Lecture 11: Solving Problems by SAT Solvers

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SAT Modelling

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- Key part of this is modelling.

Science fair puzzle

Isaac and Albert were excitedly describing the result of the Third Annual International Science Fair. There were three contestants, Louis, Rene, and Johannes.

- Isaac reported that Louis won the fair, while Rene came in second.
- Albert reported that Johannes won the fair, while Louis came in second.

In fact, neither Isaac nor Albert had given a correct report of the results of the science fair. Each of them had given one correct statement and one false statement.

What was the actual placing of the three contestants?

Set us try to solve this puzzle with SAT. Introduce propositional variables XY with the following meaning in mind:

$X \in \{L, R, J\}$ (denoting Louis, Rene, Johannes)

$Y \in \{1, 2, 3\}$ (denoting 1st, 2nd, 3rd)

Set us try to solve this puzzle with SAT. Introduce propositional variables XY with the following meaning in mind:

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For example,

$L2$:Louis came 2nd

$R1$:Rene came 1st

Recall

- Isaac reported that Louis won the fair, while Rene came in second.
- Albert reported that Johannes won the fair, while Louis came in second.

In fact, neither Isaac nor Albert had given a correct report of the results of the science fair. Each of them had given one correct statement and one false statement.
This gives clauses:

$$(L1 \leftrightarrow \neg R2) \text{ and } (J1 \leftrightarrow \neg L2)$$

What are the other clauses?

Everyone came in some position:

$$(L1 \vee L2 \vee L3), (R1 \vee R2 \vee R3), (J1 \vee J2 \vee J3)$$

Every person came in one position only:

$$\begin{gathered}(\neg L1 \vee \neg L2), (\neg L1 \vee \neg L2), (\neg L1 \vee \neg L3)\\(\neg R1 \vee \neg R2), (\neg R1 \vee \neg R2), (\neg R1 \vee \neg R3)\\(\neg J1 \vee \neg J2), (\neg J1 \vee \neg J2), (\neg J1 \vee \neg J3)\end{gathered}$$

Someone came in each position:

$$(L1 \vee R1 \vee J1), (L2 \vee R2 \vee J2), (L3 \vee R3 \vee J3)$$

Each position had one occupant only:

$$\begin{gathered}(\neg L1 \vee \neg R1), (\neg R1 \vee \neg J1), (\neg L1 \vee \neg J1)\\(\neg L2 \vee \neg R2), (\neg R2 \vee \neg J2), (\neg L2 \vee \neg J2)\\(\neg L3 \vee \neg R3), (\neg R3 \vee \neg J3), (\neg L3 \vee \neg J3)\end{gathered}$$

Finally, we can check whether this is consistent with $L1$; or $R2$.

(Binary) van der Waerden numbers

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- Denote the least number n with this property by $w(j, k)$.
- Determining $w(j, k)$ is of big interest in Maths.
- Let's construct a Boolean formula (or a clause-set) $\mathit{waerden}(j, k; n)$ such that $\mathit{waerden}(j, k; n)$ is sat iff there is an n-bit sequence with no j equally spaced 0s and no k equally spaced 1s.

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$$(\overline{x}_i \vee \overline{x}_{i+d} \vee \ldots \vee \overline{x}_{i+(k-1)d})$$

for all $d \geq 1$ and $1 \leq i \leq n - (k-1)d$

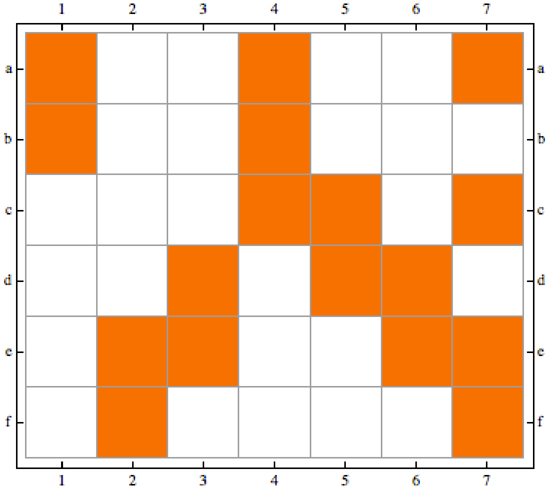
van der Waerden numbers

- It's obvious that $w(1,k) = k$ and $w(2,k) = 2k - [k \text{ even}]$
  - where $[k \text{ even}] = 1$ if k is even and $[k \text{ even}] = 0$ o/w
- When $j, k \geq 3$ the nature of $w(j,k)$ is quite mysterious
- Not many exact values are known, SAT solvers were used a lot

$k =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$W(3,k) =$	9	18	22	32	46	58	77	97	114	135	160	186	218	238	279	312	349
$W(4,k) =$	18	35	55	73	109	146	309	?	?	?	?	?	?	?	?	?	?
$W(5,k) =$	22	55	178	206	260	?	?	?	?	?	?	?	?	?	?	?	?
$W(6,k) =$	32	73	206	1132	?	?	?	?	?	?	?	?	?	?	?	?	?

Exact Cover

Given a $0-1$ matrix, find a selection of the rows that has exactly one 1 in each column.



Langford pairings

A permutation of $1, 1, 2, 2, 3, 3, \ldots, n, n$ so that the two ks are k "slots" apart.

Express as exact cover. Find a selection of the rows that has exactly one 1 in each column.

```
100010100000   1   1.1.....
100001010000   1   .1.1....
100000101000   1   ..1.1...
100000010100   1   ...1.1..
100000001010   1   ....1.1.
100000000101   1   .....1.1
010010010000    2  1..1....
010001001000    2  .1..1...
010000100100    2  ..1..1..
010000010010    2  ...1..1.
010000001001    2  ....1..1
001010001000     3 1...1...
001001000100     3 .1...1..
001000100010     3 ..1...1.
001000010001     3 ...1...1
000110000100      41....1..
000101000010      4.1....1.
000100100001      4..1....1
```

Langford pairings

- Variables: For each row, introduce a variable x_i
  - to indicate whether this row is chosen
- Constraints: should have exactly one 1 in each column
  - Col1: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$
  - Col2: $x_7 + x_8 + x_9 + x_{10} + x_{11} = 1$

    $\ldots$
  - Col5: $x_1 + x_7 + x_{12} + x_{16} = 1$
  - Col6: $x_2 + x_8 + x_{13} + x_{17} = 1$

    $\ldots$
  - Col12: $x_6 + x_{11} + x_{15} + x_{18} = 1$

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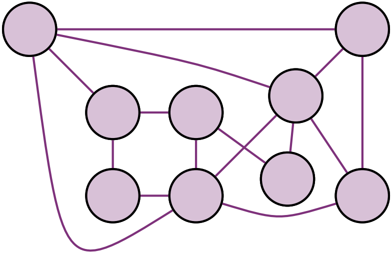
How to express $x_1 + x_2 + \ldots + x_n = 1$ as clauses?

$$(x_1 \vee x_2 \vee \ldots \vee x_n) \wedge \bigwedge_{1 \leq j < k \leq n} (\overline{x}_j \vee \overline{x}_k)$$

Graph k-colouring



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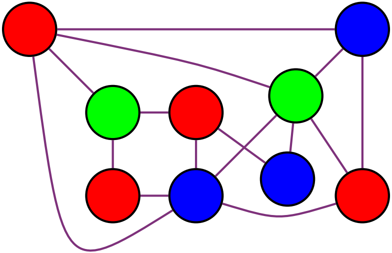

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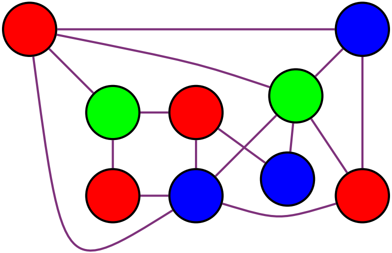

- Variables: var v_j for each vertex v and each colour j
- Clauses:
  - $(v_1 \vee v_2 \vee \ldots \vee v_k)$ – each vertex v gets some colour
  - $(\overline{u}_j \vee \overline{v}_j)$ for each edge (u, v) and each colour j
  - Optional: can say that each vertex gets exactly 1 colour