Mathematics for Computer Science
Linear Algebra

Lecture 2: Systems of Linear Equations

Andrei Krokhin

October 13, 2020

Contents for today's lecture

- Systems of linear equations;
- Elementary row operations on matrices;
- Row echelon form and reduced row echelon form;
- Solving linear systems: Gaussian elimination;
- Examples and exercises.

Systems of linear equations

- A linear equation in n variables $x_1, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

  where the a_i's and b are constants and not all a_i's are equal to 0.
- A finite set of linear equations is called a system of linear equations, or simply a linear system.
- A general linear system can be written as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

- A solution to such a system is a sequence of $s_1, \ldots, s_n$ of numbers such that the assignment $x_1 = s_1, \ldots, x_n = s_n$ satisfies every equation.
- A linear system is called consistent if it has at least one solution and it is inconsistent otherwise.

A linear system with one solution

Solve linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Eliminate x from the 2nd equation by adding -2 times the 1st equation to the 2nd.

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

We have $y = 4/3$, and from the 1st equation $x = 7/3$.
This system has one solution.

A linear system with no solutions

Solve linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Eliminate x from the 2nd equation by adding -3 times the 1st equation to the 2nd.

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The 2nd equation is contradictory. This system has no solutions.

A linear system with infinitely many solutions

Solve linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 8x - 4y &= 2 \end{aligned}$$

Eliminate x from the 2nd equation by adding -3 times the 1st equation to the 2nd.

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The 2nd equation imposes no restrictions on x and y, can be omitted.

Any pair of values for x and y that satisfies $4x - 2y = 1$ is a solution.

Solving for x, we get $x = \frac{1}{4} + \frac{1}{2}y$.

The solution set can be described as the set of all pairs of numbers of the form $x = \frac{1}{4} + \frac{1}{2}y, y$ (y is a free variable here).

This system has infinitely many solutions.

Matrix form of a linear system

A linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

can be written in a matrix form as $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix A is called the coefficient matrix of the system.

If A is (square and) invertible then the solution can be found as $\mathbf{x} = A^{-1}\mathbf{b}$.

The augmented matrix and elementary row operations

The augmented matrix of a linear system is the matrix

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

The basic method for solving a linear system is to perform algebraic operatoins on the system that (a) do not alter the solution set and (b) produce increasingly simpler systems. Typically the operations are

- Multiply an equation through by a non-zero constant;
- Interchange two equations;
- Add a constant times one equation to another.

This corresponds to the elementary row operations on the augmented matrix:

- Multiply a row through by a non-zero constant;
- Interchange two rows;
- Add a constant times one row to another.

Row echelon form

Assume that we transform the augmented matrix of a linear system in variables x, y, z to the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Then we know the solution: it's $x = 1, y = 2, z = 3$.

A matrix is in row echelon form if it has the following properties:

- If a row is not all 0s then the first non-zero number in it is 1 (the leading 1)
- The rows that are all 0s (if any) are grouped together at the bottom
- If two successive rows are not all 0s then the leading 1 of the higher row occurs further to the left than the leading 1 of the lower row.

A matrix is in reduced row echelon form if it has the above properties, plus

- Each column that contains a leading 1 has 0s everywhere else.

Strategy for solving linear systems: use elementary row operations to transform the augmented matrix to (reduced) row echelon form.

Extracting solutions from row echelon form

Assume that we have transformed the augmented matrix of a linear system to a (reduced) row echelon form.

Examples:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have the following possibilities:

- Some row has a leading 1 in the last column.
  Then the system includes equation $0 \cdot x_1 + \ldots + 0 \cdot x_n = 1$.
  Then we know the system has no solutions.
- The number of leading 1s is equal to the number of variables (and there is no leading 1 in the last column).
  Then the system has a unique solution.
- The number of leading 1s is smaller than the number of variables (and there is no leading 1 in the last column).
  Then the system has infinitely many solutions.

General solution (and an example)

Assume a matrix in reduced row echelon form is as follows:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In equations, this is

$$\begin{array}{cccccc} x_1 & -x_2 & & +2x_4 & = 2 \\ & & x_3 & -x_4 & = 5 \end{array}$$

- The variables corresponding to the leading 1s (x_1 and x_3 in the example) are the leading variables.
- The other variables are free variables.
- General solution: the leading variables expressed via free variables.
- For the above system: $x_1 = x_2 - 2x_4 + 2, x_3 = x_4 + 5$ (where x_2 and x_4 are arbitrary numbers).

Gaussian elimination procedure

Goal: Transform a matrix to row echelon form by using row operations.

Step 1. Locate the pivot column – leftmost column that contains a non-zero.

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 2. Choose a pivot - a non-zero in the pivot column and interchange the first row with another row (if necessary) to move the pivot to the top in this column

$$\begin{pmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 3. If a is the pivot, multiply the first row by $1/a$ (to get a leading 1).

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Gaussian elimination procedure, cont'd

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 4. Add suitable multiples of the first row to the rows below so that all numbers below the leading 1 are 0s.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

Step 5. Now separate the top row from the rest ("draw a line below it") and repeat Steps 1–5 for the matrix below the line.

$$\left(\begin{array}{cccccc} 1 & 2 & -5 & 3 & 6 & 14 \\ \hline 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right)$$

Gaussian elimination procedure, cont'd

$$\left(\begin{array}{rrrrrr} 1 & 2 & -5 & 3 & 6 & 14 \\ \hline 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right)$$

$$\left(\begin{array}{rrrrrr} 1 & 2 & -5 & 3 & 6 & 14 \\ \hline 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right)$$

$$\left(\begin{array}{rrrrrr} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ \hline 0 & 0 & 0 & 0 & 1/2 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrrrrr} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

The last matrix is in row echelon form, it is the output of Gaussian elimination.

Gauss-Jordan elimination

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

To find the reduced row echelon form, we need one step on top of Gaussian.

Step 6. Beginning from the last non-0 row and working upward, add suitable multiples of each row to create 0s above the leading 1s.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{added } (7/2)\times \text{ 3rd row to 2nd row}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{added } (-6)\times \text{ 3rd row to 1st row}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{added } 5\times \text{ 2nd row to 1st row}$$

Example

Solve linear system by Gauss-Jordan elimination:

$$\begin{array}{rrrrrr} & & -2x_3 & & +7x_5 & = 12 \\ 2x_1 & +4x_2 & -10x_3 & +6x_4 & +12x_5 & = 28 \\ 2x_1 & +4x_2 & -5x_3 & +6x_4 & -5x_5 & = -1 \end{array}$$

The augmented matrix of system is

$$\left(\begin{array}{rrrrr|r} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right)$$

We have already transformed the above matrix to reduced row echelon form (see four previous slides):

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

The general solution of the system is $x_1 = -2x_2 - 3x_4 + 7, x_3 = 1, x_5 = 2$.

Homogeneous linear systems

- A linear system $A\mathbf{x} = \mathbf{b}$ is homogeneous if $\mathbf{b}$ is all 0s.
- Such a system has a trivial solution: $\mathbf{x}$ is all 0s. Any other solution is called non-trivial.

Theorem

If a homogeneous linear system has n variables and the reduced row echelon form of its augmented matrix has r non-0 rows then the system has $n - r$ free variables.

The above theorem follows immediately from the shape of the reduced row echelon form.

Being consistent and having free variables implies having infinitely many solutions.

Corollary

A homogeneous linear system with more variables than equations has infinitely many solutions.