6G6Z3002 – Computational Methods for ODEs

Chapter 5 – Exercise 5 Part Solutions

Qn1:

$$y_{j+1} = y_j + h\, f_{j+1}$$

First we need to find the characteristic polynomial. Substituting $f = \lambda y$ gives

$$y_{j+1} = y_j + h\lambda y_{j+1}$$

Collecting the y terms gives:

$$(1 - h\lambda) y_{j+1} - y_j = 0$$

$$(1 - h\lambda)\xi - 1 = 0$$

$$\therefore \xi = \frac{1}{1 - h\lambda}$$

Therefore, we have a linear function (a polynomial) of degree 1, with only one root to be calculated for different values of $h\lambda$, as shown in the following table.

The interval of absolute stability is the set of $\bar{h}$ for which $\left|\xi(\bar{h})\right| \le 1$.

Now try different $\bar{h}$ values in this and close to this range:

$\bar{h}$	-10	-2	-1	-0.5	0	0.1	0.5	1.1	2	10	100
ξ	0.09	0.33	0.5	0.67	1	1.1	2	10	1	0.11	0.01

$|\xi| \le 1$ $\qquad |\xi| > 1$ $\qquad |\xi| \le 1$

Hence, the method is absolutely stable for the set of $\bar{h}$ where $\bar{h} \le 0 \;\; or \;\; \bar{h} \ge 2$.

Therefore the interval of absolute stability is $\bar{h} \notin [0, 2]$.

Qn2

$$y_{j+2} = y_j + \frac{h}{2}(f_{j+1} + 3 f_j)$$

First, we need to find the characteristic polynomial. Substituting $f = \lambda y$ gives:

$$y_{j+2} = y_j + \frac{h}{2}(\lambda y_{j+1} + 3\lambda y_j)$$

Collecting the y terms gives:

$$y_{j+2} - \frac{h\lambda}{2} y_{j+1} - (1 + \frac{3h\lambda}{2}) y_j = 0$$

$$\xi^2 - \frac{h\lambda}{2}\xi - (1 + \frac{3h\lambda}{2}) = 0$$

Therefore, we have a quadratic characteristic polynomial with two roots, to be calculated for different values of $h\lambda$, as shown in the following table:

$\bar{h}$	$-5/3$	$-4/3$	$-1/3$	**0**	**0.1**
$\lvert\xi_1\rvert$	**1.225**	**1**	**0.88**	**1**	**1.048**
$\lvert\xi_2\rvert$	**1.225**	**1**	**0.247**	**1**	**1.098**

The interval of absolute stability is the set of $\bar{h}$ for which $\left|\xi_1(\bar{h})\right| \le 1$ and $\left|\xi_2(\bar{h})\right| \le 1$.

Therefore the interval of absolute stability is $\left[-\frac{4}{3}, 0\right]$.

You can use Matlab routines to find the roots of the characteristic polynomial as shown below:

```
>> h=-5/3;
>> p=[1 -h/2 -(1+3*h/2)];
>> r=roots(p)

r =
  -0.4167 + 1.1517i
  -0.4167 - 1.1517i

>> r=abs(roots(p))

r =
    1.2247
    1.2247

>> h=-5/3;
>> h=-4/3;
>> p=[1 -h/2 -(1+3*h/2)];
>> r=roots(p)

r =
  -0.3333 + 0.9428i
  -0.3333 - 0.9428i

>> r=abs(roots(p))

r =
     1
     1
```

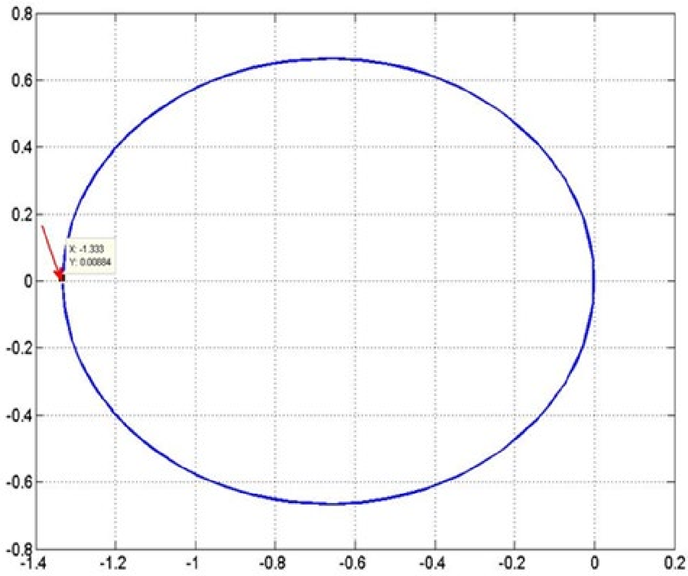
etc.

Alternatively, you can use the following Matlab program, use the coefficients of the characteristics polynomial from Qn2 and use the resulting output values to complete your table (shown for question 2).

```
%6G6Z3002 - Computational methods in ODEs
%Interval of Absolute Stability (script 1)
%Matlab script for producing a table for interval of absolute stability
%using Root Locus Method of a given LMS method- here AM2-step
%Note:the result is printed in columns, and not in rows as given in the
%lecture notes

out=[];
for h=1:-1:-7
p=[(1-5*h/12) -(1+8*h/12) (h/12)];
%r=roots(p);
r=abs(roots(p));
out=[h;r(1);r(2)];
fprintf('%6.1f %6.2f %6.2f\n',out)
end
```

A plot of the region of absolute stability for this method is shown below:



It can be seen that its region of absolute stability is the circle on the interval $\left[-\frac{4}{3},0\right]$ as diameter.

For this plot, the following Matlab programme was used.

```
%Region of absolute stability for the 2-step Adams Bashforth method
w=exp(1i*linspace(0,2*pi));
z=2*(w.^2-1)./(w+3);
plot(z)
grid on
```

Qn3

Find the interval of absolute stability for the 3rd order Adams-Bashforth method using root locus method:

$$y_{j+1} = y_j + \frac{h}{12}\left[23f_j - 16f_{j-1} + 5f_{j-2}\right]$$

Rearranging the subscripts on both sides of the equation so that the lowest subscript is j:

$$y_{j+3} = y_{j+2} + \frac{h}{12}\left[23f_{j+2} - 16f_{j+1} + 5f_j\right]$$

Method 1 of obtaining the characteristic polynomial:

A comparison with the general formula with $k = 3$

$$\sum_{i=0}^{k} \alpha_i y_{j+i} = h\sum_{i=0}^{k} \beta_i f_{j+i}, \qquad \text{with } \alpha_k = 1,$$

$$\alpha_3 = 1,\ \alpha_2 = -1,\ \alpha_1 = 0,\ \alpha_0 = 0, \qquad \beta_3 = 0,\ \beta_2 = \frac{23}{12},\ \beta_1 = -\frac{16}{12},\ \beta_0 = \frac{5}{12}.$$

The characteristic polynomial $\rho(\xi)$ is:

$$\rho(\xi) = \sum_{i=0}^{3} \alpha_i \xi^i = \alpha_0\,\xi^0 + \alpha_1\,\xi^1 + \alpha_2\,\xi^2 + \alpha_3\,\xi^3 = \xi^3 - \xi^2.$$

$$\sigma(\xi) = \sum_{i=0}^{3} \beta_i \xi^i = \beta_0\,\xi^0 + \beta_1\,\xi^1 + \beta_2\,\xi^2 + \beta_3\,\xi^3 = \frac{23}{12}\xi^2 - \frac{16}{12}\xi + \frac{5}{12}.$$

$\pi(\xi, h\lambda) = \rho(\xi) - h\lambda\sigma(\xi)$, therefore the Characteristics polynomial can be found as:

$$\left(\xi^3 - \xi^2\right) - \frac{h\lambda}{12}(23\xi^2 - 16\xi + 5) = 0$$

Collecting the like terms gives, the characteristic polynomial:

$$\xi^3 - (1 + \frac{23\bar{h}}{12})\xi^2 + \frac{16}{12}\bar{h}\xi - \frac{5}{12}\bar{h} = 0$$

Therefore, we have a cubic characteristic polynomial with three roots, to be calculated for different values of $h\lambda$. We can use the following Matlab code.

```
%6G6Z3002 - Computational methods in ODEs
%Interval of Absolute Stability (script 1)
%using Root Locus Method of a given LMS method- here AB3-step
%Note: the result is printed in columns, and not in rows as given in the
%lecture notes
out=[];
for h=0.1:-0.05:-0.6
p=[1 -(1+(23*h/12)) (16*h/12) -5*h/12];
%r=roots(p);
r=abs(roots(p));
out=[h;r(1);r(2);r(3)];
fprintf('%6.2f %6.3f %6.3f %6.3f\n',out)
end
```

The program produces the following table of results for the three roots of the cubic characteristic polynomial.

```
>> IntervalAB3
  0.10  1.105  0.194  0.194
  0.05  1.051  0.141  0.141
  0.00  0.000  0.000  1.000
 -0.05  0.951  0.173  0.126
 -0.10  0.905  0.268  0.172
 -0.15  0.861  0.353  0.205
 -0.20  0.818  0.435  0.234
 -0.25  0.516  0.777  0.260
 -0.30  0.597  0.738  0.284
 -0.35  0.677  0.698  0.308
 -0.40  0.759  0.659  0.333
 -0.45  0.841  0.617  0.361
 -0.50  0.924  0.570  0.396
 -0.55  1.008  0.477  0.477
 -0.60  1.092  0.478  0.478
```

We can now set up the horizontal table for the three roots and show the interval of absolute stability, shown in the following table:

$\bar{h}$	**-0.60**	**-0.55**	**-0.4**	**-0.20**	**-0.10**	**0.0**	**0.10**
$\lvert\xi_1\rvert$	**0.48**	**0**	**0.33**	**0.23**	**0.17**	**1**	**0.14**
$\lvert\xi_2\rvert$	**0.48**	**0**	**0.66**	**0.43**	**0.27**	**0**	**0.14**
$\lvert\xi_2\rvert$	**1.09**	**1**	**0.76**	**0.82**	**0.91**	**0**	**1.10**

Hence, the interval of absolute stability is: $\left[-0.55, 0\right]$, where all roots $\left|\xi_1(\bar{h})\right| \le 1, \left|\xi_2(\bar{h})\right| \le 1, \left|\xi_3(\bar{h})\right| \le 1$

<u>Method 2 of obtaining the characteristic polynomial:</u>

Note: you can derive your characteristic polynomial directly from the multistep formula

$$y_{j+1} = y_j + \frac{h}{12}\left[23 f_j - 16 f_{j-1} + 5 f_{j-2}\right]$$

Rearranging the subscripts on both sides of the equation so that the lowest subscript is j:

$$y_{j+3} = y_{j+2} + \frac{h}{12}\left[23 f_{j+2} - 16 f_{j+1} + 5 f_j\right]$$

Substituting $f = \lambda y$ gives:

$$y_{j+3} = y_{j+2} + \frac{h}{12}\left[23\lambda y_{j+2} - 16\lambda y_{j+1} + 5\lambda y_j\right],$$

Substituting $y_{j+1} = \xi^1$, and collecting the like terms gives, the cubic characteristic polynomial:

$$\xi^3 - (1 + \frac{23\bar{h}}{12})\xi^2 + \frac{16}{12}\bar{h}\xi - \frac{5}{12}\bar{h} = 0$$

The remaining steps for finding the interval of absolute stability will be the same as explained above.

Repeat the same process to find the interval of absolute stability for 3rd order Adams-Moulton and 4th order Adams Bashforth and Adams Moulton methods – i.e. 3 more tables and compare your results with the table on page 4, Chapter 5 lecture notes.

Next we will find the region of absolute stability for 3rd order Adams-Bashforth method:

$$y_{j+1} = y_j + \frac{h}{12}\left[23 f_j - 16 f_{j-1} + 5 f_{j-2}\right]$$

Step 1 - We need to find 1st and 2nd characteristic polynomials:

$$\rho(\xi) = \xi^3 - \xi^2$$

$$\sigma(\xi) = \frac{1}{12}(23\xi^2 - 16\xi + 5)$$

Step 2 – we write

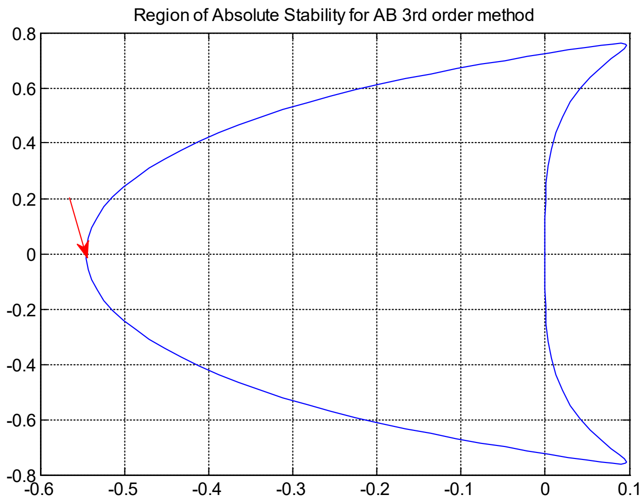
$\pi(\xi, h\lambda) = \rho(\xi) - h\lambda\sigma(\xi) = 0$, therefore $h\lambda = \dfrac{\rho(\xi)}{\sigma(\xi)} = \dfrac{12(\xi^3 - \xi^2)}{23\xi^2 - 16\xi + 5}$

Substituting $\xi = e^{i\theta}$, gives $h\lambda = \dfrac{12(e^{3i\theta} - e^{2i\theta})}{23e^{2i\theta} - 16e^{i\theta} + 5}$.

We can modify the following Matlab code and plot the region of absolute stability:

```
%Region of absolute stability for the AB2 method
w=exp(1i*linspace(0,2*pi));
z=2*(w.^2-w)./(3*w-1);
plot(z)
```

```
%Region of absolute stability for the AB3 method
w=exp(1i*linspace(0,2*pi));
z=12*(w.^3- w.^2)./(23*w.^2-
16*w+5);
plot(z)
grid on
```



Note that the region of absolute stability crosses the $Re(h\lambda)$ axis at -0.55, i.e. confirming the interval of absolute stability for the 3rd order AB method, i.e. $\left[-0.55, 0\right]$

Qn4

Rearranging the subscripts on both sides of the equation so that the lowest subscript is j gives:

$y_{j+1} = y_j + \dfrac{h}{2}\left[f_{j+1} + f_j\right]$ with characteristic polynomial $(1 - \dfrac{\bar{h}}{2})\xi - (1 + \dfrac{\bar{h}}{2}) = 0$

$\therefore \xi = \dfrac{1 + \frac{\bar{h}}{2}}{1 - \frac{\bar{h}}{2}}$. Therefore, for all negative values of $h\lambda$ from 0 to $-\infty$, the root of the characteristic polynomial ξ will be less than 1 in magnitude, except when $h\lambda = 0$ when $\xi = 1$. Hence, the interval of absolute stability is $\left[-\infty, 0\right]$.

Qn5

Follow the method as described in Qn 3. You will find that for all negative $h\lambda$ values, at least one of the roots of the associated characteristic polynomial will be greater than 1 in magnitude.