* 2×2 *matrix with complex eigenvalues* $\lambda = a \pm bi$, *where* $b \neq 0$.

Then A *is similar to* $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

What we learnt today

- Complex numbers
- Complex vector spaces
- All (complex) eigenvalues of real symmetric matrices are real
- Real 2×2 matrices with complex eigenvalues

Next time:

- Inner product spaces - generalising the dot product