Durham University

Maths for Computer Science
Calculus

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Fourier Series

Some key facts

First note:

$$\int_{-\pi}^{\pi} \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0, \qquad \forall\, m, n \in \mathbb{Z}, m \neq 0$$

Since both are periodic functions with period an exact divisor of 2π.

Recall (or look up) the trigonometric identities:

$$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$$
$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$
$$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

Hence:

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin\big((n - m)x\big) \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((n + m)x) \, dx$$
$$= 0, \qquad \forall\, m, n$$

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Some key facts

Also:

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos\big((n-m)x\big)\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\, dx$$

$$\begin{aligned} &= 0, \quad \forall\, m \neq \pm n \text{ and } m = n = 0 \\ &= \pi, \quad if\ m = n \neq 0 \\ &= -\pi, \quad if\ m = -n \neq 0 \end{aligned}$$

and

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos\big((n-m)x\big)\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\, dx$$

$$\begin{aligned} &= 0, \quad \forall\, m \neq \pm n \\ &= \pi, \quad if\ m = \pm n \neq 0 \\ &= 2\pi, \quad if\ m = \pm n = 0. \end{aligned}$$

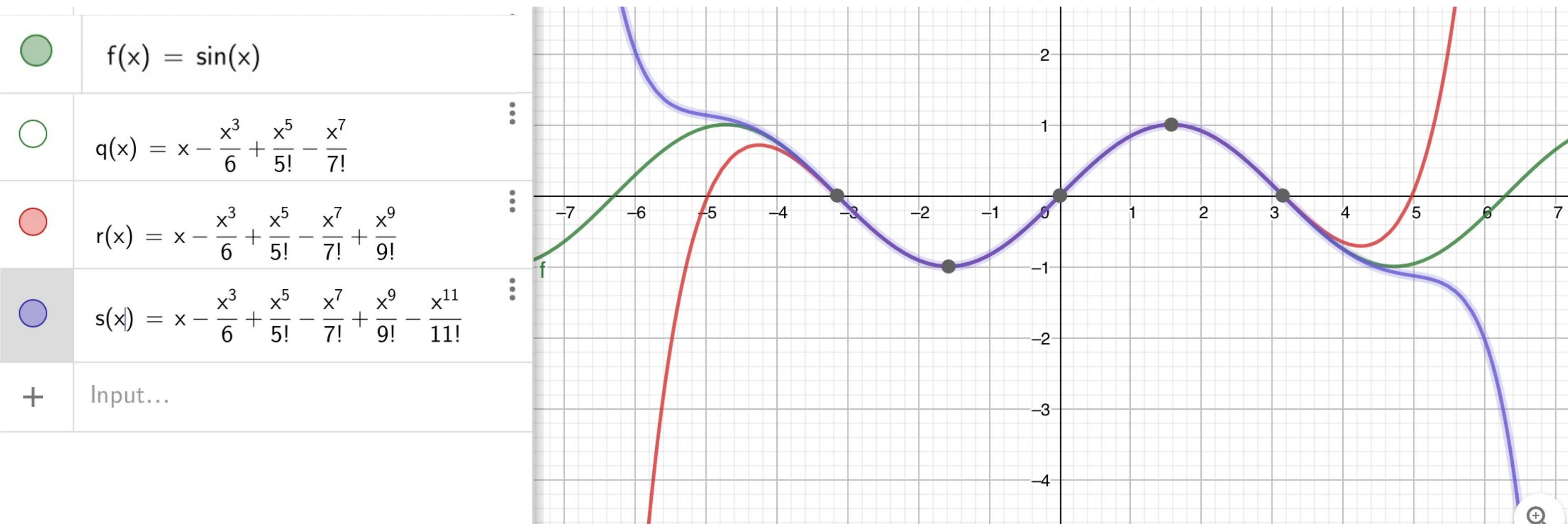
Recall the Taylor series

For any function k-times differentiable function $f(x)$, we can approximate f near a point x_0 by

$$f(x) \approx \sum_{n=0}^{k} a_n (x - x_0)^n .$$

where $a_n = \frac{f^{(n)}(x_0)}{n!}$.

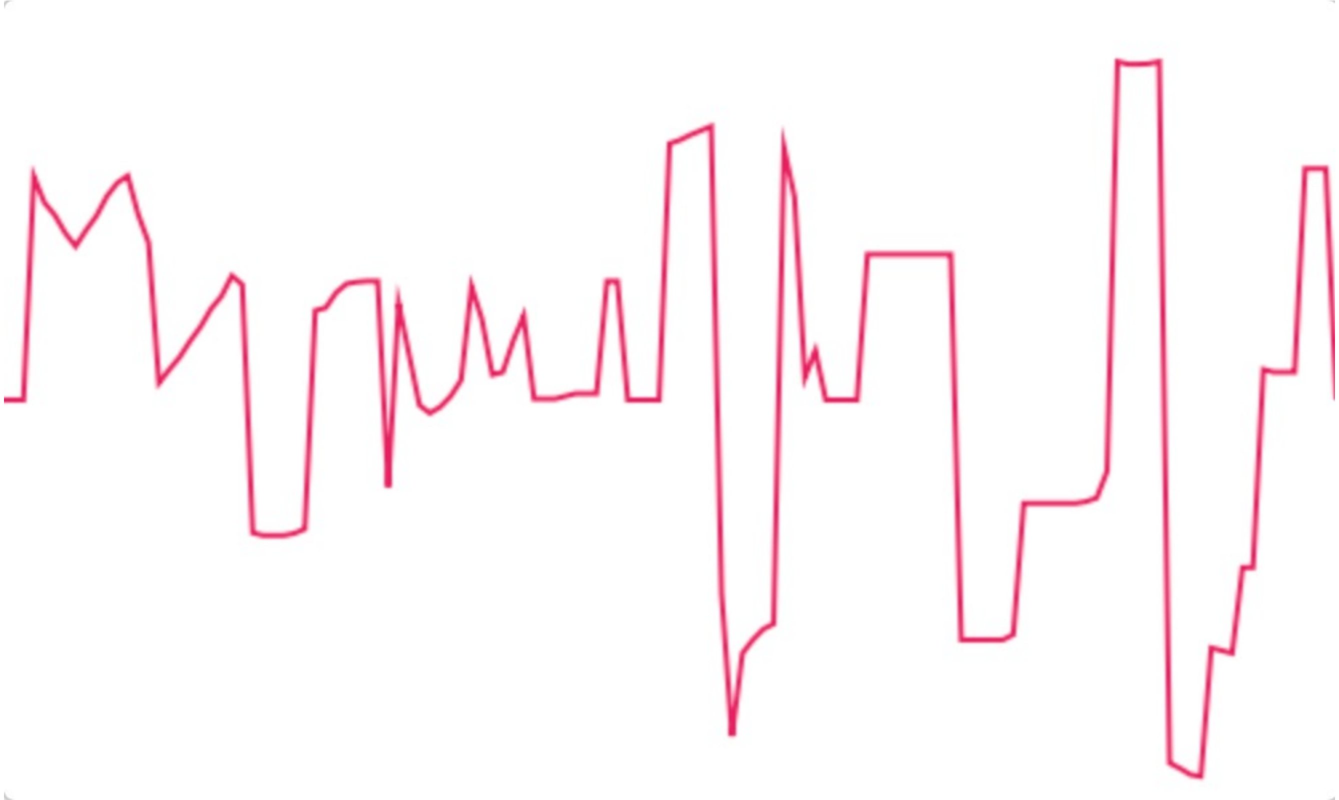
The error is given by $\frac{(x-x_0)^k}{k!} f^{(k)}(\xi)$ for some $\xi \in (x_0, x)$.



Taylor series limitations

- It only works for many-times differentiable functions (infinitely differentiable for full Taylor series).

- No luck with functions like:



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A new series proposal

For a function $f(x)$ $-\pi < x < \pi$, we will construct a series for f of the form

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

The range $[-\pi, \pi]$ is no real restriction – we can always scale an arbitrary function f on a range $[a, b]$ by setting

$$f(x) = g\left(\left(x - \frac{b+a}{2}\right)\frac{2\pi}{(b-a)}\right)$$

where g is a function on $[-\pi, \pi]$.

Fourier coefficients

Suppose first that such a series exists.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then, multiplying by $\cos mx, m \geq 0$ and integrating over the range $[-\pi, \pi]$ we get:

$$\int_{-\pi}^{\pi} f(x) \cos mx \; dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) \cos mx \, dx$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx$$

$$= \pi a_m.$$

Fourier coefficients

Likewise, still assuming there is a series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, multiplying by this time by $\sin mx$ and integrating over the range:

$$\int_{-\pi}^{\pi} f(x) \sin mx \;\, dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) \sin mx \, dx$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$= \pi b_m.$$

The Fourier Series

For a function $f(x)$ $-\pi < x < \pi$, the Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \; dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \; dx\,.$$

Theorem:

If f is piecewise continuous and $\int_{-\pi}^{\pi}[f(x)]^2 dx$ exists, then the Fourier series converges.

- For all points x at which f is continuous, the series converges to $f(x)$.
- For points y at which there is a jump discontinuity the series converges to

$$\frac{1}{2}\left(\lim_{x\to \mathrm{y}^-} f(x) + \lim_{x\to \mathrm{y}^+} f(x)\right)$$

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Example $y = x$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, dx = 0$$
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos mx \; dx = 0$$

both odd functions

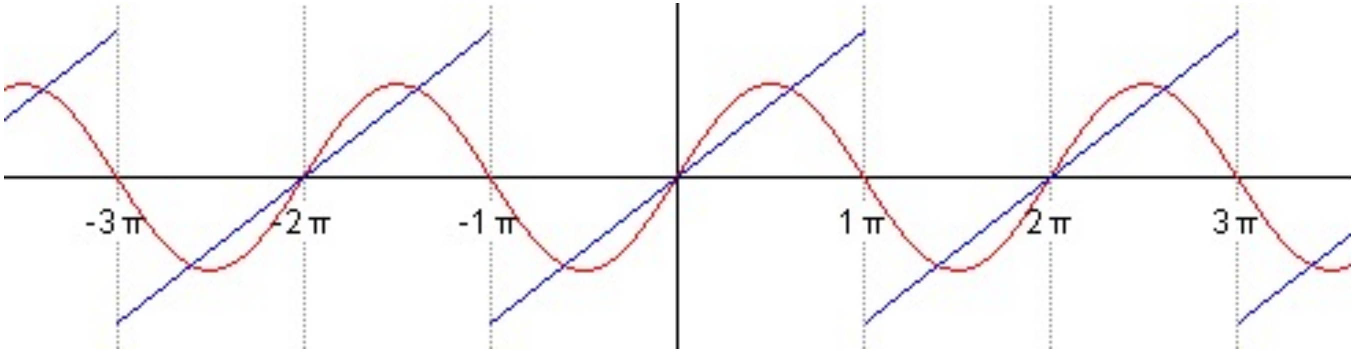
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \; dx = \left[\frac{1}{\pi} x \frac{-\cos nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} -\frac{\cos nx}{n} . 1\, dx$$

$$= \frac{1}{\pi}\pi\frac{-\cos(n\pi)}{n} - \frac{1}{\pi}(-\pi)\frac{\cos(-n\pi)}{n} = (-1)^{n+1}\frac{2}{n}.$$

So we get $\sum_{n=1}^{\infty}\frac{2}{n}\sin nx$

Shown over

1,2,3,4,5 terms:



Visualisation

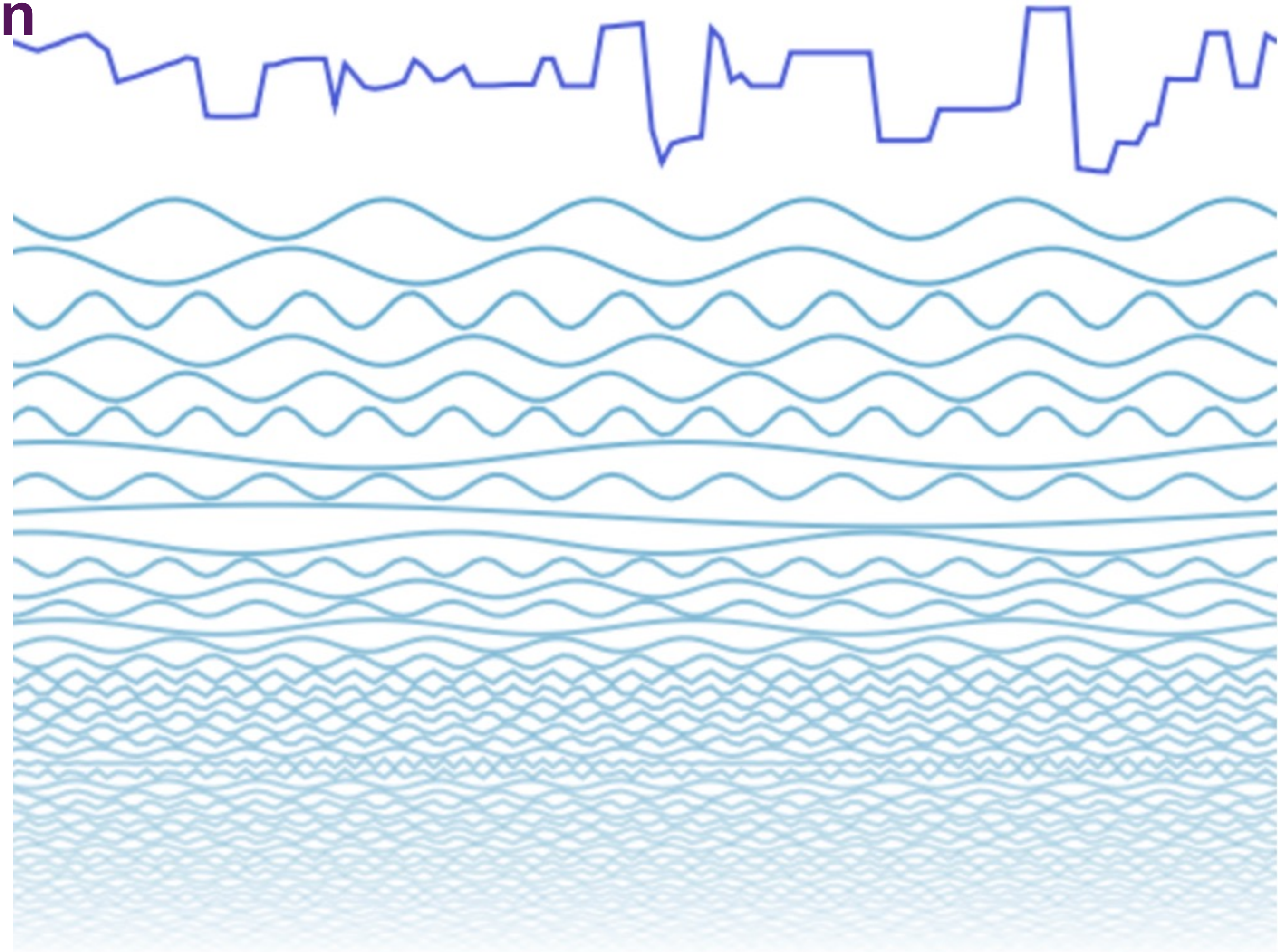


Image from http://www.jezzamon.com/fourier/

Visit the site (strongly recommended) for some beautiful and interactive examples

Linear algebra perspective

We can consider the set of all suitably integrable functions $f: [-\pi, \pi] \to \mathbb{R}$ as a vector space.

We can define the inner product of two functions f, g to be

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

Then (as noted under key facts) the functions

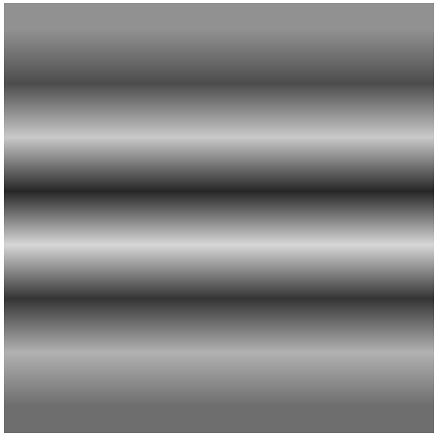
- $\sin nx\,, n > 0,$
- $\cos mx, m \geq 0$

are pairwise orthogonal.

In fact they form a basis for the vector space, which therefore has infinite dimensions.

Images and jpeg

For images, rather than using a 1D sine wave, we take vertical and horizontal waves of brightness of different frequencies:



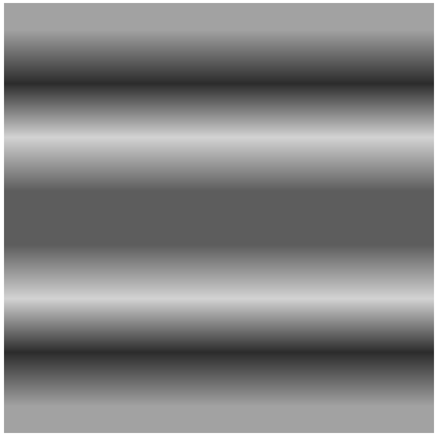


Image from http://www.jezzamon.com/fourier/

Images and jpeg

For images, rather than using a 1D sine wave, we take vertical and horizontal waves of brightness of different frequencies.

We also take the product of horizontal and vertical waves, getting kind of chess-board patterns:

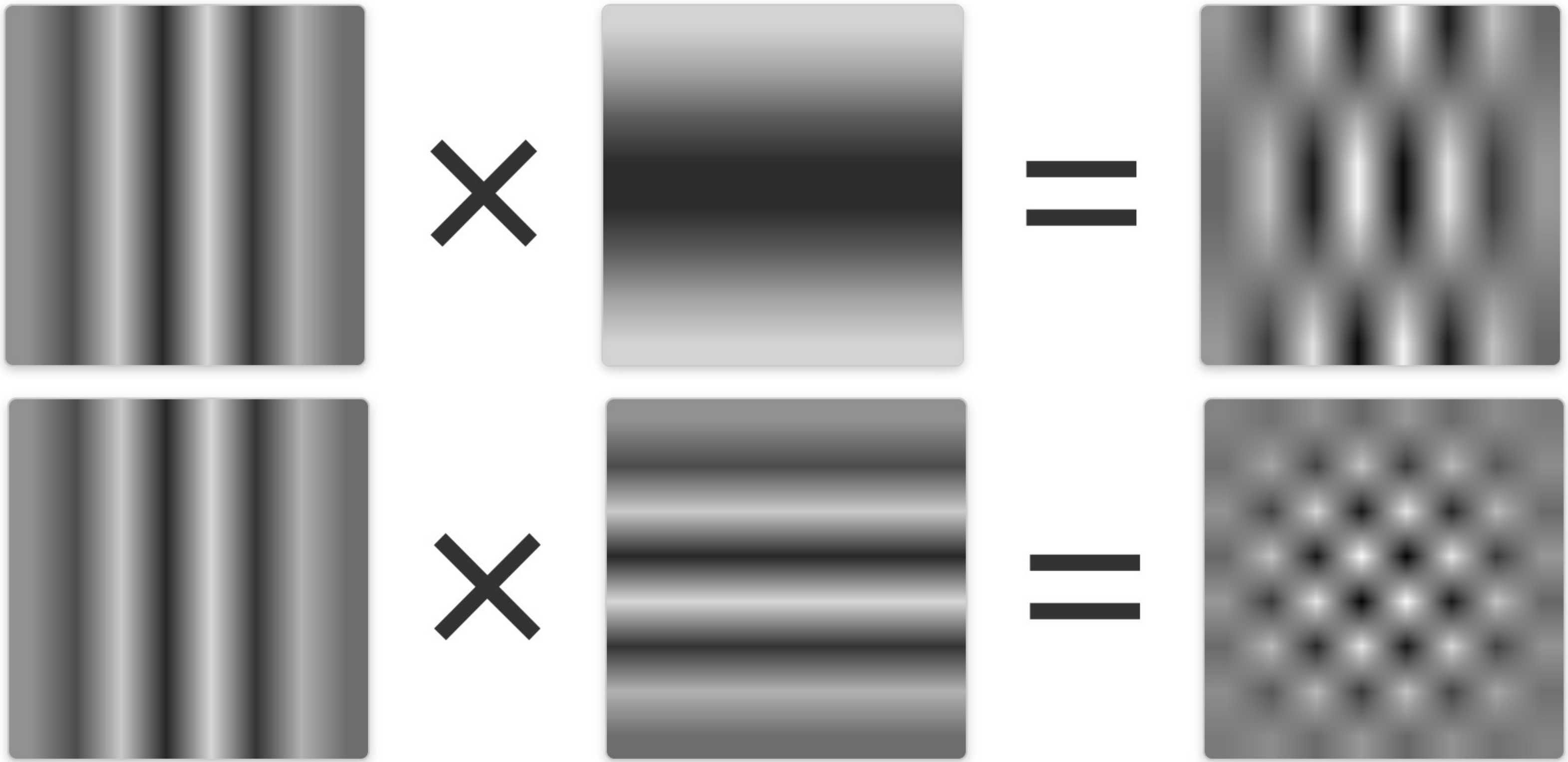


Image from http://www.jezzamon.com/fourier/

Images and jpeg

For an 8x8 pixel image we can represent any image as a combination of these tiles:

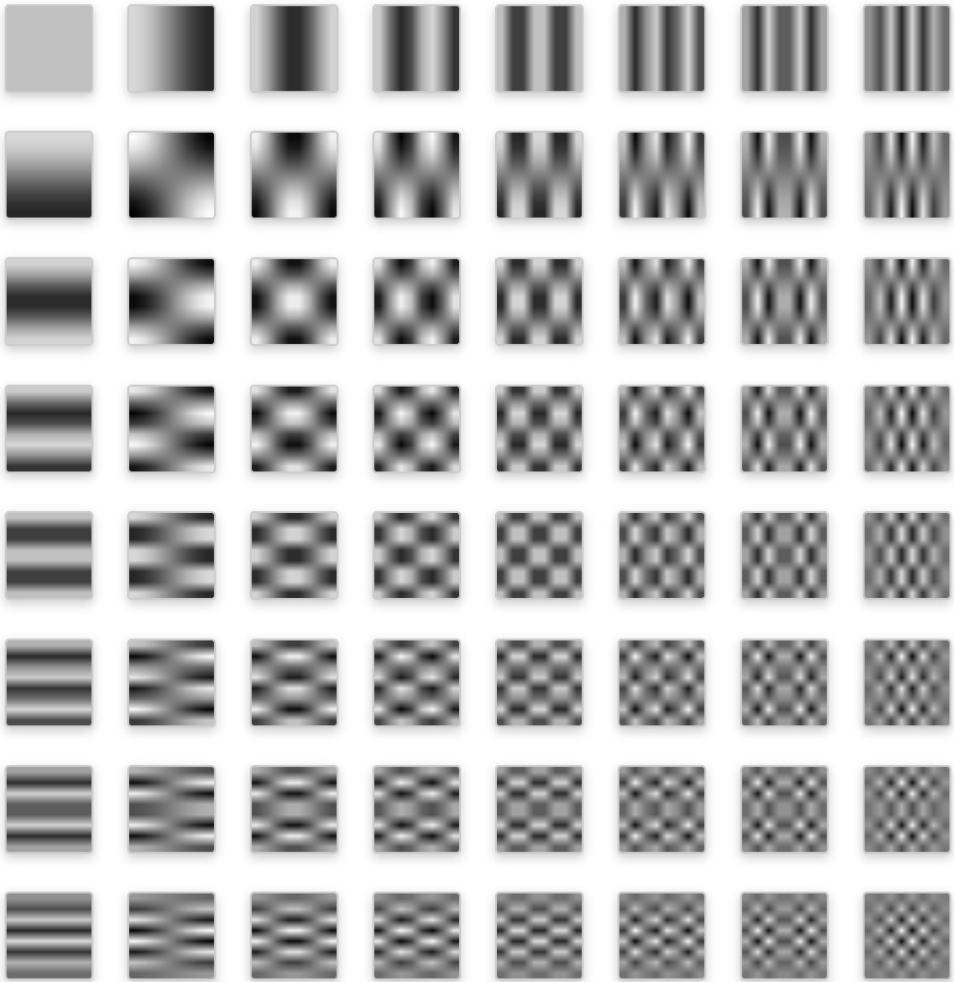


Image from http://www.jezzamon.com/fourier/

Images and jpeg

In jpeg, the image is broken down into 8x8 squares, and each is encoded as a combination of the tiles. The higher the quality setting, the more tiles are used.

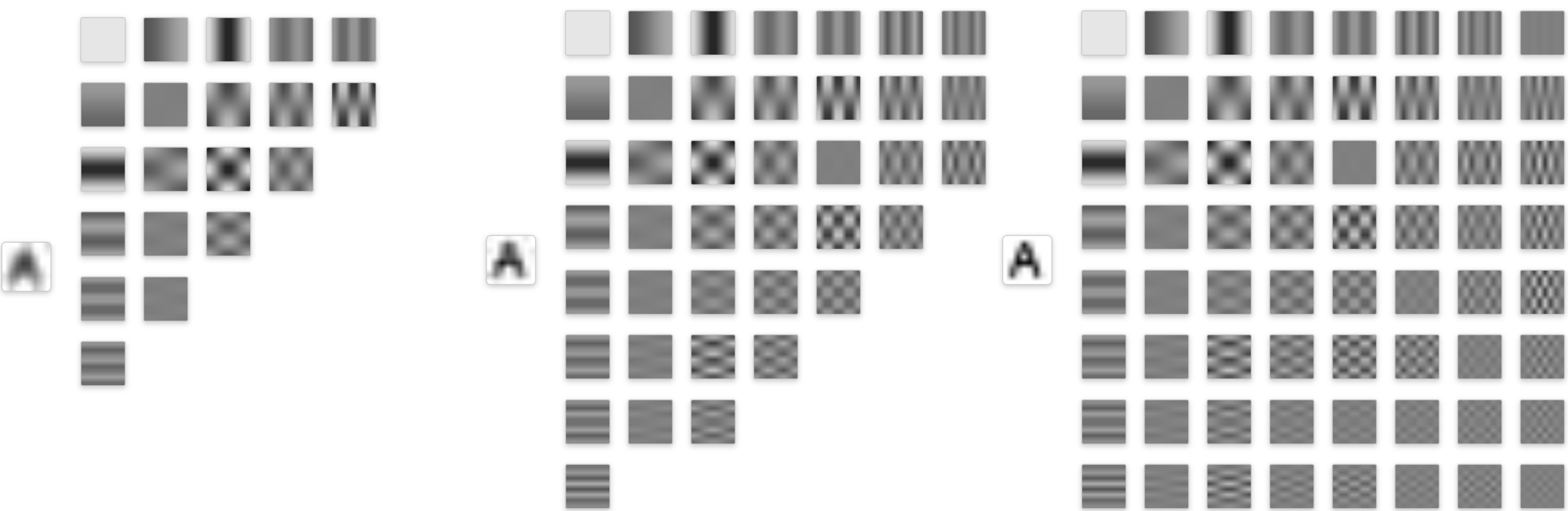


Image from http://www.jezzamon.com/fourier/

Example: a photo of my dog

1. Normal photo
2. First couple of waves
3. Only low frequency components
4. Only high frequency components

Fun tool for playing with Fourier transforms of images:

https://ejectamenta.com/imaging-experiments/fourifier/

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Low quality jpeg artifacts