Computational Thinking

Logic

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Natural Deduction for Propositional Logic

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Proof systems for propositional logic

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- So far we have not mentioned proof systems at all.
- Recall what we would like from any proof system
  - using our proof system, we should be able to prove all of the tautologies
    - completeness
  - all theorems proved by our proof system should be tautologies
    - soundness.
- A proof system defines the proofs (valid mathematical arguments) of the system
  - it is a collection of rules of inference.
- These rules of inference can be applied to infer new formulae from old.
- Henceforth, we consider propositional logic to consist only of those formulae built using the connectives $\wedge$, $\vee$, $\neg$, and $\Rightarrow$
  - with other connectives, such as $\Leftrightarrow$, abbreviations.

if rain $\Rightarrow$ me not going to school!

if I go to school, I will get a homeworks

so now it is raing $\Rightarrow$ I will not get a homework

Proof systems for propositional logic

- An argument form in propositional logic is a sequence of formulae

  $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$

  and such an argument form is valid if
  - whenever a truth assignment f is s.t. $\varphi_1, \varphi_2, \ldots, \varphi_n$ evaluate to *true* under f
    - then ψ necessarily evaluates to *true* under f.
- An argument form can also be written in the form $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$
  - when it is referred to as a sequent.
- The rule of inference corresponding to the above argument form is

  $\varphi_1, \varphi_2, \ldots, \varphi_n \Rightarrow \psi$

  and if the above argument form is valid then this rule of inference is a tautology.
- The most well-known rule of inference for propositional logic
  - modus ponens (law of detachment) $\underbrace{(p \wedge (p \Rightarrow q))}_{} \Rightarrow \underline{q}\ \psi$
  - also written as

if $p \Rightarrow q$, premise 1

p, premise 2

q? ✓

$$\frac{\varphi_1\ p \qquad \varphi_2\ p \Rightarrow q}{q}$$

Applying rules of inference

- Of course, when "applying" a rule of inference such as modus ponens

  $(p \wedge (p \Rightarrow q)) \Rightarrow q$

  we can substitute arbitrary *formulae* for p and q
  - e.g., applying modus ponens to

    $(p \wedge q) \Rightarrow \neg r$ | X

    and

    $((p \wedge q) \Rightarrow \neg r) \Rightarrow ((q \wedge r) \vee s)$ | $X \Rightarrow Y$

    yields

    $((q \wedge r) \vee s$ | Y
- Similarly, given any rule of inference $\varphi_1, \varphi_2, \ldots, \varphi_n \Rightarrow \psi$
  - we can apply this rule by substituting *any* formula for *any* propositional variable
    - so long as the same formula is substituted for the same variable
  - thus, a valid argument form yields an infinite collection of tautologies.

Other rules of inference

- There are numerous well-known rules of inference
  - *any* tautology of the form $\varphi_1, \varphi_2, \ldots, \varphi_n \Rightarrow \psi$ gives rise to a rule of inference.

- Modus tollens

$$\frac{\neg q \qquad\qquad p \Rightarrow q}{\neg p}$$

- Hypothetical syllogism

$$\frac{p \Rightarrow q \qquad\qquad q \Rightarrow r}{p \Rightarrow r}$$

- Resolution

$$\frac{p \vee q \qquad\qquad \neg p \vee r}{q \vee r}$$

  - plus many more.

Rules of inference in action

$A \wedge W \Rightarrow I$	axiom
$\neg I$	axiom
$A \vee P$	axiom
$W \vee S$	axiom
$D \Rightarrow \neg(P \vee S)$	axiom

Prove that $\neg D$ holds if the five axioms are true

True.

An alternative approach

- We could write down all possible truth assignments on A, W, I, P, S, and D, and
  - retain only those for which
    - $A \wedge W \Rightarrow I$, $A \vee P$, $W \vee S$, $\neg I$, and $D \Rightarrow \neg(P \vee S)$

    are *true*
  - then check to see that for all of these retained truth assignments
    - we have that $\neg D$ is *true*.
- However, this would mean that $2^6 = 64$ different truth assignments need to be checked.
- Consequently, the proof-theoretic approach can be significantly more efficient than the truth-table approach
  - especially when there is a large number of propositional variables.
- Of course,
  - knowing *which* rules of inference to apply to *which* formulae so that
    - we get an "speedy" proof is another difficulty that needs to be overcome!

Natural deduction

- The proof system natural deduction consists of a collection of (valid) rules of inference and is used to obtain proofs of sequents of the form
  - $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.
- We assume that we are given $\varphi_1, \varphi_2, \ldots, \varphi_n$ as premises
  - we (hope to) apply our rules of inference (from the proof system) to obtain ψ.
- Rules for conjunction

$$\frac{\varphi_1 \qquad \varphi_2}{\varphi_1 \wedge \varphi_2}\ \wedge\mathrm{i} \qquad\qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_1}\ \wedge\mathrm{e1} \qquad\qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_2}\ \wedge\mathrm{e2}$$

$\wedge$-introduction $\qquad$ $\wedge$-elimination

- Rules for double negation

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg\varphi}{\varphi}\ \neg\neg\mathrm{e}$$

$\neg\neg$-introduction $\qquad$ $\neg\neg$-elimination

- Note
  - in general, φ_1 and φ_2 are *formulae* and not necessarily propositional variables
  - all of our rules are valid.

A simple proof

- Here is a proof of the sequent $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ using the rules we have introduced so far

1.	p	premise
2.	$\neg\neg(q \wedge r)$	premise
3.	$\neg\neg p$	$\neg\neg$i 1
4.	$q \wedge r$	$\neg\neg$e 2
5.	r	$\wedge$e2 4
6.	$\neg\neg p \wedge r$	$\wedge$i 3 5

- Note that the validity of the rules means that
  - if p and $\neg\neg(q \wedge r)$ are *true* under some truth assignment
    - then $\neg\neg p \wedge r$ is necessarily *true* under this truth assignment.
- We often say that a sequent is valid if it can be proved.

More rules

- Rule for eliminating implication

$$\dfrac{\varphi_1 \qquad\qquad \varphi_1 \Rightarrow \varphi_2}{\varphi_2} \;\Rightarrow\!e$$

⇒-elimination

- Rule for introducing implication

$$\dfrac{\boxed{\begin{array}{c}\varphi_1\\ \cdots \\ \varphi_2\end{array}}}{\varphi_1 \Rightarrow \varphi_2} \;\Rightarrow\!i$$

$p \Rightarrow \neg q$

p

$\neg q$, $\Rightarrow$e

⇒-introduction

- In order to apply the rule ⇒i
  - start with the intended premise, φ_1, as the first line of a box
  - continue until we prove φ_2
  - close the box and write our implication $\varphi_1 \Rightarrow \varphi_2$.
- Thereafter, we are not allowed to use any formula in the box
  - *once a box has closed then the formulae within it are no longer available to us*.

A proof using boxes

- Here is a proof of the sequent $p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$.

1.	$p \Rightarrow q$	premise
2.	$q \Rightarrow r$	premise
3.	$\boxed{p}$	assumption
4.	q	$\Rightarrow$e 1 3
5.	r	$\Rightarrow$e 2 4
6.	$p \Rightarrow r$	$\Rightarrow$i 3-5

- Note that it is possible
  - for a proof to involve more than one box
  - for boxes to be nested within each other.
- Note that boxes cannot overlap
  - we cannot *open* a box and then *open* another box
    - then *close* the first box before *closing* the second box.

More than one box

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- Here is a proof of the sequent $(p \wedge q) \Rightarrow r \vdash p \Rightarrow (q \Rightarrow r)$.

1.	$(p \wedge q) \Rightarrow r$	premise
2.	p	assumption
3.	q	assumption
4.	$p \wedge q$	$\wedge$i 2 3
5.	r	$\Rightarrow$e 1 4
6.	$q \Rightarrow r$	$\Rightarrow$i 3-5
7.	$p \Rightarrow (q \Rightarrow r)$	$\Rightarrow$i 2-6

- Note that
  - the structure of the formula we wish to prove
    - helps to determine the structure/tactics of our proof.