Lecture 15: First-order Logic — Logical Equivalence

Barnaby Martin, 6 March 2021

Outline

- Logical equivalence (today).
- Some specific equivalences (today).
- Prenex normal form
- Resolution for first-order logic

Logical equivalence

Two formulae ϕ and ψ are logically equivalent if they are true for the same set of models, in which case we write $\phi \equiv \psi$.

D: {all men in the world}

P(x, y) : return true if x is y's Daddy

Q(x, y) : return true if x is older than y

$P(x,y) \equiv Q(x,y)$?

在一个A社会：年长的人都要叫爸爸

那么：$P(x,y) \equiv Q(x,y)$ 没问题

Logical equivalence

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$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$

All logical equivalences from propositional logic give rise to equivalences in first-order logic: for example, as

$\forall x \quad F(x, y)$

$$p \Rightarrow q \equiv \neg p \vee q, \text{ for any propositional variables } p \text{ and } q,$$

we must have that

$$\phi \Rightarrow \psi \equiv \neg\phi \vee \psi, \text{ for any first-order formulae } \phi \text{ and } \psi.$$

Logical equivalence

Note, however, that care must be taken as to exactly what an interpretation is when we "plug in" formulae as in the previous example: if

Predicate $\to$ T, $\to$ F

- ϕ is over the signature consisting of the binary relation symbol E and the constant symbol C
- ψ is over the signature consisting of the binary relation symbol E and the ternary relation symbol M

then an interpretation for $\neg\phi \vee \psi$ is over the signature consisting of the symbols E, C, and M.

ϕ: Human, eat_carrot $\to$ ~~John~~

ψ: Human, do_back_flip $\to$ ~~John~~

$\neg\phi \vee \psi$: predicate over 1 Human, 2, eat_carrot, 3. back_flip

Some tricks: renaming variables

Consider some first-order formula of the form $\forall x \phi(x)$ where y does not appear in $\phi(x)$.

$\forall x \phi(x)$: $\forall x\, \phi(x_1, x_2, \ldots x \ldots x_r)$ is true

all values of x makes $\phi(x_1, x_2 \ldots x \ldots x_r)$ True

$\phi(x_1, x_2 \ldots y \ldots y \ldots x_r)$ ✗ 不可以 rename.

$\phi(x_1) \wedge \phi(x_2) \wedge \phi(x_n) \wedge \wedge \ldots$

$\forall x$: Universal Quantifier $\quad \forall x\, \phi(x) = \text{True}$

$\psi(x_1) \vee \psi(x_2) \ldots \psi(x_n) = \text{True}$

$\exists x$: Existential Quantifier $\quad \exists x\, \psi(x)$

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Some tricks: renaming variables renaming samples

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- Let I be some interpretation for $\forall x \phi(x)$ in which $\forall x \phi(x)$ is true.

M = all human

$\phi(x)$: True when x is a boy, $x \in M$

$\phi(y)$: True when y is a boy. $y \in M$

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- Let I be some interpretation for $\forall x \phi(x)$ in which $\forall x \phi(x)$ is true.
- For every value u in the domain of I, we have that $(I, x = u) \models \phi(x)$.

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- So, for every value u in the domain of I, we have that $(I, y = u) \models \phi(y)$.
- Hence, I is an interpretation in which $\forall y \phi(y)$ is true.

Similarly, if I is an interpretation in which $\forall y \phi(y)$ is true then I is an interpretation in which $\forall x \phi(x)$ is true.

Some tricks: renaming variables

In general, and by the same reasoning, if ever we have some formula ϕ in which there is a quantification, $\forall x$, say, then we can replace

- every occurrence of x in the scope of this quantification with the variable y
- the quantification $\forall x$ by $\forall y$

so long as y does not appear in ϕ, without changing the semantics.

$\forall x\, \phi(x, \underline{y}, z\ \ w) \Rightarrow$ some ~~re~~ result.

$\forall y\, \phi(y, y, z, w)$ ✗ $\Rightarrow$ ✓ $\forall a\, \phi(a, y, z, w)$

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Of course, the same can be said of $\exists x \phi(x)$ and, more generally, any formula containing a quantification $\exists x$.

But, consider the formula $\exists x E(x, y)$.

If we simply replace x with y and $\exists x$ with $\exists y$ then we get $\exists y E(y, y)$ which is semantically very different from $\exists x E(x, y)$.

Some tricks: substitution

Consider some formula ϕ in which there is contained a sub-formula ψ.

Suppose further that ψ has free variables $x_1, x_2, \ldots, x_k$.

If ψ is logically equivalent to a formula $\chi(x_1, x_2, \ldots, x_k)$ then we can replace ψ in ϕ with the formula χ and not change the semantics.

$p \wedge q \Rightarrow \psi(x_1, x_2, x_3)$, $\psi \equiv \chi$

$\neg \chi(x_1, x_2, x_3)$ Modus Tollens (M.T.)

$\neg p \wedge q$ sub $\chi(x_1 x_2 x_3)$ with $\psi(x_1 x_2 x_3)$

Some common equivalences

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Let I be some interpretation for $\neg\forall x \phi$. We have that:

- $I \models \neg\forall x \phi$

$\neg\forall x \phi(x) : \neg\forall x \phi$

Interpretation:

$\phi'(x)$: return True if x is a ~~human~~ human.

$x \in M$, M | all bird

$I: M, \phi(x)$ $I \models \neg\forall x \phi$

is an interpretation

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Let I be some interpretation for $\neg\forall x\phi$. We have that:

- $I \models \neg\forall x\phi$ for all x in I, at least one will be a false

  if and only if it is not the case that $I \models \forall x\phi$

  if and only if it is not the case that for every value u in the domain of I, we have that $\phi(u)$ holds in I

  if and only if there exists some value u in the domain of I such that $\neg\phi(u)$ holds in I

  if and only if $I \models \exists x\neg\phi$.

($\phi(u)$ is shorthand for saying that x is to be interpreted as u.)

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($\phi(u)$ is shorthand for saying that x is to be interpreted as u.)

So, for every first-order formula $\phi(x)$

$$\neg\forall x \phi \equiv \exists x \neg\phi.$$

Some common equivalences

Consider the formula $\neg\exists x \phi$, where $\phi(x)$ is a first-order formula with free variable x.

Let I be some interpretation for $\neg\exists x \phi$. We have that:

- $I \models \neg\exists x \phi$

  There does not exist an x from I that will make $\phi(x)$ to be true.

Some common equivalences

Consider the formula $\neg\exists x\phi$, where $\phi(x)$ is a first-order formula with free variable x.

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  if and only if $I \models \forall x\neg\phi$.

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  u in the domain of I such that $\phi(u)$ holds in I

  if and only if for every value u in the domain of I, we have

  that $\neg\phi(u)$ holds in I

  if and only if $I \models \forall x \neg\phi$.

So, for every first-order formula $\phi(x)$:

$$\neg\exists x \phi \equiv \forall x \neg\phi$$

Some common equivalences

General rule: negations can be "pushed through" universal quantifiers if we change the universal quantifier to an existential quantifier.

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Example

Consider the formula $\neg\exists x\forall y(\neg E(x,y) \vee M(y,y,z,x))$. We have

$\neg\exists x\forall y\,\phi$

$$\begin{aligned}
\neg\exists x\underline{\forall y(\neg E(x,y) \vee M(y,y,z,x))} &\equiv \forall x\underline{\neg\forall y(\neg E(x,y) \vee M(y,y,z,x))} \\
&\equiv \forall x\exists y\neg(\neg E(x,y) \vee M(y,y,z,x)) \\
&\equiv \forall x\exists y(E(x,y) \wedge \neg M(y,y,z,x))
\end{aligned}$$

More complicated equivalences

Consider $\forall x \phi \wedge \exists y \psi$, where $\phi(x)$ and $\psi(y)$ are first-order formulae with free variables x and y, respectively.

By renaming bound variables (if necessary), we may assume that x does not appear in ψ and y does not appear in ϕ.

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Let I be some interpretation for $\forall x \phi \wedge \exists y \psi$.

We have that $I \models \forall x \phi \wedge \exists y \psi$ if and only if $I \models \forall x \phi$ and $I \models \exists y \psi$:

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- $I \models \forall x \phi$ if and only if no matter which value from the domain of I we give to the variable x, we have that $\phi(x)$ holds in I.
- $I \models \exists y \psi$ if and only if there exists some value from the domain of I for the variable y such that $\psi(y)$ holds in I.

More complicated equivalences continued

Thus, $I \models \forall x \phi \wedge \exists y \psi$ iff:

no matter which value we give to x, we have that $\phi(x)$ holds in I, and there exists some value for y such that $\psi(y)$ holds in I.

More complicated equivalences continued

Thus, $I \models \forall x \phi \land \exists y \psi$ iff:

no matter which value we give to x, we have that $\phi(x)$ holds in I, and there exists some value for y such that $\psi(y)$ holds in I.

Consider $\forall x \exists y (\phi \land \psi)$.

More complicated equivalences continued

Thus, $I \models \forall x \phi \wedge \exists y \psi$ iff:

no matter which value we give to x, we have that $\phi(x)$ holds in I, and there exists some value for y such that $\psi(y)$ holds in I.

Consider $\forall x \exists y (\phi \wedge \psi)$.

Suppose that $I \models \forall x \exists y (\phi \wedge \psi)$.
Choose *any* u for x. There exists a v for y such that $\phi(u) \wedge \psi(v)$ holds.
So, $I \models \forall x \phi \wedge \exists y \psi$.

More complicated equivalences continued

Hence, $\forall x \phi \wedge \exists y \psi \equiv \forall x \exists y (\phi \wedge \psi)$.

More complicated equivalences continued

Hence, $\forall x \phi \wedge \exists y \psi \equiv \forall x \exists y(\phi \wedge \psi)$.

Indeed, by the same token, $I \models \forall x \phi \wedge \exists y \psi$ if and only if $I \models \exists y \forall x(\phi \wedge \psi)$.

$\forall$: for all

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unit formula

uniformula.

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General rule: quantifications can be "pulled out" from inside logical connectives and the order of the quantifiers doesn't matter, so long as the names of the quantified variables are not used elsewhere.

$\forall x \exists y \neq \exists y \forall x$

$\forall x \exists y P(x, y)$: $y < x$

$\exists y \forall x P(x, y)$: X

$P(x, y)$: x and y

Some more complicated equivalences

Example

If we assume that

- x does not appear in ψ and χ,
- y does not appear in ϕ and χ,
- z does not appear in ϕ and ψ,

applying this general rule yields

$$\begin{aligned}(\forall x\phi \land \exists y\psi) \lor \forall z\chi &\equiv \forall x\exists y(\phi \land \psi) \lor \forall z\chi \\ &\equiv \forall x\exists y\forall z((\phi \land \psi) \lor \chi)\end{aligned}$$

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Consider the formula $(\forall x \phi \vee \forall x \psi) \wedge \exists x \chi$.

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Consider the formula $(\forall x\phi \vee \forall x\psi) \wedge \exists x\chi$.

We can rename two of the bound occurrences of x to get

$(\forall x \phi(x) \vee \forall x \psi(x)) \wedge \exists x \chi(x)$

$$(\forall x\phi(x) \vee \forall y\psi(y)) \wedge \exists z\chi(z)$$

(assuming y and z do not appear in ψ and χ, respectively).

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Now we get the equivalent formulae

$$\begin{aligned}
&(\forall x \phi(x) \vee \forall y \psi(y)) \wedge \exists z \chi(z) \\
&\qquad \equiv \quad \forall x \forall y (\phi(x) \vee \psi(y)) \wedge \exists z \chi(z) \\
&\qquad \equiv \quad \forall x \forall y \exists z (\phi(x) \vee \psi(y) \wedge \chi(z))
\end{aligned}$$

Be careful when applying general rules

Great care has to be taken when manipulating quantifiers:

- the order of quantification matters
- consider other occurrences of a quantified variable outside the scope.

Be careful when applying general rules

Example

Consider the first-order sentence $\forall x \exists y E(x, y)$.

Let I be the interpretation with domain $\{1, 2, 3, 4\}$ where
$E = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$.

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Consider the first-order sentence $\forall x \exists y E(x, y) \land \forall z \neg E(z, z)$.

Whilst $I \models \forall x \exists y E(x, y) \land \forall z \neg E(z, z)$

$I \models \forall z \forall x \exists y (E(x, y) \land \neg E(z, z))$

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$I \models \forall x \forall z \exists y (E(x, y) \wedge \neg E(z, z))$

it is not the case that $I \models \forall z \exists y \forall x (E(x, y) \wedge \neg E(z, z))$.

More on bound occurrences

Consider the first-order formula $\forall x \exists y E(x, y) \wedge \exists x U(x)$.

It does not make sense to pull the quantifiers out, as we would get $\forall x \exists y \exists x (E(x, y) \wedge U(x))$.

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$$\exists y \exists x(E(x, y) \wedge U(x))$$

(as the existentially quantified x "overwrites" the universally quantified x) which is certainly not equivalent to the sentence we started with. To see this, consider the interpretation where the domain is $\{1, 2\}$, $E = \{(1, 2)\}$ and $U = \{1\}$.

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Consider the first-order formula $\forall x \exists y E(x, y) \land \exists x U(x)$.

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We need to ensure that the two original bound occurrences of x have "nothing to do with each other". In order to ensure this, we rename one of them:

$$\begin{aligned} \forall x \exists y E(x, y) \land \exists x U(x) &\equiv \forall x \exists y E(x, y) \land \exists z U(z) \\ &\equiv \forall x \exists y \exists z(E(x, y) \land U(z)) \end{aligned}$$