6G6Z3002 – Computational Methods for ODEs

Chapter 2 – Exercise 2 – Part Solutions

Qn1 (a):

$$y_{j+2} = y_j + \frac{h}{3}(f_{j+2} + 4f_{j+1} + f_j)$$

Consider starting from the following equation:

$\int_{x_j}^{x_{j+2}} dy = \int_{x_j}^{x_{j+2}} f(x,y)dx$, this is similar to the derivation in Chapter 2, top of page 2.

$y_{j+2} - y_j = \int_{x_j}^{x_{j+2}} f(x,y)dx$, similar to equation (2.1) Chapter 2.

The RHS of the equation for y_{j+2} and y_j is already given in the appropriate form, so we do not need to integrate the RHS of the equation - we can just start from the following stage and remembering that $x = x_j + sh$ and $\frac{dx}{ds} = h, \quad so \quad dx = h\,ds$:

$$y_{j+2} = y_j + h\int_0^2 (f_j + s\Delta f_j + \frac{s(s-1)}{2!}\Delta^2 f_j)\,ds$$

Integration gives:

$$y_{j+2} = y_j + h\left[(sf_j + \frac{s^2}{2}\Delta f_j + \left(\frac{s^3}{6} - \frac{s^2}{4}\right)\Delta^2 f_j)\right]_0^2$$

$$\mathrm{y}_{j+2} = \mathrm{y}_j + \mathrm{h}\left[(2\mathrm{f}_j + \frac{2^2}{2}\Delta \mathrm{f}_j + \left(\frac{2^3}{6} - \frac{2^2}{4}\right)\Delta^2 \mathrm{f}_j)\right] - [0]$$

Substituting for Δf_j and $\Delta^2 f_j$ in terms of function values, i.e. $\Delta f_j = f_{j+1} - f_j$ and $\Delta^2 f_j = f_{j+2} - 2f_{j+1} + f_j$ (see chapter 2 Page 3), gives:

$$y_{j+2} = y_j + h\left[(2f_j + 2(f_{j+1} - f_j) + \left(\frac{8}{6} - \frac{6}{6}\right)(f_{j+2} - 2f_{j+1} + f_j)\right]$$

$$y_{j+2} = y_j + h\left[(2f_j + 2(f_{j+1} - f_j) + \left(\frac{1}{3}\right)(f_{j+2} - 2f_{j+1} + f_j)\right]$$

Collecting the like terms gives:

$$y_{j+2} = y_j + \frac{h}{3}\left[f_{j+2} + 4f_{j+1} + f_j)\right].$$

For parts (b) and (c), follow exactly the same method, and you should be able to derive the given formulae. In Chapter 3 we will continue using the same method for deriving the local truncation errors attached to the multistep methods formulae, and so we will go through the derivation again, and will cover the relevant details.

Parts (b), (c) and (d) can be solved using similar steps.

Qn2:

$$y_{j+1}=y_j+h(1+\frac{1}{2}\nabla+\frac{5}{12}\nabla^2+\frac{3}{8}\nabla^3+\frac{251}{720}\nabla^4+\frac{95}{288}\nabla^5+\cdots)f_j$$

Rearrange the above formulae in the following forms:

$$y_{j+1}=y_j+hf_j+\frac{h}{2}\nabla f_j+\frac{5h}{12}\nabla^2 f_j+\frac{3h}{8}\nabla^3 f_j+\frac{251h}{720}\nabla^4 f_j+\frac{95h}{288}\nabla^5 f_j+\cdots$$

Write down the problem with the initial conditions: $y'=-y-2x,$ $\qquad y(0)=-1,$

and consider your given difference table:

j	x	y	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0	0.0	−1.0000	1.0000				
				−0.3			
1	0.1	−0.9	0.7				
2	0.2	−0.8450	0.4450				
3							
4							
5							

Start filling in the cells in your difference table (I used red font for the calculated values):

We need to calculate the value of y_1 .

At first, we can only use this open formula up to $y_1=y_0+hf_0$ (because we have no more data), therefore:

$y_1^p=y_0+hf_0=-1.0+(0.1)(1.0)=-0.9$, then calculate f_1

$f_1=-(-0.9)-2(0.1)=0.7$, add this to the table. Now you can calculate ∇f_1

$\nabla f_1=0.7-1.000=-0.3$, add this to the table

Now use the formula to advance the solution to y_2 and can include the new ∇f term as well:

$$y_2=y_1+hf_1+\frac{h}{2}\nabla f_1=y_1+h[f_1+\frac{1}{2}\nabla f_1].$$

$$y_2=y_1+h[f_1+\frac{1}{2}\nabla f_1]$$

$$y_2=-0.9+0.1[0.7+\frac{1}{2}(-0.3)]=-0.8450 \qquad f_2=-(-0.8450)-2(0.2)=0.4450$$

Next, calculate ∇f_2 and $\nabla^2 f_2$, then use the open formula up to including $\nabla^2 f_2$ to calculate y_3 :

$y_3=y_2+h[f_2+\frac{1}{2}\nabla f_2+\frac{5}{12}\nabla^2 f_2]$, with this value of y_3 , calculate f_3,

And so on… remember with each new estimate to calculate respectively the new or updated function and differences values. …

See below the completed table computed in Excel:

	$y' = -y - 2x$	$y(0) = -1$		$y_{j+1} = y_j + h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \frac{95}{288}\nabla^5 + \cdots\right)f_j$					
j	x	y	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
0	0.00000	-1.0000	1.0000					
				-0.3000				
1	0.10000	-0.9000	0.7000		0.0450			
				-0.2550		-0.0236		
2	0.20000	-0.8450	0.4450		0.0214		0.0264	
				-0.2336		0.0027		
3	0.30000	-0.8114	0.2114		0.0241			
				-0.2095				
4	0.40000	-0.8019	0.0019					
5	0.50000	-0.8102	-0.1898					

Qn3:

$$y_{j+1} = y_j + h(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \frac{95}{288}\nabla^5 + \cdots)f_j$$

$$y_{j+1} = y_j + h(1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \frac{3}{160}\nabla^5 - \cdots)f_{j+1}$$

Rearrange the above formulae in the following forms:

$$y_{j+1} = y_j + hf_j + \frac{h}{2}\nabla f_j + \frac{5h}{12}\nabla^2 f_j + \frac{3h}{8}\nabla^3 f_j + \frac{251h}{720}\nabla^4 f_j + \frac{95h}{288}\nabla^5 f_j + \cdots$$

$$y_{j+1} = y_j + hf_{j+1} - h\frac{1}{2}\nabla f_{j+1} - h\frac{1}{12}\nabla^2 f_{j+1} - h\frac{1}{24}\nabla^3 f_{j+1} - h\frac{19}{720}\nabla^4 f_{j+1} - h\frac{3}{160}\nabla^5 f_{j+1} - \cdots$$

Write down the problem with the initial conditions: $y' = -y - 2x$, $y(0) = -1$, and consider your given difference table:

j	x	y	f	∇f	∇²f
0	0.0	−1.0000	1.0000		
				~~−0.3~~ −0.2850	
1	0.1	~~−0.9~~ −0.9150	~~0.7~~ 0.7150		0.0277
				−0.2573	
2	0.2	~~−0.8577~~ −0.8566	~~0.4577~~ 0.4566		

Start filling in the cells in your difference table (I used red font for predictor values and blue for corrector values):

We need to calculate the predictor value of y_1 .

At first we can only use the predictor formula (1) up to $y_1 = y_0 + hf_0$ (because we have no more data), therefore:

$y_1^p = y_0 + hf_0 = -1.0 + (0.1)(1.0) = -0.9$, then calculate f_1

$f_1 = -(-0.9) - 2(0.1) = 0.7$, add this to the table. Now you can calculate ∇f_1

$\nabla f_1 = 0.7 - 1.000 = -0.3$, add this to the table

Now use the corrector formula (2) up to $y_1 = y_0 + hf_1 - \frac{h}{2}\nabla f_1 = y_0 + h[f_1 - \frac{1}{2}\nabla f_1]$.

$$y_1^c = y_0 + h[f_1 - \frac{1}{2}\nabla f_1]$$

$$y_1^c = -1.0000 + 0.1[0.7 - \frac{1}{2}(-0.3)] = -0.9150$$

$\mathrm{f}_1 = -(-0.9150) - 2(0.1) = 0.7150$ $\nabla \mathrm{f}_1 = 0.7150 - (1) = -0.2850$, now we can progress to calculate y_2^p

$$y_2^p = y_1 + h[f_1 + \frac{1}{2}\nabla f_1]$$

$y_2^p = -0.915 + 0.1[0.715 + \frac{1}{2}(-0.285)] = -0.8577$ $\quad f_2 = -(-0.8577) - 2(0.2) = 0.4577$

$y_2^c = -0.915 + 0.1[0.4577 - \frac{1}{2}(-0.2573) - \frac{1}{12}(0.0277)] = -0.8566$ $\quad f_2 = -(-0.8566) - 2(0.2) = 0.4566$

And so on… remember with each new predicator or corrector estimate to calculate respectively the new or updated function and differences values.

See below the completed table computed in Excel:

0.1 $y' = -y - 2x \quad y(0) = -1$

$$y_{j+1} = y_j + hf_j + \frac{h}{2}\nabla f_j + \frac{5h}{12}\nabla^2 f_j + \frac{3h}{8}\nabla^3 f_j + \frac{251h}{720}\nabla^4 f_j + \frac{95h}{288}\nabla^5 f_j + \cdots$$

$$y_{j+1} = y_j + hf_{j+1} - \frac{h}{2}\nabla f_{j+1} - \frac{h}{12}\nabla^2 f_{j+1} - \frac{h}{24}\nabla^3 f_{j+1} - \frac{19h}{720}\nabla^4 f_{j+1} - \frac{3h}{160}\nabla^5 f_{j+1} - \cdots$$

j	x	y	f	∇f	∇^2f	∇^3f	∇^4f	∇^5f
0	0.0000	-1.0000	1.0000					
				-0.3000				
1	0.1000	-0.9000	0.7000					
				-0.2850				
1	0.1000	-0.9150	0.7150		0.0277			
				-0.2573				
2	0.2000	-0.8578	0.4578		0.0266			
				-0.2584		-0.0020		
2	0.2000	-0.8566	0.4566		0.0246			
				-0.2338		-0.0020		
3	0.3000	-0.8227	0.2227		0.0246		-0.0005	
				-0.2338		-0.0024		
3	0.3000	-0.8228	0.2228		0.0222		-0.0004	
				-0.2115		-0.0024		0.0007
4	0.4000	-0.81128	0.0113		0.0222		0.0003	
				-0.2115		-0.0021		
4	0.4000	-0.81129	0.0113		0.0201			
				-0.1914				
5	0.5000	-0.8199	-0.1801					

Qn4

(a) Solution using an Excel worksheet together with the formulae used to calculate the solutions.

$y' = x^2 - y \qquad y(0) = 1$

$h = 0.1$ and $y_{ex} = 2 - 2x + x^2 - e^{-x}$,

$$y_4^P = y_3 + \frac{0.1}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$y_4 = y_3 + \frac{0.1}{24}[9f_4^P + 19f_3 - 5f_2 + f_1]$$

	x	y	f(x,y)	y_ex	ABS(y_ex- y)
Given	0.0	1.0000000	-1.0000000	1.0000000	
Using exact	0.1	0.9051626	-0.8951626	0.9051626	
Using exact	0.2	0.8212692	-0.7812692	0.8212692	
Using Exact	0.3	0.7491818	-0.6591818	0.7491818	
Predictor	0.4	0.6896771	-0.5296771		0.0000029
Corrector	0.4	0.6896803	-0.5296803	0.6896800	0.0000003
Predictor	0.5	0.6434670	-0.3934670		0.0000024
Corrector	0.5	0.6434699	-0.3934699	0.6434693	0.0000006

Using the formulae:

x	y	f(x,y)	y_ex	ABS(y_ex- y)
0	=2-2*B5+B5^2-EXP(-B5)	=B5^2-C5	=2-2*B5+B5^2-EXP(-B5)	
0.1	=2-2*B7+B7^2-EXP(-B7)	=B7^2-C7	=2-2*B7+B7^2-EXP(-B7)	
0.2	=2-2*B9+B9^2-EXP(-B9)	=B9^2-C9	=2-2*B9+B9^2-EXP(-B9)	
0.3	=2-2*B11+B11^2-EXP(-B11)	=B11^2-C11	=2-2*B11+B11^2-EXP(-B11)	
0.4	=C11+(0.1/24)*(55*D11-59*D9+37*D7-9*D5)	=B13^2-C13		=ABS(E14-C13)
0.4	=C11+(0.1/24)*(9*D13+19*D11-5*D9+D7)	=B14^2-C14	=2-2*B14+B14^2-EXP(-B14)	=ABS(E14-C14)
0.5	=C14+(0.1/24)*(55*D14-59*D11+37*D9-9*D7)	=B16^2-C16		=ABS(E17-C16)
0.5	=C14+(0.1/24)*(9*D16+19*D14-5*D11+D9)	=B17^2-C17	=2-2*B17+B17^2-EXP(-B17)	=ABS(E17-C17)

Qn4 (b)

Using Program 1 with the function changed to:

```
function f=f(t,y);
f=t*t-y;
```

t	yp	y	F	y_ex	AbsError
0.00	0.0000000	1.0000000	-1.0000000	1.0000000	0.00000000
0.10	0.0000000	0.9051626	-0.8951626	0.9051626	0.00000000
0.20	0.0000000	0.8212692	-0.7812692	0.8212692	0.00000000
0.30	0.0000000	0.7491818	-0.6591818	0.7491818	0.00000000
0.40	0.6896771	0.6896803	-0.5296803	0.6896800	0.00000031
0.50	0.6434670	0.6434699	-0.3934699	0.6434693	0.00000056
0.60	0.6111865	0.6111891	-0.2511891	0.6111884	0.00000075
0.70	0.5934132	0.5934156	-0.1034156	0.5934147	0.00000091
0.80	0.5906699	0.5906721	0.0493279	0.5906710	0.00000103
0.90	0.6034295	0.6034315	0.2065685	0.6034303	0.00000111
1.00	0.6321200	0.6321217	0.3678783	0.6321206	0.00000117

Qn4 (c)

Using Program 2 with the function changed to:

```
function f=f(t,y);
f=t*t-y;
```

t	yp	y	F	y_ex	absError
0.00	0.0000000	1.0000000	-1.0000000	1.0000000	0.00000000
0.10	0.0000000	0.9051627	-0.8951627	0.9051626	0.00000013
0.20	0.0000000	0.8212695	-0.7812695	0.8212692	0.00000025
0.30	0.0000000	0.7491821	-0.6591821	0.7491818	0.00000037
0.40	0.6896774	0.6896806	-0.5296806	0.6896800	0.00000064
0.50	0.6434673	0.6434702	-0.3934702	0.6434693	0.00000085
0.60	0.6111868	0.6111894	-0.2511894	0.6111884	0.00000102
0.70	0.5934135	0.5934158	-0.1034158	0.5934147	0.00000115
0.80	0.5906701	0.5906723	0.0493277	0.5906710	0.00000125
0.90	0.6034297	0.6034317	0.2065683	0.6034303	0.00000131
1.00	0.6321202	0.6321219	0.3678781	0.6321206	0.00000136

Qn4 (d)

Using Program 3 with the function changed to:

```
function f=f(t,y);
f=t*t-y;
%Scriptfile to solve a single 1st Order ordinary differential equation, using
%Adam-bashforth-Moulton method ODE solver, ode113 from Matlab ODE solver routines.
function ch2ode113
y0=1.0;
tspan=[0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1];
options = odeset('RelTol',1e-6,'AbsTol',1e-4);
[t,y]=ode113(@f,tspan,y0,options);
disp('  x         y         y_ex       Abs error');
fprintf('----------------------------------------\n');
for i=1:11
yex(i)=2-2*t(i)+t(i)^2-exp(-t(i));
error(i)=abs(y(i)-yex(i));
fprintf('%5.2f %10.7f %10.7f %11.8f\n', t(i),y(i),yex(i),error(i));
end
%plot(t,y(:,1));
%title(['Solution of dy/dt = x^2-y, y(0) =' num2str(y0)]);
%xlabel('x');
%ylabel('Solution y');
%-----------------------------------------------------------------
function f=f(t,y)
f=t*t-y;
```

Using RelTol=AbsTol=1.0e-4 in the option statement, gives the following results.

t	y	y_ex	AbsError
0.00	1.0000000	1.0000000	0.00000000
0.10	0.9051626	0.9051626	0.00000005
0.20	0.8212690	0.8212692	0.00000027
0.30	0.7491815	0.7491818	0.00000023
0.40	0.6896797	0.6896800	0.00000023
0.50	0.6434691	0.6434693	0.00000021
0.60	0.6111882	0.6111884	0.00000020
0.70	0.5934145	0.5934147	0.00000019
0.80	0.5906709	0.5906710	0.00000017
0.90	0.6034302	0.6034303	0.00000015
1.00	0.6321204	0.6321206	0.00000014

Using RelTol=AbsTol=1.0e-6 in the option statement, gives the following results.

t	y	y_ex	Abserror
0.00	1.0000000	1.0000000	0.00000000
0.10	0.9051626	0.9051626	0.00000000
0.20	0.8212692	0.8212692	0.00000000
0.30	0.7491818	0.7491818	0.00000000
0.40	0.6896799	0.6896800	0.00000001
0.50	0.6434693	0.6434693	0.00000001
0.60	0.6111883	0.6111884	0.00000002
0.70	0.5934147	0.5934147	0.00000002
0.80	0.5906710	0.5906710	0.00000002
0.90	0.6034303	0.6034303	0.00000002
1.00	0.6321205	0.6321206	0.00000002

Remember to change the exact solution, compare the two set of solutions and comment on the accuracy of solutions for each case.