

Lecture 7: Depth-First Search

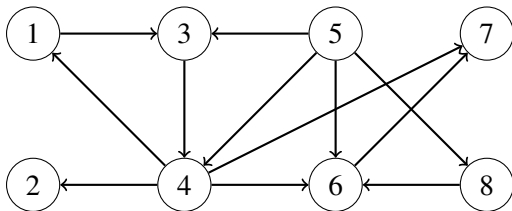
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Depth-first search

- Like BFS, **Depth-first search** explores the graph (but does not find distances to the source).
- In contrast to BFS, when a vertex is discovered it is immediately explored.
- Two **timestamps** are recorded for each vertex, d and f ; the **discovery** and **finish** times. We can also record predecessors again.
- Again colours are used:
 - white for undiscovered,
 - grey for discovered but not finished, and
 - black for finished.

Example

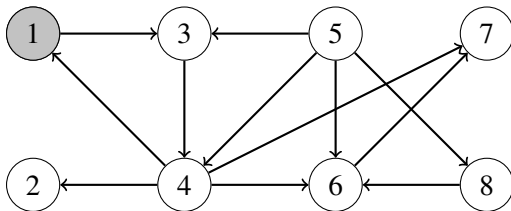


- Initialize: source vertex grey, others white; source discovered at time 1.
- Repeat:
 - Increment the time.
 - If there is a white neighbour of the current vertex, then it is coloured grey and its discovery time noted and it becomes current.
 - Else colour the current vertex black, note its finish time and return to its predecessor (**or jump** to an undiscovered vertex), or stop.

Example

	1	2	3	4	5	6	7	8
<i>d</i>	1							
<i>f</i>								

time= 1

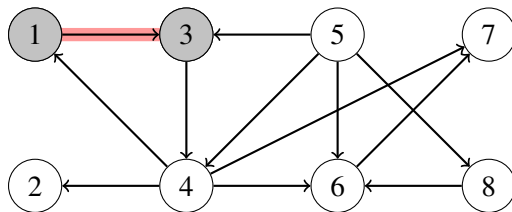


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Example

	1	2	3	4	5	6	7	8
d	1		2					
f								

time = 2

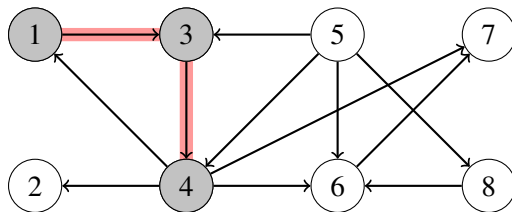


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Example

	1	2	3	4	5	6	7	8
<i>d</i>	1		2	3				
<i>f</i>								

time = 3

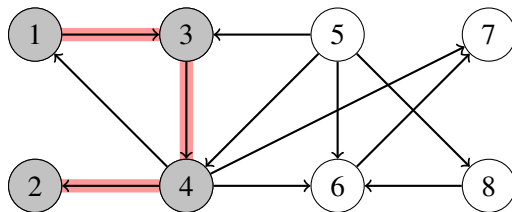


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3				
f								

time = 4

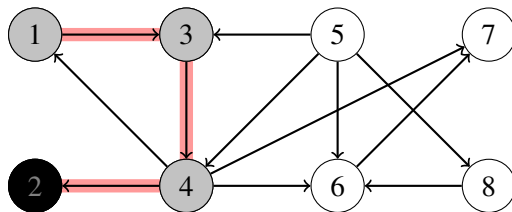


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3				
f		5						

time= 5

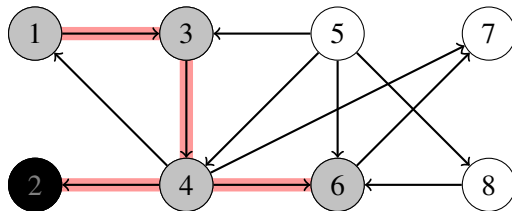


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6		
f		5						

time = 6

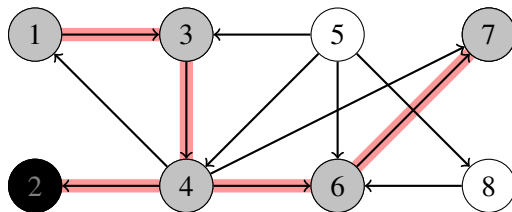


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6	7	
f		5						

time = 7

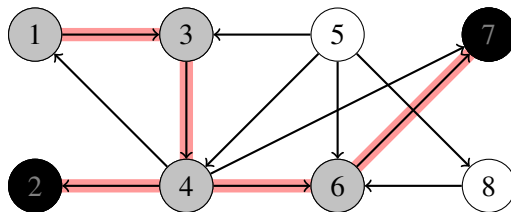


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6	7	
f		5					8	

time = 8

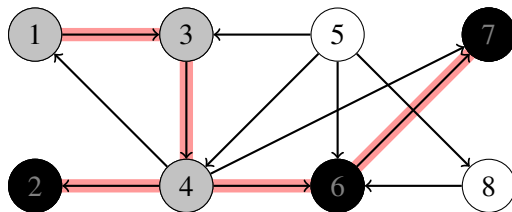


- Initialize: source vertex grey, others white; source discovered at time 1.
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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6	7	
f		5				9	8	

time = 9

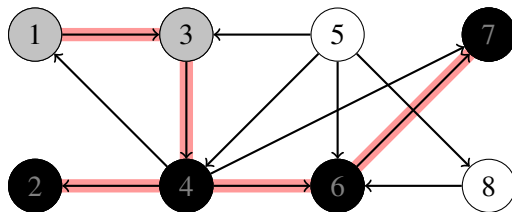


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6	7	
f		5		10		9	8	

time = 10

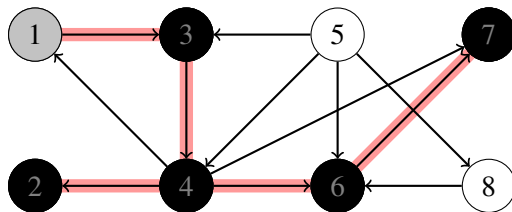


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6	7	
f		5	11	10		9	8	

time = 11

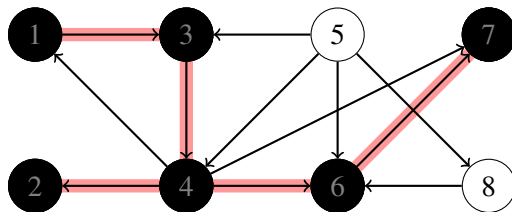


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3		6	7	
f	12	5	11	10		9	8	

time= 12

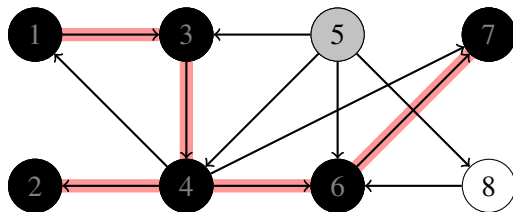


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3	13	6	7	
f	12	5	11	10		9	8	

time= 13

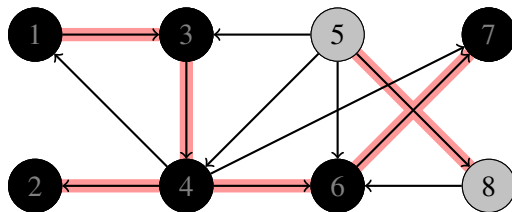


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3	13	6	7	14
f	12	5	11	10		9	8	

time = 14

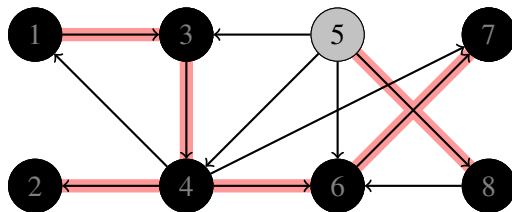


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Example

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d	1	4	2	3	13	6	7	14
f	12	5	11	10		9	8	15

time= 15

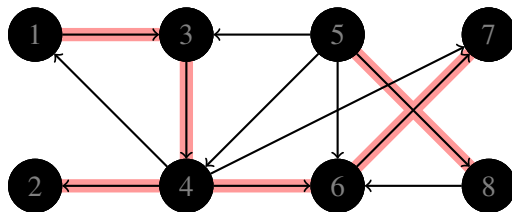


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Example

	1	2	3	4	5	6	7	8
d	1	4	2	3	13	6	7	14
f	12	5	11	10	16	9	8	15

time = 16



- Initialize: source vertex grey, others white; source discovered at time 1.
- Repeat:
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Depth-first search

DFS (G)

```
1 for each vertex  $u \in V[G]$ 
2   do colour[ $u$ ]  $\leftarrow$  WHITE
3      $\pi[u] \leftarrow$  NIL
4  $time \leftarrow 0$ 
5 for each vertex  $u \in V[G]$ 
6   do if colour[ $u$ ] = WHITE
7     then DFS-VISIT( $u$ )
```

DFS-VISIT(u)

```
1 colour[ $u$ ]  $\leftarrow$  GREY           [vertex  $u$  has just been discovered]
2  $time \leftarrow time + 1$ 
3  $d[u] \leftarrow time$ 
4 for each vertex  $v \in Adj[u]$        [explore edge  $(u, v)$ ]
5   do if colour[ $v$ ] = WHITE
6     then  $\pi[v] \leftarrow u$ 
7       DFS-VISIT( $v$ )
8 colour[ $u$ ]  $\leftarrow$  BLACK         [ $u$  has been processed]
9  $f[u] \leftarrow time \leftarrow time + 1$ 
```

Analysis

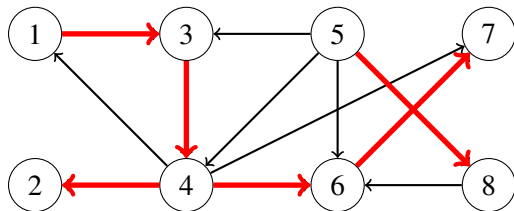
- Initialization takes time $O(V)$.
- Time $O(V)$ is spent on incrementing time, colouring vertices and updating d and f .
- Each vertex in each adjacency list is considered at most once. This takes time $O(E)$.
- Total time is $O(V + E)$.

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- Total time is $O(V + E)$.

The edges used for discovering new vertices form the depth-first tree (or forest). Again, we can find this with a [predecessor array](#).

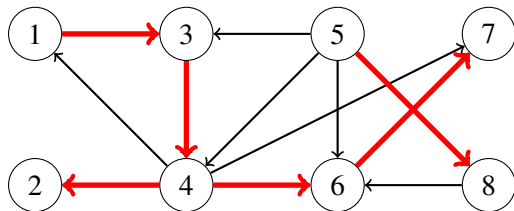
Example



Once we have run DFS on a **graph** we can construct the **predecessor subgraph**. This has the same vertex set as the graph, and for each vertex v there is an edge from the predecessor of v to v .

The predecessor subgraph is a **depth-first forest**.

Example



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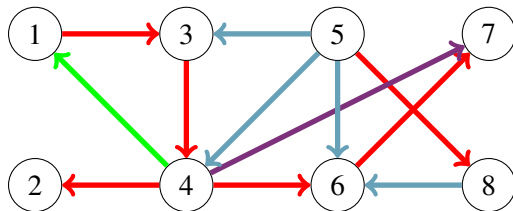
How many trees do we have in the above forest?

Classification of the edges

Once we have obtained a DFS-forest for a graph G , we can classify the edges of G .

- **Tree** edges are those edges in the DFS-forest.
- **Back** edges are edges that join a vertex to an ancestor.
- **Forward** edges are edges not in the tree that join a vertex to its descendant.
- **Cross** edges: all other edges.

Example



Tree edges →

Forward edges →

Back edges →

Cross edges →

Classification of the edges

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Let us refine the definition: suppose that e is an edge that joins a vertex u to its descendant v .

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Theorem

In an undirected graph, every edge is a tree edge or a back edge.

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Theorem

In an undirected graph, every edge is a tree edge or a back edge.

Questions: Which types of edges can be found in a directed graph?
What if BFS is used?