Lecture 14:
First-order Logic — Formal Syntax and Semantics

Barnaby Martin, 28 February 2021

Outline

- Syntax of first-order logic
- Semantics of first-order logic
- Parse trees
- Free and bound occurrences
- Parse trees

Syntax of first-order logic

Every (well-formed) formula of first-order logic is constructed from atoms (or atomic formula). We completely define the syntax of first-order logic by defining what we mean by atoms and the constructions we are allowed to use.

Syntax of first-order logic

Every (well-formed) formula of first-order logic is constructed from atoms (or atomic formula). We completely define the syntax of first-order logic by defining what we mean by atoms and the constructions we are allowed to use.

Atoms:

1. If P is a relation symbol of arity r and $y_1, \ldots, y_r$ are (not necessarily distinct) variables or constant symbols, then $P(y_1, \ldots, y_r)$ is an atom with free variables from $y_1, \ldots, y_r$ (this sequence can also contain constants and repeated items).

Syntax of first-order logic

Every (well-formed) formula of first-order logic is constructed from atoms (or atomic formula). We completely define the syntax of first-order logic by defining what we mean by atoms and the constructions we are allowed to use.

Atoms:

1. If P is a relation symbol of arity r and $y_1, \ldots, y_r$ are (not necessarily distinct) variables or constant symbols, then $P(y_1, \ldots, y_r)$ is an atom with free variables from $y_1, \ldots, y_r$ (this sequence can also contain constants and repeated items).
2. If C and D are constant symbols and x and y are variables then $C = D$, $C = x$ and $x = y$ are all atoms with, respectively, set of free variables $\emptyset$, $\{x\}$, $\{x, y\}$.

Syntax of first-order logic

Every (well-formed) formula of first-order logic is constructed from atoms (or atomic formula). We completely define the syntax of first-order logic by defining what we mean by atoms and the constructions we are allowed to use.

Atoms:

1. If P is a relation symbol of arity r and $y_1, \ldots, y_r$ are (not necessarily distinct) variables or constant symbols, then $P(y_1, \ldots, y_r)$ is an atom with free variables from $y_1, \ldots, y_r$ (this sequence can also contain constants and repeated items).
2. If C and D are constant symbols and x and y are variables then $C = D$, $C = x$ and $x = y$ are all atoms with, respectively, set of free variables $\emptyset$, $\{x\}$, $\{x, y\}$.

(The signature of the formula is its finite set of predicate (relation) and constant symbols.)

Syntax of first-order logic

A: set of integer

B: set of positive number

$\mathbb{Q}$ Rational

$A \cup B$ union: $\{A\}+\{B\}$

$A \cap B$: intersection: $\mathbb{Z}^+$

$a \in A$: is a member of

$A \setminus \{2\}$: integers without 2

Constructions:

1. If ϕ and ψ are formulae, with free variables free(ϕ) and free(ψ), then

$$\phi \vee \psi, \phi \wedge \psi, \neg\phi$$

   are formulae with, respectively, free variables free(ϕ)$\cup$ free(ψ), free(ϕ)$\cup$ free(ψ) and free(ϕ)

2. If ϕ is a formula with free variables free(ϕ) then

$$\exists x(\phi), \forall x(\phi)$$

   are formulae, both with free variables free(ϕ) $\setminus \{x\}$. The occurrence of x in both formulae is a bound occurrence.

Syntax of first-order logic

Constructions:

1. If ϕ and ψ are formulae, with free variables free(ϕ) and free(ψ), then
$$\phi \vee \psi, \phi \wedge \psi, \neg\phi$$
are formulae with, respectively, free variables free(ϕ)$\cup$ free(ψ), free(ϕ)$\cup$ free(ψ) and free(ϕ)
2. If ϕ is a formula with free variables free(ϕ) then
$$\exists x(\phi), \forall x(\phi)$$
are formulae, both with free variables free(ϕ) $\setminus \{x\}$. The occurrence of x in both formulae is a bound occurrence.

If a formula has no free variables then it is called a sentence.

Parse trees

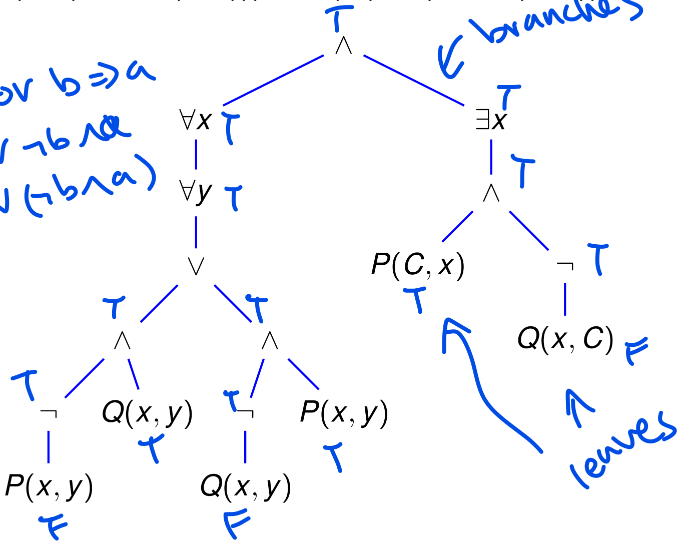
We can check that a formula is well-formed using a parse tree.
We illustrate with
$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C))$

$a \Leftrightarrow b$

$a \Rightarrow b$ or $b \Rightarrow a$

$\neg a \wedge b$ or $\neg b \wedge a$

$(\neg a \wedge b) \vee (\neg b \wedge a)$



Semantics of first-order logic

An interpretation or a structure for a first-order formula ϕ is:

- a domain of discourse D,
- a value from D for every free variable of ϕ,
- a relation over D for every relation symbol involved in ϕ,
- a value from D for every constant symbol involved in ϕ.

D: {all men in the world)

Q(x): return True if x is a mother.
x must be a woman.

$\forall x\, Q(x) \Rightarrow$ True   Non sense

$\forall x\, Q(x, \text{'Lucy'})$

Semantics of first-order logic

An interpretation or a structure for a first-order formula ϕ is:

- a domain of discourse D,
- a value from D for every free variable of ϕ,
- a relation over D for every relation symbol involved in ϕ,
- a value from D for every constant symbol involved in ϕ.

The semantics of a first-order formula in some interpretation is as follows:

$\forall \underline{x}\, \phi(x_1 .. x_n .. \underline{x} ... x_r) = \text{True}$

- we interpret atoms as propositional variables,
- we interpret $\wedge$, $\vee$, and $\neg$ as in propositional logic,
- we interpret $\forall x \phi$ as true if ϕ is true for all values for x,
- we interpret $\exists x \phi$ as true if there is at least one value for x making ϕ true.

An illustration

Consider a signature consisting of two binary relation symbols P and Q and one constant symbol C. Let ϕ be defined as

$$\forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists x(P(C, x) \wedge \neg Q(x, C)).$$

In order to decide whether ϕ evaluates to true or not, we need an interpretation.

An illustration continued

Consider the interpretation

$$\phi = \forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists x(P(C, x) \wedge \neg Q(x, C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u, v) : u, v \in \mathbb{N}, u \leq v\}$
- the relation $Q = \{(u, v) : u, v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

$P(1, 1) \rightarrow$ True

$P(2, 3) \rightarrow$ ~~False~~ True

$P(10, 9) \rightarrow$ False

$Q(10, 9) \rightarrow$ True

An illustration continued

Consider the interpretation

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u,v) : u, v \in \mathbb{N}, u \leq v\}$
- the relation $Q = \{(u,v) : u, v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

So,

- $(\mathbb{N}, P, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x,y) \Leftrightarrow \neg Q(x,y))$
  and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C,x) \wedge \neg Q(x,C))$

An illustration continued

Consider the interpretation

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u,v) : u,v \in \mathbb{N}, u \leq v\}$
- the relation $Q = \{(u,v) : u,v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

So,

- $(\mathbb{N}, P, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x,y) \Leftrightarrow \neg Q(x,y))$
  and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C,x) \wedge \neg Q(x,C))$
- if and only if for every $x, y \in \mathbb{N}$, $x \leq y \Leftrightarrow x \not> y$,
  and there exists $x \in \mathbb{N}$, such that $0 \leq x$ and $x \not> 0$.

An illustration continued

Consider the interpretation

$$\phi = \forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists x(P(C, x) \wedge \neg Q(x, C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u, v) : u, v \in \mathbb{N}, u \leq v\}$
- the relation $Q = \{(u, v) : u, v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$. ✓

$0 \geq 0 \wedge 0 \not> 0$

$x \geq 0 \wedge x \not> 0$

$\therefore x$ -ve : False

x +ve : False

$x = 0$: $0 \notin \mathbb{N}$

So,

- $(\mathbb{N}, P, Q, 0) \models \phi$ if and only if $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x, y) \Leftrightarrow \neg Q(x, y))$ and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C, x) \wedge \neg Q(x, C))$
- if and only if for every $x, y \in \mathbb{N}$, $x \leq y \Leftrightarrow x \not> y$, and there exists $x \in \mathbb{N}$, such that $0 \leq x$ and $x \not> 0$.

Both conjuncts are true. Thus, $(\mathbb{N}, P, Q, 0)$ is a model of ϕ, i.e., $(\mathbb{N}, P, Q, 0) \models \phi$

An illustration continued

Now consider the interpretation

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u,v) : u, v \in \mathbb{N}, u < v\}$
- the relation $Q = \{(u,v) : u, v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

An illustration continued

Now consider the interpretation

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u,v) : u, v \in \mathbb{N}, u < v\}$
- the relation $Q = \{(u,v) : u, v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

So,

- $(\mathbb{N}, P, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x,y) \Leftrightarrow \neg Q(x,y))$
  and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C,x) \wedge \neg Q(x,C))$

An illustration continued

Now consider the interpretation

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u,v) : u,v \in \mathbb{N}, u < v\}$
- the relation $Q = \{(u,v) : u,v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

So,

- $(\mathbb{N}, P, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x,y) \Leftrightarrow \neg Q(x,y))$
  and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C,x) \wedge \neg Q(x,C))$
- if and only if for every $x, y \in \mathbb{N}$, $x < y \Leftrightarrow x \not> y$,
  and there exists $x \in \mathbb{N}$, such that $0 < x$ and $x \not> 0$.

An illustration continued

Now consider the interpretation

$$\phi = \forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists x(P(C, x) \wedge \neg Q(x, C)).$$

where:

- the domain of discourse is the set of natural numbers $\mathbb{N}$
- the relation $P = \{(u, v) : u, v \in \mathbb{N}, u < v\}$
- the relation $Q = \{(u, v) : u, v \in \mathbb{N}, u > v\}$
- the constant $C = 0 \in \mathbb{N}$.

So,

- $(\mathbb{N}, P, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x, y) \Leftrightarrow \neg Q(x, y))$
  and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C, x) \wedge \neg Q(x, C))$
- if and only if for every $x, y \in \mathbb{N}$, $x < y \Leftrightarrow x \not> y$,
  and there exists $x \in \mathbb{N}$, such that $0 < x$ and $x \not> 0$.

Now both conjuncts are false. Thus, $(\mathbb{N}, P, Q, 0)$ is not a model of ϕ, i.e., $(\mathbb{N}, P, Q, 0) \models \neg\phi$

A subtlety

Consider a signature consisting of two binary relation symbols P and Q and one constant symbol C. Let ϕ be defined as

$$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists z(P(z,x) \wedge \neg Q(x,C)).$$

A subtlety

Consider a signature consisting of two binary relation symbols P and Q and one constant symbol C. Let ϕ be defined as

$$\forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists z(P(z, x) \wedge \neg Q(x, C)).$$

This is a perfectly legal formula of first-order logic, even though the variable x appears "differently" in the formula:

- x appears bound in the first conjunct,
- x appears free in the second conjunct.

Consequently, it is more precise to speak of "free occurrences" or "bound occurrences" of variables rather than free or bound variables.

Another subtlety

Consider the formula χ defined as

$$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \land \exists y(P(y,x) \land \neg Q(x,y)).$$

and the interpretation I for χ where:

- the domain $D = \{1,2,3\}$,
- $P = \{(1,3),(2,3),(3,1)\}$ and
  $Q = \{(1,1),(1,2),(2,1),(2,2),(3,2),(3,3)\}$,
- $x = 3$.

Not only does x appear both free and bound but y appears bound but within the scopes of two different quantifications.

Another subtlety

Consider the formula χ defined as

$$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists y(P(y,x) \wedge \neg Q(x,y)).$$

and the interpretation I for χ where:

- the domain $D = \{1,2,3\}$,
- $P = \{(1,3),(2,3),(3,1)\}$ and
  $Q = \{(1,1),(1,2),(2,1),(2,2),(3,2),(3,3)\}$,
- $x = 3$.

Not only does x appear both free and bound but y appears bound but within the scopes of two different quantifications.

We clearly have that $I \models \chi$ as

- for every $(x,y) \in D \times D$, $(x,y) \in P$ if and only if $(x,y) \notin Q$,
- there exists a $y \in D$ such that $(y,3) \in P$ and $(3,y) \notin Q$, namely y = 1.

Another subtlety

Consider the formula χ defined as

$$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists y(P(y,x) \wedge \neg Q(x,y)).$$

and the interpretation I for χ where:

- the domain $D = \{1,2,3\}$,
- $P = \{(1,3),(2,3),(3,1)\}$ and
  $Q = \{(1,1),(1,2),(2,1),(2,2),(3,2),(3,3)\}$,
- $x = 3$.

Not only does x appear both free and bound but y appears bound but within the scopes of two different quantifications.

We clearly have that $I \models \chi$ as

- for every $(x,y) \in D \times D$, $(x,y) \in P$ if and only if $(x,y) \notin Q$,
- there exists a $y \in D$ such that $(y,3) \in P$ and $(3,y) \notin Q$, namely y = 1.

If we amend the interpretation so that x is interpreted as $x = 2$ then we have that $I \models \neg\chi$.

More illustrations

Consider the well-formed formula ϕ defined as $\forall x \exists y P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

More illustrations

Consider the well-formed formula ϕ defined as $\forall x \exists y P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So,

- $(\mathbb{Z}, P) \models \forall x \exists y P(x, y)$
  if and only if for every $x \in \mathbb{Z}, (\mathbb{Z}, P) \models \exists y P(x, y)$
  if and only if for every $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ with $x > y$.

More illustrations

Consider the well-formed formula ϕ defined as $\forall x \exists y P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So,

- $(\mathbb{Z}, P) \models \forall x \exists y P(x, y)$

  if and only if for every $x \in \mathbb{Z}, (\mathbb{Z}, P) \models \exists y P(x, y)$

  if and only if for every $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ with $x > y$.

For any $x \in \mathbb{Z}$, if we take $y = x - 1$ then this value of y witnesses that $x > y$; hence, $(\mathbb{Z}, P) \models \phi$.

More illustrations

Consider the well-formed formula ϕ defined as $\forall x \exists y P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So,

- $(\mathbb{Z}, P) \models \forall x \exists y P(x, y)$

  if and only if for every $x \in \mathbb{Z}, (\mathbb{Z}, P) \models \exists y P(x, y)$

  if and only if for every $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ with $x > y$.

For any $x \in \mathbb{Z}$, if we take $y = x - 1$ then this value of y witnesses that $x > y$; hence, $(\mathbb{Z}, P) \models \phi$.

If we restrict the domain to the natural numbers $\mathbb{N}$ and where $P = \{(u, v) : u, v \in \mathbb{N}, u > v\}$, i.e, we have the restriction of $(\mathbb{Z}, P)$ to $\mathbb{N}$, then $(\mathbb{N}, P) \models \neg\phi$.

More illustrations

Consider the well-formed formula ϕ defined as $\exists y \forall x P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

More illustrations

Consider the well-formed formula ϕ defined as $\exists y \forall x P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So,

- $(\mathbb{Z}, P) \models \exists y \forall x P(x, y)$
  if and only if there exists $y \in \mathbb{Z}$ such that $(\mathbb{Z}, P) \models \forall x P(x, y)$
  if and only if there exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $x > y$.

More illustrations

Consider the well-formed formula ϕ defined as $\exists y \forall x P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So,

- $(\mathbb{Z}, P) \models \exists y \forall x P(x, y)$

  if and only if there exists $y \in \mathbb{Z}$ such that $(\mathbb{Z}, P) \models \forall x P(x, y)$

  if and only if there exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $x > y$.

No matter which $y \in \mathbb{Z}$ we choose, putting $x = y - 1$ results in $x \leq y$.

More illustrations

Consider the well-formed formula ϕ defined as $\exists y \forall x P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So, There exist y, forall x. x > y

- $(\mathbb{Z}, P) \models \exists y \forall x P(x, y)$

  if and only if there exists $y \in \mathbb{Z}$ such that $(\mathbb{Z}, P) \models \forall x P(x, y)$

  if and only if there exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $x > y$.

No matter which $y \in \mathbb{Z}$ we choose, putting $x = y - 1$ results in $x \leq y$.

永远找不到

最小的整数

Hence, $(\mathbb{Z}, P) \models \neg \exists y \forall x P(x, y)$.

More illustrations

Consider the well-formed formula ϕ defined as $\exists y \forall x P(x, y)$.

And consider the interpretation of ϕ where

- the domain of discourse is the set $\mathbb{Z}$ of integers,
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$.

So,

- $(\mathbb{Z}, P) \models \exists y \forall x P(x, y)$

  if and only if there exists $y \in \mathbb{Z}$ such that $(\mathbb{Z}, P) \models \forall x P(x, y)$

  if and only if there exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $x > y$.

No matter which $y \in \mathbb{Z}$ we choose, putting $x = y - 1$ results in $x \leq y$.

Hence, $(\mathbb{Z}, P) \models \neg \exists y \forall x P(x, y)$.

$\boxed{\text{Take care with the order of quantifiers.}}$