Mathematics for Computer Science
Linear Algebra

Lecture 16: The Gram-Schmidt process and QR-decomposition

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Reminder from the last lecture

- An inner product space is a (real) vector space V equipped with an inner product — a function that associates to each pair $\mathbf{u}, \mathbf{v} \in V$ a real number denoted $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$. This function must satisfy four axioms:
    - Symmetry, additivity, homogeneity, positivity.
- The norm of a vector is defined as $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

  A vector $\mathbf{v}$ with $||\mathbf{v}|| = 1$ is called a unit vector. Each non-zero vector can be normalised (scaled to become a unit vector): $\mathbf{v} \mapsto \frac{1}{||\mathbf{v}||}\mathbf{v}$.
- Vectors $\mathbf{u}$ and $\mathbf{v}$ are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
- For a subspace W of an inner product space V, can define the orthogonal complement

$$W^{\perp} = \{\mathbf{x} \in V \mid \langle \mathbf{u}, \mathbf{x} \rangle = 0 \text{ for all } \mathbf{u} \in W\}$$

- A vector space can be equipped with different inner products — the notions of norm and orthogonality depend on the choice of inner product.

Contents for today's lecture

- Orthogonal and orthonormal bases in an inner vector space
- Constructing such bases - the Gram-Schmidt process
- QR-decomposition of matrices

Orthogonal and orthonormal sets of vectors

Definition

A set of vectors in an inner product space is called orthogonal if all pairs of distinct vectors in it are orthogonal. An orthogonal set consisting of unit vectors is called orthonormal.

Example in $\mathbb{R}^3$ (with the Euclidean inner product): Let

$$\mathbf{v}_1 = (0, 1, 0),\ \mathbf{v}_2 = (1, 0, 1),\ \mathbf{v}_3 = (1, 0, -1).$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal, since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$.

The norms of the vectors are:

$$||\mathbf{v}_1|| = 1,\ ||\mathbf{v}_2|| = \sqrt{2},\ ||\mathbf{v}_3|| = \sqrt{2}.$$

By normalising (i.e. setting $\mathbf{q}_i = \frac{1}{||\mathbf{v}_i||}\mathbf{v}_i$), we get an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$

$$\mathbf{q}_1 = (0, 1, 0),\ \mathbf{q}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}),\ \mathbf{q}_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}).$$

Orthogonal sets are linearly independent

Theorem

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of <u>non-zero</u> vectors in an inner product space then S is linearly independent.

Proof.

Assume that $k_1\mathbf{v_1} + \ldots + k_n\mathbf{v}_n = \mathbf{0}$ and prove that $k_1 = \ldots = k_n = 0$.

Pick any $\mathbf{v}_i$ and take the product of both sides of the above equation with this $\mathbf{v}_i$:

$$\langle k_1\mathbf{v_1} + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i\rangle = \langle \mathbf{0}, \mathbf{v}_i\rangle = 0$$

By using the linearity (i.e. additivity + homogeneity) of the inner product, as well as orthogonality of S (i.e. $\langle \mathbf{v}_j, \mathbf{v}_i\rangle = 0$ for $j \neq i$), we get

$$\langle k_1\mathbf{v_1} + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i\rangle = k_1\langle \mathbf{v}_1, \mathbf{v}_i\rangle + \ldots + k_n\langle \mathbf{v}_n, \mathbf{v}_i\rangle = k_i\langle \mathbf{v_i}, \mathbf{v}_i\rangle.$$

Thus, $k_i\langle \mathbf{v_i}, \mathbf{v}_i\rangle = 0$, and, since $\mathbf{v}_i$ is non-zero, it follows that $k_i = 0$. □

Orthogonal and orthonormal bases

An orthogonal (resp. orthonormal) basis in an inner product space is a basis, which is an orthogonal (resp. orthonormal) set. For example,

$\{\mathbf{v}_1 = (0,1,0), \mathbf{v}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), \mathbf{v}_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})\}$ is an orthonormal basis in $\mathbb{R}^3$

Theorem

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis in inner product space V then for any $\mathbf{u}$

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{||\mathbf{v}_n||^2}\mathbf{v}_n.$$

Moreover, if S is an orthonormal basis in V then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Proof.

If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ then, since $\langle \mathbf{v}_j, \mathbf{v}_i \rangle = 0$ for all $j \neq i$, we have that, for each i, $\langle \mathbf{u}, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i ||\mathbf{v}_i||^2$, so $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{||\mathbf{v}_i||^2}$, as required. □

Orthogonal projections

Theorem (Projection Theorem)

If W is a subspace in a finite-dimensional inner product space V then every vector $\mathbf{u} \in V$ can be uniquely expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.

If $\mathbf{u}, \mathbf{w}_1$ and $\mathbf{w}_2$ are as above then $\mathbf{w}_1$ is the orthogonal projection of $\mathbf{u}$ onto W.
Notation: $\mathbf{w}_1 = \mathrm{proj}_W \mathbf{u}$ and $\mathbf{w}_2 = \mathrm{proj}_{W^\perp} \mathbf{u}$.

How to compute orthogonal projections? Pretty much the same idea as before.

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W and $\mathbf{w}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r$, use $\langle \mathbf{u}, \mathbf{v}_i\rangle = \langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{v}_i\rangle = \langle \mathbf{w}_1, \mathbf{v}_i\rangle + \langle \mathbf{w}_2, \mathbf{v}_i\rangle = \langle \mathbf{w}_1, \mathbf{v}_i\rangle = c_i\langle \mathbf{v}_i, \mathbf{v}_i\rangle$ to find the c_i's:

$$\mathrm{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_r\rangle}{||\mathbf{v}_r||^2}\mathbf{v}_r.$$

Moreover, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis in W then

$$\mathrm{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r\rangle\mathbf{v}_r.$$

The Gram-Schmidt (orthogonalisation) process

Theorem

Every non-zero finite-dimensional inner product space V has an orthonormal basis.

- Let W be a subspace of V and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis of W.
  Consider the subspaces $W_r = span(\mathbf{u}_1, \ldots, \mathbf{u}_r)$, $r = 1, \ldots, n$, in W.
  Note that $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_n = W$.
- The Gram-Schmidt process inductively constructs orthogonal bases for the subspaces W_i, eventually constructing an orthogonal basis for $W_n = W$.

  Once we have an orthogonal basis for W, we can normalise all vectors in it.

The Gram Schmidt process:

Step 1. Let $\mathbf{v}_1 = \mathbf{u}_1$. Clearly, $\{\mathbf{v}_1\}$ is an orthogonal basis for $W_1 = span(\mathbf{u}_1)$.

Step r ($2 \leq r \leq n$). Assuming that we have an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ for W_{r-1}, add a vector $\mathbf{v}_r$ to it to get an orthogonal basis for W_r.

The Gram-Schmidt (orthogonalisation) process

Step r: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1} = \mathit{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, find a vector $\mathbf{v}_r$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ is an orthogonal basis for W_r.

Apply the Projection theorem to $\mathbf{u}_r \in W_r$ and W_{r-1} (as a subspace of W_r):

$$\mathbf{u}_r = \mathrm{proj}_{W_{r-1}} \mathbf{u}_r + \mathrm{proj}_{W_{r-1}{}^\perp} \mathbf{u}_r.$$

(Note that the orthogonal complement $W_{r-1}{}^\perp$ here is taken in W_r).

Recall that, since $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for W_{r-1}, we have

$$\mathrm{proj}_{W_{r-1}} \mathbf{u}_r = \frac{\langle \mathbf{u}_r, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}_r, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2} \mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1} \rangle}{||\mathbf{v}_{r-1}||^2} \mathbf{v}_{r-1}.$$

Set

$$\mathbf{v}_r = \mathrm{proj}_{W_{r-1}{}^\perp} \mathbf{u}_r = \mathbf{u}_r - \frac{\langle \mathbf{u}_r, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_r, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2} \mathbf{v}_2 - \ldots - \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1} \rangle}{||\mathbf{v}_{r-1}||^2} \mathbf{v}_{r-1}.$$

Since $\mathbf{v}_r \in W_{r-1}^\perp$, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ is orthogonal (and so linearly indep.)

$W_r = \mathit{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \mathbf{u}_r) = \mathit{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{u}_r) = \mathit{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r)$.

The Gram-Schmidt process: Summary

To convert a (linearly independent) set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ into an orthogonal basis for *span*(S), do the following:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$.

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1$.

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \mathrm{proj}_{W_3}\mathbf{u}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{||\mathbf{v}_3||^2}\mathbf{v}_3$

$\vdots$

(continue for n steps)

Optional step. Normalise all vectors $\mathbf{v}_i$ if an orthonormal basis is needed.

Example: Using the Gram-Schmidt process

Task: Consider $\mathbb{R}^3$ with the Euclidean inner product. Find an orthonormal basis of $W = span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (0,1,1), \mathbf{u}_3 = (0,0,1)$.

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$.

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 =$

$$(0,1,1) - \frac{2}{3}(1,1,1) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}).$$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 =$

$$(0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (0, -\frac{1}{2}, \frac{1}{2}).$$

$\{\mathbf{v}_1 = (1,1,1),\ \mathbf{v}_2 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}),\ \mathbf{v}_3 = (0, -\frac{1}{2}, \frac{1}{2})\}$ is an orthogonal basis for W.

Since $||\mathbf{v}_1|| = \sqrt{3},\ ||\mathbf{v}_2|| = \frac{\sqrt{6}}{3},\ ||\mathbf{v}_3|| = \frac{1}{\sqrt{2}}$, we have an orthonormal basis for W:

$\{\mathbf{q}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}),\ \mathbf{q}_2 = (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),\ \mathbf{q}_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

Example: Using the Gram-Schmidt process

Consider the space $C[-1,1]$. Find an orthogonal basis of $W = span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = 1, \mathbf{u}_2 = x, \mathbf{u}_3 = x^2$.

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = 1$.

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1$. We have

$$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = \int_{-1}^{1} x\,dx = 0 \quad \text{and} \quad ||\mathbf{v}_1||^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} 1\,dx = 2$$

so $\mathbf{v}_2 = \mathbf{u}_2 - 0\mathbf{v}_1 = x$.

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2$. We have

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \quad \text{and} \quad \langle \mathbf{u}_3, \mathbf{v}_2 \rangle = \int_{-1}^{1} x^3\,dx = 0.$$

Hence, $\mathbf{v}_3 = x^2 - \frac{2/3}{2}\mathbf{v}_1 - 0\mathbf{v}_2 = x^2 - \frac{1}{3}$.

So, $\{\mathbf{v}_1 = 1,\ \mathbf{v}_2 = x,\ \mathbf{v}_3 = x^2 - \frac{1}{3}\}$ is an orthogonal basis for W.

Extending an orthogonal set to an orthogonal basis

Theorem

If V is a finite-dimensional inner product space then

1. *Any orthogonal set of vectors in V can be extended to an orthogonal basis.*
2. *Any orthonormal set of vectors in V can be extended to an orthonormal basis.*

Proof.

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal set in V and $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_{k+n}\}$ some basis in V.

- Apply Gram-Schmidt to the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{k+n}\}$.
- Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal set, we will have $\mathbf{v}_i = \mathbf{u}_i$ for $1 \leq i \leq k$.
- If $\mathbf{v}_r = \mathbf{0}$ at any Step r (with $r > k$), do not add it to the output set.

  (This happens iff $\mathbf{u}_r \in span(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, i.e. if $W_{r-1} = W_r$.)

The final set will extend $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, it will be orthogonal (and hence linearly independent), and its span will be $span(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{k+n}) = V$.
For item (2), normalise all vectors in the final set. □

QR-decomposition

Let A be an $m \times n$ matrix with linearly independent columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$

Let $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ be the orthonormal set obtained by applying Gram-Schmidt to $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. How does $A = [\mathbf{u}_1 | \ldots | \mathbf{u}_n]$ relate to the matrix $Q = [\mathbf{q}_1 | \ldots | \mathbf{q}_n]$?

Since $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ is an orthonormal basis for $span(\mathbf{u}_1, \ldots, \mathbf{u}_n)$, we have

$$\begin{aligned} \mathbf{u}_1 &= \langle \mathbf{u}_1, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_1, \mathbf{q}_2 \rangle \mathbf{q}_2 + \ldots + \langle \mathbf{u}_1, \mathbf{q}_n \rangle \mathbf{q}_n \\ \mathbf{u}_2 &= \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \mathbf{q}_2 + \ldots + \langle \mathbf{u}_2, \mathbf{q}_n \rangle \mathbf{q}_n \\ &\vdots \\ \mathbf{u}_n &= \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \mathbf{q}_2 + \ldots + \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n \end{aligned}$$

or, in the matrix form,

$$A = [\mathbf{u}_1 | \ldots | \mathbf{u}_n] = [\mathbf{q}_1 | \ldots | \mathbf{q}_n] \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n \rangle & \langle \mathbf{u}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{pmatrix} = QR.$$

QR-decomposition

What can we say about the matrix R?

From Gram-Schmidt, for each $j \geq 2$, $\mathbf{q}_j$ is orthogonal to $\mathbf{u}_1, \ldots, \mathbf{u}_{j-1}$. Hence R is

$$\begin{pmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n \rangle & \langle \mathbf{u}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{pmatrix} = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{pmatrix}.$$

From Gram-Schmidt, $\langle \mathbf{u}_i, \mathbf{q}_i \rangle = \langle \mathbf{u}_i, \frac{1}{||\mathbf{v_i}||} \mathbf{v}_i \rangle = \frac{1}{||\mathbf{v_i}||} \langle \mathbf{u}_i, \mathbf{v}_i \rangle = \frac{1}{||\mathbf{v_i}||} \langle \mathbf{v}_i, \mathbf{v}_i \rangle > 0$.

Theorem (QR-decomposition)

If A is an $m \times n$ matrix with linearly independent column vectors, then it can be factored as

$$A = QR$$

where Q has orthonormal columns and R is an invertible upper triangular matrix.

For $m = n$, this theorem says that every invertible matrix has a QR-decomposition.

What we learnt today

- Orthogonal and orthonormal bases in an inner vector space
- Constructing such bases - the Gram-Schmidt process
- QR-decomposition of matrices

Next time:

- Least squares - solving inconsistent linear systems