Durham University

Maths for Computer Science *Calculus*

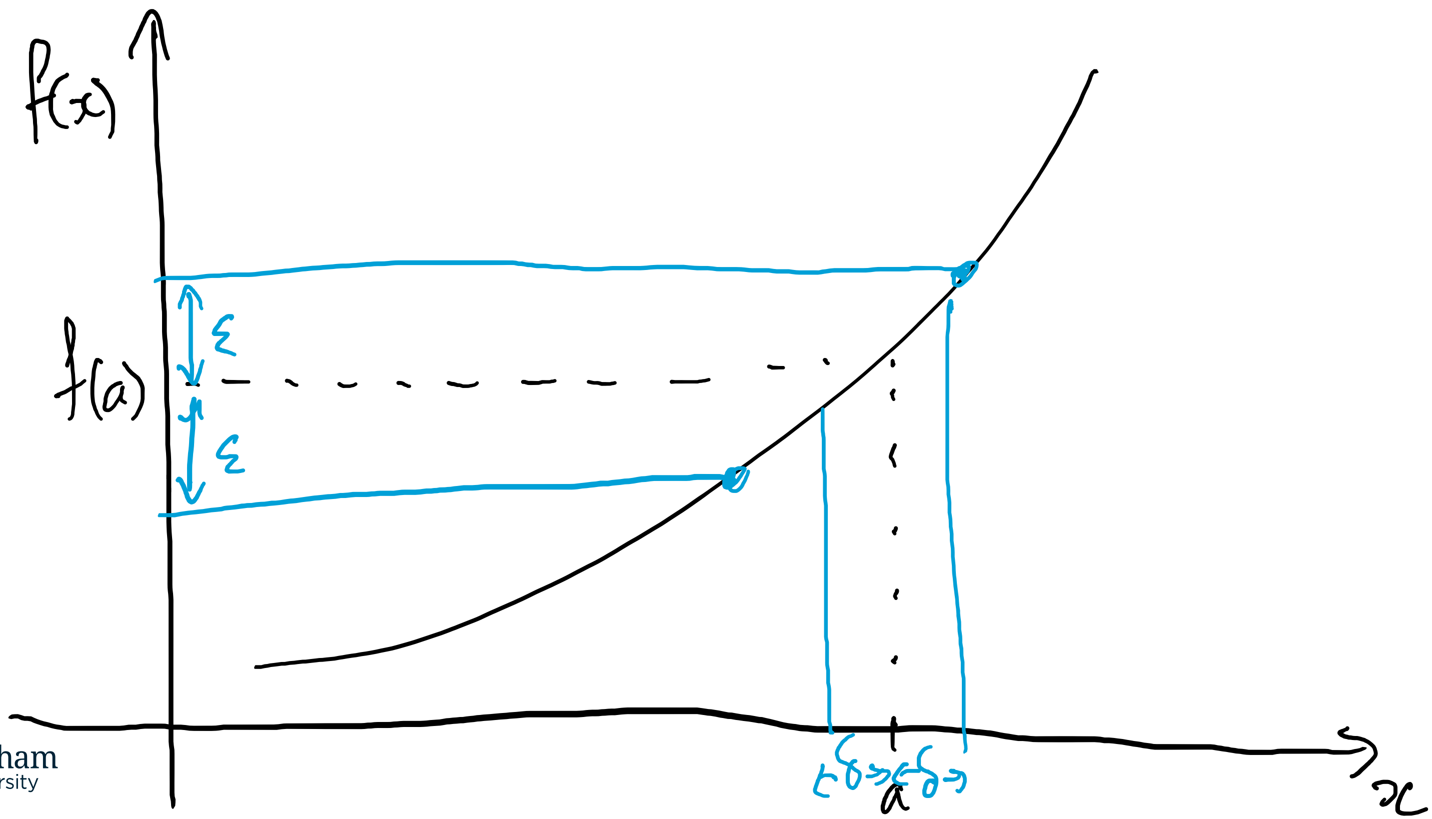
Prof. Magnus Bordewich

Functions and limits

Limits of functions

A function $f: \mathbb{R} \to \mathbb{R}$ is said to tend to the limit L as x tends to a if, and only if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\forall x:\ 0 < |x - a| < \delta \text{ we have } |f(x) - L| < \epsilon.$$

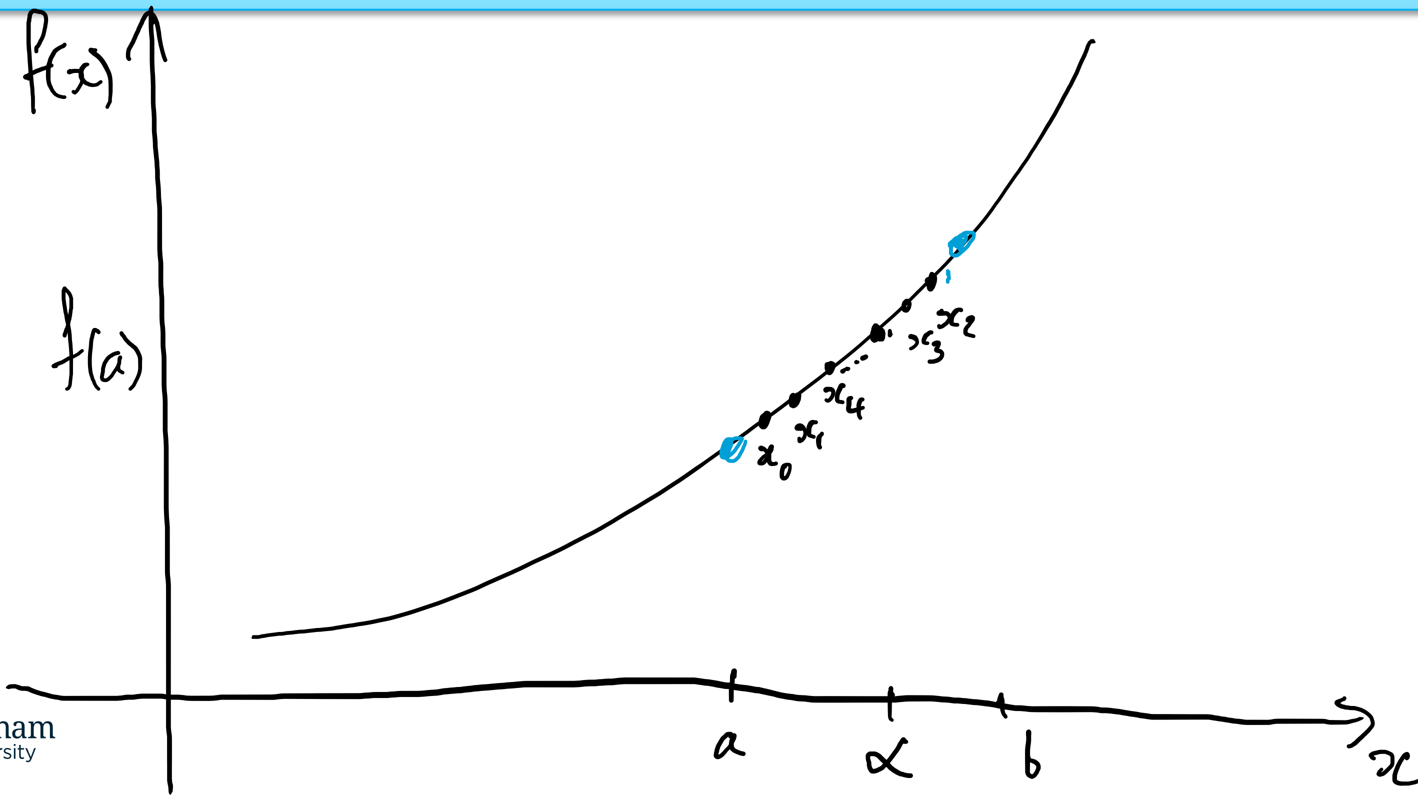


Durham University

Connecting sequences and functions

Let $f(x)$ be a function defined on some real interval (a, b), and let $\{x_n\}$ be a sequence of real values such that $\lim_{n\to\infty} x_n = \alpha \in (a, b)$.

Then $\lim_{x\to\alpha} f(x) = L$ exists if and only if for every such sequence $\{x_n\}$ the sequence $\{f(x_n)\}$ converges to the limit L.

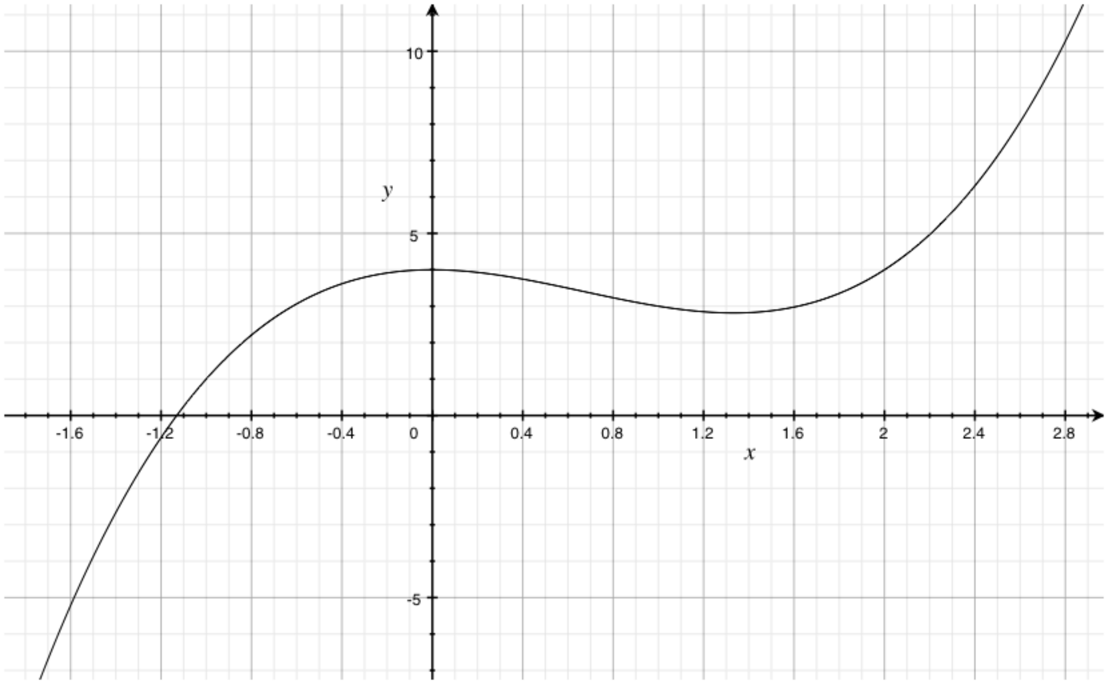


Durham University

Limits of functions

For “nice” functions this may all seem redundant.

E.g. $f(x) = x^3 - 2x^2 + 4$



$f(2) = 8 - 8 + 4 = 4$

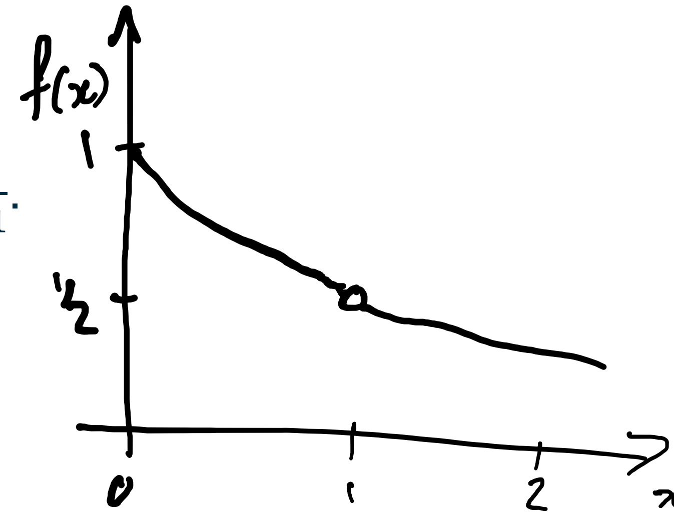
As $x \to 2$ from below, $f(x)$ gets smoothly closer and closer to 4. Likewise, as $x \to 2$ from above, $f(x)$ gets smoothly closer and closer to 4.

We say $\lim_{x \to 2} f(x) = 4$.

But what do we mean by a “nice” function?

Limits of functions

E.g. $f(x) = \frac{\sqrt{x}-1}{x-1}$, for $x \neq 1$.



We can rewrite $f(x)$ as $f(x) = \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$.

from which it is clear that $\lim_{x\to 1} f(x) = \frac{1}{2}$,

even though $f(1)$ is not even defined.

Note that the definition of the limit at a is about the behaviour of f close to a but not actually at a.

E.g. we could define $f(x) = \begin{cases} \frac{\sqrt{x}-1}{x-1}, & x \neq 1 \\ 5, & x = 1 \end{cases}$ then $\lim_{x\to 1} f(x) = \frac{1}{2} \neq f(1)$.

Durham University

An important limit

Consider $f(x) = \frac{\sin(x)}{x}$ as $x \to 0$.

Clearly the area of PQR is larger than the area of the segment PSR which is larger than the area of the triangle PSR:

$$½ \tan x > ½ x > ½ \sin x$$

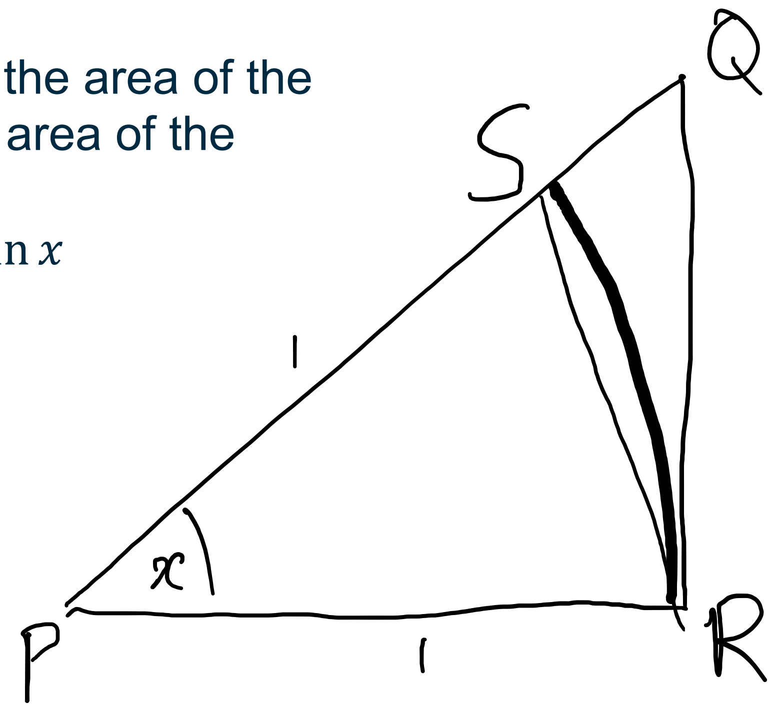
(x is in radians)

Hence $1/\cos x > x/\sin x > 1$.

As x tends to zero $1/\cos x$ tends to 1,

so $x/\sin x$ is squeezed between two things both tending to 1 and must converge to 1.

Therefore $\sin x/x$ also tends to 1.



General squeezing theorem

Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences such that $\lim_{n\to\infty} u_n = L = \lim_{n\to\infty} v_n$ and $u_i \leq w_i \leq v_i$ for all i.

Then $\lim_{n\to\infty} w_n = L$ also.

Let f, g and h be functions such that $\lim_{x\to a} f(x) = \alpha = \lim_{x\to a} g$ and $f(x) \leq h(x) \leq g(x)$ for all x some neighbourhood of a.

Then $\lim_{x\to a} h(x) = \alpha$ also.

Durham University

Limits: arithmetic

Let $\{u_n\}$ and $\{v_n\}$ be sequences such that $\lim_{n\to\infty} u_n = L$ and $\lim_{n\to\infty} v_n = M$.

Then

- $\{u_n + v_n\}$ is a sequence such that $\lim_{n\to\infty}(u_n + v_n) = L + M$.
- $\{u_n v_n\}$ is a sequence such that $\lim_{n\to\infty}(u_n v_n) = LM$.
- $\left\{\frac{u_n}{v_n}\right\}$ is a sequence such that, provided $M \neq 0$, $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{L}{M}$.

Durham University

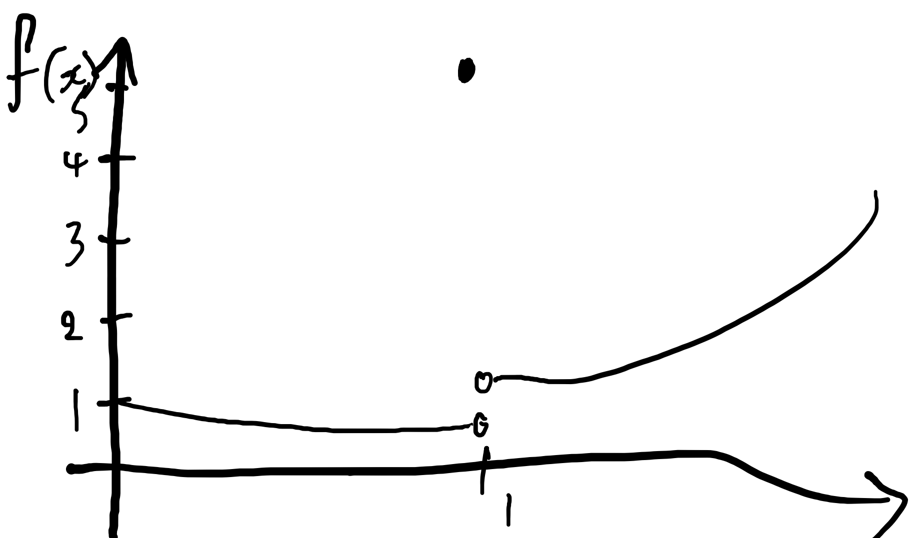
One sided limits of functions

We could have a perfectly valid function that has a break in it.

E.g. $f(x) = \begin{cases} \frac{\sqrt{x}-1}{x-1}, & x < 1 \\ 5, & x = 1 \\ x^2, & x > 1 \end{cases}$

then the limit as x approaches 1 from below is different from the limit as x approaches 1 from above. And both are different from $f(1)$.

We write $\lim\limits_{x\to 1^-} f(x) = \frac{1}{2}$, $\lim\limits_{x\to 1^+} f(x) = 1$ for the limits ‘from below’ and ‘above’.



One sided limits of functions

A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a left hand limit L_- as x tends to a from below if, and only if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that:

$\forall x$: $0 < a - x < \delta$ we have $|f(x) - L_-| < \epsilon$.

A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a right hand limit L_+ as x tends to a from above if, and only if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that:

$\forall x$: $0 < x - a < \delta$ we have $|f(x) - L_+| < \epsilon$.

Continuity

A function $f: \mathbb{R} \to \mathbb{R}$ is said be continuous at $x = a$ if

a) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$, and

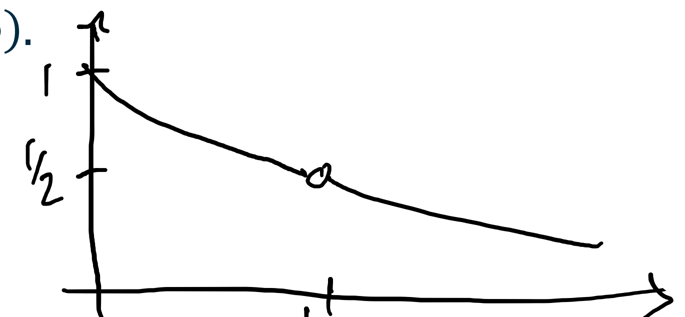
b) $f(a) = L$.

A function $f: \mathbb{R} \to \mathbb{R}$ is said be continuous on an interval (a, b) if it is continuous at all points in the interval.

Example: $f(x) = \frac{\sqrt{x}-1}{x-1}$, for $x \neq 1$.

Continuous on $(0,1)$ and on $(1, \infty)$. Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \frac{1}{2}$, we can define $f(1) = \frac{1}{2}$ and then $f(x)$ is continuous on $(0, \infty)$.

This is called a **removable discontinuity**.



Continuity

A function $f: \mathbb{R} \to \mathbb{R}$ is said be continuous at $x = a$ if

a) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$, and

b) $f(a) = L$.

A function $f: \mathbb{R} \to \mathbb{R}$ is said be continuous on an interval (a, b) if it is continuous at all points in the interval.

Example: $f(x) = |x|$. Continuous on $(-\infty, \infty)$. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$, and $f(0) = 0$.

Example: $g(x) = \frac{d|x|}{dx}$. I.e. $\mathrm{g(x)} = \begin{cases} -1, & x < 0 \\ undefined, & x = 0 \\ 1, & x > 0 \end{cases}$

Continuous everywhere except $x = 0$. $\lim_{x \to 0^-} g(x) = -1 \neq 1 = \lim_{x \to 0^+} g(x)$.

So this is **not** removable.

Durham University

Continuity and limit arithmetic

Let f and g be functions such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Then

- $\lim_{x\to a}(f(x) + g(x)) = L + M$.
- $\lim_{x\to a}(f(x)g(x)) = LM$.
- Provided $M \neq 0$, $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$.

Let f and g be functions continuous at $x = a$.

Then

- $\big(f(x) + g(x)\big), f(x)g(x)$, and, provided $M \neq 0$, $\frac{f(x)}{g(x)}$ are continuous at $x = a$.

Intermediate Value Theorem

Let $f(x)$ be a function that is **continuous** on an interval (a, b), then for any value v that lies between $f(a)$ and $f(b)$, there is some value $x^* \in (a, b)$ such that $f(x^*) = v$.

Proof: Consider the point $x_1 = \frac{1}{2}(a + b)$. If $f(x_1) = v$ we are done. If not, either v is between $f(a)$ and $f(x_1)$ or it is between $f(x_1)$ and $f(b)$. In the former case define $a_1 = a, b_1 = x_1$, in the latter, $a_1 = x_1, b_1 = b$.

Now f is continuous on (a_1, b_1) and v that lies between $f(a_1)$ and $f(b_1)$. We repeat the process for intervals $(a_2, b_2), (a_3, b_3)$, ... and observe that $\{a_n\}$ is an increasing sequence bounded above and $\{b_n\}$ is a decreasing sequence bounded below. Also $\{(b_n - a_n)\}$ converges to 0, so

$\lim_{n\to\infty} \{a_n\} = x^* = \lim_{n\to\infty} \{b_n\}$, and $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x^*)$.

Since v is between $f(a_n)$ and $f(b_n)$ for all n, $f(x^*) = v$.

Limit at $\pm\infty$

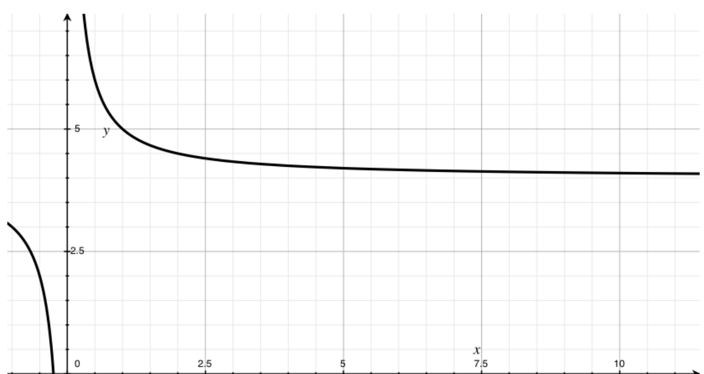
A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have a limit L as x tends to $+\infty$ if and only if for any $\epsilon > 0$, there exists a $N > 0$ such that:

$$\forall x\colon N < x \text{ we have } |f(x) - L| < \epsilon.$$

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have a limit L as x tends to $-\infty$ if and only if for any $\epsilon > 0$, there exists a $N < 0$ such that:

$$\forall x\colon x < N \text{ we have } |f(x) - L| < \epsilon.$$

Example: $f(x) = \left(4 - \frac{1}{x}\right)$ we say $\lim_{x\to\infty} \left(4 + \frac{1}{x}\right) = 4$.



Limit of $\pm\infty$

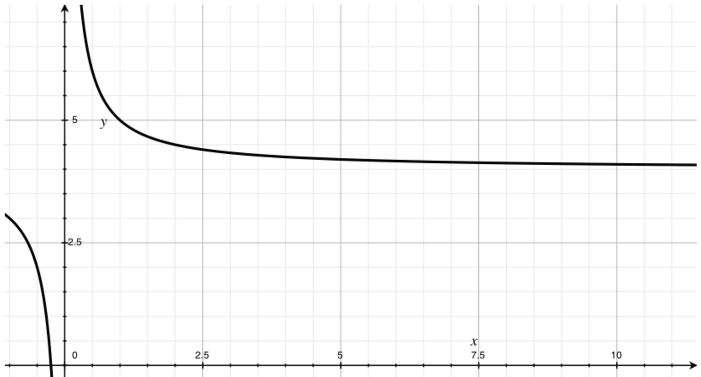
A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a limit $+\infty$ at $x = a$ if and only if for any $N > 0$, there exists a $\delta > 0$ such that:

$$\forall x:\ 0 < |a - x| < \delta \text{ we have } f(x) > N.$$

A function $f: \mathbb{R} \to \mathbb{R}$ is said to have a limit $-\infty$ at $x = a$ if and only if for any $N > 0$, there exists a $\delta > 0$ such that:

$$\forall x:\ 0 < |a - x| < \delta \text{ we have } f(x) < -N.$$

Example: $f(x) = \left(4 - \frac{1}{x}\right)$ we say $\lim_{x\to 0^+} \left(4 + \frac{1}{x}\right) \to \infty$, $\lim_{x\to 0^-} \left(4 + \frac{1}{x}\right) \to -\infty$.



Durham University