Durham University

Maths for Computer Science
Calculus

Prof. Magnus Bordewich

Integration II

The Fundamental Theorem of Calculus

Let f be a continuous function on $[a,b]$, and let

$$F(x) = \int_a^x f(t)\,dt$$

then F is continuous and differentiable on (a,b) and

$$F'(x) = f(x) \text{ for all } x \in (a,b).$$

Integrals

A function F such that $F'(x) = f(x)$ is called an antiderivative of f.

An **indefinite integral** is one with no specific bounds:

$$\int f(t)\,dt = F(x) + C, \qquad or \qquad \int_a^x f(t)\,dt = F(x) + C,$$

where F is an antiderivative of f, to denote an indefinite integral.

A **definite integral** is one with specific bounds, and therefore a value:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Example

$$\int_1^5 \frac{1}{x^2}\,dx$$

We can guess an anti-derivative: if $F(x) = -\frac{1}{x}$, then $F'(x) = \frac{1}{x^2}$.

So

$$\int_1^5 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^5 = -\frac{1}{5} - \left(-\frac{1}{1}\right) = \frac{4}{5}.$$

What about

$$\int_0^5 \frac{1}{x^2}\,dx?$$

We are in trouble here since the integrand is infinite at $x = 0$.

Improper integrals: infinity of integrand

We can deal with a function with a singularity – i.e. an isolated point at which the function tends to $\pm$infinity, the same way as discontinuities:

If f is continuous **and finite** on $[a, b]$ except at some point c where $\lim_{x \to c} f(x) \to \infty$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon' \to 0} \int_{c+\epsilon'}^b f(x)\,dx$$

where both limits exist.

Or if f is continuous and finite on $[a, b]$ except at b where $\lim_{x \to b} f(x) \to \infty$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)\,dx$$

We can deal with a singularity at a similarly.

Divergent example

Now

$$\int_0^5 \frac{1}{x^2}\,dx = \lim_{\epsilon\to 0}\int_\epsilon^5 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_\epsilon^5 = -\frac{1}{5} + \frac{1}{\epsilon} \to \infty$$

as $\epsilon \to 0$, so the integral **diverges**.

Convergent example

The denominator in the following integrand is undefined when $x = 0$:

$$\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$$

So we split the interval at 0 and take both limits:

$$\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx = \lim_{\epsilon\to 0} \int_{-1}^{-\epsilon} \frac{1}{\sqrt{-x}}\,dx + \lim_{\epsilon'\to 0} \int_{+\epsilon'}^{2} \frac{1}{\sqrt{x}}\,dx$$

Now guessing an antiderivative $F(x) = -2\sqrt{-x}$ for the first integral and $G(x) = 2\sqrt{x}$ for the second, we obtain

$$\begin{aligned}
\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx &= \lim_{\epsilon\to 0} \left[-2\sqrt{-x}\right]_{-1}^{-\epsilon} + \lim_{\epsilon'\to 0} \left[2\sqrt{x}\right]_{\epsilon'}^{2} \\
&= \lim_{\epsilon\to 0}\left[-2\sqrt{\epsilon} + 2\sqrt{1}\right] + \lim_{\epsilon'\to 0}\left[2\sqrt{2} - 2\sqrt{\epsilon'}\right] \\
&= 2 + 2\sqrt{2}
\end{aligned}$$

Troubling example

Consider

$$\int_{-2}^{3} \frac{1}{x^3}\,dx = \lim_{\epsilon\to 0}\left[\int_{-2}^{-\epsilon} \frac{1}{x^3}\,dx + \int_{\epsilon}^{3} \frac{1}{x^3}\,dx\right]$$

Guess the antiderivative $F(x) = -\frac{1}{2x^2}$, then

$$\int_{-2}^{3} \frac{1}{x^3}\,dx \;= \lim_{\epsilon\to 0}\left[\left[\frac{-1}{2x^2}\right]_{-2}^{-\epsilon} + \left[\frac{-1}{2x^2}\right]_{\epsilon}^{3}\right]$$

$$= \lim_{\epsilon\to 0}\left[\frac{-1}{2\epsilon^2} - \frac{-1}{8} + \frac{-1}{18} - \frac{-1}{2\epsilon^2}\right] = \lim_{\epsilon\to 0}\left[\frac{1}{8} - \frac{1}{18}\right] = \frac{1}{8} - \frac{1}{18}.$$

NO!

The limits must independently exist:

$$\int_{-2}^{3} \frac{1}{x^3}\,dx \;= \lim_{\epsilon\to 0}\left[\left[\frac{-1}{2x^2}\right]_{-2}^{-\epsilon}\right] + \lim_{\epsilon'\to 0}\left[\left[\frac{-1}{2x^2}\right]_{\epsilon'}^{3}\right]$$

$$= \lim_{\epsilon\to 0}\left[\frac{-1}{2\epsilon^2} - \frac{-1}{8}\right] + \lim_{\epsilon'\to 0}\left[\frac{-1}{18} - \frac{-1}{2\epsilon'^2}\right]$$

neither of which exist.

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Improper integrals: infinity of range

We can deal with an infinite range also by taking limits:

If f is piecewise continuous on $[a, \infty]$ then we define

$$\int_a^{\infty} f(x)\, dx = \lim_{N \to \infty} \int_a^N f(x)\, dx$$

where the limit exists.

Likewise

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{N \to \infty} \int_{-N}^{b} f(x)\, dx$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{N \to \infty} \int_{-N}^{a} f(x)\, dx + \lim_{N' \to \infty} \int_a^{N'} f(x)\, dx$$

where the limits exist.

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Examples

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty}\int_2^N \frac{1}{x^2}\,dx = \lim_{N\to\infty}\left[-\frac{1}{x}\right]_2^N = \lim_{N\to\infty}\left(-\frac{1}{N}+\frac{1}{2}\right) = \frac{1}{2}.$$

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty}\int_2^N \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty}\left[2\sqrt{x}\right]_2^N = \lim_{N\to\infty}\left(2\sqrt{N}-2\sqrt{2}\right) \to \infty.$$

Seems familiar?

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

Integral test for series convergence:

Indeed if f is a continuous, decreasing, positive function on $[m,\infty]$ then

- if $\int_m^\infty f(x)\,dx$ is convergent, then so is $\sum_{n=m}^{\infty} f(m)$
- if $\int_m^\infty f(x)\,dx$ is divergent, then so is $\sum_{n=m}^{\infty} f(m)$

Systematic Integration

Integration of elementary functions

Deduced from knowledge of derivatives:

$f(x)$	$F(x)$
a (constant)	ax
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
x^{-1}	$\ln x$
e^{ax}	$\frac{1}{a}e^{ax}$
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sinh(ax)$	$\frac{1}{a}\cosh(ax)$
$\cosh(ax)$	$\frac{1}{a}\sinh(ax)$

Integration of elementary functions

Many more complex examples can deduced from knowledge of derivatives:

E.g.:

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

Integration by substitution

Recall the chain rule: if u is a function of x and g is a function of u, then

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}.$$

If we spot an integrand of the form $f\big(u(x)\big)u'(x)$ and take $F(u)$ to be the anti-derivative of $f(u)$ wrt u, then

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = f(u)u'(x)$$

i.e. our original integrand! Hence $F(u(x))$ is the antiderivative of $f\big(u(x)\big)u'(x)$ wrt x. Therefore

$$\int f\big(u(x)\big)u'(x)\,dx = F\big(u(x)\big) + C = F(u) + C = \int f(u)\,du$$

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Integration by substitution

This is normally presented as:

$$\int f\big(u(x)\big)u'(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

Example:

$$\int \frac{4x}{\sqrt{2x^2+1}}\,dx$$

Recognise that $\frac{4x}{\sqrt{2x^2+1}} = \frac{1}{\sqrt{2x^2+1}}4x = f(u)\frac{du}{dx}$ where $u = 2x^2+1$, $f(u) = \frac{1}{\sqrt{u}}$.

So

$$\int \frac{4x}{\sqrt{2x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C = 2\sqrt{2x^2+1} + C.$$

Integration by parts

Recall the product rule: if u and v are functions of x then

$$\frac{duv}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Thus uv is the anti-derivative of $u'v + uv'$. I.e.

$$\int u'v\,dx + \int uv'\,dx = uv + C$$

Rearranging

$$\int uv'\,dx = uv - \int u'v\,dx$$

Integration by parts

Example:

$\int x^k \ln x \, dx$ for some $k \in \mathbb{N}$.

We see a product of terms and wonder if we can select u, v in a way that will make things simpler.

Here if we select $u = x^k, v' = \ln x$, then it is hard to proceed: we need $v = \int \ln x$

Instead select $u = \ln x, v' = x^k$, then $v = \int x^k = \frac{x^{k+1}}{k+1}$

So

$$\int x^k \ln x \, dx = \int uv' \, dx = uv - \int u'v \, dx = \frac{x^{k+1}}{k+1} \ln x - \int \frac{1}{x} \frac{x^{k+1}}{k+1} dx$$

Which simplifies to

$$\frac{x^{k+1}}{k+1} \ln x - \int \frac{x^k}{k+1} dx = \frac{x^{k+1}}{k+1} \ln x - \frac{x^{k+1}}{(k+1)^2} + C.$$

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Other techniques

Integration by partial fractions:

$$\int \frac{x^3}{x^2-4}\,dx = \int \frac{x(x^2-4)}{x^2-4} + \frac{4x}{x^2-4}\,dx$$

$$= \int x + \frac{2}{x-2} + \frac{2}{x+2}\,dx$$

$$= \frac{x^2}{2} + 2\ln|x-2| + 2\ln|x+2| + C$$

Useful for quotients of polynomials.

Other techniques

Reduction formulae:

$$I_m = \int \cos^m \theta \, d\theta = \int \cos^{m-1} \theta \frac{d \sin \theta}{d\theta} d\theta$$

$$= \cos^{m-1} \theta \sin \theta - \int \sin \theta \,.\, (m-1) \cos^{m-2} \theta \, (-\sin \theta) \, d\theta$$

$$= \cos^{m-1} \theta \sin \theta + (m-1) \int \cos^{m-2} \theta \, d\theta - (m-1) \int \cos^m \theta \, d\theta$$

$$= \cos^{m-1} \theta \sin \theta + (m-1) I_{m-2} - (m-1) I_m$$

So

$$I_m = \frac{1}{m} \cos^{m-1} \theta \sin \theta + \frac{(m-1)}{m} I_{m-2}$$

E.g. $I_7 = \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{7} I_5$

$$= \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{35} \cos^4 \theta \sin \theta + \frac{24}{35} I_3$$

$$= \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{35} \cos^4 \theta \sin \theta + \frac{8}{35} \cos^2 \theta \sin \theta + \frac{16}{35} I_1$$

$$= \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{35} \cos^4 \theta \sin \theta + \frac{8}{35} \cos^2 \theta \sin \theta + \frac{16}{35} \sin \theta + C$$