Mathematics for Computer Science
Linear Algebra

Lecture 15: Inner product spaces

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Contents for today's lecture

- Inner product spaces – definition, norm, orthogonality
- Important examples

Inner product: the definition

Recall: The dot product (aka Euclidean inner product) of vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

Using it, one can define norm (aka length), distance, angles, orthogonality in $\mathbb{R}^n$

Definition

Let V be a (real) vector space. An inner product on V is a function that

associates to each pair $\mathbf{u}, \mathbf{v} \in V$ a real number $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$,

satisfying the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $k \in \mathbb{R}$.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$ [Positivity axiom]

Norm and distance

Generalising from $\mathbb{R}^n$ to an arbitrary inner product space (i.e. a vector space equipped with an inner product), we can define norm and distance as

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \text{ and } d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

The following properties of norm and distance follow directly from defintions:

- $||\mathbf{v}|| \geq 0$, and $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$
- $||k\mathbf{v}|| = |k|\,||\mathbf{v}||$
- $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
- $d(\mathbf{u}, \mathbf{v}) = 0$ iff $\mathbf{u} = \mathbf{v}$.

A vector $\mathbf{v}$ with $||\mathbf{v}|| = 1$ is called a unit vector. Each non-zero vector can be normalised (scaled to become a unit vector): $\mathbf{v} \mapsto \frac{1}{||\mathbf{v}||}\mathbf{v}$.

Orthogonality and orthogonal complement

Definition

Vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

In $\mathbb{R}^n$ with the dot product, this is the same notion as before.

Definition

Let W be a subspace in an inner product space V. Then the set

$$W^{\perp} = \{\mathbf{x} \in V \mid \langle \mathbf{u}, \mathbf{x} \rangle = 0 \text{ for all } \mathbf{u} \in W\}$$

is called the orthoginal complement of W.

Example: Take $\mathbf{u} = (2, -3, 5, 4)$ and $\mathbf{v} = (0, 1, -4, 7)$ in $\mathbb{R}^4$ (with the dot product) and let $W = \mathit{span}(\mathbf{u}, \mathbf{v})$. Then $W^{\perp}$ is the solution space of the linear system

$$\begin{array}{rcl} 2x_1 - 3x_2 + 5x_3 + 4x_4 & = & 0 \qquad (\langle \mathbf{u}, \mathbf{x} \rangle = 0) \\ x_2 - 4x_3 + 7x_4 & = & 0 \qquad (\langle \mathbf{v}, \mathbf{x} \rangle = 0) \end{array}$$

Example: weighted Euclidean inner product

- Let $w_1, \ldots, w_n \in \mathbb{R}$ be arbitrary *positive* numbers, which we'll call *weights*.
- The weighted Euclidean inner product (with weights $w_1, \ldots, w_n$) on $\mathbb{R}^n$ is defined as follows: for vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n.$$

- Easy to check that all four axioms of inner product are satisfied.
- If all $w_i = 1$, this becomes the standard dot product.

Example: Consider $\mathbb{R}^2$ equipped with the weighted Euclidean inner product with weights $w_1 = 3, w_2 = 2$, i.e define $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$.

- The norm of $\mathbf{e}_1 = (1, 0)$ is $||\mathbf{e}_1|| = \sqrt{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} = \sqrt{3 \cdot 1^2 + 2 \cdot 0^2} = \sqrt{3}$.
- $\mathbf{u} = (1, -3)$ and $\mathbf{v} = (2, 1)$ are orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = 3 \cdot 1 \cdot 2 + 2 \cdot (-3) \cdot 1 = 0$.

Norms, distances and orthogonality depend on the choice of inner product!

Working with orthogonal complement

Theorem

For any subspace W in an inner product space V, the set $W^\perp$ is also a subspace.

- Take $\mathbf{u} = (2, -3, 5, 4)$ and $\mathbf{v} = (0, 1, -4, 7)$ in $\mathbb{R}^4$ and let $W = span(\mathbf{u}, \mathbf{v})$. If our inner product on $\mathbb{R}^4$ is the dot product, $W^\perp$ is the solution space of

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 + 4x_4 &= 0 \qquad (\langle \mathbf{u}, \mathbf{x} \rangle = 0) \\ x_2 - 4x_3 + 7x_4 &= 0 \qquad (\langle \mathbf{v}, \mathbf{x} \rangle = 0) \end{aligned}$$

- If we change the inner product to the weighted Euclidean inner product with weights $w_1 = 2$, $w_2 = 1$, $w_3 = 3$, $w_4 = 1$. Then $W^\perp$ is the solution space of

$$\begin{aligned} 4x_1 - 3x_2 + 15x_3 + 4x_4 &= 0 \qquad (\langle \mathbf{u}, \mathbf{x} \rangle = 0) \\ x_2 - 12x_3 + 7x_4 &= 0 \qquad (\langle \mathbf{v}, \mathbf{x} \rangle = 0) \end{aligned}$$

Finding a basis for $W^\perp$ = finding a basis in the solution space of linear system

Example: Matrix inner product on $\mathbb{R}^n$

Let A be an *invertible* $n \times n$ matrix.

Considering vectors in $\mathbb{R}^n$ as column vectors, define

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}, \text{ (or, equivalently, } \langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u} = \mathbf{v}^T A^T A\mathbf{u})$$

where the right-hand side uses the standard dot product in $\mathbb{R}^n$.

This is an inner product (can check all axioms), called the inner product on $\mathbb{R}^n$ generated by A.

Examples:

- The dot product on $\mathbb{R}^n$ is the inner product generated by the identity matrix
- The weighted Euclidean inner product on $\mathbb{R}^n$ with weights $w_1, \ldots, w_n$ is the inner product generated by $A = diag(\sqrt{w_1}, \ldots, \sqrt{w_n})$.

  For the earlier example of a weighted inner product on $\mathbb{R}^2$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Example: Standard inner product on $\mathbb{P}_n$

Recall: $\mathbb{P}_n$ is the space of all polynomials of degree at most n.

For vectors $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$ and $\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n$ in $\mathbb{P}_n$, define

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n.$$

This is an inner product, called the standard inner product on $\mathbb{P}_n$.

- It is easy to see that each vector $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{P}_n$ can be identified with the corresponding vector $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$.
- Then the standard inner product on $\mathbb{P}_n$ = the dot product on $\mathbb{R}^{n+1}$.

Example: Evaluation inner product on $\mathbb{P}_n$

Fix *distinct* points $x_0, x_1, \ldots, x_n \in \mathbb{R}$ (called *sample points*).

For vectors $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ in $\mathbb{P}_n$, define

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \ldots + p(x_n)q(x_n).$$

This inner product on $\mathbb{P}_n$ is called the evaluation inner product at $x_0, x_1, \ldots, x_n$.

- It is easy to see that one can identify each vector $\mathbf{p} = p(x) \in \mathbb{P}_n$ with the corresponding vector $(p(x_0), p(x_1), \ldots, p(x_n)) \in \mathbb{R}^{n+1}$.
- Then the standard inner product on $\mathbb{P}_n$ = the dot product in $\mathbb{R}^{n+1}$.
- One subtlety: need to check that $\langle \mathbf{p}, \mathbf{p} \rangle = 0$ iff $\mathbf{p} = 0$. This holds because

$$\langle \mathbf{p}, \mathbf{p} \rangle = [p(x_0)]^2 + [p(x_1)]^2 + \ldots + [p(x_n)]^2 = 0 \qquad \Rightarrow$$
$$p(x_0) = p(x_1) = \ldots = p(x_n) = 0 \qquad \Rightarrow \qquad \mathbf{p} = 0$$

The last implication follows from the fundamental theorem of algebra: a non-0 polynomial of degree $\leq n$ can have at most n distinct roots.

Working with evaluation inner product on $\mathbb{P}_n$

Consider $\mathbb{P}_2$ with evaluation inner product at $x_0 = -2, x_1 = 0, x_2 = 2$, i.e.

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) = p(-2)q(-2) + p(0)q(0) + p(2)q(2)$$

Consider two vectors $\mathbf{p} = x^2$ and $\mathbf{q} = x + 1$. Then

$$||\mathbf{p}|| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{[p(x_0)]^2 + [p(x_1)]^2 + [p(x_2)]^2} =$$
$$\sqrt{[p(-2)]^2 + [p(0)]^2 + [p(2)]^2} = \sqrt{4^2 + 0^2 + 4^2} = 4\sqrt{2}.$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2) = (4)(-1) + (0)(1) + (4)(3) = 8$$

If we normalise $\mathbf{p}$, we get vector $\mathbf{p}' = \frac{1}{||\mathbf{p}||}\mathbf{p} = \frac{1}{4\sqrt{2}}x^2 \in \mathbb{P}_2$.

Example: Inner product on the space $C[a, b]$

- Recall: $C[a, b]$ consists of all functions that are continuous on interval $[a, b]$.
- The operations in $C[a, b]$ are defined point-wise: if $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ then $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\mathbf{f})(x) = kf(x)$.
- Recall: any function continuous on $[a, b]$ is integrable on $[a, b]$.
- For $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ in $C[a, b]$, define

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\, dx.$$

- This formula defines an inner product on $C[a, b]$, let's check the 4 axioms. The first three are straightforward, let's check the last one (positivity): Clearly, $\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b f^2(x)\, dx \geq 0$. Moreover, the integral is 0 only if $f = 0$ (because f is continuous on $[a, b]$).
- Each polynomial is a continuous function: so $\mathbb{P}_n$ is a subspace of $C[a, b]$, and this inner product works on $\mathbb{P}_n$ too.

Working with inner product on $C[a, b]$

Consider $\mathbb{P}_2$ or $C[-1, 1]$ with inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Consider two vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$||\mathbf{p}|| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_{-1}^{1} xx\,dx} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{2}{3}}$$

$$||\mathbf{q}|| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle} = \sqrt{\int_{-1}^{1} x^2x^2\,dx} = \sqrt{\int_{-1}^{1} x^4\,dx} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2\,dx = \int_{-1}^{1} x^3\,dx = 0$$

In particular, $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal w.r.t. this inner product

Standard (in)equalities

The standard (in)equalities for the dot product work for general inner products (and the proofs are the same):

Theorem (Pythogoras' theorem)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in an inner product space then* $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Theorem (Cauchy-Schwarz inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in an inner product space then* $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \, ||\mathbf{v}||$.

Corollary (Triangle inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in an inner product space then* $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

Example: Cauchy-Schwarz inequality in $C[a, b]$

$$|\int_a^b f(x)g(x)\,dx| \leq \sqrt{\int_a^b f^2(x)\,dx}\sqrt{\int_a^b g^2(x)\,dx}.$$

What we learnt today

- Inner product spaces
- Norm and orthogonality in these spaces
- Important examples

Next time:

- The Gram-Schmidt orthogonalisation process
- QR-decomposition of matrices