Computational Thinking

Logic

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Lecture 5

More on Natural Deduction for Propositional Logic

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More rules

p is true   q True False.

$p \vee q$ = True

$\Rightarrow p \vee q \vee w \vee v \vee z \vee a \vee b \vee c$

- Rules for introducing disjunction

$$\frac{\varphi}{\varphi \vee \psi}\ \vee\text{i1} \qquad\qquad \frac{\varphi}{\psi \vee \varphi}\ \vee\text{i2}$$

$\vee$-introduction

we usually just write $\vee$i

- Rule for eliminating disjunction

Tru

$$\frac{\varphi \vee \psi \qquad \boxed{\begin{matrix}\varphi \\ \vdots \\ \chi\end{matrix}} \qquad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\ \vee\text{e}$$

$\vee$-elimination

- In order to apply the rule $\vee$e, we uses boxes as previously
  - but now there is a box starting with each given disjunct, φ and ψ
  - each box needs to end with the same intended formula, χ.

A proof using $\vee$-elimination

$q \Rightarrow r$ $(q \vee a \vee b \vee c) \Rightarrow (r \vee a \vee b \vee c)$

- Here is a proof of the sequent $q \Rightarrow r \vdash (p \vee q) \Rightarrow (p \vee r)$

1.	$q \Rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee r$	$\vee$i 3
5.	q	assumption
6.	r	$\Rightarrow$e 1 5
7.	$p \vee r$	$\vee$i 6
8.	$p \vee r$	$\vee$e 2-7
9.	$(p \vee q) \Rightarrow (p \vee r)$	$\Rightarrow$i 2-8

A proof using $\vee$-elimination

- Here is a proof of the sequent $p \vee (q \vee r) \vdash (p \vee q) \vee r$

1.	$p \vee (q \vee r)$	premise
2.	p	assumption
3.	$p \vee q$	$\vee$i 2
4.	$(p \vee q) \vee r$	$\vee$i 3
5.	$q \vee r$	assumption
6.	q	assumption
7.	$p \vee q$	$\vee$i 6
8.	$(p \vee q) \vee r$	$\vee$i 7
9.	r	assumption
10.	$(p \vee q) \vee r$	$\vee$i 9
11.	$(p \vee q) \vee r$	$\vee$e 5-10
12.	$(p \vee q) \vee r$	$\vee$e 2-11

More rules

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- Rules for negation

$$\frac{\bot}{\varphi}\ \bot e \qquad\qquad \frac{\varphi \qquad \neg\varphi}{\bot}\ \neg e$$

$\bot$-elimination — $\neg$-elimination

- The symbol $\bot$, known as bottom, represents a contradiction
  - in natural deduction if one has a contradiction then one can infer *any* formula.
- Rule for introducing negation

$$\frac{\boxed{\begin{array}{c}\varphi \\ \cdots \\ \bot\end{array}}}{\neg\varphi}\ \neg i$$

$\neg$-introduction

A proof using the rules for negation

$\vdash$

$x \Rightarrow (y \Rightarrow z) \wedge x \wedge \neg z \Rightarrow \neg y$

- Here is a proof of the sequent $x \Rightarrow (y \Rightarrow z), x, \neg z \vdash \neg y$.

1.	$x \Rightarrow (y \Rightarrow z)$	premise
2.	x	premise
3.	$\neg z$	premise
4.	y	assumption
5.	$y \Rightarrow z$	$\Rightarrow$e 1 2
6.	z	$\Rightarrow$e 4 5
7.	$\perp$	$\neg$e 3 6
8.	$\neg y$	$\neg$i 4-7

Another proof using rules for negation

- Here is a proof of the sequent $x \vee \neg y \vdash y \Rightarrow x$.

1.	$x \vee \neg y$	premise
2.	x	assumption
3.	y	assumption
4.	x	copy 2
5.	$y \Rightarrow x$	$\Rightarrow$i 3-4
6.	$\neg y$	assumption
7.	y	assumption
8.	$\perp$	$\neg$e 6 7
9.	x	$\perp$e 8
10.	$y \Rightarrow x$	$\Rightarrow$i 7-9
11.	$y \Rightarrow x$	$\vee$e 1 2-5 6-10

$\neg y \vee x \quad y \Rightarrow x$

$y \Rightarrow x \vdash y \Rightarrow x$ proved

$a \Rightarrow b$

a	b	$a \Rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

True $\Rightarrow \psi \Leftrightarrow$ True.

$\psi \Rightarrow$ True is true

A derived rule

- We can derive other rules in natural deduction.
- Consider modus tollens $\varphi \Rightarrow \psi, \neg\psi \vdash \neg\varphi$.

1.	$\varphi \Rightarrow \psi$	premise
2.	$\neg\psi$	premise
3.	φ	assumption
4.	ψ	$\Rightarrow$e 1 3
5.	$\perp$	$\neg$e 2 4
6.	$\neg\varphi$	$\neg$i 3-5

- Note that we can use derived rules just as if they were rules of natural deduction
  - e.g., in a proof with
    - a line reading $\varphi \Rightarrow \psi$
    - and another line reading $\neg\psi$

    we could immediately infer $\neg\varphi$
    - and write "modus tollens" or "MT" as an explaining remark.

More derived rules

$\neg\varphi \Rightarrow \bot \vdash \varphi$

- Reductio ad absurdum or proof by contradiction is the principle
  - "if from $\neg\varphi$ I can prove $\bot$ then I can deduce φ".
- Here is a proof that this principle can be applied in natural deduction.

1.	$\neg\varphi \Rightarrow \bot$	premise
2.	$\boxed{\neg\varphi}$	assumption
3.	$\bot$	$\Rightarrow$e 1 2
4.	$\neg\neg\varphi$	$\neg$i 2-3
5.	φ	$\neg\neg$e 4

- We denote reductio ad absurdum by RAA.

More derived rules

- The law of the excluded middle states that either φ is true or $\neg\varphi$ is true.
- Here is a proof of it.

1.	$\neg(\varphi \vee \neg\varphi)$	assumption
2.	φ	assumption
3.	$\varphi \vee \neg\varphi$	$\vee$i 2
4.	$\bot$	$\neg$e 1 3
5.	$\neg\varphi$	$\neg$i 2-4
6.	$\varphi \vee \neg\varphi$	$\vee$i 5
7.	$\bot$	$\neg$e 1 6
8.	$\neg\neg(\varphi \vee \neg\varphi)$	$\neg$i 1-7
9.	$\varphi \vee \neg\varphi$	$\neg\neg$e 8

$\cancel{\varphi \vee \neg\varphi \equiv \text{True}}$

$\vdash \varphi \vee \neg\varphi \Rightarrow \text{True}$

Theorem

- We denote the law of the excluded middle by LEM.

Some facts about Natural Deduction

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human language is ~~ambiguous~~ ambiguous

$\vee$ $\neg$ $\wedge$ $\Rightarrow$ $\Leftrightarrow$ RAA LEM . . .

- Natural deduction is sound and complete.
- Let $\varphi_1, \varphi_2, \ldots, \varphi_m$ and ψ be formulae.
- Soundness
  - if the sequent $\varphi_1, \varphi_2, \ldots, \varphi_m \vdash \psi$ is provable
    - then the formula $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_m \Rightarrow \psi$ is a tautology.
- Completeness
  - if $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_m \Rightarrow \psi$ is a tautology
    - then the sequent $\varphi_1, \varphi_2, \ldots, \varphi_m \vdash \psi$ is provable.
- A theorem is a formula ψ for which the sequent $\vdash \psi$ is provable
  - thus, the soundness and completeness of natural deduction tells us that
    - every theorem is a tautology and every tautology is a theorem.

Proving theorems

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- Here is a proof that the sequent $(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$ is a theorem.

1.	$q \vee \neg q$	LEM
2.	q	assumption
3.	$\neg p \vee q$	$\vee$i 2
4.	p	assumption
5.	$\neg p \vee q$	copy 3
6.	$p \Rightarrow (\neg p \vee q)$	$\Rightarrow$i 4-5
7.	$(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$	$\vee$i 6
8.	$\neg q$	assumption
9.	p	assumption
10.	$\neg q$	copy 8
11.	$p \Rightarrow \neg q$	$\Rightarrow$i 9-10
12.	$(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$	$\vee$i 11
13.	$(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$	$\vee$e 1 2-7 8-12