Lecture 13:
An Overview of First-Order Logic

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Outline

- Limitations of propositional logic
- Predicates
- Atomic formulae
- Quantifier-free formulae
- Quantifiers
- Summary

Predicates and atomic formulae

Whereas the fundamental building block in propositional logic is the propositional variable, within first-order logic it is the predicate (we have already been introduced to predicates when we studied relations).

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Given a predicate symbol P of arity r and some variables $x_1, x_2, \dots, x_r$ (where it might be the case that some of these variables are the same), the formula

True

$\{x\}$

$$P(x_1, x_2, \dots, x_r)$$

False

is an atomic formula of first-order logic.

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$$P(x_1, x_2, \ldots, x_r)$$

is an atomic formula of first-order logic.

In order to know whether this atomic formula is true or false, we need to be given an r-ary relation P', over some domain D, say, and values $v_1, v_2, \ldots, v_r$ from D for $x_1, x_2, \ldots, x_r$.

Atomic formula: an example

Suppose T is a ternary relation symbol. Then

$$T(x, y, x)$$

is an atomic formula.

unary 1

binary 2

ternary 3

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In this case, we write $(T', x = 6, y = 3) \models T(x, y, x)$ or sometimes $(\mathbb{N}, T', x = 6, y = 3) \models T(x, y, x)$.

Building formulae

Given some atomic formulae, we can build more complicated formulae from these atomic formulae by using the usual connectives of propositional logic, namely $\neg$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$. For example,

$T \Rightarrow F$

$$\underset{F}{E(\overset{3}{x_1}, \overset{2}{x_2})} \underset{\vee}{\vee} (\underset{T}{T(\overset{3}{x_1}, \overset{3}{x_1}, \overset{9}{x_3})} \underset{\Rightarrow}{\Rightarrow} \underset{F}{\underline{\neg E(\overset{2}{x_2}, \overset{9}{x_3})}})$$

is a formula of first-order logic, where E is a predicate symbol of arity 2, T is a predicate symbol of arity 3, and x_1, x_2 and x_3 are variables.

In order to interpret this formula, we need a binary relation for E, a ternary relation for T and values for x_1, x_2 and x_3. The domains of the relations for E and T must be the same.

Is the following interpretation true?
$E = \{(u_1, u_2) \in \mathbb{N}^2 : u_1 \leq u_2\}$, $T = \{(u_1, u_2, u_3) \in \mathbb{N}^3 : u_1 \cdot u_2 = u_3\}$
and $x_1 = 3$, $x_2 = 2$ and $x_3 = 9$. this is false

Building formulae

Not only do we allow formulae such as $P(x_1, x_2, \ldots, x_r)$ as atomic formulae but we are also allowed formulae of the form $x = y$, where x and y are variables (this constitutes all atomic formulae).

The semantics of $x = y$ is that this atomic formula is true only if the value of x is equal to the value of y (in an interpretation).

$P(x_1, x_2)$

$x_1 = 3, x_2 = 4, P' (x_1 = x_2) \models P(x_1, x_2)$

$\downarrow$

False

$(P', x_1 = 5, x_2 = 5) \to$ True.

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Example

Let E be a binary predicate symbol. Consider the formula
$(E(x, y) \wedge E(y, z)) \Rightarrow \neg(x = z)$
(we sometimes abbreviate $\neg(x = z)$ by $x \neq z$).
If E is interpreted as
$E = \{(x, y) \in \mathbb{N}^2 : x < y\}$ and $x = 5$, $y = 7$ and $z = 11$ then is the formula true in this interpretation?

$E(5,7) \wedge E(7,11) \Rightarrow \neg(5=7)$

True $\wedge$ True $\Rightarrow$ True

$\Downarrow$

True

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Formulae built from atomic formulae are called quantifier-free formula and the free variables are those variables appearing in a formula.

Quantifiers

Given a formula with free variables, we can now “quantify” over these variables using the universal quantifier (or the for-all quantifer) $\forall$ and the existential quantifier (or the exists quantifier) $\exists$.

$\forall x \in \mathbb{N} : x \% 2 == 0$ ✗

$\forall x \in \mathbb{N} : x \geqslant 0$

$\exists x \in \mathbb{N} : x \% 2 == 0$ ✓

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Suppose that $\phi(x)$ is a quantifier-free formula with one free variable x. Then $\forall x \phi(x)$ is a formula of first-order logic and has no free variables. The variable x is a bound variable in $\forall x \phi(x)$.

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Example

Suppose that Q is a unary relation symbol. Consider the formula $\forall x Q(x)$. Is it true for the following interpretations?

- Interpret Q as the relation $Q = \{u \in \mathbb{N} : u \text{ is even}\}$. False
- Interpret Q as the relation $Q = \{u \in \mathbb{N} : u \text{ is a square root}\}$. True

More complicated formulae

We can apply quantifiers to quantifier-free formula even when there is more than one free variable in the formula.

Let $\phi(x_1, x_2, \ldots, x_r)$ be a quantifier-free formula with free variables $x_1, x_2, \ldots, x_r$. Then the following are two examples of formulae of first order logic.

$$\forall x_1 \phi(x_1, x_2, \ldots, x_r) \qquad \exists x_3 \phi(x_1, x_2, \ldots, x_r)$$

$\wedge$ for all x_1, $\phi(\cdot\cdot)$ is true T T T T T T T $\Rightarrow$ T

$\vee$ There exists a x_3 such that $\phi(\cdots)$ is true

F F F F F T F F F F = T

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The first has free variables $x_2, x_3, \ldots, x_r$ and bound variable x_1; and the second has free variables $x_1, x_2, x_4, \ldots, x_r$ and bound variable x_3.

The interpretation of such formulae are as before except that relations and values for the free variables have to be supplied in order for any interpretation to make sense.

More complicated formulae: examples

- If $\phi(x)$ is the formula $\forall y(x = y \vee E(x,y))$ and $E = \{(u,v) \in \mathbb{N}^2 : u < v\}$ then

$$(E, x = 0) \models \phi(x)$$

but

$$(E, x = v) \models \neg\phi(x) \text{ whenever } v \neq 0$$

$(E, x=0) \models \phi(x)$?

$\forall y(y=0 \vee 0<y) = \mathbb{N}$, $\forall y(y=0 \vee y>0)$ ✓

$(E, x=2) \models \phi(x)$?

$\forall y(\underline{y=2 \vee y>2})\ \mathbb{N}$ what if $y=1$?

$y=2$: False. $y>2$: False.

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- If $\phi(x)$ is the formula $\exists y E(y, x)$ and $E = \{(u, v) \in \mathbb{N}^2 : u < v\}$ then we have

$$(E, x = 0) \models \neg\phi(x)$$

$\exists y E(y, 0), \mathbb{N}$: X

but

$$(E, x = v) \models \phi(x) \text{ whenever } v \neq 0$$

$\exists y E(y, v) \mathbb{N}$. ✓

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More complicated formulae

We can also apply quantifiers to formulae already involving quantifiers.

Consider the formula $\forall y(x = y \vee E(x, y))$. There is one free variable and we can quantify over this free variable; like this

$$\exists x \forall y(x = y \vee E(x, y))$$

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$$\exists x \forall y(x = y \vee E(x, y))$$

Let the binary relation $E = \{(u, v) \in \mathbb{N}^2 : u < v\}$.

For formula above to be true in this interpretation, we need that there exists some value $u \in \mathbb{N}$ for x such that for any value $v \in \mathbb{N}$ for y, we have that $u = v \vee E(u, v)$; that is, either $u = v$ or $u < v$.

$\exists x \forall y (x = y \vee E(x, y))$.

有一个 x such that $(x = y \vee E(x, y))$ is true for all y

$\forall y \exists x (x = y \vee E(x, y))$

每一个 y 都可以找到一个 x, S.T $(x = y \vee E(x, y))$

More complicated formulae

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$$\exists x \forall y(x = y \vee E(x, y))$$

$\exists x \forall y (x = y) \vee \exists x \forall y (x < y)$

Let the binary relation $E = \{(u, v) \in \mathbb{N}^2 : u < v\}$.

For formula above to be true in this interpretation, we need that there exists some value $u \in \mathbb{N}$ for x such that for any value $v \in \mathbb{N}$ for y, we have that $u = v \vee E(u, v)$; that is, either $u = v$ or $u < v$.

$\exists x \underline{\forall y (x < y)}$ $x = -100$ $x = -1000000000$

There clearly does exist such a value u, namely $u = 0$.

However, if $E = \{(u, v) \in \underline{\mathbb{Z}^2} : u < v\}$ then the formula is <u>false</u> as given any value for x, there is always some integer that is strictly less than this value for x.

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We can also build new formulae, using the usual propositional connectives, from existing formulae that involve quantifiers. Consider the formulae

$$\exists x \forall y (x = y \vee E(x, y)), \text{ and } \exists x \forall w (x = w \vee E(w, x)).$$

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$$\exists x \forall y (x = y \vee E(x, y)), \text{ and } \exists x \forall w (x = w \vee E(w, x)).$$

If we interpret E as $\{(u, v) \in \mathbb{N}^2 : u < v\}$ then is the following formula true?

True $\wedge$ False = False

$$\exists x \forall y (x = y \vee E(x, y)) \wedge \exists x \forall w (x = w \vee E(w, x)).$$

What if we interpret E as
$\{(u, v) \in \{0, 1, \ldots, 9\} \times \{0, 1, \ldots, 9\} : u < v\}$?

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False if $y=0$ False if $x=9$

$$\exists x \forall y (x = y \vee E(x, y)), \text{ and } \exists x \forall w (x = w \vee E(w, x)).$$

If we interpret E as $\{(u, v) \in \mathbb{N}^2 : u < v\}$ then is the following formula true?

$\exists x \forall x$ X wrong

$$\exists x \forall y (x = y \vee E(x, y)) \wedge \exists x \forall w (x = w \vee E(w, x)).$$

What if we interpret E as
$\{(u, v) \in \{0, 1, \ldots, 9\} \times \{0, 1, \ldots, 9\} : u < v\}$?

Notice how the same variable, x, is quantified twice in the same formula yet the two quantifications are entirely separate!

Summary

We have given an informal introduction to first-order logic and seen how we can use quantifiers to express statements that are not expressible in propositional logic.

However, there are some subtleties that we still have to deal with.

- How do we deal with the fact that the same variable can be quantified in more than one place in a formula?
- Can the same variable be both free and bound in a formula?
- We haven't as yet actually explicitly defined the syntax of first-order logic.
- We haven't as yet actually explicitly defined the semantics of first-order logic.
- We haven't as yet considered proof systems for first-order logic.

We shall address the above issues next.