Lecture 1: The Basics of Graph Theory

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Contents for today's lecture

- Graphs and types of graphs;
- Graph models;
- Basic terminology;
- Classes of graphs;
- Examples and exercises.

What is a graph?

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- A mathematical model (central for Computer Science)
- A representation of objects and relations between them
- The objects can be 'anything'
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Example

Objects: lecturers and modules (or machines and jobs)
Relation: capability/availability

Formal definitions

Definition

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a nonempty set of vertices (or nodes) and $E(G)$ is a set of unordered pairs $\{u, v\}$ with $u, v \in V(G)$ and $u \neq v$, called the edges of G.

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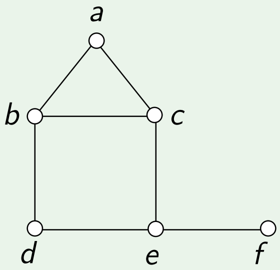
- $V(G)$ can be infinite, but all our graphs here will be finite.
- If no confusion can arise, we write uv instead of $\{u, v\}$.
- If the graph G is clear from the context, we write V and E instead of $V(G)$ and $E(G)$.
- It often helps to draw graphs:
  - represent each vertex by a point, and
  - each edge by a line or curve connecting the corresponding points;
  - only endpoints of lines/curves matter, not the exact shape.

A drawing of a graph

Example



This is a drawing of the graph $G = (V, E)$ with $V = \{a, b, c, d, e, f\}$ and $E = \{ab, ac, bc, bd, ce, de, ef\}$.

Of course the drawing is not unique.

Types of graphs

Possible variations in definition:

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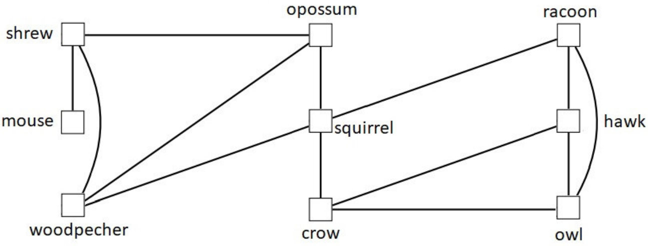
By default, all our graphs are simple undirected or simple directed graphs (sometimes edge-weighted too), i.e. no multiple edges, no loops.

Types of graphs

Case study example: system of species

- undirected edge between two vertices: two species compete for the same food



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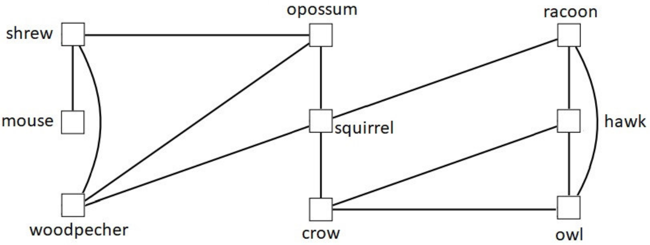

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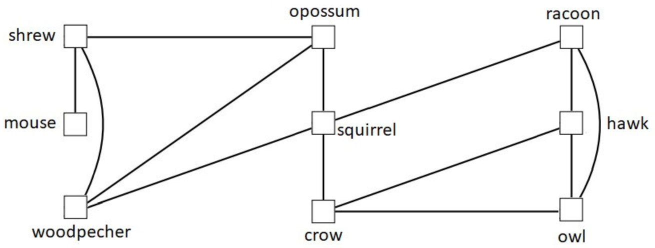


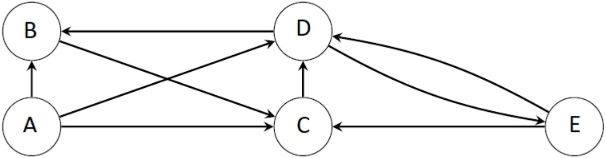
Possible questions:

- "independent set": largest set of non-competing species
  (to live together in a zoo)
- "minimum coloring": partition into the smallest number of independent sets
  (smallest number of rooms in the zoo)

Types of graphs

Case study example: social network

- vertices are persons
- directed edge from x to y: person x influences person y



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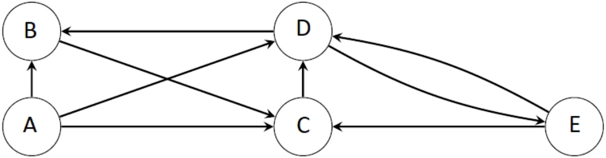

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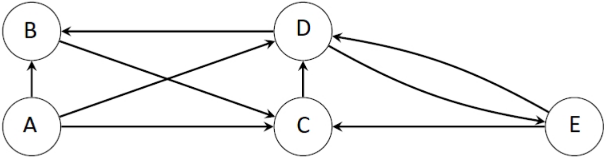

Possible question:

- "dominating set": smallest number of persons, which collectively influence all others (best influencer set), e.g. $\{A, E\}$, $\{A, D\}$, ...

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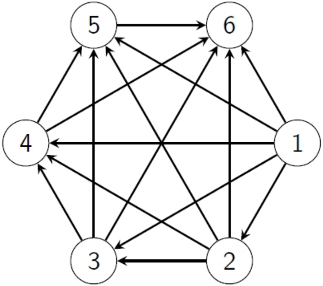
Case study example: sport tournament

- vertices are teams
- directed edge from x to y: team x wins over team y



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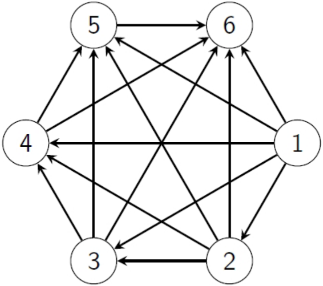

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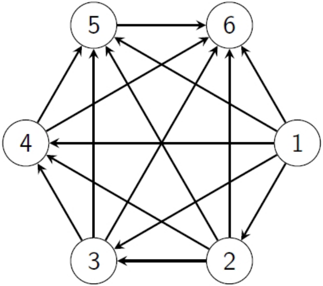

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Q: does always an absolute winner / looser exist?

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- Vertices: states in a game, Edges: transitions between states.
- Vertices: steps in a solution, Edges: transitions between steps.

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Example of graph problems for finding "good strategies" in a game (e.g. chess):

- (Directed) transition edges of player 1 are red, while edges of player 2 are blue
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- Question 1 is relatively simple to answer (a type of "reachability problem"), if the graph of game states is small. However, usually this graph is huge!
- Question 2 is among the hardest questions that one can ask, even when the graph is small. Imagine when the graph is huge (as in a graph of game states)...

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- u and v are called endpoints of the edge uv
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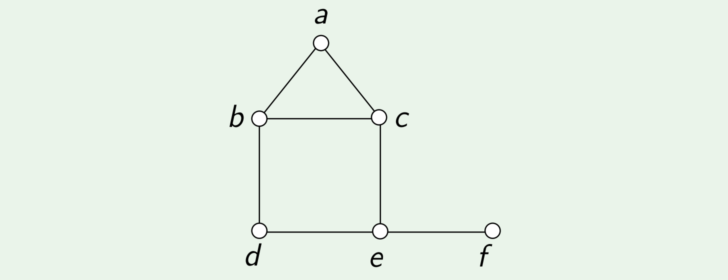

More terminology

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Let $G = (V, E)$ be a graph. The neighbourhood of a vertex $v \in V$, notation $N(v)$, is the set of neighbours of v, i.e., $N(v) = \{ u \in V \mid uv \in E \}$.

The degree of a vertex $v \in V$, notation $deg(v)$, is the number of neighbours of v, i.e. $deg(v) = |N(v)|$.

With $\delta(G)$ or δ we denote the smallest degree in G, and with $\Delta(G)$ or Δ the largest degree.

A vertex with degree 0 will be called an isolated vertex.

A vertex with degree 1 an end vertex or a pendant vertex.

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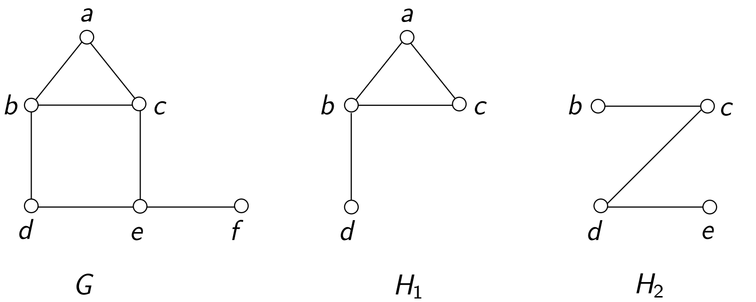
It is called induced subgraph if E' contains all edges of E between vertices of V', i.e. it is obtained by just removing from G all vertices of $V \setminus V'$ (and their edges).

Examples of the above concepts



Examples

The graph H_1 is a subgraph of G, but not a spanning subgraph, so it is also a proper subgraph of G.
H_2 is not a subgraph of $G : cd \notin E(G)$.
The pair $(\{a, b, c\}, \{ab, bd\})$ is no subgraph of G either, since it is not a graph.

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Theorem (Handshaking Lemma)

Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} deg(v) = 2|E|$.

How to prove this?

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Proof.

Every edge has two endpoints and contributes one to each of their degrees, so contributes two to the sum of the degrees of all the vertices of V. $\square$

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This simple relationship can be useful for proving non-existence of graphs with certain properties.

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Corollary

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Let $G = (V, E)$. Partition V to two subsets:

- $V_{odd} = \{v : \deg(v) \text{ is odd}\}$
- $V_{even} = \{v : \deg(v) \text{ is even}\}$

Clearly, $\sum_{v \in V_{even}} \deg(v)$ is even. By the Handshaking Lemma it follows that:

$$\sum_{v \in V_{odd}} \deg(v) = 2 \cdot |E| - \sum_{v \in V_{even}} \deg(v)$$

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Thus there is an even number of vertices with odd degree. $\square$

The most basic graph classes

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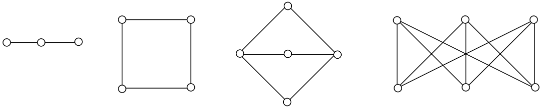

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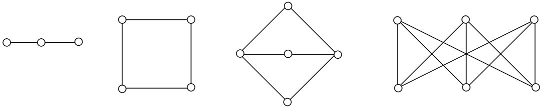

Figure: Special graph classes

The first graph is often denoted by P_3, and in general we define P_n as the path on n vertices, i.e. a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

So, P_n has ??? edges.

Definition

A path in a graph G is a subgraph of G which is *(isomorphic to)* the graph P_k, for some integer $k \geq 1$. Sometimes a path is also called a simple path.

The most basic graph classes

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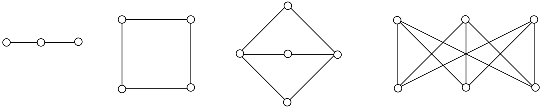

Figure: Special graph classes

The first graph is often denoted by P_3, and in general we define P_n as the path on n vertices, i.e. a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n\}$.

So, P_n has $n - 1$ edges.

Definition

A path in a graph G is a subgraph of G which is *(isomorphic to)* the graph P_k, for some integer $k \geq 1$. Sometimes a path is also called a simple path.

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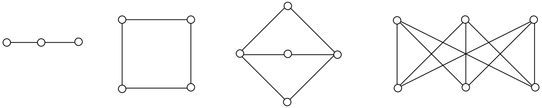

Figure: Special graph classes

The second graph is often denoted by C_4, the cycle on 4 vertices. In general a C_n on n vertices is defined similarly to the P_n, but now with an additional edge between v_n and v_1. So, C_n has ??? edges.

Definition

A cycle in a graph G is a subgraph of G which is *(isomorphic to)* the graph C_k, for some integer $k \geq 3$. Sometimes a cycle is also called a simple circuit.

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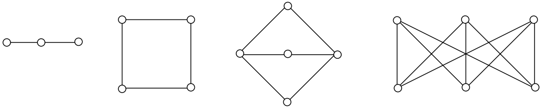

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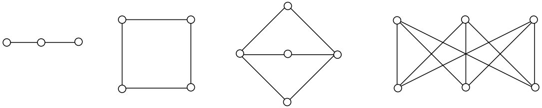

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How many cycles does the third graph have?

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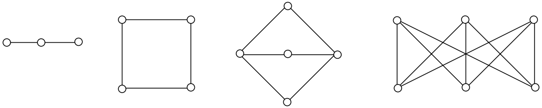

Figure: Special graph classes

All four of these graphs can be described as a $K_{p,q}$: a graph consisting of two disjoint vertex sets on p and on q vertices, and all possible edges between these two vertex sets (and no other edges). So, $K_{p,q}$ has ??? edges.

Definition

$K_{p,q}$ is called a complete bipartite graph. Any subgraph of $K_{p,q}$ is called a bipartite graph.

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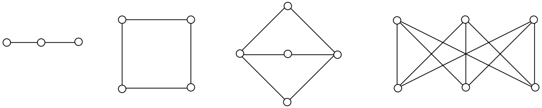

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All four of these graphs can be described as a $K_{p,q}$: a graph consisting of two disjoint vertex sets on p and on q vertices, and all possible edges between these two vertex sets (and no other edges). So, $K_{p,q}$ has $p \cdot q$ edges.

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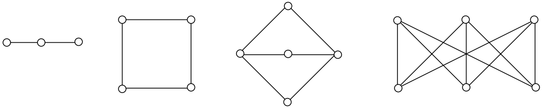

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$K_{p,q}$ is called a complete bipartite graph. Any subgraph of $K_{p,q}$ is called a bipartite graph.
So a graph is bipartite if and only if we can partition its vertex set to two vertex sets such that every edge has one endpoint in each set.

Bipartite graphs play an eminent role in scheduling and assignment problems.

Some graph classes

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The (n-dimensional) hypercube or n-cube Q_n $(n \geq 1)$ is the graph with

$$V = \{\,(e_1, \ldots, e_n) \mid e_i \in \{0,1\}\ (i = 1, \ldots, n)\,\},$$

in which two vertices are neighbours if and only if the corresponding rows differ in exactly one entry.

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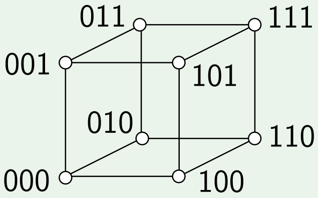
Examples

$Q_1 = P_2 = K_2$; $Q_2 = C_4$. For $n = 3$ the set V consists of $2^3 = 8$ elements, namely all rows (in short hand notation) 000, 001, 010, 011, 100, 101, 110, 111.



Q_3

More on n-cubes

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How to prove this?

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- This is clearly a partition of V into two disjoint sets.
- It is easy to see that each edge has one endpoint in each of the sets.
- So it proves that all n-cubes are bipartite.

□

Exercise

Exercise 1: A graph is called k-regular if all of its vertices have degree k. Which of the graphs P_n, C_n, $K_{p,q}$, K_n, Q_n are k-regular (for some k)?

Exercise 2: Find the number of edges in Q_n.

Exercise 3: Which of the graphs P_n, C_n, K_n are bipartite?