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# Optimal reserve prices in weighted GSP auctions ☆

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ABSTRACT

Most search engines use the weighted Generalized Second Price (wGSP) auction to sell keyword-based text ads, generating billions of dollars of advertising revenue every year. Designing and implementing near-optimal reserve prices for these wGSP auctions are naturally important problems for both academia and industry.

In this paper, we show how to calculate and implement the near-optimal reserve price of the wGSP mechanism in realistic settings. Unlike reserve prices in standard single-item auctions, optimal reserve prices in wGSP auctions are discriminatory, different even for advertisers bidding on the same keyword. The optimal reserve price results can be extended to support CPA/CPC/CPM[1] hybrid auctions.

Our simulations indicate that setting a proper reserve price will transfer some bidder utility (payoff) to auctioneer utility, resulting in higher revenue for the search engine. We describe a practical methodology to implement optimal reserve prices in production systems.



## 1. Introduction

Search advertising places keyword-based text ads alongside search results, generating billions of dollars of advertising revenue annually, and is the primary factor in the successful commercialization of search engines (Edelman et al. 2007, Varian 2007). Keyword-based text ads typically contain a title, a text description, and a display URL. Advertisers generate revenue when search engine users click on keyword ads and subsequently purchase advertised goods or services. Search providers conduct auctions to allocate advertising positions and decide per-click prices. Early keyword auctions implemented the Generalized First Price (GFP) mechanism, which was pioneered by Overture (later acquired by Yahoo!). The major stages of keyword auction evolution (Thompson and Leyton-Brown 2009) are as follows:

1. **GFP**: The unweighted Generalized First Price auction. Agents are ranked by their bids and each bidder who wins a slot pays her bid per click. This mechanism was first used by Overture, 1997–2002.
2. **uGSP**: The unweighted Generalized Second Price auction. Agents are ranked by their bids and each bidder who wins a slot pays the next highest bid per click. This mechanism was first used by Yahoo!, 2002–2007.
3. **wGSP**: The weighted Generalized Second Price auction. The mechanism assigns a weight $w_i$ to each bidder; agent $i$ is ranked by the product of her bid $b_i$ and her weight $w_i$, winner $i$ pays $p_i = b_{i+1} \cdot w_{i+1}/w_i$ per click (where $b_{i+1}$ and $w_{i+1}$ are the next highest bidder's bid and weight). This mechanism was first used by Google and adopted by all major search engines gradually.

The wGSP mechanism is widely used in the industry nowadays and naturally is the focus in both academia and industry. Instead of rank-by-bid in uGSP, we refer to wGSP as rank-by-revenue. When users search for the keyword, the search engine will rank the ads in descending order of the product of each advertiser's weight (ad quality) and bid (per-click). Under the wGSP mechanism, search engines charge advertisers according to their bids as well as their ad quality factors (primarily the estimated click-through rate). Our work focuses on wGSP auctions and our results allow the search engine to set discriminatory reserve prices for each advertiser and each keyword. Our settings are as follows: each bidder knows her private value and her expected click-through rate (eCTR) of the keyword, and only bidders who bid greater than their reserve prices can participate in the auction.

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[1] CPM: cost per mille (impressions), CPA: cost per action, CPC: cost per click.

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### 1.1. Related work

Myerson (1981) proposed the general optimal auction framework in a Bayesian setting known as the classical Myerson optimal auction, which maximizes auctioneer revenue. Myerson proved that the optimal auction problem is equivalent to the problem of maximizing bidders' expected virtual surplus, which can be solved by an efficient auction with respect to virtual surplus under the technical condition of monotone non-decreasing hazard rate (MHR) on the private value distribution. Thus, Myerson converted the problem of optimal auction design into one of efficient auction design on a perturbed problem.

In the case of single-item auctions, the Myerson optimal auction is equivalent to the Vickrey auction with a reserve price. For sponsored search auctions, the goal is to compute the allocation and payment rules that result in a revenue optimal mechanism for the multi-item auction. This calls for extending the Myerson optimal auction to the case of the sponsored search auction.

Varian (2007) and Edelman et al. (2007) modeled the keyword auction problem as a Nash equilibrium problem with complete information about all the bids. They defined a refined Nash equilibrium called *locally envy-free equilibrium* (Edelman et al. 2007) or *symmetric Nash equilibrium* (Varian 2007). These two equilibria are actually equivalent: if each advertiser bids at an envy-free point, then she is exactly indifferent between remaining in her current position and trading places with a bidder above her.

Gomes and Sweeney (2009) characterized the efficient Bayes-Nash equilibrium in GSP auctions with a Bayesian setting. Their incomplete information assumption was different from the above-mentioned full-information setting. They assumed that each bidder can only estimate the overall bidders' value distribution, instead of knowing the details of others' bids. Their assumption was closer to actual conditions in keywords auctions, since most search engines run sealed-bid auctions nowadays and bidders cannot observe others' bids. Their results provided the foundation for our study: assume that bidders' values were independently drawn from a distribution $F$, each bidder privately observed her value $v_i$ and simultaneously submitted her bid to the search engine following the same function $b_i = \beta(v_i, F)$.

Edelman and Schwarz (2010) generalized Myerson's theory to the multi-unit environment and simulated uGSP auctions. They proved that the uGSP auction with reserve price $r^*$ is an optimal mechanism. Ostrovsky and Schwarz (2011) followed their work and conducted a large scale experiment at Yahoo! Search with a new reserve price system. Below are excerpts from their paper:

> "The theoretically optimal reserve prices were computed under the assumption that all bidders have the same quality factors and are ranked solely on the basis of their bids, which is a simplification. In practice, the ads on Yahoo! are ranked based largely on the product of their quality factors and bids, and the amount each advertiser pays is lower when his ad's quality is higher. Thus, in order to keep the implementation of reserve prices consistent with the company's ranking and pricing philosophy, the theoretical reserve prices were converted into advertiser-specific reserve prices that reflected the quality factors of the ads: ads with higher quality factors faced lower per-click reserve prices, and vice versa."
> *Note that this is a deviation from the theoretically optimal auction design with asymmetric bidders.*" (Ostrovsky and Schwarz 2011).

Their new reserve price was the average of the theoretically optimal reserve price and the original uniform reserve price (10 cents), and it was reported that the new reserve price improved the average revenue per search by almost 13%.

Liu et al. (2010) studied a scheme where the auctioneer can weigh advertisers' bids differently and require different minimum bids based on advertisers' quality factors. They studied the impact and design of such weighting schemes and minimum-bid policies. They proposed the revenue-maximizing optimal reserve price for wGFP auctions, where all bidders' quality factors $e_i$ are discrete numbers. For example, $n$ bidders bid on one keyword, and there are only 2 possible values of $e_i$: the $h$-type bidder with $e_h$, and $l$-type bidder with $e_l$. In the setting without considering the distortion effect, this requires the revenue-maximizing reserve prices to satisfy

$$\underline{b_l} - \frac{1 - F_l(\underline{b_l})}{f_l(\underline{b_l})} = 0 \qquad \text{and} \qquad \underline{b_h} - \frac{1 - F_h(\underline{b_h})}{f_h(\underline{b_h})} = 0 \tag{1}$$

While Liu et al. (2010) proved the revenue equivalence between wGFP and wGSP under their settings, their results cannot extend to the setting where $e_i$ follows a continuous distribution. In realistic wGSP auctions, bidders' quality factors $e_i$ are real numbers from $(0, 1]$, and it is possible that all $e_i$ are distinct from each other. For example, there are 30 bidders bidding on one keyword. Since there are 30 different $e_i$, 30 distributions $F$ need to be estimated and 30 equations need to be solved. But there is only one sample for each distribution estimation, implementation is thus impossible. Even if we reduce those $e_i$ to 10 discrete intervals approximately, it is hard to estimate well each distribution $F$ from 3 samples on average. Obviously, these results could not work in production systems.

Recent related work of Thompson and Leyton-Brown (2013) analyzed a more general GSP ranking scheme, where bidders were ranked by $b_i \cdot q_i^{\alpha}$ with $\alpha \in [0, 1]$ a constant. They simulated the revenue optimal parameters (coefficient $\alpha$ and reserve prices) by searching all possible parameter values, for both single-slot and multi-slot cases.

### 1.2. Our contributions

Obviously, the optimal reserve prices in wGSP auctions were still an unsolved issue when Yahoo! did the field experiment, and the previous theoretically optimal reserve price is not directly applicable to the case where bidders have different quality factors. Thus, Ostrovsky and Schwarz (2011) implemented it with a compromise between the theoretically optimal uGSP results and practical wGSP auctions, even though it was a deviation from the theoretically optimal results.

In this paper, we show how to calculate provably optimal reserve prices in wGSP auctions where bidders have *different* quality factors drawn from a *continuous* interval (such as $(0, 1]$). This model is based on the realistic keyword auctions implemented by most search engines nowadays. We also show that this optimal reserve price can support CPA/CPC/CPM hybrid auctions.

The rest of the paper is organized as follows: Section 2 introduces the details of the wGSP mechanism and our model. Section 3 discusses how to compute the optimal reserve price, and extends this optimal reserve price to CPA/CPC/CPM hybrid auctions. Section 4 presents simulation results from estimated distributions, considering the search engine revenue, bidders' payments and utilities. Section 5 describes a practical technique to compute the optimal reserve price for each keyword and each advertiser.

## 2. Preliminaries

### 2.1. Ranking and pricing in wGSP auctions

There are $k$ positions to be allocated among $n$ bidders, with $k \leqslant n$. We denote the ad positions as $t \in \{1, \ldots, k\}$, and the bidders as $i \in \{1, \ldots, n\}$. We also denote $v_i$ and $b_i$ as the private value and bid of bidder $i$. For a specific bidder $i$, his valuation $v_i$ for one click is identical through all positions.

We define a position factor $x_t \in (0, 1]$ and an ad quality factor $e_i \in (0, 1]$, where $x_t$ denotes the probability that an ad in position $t$ will be noticed by users, assuming that the position factor is decreasing by $t$, i.e., $x_1 > x_2 > \cdots > x_k > 0$. $e_i$ denotes the probability that ad $i$ will be clicked if noticed, and it is equal to the expected click-through rate (eCTR) if ad $i$ is placed at the first slot.[2] We assume that the click-through rate (CTR) of ad $i$ in position $t$ is $e_i \cdot x_t$, which is separable into the ad quality factor $e_i$ and position factor $x_t$ (Aggarwal et al. 2006).

All $x_t$ are common knowledge to all bidders, but $v_i$ and $b_i$ are private information of bidder $i$. For $e_i$, most search engines will disclose the quality factor $e_i$ to advertiser $i$, in order to encourage advertisers to improve their ads' performance. Hence, *only bidder i knows* $e_i$.

We analyze a typical ranking rule where ads are ranked in descending order of *rank score* $e_i \cdot b_i$ $(e_1 b_1 > e_2 b_2 > \cdots > e_n b_n)$. The price-per-click $p_i$ of winner $i$ depends on the next bidder's rank score and his $e_i$:

$$p_i = \frac{e_{i+1} \cdot b_{i+1}}{e_i} \tag{2}$$

Obviously, this pricing schedule guarantees any winner's price-per-click $p_i$ less than his bid $b_i$,

$$\frac{e_{i+1} \cdot b_{i+1}}{e_i} < \frac{e_i \cdot b_i}{e_i} \Rightarrow p_i < b_i \tag{3}$$

### 2.2. Joint distribution of value $v_i$ and quality factor $e_i$

The quality factor $e_i$ has a strong impact on the revenue that the search engine receives from the ads. It is important to be able to accurately estimate quality factors of ads. The search engine can record ads' past CTR performance, and estimate their eCTR for those ads which have been displayed repeatedly (Regelson and Fain 2006). Even for new ads, quality factor $e_i$ can be estimated through their features and terms (Richardson et al. 2007). In summary, the search engine can learn a model that accurately predicts the quality factors for all ads, and we assume the estimated $e_i$ is accurate and unbiased in comparison with the a posteriori data. Both $e_i$ and $b_i$ are parts of the information set of the mechanism designer, but $v_i$ is not.

We assume that each bidder's value-per-click $v_i$ is linearly independent of his quality factor $e_i$. (from industrial data analysis in Appendix A, we can validate this assumption). Otherwise, if $e_i$ and $v_i$ are correlated, the search engine can estimate each bidder's value through his quality factor. In this case, the full revenue extraction is possible.

Thus, we denote bidder's *score* $s_i = e_i \cdot v_i$, and assume that $s_i$ is drawn from a continuous distribution with cumulative distribution function $F(s)$ and probability density function $f(s)$. For clarification, the *score* $s_i = e_i \cdot v_i$ is different from the rank score $e_i \cdot b_i$.

### 2.3. Bidding strategy in wGSP auctions

In our model, there is no need to assume all bidders obey a particular bidding strategy or equilibrium. We just assume that all bidders are individually rational, i.e., a bidder always chooses his expected utility-maximizing strategy. Liu et al. (2010) analyzed a similar case for GFP auctions, where bidders obey a bidding strategy and there are two types of advertisers: $h$-type advertiser with high quality factor $e_h$ and $l$-type advertiser with low quality factor $e_l$. An $l$-type advertiser with value-per-click $v$ matches an $h$-type advertiser with value-per-click $\omega v$ in equilibria. Formally,

$$b_h(\omega v) = \omega b_l(v) \qquad \forall v, \omega \in [0, 1] \quad \text{where} \quad \omega = e_l/e_h$$

In this section, we extend the lemma to represent the bidding relations between bidders in wGSP auctions, where quality factors are drawn from a continuous interval.

**Lemma 1.** *In a wGSP auction, given a bidder i with value-per-click $v_i$ and quality factor $e_i$, if his bid $b_i$ maximizes his utility, $b_i$ satisfies*

$$b_i = \frac{\beta(s_i)}{e_i}$$

*where $\beta(s_i)$ is the utility-maximizing bidding strategy of all bidders in the auction and $s_i = e_i v_i$.*

**Proof.** The proof for Lemma 1 is in Appendix B. □

We conjecture that the bidding strategy $\beta(s_i)$ is a strictly increasing function of $s_i$ (Gomes and Sweeney 2009). Since wGSP ranks the ads by $e_i b_i = \beta(s_i)$, all ads were actually ranked by $s_i$ when all bidders follow the utility-maximizing strategy. Since most search engines offer tools to estimate bidders' positions with varying bids, each bidder can estimate the overall distribution of $s_i$ from the repeated auction. Thus, even though each bidder cannot get the information of others' $v_i$ and $e_i, F(s)$ and $f(s)$ are common knowledge to each bidder.

### 2.4. Expected position factors and payments in equilibria

When $n$ bidders compete for $k$ ad positions, bidder $i$ should win each position $t$ between 1 and $\min(n, k)$ with some probability. We define a function $x^i = \tilde{q}(e_i b_i)$ to represent bidder $i$'s expected position factor, which is the sum of each position factor times the corresponding probability:

$$x^i = \tilde{q}(e_i \cdot b_i) = \sum_{t=1}^{\min(n,k)} x_t \cdot Pr\langle e_i b_i \text{ ranks } t\text{th}\rangle \tag{4}$$

When bidder $i$ follows the utility-maximizing strategy, his rank score $e_i b_i = \beta(s_i)$. It yields

$$x^i = \tilde{q}(\beta(s_i)) = \sum_{t=1}^{\min(n,k)} x_t \cdot Pr\langle \beta(s_i) \text{ ranks } t\text{th}\rangle \tag{5}$$

Similar to $x^i$, we denote bidder $i$'s expected payment by a function $\tilde{m}(\cdot)$ of bidder $i$'s $e_i b_i$, and expand each bidder's payment to the sum of expected payments at each position times the corresponding position probability:

$$\text{Payment}_i = \tilde{m}(e_i \cdot b_i) = \sum_{t=1}^{\min(n,k)} \left[\frac{e_{t+1} b_{t+1}}{e_i} \cdot e_i x_t\right] \cdot Pr\langle e_i b_i \text{ ranks } t^{\text{th}}\rangle \tag{6}$$

When all bidders follows the utility-maximizing strategy, $\frac{e_{t+1} b_{t+1}}{e_i} \cdot e_i = e_{i+1} b_{i+1} = \beta(s_{i+1})$, his rank score $e_i b_i = \beta(s_i)$. It yields

$$\text{Payment}_i = \tilde{m}(\beta(s_i)) = \sum_{t=1}^{\min(n,k)} \beta(s_{t+1}) \cdot x_t \cdot Pr\langle \beta(s_i) \text{ ranks } t^{\text{th}}\rangle \tag{7}$$

Here $\beta(s_{t+1})$ is a parameter to bidder $i$. When $F(s), \min(n, k), t$ are fixed, $s_{t+1}$ equals to the expected value of the $(t+1)$-th largest number when selecting $m = \min(n, k)$ numbers from distribution $F(s)$.

$$s_t = \int_{-\infty}^{\infty} s \cdot m \cdot \binom{m-1}{t-1} F(s)^{m-t} [1 - F(s)]^{t-1} f(s) ds \tag{8}$$

[2] Most search engines may adjust the quality factor $e_i$ to incorporate other metrics of their ads (relevance, landing page quality, etc.), but this adjustment is usually small and affects limited numbers of ads. Thus for simplicity, we treat $e_i$ as equal to the raw eCTR in this paper.

Thus, each bidder's expected position factor $x^i$ and expected $\text{Payment}_i$ are continuous functions of his $s_i$, when all bidders are in an equilibrium of Bayesian setting. The functions allow differentiation operations under appropriate smoothness assumptions, and it will not affect our results on the revenue analysis.

## 3. The optimal reserve price

An adjustment in the reserve price can immediately lead to changes in the advertisers' bids who were previously bidding around the new reserve price, as well as those previous bids above, and finally lead to a new equilibrium.

When an auction scheme and bidders' valuation distribution were changed, we adjust the reserve price, this will result in a change of expected revenue. Thus, the total expected revenue from bidders can be seen as a function of the assigned reserve price $r$, and the optimal reserve price $r^*$ is the one which maximizes the expected revenue.

**Theorem 1.** *In wGSP auctions, for a given keyword, suppose the product of each bidder' quality factor $e_i$ and value-per-click $v_i$ follows a common cdf $F(s)$ and pdf $f(s)$. When $F(s)$ is a common knowledge to all bidders, the revenue optimal reserve price $p_i^*$ for bidder i is $\underline{s}^*/e_i$, where $\underline{s}^*$ satisfies:*

$$\underline{s}^* - \frac{1 - F(\underline{s}^*)}{f(\underline{s}^*)} = 0 \tag{9}$$

**Proof.** The proof for Theorem 1 is in Appendix C. □

This result shows that the minimal score $\underline{s}^*$ depends only on the joint distribution of $v_i$ and $e_i$. Subsequently, we denote $\underline{s}^*$ the *optimal reserve score*, and $p_i^*$ the *optimal reserve price*.

For each bidder with quality factor $e_i$, its bid should be greater than $p_i^*$, otherwise he is not allowed to participate in the auction. For bidders bidding on the same keyword, even though their optimal reserve scores $\underline{s}^*$ are the same, since they have different quality factors, their optimal reserve prices $p_i^*$ are different. Obviously Eq. (9) is a discriminatory reserve price.

### 3.1. Extension to hybrid auctions

Besides CPC, two other price mechanisms are widely used in keyword auctions:

- **CPM**: cost per mille (impressions), the search engine charges the advertiser for every impression of an ad shown to a user;
- **CPA**: cost per action, the search engine charges the advertiser when a pre-defined action happens, such as purchase or transaction.

Goel and Munagala (2009) showed that these three models are equivalent when eCTR is estimated precisely and the action conversion rate per-impression is known.

Some search engines allow CPC/CPM/CPA hybrid bidding, in which each advertiser may choose whether to bid on per-click (CPC), per impression (CPM), or per-action (CPA). For example, Google allows advertisers to bid on conversions in AdWords, besides CPC and CPM. Bids of all three types compete for the same ad positions. For a CPC bid $b_i$, the search engine can convert it into an estimated CPM bid $m_i = b_i * e_i$, by multiplying his bid $b_i$ with the quality factor $e_i$. For a CPA bid $a_k$, similarly it can be converted to an estimated CPM bid $m_k = a_k * \alpha_k$ where $\alpha_k$ is the estimated conversion rate given an impression. All those CPC and CPA bids are converted into the equivalent CPM bids, and compete together with those CPM bidders $j$ who bid $m_j$ per-impression. Bidders will be ranked in descending order of their equivalent CPM, and charged depending on their bidding types: $\text{Price}_i = p_i/e_i$ per-click, $\text{Price}_j = p_j$ per impression, or $\text{Price}_k = p_k/\alpha_k$ per-action, where $p_i, p_j, p_k$ denotes the next bidder's equivalent CPM ($b_{i+1} \cdot e_{i+1}$ or $m_{j+1}$ or $a_{k+1} * \alpha_{k+1}$). With this prior knowledge, we can extend our optimal reserve price to those hybrid auctions.

**Corollary 1.** *In a CPC/CPA/CPM hybrid keyword auction, given a specific keyword, assume the distribution of all bidders' equivalent value score (per-impression) $s_i$ is $F(s)$. When $F(s)$ is a common knowledge to each bidder, the optimal reserve price $p_i^*$ is*

- *$p_i^* = \underline{s}^*$ for CPM bidders,*
- *$p_i^* = \underline{s}^*/e_i$ for CPC bidders, where $e_i$ is the expected CTR,*
- *$p_i^* = \underline{s}^*/\alpha_i$ for CPA bidders, where $\alpha_i$ is the expected conversion rate,*

*and the optimal reserve score $\underline{s}^*$ satisfies*

$$\underline{s}^* - \frac{1 - F(\underline{s}^*)}{f(\underline{s}^*)} = 0 \tag{10}$$

**Proof.** Zhu and Wilbur (2011) studied this case, and showed that any GSP equilibrium can be supported in the hybrid auction context. We assume that the estimated quality factor $e_i$ is unbiased and accurate enough, and with the proposition, the bidders in this hybrid auction satisfy the symmetric bidding strategy $\beta(s_i)$, where $m_i = \beta(s_i)$ is the strategy for CPM bidders, $b_i = \beta(s_i)/e_i$ for CPC bidders, and $a_i = \beta(s_i)/\alpha_i$ for CPA bidders.

Given the proliferation of hybrid pricing scheme of CPC/CPM/CPA, De Liu and Siva Viswanathan Liu and Viswanathan (2010) study the optimal choices for advertising providers. They highlight the role of two-way information asymmetry in the choice of pricing schemes.

The optimal reserve score depends only on the distribution of $s_i$. With those conditions, the optimal reserve score $\underline{s}^*$ satisfies Eq. (10), where $\underline{s}^*$ is also the reserve price for the CPM bidders, and $p_i^* = \underline{s}^*/e_i$ is the reserve price for the CPC bidders, and $p_i^* = \underline{s}^*/\alpha_i$ for CPA bidders.

In summary, the optimal reserve price result can be extended to this CPA/CPC/CPM hybrid auction. □

## 4. Simulations

Search engines can increase revenue significantly by adjusting reserve scores, and to maximize their profits. We simulate this auction process via Monte-Carlo method with different numbers of bidders. We consider the equilibrium status in wGSP mechanisms described in Section 2.

Our evaluation indicators are similar to Edelman and Schwarz (2010), including the search engine's expected revenue, each bidder's payment, all bidders' payoff, and the efficiency of the auction. However, different from their measurement in per-click metrics, we measure these indicators per search (or per-impression). Since raising reserve prices may decrease overall clicks while the search volume of each keyword remains constant, thus measuring per-click metrics is more biased than measuring per search (or per-impression) metrics.

Since the results are only related to bidders' scores $s_i$ for a specific keyword, we treat bidders' scores $s_i$ as the input directly, and disregard the details of $v_i$ and $e_i$. In our simulation, only bidders with $s_i > \underline{s}$ can participate in the auction, which is equivalent to the condition $b_i > s_i/e_i$.

Fix one keyword, we set all bidders' scores drawn from the estimated distribution in Section 5.2. It is a *lognormal distribution*, derived from a normal distribution with mean $\mu = 1.053$ and standard deviation $\sigma = 0.882$. We use these parameters to approximate the real-world distribution and get realistic revenue improvement estimates in search engines. The theoretically optimal reserve score for this distribution should be $\underline{s}^* = 3.21$, which is shown as a dashed line in Figs. 1–3.

The optimal reserve score is independent of the specific equilibrium, thus we simply assume that each bidder bids at the envy-free point, the same as the lower bound of SNE. We also assume the position factor ratio $x_{t+1}/x_t = 0.7$ as the same settings of Varian (2007), which is a common knowledge for all bidders. A randomizer generated the samples in our simulation, and bidders' values-per-impression are generated according to the *pdf* function of the lognormal distribution:

$$f(s) = \frac{1}{\sqrt{2\pi}s\sigma} e^{\frac{-(\ln s-\mu)^2}{2\sigma^2}} \tag{11}$$

We vary the reserve score $\underline{s}$ from 0 to 15 with 0.01 increments. With the settings above, we compute the average of the expected revenue-per-impression for each reserve price with 1000,000 iterations.

### 4.1. Revenue effect

Revenue-per search (RPS) is one of the most critical indicators in paid search. Fig. 1 shows the RPS variation on one keyword auction with different numbers of bidders. Each curve in the figure reflects the per search revenue as a function of reserve score $\underline{s}$, from 1 bidder to 5 bidders participating in the auction, corresponding to bottom-to-top curves.

These curves have the same optimal reserve score (around 3.21), which corroborates that the optimal reserve score for one keyword is independent of the number of bidders, and depends only on the distribution of bidders' scores, consistent with our theoretical results.

When few bidders bid on this keyword, adjusting the reserve score from the baseline to the optimal score has substantial increase of RPS, while the increase is less pronounced when more bidders participate in. Table 1 shows that with the increase of the number of bidders, the revenue increases while the increase rate declines.

### 4.2. Bidder's payment

Fig. 2 shows each bidder's expected payment as a function of reserve score when five bidders bid on one keyword. It actually decomposes the top (5 bidders) revenue curve in Fig. 1 into each

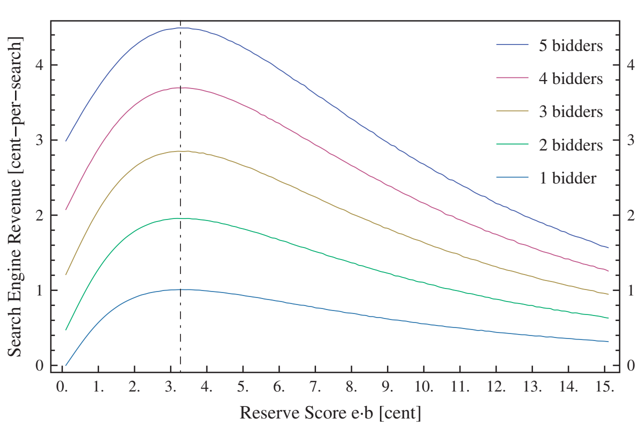


**Fig. 1.** Revnue-per search vs. reserve score.

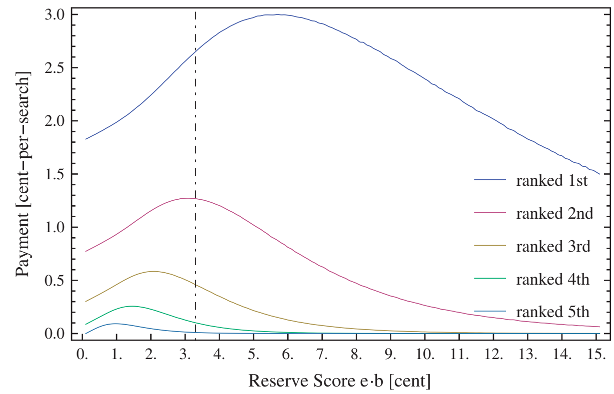


**Fig. 2.** Each bidder's payment (per search).

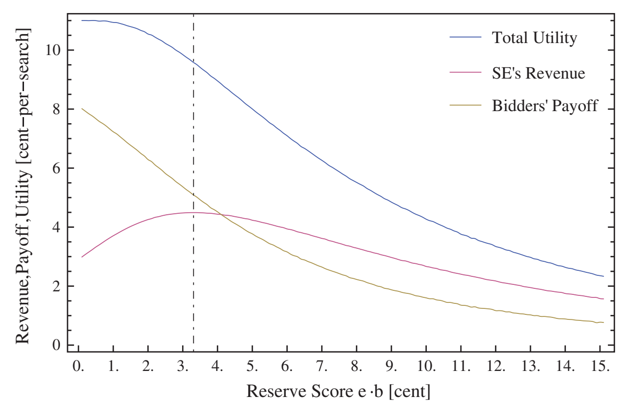


**Fig. 3.** Search engine revenue, efficiency, and bidders' utility (per search) with 5 bidders.

bidder's payment, and demonstrates how the reserve score impacts each bidder's expected payment in this auction.

From the top to the bottom curves in Fig. 2, each curve represents one bidder's variation of expected payments, from the topmost ranked to the lowest ranked. The bidder with the highest $s_i$ is ranked at the highest position and pays the most at any reserve price.

Each bidder's expected payment is impacted by the increase of the reserve score. Each represents a similar trend: a bell-shaped curve that reaches a payment peak and then declines to zero. The lowest bidder is most price-sensitive and reaches the peak first, followed by the second lowest, third lowest and the fourth lowest one. The top bidder is least price-sensitive and has not reached the peak even after the reserve price exceeds 3.21, the optimal reserve price.

Table 2 demonstrates each bidder's expected payment variation. Each row corresponds to one curve in Fig. 2, with their values at reserve scores 1.0 and 3.21 respectively. It is amazing that top

**Table 1**
Revenue per search increase.

| Total # of bidders | Reserve score | | Increase percent |
|---|---|---|---|
| | 1.0 (baseline) | 3.21 (optimal) | |
| 1 | 0.61874 | 1.00707 | 62.76% |
| 2 | 1.35143 | 1.95661 | 44.78% |
| 3 | 2.14331 | 2.85278 | 33.10% |
| 4 | 2.95813 | 3.69678 | 24.97% |
| 5 | 3.77471 | 4.49293 | 19.03% |

**Table 2**
Expected payments changes from baseline to optimal reserve prices.

| Bidder's rank | Reserve score | | Variation percent |
|---|---|---|---|
| | 1.0 (baseline) | 3.21 (optimal) | |
| 1 | 2.008890 | 2.65471 | +32.15% |
| 2 | 0.954742 | 1.26905 | +32.92% |
| 3 | 0.479157 | 0.45851 | −4.31% |
| 4 | 0.241240 | 0.10080 | −58.22% |
| 5 | 0.090683 | 0.00986 | −89.13% |
| (Sum) | 3.774712 | 4.49293 | +19.03% |

**Table 3**
An example of altered keywords of "childrens pajamas".

| ID | Keywords | Similarity rank | Bids |
|---|---|---|---|
| $K_0$ | children's pajamas | 0 | 2 |
| $K_1$ | kids pyjamas | 1 | 9 |
| $K_2$ | children sleepwear | 2 | 11 |
| $K_3$ | juniors sleepwear | 3 | 5 |

two bidders' expected payments increased significantly at the optimal reserve score comparing to the baseline reserve price of 1.0. The other three bidders actually had expected payments reduced, since some of them may give up bidding when the bidder-specific reserve price is greater than their value-per-click. For those price sensitive bidders, the percentage of payment reduced is very significant; however, the sum of payments (the search engine revenue) increased 19.03%. The contributions are totally from top two bidders, contrary to the intuition that increasing reserve score can force lower bidders to bid and pay more. Actually, those top bidders' bids are higher in the new equilibrium, and the increased RPS is from the indirect effect, rather than from the bidders in lower positions whose bids are close to their reserve prices (see Table 3).

### 4.3. Compare revenue, efficiency and utility

Fig. 3 shows the trend and relationship among the total utility, search engine revenue, and bidders' payoff. The top-most curve is total utility, also called the auction's efficiency (Lahaie and Pennock 2007), which is equal to all bidders' payoff plus their payments (the search engine's revenue). The bell-shaped curve is the revenue curve, which is the same as the top one (5 bidders) at Fig. 1, and the remaining one is the total payoff of all bidders. Both the bidders' payoff and total utility decrease with the increase of the reserve score.

Increasing reserve score is a non-zero-sum game among the search engine and bidders, which decreases auction's efficiency and forces bidders to transfer more payoff to payment simultaneously, while maximizing the search engine's revenue at one unique point. If we set the reserve price over the optimal one, then all three metrics (revenue, efficiency and utility) are worse, thus in practice it is safer to set the reserve price slightly smaller than the theoretically optimal one.

## 5. Implementation methodology

Realistic keyword auctions are complex, e.g., in terms of ads selection and matching algorithms. Search engines also consider advertisers budgets and other restrictions to determine rankings. Our theory on the optimal reserve price for wGSP auctions is a simplified model of the realistic auction process. Prior work such as Walsh et al. (2008) provides a scalable way to compute appropriate reserve prices in a complex and realistic setting. Similarly, only the accessible data in production systems can be treated as our input, including bids $b_i$ and quality factors $e_i$ and bidding descriptions (title, website link etc.), but bidders' values $v_i$ and scores $s_i$ are unknown.

We collected a large data set from Baidu, the dominant search engine in China. Our data set includes more than 40 million keywords with bids. Their quality factors $e_i$ were already computed and we assume they are unbiased. Those keywords were classified into dozens of disjoint categories according to their semantics and corresponding bidding advertisers' industry classifications. For each keyword, we collect all bidders' bids and quality factors.

This section describes a practical method for optimal reserve price computation. The process includes 3 steps: keyword semantic clustering, estimating the value distribution of bidders, and setting the reserve price.

### 5.1. Keywords semantic clustering

Since bidders' true private values are not known, we can only estimate the score distribution parameters from bids of each keyword. We observe that the average number of bids for each keyword (called market depth in Edelman and Schwarz (2010)) is very low, and many keywords have only one bid.

Statistical methods for distribution estimation do not work if only one sample is available, and it is not reliable with few samples. However, the simulation in Section 4 indicates that the revenue increase in those keywords with few bids is more remarkable. We face an additional challenge to estimate the value distribution when the bidding sample is sparse. Regelson and Fain (2006) proposed a method to estimate click-through-rates via ads clustering, and this idea motivates us to use a similar keywords semantic clustering as a pre-treatment before doing the bids statistics.

For example, bidders who bid on the keyword "kids sleepwear" may also be shown on the results page of query "children's pajamas" due to broad match techniques, based on users intention analysis and ads semantic matching. Vice versa, the value distribution of keyword "children's pajamas" should apply to semantically related keywords, such as bids on "kids sleepwear", because they compete with not only bidders bidding for the same keyword, but also more bidders bidding on a group of similar keywords. Similarly, for keywords in the same category, those bidders with similar descriptions and semantic attributes have similar competitors and reflect similar values, which is our fundamental idea.

Our algorithm computes the similarity degree of their descriptions and attributes between each candidate keyword pair in the keyword pool, following this general approach:

1. **Word-segments:** Separates the string into individual words based on word boundaries. For example, "children's pajamas" is separated into "children's" and "pajamas".
2. **Stemming:** Generates stems of the words. For example, "children's" generates "children" or "child", "pajamas" generates "pajama".
3. **Keyword Expansion:** Identifies a list of altered word based on matches in the thesaurus. For example, "children" expands to "kid" and "junior", "pajama" expands to "pyjama", "sleepwear", and "sleepcoat".

With thesaurus and inverted index technology, we can find the altered word list for each target keyword efficiently. The similarity degree (or confidence) of each altered word can be evaluated via the OKAPI BM25 ranking algorithm (Jones et al. 2000), with their ads description, title and bidding keywords. For those keywords with only a few bids, we select the top $n$ altered words in the keywords set from the same category, according to their similarity degree. Here we demonstrate an example of the altered keyword expansion of "children's pajamas."

To determine $n$, we make sure that the number of bidders bidding on these $n$ keywords (include the original keyword) exceeds a pre-determined threshold value.[3] In the example above, bids on the first 3 keywords just exceed 20, so we select $n = 3$ and use those bids on $K_0, K_1, K_2$ to estimate the value distribution of $K_0$. Thus, those bids are reliable enough to estimate the value distribution of the target keyword $K_0$ "children's pajamas."

### 5.2. Estimating the distribution parameters

We picked up millions of keywords from dozens of categories, and for each keyword, we collected the rank scores $e_i b_i$ on each keyword if the keyword has enough biddings. Otherwise, collect more rank scores from altered keywords via the above approach. We assume that bidders were playing a perfect SNE. Following Varian (2007)'s method, we estimated their scores $(e_i v_i)$ from rank scores $(e_i b_i)$ which we knew. For the position factors $x_t$, they can be calculated from the click and impression logs.

The next step is to fit a publicly known distribution from these $e_i v_i$. For different keywords, we assume that the distributions of bidders' $e_i v_i$ fall into the same kind of continuous distributions, but *each keyword has different parameters*. Since the bidders' $s_i$ are collected when a reserve score $\underline{s}$ is already set, those samples just reflects the distribution at $[\underline{s}, \infty)$. Thus, this sub-problem is a truncated distribution estimation problem.[4] Reserve prices should also be updated periodically, which is actually an iterative process in distribution estimation.

**Proposition 1.** *Considering a family of distributions with probability density function $p(x, \theta)$ and the cumulative density function is $F(x, \theta), \theta \in \Theta, \Theta$ is the parameter space. Random variable $X \sim p(x, \theta_0)$ and another random variable $X\prime$ follows the corresponding truncated distribution $p_s(x, \theta_0)$ with threshold s. There are N observations of $X\prime : x_1, x_2, \ldots, x_N$. When N is large enough, we can estimate the unbiased distribution parameters for any given truncated value s with Maximum-Likelihood Estimation (MLE) if $p(x, \theta)$ is second degree differentiable with respect to $\theta$.*

**Proof.** The proof for Propostion 1 is in Appendix E. □

For most keywords, the realistic distribution may be too complex to be computed. Thus, we simplify the problem and assume bidders' score $s_i$ on each keyword follow some publicly known distributions. We select several candidate distributions $\mathrm{dist}_1, \mathrm{dist}_2, \ldots, \mathrm{dist}_d$ whose definition domains are in $[0, \infty)$, then compute their distribution parameters $\mathrm{dist}_j(\mu, \sigma, \ldots)$ via the MLE method above (Sun et al. 2012). Then we compare the sample and each candidate distribution $\mathrm{dist}_j(\mu, \sigma, \ldots)$ via a distribution fitting test method in $[\underline{s}, \infty)$ (e.g. the Pearson $\chi^2$ test, Anderson–Darling test, Cramér-von Mises test), and finally selected the most likely distribution type and its parameters as the estimated distribution.

Table 4 demonstrates an example of keyword "Company Registered Agent" (in Chinese) with Cramér-von Mises test (Darling 1957). The 3rd column shows the statistic error, while the 4th column shows the $p$-value. A small $p$-value suggests that the data is unlikely to come from the assumed distribution.

*Empirically we found the lognormal distribution fits better than other well-known distributions for most keywords.* Fig. 4 draw the histogram of bidders' scores $s_i$ and the fitted lognormal distribution for an example keyword.

**Table 4**
An example of keyword "Company Registered Agent" with test.

| Distribution type | Parameters | Statistic err. | $p$-Value |
|---|---|---|---|
| Lognormal dist. | [1.053, 0.882] | 0.13945 | 0.4234 |
| Weibull dist. | [1.179, 4.462] | 0.39886 | 0.0728 |
| Gamma dist. | [1.460, 2.870] | 0.42102 | 0.0636 |
| Exponential dist. | [0.23864] | 0.56207 | 0.0278 |

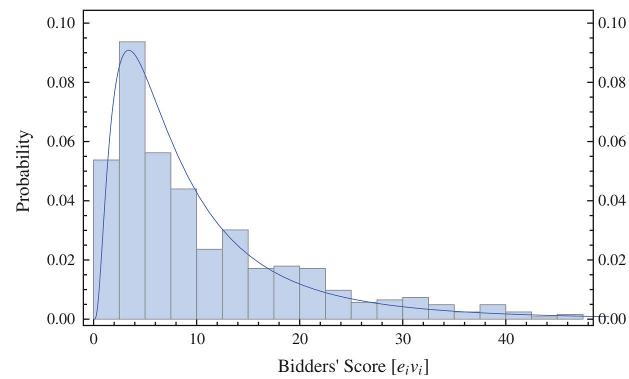


**Fig. 4.** Histogram of $s_i$ and fitted distribution.

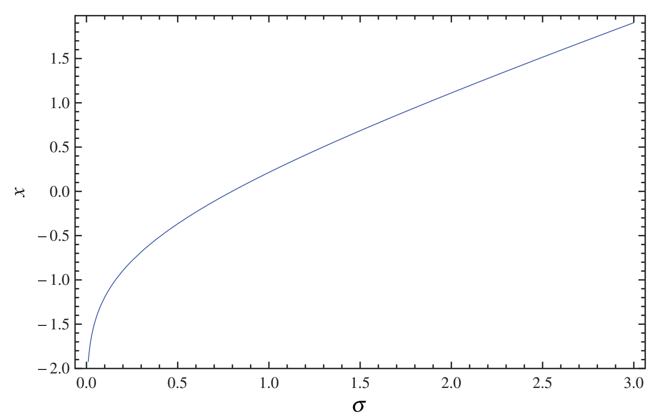


**Fig. 5.** A graph of $x^*$ as a function of $\sigma$.

### 5.3. Setting the reserve price

With those prior distribution parameters, the final step is to set the optimal reserve score for each keyword, and calculate the optimal reserve price for each bidder. By testing sample keywords in each industry, we fit bidders' scores $s_i$ on each keyword with a lognormal distribution, and each keyword has its unique distribution parameters.

Combining with the lognormal expression (Eq. (11)), the optimal reserve score $\underline{s}^*$ should satisfy $\underline{s}^* \cdot f(\underline{s}^*) = 1 - F(\underline{s}^*)$, which is simplified to

$$\underline{s}^* = e^{\mu + \sqrt{2}\sigma x^*}, \text{ where } \int_{x^*}^{\infty} e^{-t^2} dt = \frac{e^{-(x^*)^2}}{\sqrt{2}\sigma} \tag{12}$$

$\underline{s}^*$ can be solved by numerical methods when the distribution parameters $\mu, \sigma$ are known. An efficient approach is using a pre-computed mapping table to record the value of $x^*$ as a function of $\sigma$. The graph of the function shows as Fig. 5.

[3] 20 in our experiments, and Sun et al. (2012) studied the accuracy problem.
[4] In Yahoo!'s experiment, they estimated distributions with the method of moment. Appendix E shows MLE is an unbiased method to estimate distribution parameters from truncated samples.

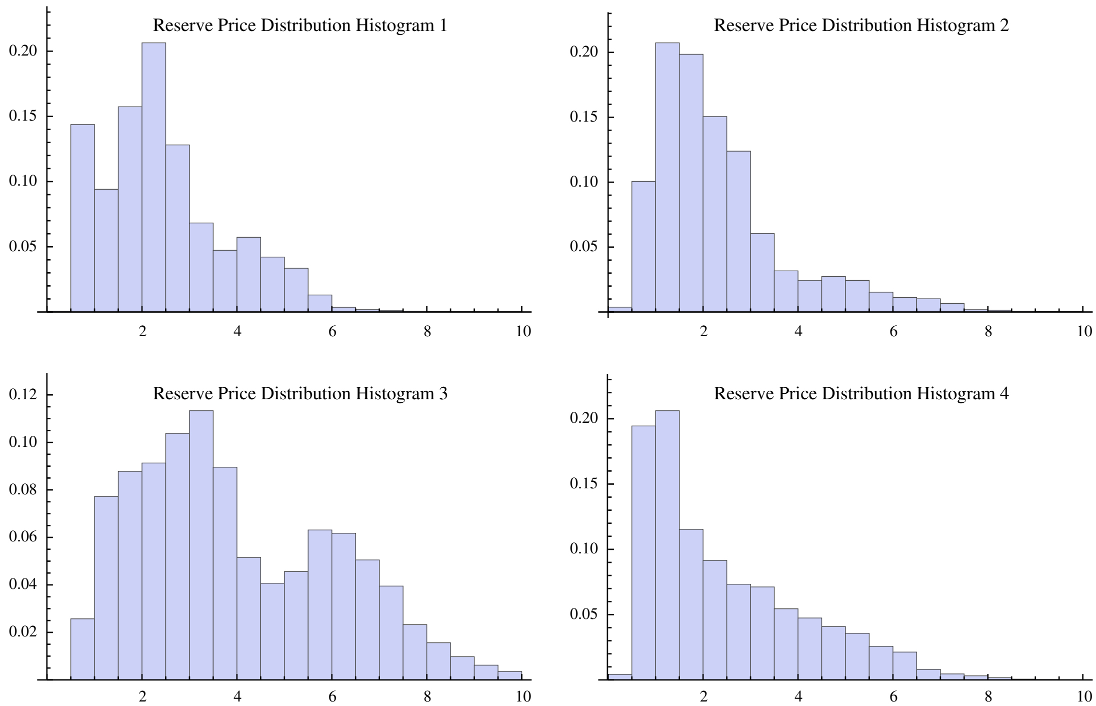


**Fig. 6.** The estimated optimal reserve prices histogram of 4 categories' data.

For each individual bidder $i$ with quality factor $e_i$, his optimal reserve price $p_i^*$ is set to $\underline{s}^*/e_i$.

In practice, search engines may consider other impacts of rising reserve prices, thus they will use a weighted (reduced) reserve price instead the optimal one. For example, Ostrovsky and Schwarz (2011) used the median price between the theoretical result and 10 cents for the Yahoo! experiment. Sun et al. (2012) also showed that the reduced reserve prices are more robust and converge faster.

### 5.4. Results histogram from actual bidding data

We followed the method above to compute the optimal reserve price with our data set from Baidu. After optimizing the numerical computation code and the database, the computation is highly efficient in our prototype system.[5] Since we have already classified those keywords into categories, we show the distribution of optimal reserve prices for a few categories in Fig. 6.

Each histogram shows the distribution of optimal reserve prices in each category, and each category contains millions of keywords.[6] Those histograms reveal different patterns from one category to another. While we cannot describe it in a publicly known distribution type, however the number of keywords with relatively low reserve prices is much more than those with relatively high reserve prices overall. It is a typical long tail pattern in Internet activities.

## 6. Conclusions

Keyword auctions are a key part of the business model for most search engines, generating billions of dollars of advertising revenue for these companies annually. Examining the optimal reserve price for these auctions is a very important problem. Our work generalizes the optimal reserve price for Myerson auctions to wGSP auctions in realistic contexts. We prove that the optimal reserve price in wGSP auctions depends only on the joint distribution of bidders' valuations $v_i$ and quality factors $e_i$, independent of the number of bidders. We further extend this theory to the CPA/CPC/CPM hybrid auction, and show that the optimal reserve price can be calculated similarly.

Simulations shed additional light on our theory and reveal a significant revenue increase by adjusting reserve prices to the optimal one. Those simulations indicate that increasing the reserve price leads to an indirect positive change to the top bidders' expected payments, while the lower position bidders' contributions are negative. Increasing reserve price decreases auction efficiency, and it is not a zero-sum game, however, it transfers bidders' utilities into payments and finally maximizes the search engine's revenue.

Besides theory and simulation, we also propose a practical implementation method to compute the optimal reserve price in realistic contexts for search engines, and resolve some key issues via statistical techniques, such as keyword semantic clustering, value distribution estimation from truncated samples, etc.

## Appendix A. Independence between $_i$ and $e_i$ – validation from actual data analysis

While we do not know bidder's private values $v_i$, we can analyze the correlation between bids $b_i$ and quality factors $e_i$ for each keyword. Assume all bidders bid $b_i$ symmetrically based on their values and quality factors, if their values $v_i$ are correlated with their quality factors $e_i$, $b_i$ must be correlated with $e_i$.

We examine correlation coefficients of each keyword's bids $b_i$ and quality factors $e_i$ using bidding data of 500,000 keywords which is a subset of the dataset used in Section 5. Those bids constitute nearly perfect competition, since we only select keywords with at least 10 bidders. Finally, we draw those correlation coefficients in a histogram as in Fig. A7.

[5] It takes nearly 30 min to compute a category which contains about 10 million keyword in a small cluster, equivalent to 5000 keywords/s. Keywords semantic clustering takes most computing resource.

[6] We made a linear transformation, transforming all keywords' reserve prices into an interval of [0, 10], thus the histogram still shows its original pattern while the absolute values are hidden.

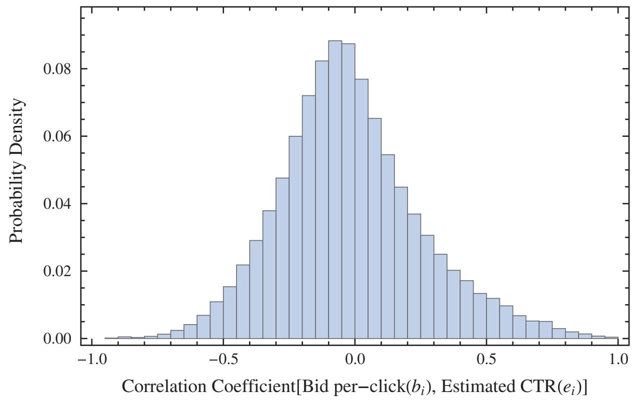


**Fig. A7.** Correlation distribution between bids $b_i$ and quality factors $e_i$.

Those correlation coefficients are symmetrically distributed around 0, with their average −0.019. In summary, the expected correlation coefficient of $b_i$ and $e_i$ is almost 0 for any given keyword. This suggests that for any keyword, bidders' values $v_i$ and quality factors $e_i$ are linear independent.

## Appendix B. The Proof of Lemma 1

According to the ranking and pricing schemes in wGSP auctions, given bidder $i$ with value-per-click $v_i$, quality factor $e_i$, and bid $b_i$, his expected utility is:

$$\begin{aligned} U_i(v_i, b_i, e_i) = & \sum_{j=1}^{k} [(v_i - \langle \text{ price at } j\text{th} \rangle) \cdot e_i \cdot x_j] \\ & \cdot Pr\langle b_i e_i \text{ ranks } j\text{th} \rangle \end{aligned} \tag{B.1}$$

Let $v_i = v$ and $e_i = e$, and consider another bidder $h$ with value-per-click $v_h = e\dot{v}$ and quality factor $e_h = 1$, here $e \in (0, 1]$. Suppose bidder $i$ bids $b$ and bidder $h$ bids $eb$, thus both advertisers get the same rank score $eb$. Their expected utilities are:

$$\begin{aligned} U_i(v, b, e) &= \sum_{j=1}^{k} \left[ \left( v - \frac{b_{j+1} e_{j+1}}{e} \right) \cdot e \cdot x_j \right] \cdot Pr\langle eb \text{ ranks } j\text{th} \rangle \\ &= \sum_{j=1}^{k} [(ev - b_{j+1} e_{j+1}) \cdot x_j] \cdot Pr\langle eb \text{ ranks } j\text{th} \rangle \end{aligned} \tag{B.2}$$

$$\begin{aligned} U_h(ev, eb, 1) &= \sum_{j=1}^{k} \left[ \left( ev - \frac{b_{j+1} e_{j+1}}{1} \right) \cdot 1 \cdot x_j \right] \cdot Pr\langle eb \text{ ranks } j\text{th} \rangle \\ &= \sum_{j=1}^{k} [(ev - b_{j+1} e_{j+1}) \cdot x_j] \cdot Pr\langle eb \text{ ranks } j\text{th} \rangle \end{aligned} \tag{B.3}$$

Here $b_{j+i}$ and $e_{j+1}$ are the next bidder's bid and quality factor when the bidder wins the $j$-th slot. It is easy to get that:

$$U_i(v, b, e) = U_h(ev, eb, 1) \tag{B.4}$$

We define $b_i = \delta(v_i, e_i)$ to be the equilibrium bidding function at any $v_i$ and $e_i$. Let $b_h = \delta(ev, 1)$ maximize $U_h$ and $b_i = \delta(v, e)$ maximize $U_i$ respectively. Eq. (B.4) suggests that if bidding $b_i = b$ is the best for bidder $i$ with value-per-click $v$, bidding $b_h = eb$ must be the best for bidder $h$ with value-per-click $ev$. Thus, we derive:

$$e \cdot b_i = b_h \Rightarrow e \cdot \delta(v, e) = \delta(ev, 1) \tag{B.5}$$

Finally, bidder $i$'s score $s_i = e_i v_i = ev$. Let $\beta(s_i) = \delta(ev, 1)$, we rewrite bidder $i$'s bidding strategy as:

$$b_i = \delta(v_i, e_i) = \frac{\beta(s_i)}{e_i} \tag{B.6}$$

## Appendix C. The Proof of Theorem 1

We assume individually rational (utility-maximizing) bidders with the same bidding strategy $b_i = \frac{\beta(s_i)}{e_i}$ in equilibria. Bidder $i$'s expected CTR is

$$\text{eCTR}_i = e_i \cdot x^i = e_i \cdot \tilde{q}(e_i \cdot b_i)$$

Thus, the bidder's utility formula:

$$U_i(b_i) = v_i \cdot eCTR_i - \text{Payment}_i = v_i \cdot e_i \cdot \tilde{q}(e_i \cdot b_i) - \tilde{m}(e_i \cdot b_i)$$

The derivative of $U_i$ with variable $b_i$ is:

$$\frac{\partial U_i(b_i)}{\partial b_i} = v_i \cdot e_i^2 \cdot \tilde{q}'(e_i \cdot b_i) - e_i \cdot \tilde{m}\prime(e_i \cdot b_i) \tag{C.1}$$

Since the bidding $b_i = \frac{\beta(s_i)}{e_i}$ maximizes his utility, thus $b_i$ is an equilibrium bidding, i.e.,

$$\frac{\partial U_i(b_i)}{\partial b_i}\Big|_{b_i = \frac{\beta(s_i)}{e_i}} = v_i \cdot e_i^2 \cdot \tilde{q}'(\beta(s_i)) - e_i \cdot \tilde{m}'(\beta(s_i)) = 0 \tag{C.2}$$

Let $\tilde{q}(\beta()) = q(), \tilde{m}(\beta()) = m()$, then

$$s_i \cdot q'(s_i) - m'(s_i) = 0$$

If the minimum score of all ads is $\underline{s}$, then its utility is zero, and it satisfies the restraint boundary condition:

$$U(\underline{s}) = \underline{s} \cdot q(\underline{s}) - m(\underline{s}) = 0 \tag{C.3}$$

Solving Eqs. (C.2) and (C.3) yields

$$m(s) = s \cdot q(s) - \int_{\underline{s}}^{s} q(r)dr, \tag{C.4}$$

where $s$ is a random variable drawn from a known distribution $F(s)$, with the probability density function $f(s) = F'(s)$ in $(0, \infty)$ (Bulow and Roberts 1989). The payment of a bidder is determined by his score $s$ only: when $s < \underline{s}$, his utility is 0; and when $s \geqslant \underline{s}$, his utility is computed by Eq. (C.4). The expected payment of bidder $i$ is:

$$\begin{aligned} E[m(s)] &= \int_{0}^{\underline{s}} 0 \cdot f(s)ds + \int_{\underline{s}}^{\infty} m(s) \cdot f(s)ds \\ &= \int_{\underline{s}}^{\infty} s \cdot q(s)f(s)ds - \int_{\underline{s}}^{\infty} \left[ \int_{\underline{s}}^{s} q(r)dr \right] f(s)ds \\ &= \int_{\underline{s}}^{\infty} s \cdot q(s) \cdot f(s)ds - \left[ \int_{\underline{s}}^{s} q(r)dr \cdot F(s) \right]_{\underline{s}}^{\infty} + \int_{\underline{s}}^{\infty} F(s) \cdot q(s)ds \\ &= \int_{\underline{s}}^{\infty} [s \cdot f(s) - 1 + F(s)] \cdot q(s)ds \\ &= \int_{\underline{s}}^{\infty} \left[ s - \frac{1 - F(s)}{f(s)} \right] \cdot q(s) \cdot f(s)ds \end{aligned} \tag{C.5}$$

There are $n$ bidders, each bidder's expected payment is given by the above expression, and the sum of their expected payments is the expected revenue of the search engine:

$$n \cdot E[m(s)] = n \int_{\underline{s}}^{\infty} \left[ s - \frac{1 - F(s)}{f(s)} \right] \cdot q(s) \cdot f(s)ds \tag{C.6}$$

With probability $F^n(\underline{s})$, nobody wins any ad slot and the search engine gains a value $t_0$ of better user experience (no ads, Jun Li et al. study this case in Li et al. (2012)). Therefore, the expected revenue per search of the search engine is:

$$T(\underline{s}) = F^n(\underline{s}) \cdot t_0 + n \int_{\underline{s}}^{\infty} \left[ s - \frac{1 - F(s)}{f(s)} \right] \cdot q(s) \cdot f(s)ds$$

If the minimum score $\underline{s}$ maximizes the expected revenue $T(\underline{s})$, it satisfies:

$$\frac{\partial T(\underline{s})}{\partial \underline{s}} = n \cdot F^{n-1}(\underline{s}) f(\underline{s}) t_0 - n\left[\underline{s} - \frac{1 - F(\underline{s})}{f(\underline{s})}\right] q(\underline{s}) \cdot f(\underline{s}) = 0$$

Then the optimal value of the minimum score $\underline{s}^*$ satisfies

$$\underline{s}^* - \frac{1 - F(\underline{s}^*)}{f(\underline{s}^*)} = \frac{F^{n-1}(\underline{s}^*)}{q(\underline{s}^*)} \cdot t_0 \tag{C.7}$$

where $q(s)$ is the expected position factor when a bidder's score is $s$. When $t_0 = 0$, Eq. (C.7) is simplified into:

$$\underline{s}^* - \frac{1 - F(\underline{s}^*)}{f(\underline{s}^*)} = 0 \tag{C.8}$$

When $\underline{s}^*$ is solved via Eq. (C.8), for bidder $i$ with quality factor $e_i$, his optimal reserve price $p_i^*$ for this keyword is

$$p_i^* = \frac{\underline{s}^*}{e_i} \tag{C.9}$$

## Appendix D. The Proof of Proposition 1

Assume there is a family of probability distributions with probability density function $p(x, \theta)$ and the corresponding cumulative density function $F(x, \theta), \theta \in \Theta, \Theta$ is the parameter space. Random variable $X \sim p(x, \theta)$ and for another random variable $X'$ which follows the truncated distribution with threshold $s$, its probability density function $p_s(x, \theta)$ is:

$$p_s(x, \theta) = \frac{p(x, \theta)}{1 - F(s, \theta)} \qquad (x \geqslant s) \tag{D.1}$$

Given $N$ observations $x_1, x_2, \ldots, x_N$ of $X'$, the likelihood function for distribution estimation $L(\theta)$ is defined as:

$$L(\theta) = \prod_{i=1}^{N} p_s(x, \theta) = \frac{\prod_{i=1}^{N} p(x_i, \theta)}{(1 - F(s, \theta))^N} \tag{D.2}$$

Correspondingly, the log-likelihood is:

$$logL(\theta) = \sum_{i=1}^{N} \log p(x_i, \theta) - N \log(1 - F(s, \theta)) \tag{D.3}$$

We say $\hat{\theta} = \hat{\theta}(x_1, x_2, \ldots, x_N)$ is the maximum likelihood estimate if $\hat{\theta}$ satisfies that:

$$\log L(\hat{\theta}) = \max_{\theta \in \Theta} \log L(\theta) \tag{D.4}$$

To find the parameter values $\hat{\theta}$ maximizing $logL(\theta), \hat{\theta}$ should satisfy $\frac{d \log L(\theta)}{d\theta} = 0$. Given the strong connectivity of the probability density function, it is equivalent to solve the following equation

$$\frac{1}{N} \frac{d \log L(\theta)}{d\theta} = \frac{1}{N} \sum_{i=1}^{N} \frac{\frac{\partial p}{\partial \theta}(x_i, \theta)}{p(x_i, \theta)} + \frac{\int_{-\infty}^{s} \frac{\partial p}{\partial \theta}(x, \theta) dx}{1 - F(s, \theta)} = 0 \tag{D.5}$$

Given that $N$ is large enough, we have:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\frac{\partial p}{\partial \theta}(x_i, \theta)}{p(x_i, \theta)} = \int_{s}^{+\infty} \frac{\frac{\partial p}{\partial \theta}(x, \theta)}{p(x, \theta)} \frac{p(x, \theta_0)}{1 - F(s, \theta_0)} dx \tag{D.6}$$

Therefore,

$$\lim_{N \to \infty} \frac{1}{N} \frac{d \log L(\theta)}{d\theta} = \int_{s}^{\infty} \frac{\frac{\partial p}{\partial \theta}(x, \theta)}{p(x, \theta)} \frac{p(x, \theta_0)}{1 - F(s, \theta_0)} dx + \frac{\int_{-\infty}^{s} \frac{\partial p}{\partial \theta}(x, \theta) dx}{1 - F(s, \theta)} \tag{D.7}$$

If $\theta_0$ is the maximum likelihood estimate, the corresponding derivative will approach zero as $N$ approximates infinity. By replacing $\theta$ with $\theta_0$, we have:

$$\lim_{N \to \infty} \frac{1}{N} \frac{d \log L(\theta)}{d\theta} = \frac{\int_{-\infty}^{+\infty} \frac{\partial p}{\partial \theta}(x, \theta)|_{\theta=\theta_0} dx}{1 - F(s, \theta_0)} \tag{D.8}$$

Thus, once if $\int_{-\infty}^{+\infty} \frac{\partial p}{\partial \theta}(x, \theta)|_{\theta=\theta_0} dx = 0$ is always true, we can conclude that the estimated parameter value $\hat{\theta}$ always equals to $\theta_0$. Given the assumption that $p(x, \theta)$ is second degree differentiable with respect to $\theta$, we can interchange integration and differentiation for the right side term of the above equation, leading to:

$$\int_{-\infty}^{+\infty} \frac{\partial p}{\partial \theta}(x, \theta)|_{\theta=\theta_0} dx = \left[\frac{d}{d\theta} \int_{-\infty}^{+\infty} p(x, \theta) dx\right]|_{\theta=\theta_0} \tag{D.9}$$

Since $\int_{-\infty}^{+\infty} p(x, \theta) dx = 1$, for $\forall \theta$, we have the equation

$$\frac{d}{d\theta} \int_{-\infty}^{+\infty} p(x, \theta) dx = 0 \tag{D.10}$$

Consequently, $\lim_{N \to \infty} \frac{1}{N} \frac{d \log L(\theta)}{d\theta}|_{\theta=\theta_0} = 0, \theta_0$ is a potential candidate of maximum likelihood estimate.

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