Maths for Computer Science
Calculus

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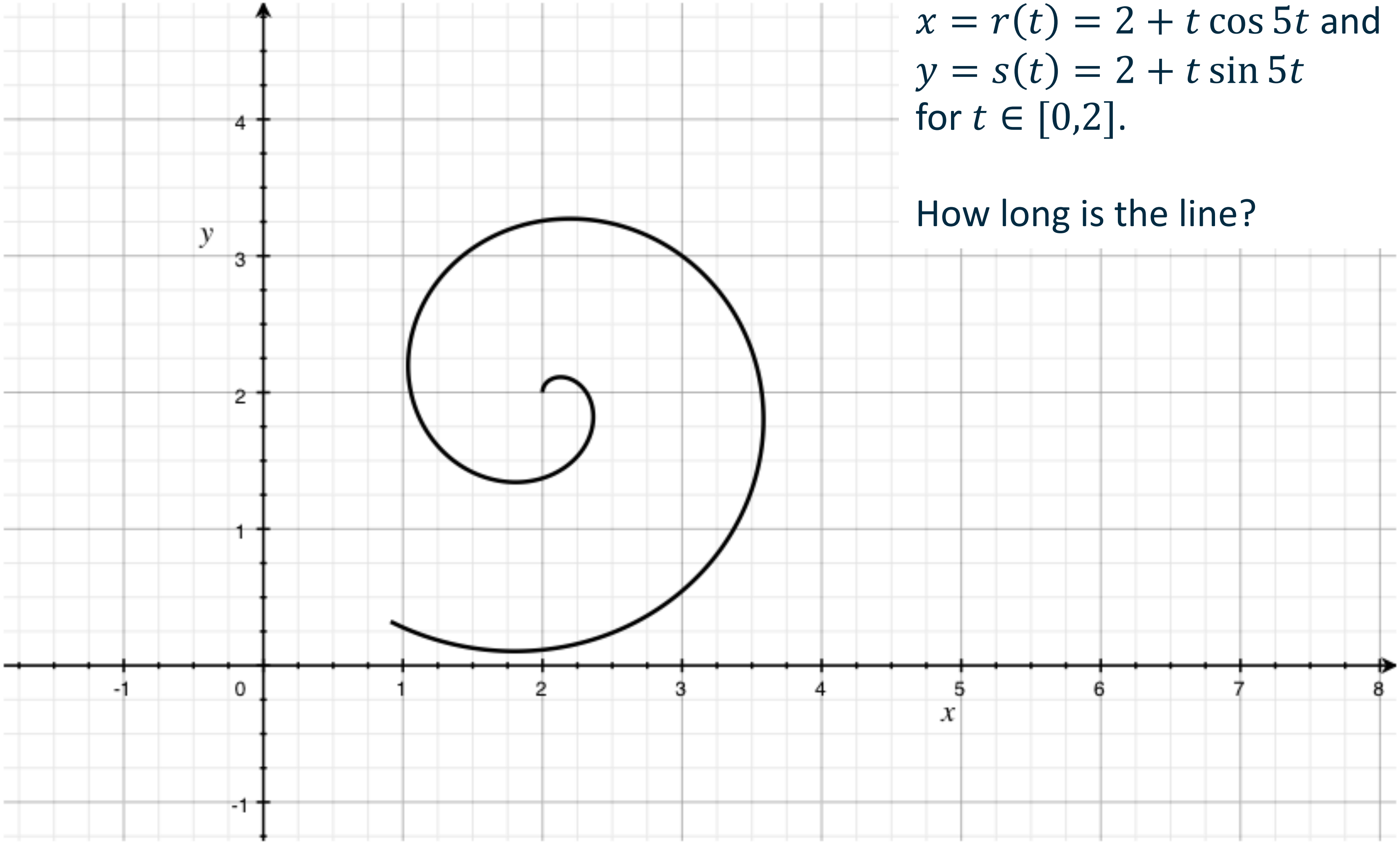
Applications of Integration

Parameterised curves



$x = r(t) = 2 + t\cos 5t$ and
$y = s(t) = 2 + t\sin 5t$
for $t \in [0,2]$.

How long is the line?

Arc length integrals

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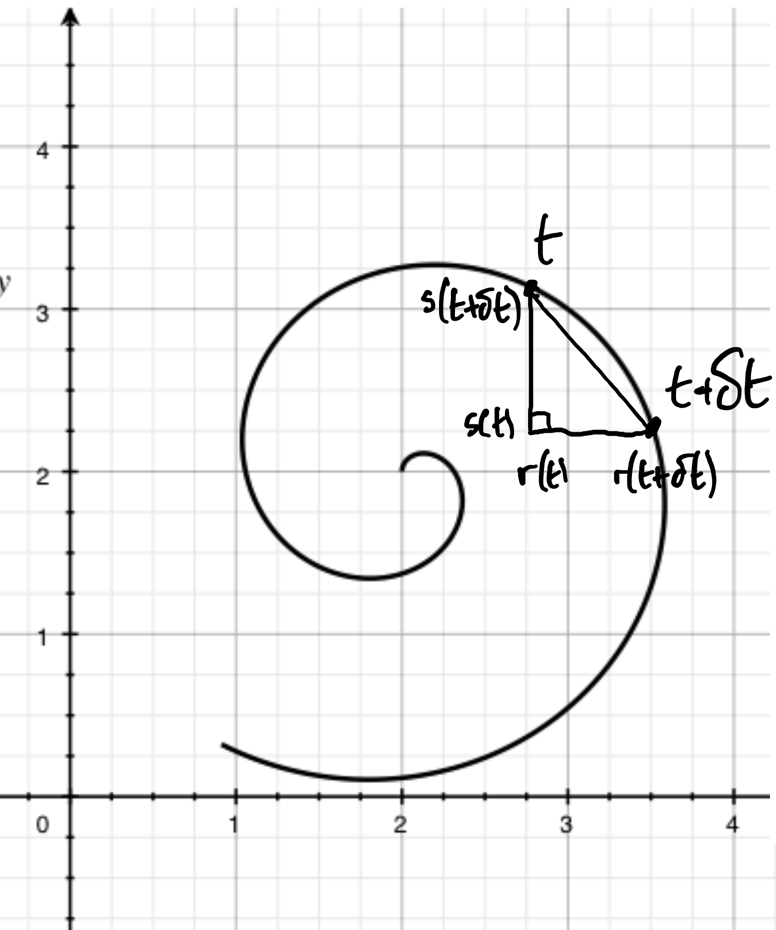


Consider the segment from t to $t + \delta t$.
It has length:

$$\sqrt{\big(r(t+\delta t) - r(t)\big)^2 + \big(s(t+\delta t) - s(t)\big)^2} = \sqrt{\left(\frac{r(t+\delta t) - r(t)}{\delta t}\right)^2 + \left(\frac{s(t+\delta t) - s(t)}{\delta t}\right)^2}\,.\,\delta t$$

So we can approximate the total length by

$$\sum_{i=0}^{n-1} \sqrt{\left(\frac{r(t_i+\delta t) - r(t_i)}{\delta t}\right)^2 + \left(\frac{s(t_i+\delta t) - s(t_i)}{\delta t}\right)^2}\,\delta t$$

where $\delta t = \frac{2}{n}$ and $t_i = \frac{2}{n} i$.

Arc length integrals

$x = r(t) = 2 + t\cos 5t$ and
$y = s(t) = 2 + t\sin 5t$ for $t \in [0,2]$.

We define the length to be the limit as $n \to \infty$

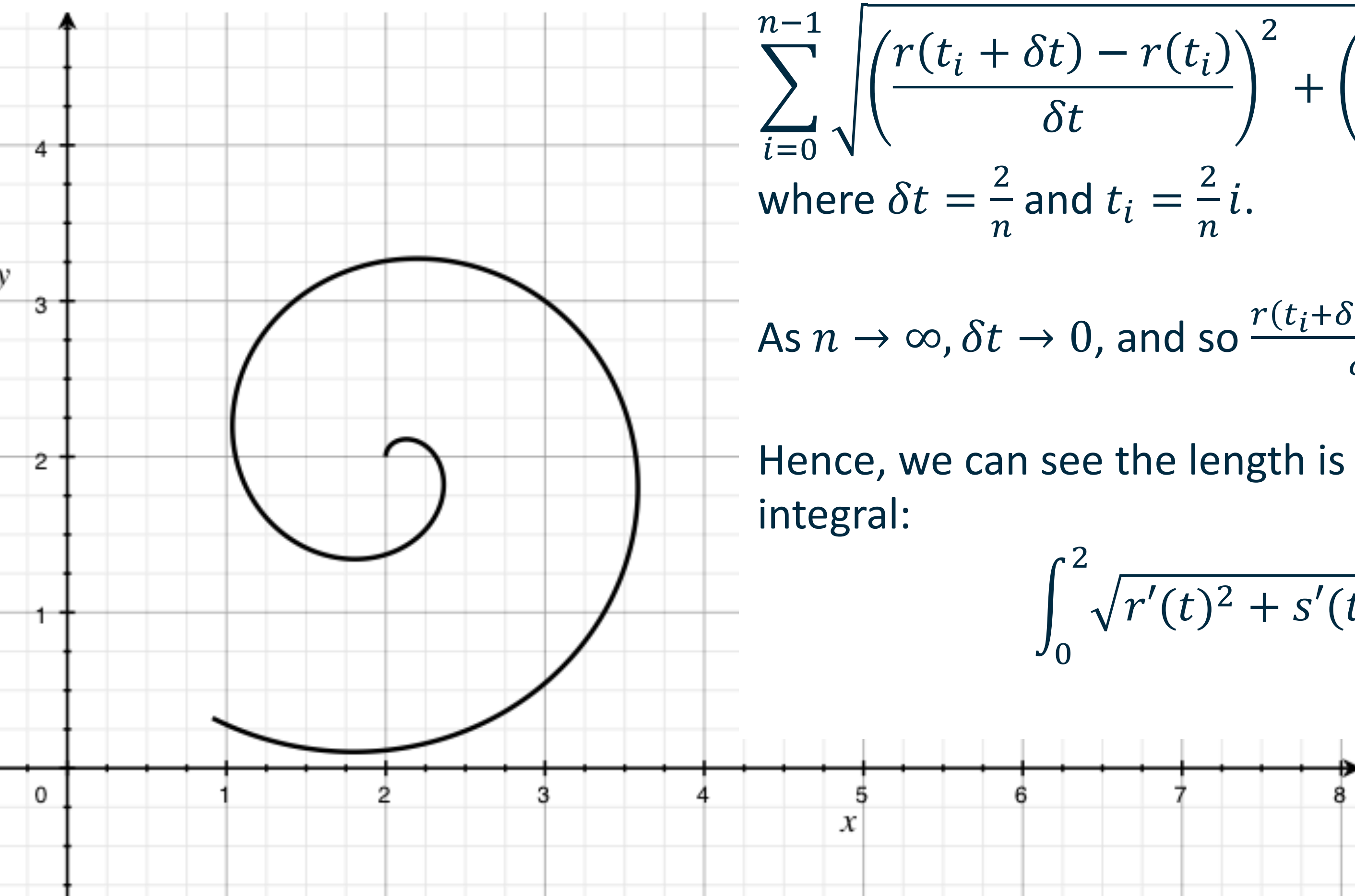
$$\sum_{i=0}^{n-1} \sqrt{\left(\frac{r(t_i + \delta t) - r(t_i)}{\delta t}\right)^2 + \left(\frac{s(t_i + \delta t) - s(t_i)}{\delta t}\right)^2}\, \delta t$$

where $\delta t = \frac{2}{n}$ and $t_i = \frac{2}{n} i$.

As $n \to \infty, \delta t \to 0$, and so $\frac{r(t_i+\delta t)-r(t_i)}{\delta t} \to r'(t)$.

Hence, we can see the length is exactly the definite integral:

$$\int_0^2 \sqrt{r'(t)^2 + s'(t)^2}\, dt$$



Arc length integrals

$x = r(t) = 2 + t\cos 5t$ and
$y = s(t) = 2 + t\sin 5t$ for $t \in [0,2]$.

$r'(t) = \cos 5t - 5t\sin 5t$ and $s'^{(t)} = \sin 5t + 5t\cos 5t$
So

$$r'(t)^2 + s'(t)^2$$
$$= \cos^2 5t + 25t^2 \sin^2 5t - 5t\cos 5t\sin 5t + \sin^2 5t + 25t^2\cos^2 5t + 5t\cos 5t\sin 5t$$
$$= 1 + 25t^2$$

The length is therefore the definite integral:

$$\int_0^2 \sqrt{(1 + 25t^2)}dt$$

Using substitution and repeated integration by parts:

$$\int_0^2 \sqrt{(1 + 25t^2)}dt = \left[\frac{1}{2}\ln\left(5t + \sqrt{1 + 25t^2}\right) + 5t\sqrt{1 + 25t^2}\right]_0^2 = 10.35$$

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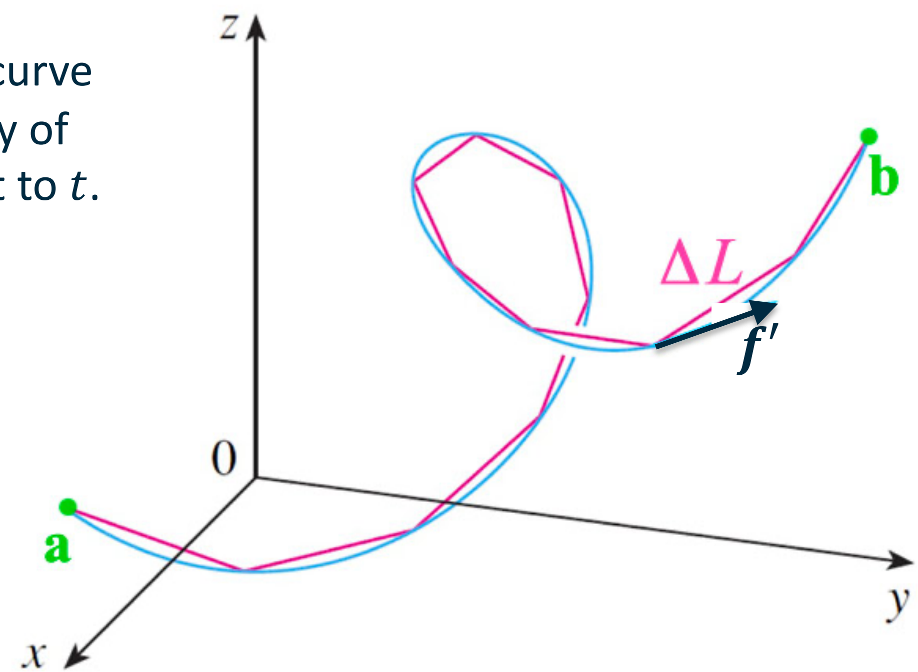
Arc length integrals

In general, if $\boldsymbol{f}(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big), t \in [a, b]$ is a parameterised curve in $\mathbb{R}^n$, then its length is

$$\int_a^b \left(\sqrt{\sum_{i=1}^{n} x_i'(t)^2} \right) dt = \int_a^b |\boldsymbol{f}'(t)| dt$$

where $\boldsymbol{f}'(t) = \big(x_1'(t), x_2'(t), \dots, x_n'(t)\big)$.

Indeed $\boldsymbol{f}'(t)$ is a tangent vector to the curve at $\boldsymbol{f}(t)$, and its magnitude is the velocity of movement along the curve with respect to t.



Line integrals

Suppose we have some scalar function f defined at every point of a curve C (or indeed f is defined at every point in $\mathbb{R}^n$, but we are only interested in points on the curve).

We can integrate the values of f along the curve C as follows.

Let $C(t) = \big(x(t), y(t), z(t)\big)$ for some range $t \in [a, b]$.

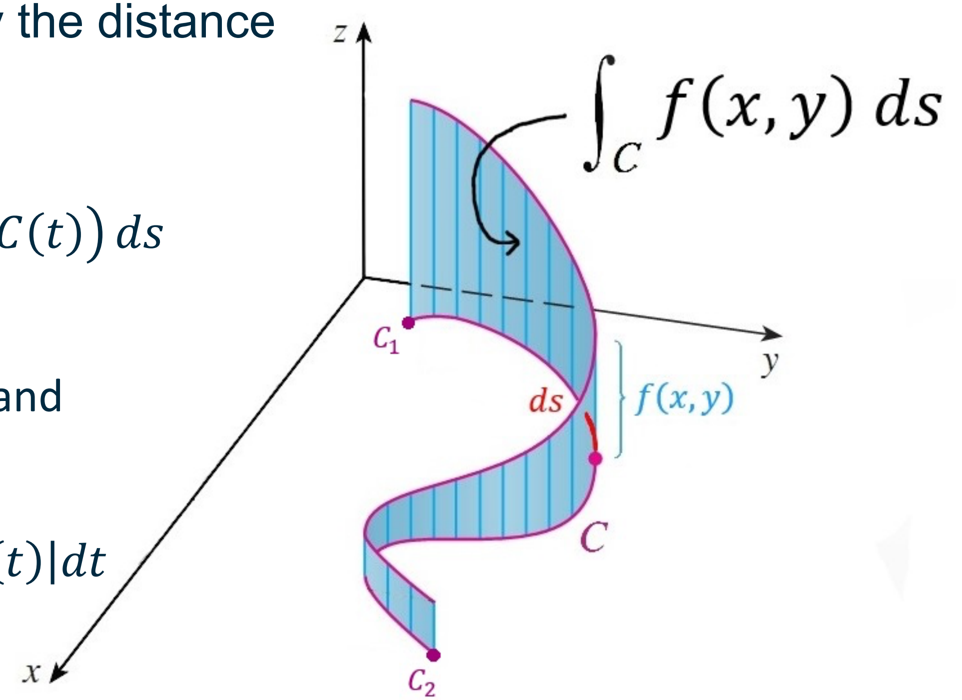
Then we can imagine a graph of height $f(t)$ as t ranges from a to b, but the area under this curve will depend on the parameterisation!

We need to scale the horizontal axis by the distance along the curve $s(t)$ (arc length).
I.e. we want the integral

$$\int_{t=a}^{t=b} f\big(C(t)\big)\frac{ds}{dt}\,dt = \int_{t=a}^{t=b} f\big(C(t)\big)\,ds$$

But $s(t) = \int_a^t |C'(w)|dw$, so $\frac{ds}{dt} = |C'(t)|$ and

$$\int_C f\,ds = \int_{t=a}^{t=b} f\big(C(t)\big)\,|C'(t)|dt$$



Line integral example

Evaluate $\int_C xyz\, ds$ where C is the helix $C(t) = (\cos t\,, \sin t\,, 3t), 0 \leq t \leq 4\pi$.

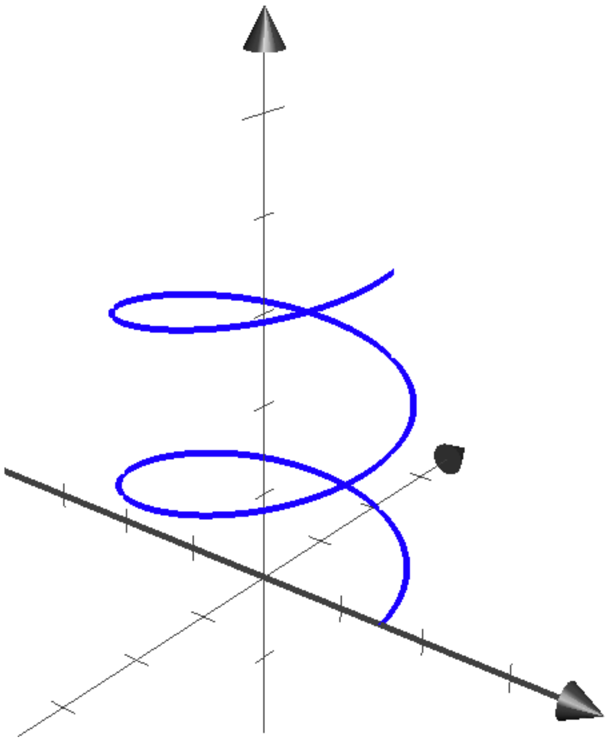
$$\int_C xyz\, ds = \int_0^{4\pi} 3t\cos(t)\sin(t)\sqrt{\sin^2 t + \cos^2 t + 9}\, dt$$

$$= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin(2t)\right)\sqrt{1+9}\, dt$$

$$= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin(2t)\, dt$$

$$= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin(2t) - \frac{t}{2}\cos(2t)\right)\Big|_0^{4\pi}$$

$$= -3\sqrt{10}\,\pi$$



Double integrals

To determine the volume under a surface we can consider little cuboids instead of rectangles.

Let $z = f(x, y) \geq 0$ be the height of a surface over a region S of the xy-plane bounded by a curve Γ. Assume f is continuous in S.

If for all $(x, y) \in S \quad a \leq x \leq b$, consider a partition $P_n = \{x_0 = a, x_1, x_2, \dots, x_n = b\}$.

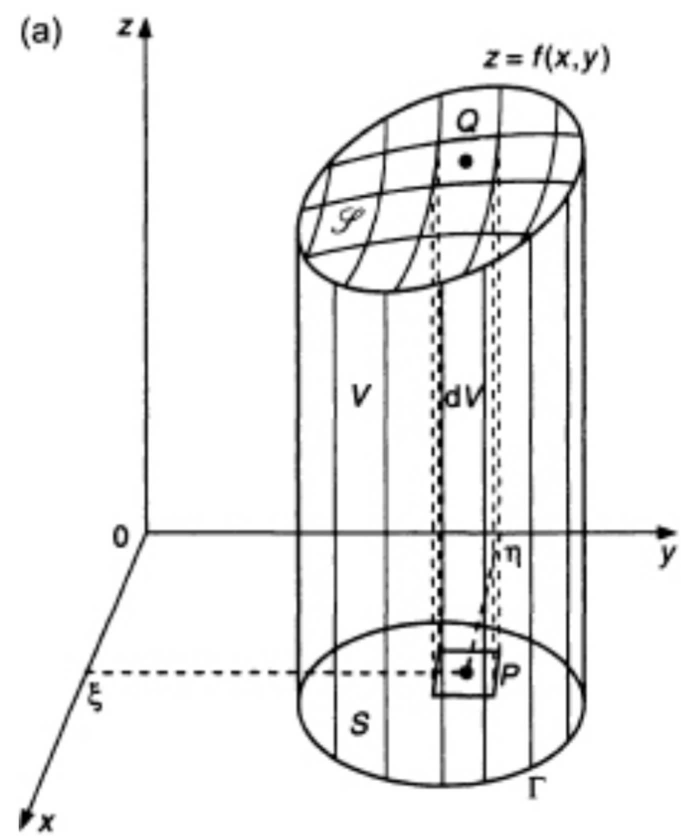
Let g_1, g_2 be functions such that for any $x \in (a, b)$ the points $(x, g_1(x))$ and $(x, g_2(x))$ are the two points on Γ and $g_1(x) \leq g_2(x)$.

Let $Q_m = \{y_0 = g_1(x), y_1, \dots, y_m = g_2(x)\}$ be a partition of $[g_1(x), g_2(x)]$.

Then the volume under the surface is approximately

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \Delta_{y_j} \Delta_{x_i}$$

where $\xi_i \in (x_{i-1}, x_i), \eta_j \in (y_{j-1}, y_j), \Delta_{x_i} = x_i - x_{i-1}$ and $\Delta_{y_j} = y_j - y_{j-1}$.



Double integrals

If we now take the limit as n and m tend to infinity we get the double Riemann integral:

$$\iint_S f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i,\eta_j)\Delta_{y_j}\Delta_{x_i}$$

Properties:

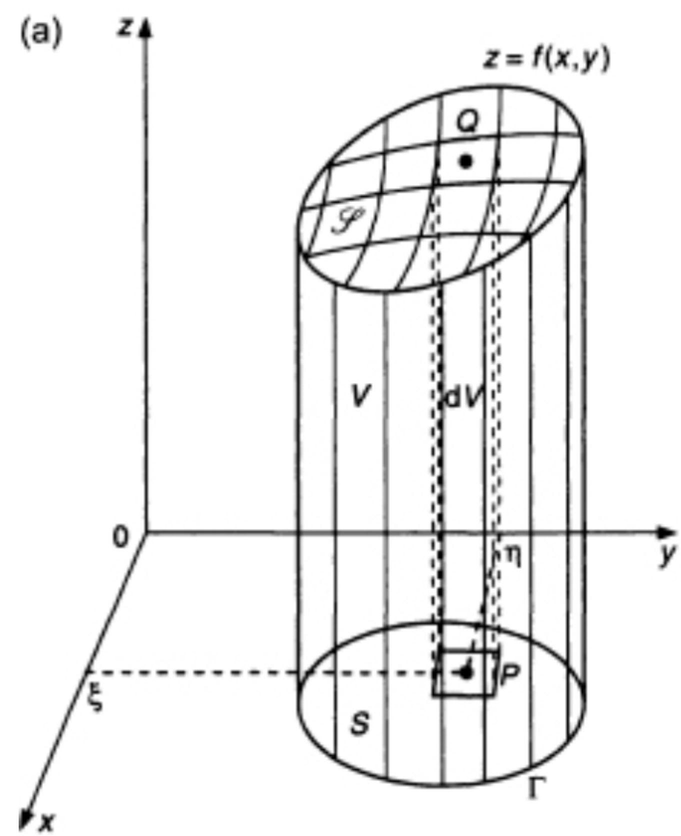
If f, g are continuous on S, and $\alpha, \beta \in \mathbb{R}$, then

$$\iint_S \alpha f(x,y) + \beta g(x,y)dA$$

$$= \alpha \iint_S f(x,y)dA + \beta \iint_S g(x,y)dA$$

If f experiences a jump discontinuity across a curve γ which divides S into two parts S_1, S_2 then

$$\iint_S f(x,y)dA = \iint_{S_1} f(x,y)dA + \iint_{S_2} f(x,y)dA$$



Order of integration

We bounded x and defined functions g_1, g_2 to limit the range of y.

We could equally have bounded $y \in [c, d]$ and defined the functions h_1, h_2 such that for any $y \in (c, d)$ the points $(h_1(y), y)$ and $(h_2(y), y)$ are the two points on Γ and $h_1(y) \leq h_2(y)$.

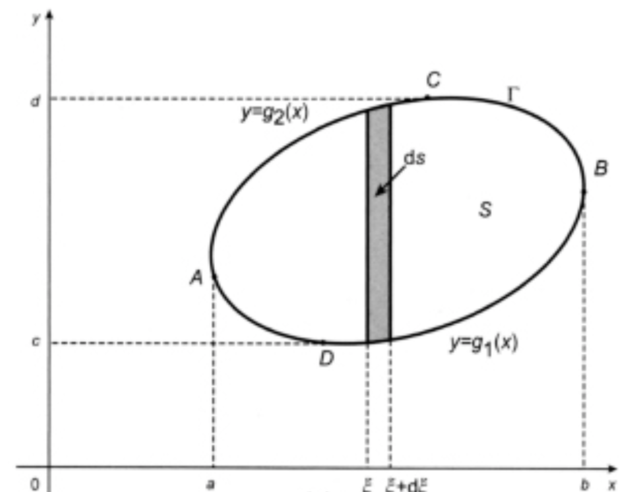
Then the volume under the surface is approximately

$$\sum_{j=1}^{m} \sum_{i=1}^{n} f(\xi_i, \eta_j) \Delta_{x_i} \Delta_{y_j}$$

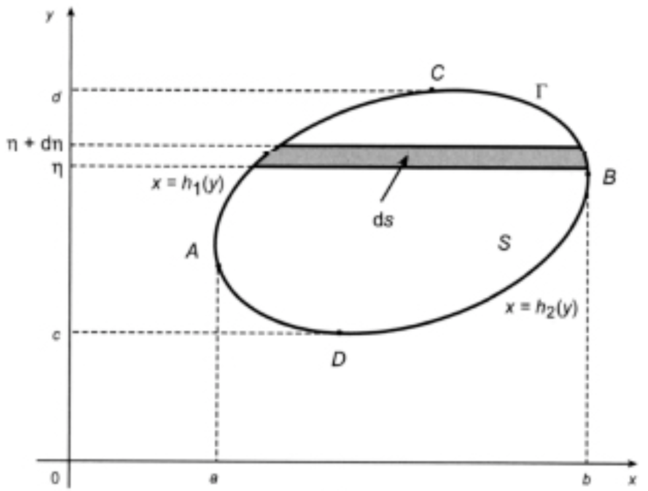
i.e. the order of summation is reversed.

Hence we obtain

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy\, dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx\, dy$$

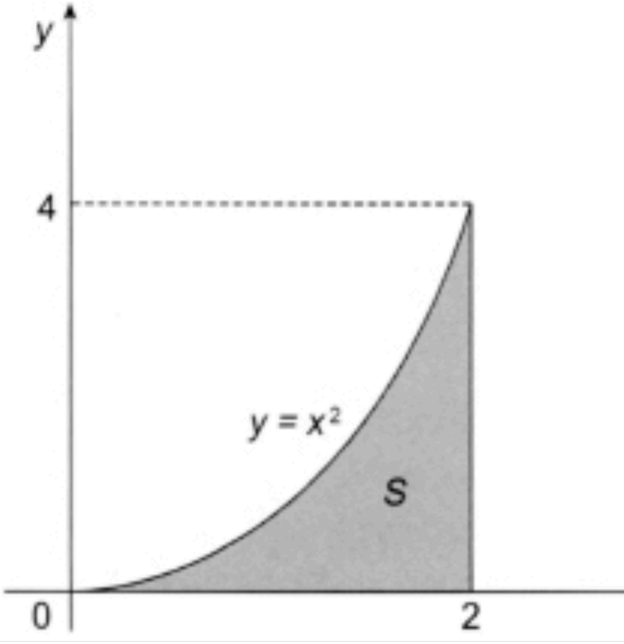


(a)



(b)

Example



Evaluate $\iint_S (x^2 + y^2)dA$ where S is the area bounded by the x-axis, the parabola $y = x^2$ and the line $x = 2$.

$$\int_0^2 \int_0^{x^2} (x^2 + y^2)dy\, dx = \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^2 x^4 + \frac{x^6}{3} dx = \left[\frac{x^5}{5} + \frac{x^7}{21}\right]_0^2 = \frac{32}{5} + \frac{128}{21} = \frac{1312}{105}$$

Or

$$\int_0^4 \int_{\sqrt{y}}^2 (x^2 + y^2)dx\, dy = \int_0^4 \left[\frac{x^3}{3} + xy^2\right]_{\sqrt{y}}^2 dy = \int_0^4 \frac{8}{3} + 2y^2 - \frac{y^{\frac{3}{2}}}{3} - y^{\frac{5}{2}}\, dy$$

$$= \left[\frac{8}{3}y + \frac{2}{3}y^3 - \frac{2}{15}y^{\frac{5}{2}} - \frac{2}{7}y^{\frac{7}{2}}\right]_0^4 = \frac{32}{3} + \frac{128}{3} - \frac{64}{15} - \frac{256}{7} = \frac{1312}{105}$$

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Example 2: order of integration can help!

Let S be the region $0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 1$.

Consider the integral $I = \iint_S \ x\cos(xy)\, dA$.

$$I = \int_0^1 \int_0^{\frac{\pi}{2}} x\cos(xy)\,dx\,dy = \int_0^1 \left[\frac{xy\sin(xy) + \cos xy}{y^2}\right]_0^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \frac{\pi y \sin\left(\frac{\pi y}{2}\right) + 2\cos\left(\frac{\pi y}{2}\right) - 2}{2y^2} dy \ = \quad ugh. I'm\ stuck.$$

But the other way round:

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 x\cos(xy)\,dy\,dx = \int_0^{\frac{\pi}{2}} [\sin xy]_0^1 dx = \int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$