Mathematics for Computer Science
Linear Algebra

Lecture 11: Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Andrei Krokhin

April 18, 2021

Reminder from two previous lectures

- A vector space is a set V equipped with operations of "addition" and "multiplication by scalars"
  - Examples: $\mathbb{R}^n$ (n-tuples of reals), $\mathbb{M}_{mn}$ (matrices of size $m \times n$).
- Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are called linearly independent iff

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0} \;\Rightarrow\; k_1 = k_2 = \ldots = k_r = 0.$$

- $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V if S spans V and is linearly independent.
- Every vector $\mathbf{v} \in V$ can be represented as a linear combination of vectors in a basis $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$ in exactly one way.

Contents for today's lecture

- Linear maps and matrix transformations;
- Linear operators on $\mathbb{R}^2$.

Linear maps

Definition

Let V and W be vector spaces. A function $T: V \to W$ is called a linear map, or a linear transformation from V to W if, for all $\mathbf{u}, \mathbf{v} \in V, k \in \mathbb{R}$,

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = k \cdot T(\mathbf{u}).$$

If $V = W$ then T is called a linear operator on V.

Example:

If A is an $m \times n$ matrix then the map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(\mathbf{x}) = A\mathbf{x}$ is called a matrix transformation. (Here $\mathbf{x}$ and $A\mathbf{x}$ are column vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, resp.)

Every matrix transformation is linear. Indeed, we have

$$T_A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v})$$

and

$$T_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kT_A(\mathbf{u}).$$

Matrix transformations

Let A is a $m \times n$ matrix and let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ be the columns of A, i.e.

$$A = [\mathbf{a}_1 | \mathbf{a}_2 | \ldots | \mathbf{a}_n].$$

Then, for any $\mathbf{x} = (x_1, x_2 \ldots, x_n) \in \mathbb{R}^n$, we have

$$T_A(\mathbf{x}) = A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

For example, if T_A is a matrix transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ where

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 0 & 1 \end{pmatrix}$$

then

$$T_A(\mathbf{x}) = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Example in $\mathbb{R}^2$

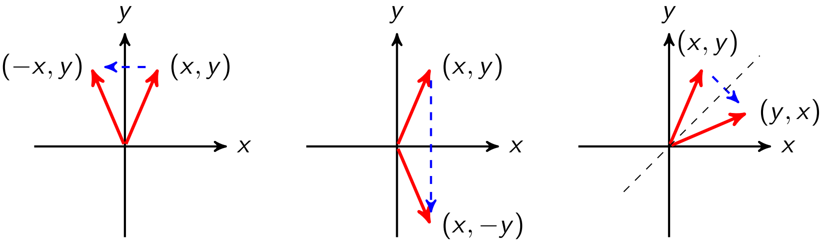
Consider the transformations T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The transformations T_A satisfy

$$T_A(x,y) = (-x,y),\ T_A(x,y) = (x,-y),\ T_A(x,y) = (y,x),\ \text{respectively.}$$

They correspond to reflections of $\mathbb{R}^2$ about y-axis, x-axis, and line $x = y$, resp.



Example in $\mathbb{R}^2$

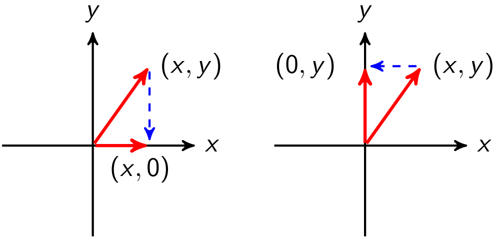
Consider the transformations T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The transformations T_A satisfy

$$T_A(x, y) = (x, 0) \text{ and } T_A(x, y) = (0, y), \text{ respectively.}$$

They correspond to orthogonal projections of $\mathbb{R}^2$ onto x-axis and y-axis, resp.



Example in $\mathbb{R}^2$

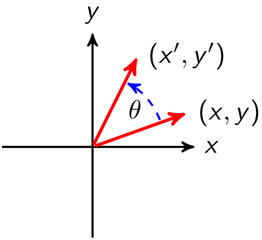
Consider the transformation T_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The transformation T_A satisfies

$$T_A(x, y) = (x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

This corresponds to the rotation of $\mathbb{R}^2$ by angle θ counterclock-wise.



Example in $\mathbb{R}^2$

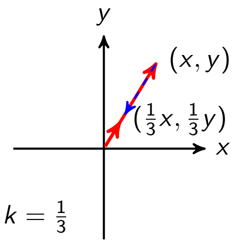
Consider the transformation T_A on $\mathbb{R}^2$ where A is the following matrix:

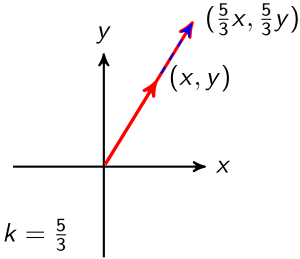
$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

The transformation T_A satisfies

$$T_A(x, y) = (kx, ky).$$

This is contraction (if $0 < k < 1$) or dilation (if $k > 1$) of $\mathbb{R}^2$.





Example in $\mathbb{R}^2$

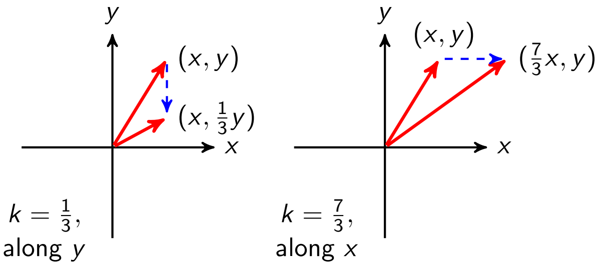
Consider the transformations T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.$$

The transformations T_A satisfy

$$T_A(x,y) = (kx, y) \text{ and } T_A(x,y) = (x, ky), \text{ respectively.}$$

They correspond to compressions (if $0 < k < 1$) and expansions (if $k > 1$) of $\mathbb{R}^2$ along x-axis and y-axis, resp.



Example in $\mathbb{R}^2$

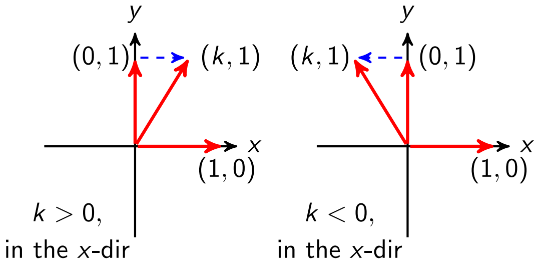
Consider the transformations T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

The transformations T_A satisfy

$$T_A(x, y) = (x + ky, y) \text{ and } T_A(x, y) = (x, kx + y), \text{ respectively.}$$

They correspond to shears of $\mathbb{R}^2$ in the x-direction and y-direction, respectively, with factor k.



Composing matrix transformations

- If T_A is a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$ and T_B is a matrix transformation from $\mathbb{R}^k$ to $\mathbb{R}^m$, one can consider the composition of maps:

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})).$$

- The map $T_B \circ T_A$ is equal to the matrix transformation T_{BA} from $\mathbb{R}^n$ to $\mathbb{R}^m$.
  - Composition of matrix transformations $\leftrightarrow$ matrix multiplication.
- This works for composition of several transformations, e.g. $T_C \circ T_B \circ T_A = T_{CBA}$

Theorem

If A is an invertible 2×2 matrix then the linear operator T_A on $\mathbb{R}^2$ is a composition of shears, compressions, expansions, and reflections.

Idea for a proof: Use (i) the fact that every invertible matrix is a product of elementary matrices and (ii) the examples on the previous slides for operators corresponding to elementary 2×2 matrices.

Linear maps are matrix transformations

Theorem

For every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, there is a unique $m \times n$ matrix A such that $T = T_A$, i.e. $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof:

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.
- Consider the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ and let $A = [T(\mathbf{e}_1)| \ldots |T(\mathbf{e}_n)]$ be the $m \times n$ matrix whose columns are vectors $T(\mathbf{e}_i) \in \mathbb{R}^m$.
- This matrix A is called the (standard) matrix of linear map T.
- Note that $T(\mathbf{e}_i) = A\mathbf{e}_i = T_A(\mathbf{e}_i)$ for all i. For example,

$$T(\mathbf{e}_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = A\mathbf{e}_2 = T_A(\mathbf{e}_2).$$

Proof continued

- Note that $T(\mathbf{e}_i) = A\mathbf{e}_i = T_A(\mathbf{e}_i)$ for all i.
- Choose any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We have $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$.
- We have

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n) = x_1 T(\mathbf{e}_1) + \ldots + x_n T(\mathbf{e}_n)$$

  and

$$T_A(\mathbf{x}) = A(x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n) = x_1 A\mathbf{e}_1 + \ldots + x_n A\mathbf{e}_n = x_1 T_A(\mathbf{e}_1) + \ldots + x_n T_A(\mathbf{e}_n)$$

- Since $T(\mathbf{e}_i) = T_A(\mathbf{e}_i)$ for all i, we have $T(\mathbf{x}) = T_A(\mathbf{x})$.
- To see that A is a unique matrix such that $T = T_A$, let $B \neq A$ be any other matrix - say, they differ in i-th column. Then $T(\mathbf{e}_i) = A\mathbf{e}_i \neq B\mathbf{e}_i = T_B(\mathbf{e}_i)$. $\square$

- Thus, linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are in 1-to-1 correspondence with $m \times n$ matrices. (The same works for any pair of finite-dimensional spaces.)

Example

It is easy to check that the map $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4,\ x_2 + x_3,\ -x_1)$$

is linear. Find its standard matrix.

Solution: we have

$$\begin{array}{rcl} T(1,0,0,0) & = & (7,0,-1) \\ T(0,1,0,0) & = & (2,1,0) \\ T(0,0,1,0) & = & (-1,1,0) \\ T(0,0,0,1) & = & (1,0,0) \end{array}$$

Hence the standard matrix is

$$\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

What we learnt today

- Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- "Special" linear maps: Matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.
- Examples of matrix operators on $\mathbb{R}^2$.

Next time:

- Eigenvalues and eigenvectors of matrices.