Mathematics for Computer Science
Linear Algebra

Lecture 3: Elementary matrices

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Contents for today's lecture

- Elementary matrices and their properties
- A characterisation of invertible matrices
- An algorithm for finding A^{-1};
- Examples and exercises
- Lots of proofs

Reminder: Elementary row operations

The three elementary row operations on a matrix are:

- Multiply a row by a non-zero constant c;
- Interchange two rows;
- Add a constant c times one row R_1 to another row R_2.

Obvious fact: If B is obtained from A by using an elementary row operation then A can be obtained from B by using the inverse elementary row operation:

- Multiply the same row by a non-zero constant $1/c$;
- Interchange the same two rows;
- Add $-c$ times row R_1 to row R_2.

Elementary matrices

An $n \times n$ matrix is called an elementary matrix if it is obtained from the identity matrix I_n by performing a single elementary row operation.

Examples of elementary matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

This matrix is not elementary:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Elementary row operations via elementary matrices

Lemma

Let E be an elementary matrix obtained from I_m by performing some elementary row operation. Then, for any $m \times n$ matrix A, the following matrices are equal:

- *the matrix obtained from A by performing the same row operation, and*
- *the product matrix EA.*

Thus, preforming an elementary row operation has the same effect as multiplying by the corresponding elementary matrix (from the left).

Exercise: Prove this lemma (by considering the three operations in turn).

Examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ -6 & -8 & -10 \\ 6 & 7 & 8 \end{pmatrix} \quad -2 \times R_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 6 & 7 & 8 \\ 3 & 4 & 5 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 5 & 4 \end{pmatrix} \quad +(-2 \times R_1) \to R_3$$

Elementary matrices are invertible

Lemma

Every elementary matrix E is invertible, and the inverse E^{-1} is also elementary.

Proof.

We know that (multiplying by) E corresponds to some elementary row operation. It is the same row operation as used to obtain E from I. This row operation has an inverse elementary row operation.

If E' is the matrix corresponding to this inverse elementary row operation then $I = E'EI = E'E$ and similarly $I = EE'I = EE'$. Hence $E' = E^{-1}$.

$\square$

Theorem about invertible matrices

Theorem (Theorem about invertible matrices, Version 1)

If A is an $n \times n$ matrix then TFAE (The Following Are Equivalent, i.e. all true or all false):

1. *A is invertible.*
2. *The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *The reduced row echelon form of A is I_n.*
4. *A can be expressed as a product of elementary matrices.*

Proof: We will prove implications (1)⇒(2)⇒(3)⇒(4)⇒(1). Hence, if one of the conditions is true (for some matrix A), all of them must be true (for this A).

(1)⇒(2). Assume that A is invertible. If $A\mathbf{x} = \mathbf{0}$ then

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} = \mathbf{0}.$$

Theorem about invertible matrices

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If A is an $n \times n$ matrix then TFAE (The Following Are Equivalent, i.e. all true or all false):

1. *A is invertible.*
2. *The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *The reduced row echelon form of A is I_n.*
4. *A can be expressed as a product of elementary matrices.*

Proof: (2)$\Rightarrow$(3). The linear system $A\mathbf{x} = \mathbf{0}$ is homogenous (i.e. RHS is $\mathbf{0}$). As proved in lecture 2, if it has n variables and the reduced row echelon form of its augmented matrix $[A|\mathbf{0}]$ has r non-0 rows then the system has $n - r$ free variables.

Since our system has only one solution, it cannot have free variables, so $n = r$. Hence, each row in the reduced row echelon form of $[A|\mathbf{0}]$ has a leading 1.

The reduced row echelon form must then be $[I_n|\mathbf{0}]$, which immediately implies (3).

Theorem about invertible matrices, cont'd

Theorem (Theorem about invertible matrices)

If A is an $n \times n$ matrix then TFAE:

1. *A is invertible.*
2. *The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *The reduced row echelon form of A is I_n.*
4. *A can be expressed as a product of elementary matrices.*

Proof: (3)$\Rightarrow$(4). If I_n is obtained from A by a sequence of elementary row operations then there are elementary matrices $E_1, \ldots E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

We proved today that each E_i is invertible and each E_i^{-1} is elementary. Hence

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

Theorem about invertible matrices, cont'd

Theorem (Theorem about invertible matrices)

If A is an $n \times n$ matrix then TFAE:

1. *A is invertible.*
2. *The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *The reduced row echelon form of A is I_n.*
4. *A can be expressed as a product of elementary matrices.*

Proof: (4)$\Rightarrow$(1). We proved today that each elementary matrix is invertible. The product of invertible matrices is also invertible (see lecture 1).

$\square$

A corollary

By definition, a matrix B is an inverse of A if we have *both* $AB = I$ *and* $BA = I$.

We now apply the above theorem to show that *either* condition suffices.

Corollary

Let A be a square matrix.

1. *If B is a square matrix with $BA = I$ then $B = A^{-1}$.*
2. *If B is a square matrix with $AB = I$ then $B = A^{-1}$.*

Proof.

We prove (1) and leave (2) as an exercise.
Enough to prove that $BA = I$ implies that A is invertible because then

$$BA = I \;\Rightarrow\; BAA^{-1} = IA^{-1} \;\Rightarrow\; BI = A^{-1} \Rightarrow\; B = A^{-1}.$$

We show that $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = \mathbf{0}$, and then apply the theorem to conclude that A is invertible. Let $\mathbf{x}_0$ be any solution of $A\mathbf{x} = \mathbf{0}$. Then

$$A\mathbf{x}_0 = \mathbf{0} \;\Rightarrow\; BA\mathbf{x}_0 = B\mathbf{0} \;\Rightarrow\; I\mathbf{x}_0 = \mathbf{0} \;\Rightarrow\; \mathbf{x}_0 = \mathbf{0}.$$

$\square$

Inversion algorithm

As an application of the above theorem, we give an algorithm which

1. decides whether a given matrix A is invertible,
2. and, if so, finds the inverse A^{-1}.

If A is invertible, condition (3) of the theorem implies that $E_k \cdots E_2 E_1 A = I_n$ for some elementary matrices E_i. Multiplying by A^{-1}, get $E_k \cdots E_2 E_1 I_n = A^{-1}$.

Therefore, if a sequence of elementary row operations transforms A to I_n then the same sequence transforms I_n to A^{-1}.

Inversion algorithm:

1. Write the $n \times 2n$ matrix $[A|I_n]$.
2. Apply elementary row operations to the whole matrix to transform its left half (i.e. A) to reduced row echelon form.
3. If this form (of the left half) is not I_n then the matrix is not invertible.
4. Otherwise, the obtained matrix is $[I_n|A^{-1}]$.

Example

Find the inverse (if it exists) of the matrix $A = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{array}\right)$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

We have $A^{-1} = \left(\begin{array}{ccc} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{array}\right)$.

What we learnt today

- Elementary matrices: their properties and applications
- A characterisation of invertible matrices: 4 equivalent conditions
- An algorithm for finding A^{-1} via elementary row operations