Durham University

Maths for Computer Science
Calculus

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Power Series

Series vs Sequences

A **sequence** is an ordered countably infinite set of numbers $\{a_n\}$.

E.g. $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}, \ldots$

A **series** is the sum of members of a sequence:

E.g. $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2 = 1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$.

This is a **finite series.**

E.g. $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

This in an **infinite series.**

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Comparison test

Convergence:

Let $\sum_{n=1}^{\infty} b_n$ be a convergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and N be an integer such that

$$|a_n| \leq b_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

Divergence:

Let $\sum_{n=1}^{\infty} b_n$ be a divergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and N be an integer such that

$$0 \leq b_n \leq a_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

The Ratio test

Convergence:

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

Proof a):

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then we can pick $r \colon L < r < 1$ and N_r such that for all $n > N_r$

$\left|\frac{a_{n+1}}{a_n}\right| < r$ i.e. $|a_{n+1}| < r|a_n|$.

Then $|a_{n+2}| < r|a_{n+1}| < r^2|a_n|, |a_{n+3}| < r|a_{n+2}| < r^3|a_n|$, etc.

So $R_{n+m} = a_{n+m} + a_{n+m+1} + \cdots < |a_n|r^{m-1}(1 + r + r^2 + r^3 + \cdots) = a_n \frac{r^{m-1}}{1-r} \to 0$ as $m \to \infty$. Hence the series converges.

Alternating series test

The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

- $a_n > 0$,
- $a_{n+1} \leq a_n$ for all n, and
- $\lim_{n\to\infty} a_n = 0$.

Proof:

Consider the partial sum

$$S_{2r} = a_1 - a_2 + a_3 - a_4 + a_5 \ldots - a_{2r-2} + a_{2r-1} - a_{2r}.$$

First note that

$$S_{2r} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2r-1} - a_{2r}) \geq 0$$

Also

$$S_{2r} = a_1 - (a_2 - a_3) - (a_4 - a_5) \ldots - (a_{2r-2} - a_{2r-1}) - a_{2r} < a_1$$

So $\{S_{2r}\}$ is bounded and monotonically increasing, hence converges to S.

But $S_{2r+1} = S_{2r} + a_{2r+1}$, and so $\lim_{n\to\infty} S_{2r+1} = \lim_{n\to\infty} S_{2r} + \lim_{n\to\infty} a_{2r+1} = S + 0 = S$.

And as $\{S_{2r}\}$ and $\{S_{2r+1}\}$ both tend to S, $\{S_n\}$ does also.

Grouping and rearrangement of series

For any finite sum we can group and rearrange terms as much as we like.

$$1+3-5-6=(1-5)-(6-3)$$

For an infinite sum we need to be careful!

Consider $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

By the alternating series test this is convergent, to some value S.

So

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

Rearranging

$$S=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}\ldots+\left(\frac{1}{2k+1}-\frac{1}{4k+2}\right)-\frac{1}{4k+4}+\cdots$$

$$S=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)=\frac{1}{2}S$$

So as $S=\frac{1}{2}S$ we must have $S=0$, but actually $S>\frac{1}{2}$. (Exercise).

Grouping and rearrangement of series

If $\sum_{n=1}^{\infty} a_n$ converges, then we can insert brackets/groupings without altering the sum.

Proof:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

Consider bracketing $(a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + \cdots$ etc.

Define b_n to be the n^{th} bracketed term.

Now $\sum_{n=1}^{\infty} b_n = (a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + \cdots$ etc.

But the partial sums T_n of this series are a subsequence of the partial sums S_n of $\sum_{n=1}^{\infty} a_n$.

Since $\lim_{n\to\infty} S_n = S$, it follows that $\lim_{n\to\infty} T_n = S$ and so $\sum_{n=1}^{\infty} b_n$ converges.

Grouping and rearrangement of series

If $\sum_{n=1}^{\infty} a_n$ absolutely converges, then we can reorder the terms without altering the sum.

Proof:

Let $\sum_{n=1}^{\infty} b_n$ be a reordering of $\sum_{n=1}^{\infty} a_n$.

Since $\sum_{n=1}^{\infty} |a_n|$ converges, say to S, and the partial sums $T_n = \sum_{r=1}^{n} |b_r|$ are monotonic increasing and bounded above by S, $\sum_{r=1}^{n} |b_r|$ also converges.

Since $\sum_{r=1}^{n} |b_r|$ converges, $\sum_{r=1}^{n} b_r$ also converges.

Let $\sum_{r=1}^{n} a_r = S_n'$ and $\sum_{n=1}^{\infty} a_n = S'$. Let $\sum_{r=1}^{n} b_r = T_n'$ and $\sum_{n=1}^{\infty} b_n = T'$

Then given $\epsilon > 0, \exists\, N$: $|S_n' - S'| < \epsilon$.

Then for large enough m, $T_m' = S_n' + a_{i_1} + a_{i_2} + \cdots + a_{i_k}$ for some a_{i_j}.

Then $|T_m' - S'| \leq |S_n' - S'| + |a_{i_1}| + |a_{i_2}| + \cdots + |a_{i_k}| < 2\epsilon$. So $T_m' \to S'$ as $n \to \infty$.

Power series

A **power series** is a series involving a variable x and a constant x_0 of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

At any specific value of x this becomes a normal series and we can use the tests we have discussed to determine convergence.

For example, earlier we saw the power series for e^x given as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

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Radius of convergence

For an arbitrary power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, we use the ratio test:

Convergence:

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

Now we must remember to include $(x-x_0)^n$ in our ratio:

Radius of convergence

For an arbitrary power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, we use the ratio test:

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x - x_0| = L < 1$ we have convergence, if $L > 1$, divergence.

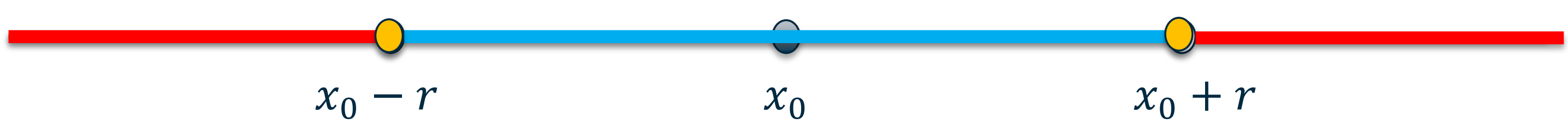
But for a specific value of x, $|x - x_0|$ is just a number, so

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{|x-x_0|}$ we have convergence, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > \frac{1}{|x-x_0|}$: divergence.

Equivalently, if it exists we call $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = r$ the **radius of convergence**, and:

- if $|x - x_0| < r$ we have absolute convergence,
- if $|x - x_0| > r$ we have divergence.

At $x_0 \pm r$ we cannot tell with the ratio test.



For x in this range the power series converges

Outside this range the power series diverges

Example: e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

First we evaluate the limit $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1/n!}{1/(n+1)!} = n + 1 \to \infty$ as $n \to \infty$.

So the radius of convergence is unbounded and $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real numbers of x.

Example: $\ln(x+1)$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad [= \ln(x+1)]$$

First we evaluate the limit $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1/n}{1/(n+1)} = \frac{n+1}{n} \to 1$ as $n \to \infty$.

So the radius of convergence is 1:

For $|x| < 1$, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ converges absolutely, for $|x| > 1$ it diverges.

What about $x = 1$?

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent by the alternating series test.

What about $x = -1$?

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n} \to -\infty$ is divergent (harmonic series).

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Example: $r = 0$

$$\sum_{n=0}^{\infty}(nx)^n = 1 + x + (2x)^2 + \cdots$$

First we evaluate the limit $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{n^n}{(n+1)^{n+1}} < \frac{1}{n} \to 0$ as $n \to \infty$.

So the radius of convergence is 0:

For any $|x| > 0$, this series diverges.

What about $x = 0$?

$\sum_{n=0}^{\infty} a_n x^n = a_0$ is always trivially convergent.

Differentiation (and integration) of power series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $r > 0$.

Then within the interval $(-r, r)$:

- $f(x)$ is continuous
- $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ (term by term differentiation)
- $\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ (term by term integration).

Example

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$$

First we evaluate the limit $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)(n+2)}{n(n+1)} = \frac{n+2}{n} \to 1$ as $n \to \infty$.

So the radius of convergence is 1.

When $x = 1$, by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we can see the series converges.

When $x = -1$, by the alternating series test, the series converges.

Hence, for $|x| < 1$

$$f'^{(x)} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n+1)}$$

This again has radius of convergence 1, and converges at $x = -1$.

But at $x = 1$ this is the harmonic series and diverges.

Power series from functions

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

First observe that by setting $x = 0$, we must have $a_0 = f(0)$.

Now assume $r > 0$ and differentiate once: for $-r < x < r$

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} , i.e. \quad f'(0) = a_1$$

Differentiating again: for $-r < x < r$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} , i.e. \quad f''(0) = 2a_2$$

Proceeding systematically:

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1) \ldots (n-m+1) a_n x^{n-m} , whence \quad f^{(m)}(0) = m!\, a_m.$$

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Power series from functions: Maclaurin series

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Putting these values in, we see that if such a power series exists then

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

This is called the **Maclaurin series for f**.

To use a Maclaurin series for some function f we must have that

- f can be differentiated an infinite number of times ($f \in C^{\infty}$)
- the power series converges.

Power series from functions: Taylor series

Let us recentre our power series on x_0.

Define a new function $g(x) = f(x + x_0)$.

Then $g'(x) = f'(x + x_0)$, so $g'(0) = f'(x_0)$.

Repeatedly differentiating $g^{(n)}(x) = f^{(n)}(x + x_0)$, so $g^{(n)}(0) = f^{(n)}(x_0)$.

So the Maclaurin series for g gives us:

$$f(x + x_0) = g(x) = \sum_{n=0}^{\infty} g^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{x^n}{n!}$$

Or

$$f(x) = g(x - x_0) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}.$$

This form is called the **Taylor series expansion of f**.

Example: $x\ln(x)$

Let $f(x) = x\ln(x)$.

Differentiating:

$$f'(x) = 1 + \ln x\,, f''(x) = \frac{1}{x}, f'''(x) = -\frac{1}{x^2}, \dots, f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}$$

Note: at 0 these are undefined! So we can't make a Maclaurin series.

But we can make the Taylor series at $x_0 = 1$.

Then $\mathrm{f}(1) = 0, f'(1) = 1, f''(1) = 1, \dots, f^{(n)}(1) = (-1)^n(n-2)!$

Hence the Taylor series for f is

$$x\ln(x) = (x-1) + \frac{(x-1)^2}{1.2} - \frac{(x-1)^3}{2.3} + \frac{(x-1)^4}{3.4} - \frac{(x-1)^5}{4.5} \dots$$

$$x\ln(x) = (x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}(x-1)^n$$

Which converges on $[0,2]$.