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Digital Electronics

Boolean Algebra

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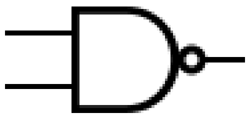
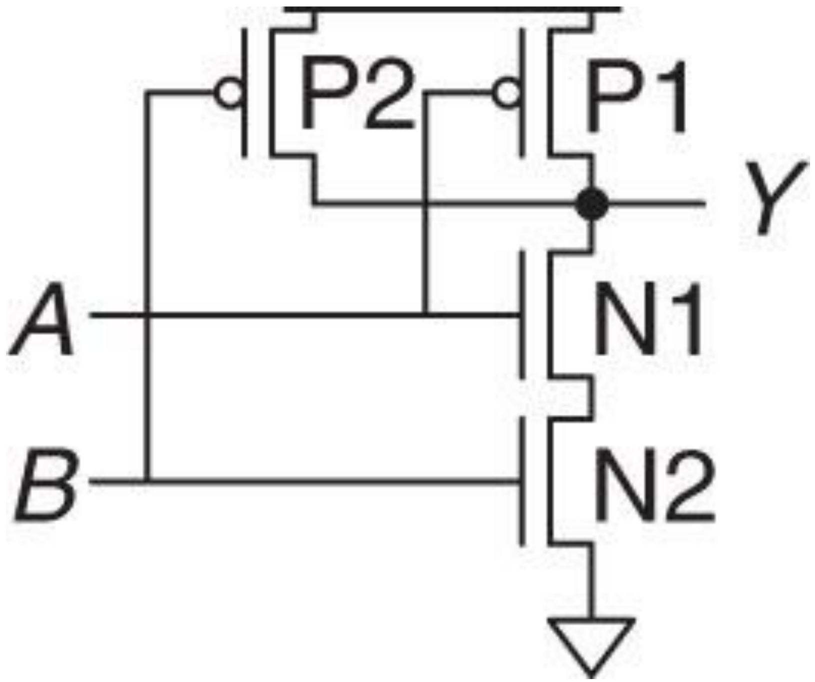
Overview of today's lecture

- Functionally complete sets
- Intro to Combinational Logic and Circuits
- Sum of Products vs Product of Sums
- Boolean Algebra
  - Axioms
  - Theorems

From transistors to gates

A **NAND** gate: 



A	B	P1	P2	N1	N2	Y
0	0	on	on	off	off	1
0	1	on	off	off	on	1
1	0	off	on	on	off	1
1	1	off	off	on	on	0

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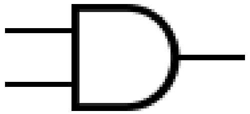
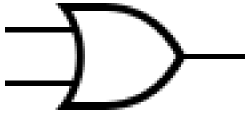
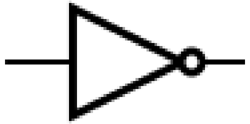
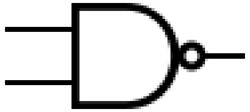
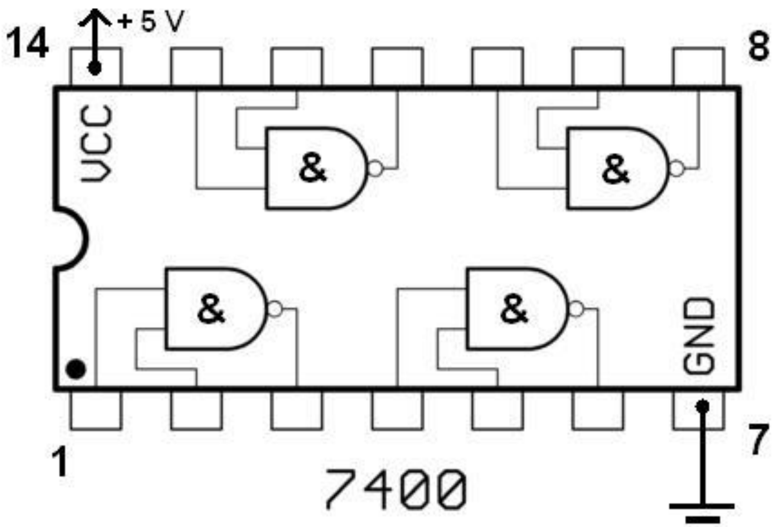
Gates we have seen

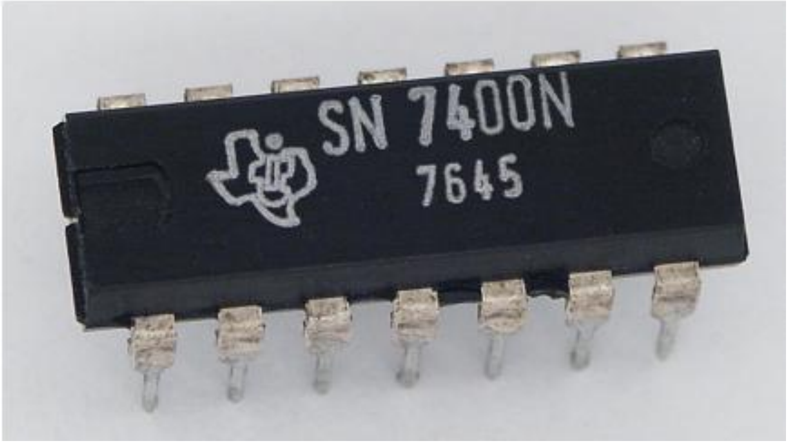
AND gate: 

OR gate: 

NOT gate: 

NAND gate: 





More Boolean operations

A	B	Y
0	0	**?**
0	1	**?**
1	0	**?**
1	1	**?**

There are 2^{2^k} possible Boolean operations on k inputs – *i.e.* 16 on 2 inputs.

The trivial operations are: 0, 1, A, B

We have seen $\overline{A}$, $\overline{B}$, A·B, A+B

What else is there?

Adding rule:

0 + 0 = 0

0 + 1 = 1

1 + 0 = 1

1 + 1 = 0   (with Carry)

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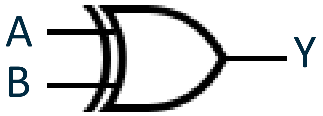
Truth tables: Exclusive OR (XOR)

- **XOR** gate: 

  Algebraic expression: $Y = A \oplus B$

A	B	Y
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**0**

Linear truth table

		B	
	AND	**0**	**1**
A	**0**	**0**	**1**
	1	**1**	**0**

Rectangular/Coordinate table

Exclusive OR (XOR)

We can construct XOR using AND, OR and NOT:

$$A \oplus B = (A + B) \bullet (\overline{A \bullet B})$$

Check:

A	B	Y
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**0**

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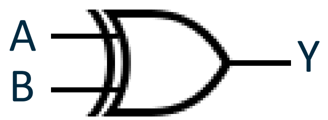
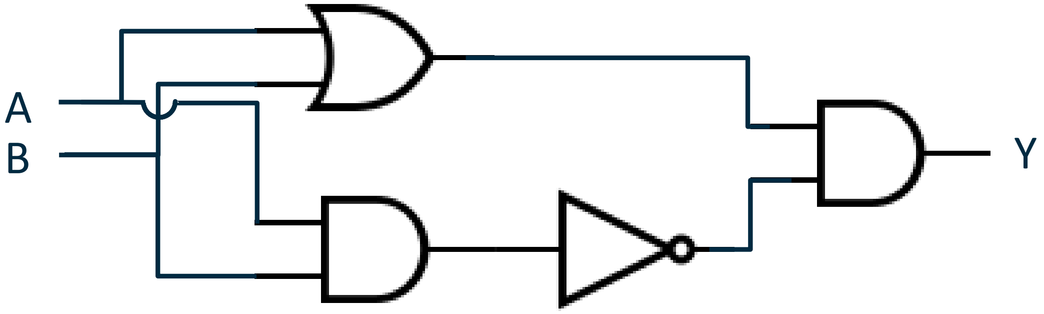
Exclusive OR (XOR)



is the same as



Functionally Complete Sets

In logic, a **functionally complete set of Boolean operators** is one which can be used to *express all possible truth tables* by combining members of the set into a Boolean expression.

AND

0	0	0
0	1	0
1	0	0
1	1	1

NOT

0	1
1	0

OR

0	0	0
0	1	1
1	0	1
1	1	0

XOR

NAND

NOR

$A \cdot \overline{B}$

0	1
0	1
0	1
1	0

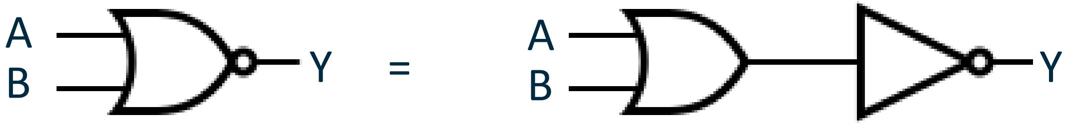
Functionally Complete Sets

Any logic circuit can be constructed with just these three operators

- **AND, OR,** and **NOT**
- They form a **functionally complete set.**

Charles Sanders Peirce (1880) showed that **NOR gates alone** form a functionally complete set.

NOR: inverse of OR, $Y = \overline{A + B}$



A	B	Y
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**0**

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NOR gates

AND: $$A \cdot B = \overline{\overline{(A+A)} + \overline{(B+B)}}$$

OR: $$A + B = \overline{\overline{(A+B)} + \overline{(A+B)}}$$

NOT: $$\overline{A} = \overline{A+A}$$

(A NOR A) NOR (B NOR B)

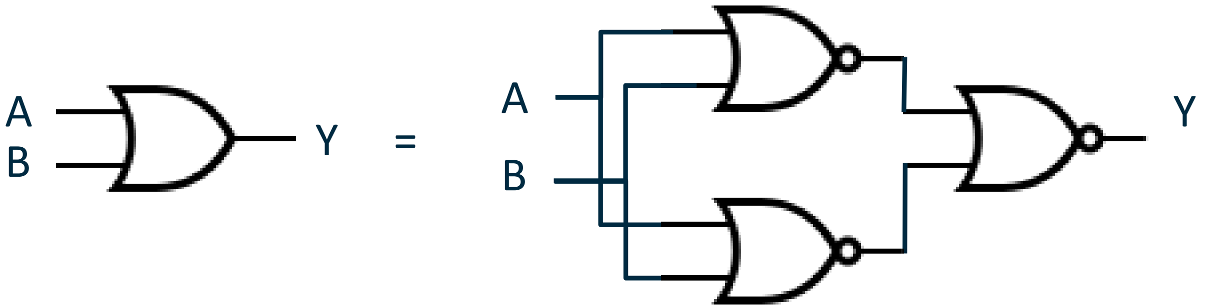
A NOR A

0	1
1	0

B NOR B

0	1
1	0

A	B	A+A (α)	B+B (β)	$\alpha + \beta$
0	0	1	1	0
0	1	1	0	0
1	0	0	1	0
1	1	0	0	1



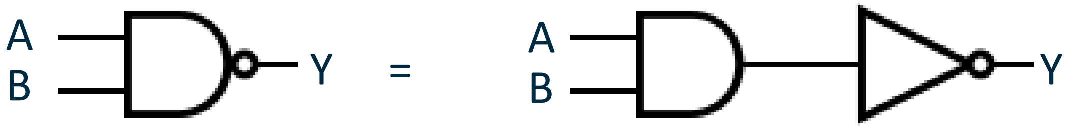
Functionally Complete Sets

Any logic circuit can be constructed with just these three operators

- **AND, OR,** and **NOT**
- They form a **functionally complete set.**

Henry M. Sheffer (1913) showed that the **NAND gates alone** form a functionally complete set.

NAND: inverse of AND, $Y = \overline{A \cdot B}$



A	B	Y
0	0	**1**
0	1	**1**
1	0	**1**
1	1	**0**

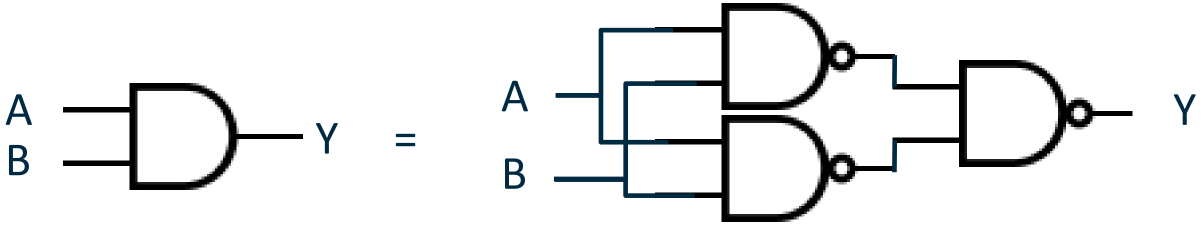
NAND gates

AND: $$A \cdot B = \overline{\overline{(A \cdot B)} \cdot \overline{(A \cdot B)}}$$

OR: $$A + B = \overline{\overline{(A \cdot A)} \cdot \overline{(B \cdot B)}}$$

NOT: $$\overline{A} = \overline{A \cdot A}$$

		α	β	γ	AND	OR	
A	B	A<B	A<A	B<B	α<α	β<γ	β
0	0						
0	1						
1	0						
1	1						



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Digital design principles

Digital design is all about **managing the complexity** of huge numbers of interacting elements. Some principles help humans do this:

Abstraction: Hiding details when they aren't important.

Discipline: Restricting design choices to make things easier to model, design and combine. E.g. the logic families and the digital abstraction.

The three –y's:

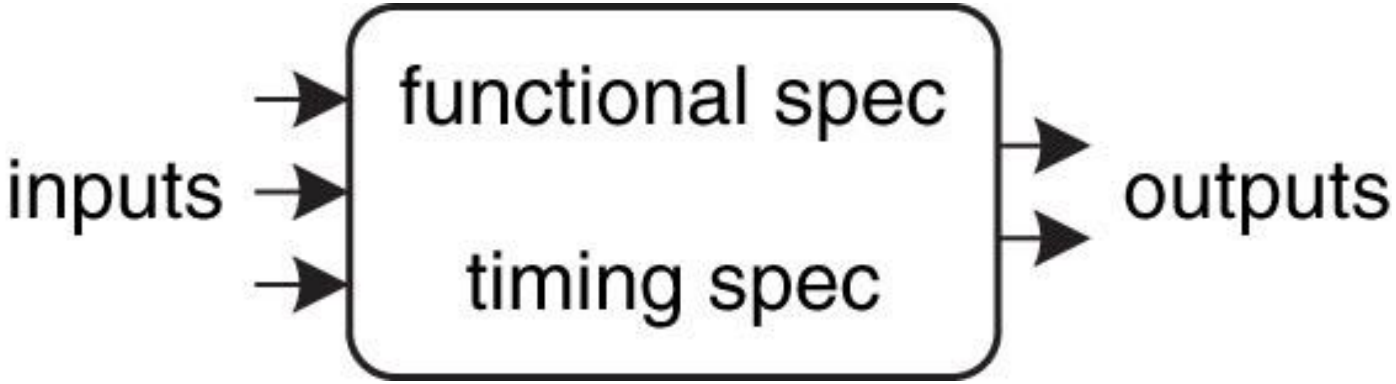
> **Hierarchy**: dividing a system into modules and submodules
>
> **Modularity**: well-defined functions and interfaces for modules
>
> **Regularity**: encouraging uniformity so modules can be swapped or reused.

Circuits

Network that processes discrete-valued variables.

A **circuit** has:

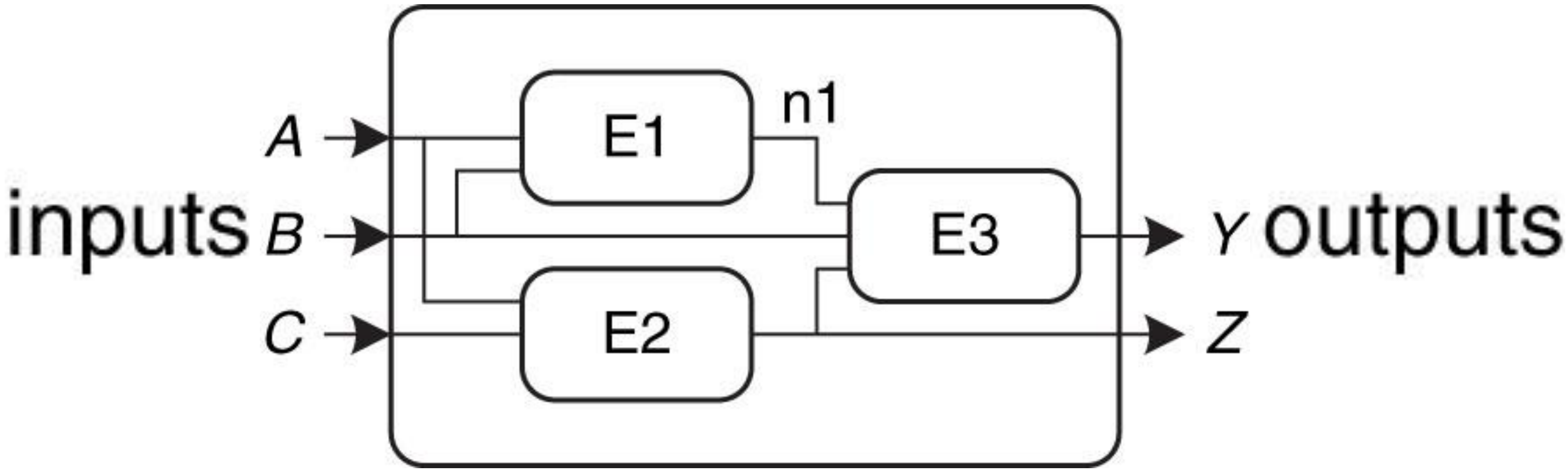
- one or more discrete valued input terminals
- one or more discrete valued output terminals
- a specification of the relationship between inputs and outputs
- a specification of the delay between inputs changing and outputs responding



Circuits

The circuit is made up of **elements** and **nodes:**

- An **element** is itself a circuit with inputs, outputs and specs.
- A **node** is a wire joining elements, whose voltage conveys a discrete valued variable.



Combinational logic

We wish to design **very large** circuits to perform functions for us.

We will restrict what we allow, for now, firstly to **combinational logic*** and circuits.

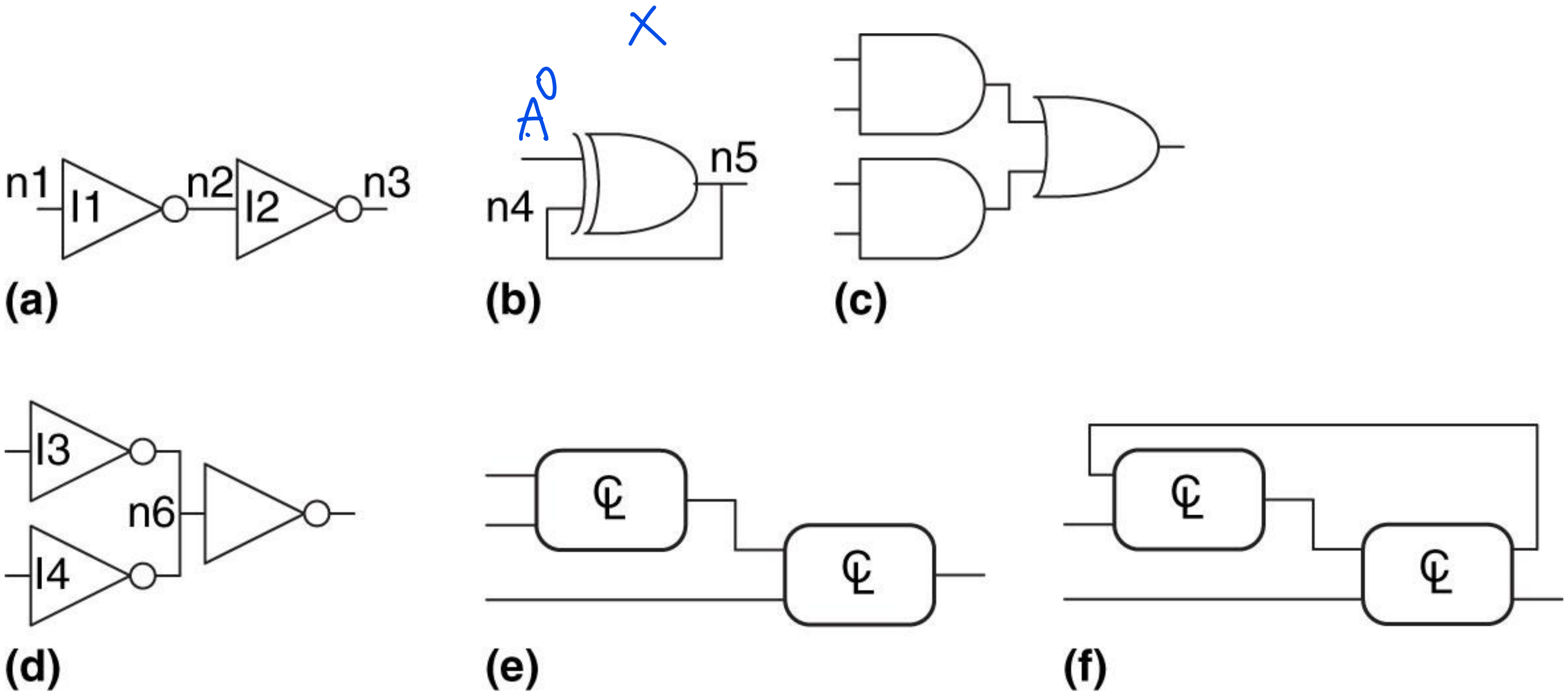
Combinational logic rules:

- **Individual gates** are combinational circuits.
- Every circuit **element** must be a combinational circuit.
- Every node is either an **input** to the circuit or connecting **to exactly one output** of a circuit element
- The circuit has **no cyclic paths** – every path through the circuit visits any node at most once.

*And later **sequential logic.**

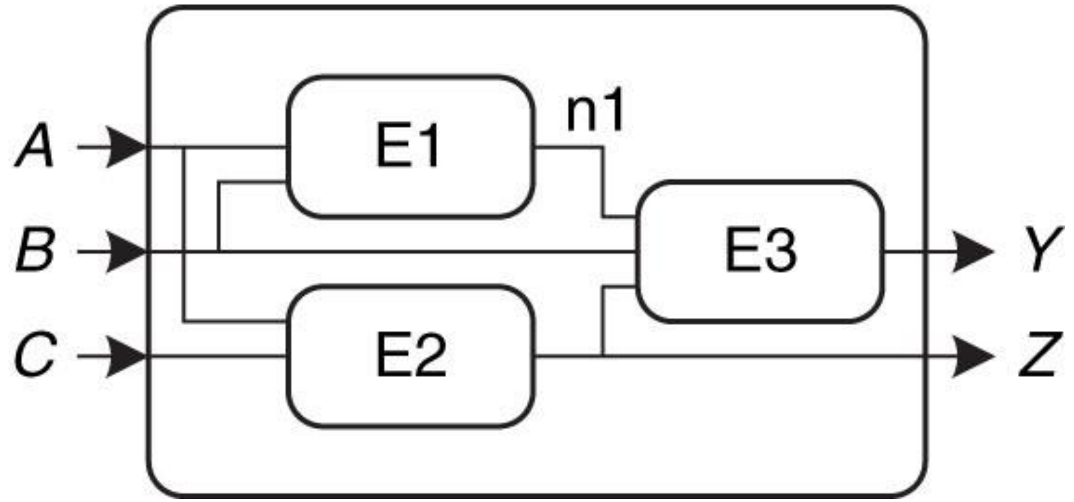
Combinational logic

Which of these are combinational circuits and why?



Boolean Algebra

- The algebra of 0/1 variables.
- Used for specifying the function of a combinational circuit
- Used to analyse and simplify the circuits required to give a specified truth table.



Boolean Algebra

$A \cdot A = A$

- **Variables** are represented by letters, e.g. $A, B, C \ldots$
  The **complement** or **inverse** of a variable is written with a bar, e.g. $\overline{A}$.

- A variable or its complement is called a **literal**, e.g. $A, \overline{A}, B$ or $\overline{B}$.

$AA\overline{B}C = A\overline{B}C$

A	A	A
0	0	0
1	1	1

- The AND of several literals is called a **product**, e.g. $A\overline{B}C$ or $A\overline{C}$,
  Products may be written $A \cdot B \cdot C$, ABC, $A \cap B \cap C$ or $A \wedge B \wedge C$.
  A **minterm** is a product in which **all** the inputs to a function appear once each (either in its complemented or uncomplemented form).

$1 \cdot 0 \cdot 0 = 0$

$1 + 1 + 0 \qquad 0 + 0 + 0 = 0$

- The OR of several literals is called a **sum** or **implicant**, e.g. $A + \overline{B} + C$ or $A + C$
  Sums may be written $A + B + C$, $A \cup B \cup C$ or $A \vee B \vee C$.
  A **maxterm** is a sum in which **all** the inputs to a function appear once each (either in its complemented or uncomplemented form).

Truth table to Boolean eqn.

X	Y	Z	F(X,Y,Z)
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Total Variable = 5

$\bar{X}(\bar{Y}\bar{Z}+YZ)$

2^5

$X(\bar{Y}Z+Y\bar{Z})$

$\bar{X}\bar{Y}\bar{Z}+\bar{X}YZ+X\bar{Y}Z+XY\bar{Z}$

$\begin{cases} 1 & \text{if my input is one of} \\ 0 & \end{cases}$

$(X+Y+\bar{Z})(\bar{X}+Y+\bar{Z})(X+\bar{Y}+\bar{Z})+(\bar{X}+\bar{Y}+\bar{Z})$

POS

$AB+AC = A(B+C)$

Truth table to Boolean eqn.

"Sum of products" form

Every Boolean expression can be written as minterms **ORed together**:

$$(A \cdot B \cdot C) + (A \cdot \overline{B} \cdot \overline{C}) + (\overline{A} \cdot B \cdot C)$$

"Product of sums" form

Also every Boolean expression can be written as maxterms **ANDed together**:

$$(\overline{A} + \overline{B} + \overline{C}) \cdot (\overline{A} + B + C) \cdot (A + B + \overline{C})$$

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Truth table to SOP

X	Y	Z	F(X,Y,Z)
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Each row can be represented by a minterm that is true:

$$\overline{X} \cdot \overline{Y} \cdot \overline{Z}$$

FGPA

$$\overline{X} \cdot Y \cdot \overline{Z}$$

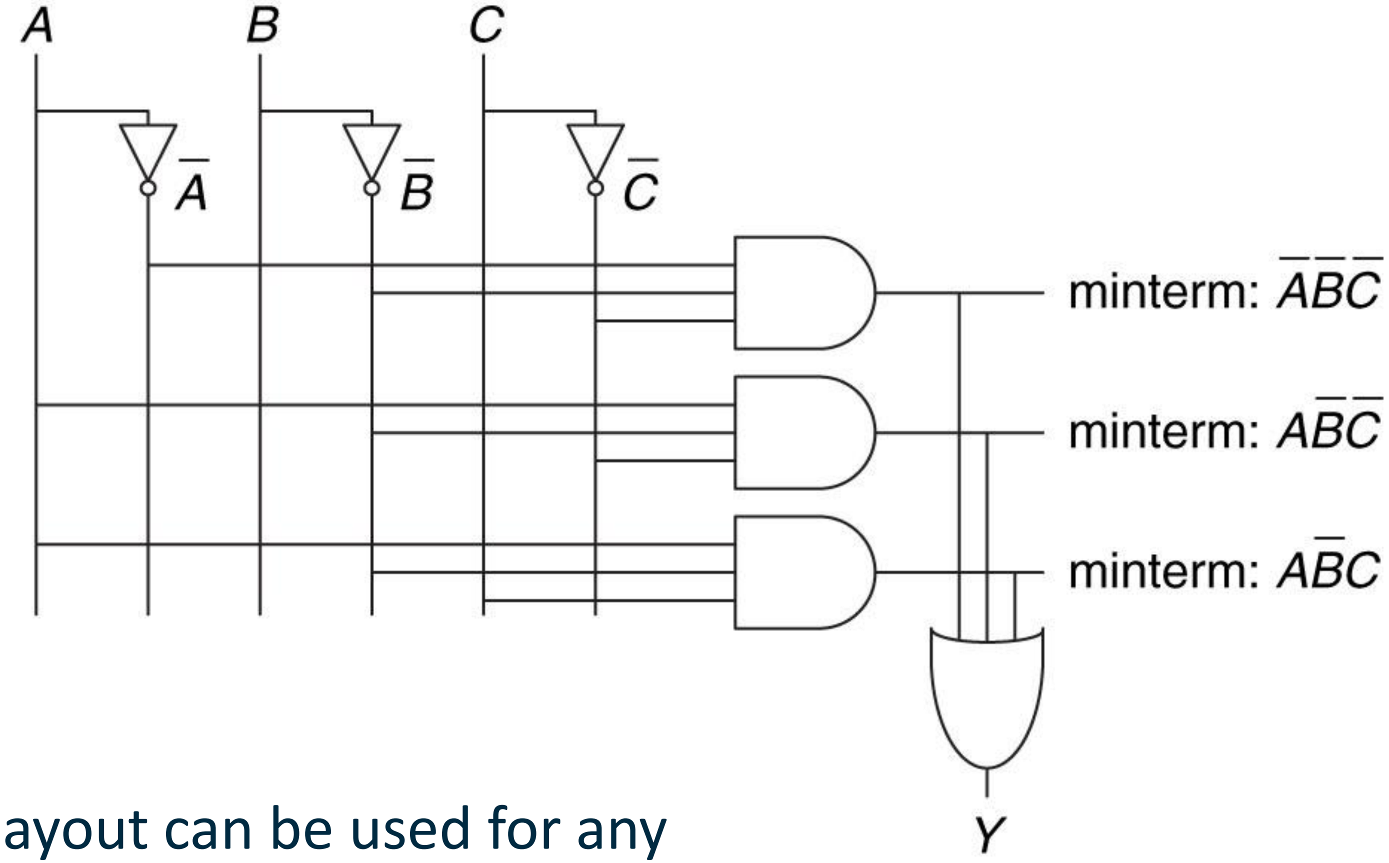
$$X \cdot \overline{Y} \cdot Z$$

OR together the **1 values** of the function, to give SOP form:

$$F(X,Y,Z) = \overline{X} \cdot \overline{Y} \cdot \overline{Z} + \overline{X} \cdot Y \cdot Z + X \cdot \overline{Y} \cdot Z + X \cdot Y \cdot \overline{Z}$$

Example

$$Y = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$$



This layout can be used for any sum-of-products expression.

Truth table to POS

X	Y	Z	F(X,Y,X)
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Each row can be represented by a maxterm that is false:

$$X+Y+Z$$

$$X+\overline{Y}+Z$$

$$\overline{X}+Y+\overline{Z}$$

AND together the **0 values** of the function, to give POS for F:

$$F(X,Y,Z)=(X+Y+\overline{Z})(X+\overline{Y}+Z)(\overline{X}+Y+Z)(\overline{X}+\overline{Y}+\overline{Z})$$

Compare to $F(X,Y,Z)=\overline{X}\cdot\overline{Y}\cdot\overline{Z}+\overline{X}\cdot Y\cdot Z+X\cdot\overline{Y}\cdot Z+X\cdot Y\cdot\overline{Z}$

Boolean Algebra

Two equivalent expression for the same logical formula:

$$F(X,Y,Z) = (X+Y+\overline{Z})(X+\overline{Y}+Z)(\overline{X}+Y+Z)(\overline{X}+\overline{Y}+\overline{Z})$$
$$F(X,Y,Z) = \overline{X}\cdot\overline{Y}\cdot\overline{Z} + \overline{X}\cdot Y\cdot Z + X\cdot\overline{Y}\cdot Z + X\cdot Y\cdot\overline{Z}$$

Which is simpler?

Is there another equivalent expression that is simpler than either?

We will use **Boolean algebra** and **Karnaugh maps** to produce the simplest equivalent expression that can then be turned into circuitry.

Axioms of Boolean Algebra

	Axiom		Dual axiom	Name
A1	$B = 0 \; if \; B \neq 1$	A1’	$B = 1 \; if \; B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2’	$\overline{1} = 0$	NOT
A3	$0 \cdot 0 \; = \; 0$	A3’	$1 + 1 \; = \; 1$	AND/OR
A4	$1 \cdot 1 \; = \; 1$	A4’	$0 + 0 \; = \; 0$	AND/OR
A5	$0 \cdot 1 \; = \; 1 \cdot 0 \; = \; 0$	A5’	$1 + 0 = 0 + 1 = 1$	AND/OR

Axioms cannot be proven – they are defined or assumed.
Each axiom has a dual obtained by interchanging AND and OR, and 0 and 1.

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Theorems of one variable

	Theorem		Dual theorem	Name
T1	$B \cdot 1 = B$	T1’	$B + 0 = B$	Identity
T2	$B \cdot 0 = 0$	T2’	$B + 1 = 1$	Null element
T3	$B \cdot B = B$	T3’	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$			Involution
T5	$B \cdot \overline{B} = 0$	T5’	$B + \overline{B} = 1$	Complements

Theorems can be proved by applying the axioms and checking cases

B	$B \cdot \overline{B}$
0	$0 \cdot 1 = 0$
1	$1 \cdot 0 = 0$

B	$B + \overline{B}$
0	$0 + 1 = 1$
1	$1 + 0 = 1$

Theorems of several variables

$B\cdot(B+C)$

$B\cdot B+B\cdot C$

$B+B\cdot C$

	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B+C = C+B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B+C)+D = B+(C+D)$	Associativity
T8	$B \bullet (C+D) = B \bullet C + B \bullet D$	$(B+C) \bullet (B+D) = B+(C \bullet D)$	Distributivity
T9	$B \bullet (B+C) = B$	$B+(B \bullet C) = B$	Covering
T10	$B\cdot C+B\cdot \overline{C} = B$	$(B+C)\cdot(B+\overline{C}) = B$	Combining
T11	$B\cdot C+\overline{B}\cdot D+C\cdot D = B\cdot C+\overline{B}\cdot D$		Consensus
T11’	$(B+C)\cdot(\overline{B}+D)\cdot(C+D) = (B+C)\cdot(\overline{B}+D)$		
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots} = \overline{B_0}+\overline{B_1}+\overline{B_2}\ldots$		De Morgan’s
T12’	$\overline{B_0+B_1+B_2 \ldots} = \overline{B_0}\cdot\overline{B_1}\cdot\overline{B_2}\ldots$		

$\overline{A\cdot B} = \bar{A}+\bar{B}$

$\overline{A\cdot B\cdot C\cdot D} = \bar{A}+\bar{B}+\bar{C}+\bar{D}$

Theorems of several variables

Theorem		Dual	Name
Key principle for simplification: use T10 and T11 to remove variables or terms; use others to rearrange so that T10 or T11 can be applied			
T10	$B \cdot C + B \cdot \overline{C} = B$	$(B + C) \cdot (B + \overline{C}) = B$	Combining
T11	$B \cdot C + \overline{B} \cdot D + C \cdot D = B \cdot C + \overline{B} \cdot D$		Consensus
T11'		$(B + C) \cdot (\overline{B} + D) \cdot (C + D) = (B + C) \cdot (\overline{B} + D)$	

General form of T10: for any implicant (i.e., product or sum) P and variable A, $PA + P\overline{A} = P$ $P(A+\overline{A}) = P \cdot 1 = P$

DeMorgan's Theorem

Proof of two variable case:

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

Proof:

A	B	$A \cdot B$	$\overline{A \cdot B}$	$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$
0	0	**0**	**1**	**1**	**1**	**1**
0	1	**0**	**1**	**1**	**0**	**1**
1	0	**0**	**1**	**0**	**1**	**1**
1	1	**1**	**0**	**0**	**0**	**0**

Example

Write out the sum of products form

A	B	C	Y
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Sum of products: $Y = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$

$\Rightarrow \bar{B}\bar{C} \quad \Rightarrow A\bar{B}$

Now minimise this equation.

$$\bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} = \bar{B}\bar{C}(\overbrace{\bar{A}+A}^{1}) = \bar{B}\bar{C}$$

$$Y = (\bar{A}+A)\bar{B}\bar{C} + A\bar{B}C$$
$$Y = (1)\bar{B}\bar{C} + A\bar{B}C$$
$$Y = \bar{B}\bar{C} + A\bar{B}C$$

Example

Write out the sum of products form

A	B	C	Y
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Sum of products: $Y = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + A\bar{B}\bar{C}$

$A\bar{B}\bar{C} = A\bar{B}\bar{C} + A\bar{B}\bar{C}$

Now minimise this equation.

$$Y = (\bar{A} + A)\bar{B}\bar{C} + A\bar{B}(C + \bar{C})$$

$$Y = \bar{B}\bar{C} + A\bar{B}$$

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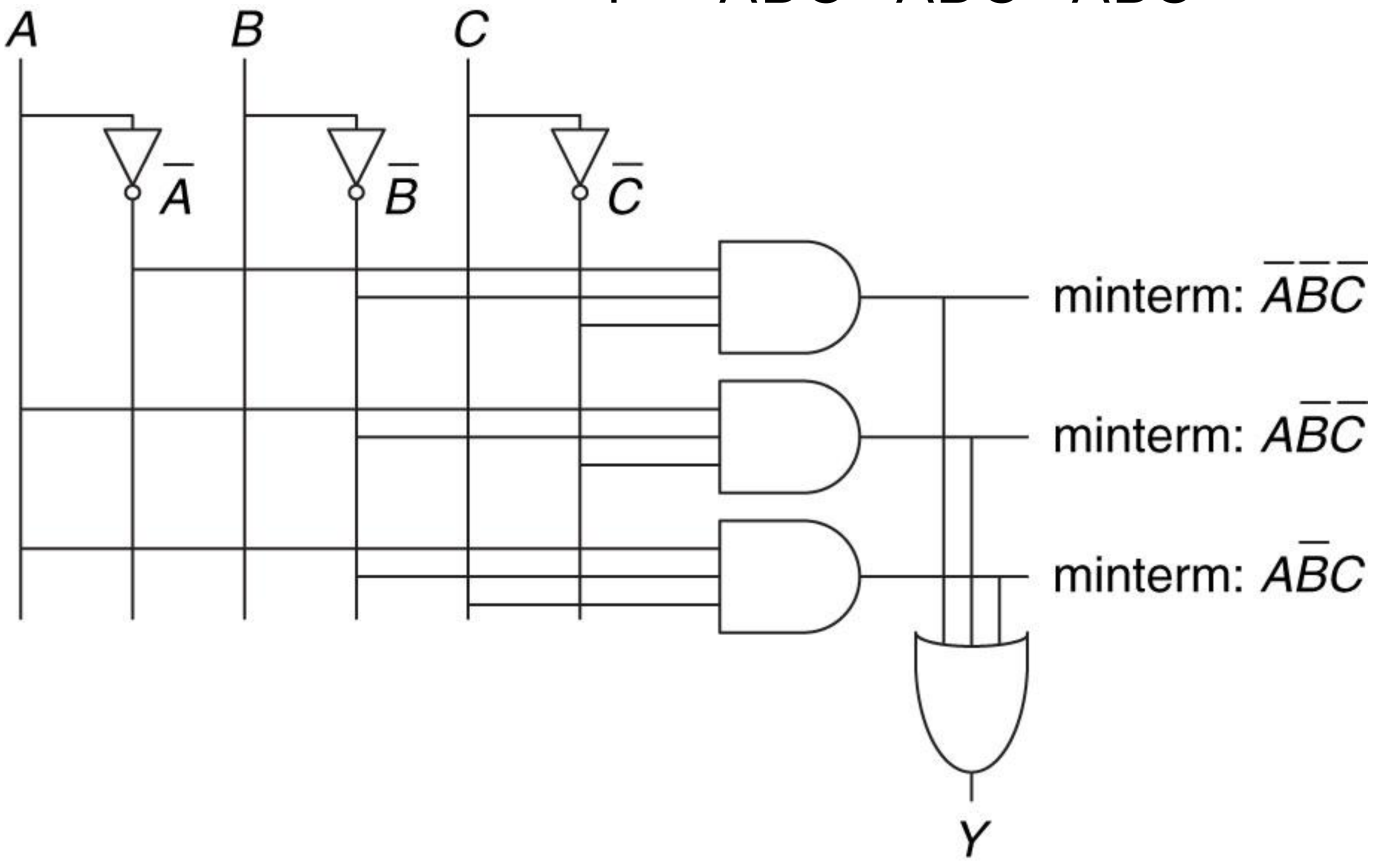
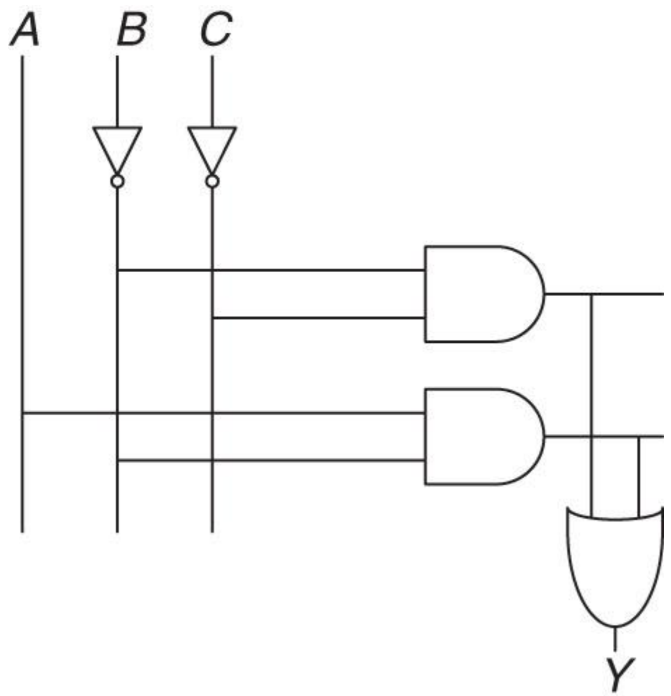
Example

$$Y = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$$



Example

$$Y = \overline{B}\,\overline{C} + A\overline{B}$$



The simplified expression gives the same logical output with much less hardware.