Mathematics for Computer Science
Linear Algebra

Lecture 6: Determinants

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Contents for today's lecture

- Determinants via cofactor expansion;
- Determinants via elementary row operations;
- Invertible matrices and determinants;
- A new way to invert a matrix.

What are determinants?

- Each square matrix A has a determinant, which is a *number* computed from A
- Notation: $det(A)$ or $|A|$

- Determinants are mainly a technical tool with useful properties
- We will use them later to study eigenvalues of matrices
- Determinants can be visualised:
  - view a matrix as a transformation of a vector space (we'll do this in term 2)
  - then the determinant measures how the transformation scales the area/volume
  - watch a nice YouTube video (by 3B1B), visualising this for 2D and 3D spaces: https://youtu.be/Ip3X9LOh2dk

Finding the inverse of a 2×2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant of A is the number $det(A) = ad - bc$.

This number is also denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Theorem

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is invertible iff* $det(A) \neq 0$, *in which case*

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example: Let $A = \begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix}$. Then $det(A) = 6 \cdot 2 - 5 \cdot 1 = 7$, so A is invertible.

We have

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 2 & -1 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Minors and cofactors

- We defined determinants of 2×2 matrices (and they turned out to be important), so will now define them for general square matrices.
- Assume we can compute determinants of square matrices of order $n-1$.
- If A is a square matrix of order n, then the minor of the entry a_{ij}, denoted by M_{ij}, is the determinant of the matrix (of order $n-1$) obtained from A by removing its i-th row and j-th column.
- The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of a_{ij}.

Example: Let

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}.$$

The minor of a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} \cdot 26 = -26.$$

Determinants

If A is an $n \times n$ matrix then the determinant of A can be computed by any of the following cofactor expansions along the i-th row and along the j-th column, respectively:

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Theorem

The above expressions for $det(A)$ all give the same result.

Proof idea: Unfold the recursive definitions and check that the results are equal. Details omitted.

- Easy to see: If A has a row of 0s or a column of 0s then $det(A) = 0$.
- Easy to see: It holds that $det(A) = det(A^T)$.

Example

Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}.$$

Compute $det(A)$ by cofactor expansion along the first row.

Recall that $det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ and $C_{ij} = (-1)^{i+j} \cdot M_{ij}$.

$$\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \cdot \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & -4 \\ 5 & -4 \end{vmatrix} =$$

$$3 \cdot (-4) - 1 \cdot (-11) + 0 = -1$$

Smart choice of row or column

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

- When computing a determinant by expanding along a row or column, we have a choice - which row or column to expand along.
- What is the best choice for the following matrix?

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Computing (large) determinant by cofactor expansion

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

- In the above formula, each C_{1j} is a $(\pm 1)\times$ determinant of order $n-1$, so can be expressed similarly via determinants of order $n-2$.
- If we fully expand the definition, we get a sum of products (with a sign ± 1).
- There will be n numbers in each product, e.g. for $n = 3$

$$a_{11}C_{11} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

- How many products will the whole sum have? Let $P(n)$ denote this number.
- $det(A)$ contains $P(n)$ products, and each C_{1j} has $P(n-1)$ products. Hence,

$$P(n) = n \cdot P(n-1) = n(n-1)P(n-2) = \ldots = n(n-1)(n-2)\cdots 2 \cdot 1$$

- The number $n(n-1)(n-2)\cdots 2 \cdot 1$ is denoted by $n!$ and called "n factorial"
- How fast does $n!$ grow?

Determinants and elementary row operations

How do elementary row operations affect the determinant of a square matrix?

Theorem

Let A be an $n \times n$ matrix.

- *If B is obtained form A by multiplying a row by a constant k then $det(B) = k \cdot det(A)$.*
- *If B is obtained from A by interchanging two rows then $det(B) = -det(A)$.*
- *If B is obtained from A by adding a multiple of one row to another row then $det(B) = det(A)$.*

Proof: (1) follows directly from definition, proofs of (2) and (3) are omitted.

Lemma

If $A = (a_{ij})$ is a triangular matrix then $det(A) = a_{11} \cdot a_{22} \cdots a_{(n-1)(n-1)} \cdot a_{nn}$.

Exercise: Prove this lemma by induction on n.

Computing determinants by row reduction

The previous slide suggests a strategy for computing the determinant of a matrix:

- Use elementary row operations to transform the matrix to row echelon form.
- Record how the determinant changes during the transformation.
- The row echelon form is upper triangular, its determinant is easy to find.

Example:

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} =$$

$$-3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3)\cdot(-55) = 165$$

We now have two ways of computing determinants:
by cofactor expansion (i.e. by definition) and by row reduction (as above).

They can be mixed: create many 0s by row reduction and use cofactor expansion.

Determinants of elementary matrices

We have $det(I_n) = 1$. The following is a special case of the previous theorem.

Corollary

Let E be an $n \times n$ elementary matrix.

- *If E is obtained from I_n by multiplying a row by a constant k then $det(E) = k$.*
- *If E is obtained from I_n by interchanging two rows then $det(E) = -1$.*
- *If E is obtained from I_n by adding a multiple of one row to another row then $det(E) = 1$.*

Lemma

If E and B are $n \times n$ matrices and E is elementary then $det(EB) = det(E)\det(B)$.

Proof: We consider only the 1st case from the above corollary, the other two are similar. If E is obtained from I_n by multiplying a row by k, then EB is obtained from B by the same operation, so $det(EB) = k \cdot det(B) = det(E)det(B)$.

Invertibility criterion

We can now add a useful condition to the theorem about invertible matrices.

Theorem

A square matrix A is invertible iff (="if and only if") $det(A) \neq 0$.

Proof.

Let R be the reduced row echelon form of A. We have the following facts:

- Either $R = I$ (and $det(R) = 1$) or R contains a row of 0s (and $det(R) = 0$).
- A is invertible iff $R = I$, by the theorem about invertible matrices (1) $\Leftrightarrow$ (3).
- We know that $R = E_r \cdots E_2 E_1 A$ for some elementary matrices E_i.
- $det(R) = det(E_r) \cdots det(E_2) det(E_1) det(A)$, by the previous lemma.
- $det(E_i) \neq 0$ for all i, so $det(R)$ and $det(A)$ are either both 0 or both non-0.
- Finally, A is invertible $\Leftrightarrow R = I \Leftrightarrow det(R) \neq 0 \Leftrightarrow det(A) \neq 0$.

$\square$

Properties of determinants

Theorem

If A and B are square matrices of the same size then $det(AB) = det(A)det(B)$.

Proof.

- It can be shown that if A is not invertible then neither is AB. In this case, $det(A) = det(AB) = 0$.
- Assume that A is invertible, then $A = E_1 E_2 \cdots E_r$ for some elementary E_i.
- Then $AB = E_1 E_2 \cdots E_r B$ and $det(AB) = det(E_1)det(E_2) \cdots det(E_r)det(B)$.
- Since $det(A) = det(E_1)det(E_2) \cdots det(E_r)$, we have the required equality.

$\square$

Applying the above theorem to the case when A is invertible and $B = A^{-1}$, we get

Corollary

If A is invertible then $det(A^{-1}) = 1/det(A)$.

Note that $det(A + B) \neq det(A) + det(B)$ in general. Try $A = I_2$ and $B = -I_2$.

Inverting a matrix via cofactors/adjoint

- If A is a square matrix of order n, then the minor of the entry a_{ij}, denoted by M_{ij}, is the determinant of the matrix (of order $n-1$) obtained from A by removing its i-th row and j-th column.
- The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of a_{ij}.
- The matrix

$$cof(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

  is called the matrix of cofactors of A.
- The matrix $(cof(A))^T$ is the adjoint matrix of A, denoted by $adj(A)$.

Theorem

If A is an invertible matrix then $A^{-1} = \frac{1}{det(A)} \cdot adj(A)$.

Exercise: Prove this theorem by showing that $a_{i1}C_{j1} + a_{i2}C_{j2} + \ldots + a_{in}C_{jn}$ is equal to $det(A)$ if $i = j$ and to 0 if $i \neq j$ (the latter is moderately hard).

Example

Find the inverse (if it exists) of the following matrix $A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$.

We have computed $det(A) = 165$ earlier, so the inverse exists.

We have

$$cof(A) = \begin{pmatrix} -60 & 15 & 30 \\ 29 & -10 & 2 \\ 39 & 15 & -3 \end{pmatrix}, \text{ so } adj(A) = \begin{pmatrix} -60 & 29 & 39 \\ 15 & -10 & 15 \\ 30 & 2 & -3 \end{pmatrix}.$$

Therefore,

$$A^{-1} = \frac{1}{165} \begin{pmatrix} -60 & 29 & 39 \\ 15 & -10 & 15 \\ 30 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -60/165 & 29/165 & 39/165 \\ 15/165 & -10/165 & 15/165 \\ 30/165 & 2/165 & -3/165 \end{pmatrix}.$$

What we learnt today

Determinants:

- What they are
- How to compute them by cofactor expansion
- How to compute them by row reduction
- How to use them to decide whether a matrix is invertible
- How to use them to invert matrices

Next time:

- Euclidean vector spaces