Maths for Computer Science
Calculus

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Taylor's Theorem

Quadratic approximation from function

Suppose we start with some function $f(x)$ and we wish to determine a quadratic form such that near x_0

$$f(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)^2.$$

What do we mean by $\approx$?

Let's at least demand that f and our approximation have

- the same value at x_0
- the same slope (derivative) at x_0
- the same curvature (2nd derivative) at x_0

Quadratic approximation from function

Suppose we start with some function $f(x)$ and we wish to determine a quadratic form such that near x_0

$$f(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)^2.$$

First observe that by setting $x = x_0$, we must have $f(x_0) = a_0$.

Now differentiate once and set $x = x_0$:

$$f'(x_0) = a_1 + 2a_2(x_0 - x_0) = a_1, \qquad i.e. \quad f'(x_0) = a_1$$

Differentiating again:

$$f''(x_0) = 2a_2, \qquad i.e. \quad f''(x_0) = 2a_2$$

Putting it together:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

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Taylor series from function

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Now we ask for exact equality.

As before, by setting $x = x_0$, we must have $a_0 = f(x_0)$.

Now assume $r > 0$ and differentiate once: for $-r < x < r$

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots,$$

$$i.e. \quad f'(x_0) = a_1$$

Differentiating again: for $-r < x < r$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = 2a_2 + 6a_3(x - x_0) + 12a_4(x - x_0)^2 \ldots,$$

$$i.e. \quad f''(x_0) = 2a_2$$

Proceeding systematically:

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1) \ldots (n - m + 1) a_n (x - x_0)^{n-m}, whence \;\; f^{(m)}(x_0) = m!\, a_m.$$

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Taylor series from function

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.

So putting it together, **if** f is equal to a power series then the power series must be:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}.$$

This form is called the **Taylor series expansion of f**.

Recall the Maclaurin expansion is just the special case when $x_0 = 0$.

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Example: $\ln(x+1)$

Let $f(x) = \ln(x+1)$.

Note: $f(0) = 0$

Then $f'(x) = \frac{1}{x+1}, f''(x) = -\frac{1}{(x+1)^2}, \ldots, f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(x+1)^n}$

So $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

Hence the Maclaurin series for f is

$$\ln(x+1) = \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\frac{x^n}{n!} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

We have already seen this converges on $(-1,1]$. Hence this expansion is valid in this region.

Example: $x\ln(x)$

Let $f(x) = x\ln(x)$.

Differentiating:

$$f'(x) = 1 + \ln x\,, f''(x) = \frac{1}{x}, f'''(x) = -\frac{1}{x^2}, \ldots, f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}$$

Note: at 0 these are undefined! So we can't make a Maclaurin series.

But we can make the Taylor series at $x_0 = 1$.

Then $\mathrm{f}(1) = 0, f'(1) = 1, f''(1) = 1, \ldots, f^{(n)}(1) = (-1)^{n-1}(n-2)!$

Hence the Taylor series for f is

$$x\ln(x) = (x-1) + \frac{(x-1)^2}{1.2} - \frac{(x-1)^3}{2.3} + \frac{(x-1)^4}{3.4} - \frac{(x-1)^5}{4.5} \ldots$$

$$x\ln(x) = (x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}(x-1)^n$$

Which converges on $[0,2]$.

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Taylor's Theorem

So far we have said: ***if*** a series exists that is equal to f, then it must have these coefficients. This is not quite the same as proving that the series does indeed converge to the correct value.

Taylor's Theorem says the following:

Suppose a function f is n times differentiable on $[a, x]$, then for then there is some $\xi \in (a, x)$ such that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi)$$

Proof of Taylor's Theorem

Suppose a function f is n times differentiable on $[a, x]$.

Define a constant k such that

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k$$

And the function $F(y)$ by

$$F(y) = f(x) - f(y) - (x - y)f'(y) - \cdots - \frac{(x-y)^{n-1}}{(n-1)!}f^{(n-1)}(y) - \frac{(x-y)^n}{n!}k$$

Then clearly $F(x) = 0$ (i.e. $F(y)$ evaluated at $y = x$ is 0), and also

$$F(a) = f(x) \ - [f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k\ ]$$

$$= f(x) - f(x) = 0.$$

So we can apply Rolle's theorem to F on $[a, x]$, i.e. there is some $\xi \in (a, x)$ such that $F'(\xi) = 0$.

Proof of Taylor's Theorem

$$F(y) = f(x) - f(y) - (x-y)f'(y) - \frac{(x-y)^2}{2!}f''(y) - \cdots - \frac{(x-y)^{n-1}}{(n-1)!}f^{(n-1)}(y) - \frac{(x-y)^n}{n!}k$$

So differentiating w.r.t. y

$$F'(y) = 0 - f'(y) + f'(y) - (x-y)f''(y) + (x-y)f''(y) - \cdots - \frac{(x-y)^{n-1}}{(n-1)!}f^{(n)}(y) + \frac{(x-y)^{n-1}}{(n-1)!}k$$

I.e.

$$F'(y) = \frac{(x-y)^{n-1}}{(n-1)!}\left(k - f^{(n)}(y)\right)$$

Since $F'(\xi) = 0$ we have

$$F'(\xi) = \frac{(x-\xi)^{n-1}}{(n-1)!}\left(k - f^{(n)}(\xi)\right) = 0$$

But as $\xi \in (a, x)$, i.e. $(x-\xi) \neq 0$, this can only happen if $f^{(n)}(\xi) = k$, so

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi).$$

Example: $\sin x$

Let $f(x) = \sin x$.

Then $f(0) = 0$,

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$f^{(2k)}(0) = 0$

$f^{(2k+1)}(0) = (-1)^k$

Since $f \in C^\infty$, for any x, n there is some $\xi \in (0, x)$ such that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots - \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} \sin^{(\mathrm{n})} \xi$$

Since $|\sin^{(n)} \xi\,| \leq 1$ we have $\lim_{n\to\infty} \frac{x^n}{n!} \sin \xi = 0$, so the series does converge to $\sin x$.

Example: sin x

$f(x) = \sin(x)$

$g(x) = x$

$h(x) = x - \frac{x^3}{6}$

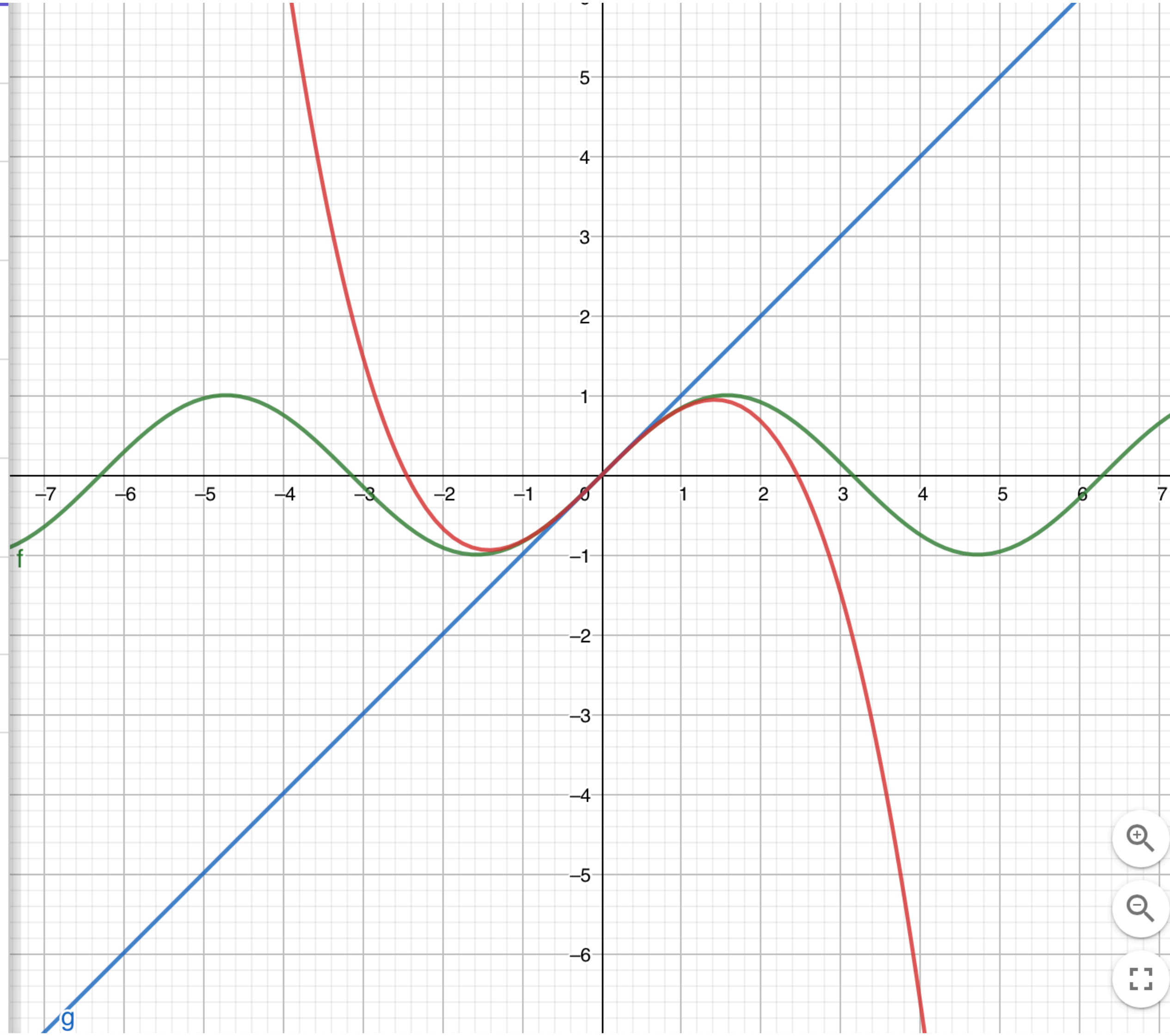
$p(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$

$q(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$r(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$s(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

Input…



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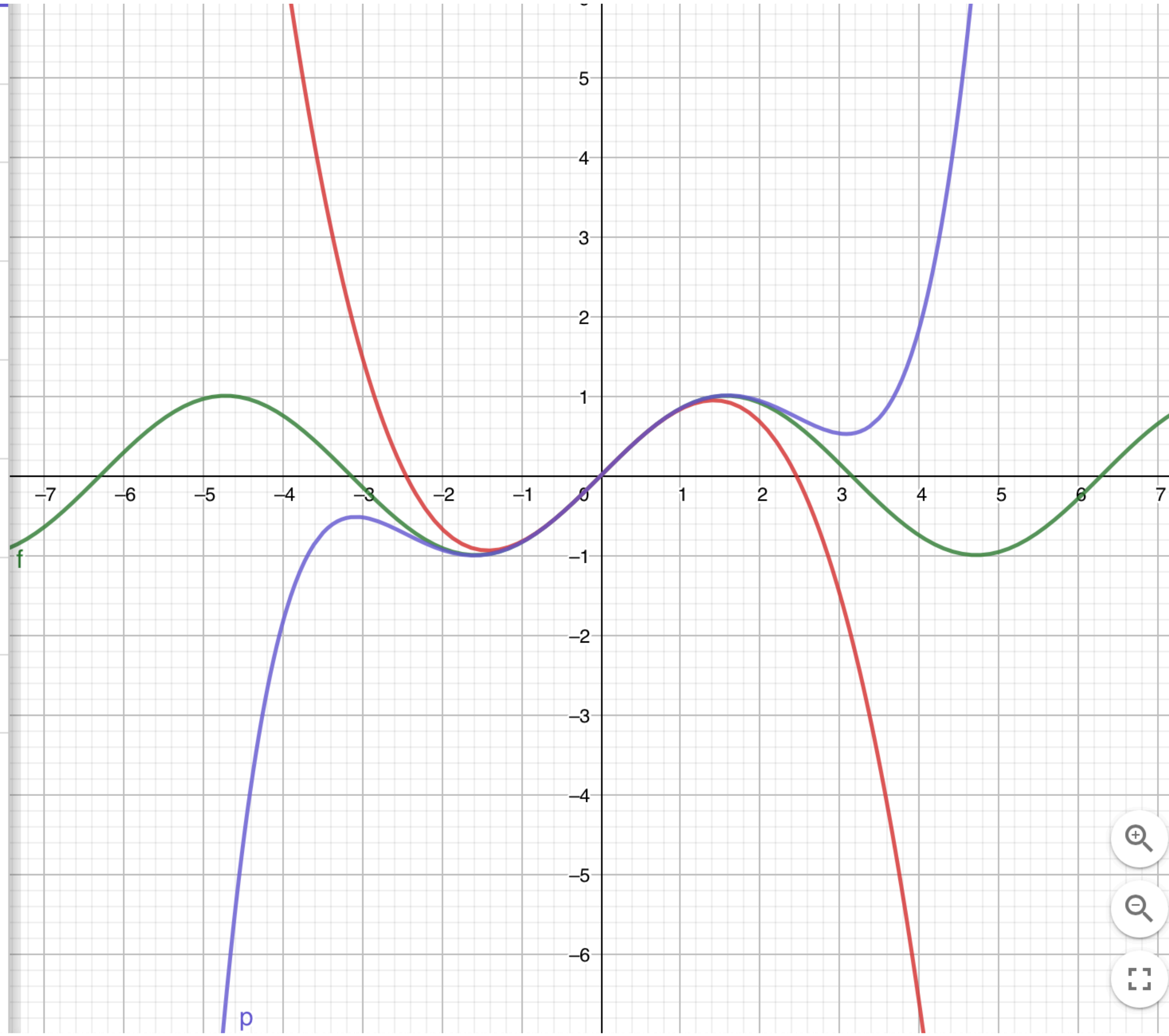

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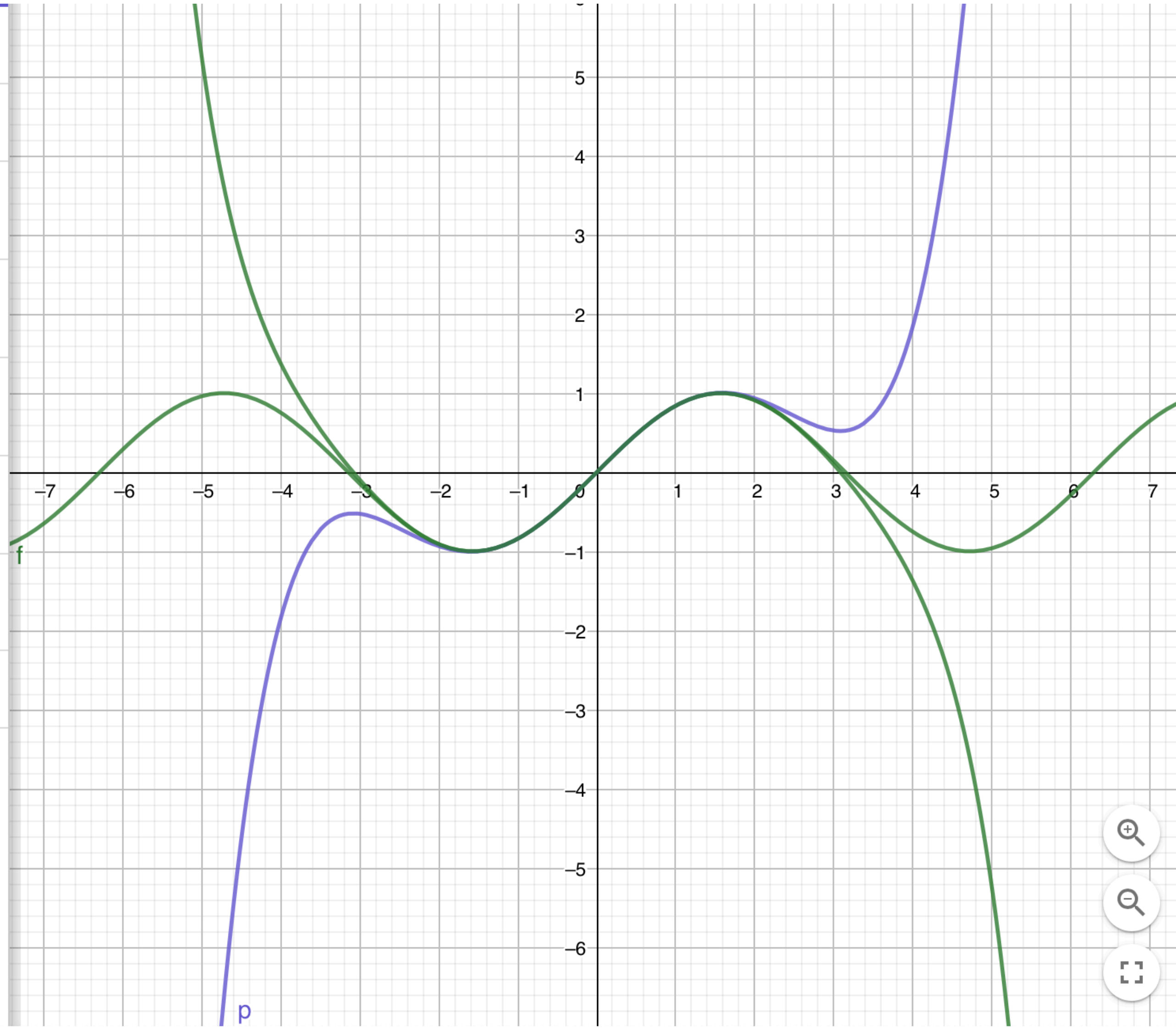

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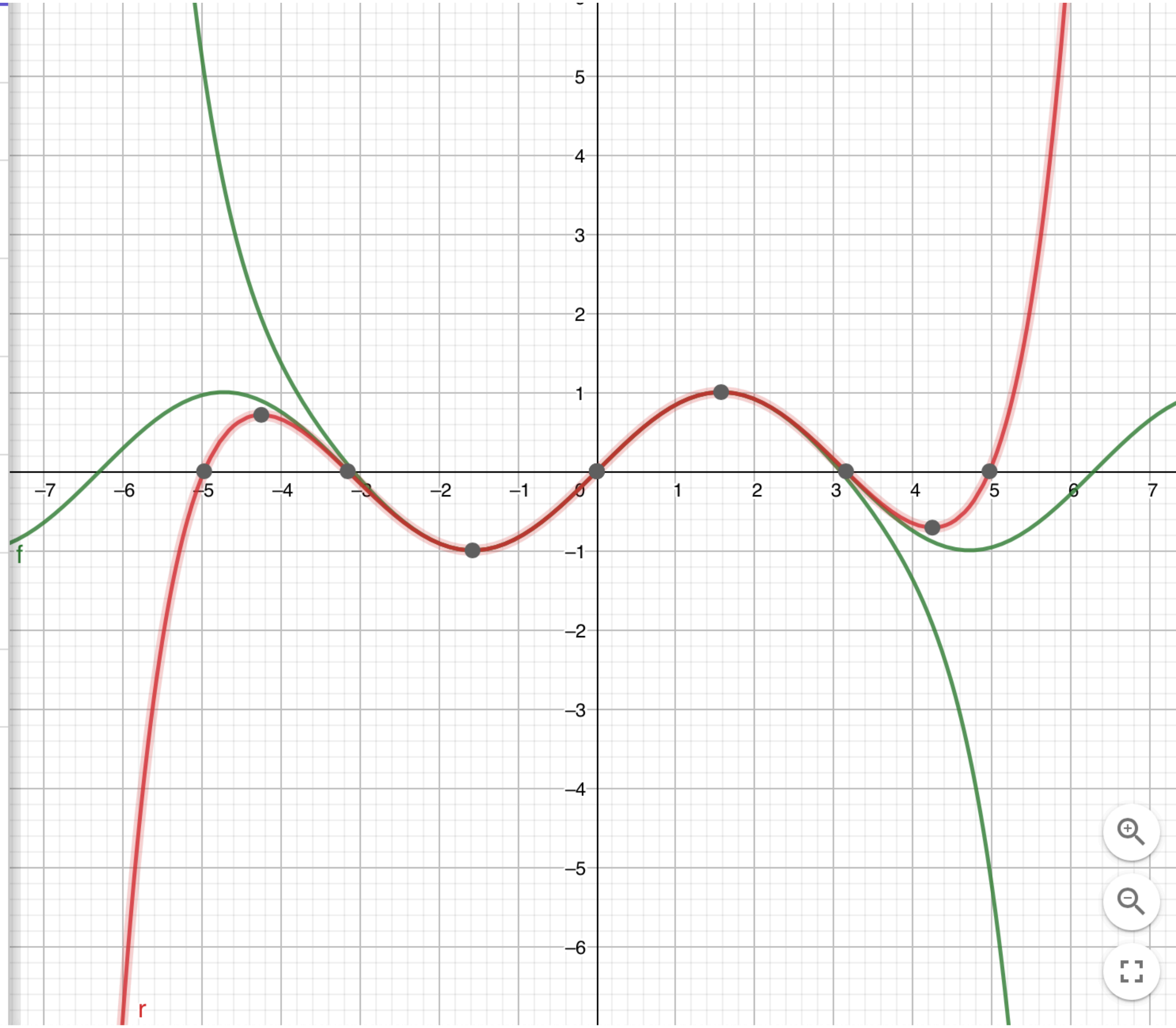

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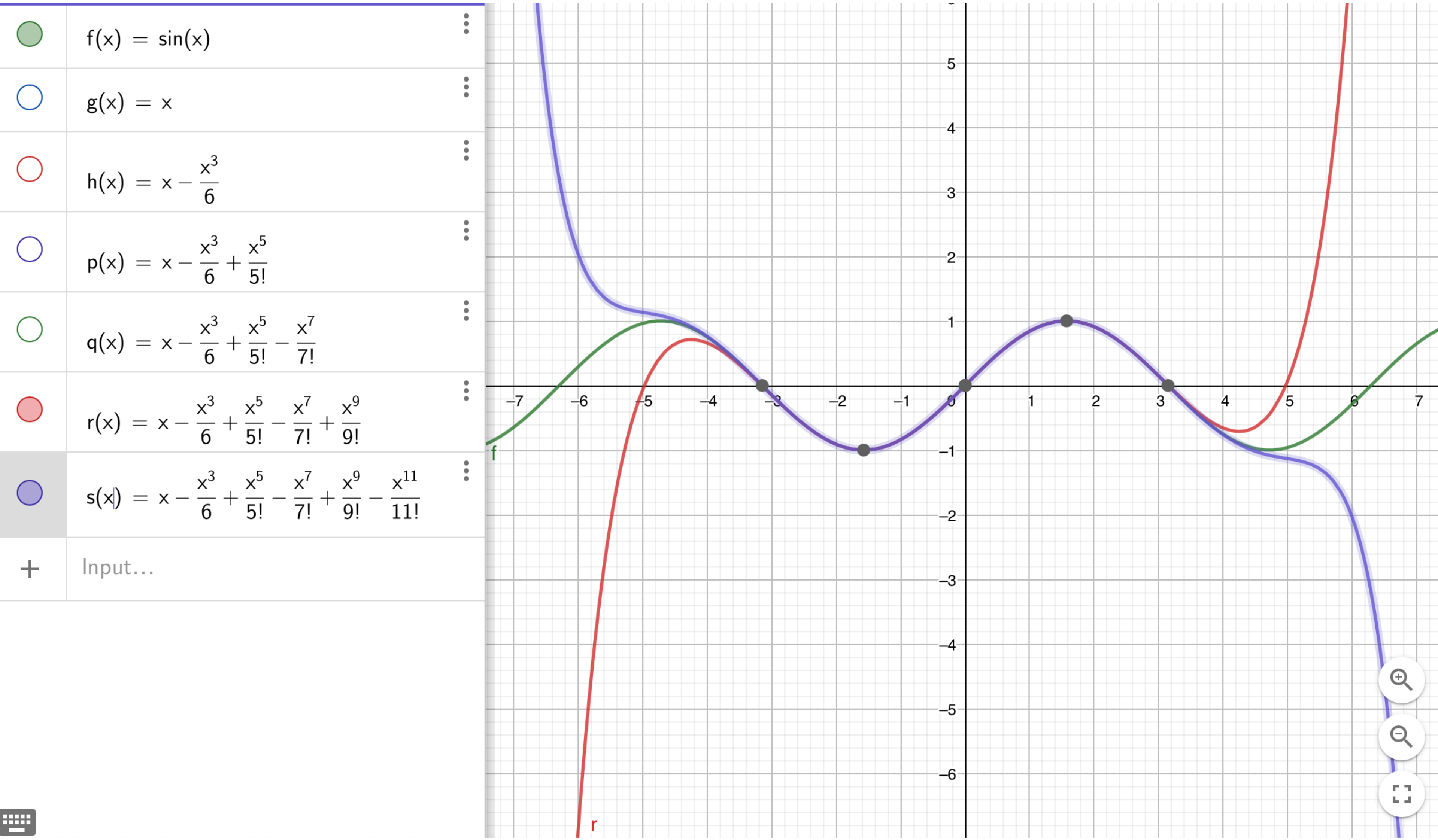

Example: $\cos x$

Let $f(x) = \cos x$.

Then $f(0) = 1$,

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f^{(2k)}(0) = (-1)^k$

$f^{(2k+1)}(0) = 0$

Since $f \in C^\infty$, for any x,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

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Example: $\cos x, e^x$, etc.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

Likewise

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Recall derivatives of sinh and cosh don't get the negatives, so:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

And

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + \frac{x^{2k}}{2k!} + \cdots$$

And again we see that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sinh x + \cosh x$$

Example: complex $\cos x, e^x$, etc.

These expansions hold for complex values of x too.

Let's try setting $x = i\theta$.

$$\sin i\theta = i\theta - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} - \frac{(i\theta)^7}{7!} + \frac{(i\theta)^9}{9!} - \cdots$$

$$\sin i\theta = i\left(\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots\right) = i \sinh\theta$$

$$\cos i\theta = 1 - \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} - \frac{(i\theta)^6}{6!} + \cdots = \cosh\theta$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \ldots$$

$$= \cos\theta + i\sin\theta$$

In particular: $e^{i\pi} = \cos\pi + i\sin\pi = -1$ (Euler's identity)

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Application: extended L'Hôpital's Rule

Let f and g be n-times differentiable functions such that

- $f(a) = g(a) = 0$,
- $f^{(r)}(a) = g^{(r)}(a) = 0$ for $1 \leq r \leq n-1$,
- $f^{(n)}(a),\ g^{(n)}(a)$ are not both zero.

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f^n(x)}{\lim\limits_{x\to a} g^n(x)}$$

Proof:

By Taylor's theorem there exist $\xi_1, \xi_2 \in (a, a+h)$ such that

$$\frac{f(a+h)}{g(a+h)} = \frac{f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^{(n)}(\xi_1)}{g(a) + gf'(a) + \cdots + \frac{h^n}{n!}g^{(n)}(\xi_2)} = \frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_2)}$$

Now if $h \to 0$ then $\xi_1, \xi_2 \to a$, so

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{f(a+h)}{g(a+h)} = \frac{\lim\limits_{\xi_1\to a} f^{(n)}(\xi_1)}{\lim\limits_{\xi_2\to a} g^{(n)}(\xi_2)}.$$

Application: classifying extrema

1. A necessary and sufficient condition for a suitably differentiable function $f(x)$ to have a local maximum at $x = a$ is that the first derivative $f^{(n)}(x)$ with a non-zero value at $x = a$ is of even order (i.e. n is even) and $f^{(n)}(a) < 0$.
2. A necessary and sufficient condition for a suitably differentiable function $f(x)$ to have a local minimum at $x = a$ is that the first derivative $f^{(n)}(x)$ with a non-zero value at $x = a$ is of even order (i.e. n is even) and $f^{(n)}(a) > 0$.
3. If the first derivative $f^{(n)}(x)$ with a non-zero value at $x = a$ is of odd order and $n > 1$, then f has a stationary point of inflection at $x = a$.

Proof:

By Taylor's theorem there exists $\xi \in (a, a+h)$ such that

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^{(n)}(\xi) = f(a) + \frac{h^n}{n!}f^{(n)}(\xi)$$

Now if n is even then $f(a+h) - f(a) = \frac{h^n}{n!}f^{(n)}(\xi)$, which by continuity of $f^{(n)}$ has the sign of $f^{(n)}(a)$ for small enough h.

If n is odd then $f(a+h) - f(a) = \frac{h^n}{n!}f^{(n)}(\xi)$ has dependent on sign of h.

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