Sat Solvers: Introduction to SAT-solving

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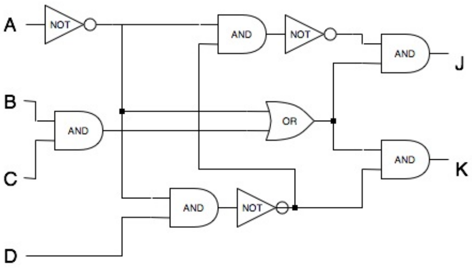
Modern computer systems are unreliable.

Example



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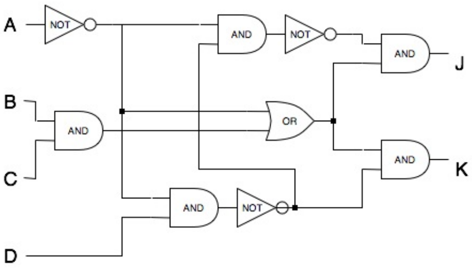

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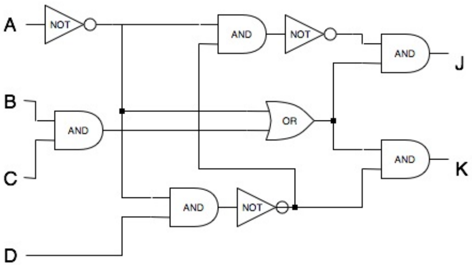

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- We want to be sure that C2 is correct, that is, it will behave according to some specification.
- If we know that C1 is correct, it is sufficient to prove that C2 is functionally equivalent to C1.

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- A natural tool for this whole process is logic.

Automated Reasoning

- Formal proofs proceed via reasoning
  - *Reasoning = deriving conclusions from facts.*
- This course is about automation of (i.e. algorithms for) reasoning.

Applications of Automated Reasoning

- SAT solvers
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- Circuit design
- Cryptography
- Databases
- Theorem proving in mathematics

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In the course we will mainly consider *propositional logic*.

Reminder: Propositional logic

- Propositional variables: $x, y, z, \ldots$ take values $0/1$ (or `false`/`true`)
- Connectives: $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\oplus$

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- Truth assignment (TA) = assignment of values to variables in a formula
- Obvious way to evaluate a formula on a given TA
- TA t satisfies a formula Φ if Φ evaluates to 1 on t.
- Φ is satisfying if some TA satisfies it.

Satisfaction vs. Consequence

- Let Ψ and Φ be two propositional formulas. Then Φ is a logical consequence of Ψ, or Ψ entails Φ (denoted $\Psi \models \Phi$) if every TA that satisfies Ψ also satisfies Φ.
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- Similarly, $\Psi_1, \ldots, \Psi_n \models \Phi$ if and only if $\Psi_1 \wedge \ldots \wedge \Psi_n \wedge \neg\Phi$ is unsatisfiable
- Thus, knowing how to check for entailment is the same as knowing how to check for (un)satisfiability.

Clauses

- Literals: variables $x, y, z, \ldots$ and their negations $\overline{x}, \overline{y}, \overline{z}, \ldots$
- Clauses: formulas of the form $\ell_1 \vee \ldots \vee \ell_n$ where each ℓ_i is a literal
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- Conjunctive normal form: $C_1 \wedge \ldots \wedge C_m$ where each C_i is a clause
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- Enough to know how to check satisfiability of clause-sets

Example

- $F = \{\{u, \overline{v}, \overline{y}\}, \{\overline{u}, z\}, \{\overline{v}, \overline{w}\}, \{w, \overline{x}\}, \{x, y, \overline{z}\}\}$
- $var(F) = \{u, v, w, x, y, z\}$
- Since F is a *set*, we have $F = \{\{\overline{u}, z\}, \{\overline{v}, \overline{w}\}, \{u, \overline{v}, \overline{y}\}, \{w, \overline{x}\}, \{\overline{u}, z\}, \{x, y, \overline{z}\}\}$.

The DIMACS format

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- Variables are indicated as positive integers, negative literals by negative integers.
- Any line starting with a `c` is a comment.
- At the top of the file is a line of form
  `p cnf N M`
  where N is the number of the largest variable, and M is the total number of clauses in the clause-set.
- Clauses are separated by a zero ("0").

Example

Clause-set $F = \{\{u, \overline{v}, \overline{y}\}, \{\overline{u}, z\}, \{\overline{v}, \overline{w}\}, \{w, \overline{x}\}, \{x, y, \overline{z}\}\}$ written in DIMACS format looks like this:

```
c This is an example
c NOTE: Satisfiable
c
p cnf 6 5
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-1 6 0
-2 -3 0
3 -4 0
-4 5 -6 0
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Try solving F by this online SAT solver

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- Standard procedure for converting a formula Φ to CNF (clause-set) $F_{\mathrm{CNF}(\Phi)}$:
  1. Express all connectives using $\vee$, $\wedge$, and $\neg$
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  3. Use De Morgan's Laws to convert to CNF
     See slides at the end for more details and examples

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- For $n > 0$ let Ψ_n be the formula

$$(x_1 \wedge y_1) \vee \cdots \vee (x_n \wedge y_n).$$

- It turns out that $F_{\text{CNF}(\Psi_n)}$ is rather huge!

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- CNF(Ψ_n) contains all 2^n possible terms $(z_1 \vee \cdots \vee z_n)$ for $z_i \in \{x_i, y_i\}$.
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- Thus $F_{\mathrm{CNF}(\Psi_n)}$ contains 2^n clauses.
- It is not difficult to show that one cannot do it with fewer clauses.

Fact

For every $n > 1$ there exists a propositional formula of length $2n$ for which any logical equivalent clause-set contains at least 2^n clauses.

Fast Transformation

Fast Transformation

A faster transformation was proposed by Tseitin in 1968.

- **Idea**: Transformation needs not preserve logical equivalence, equivalence w.r.t. satisfiability is enough.

Definition (equisatisfiability)

Two propositional formulas or clause-sets are equisatisfiable if either both are satisfiable or both are unsatisfiable.

- We may introduce new variables (“extension variables”)

Tseitin's Algorithm

Tseitin's Algorithm

- **Input**: a propositional formula Ψ; assume that double negations are eliminated. Initially, put $F = \emptyset$.
- While Ψ has a subformula $\ell \circ m$ where $\circ$ is any binary logical connective and ℓ, m are literals
  - replace in Ψ the subformula $\ell \circ m$ with a new variable $x_{\ell \circ m}$,
  - extend the clause set: $F \cup F_{\ell \circ m}$ and use this as a new F
    $F_{\ell \circ m}$ is a clause-set logically equivalent to the propositional formula $x_{\ell \circ m} \leftrightarrow (\ell \circ m)$.
- Now Ψ is a single literal ℓ. Add the unit clause $\{\ell\}$ to F and halt.
- **Output**: clause-set F.
- Note we can stop as soon as Ψ is in CNF and add the clauses of F_Ψ to F.

Defining clause-set $F_{\ell \wedge m}$

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- The clause-set $F_{\ell \wedge m}$ needs to be logically equivalent to the formula $x \equiv \ell \wedge m$.
- We derive $F_{\ell \wedge m}$ using the "slow" method:
  - $x \equiv \ell \wedge m$
  - $[x \to (\ell \wedge m)] \wedge [(\ell \wedge m) \to x]$
  - $[\neg x \vee (\ell \wedge m)] \wedge [\neg(\ell \wedge m) \vee x]$
  - $[(\neg x \vee \ell) \wedge (\neg x \vee m)] \wedge [(\neg \ell \vee \neg m) \vee x]$
  - $(\neg x \vee \ell) \wedge (\neg x \vee m) \wedge (\neg \ell \vee \neg m \vee x)$
- Hence $F_{\ell \wedge m} = \{\{\overline{x}, \ell\}, \{\overline{x}, m\}, \{\overline{\ell}, \overline{m}, x\}\}$.

List of defining clause-sets

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$$
\begin{aligned}
F_{\ell\wedge m} &= \{\{\overline{\ell}, \overline{m}, x_{\ell\wedge m}\}, \{\ell, \overline{x_{\ell\wedge m}}\}, \{m, \overline{x_{\ell\wedge m}}\}\} \\
F_{\ell\vee m} &= \{\{\overline{\ell}, x_{\ell\vee m}\}, \{\overline{m}, x_{\ell\vee m}\}, \{\ell, m, \overline{x_{\ell\vee m}}\}\} \\
F_{\ell\to m} &= \{\{\ell, x_{\ell\to m}\}, \{\overline{m}, x_{\ell\to m}\}, \{\overline{\ell}, m, \overline{x_{\ell\to m}}\}\} \\
F_{\ell\leftrightarrow m} &= \{\{\ell, m, x_{\ell\leftrightarrow m}\}, \{\overline{\ell}, \overline{m}, x_{\ell\leftrightarrow m}\}, \\
&\quad \{\overline{\ell}, m, \overline{x_{\ell\leftrightarrow m}}\}, \{\ell, \overline{m}, \overline{x_{\ell\leftrightarrow m}}\}\} \\
F_{\ell\oplus m} &= \{\{\overline{\ell}, m, x_{\ell\oplus m}\}, \{\ell, \overline{m}, x_{\ell\oplus m}\}, \\
&\quad \{\ell, m, \overline{x_{\ell\oplus m}}\}, \{\overline{\ell}, \overline{m}, \overline{x_{\ell\oplus m}}\}\}
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- Input: $\Psi = \neg(x \rightarrow (y \rightarrow x))$.

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- Input: $\Psi = \neg(x \to (y \to x))$.
- $\Psi = \neg(x \to (y \to x))$, $F = \emptyset$.
- $\Psi = \neg(x \to u)$, $F = \{\{y, u\}, \{\overline{x}, u\}, \{\overline{y}, x, \overline{u}\}\}$.
- $\Psi = \neg v$,
  $F = \{\{y, u\}, \{\overline{x}, u\}, \{\overline{y}, x, \overline{u}\}, \{x, v\}, \{\overline{u}, v\}, \{\overline{x}, u, \overline{v}\}\}$.
- Output: $F = \{\{y, u\}, \{\overline{x}, u\}, \{\overline{y}, x, \overline{u}\}, \{x, v\}, \{\overline{u}, v\}, \{\overline{x}, u, \overline{v}\}, \{\overline{v}\}\}$.

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- $\Psi_n = (x_1 \wedge y_1) \vee \cdots \vee (x_n \wedge y_n)$, $F = \emptyset$
- $x_{x_1 \wedge y_1} \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$, $F = F_{x_1 \wedge y_1}$.
- $x_{x_1 \wedge y_1} \vee x_{x_2 \wedge y_2} \vee (x_3 \wedge y_3) \vee \cdots \vee (x_n \wedge y_n)$,
  $F = F_{x_1 \wedge y_1} \cup F_{x_2 \wedge y_2}$.
- etc.
- $x_{x_1 \wedge y_1} \vee x_{x_2 \wedge y_2} \vee \cdots \vee x_{x_n \wedge y_n}$, $F = F_{x_1 \wedge y_1} \cup \cdots \cup F_{x_n \wedge y_n}$.
- $F = \{\{x_{x_1 \wedge y_1}, \ldots, x_{x_n \wedge y_n}\}\} \cup F_{x_1 \wedge y_1} \cup \cdots \cup F_{x_n \wedge y_n}$.

Results

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Let Ψ be a propositional formula and $T(\Psi)$ the clause-set obtained using the fast transformation.

- Ψ and $T(\Psi)$ are equisatisfiable.
- Ψ is a tautology if and only if $T(\neg\Psi)$ is unsatisfiable.

Results

Let Ψ be a propositional formula and $T(\Psi)$ the clause-set obtained using the fast transformation.

- Ψ and $T(\Psi)$ are equisatisfiable.
- Ψ is a tautology if and only if $T(\neg\Psi)$ is unsatisfiable.
- Let n be the number of occurrences of binary connectives in Ψ. Then $T(\Psi)$ contains at most $4n + 1$ clauses, each clause contains at most 3 literals. Thus $|T(\Psi)| = O(n)$, i.e., the number of clauses in $T(\Psi)$ is linear in the number n of occurrences of binary connectives in Ψ.
- Note, however, that Ψ and $T(\Psi)$ are equisatisfiable but not logically equivalent. (What does this mean?)

A search algorithm

A search algorithm

- Algorithm A(F)
- outputs *sat* or *unsat*
- Pseudocode:

if $\mathrm{var}(F) = \emptyset$ then

if $F = \emptyset$ then exit(sat).

else exit(unsat). /* $F = \{\Box\}$ where $\Box$ is empty clause */

else choose $x \in \mathrm{var}(F)$,

/* x is the branching variable */

if A($F[x = 0]$) = sat then exit(sat).

else if A($F[x = 1]$) = sat then exit(sat).

else exit(unsat).

Pure Literals

- A literal x of a clause-set F is a pure literal of F if some clauses of F contain x but no clause of F contains $\overline{x}$.
- Example: $F = \{\{x, y\}, \{\overline{y}, \overline{z}\}, \{z, x\}, \{u, \overline{z}\}\}$.
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  x and u are the pure literals of F.
- Let F' be the clause-set obtained from F by removing all clauses that contain pure literals.

  Then F and F' are equisatisfiable (why?).

  We say that F' is obtained from F by pure literal elimination.

Applying Pure Literal Elimination

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- For a clause-set F we denote by $PL(F)$ the smallest clause-set that can be obtained from F by (possibly repeated) applications of pure literal elimination.
- In the above example we have $PL(F) = \emptyset$.

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- F and $PL(F)$ are always equisatisfiable.

Unit Propagation

- When a clause-set F contains a unit clause $\{\ell\}$, we can obtain by unit propagation the clause-set $F[\ell = 1]$ from F.
- In that case F and $F[\ell = 1]$ are equisatisfiable (why?)
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- We write $UP(F)$ for the clause-set obtained from F by applying unit propagation as often as possible.
- Example: Let $F = \{\{x, y\}, \{\overline{y}\}, \{z, \overline{x}, v\}\}$.
  - From F we obtain by unit propagation the clause set $F' = \{\{x\}, \{z, \overline{x}, v\}\}$.
  - With a second step of unit propagation we obtain from F' the clause-set $F'' = \{\{z, v\}\}$.

  Consequently $UP(F) = \{\{z, v\}\}$.
- F and $UP(F)$ are always equisatisfiable.

First UP and then PL, or vice versa?

Fact

For any clause-set F, we have $UP(PL(UP(F))) = PL(UP(F))$.

Proof: exercise (Question 3a.i in 2009 Advanced AI exam)

Fact

There is a clause-set F such that $PL(UP(PL(F))) \neq UP(PL(F))$.

Proof: exercise (Question 3a.ii in 2009 Advanced AI exam)

So, first UP and then PL, or the other way around?

The DPLL algorithm

- `Algorithm DPLL`(F)
- `outputs` *sat* `or` *unsat*

$F := UP(F)$.

$F := PL(F)$.

`if` $\mathrm{var}(F) = \emptyset$ `then`

    `if` $F = \emptyset$ `then exit(sat).`

    `else exit(unsat).   /*` $F = \{\square\}$ `*/`

`else choose` $x \in \mathrm{var}(F)$,

`/*` x `is the branching variable */`

`if A`$(F[x = 0]) =$ `sat then exit(sat).`

`else if A`$(F[x = 1]) =$ `sat then exit(sat).`

`else exit(unsat).`

Some Well-Known DPLL-based SAT Solvers

- Important SAT solvers:
  - Grasp (Marques-Silva & Sakallah 1996)
  - Relsat (Bayardo Jr. & Schrag 1997)
  - chaff (Moskewicz et al 2001), zChaff (Zhang 2001)
  - Minisat (Een & Sörensson 2003)
  - RSat (Pipatsrisawat & Darwiche 2007)
  - Glucose (Audemard & Simon 2009)
  - CryptoMiniSat (Soos 2010)
  - Lingeling, PicoSAT (Biere 2010)
- Many of them are available for free and continually improved.

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- Many of them are available for free and continually improved.
- Key improvements to DPLL that boosted the performance:
  - combination of clever branching heuristics with
  - clause learning,
  - non-chronological backtracking,
  - restart strategies,
  - implementation of propagation.
- Check *www.satlive.org* for more info on SAT solvers!