Computational Thinking Logic

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Lecture 5

More on Natural Deduction for Propositional Logic



More rules

9 = True

9 = True

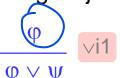
9 - True

10 - True

10



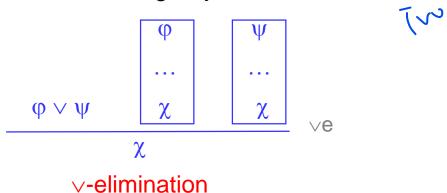
Rules for introducing disjunction





∨-introduction

Rule for eliminating disjunction



we usually just write vi

- In order to apply the rule ve, we uses boxes as previously
 - but now there is a box starting with each given disjunct, φ and ψ
 - each box needs to end with the same intended formula, χ .

A proof using v-elimination

• Here is a proof of the sequent $q \Rightarrow r \mid (p \lor q) \Rightarrow (p \lor r)$

- 1.
- $p \vee q$
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.

- $q \Rightarrow r$
- $p \vee r$
- q
- $p \vee r$
- $p \vee r$
- $\overline{(p \vee q)} \Rightarrow (p \vee r)$

- premise
- assumption
- assumption
- √i 3
- assumption
- ⇒e 1 5
- √i 6
- ∨e 2-7
- ⇒i 2-8

A proof using v-elimination

 $\rightarrow p \vee (q \vee r)$

 $p \vee q$

 $\overline{q \vee r}$

 $p \vee q$

 $(p \lor q) \lor$

 $(p \lor q) \lor r$

 $(p \lor q) \lor r$



• Here is a proof of the sequent $p \vee (q \vee r) + (p \vee q) \vee r$

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.

- premise
- assumption
- √i 2
- √i 3
- assumption
- assumption
- √i 6
- √i 7
- assumption
- √i 9
- ∨e 5-10
- ∨e 2-11

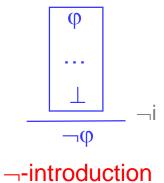
More rules



Rules for negation



- The symbol ⊥, known as bottom, represents a contradiction
 - in natural deduction if one has a contradiction then one can infer *any* formula.
- Rule for introducing negation



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A proof using the rules for negation



• Here is a proof of the sequent $x \Rightarrow (y \Rightarrow z)$, x, $\neg z \vdash \neg y$.

- 1. $x \Rightarrow (y \Rightarrow z)$
- 2. **x**
- 3. ¬*z*
- 4. <u>y</u>
- 5. $y \Rightarrow z$
- 6.
- 7. _______
- 8. ¬*y*

- premise
- premise
- premise
- assumption
- ⇒e 1 2
- ⇒e 4 5
- –e 3 6
- -i 4-7

Another proof using rules for negation



Here is a proof of the sequent $x \vee \neg y \vdash y \Rightarrow x$.

- 1.
- $(x) / \neg y$

- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.

X

- X
- $y \Rightarrow x$
- $\neg y$
 - - - $y \Rightarrow x$

premise

- assumption
- assumption
- copy 2
- ⇒i 3-4
- assumption assumption
- –e 6 7
- ⊥e 8
- ⇒i 7-9
- ∨e 1 2-5 6-10

メンマア 4=>x + 1=>x brosed (コラ) True > 4 & True.

A derived rule



- We can derive other rules in natural deduction.
- Consider modus tollens $\phi \Rightarrow \psi$, $\neg \psi \vdash \neg \phi$.
 - 1. $\phi \Rightarrow \psi$
 - 2. ¬₩
 - 3. φ
 - 1. ι
 - 5. <u>⊥</u>
 - 6. ¬φ

- premise
- premise
- assumption
- ⇒e 1 3
- –e 2 4
- −i 3-5
- Note that we can use derived rules just as if they were rules of natural deduction
 - e.g., in a proof with
 - a line reading $\phi \Rightarrow \psi$
 - and another line reading ¬ψ

we could immediately infer $\neg \varphi$

and write "modus tollens" or "MT" as an explaining remark.

More derived rules



$$\neg \varphi \Rightarrow \bot \vdash \varphi$$

- Reductio ad absurdum or proof by contradiction is the principle
 - "if from $\neg \varphi$ I can prove \bot then I can deduce φ ".
- Here is a proof that this principle can be applied in natural deduction.
 - 1.
 - 2.
 - $\neg \phi$ 3.
 - 4.
 - 5.

 $\neg \phi \Rightarrow \bot$

 $\neg\neg 0$

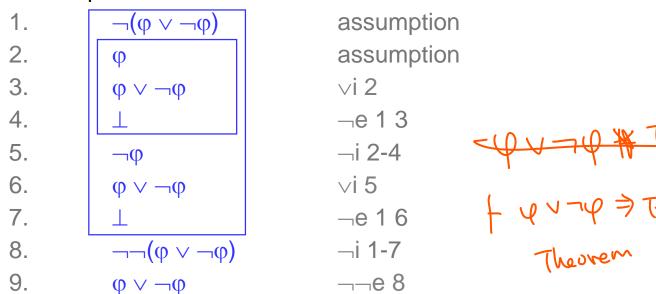
φ

- premise
- assumption
- ⇒e 1 2
- −i 2-3
- ---e 4
- We denote reductio ad absurdum by RAA.

More derived rules

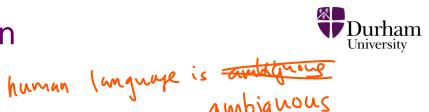


- The law of the excluded middle states that either φ is true or $\neg \varphi$ is true.
- Here is a proof of it.



We denote the law of the excluded middle by LEM.

Some facts about Natural Deduction



- Natural deduction is sound and complete.
- Let $\varphi_1, \varphi_2, ..., \varphi_m$ and ψ be formulae.
- Soundness
 - if the sequent $\varphi_1, \varphi_2, ..., \varphi_m \models \psi$ is provable
 - then the formula $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_m \Rightarrow \psi$ is a tautology.
- Completeness
 - if $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_m \Rightarrow \psi$ is a tautology
 - then the sequent $\varphi_1, \varphi_2, ..., \varphi_m \vdash \psi$ is provable.
- A theorem is a formula ψ for which the sequent $+\psi$ is provable
 - thus, the soundness and completeness of natural deduction tells us that
 - every theorem is a tautology and every tautology is a theorem.

Proving theorems



• Here is a proof that the sequent $(p \Rightarrow (\neg p \lor q)) \lor (p \Rightarrow \neg q)$ is a theorem.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.

- $q \vee \neg q$
- q
- $\neg p \lor q$
- $\neg p \lor q$
- $p \Rightarrow (\neg p \lor q)$
- $(p \Rightarrow (\neg p \lor q)) \lor (p \Rightarrow \neg q)$
- $\neg q$
 - p
 - $\neg q$
 - $p \Rightarrow \neg q$
 - $(p \Rightarrow (\neg p \lor q)) \lor (p \Rightarrow \neg q)$
 - $(p \Rightarrow (\neg p \lor q)) \lor (p \Rightarrow \neg q)$

- LEM
- assumption
- √i 2
- assumption
- copy 3
- ⇒i 4-5
- √i 6
- assumption
- assumption
- copy 8
- ⇒i 9-10
- √i 11
- ∨e 1 2-7 8-12