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Maths for Computer Science
Calculus

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Limit at $\pm\infty$

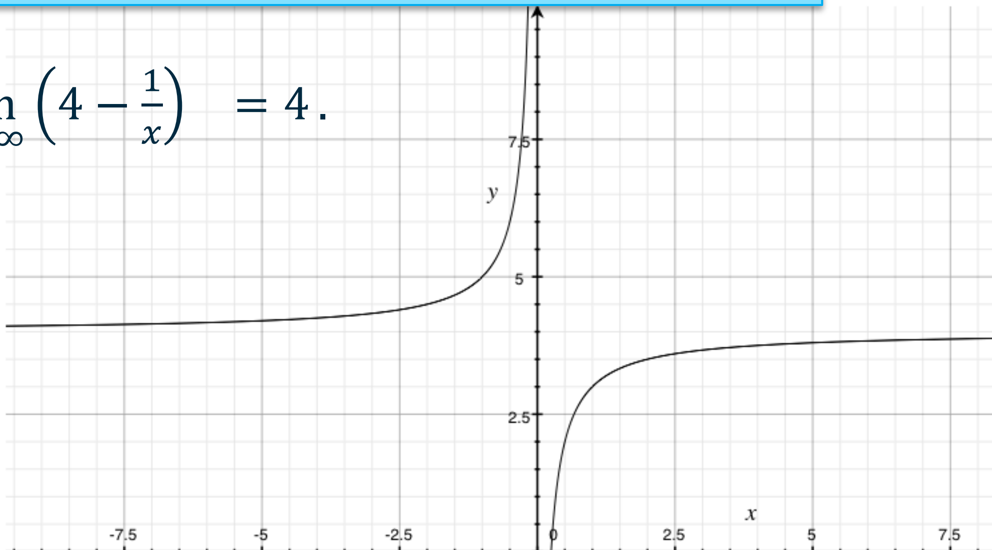
A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a limit L as x tends to $+\infty$ if and only if for any $\epsilon > 0$, there exists a $N > 0$ such that:

$$\forall x: N < x \text{ we have } |f(x) - L| < \epsilon.$$

A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a limit L as x tends to $-\infty$ if and only if for any $\epsilon > 0$, there exists a $N < 0$ such that:

$$\forall x: x < N \text{ we have } |f(x) - L| < \epsilon.$$

Example: $f(x) = \left(4 - \frac{1}{x}\right)$ we say $\lim_{x\to\infty}\left(4 - \frac{1}{x}\right) = 4$.



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Limit of $\pm\infty$

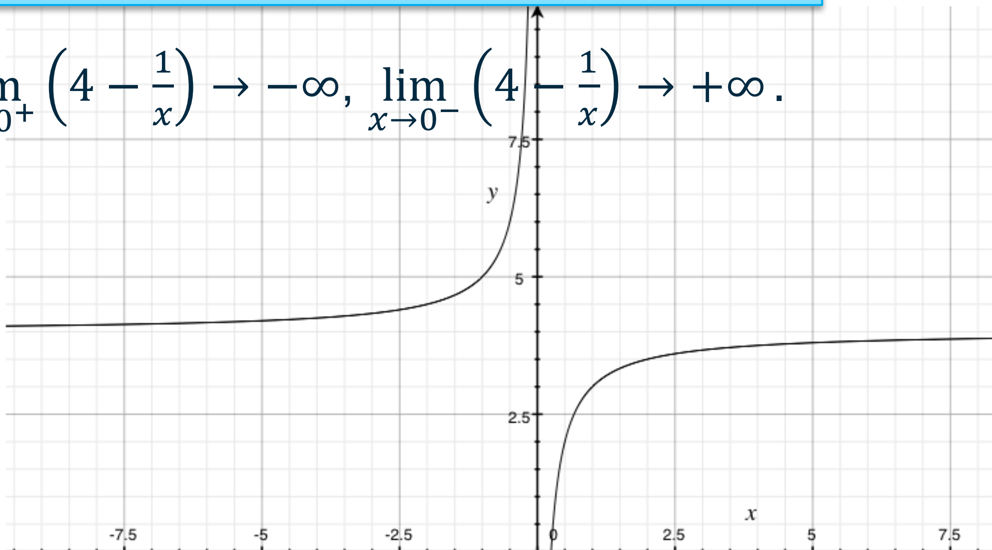
A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a limit $+\infty$ at $x = a$ if and only if for any $N > 0$, there exists a $\delta > 0$ such that:

$$\forall x:\ 0 < |a - x| < \delta \text{ we have } f(x) > N.$$

A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a limit $-\infty$ at $x = a$ if and only if for any $N > 0$, there exists a $\delta > 0$ such that:

$$\forall x:\ 0 < |a - x| < \delta \text{ we have } f(x) < -N.$$

Example: $f(x) = \left(4 - \frac{1}{x}\right)$ we say $\lim_{x \to 0^+} \left(4 - \frac{1}{x}\right) \to -\infty$, $\lim_{x \to 0^-} \left(4 - \frac{1}{x}\right) \to +\infty$.



Continuity and limit arithmetic

Let f, g and h be functions such that

- $|f(x)| \leq L$ for $x \in (a, b)$
- $\lim_{x \to x^*} g(x) \to \infty$ for some $x^* \in (a, b)$
- $\lim_{x \to x^*} h(x) = 0$ for some $x^* \in (a, b)$.

Then

- $\lim_{x \to x^*} (f(x) + g(x)) \to \infty$.
- $\lim_{x \to x^*} \left(\frac{f(x)}{g(x)}\right) = 0$.
- $\lim_{x \to x^*} (f(x)h(x)) = 0$.

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Differentiation

Differentiability

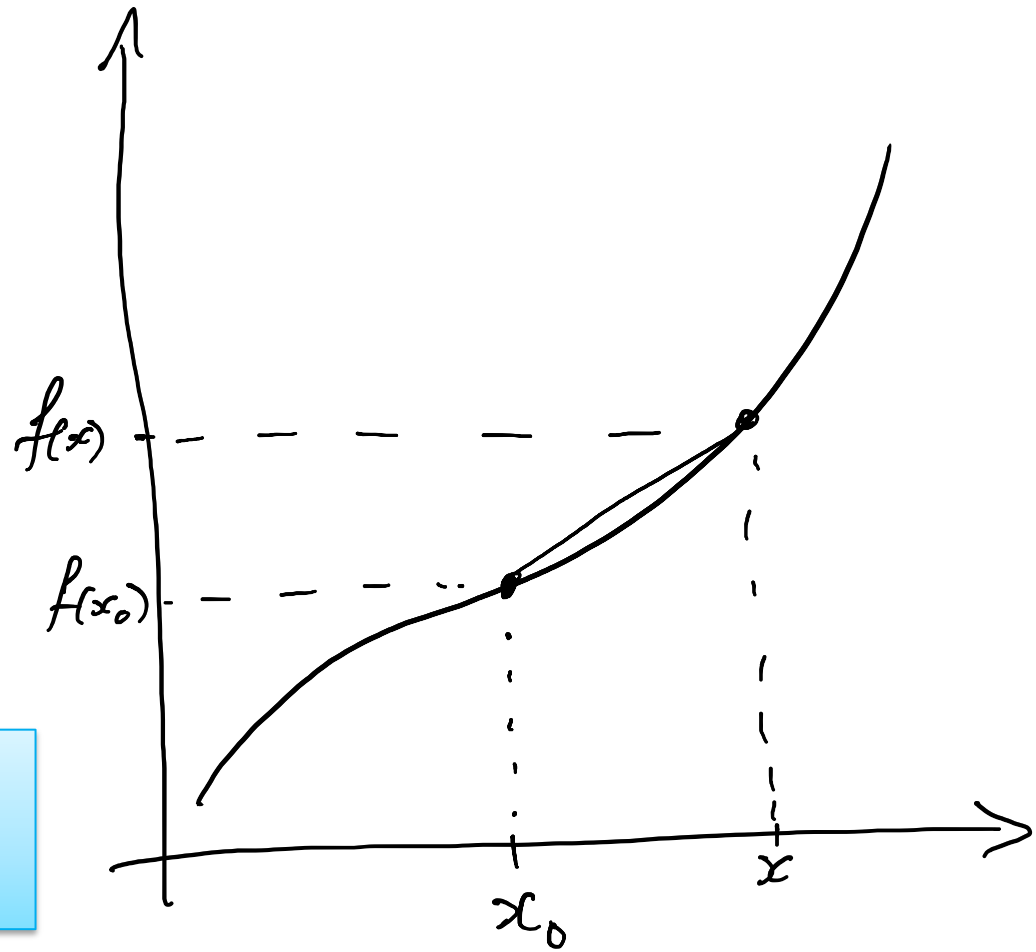
Intuitively the derivative of a function $f(x)$ at a point $x = x_0$ is the instantaneous rate of change (gradient) of f at the point x_0.

The gradient at x_0 may be approximated by $\frac{f(x)-f(x_0)}{x-x_0}$.

The closer we take x to x_0, the better the approximation will be.

Formally we define:

f is differentiable at $x = x_0$ if and only if $\lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists.



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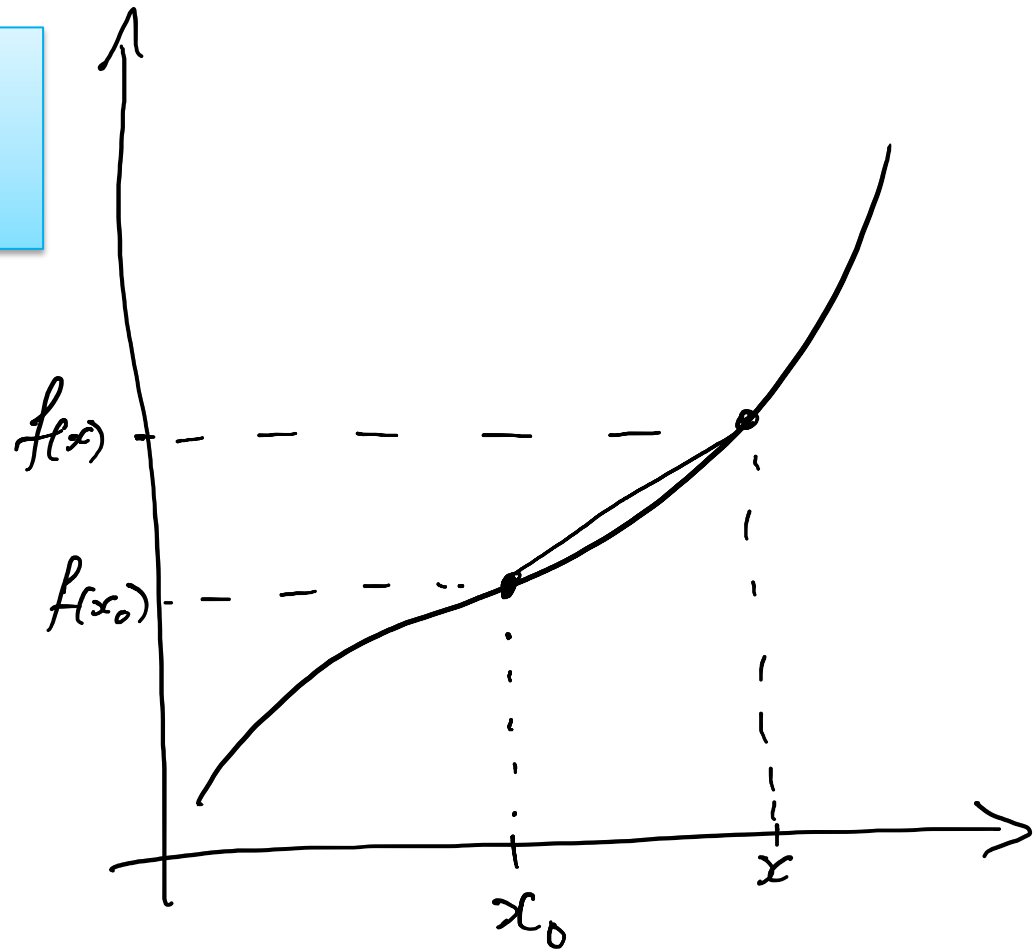
Derivatives

If f is differentiable at x_0 we call $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ the derivative of f at x_0.

The derivative at x_0 is denoted $f'(x_o)$ or $\frac{df}{dx}(x_0)$.

If f is differentiable at all points in an interval (a, b), then the derivative function $f'(x)$ is the function that maps a point. $x \in (a, b)$ to the derivative of f at x.



Example: $x\ \sin\frac{1}{x}$

Define a function $f(x) = \begin{cases} x\ \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

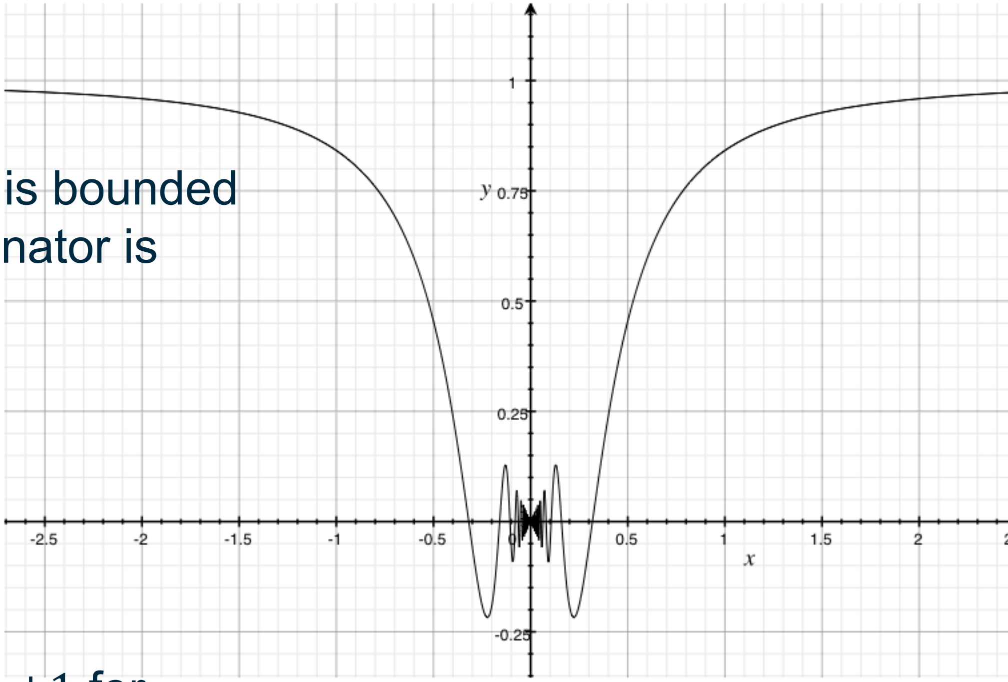
For $x \neq 0$ we can write $f(x) = \frac{\sin\frac{1}{x}}{\frac{1}{x}}$.

So when x tends to 0, the numerator is bounded between -1 and $+1$, and the denominator is unbounded, so $\lim_{x\to 0} f(x) = 0$.

Hence f is continuous on $(-\infty, \infty)$.

For $x_0 = 0$, the derivative is

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x\ \sin\frac{1}{x}}{x} = \lim_{x\to 0} \sin\frac{1}{x}.$$



Since this oscillates between -1 and $+1$ for arbitrarily small x, the limit does not exist.

Hence f is not differentiable at 0.

Basic derivatives

If $f(x) = c$ for some constant c, then $f'(x) = 0\ \forall x$.

By definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$.

Basic derivatives

If $f(x) = c$ for some constant c, then $f'(x) = 0\ \forall x$.

If $f(x) = x^n$ for some $n \in \mathbb{N}^{>0}$, then $f'(x) = nx^{n-1}$

By definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + \binom{n}{n} h^n$ so

$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + h\left(\binom{n}{2} x^{n-2} + \cdots + \binom{n}{n} h^{n-1}\right)$ and

$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}$.

Basic derivatives

If $f(x) = x^{-n}$ for some $n \in \mathbb{N}^{>0}$, then $f'(x) = -nx^{-n-1}$ for $x \neq 0$.

By definition $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h)^{-n}-x^{-n}}{h}$

$\frac{(x+h)^{-n}-x^{-n}}{h} = \frac{x^n-(x+h)^n}{h} . \frac{1}{(x+h)^n x^n}$ and

$\lim\limits_{h \to 0} - \frac{(x+h)^n - x^n}{h}$ = $-nx^{n-1}$ and also for $x \neq 0$, $\lim\limits_{h \to 0} \frac{1}{(x+h)^n x^n} = \frac{1}{x^{2n}}$ so

$f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$.

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Basic derivatives

If $f(x) = x^{1/n}$ for some $n \in \mathbb{N}^{>0}$, then $f'(x) = \frac{1}{n}x^{1/n-1}$ for $x > 0$.

By definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{1/n}-x^{1/n}}{h}$

Note that $(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$.

If we set $a = (x+h)^{1/n}, b = x^{1/n}$ and $\mathrm{C} = (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$

We can "rationalise the numerator" by multiplying by C.

$$\frac{(x+h)^{1/n} - x^{1/n}}{h} . \frac{C}{C} = \frac{(x+h) - x}{h} . \frac{1}{C} = \frac{1}{C}$$

Now C is a sum of n terms of the form $\left((x+h)^{\frac{1}{n}}\right)^{n-i} \left(x^{\frac{1}{n}}\right)^{i-1}$

Each of which tends to $\left(x^{\frac{1}{n}}\right)^{n-i} \left(x^{\frac{1}{n}}\right)^{i-1} = x^{\frac{n-1}{n}}$ as $h \to 0$.

So $f'(x) = \lim_{h\to 0} \frac{1}{C} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}$.

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Basic derivatives

If $f(x) = \sin \alpha x$ for some $\alpha \in \mathbb{R}$, then $f'(x) = \alpha \cos \alpha x$.

By definition $f'(x) = \lim_{h\to 0} \frac{\sin \alpha(x+h) - \sin \alpha x}{h} =$

$$= \lim_{h\to 0} \frac{\sin \alpha x \cos \alpha h + \cos \alpha x \sin \alpha h \ - \sin \alpha x}{h}$$

$$= \lim_{h\to 0} \sin \alpha x \left(\frac{\cos \alpha h - 1}{h}\right) + \lim_{h\to 0} \cos \alpha x \left(\frac{\sin \alpha h}{h}\right)$$

$$= \alpha \sin \alpha x \lim_{h\to 0} \left(\frac{\cos \alpha h - 1}{\alpha h}\right) + \alpha \cos \alpha x \lim_{h\to 0} \left(\frac{\sin \alpha h}{\alpha h}\right)$$

$$= 0 + \alpha \cos \alpha x$$

Note: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
and $\lim_{h\to 0} \left(\frac{\cos \alpha h - 1}{\alpha h}\right) = 0$

Differentiation of products

If $f(x)$ and $g(x)$ are differentiable at x_0 then so is $f(x)g(x)$ and $\frac{df(x)g(x)}{dx}$ at x_0 is equal to $f'(x)g(x) + f(x)g'(x)$.

Proof:

$$\frac{df(x)g(x)}{dx} = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} g(x) + \frac{g(x)-g(x_0)}{x-x_0} f(x_0)$$

So $\frac{df(x)g(x)}{dx} = f'(x_0) \lim_{x \to x_0} g(x) + g'(x_0) f(x_0)$.

All we need now is that $\lim_{x \to x_0} g(x) = g(x_0)$, i.e. that g is **continuous** at x_0.

But if $\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g(x_0)$ exists, then $g(x) - g(x_0) = (x - x_0)g(x_0)$, and so $\lim_{x \to x_0} g(x) - g(x_0) = \lim_{x \to x_0} (x - x_0)g(x_0) = 0$ which implies $\lim_{x \to x_0} g(x) = g(x_0)$.

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Chain Rule

If $g(x)$ and $f(x)$ are differentiable at x_0 and at $g(x_0)$ respectively, then the composite $f \circ g(x)$ is differentiable at x_0 and $\frac{df \circ g(x)}{dx}$ at x_0 is equal to $f'(g(x_0)\,)g'(x_0)$.

Proof:

$$\frac{df \circ g(x)}{dx} = \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} = \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} . \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}.$$

Since $g(x)$ is differentiable at x_0, it is continuous at x_0.

So as $x \to x_0$, $g(x) \to g(x_0)$ and $\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = \lim_{g(x) \to g(x_0)} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)}$.

I.e. $\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0))$.

The result follows.

Using the chain rule

For a composition of functions $f \circ g(x)$, set a new variable $u = g(x)$.

So $f \circ g(x) = f(u)$. Then the chain rule can be written

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

And if u is itself a composite function we can apply the chain rule again, setting $u = u(v(x))$:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dv}\frac{dv}{dx}$$

And so on.

Examples

Differentiate $\sin(x^2+3)$.

Set $f(u)=\sin(u)$ and $u(x)=x^2+3$.

Then $f'(u)=\cos u$, and $u'(x)=2x$.

Then by the chain rule

$$\frac{df}{dx}=\frac{df}{du}\frac{du}{dx}=\cos(u)\,.\,2x=2x.\cos(x^2+3)$$

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Examples

Differentiate $\sin\sqrt{x^2+1}$.

Set $f(u) = \sin(u)$, $u(v) = \sqrt{v}$ and $v(x) = x^2 + 1$.

Then $f'(u) = \cos u$, $u'(v) = \frac{1}{2}v^{-\frac{1}{2}}$ and $v'(x) = 2x$.

Then by the chain rule

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dv}\frac{dv}{dx} = \cos(u).\frac{1}{2}v^{-\frac{1}{2}}.2x$$

$$= \cos\sqrt{x^2+1}.\frac{1}{2\sqrt{x^2+1}}.2x$$

$$= \frac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

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Differentiation of a quotient

We can use the chain rule to derive the quotient rule. Given functions f and g both differentiable at x_0 and with $g(x_0) \neq 0$, then $\left(\frac{f(x)}{g(x)}\right)$ is differentiable at x_0 and

$$\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

Proof:

$$\frac{d\left(\frac{f}{g}\right)}{dx} = f'(x)\left(\frac{1}{g(x)}\right) + f(x)\frac{d\left(\frac{1}{g}\right)}{dx}$$

By the chain rule, setting $u = g(x)$,

$$\frac{d\left(\frac{1}{g}\right)}{dx} = \frac{d\left(\frac{1}{u}\right)}{du}\frac{du}{dx} = -\frac{1}{u^2}g'(x) = -\frac{g'(x)}{g(x)^2}.$$

Putting these together give the result.

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Example

Differentiate $h(x) = \frac{3x+1}{x^2-2}$.

Using the quotient rule with $f(x) = 3x + 1$

and $g(x) = x^2 - 2$,

$$h'(x) = \frac{3(x^2 - 2) - (3x + 1)2x}{(x^2 - 2)^2} = \frac{-3x^2 - 2x - 6}{(x^2 - 2)^2}$$

when $x \neq \pm\sqrt{2}$.

Extrema

Let $f(x)$ be a function defined on an interval $[a, b]$.

A point $x_0 \in [a, b]$ is:

- an absolute maximum if $f(x_0) \geq f(x) \;\; \forall\, x \in [a, b]$
- an absolute minimum if $f(x_0) \leq f(x) \;\; \forall\, x \in [a, b]$
- a local maximum if $\exists \delta > 0\colon\; f(x_0) \geq f(x_0 + h) \;\; \forall\, |h| < \delta$
- a local minimum if $\exists \delta > 0\colon\; f(x_0) \leq f(x_0 + h) \;\; \forall\, |h| < \delta$

Example: Most AI boils down to the following.

For a function AI(input, parameters) = output, we want to set the parameters so that the outputs are close to some ground truth for each input.

I.e. We have a function

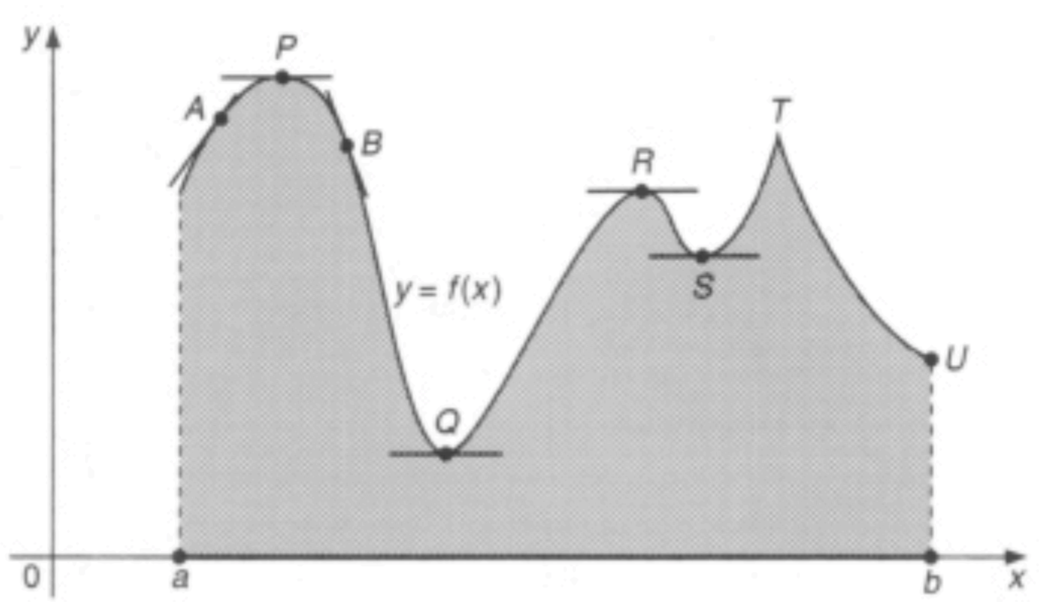
$$\mathrm{error}(params) = \sum_{inputs} (AI(input, params) - groundtruth(input))$$

Extrema

Let $f(x)$ be a function defined on an interval $[a, b]$.

A point $x_0 \in [a, b]$ is:

- an absolute maximum if $f(x_0) \geq f(x) \;\; \forall\, x \in [a, b]$
- an absolute minimum if $f(x_0) \leq f(x) \;\; \forall\, x \in [a, b]$
- a local maximum if $\exists \delta > 0$: $f(x_0) \geq f(x_0 + h) \;\; \forall\, |h| < \delta$
- a local minimum if $\exists \delta > 0$: $f(x_0) \leq f(x_0 + h) \;\; \forall\, |h| < \delta$



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Extrema

Let $f(x)$ be a function defined on an interval $[a, b]$ and differentiable at a point $x_0 \in [a, b]$.

Then if x_0 is a maximum (or minimum) of f, $f'(x_0) = 0$.

Proof (maximum case):

For x sufficiently close to x_0 and $x > x_0$, $\frac{f(x)-f(x_0)}{x-x_0} < 0$, so $\lim\limits_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$.

For x sufficiently close to x_0 and $x < x_0$, $\frac{f(x)-f(x_0)}{x-x_0} > 0$, so $\lim\limits_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$.

Hence as $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists, it must be exactly 0.

Points where $f'(x) = 0$ are called **stationary points**.

Example

$f(x) = \frac{x^3}{3} + 2x^2 + 3x + 1.$

f is continuous and differentiable on $(-\infty, \infty)$ so stationary points when $f'(x) = 0.$

$f'(x) = \frac{3x^2}{3} + 4x + 3 = (x + 1)(x + 3).$

So f has extrema at $x = -1$ and $x = -3.$

What form do these have?

Consider f' very close to $-1,$ i.e. at $x = -1 + h$ for some small $h.$

$f'(-1 + h) = (h)(h + 2).$ For small h this is positive for $h > 0$ and negative for $h < 0.$

I.e. f is sloping down to the left of -1 and up to the right of -1, so -1 is a minimum.

Near -3, $f'(-3 + h) = (-2 + h)(h)$ which is positive for $h < 0$ and negative for $h > 0,$ so we get a maximum.

Rolle's theorem

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).

If $f(a) = f(b)$ then there exists some $\xi \in (a, b)$ such that $f'(\xi) = 0$.

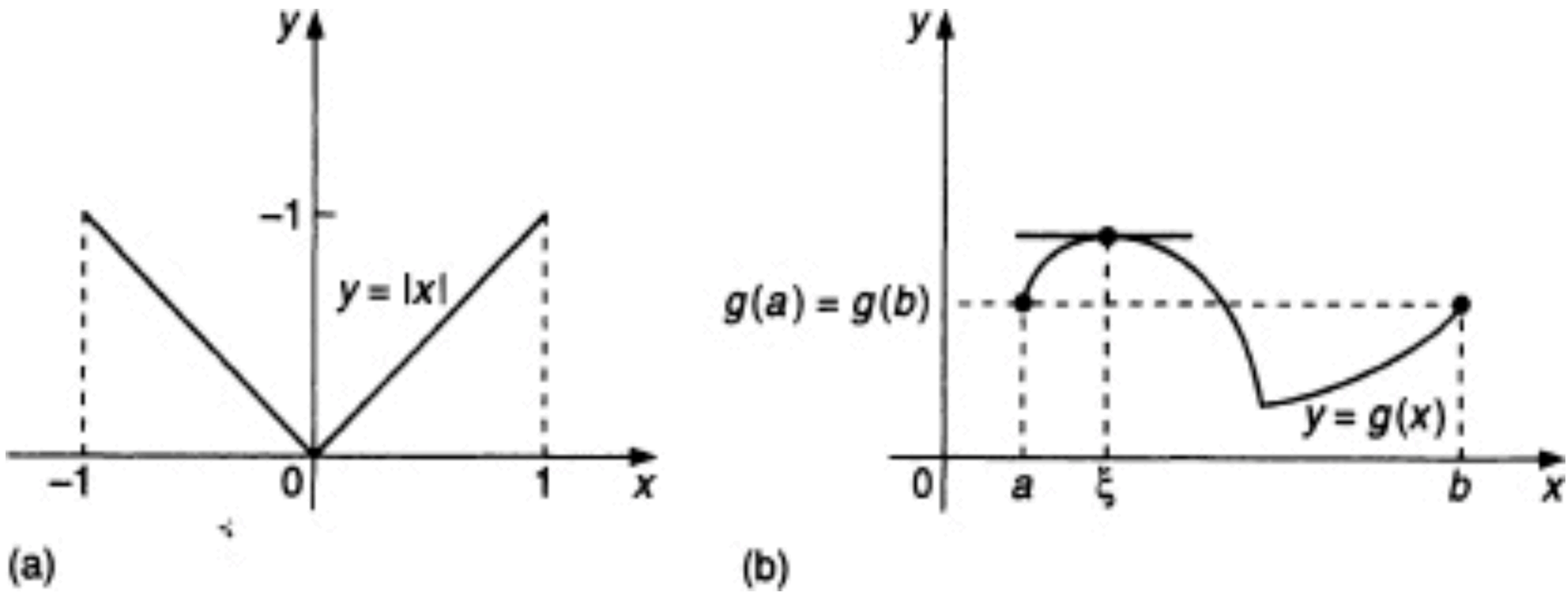
Proof:

Let m be the minimum of f and M the maximum of f on $[a, b]$.

One of the following must occur:

- $m = M = f(a) = f(b)$. Then f is constant and $f'(x) = 0$. $\forall x \in (a, b)$.
- $M > f(a)$. Then the maximum occurs at some point $\xi \in (a, b)$.
  Since ξ is a maximum of f, it must be that $f'(\xi) = 0$.
- $m < f(a)$. Then the minimum occurs at some point $\xi \in (a, b)$.
  Since ξ is a minimum of f, it must be that $f'(\xi) = 0$.

Two cases where Rolle's theorem does not apply



Mean value theorem for derivatives

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).

Then there exists some $\xi \in (a, b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a} = m$.

Proof:

Let $g(x) = f(x) - m(x - a)$.

By Rolle's Theorem there is some ξ such that $g'(\xi) = 0$.

But $g'(x) = f'(x) - m$ so $f'(\xi) = m$.

MVT example: $f(x) = (x+1)^3$ on $[-1,1]$

$f(-1) = 0, f(1) = 8$ so $m = 4$.

$f'(x) = 3(x+1)^2$ so we are looking for ξ such that $3(\xi + 1)^2 = 4$.

We can solve this quadratic:

$$(\xi + 1)^2 = 4/3$$

$$(\xi + 1) = \pm \frac{2}{\sqrt{3}}$$

$$\xi = \pm \frac{2}{\sqrt{3}} - 1$$

Taking the value in $[-1,1]$ we get $\xi = \frac{2}{\sqrt{3}} - 1$.

Extended mean value theorem (Cauchy)

Let f, g be functions that are continuous on $[a, b]$ and differentiable on (a, b).

Then there exists some $\xi \in (a, b)$ such that $\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Proof:

Let $h(x) = f(a)g(a) - f(b)g(a) + [g(a) - g(b)]f(x) - [f(a) - f(b)]g(x)$.

h is continuous on $[a, b]$ and differentiable on (a, b).

$$\begin{aligned} h(a) &= f(a)g(a) - f(b)g(a) + [g(a) - g(b)]f(a) - [f(a) - f(b)]g(a) \\ &= f(a)g(a) - f(b)g(a) + f(a)g(a) - f(a)g(b) - f(a)g(a) + f(b)g(a) \\ &= f(a)g(a) - f(a)g(b). \end{aligned}$$

Also

$$\begin{aligned} h(b) &= f(a)g(a) - f(b)g(a) + [g(a) - g(b)]f(b) - [f(a) - f(b)]g(b) \\ &= f(a)g(a) - f(b)g(a) + f(b)g(a) - f(b)g(b) - f(a)g(b) + f(b)g(b) \\ &= f(a)g(a) - f(a)g(b). \end{aligned}$$

Extended mean value theorem (Cauchy)

Let f, g be functions that are continuous on $[a, b]$ and differentiable on (a, b).

Then there exists some $\xi \in (a, b)$ such that $\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Proof:

Let $h(x) = f(a)g(a) - f(b)g(a) + [g(a) - g(b)]f(x) - [f(a) - f(b)]g(x)$.

h is continuous on $[a, b]$ and differentiable on (a, b).

So $h(a) = h(b)$, and hence we can apply Rolle's Theorem.

By Rolle's Theorem there is some ξ such that $h'(\xi) = 0$.

I.e. $[g(a) - g(b)]f'(\xi) - [f(a) - f(b)]g'(\xi) \ = 0$, whence

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

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L'Hôpital's Rule

Let f, g be functions that are differentiable at x_0. If

1. $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, or $\lim_{x \to x_0} |f(x)| \to \infty$ and $\lim_{x \to x_0} |g(x)| \to \infty$,
2. $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$ exists, or $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \to \pm\infty$
3. $g'(x) \neq 0$ in some region around (but not at) x_0.

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Proof (case 0/0):

By the extended MVT there is some $\xi \in (x_0, x)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f(x)}{g(x)}.$$

Then as $x \to x_0$ also $\xi \to x_0$, so

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)}.$$

L'Hôpital's Rule Examples

$\lim_{x\to 0}\frac{\sin \alpha x}{x}$? Type $\frac{0}{0}$ so = $\lim_{x\to 0}\frac{\alpha\ \cos \alpha x}{1} \to \alpha$.

$\lim_{x\to 0}\frac{\sin \alpha x}{x^2}$? Type $\frac{0}{0}$ so = $\lim_{x\to 0}\frac{\alpha\ \cos \alpha x}{2x} \to \infty$. Correct.

It would be incorrect to apply L'Hôpital again: $\lim_{x\to 0}\frac{\alpha\ \cos \alpha x}{2x} \neq \lim_{x\to 0}\frac{-\alpha^2\ \sin \alpha x}{2} = 0$.

$\lim_{x\to 0}\left(\frac{1}{\sin 2x} - \frac{1}{2x}\right)$? Type $\infty - \infty$. Reformat as $\lim_{x\to 0}\left(\frac{2x-\sin 2x}{2x \sin 2x}\right)$, now type $\frac{0}{0}$.

So = $\lim_{x\to 0}\frac{2-2\cos 2x}{2\sin 2x+4x\cos 2x} \to \alpha$ still type $\frac{0}{0}$ so = $\lim_{x\to 0}\frac{4\sin\ 2x}{8\cos 2x-8x\sin 2x} \to 0$.

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