RATIONALITY OF FORMS OF $\overline{\mathcal{M}}_{0,n}$

BRENDAN HASSETT, YURI TSCHINKEL, AND ZHIJIA ZHANG

Abstract. We study equivariant geometry and rationality of moduli spaces of points on the projective line, for twists associated with permutations of the points.

1. Introduction

In this note, we strengthen a theorem of Florence–Reichstein [FR18] concerning rationality of moduli spaces. They consider *forms* of $\overline{\mathcal{M}}_{0,n}$, i.e., varieties over nonclosed fields F which are isomorphic to the moduli space of n points on $\mathbb{P}^1$ over an algebraic closure of F. These forms are obtained by twisting via Galois actions permuting the points over F. The main results of [FR18] are:

- if $n \geq 5$ is odd, and F is infinite of characteristic $\neq 2$, then every form over F is rational;
- if $n \geq 6$ is even, and F has nontrivial 2-torsion in its Brauer group and contains fourth roots of unity, then there exists a form X of $\overline{\mathcal{M}}_{0,n}$ over F such that X is not retract rational over F.

These were inspired by a classical theorem of Enriques, Manin, and Swinnerton-Dyer concerning rationality of twists of $\overline{\mathcal{M}}_{0,5}$, a del Pezzo surface of degree 5, over any field F. The proof for $n \geq 5$ uses (a twisted form of) the Gelfand-MacPherson correspondence, and techniques developed in connection with Noether's problem for twisted forms of the groups in question.

By [FR18], every form over an infinite field F is unirational over F. It is known that every form of $\overline{\mathcal{M}}_{0,6}$ over $\mathbb{R}$ is rational [Avi20, Proposition 2.9]; see Corollary 20 for generalizations.

Here, we strengthen their conclusions in two directions: we prove rationality in several situations not addressed in [FR18]. On the other hand, we show failure of rationality via Galois cohomology in instances not covered by [FR18], e.g., where the Brauer group of F is trivial.

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An important ingredient throughout is a theorem of [BM13]:

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}) = \mathfrak{S}_n, \quad n \geq 5,$$

acting via permutations of the n points on $\mathbb{P}^1$. In particular, Galois twists of $\overline{\mathcal{M}}_{0,n}$ factor through subgroups of $\mathfrak{S}_n$, and there is a close link between rationality of twists and linearizability of G-actions on $\overline{\mathcal{M}}_{0,n}$; see [DR15] for a general discussion of such connections. In both situations, there is an action of a finite group on the geometric Picard group

$$\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}),$$

via a subgroup of $\mathfrak{S}_n$.

We present several stable rationality and linearizability results, including Propositions 2 and 4 (based on the Kapranov construction) and Theorem 23 (using torsors and quotients). Section 3 focuses on geometric constructions. One rationality construction uses Schubert calculus and the geometry of Grassmannians; Theorem 13 extends results of [FR18] to small fields (Corollary 15) and some point configurations in higher-dimensional projective spaces (Corollary 16). Another relies on fibration structures; see Theorem 19. We close with a comprehensive discussion of the $n = 6$ case (Theorem 33).

For nonrationality/nonlinearizability, we focus on situations where the twisted moduli spaces are toric via the Losev-Manin construction [LM00]. We utilize cohomological **(H1)** and **(SP)**-obstructions (see Section 5): In the arithmetic context, the group is replaced by the absolute Galois group of the ground field F and the Picard module by the geometric Picard module. We focus on *even* n:

Theorem 1 (Corollary 28 and Theorem 29)**.** *For every even $n \geq 6$ there exists a subgroup $G = C_2^2 \subset \mathfrak{S}_n$ such that*

$$\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})) = \mathbb{Z}/2.$$

In particular,

- *for all subgroups of $\mathfrak{S}_n$ containing G, the corresponding action is not stably linearizable,*
- *for all fields F admitting a Galois extension L/F with Galois group $\mathrm{Gal}(L/F) \simeq G$ there exists a form X of $\overline{\mathcal{M}}_{0,n}$ over F such that X is not retract rational over F.*

Indeed, nonvanishing group cohomology is an obstruction to (stable) linearizability, see, e.g., [BP13, Corollary 2.5.2.]. In the context of nonclosed fields, one can find a twist X of $\overline{\mathcal{M}}_{0,n}$ over F so that the corresponding Galois action on the geometric Picard group of X factors through the prescribed action of G. This yields nontrivial Galois cohomology, which in turn obstructs retract rationality of X over F. In particular, our result applies to fields F with *trivial* Brauer group, e.g., $F = \mathbb{C}(t)$.

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2. $\mathfrak{S}_n$-EQUIVARIANT GEOMETRY

We recall some terminology: Let G be a finite group acting regularly on a projective variety X. Assume the action is generically free. The action is *linearizable* if X is equivariantly birational to the projectivization $\mathbb{P}(V)$ of a linear representation V of G on a vector space. It is *stably linearizable* if $X \times \mathbb{P}^r$ – where G acts trivially on the second factor – is linearizable.

Kapranov blowup. We make use of the Kapranov blowup realization

$$\beta_n : \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}, \quad n \geq 4,$$

where β_n is an iterated blowup of $n-1$ general points on $\mathbb{P}^{n-3}$, lines through pairs of points, etc., see, e.g., [HT02, Section 3.1]. Precisely, we regard

$$\mathbb{P}^{n-3} = \mathbb{P}(k[\mathfrak{S}_{n-1}]/(1, \dots, 1)),$$

so that the $\mathfrak{S}_{n-1}$-action is linear. Boundary divisors D_I are labeled by partitions

$$[1, \dots, n] = I \sqcup I^c, \quad |I|, |I^c| \geq 2.$$

Recall that the Picard group $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ has rank $2^{n-1} - \binom{n}{2} - 1$, and an explicit basis is given by

$$\{H, E_{i_1}, E_{i_1,i_2}, \dots, E_{i_1,\dots,i_{n-4}}\},$$

where H is the (pullback of the) hyperplane class on $\mathbb{P}^{n-3}$, and the other elements are (classes of) exceptional divisors from blowups of points, lines, etc. The boundary divisors D_I expressed in this basis are

$$D_{i_1,\dots,i_k,n} = E_{i_1,\dots,i_k}, \quad \{i_1, \dots, i_k\} \subset \{1, \dots, n-1\}, \quad k \leq n-4,$$

and

$$[D_{i_1,\dots,i_{n-3},n}] = L - E_{i_1} - E_{i_2} - \dots - E_{i_1,\dots,i_{n-4}} - E_{i_2,\dots,i_{n-3}}.$$

The $\mathfrak{S}_n$-action on $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ is best understood in terms of the natural $\mathfrak{S}_n$-action on the boundary divisors via permutations of indices of D_I. In particular, there is a distinguished $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ acting via permutation of indices on E_i, for $i \in \{1, \dots, n-1\}$.

The Kapranov construction has applications to linearizability:

Proposition 2. *Suppose that $G \subseteq \mathfrak{S}_{n-1}$ acts on $\overline{\mathcal{M}}_{0,n}$ leaving the nth point invariant. Then the action of G is linearizable.*

For $n = 2m+1$ and $G \subseteq \mathfrak{S}_{2m+1}$, the G-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.

More generally, for $G \subseteq \mathfrak{S}_n$ leaving an odd cycle invariant, the G-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.

Proof. The first assertion reflects the fact that the Kapranov morphism β_n is $\mathfrak{S}_{n-1}$ invariant and the $\mathfrak{S}_{n-1}$-action on $\mathbb{P}^{n-3}$ is linear. The second assertion is a special case of the third. For the third statement, consider the universal curve

$$\overline{\mathcal{C}}_{0,n} \to \overline{\mathcal{M}}_{0,n}.$$

Lemma 3. *Let $G \subset \mathfrak{S}_n$ act on $\overline{\mathcal{M}}_{0,n}$ by permutation of the marked points. Then there is a canonical lift of the action to the universal curve*

$$\phi : \overline{\mathcal{C}}_{0,n} \to \overline{\mathcal{M}}_{0,n}.$$

We prove the lemma. Interpreting $\overline{\mathcal{C}}_{0,n} = \overline{\mathcal{M}}_{0,n+1}$, we have

$$\mathrm{Aut}(\overline{\mathcal{C}}_{0,n}) = \mathfrak{S}_{n+1} \supset \mathfrak{S}_n \hookrightarrow \mathrm{Aut}(\overline{\mathcal{M}}_{0,n}),$$

with the last inclusion an equality when $n \geq 5$. The induced action on $\mathrm{Aut}(\overline{\mathcal{C}}_{0,n})$ is equivariant under forgetting the $(n+1)$st point.

Returning to the Proposition, we assume that G leaves an odd cycle invariant. Then the forgetting morphism ϕ – an étale $\mathbb{P}^1$-bundle over $\mathcal{M}_{0,n}$ – admits a multisection of odd degree. It must therefore be the projectivization of a rank-two G-equivariant vector bundle over $\mathcal{M}_{0,n}$. However, we have already seen that the G-action on $\overline{\mathcal{C}}_{0,n} = \overline{\mathcal{M}}_{0,n+1}$ is linearizable. We conclude then that $\overline{\mathcal{M}}_{0,n}$ is stably linearizable. $\square$

A similar argument yields dividends for the Galois-theoretic question:

Proposition 4. *Let L/F be a Galois extension with Galois group Γ. Fix a representation*

$$\rho : \Gamma \to \mathfrak{S}_n$$

and let ${}^\rho\overline{\mathcal{M}}_{0,n}$ denote the corresponding twist of $\overline{\mathcal{M}}_{0,n}$ defined over F.

- *If ρ factors through an $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ then ${}^\rho\overline{\mathcal{M}}_{0,n}$ is rational over F.*
- *If n is odd then $\mathbb{P}^1 \times {}^\rho\overline{\mathcal{M}}_{0,n}$ is rational. The same holds if ρ leaves an odd cycle invariant.*

This gives a weaker version of [FR18, Theorem 1.2]; however, our statement is valid over a finite field as well. See Remark 21 below for a related result.

Proof. The Kapranov morphism $\beta : \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}$ is equivariant for $\mathfrak{S}_{n-1}$, which acts linearly on the target. Thus it descends to

$${}^\rho\overline{\mathcal{M}}_{0,n} \stackrel{\sim}{\to} \mathbb{P}^{n-3}$$

over F, proving rationality. For the second assertion, the Kapranov construction yields

$${}^\rho\overline{\mathcal{C}}_{0,2m+1} \stackrel{\sim}{\to} \mathbb{P}^{2m-1};$$

moreover

$${}^\rho\overline{\mathcal{C}}_{0,2m+1} \to {}^\rho\overline{\mathcal{M}}_{0,2m+1}$$

is a $\mathbb{P}^1$-bundle over a Zariski open subspace of the base. (The generic fiber is a smooth genus zero curve with a cycle of odd degree.) In particular, $\mathbb{P}^1 \times {}^\rho\overline{\mathcal{M}}_{0,2m+1}$ is rational over F. □

Example 5. Let $\mathfrak{S}_n$ act on $\overline{\mathcal{M}}_{0,n}$, for $n \geq 5$. This action is not linearizable since $\mathfrak{S}_n$ does not act linearly and generically freely on $\mathbb{P}^{n-3}$. Indeed, the smallest faithful representation of $\mathfrak{S}_n$ has dimension $n-1$. When $n = p$ is a prime, then even the action of the Frobenius subgroup $\mathfrak{F}_p = \mathrm{Aff}_1(\mathbb{F}_p) \subset \mathfrak{S}_p$ is not linearizable, for the same reason.

The Losev-Manin construction. This construction [LM00], [Has03, Section 6.4] is a distinguished factorization

$$\beta_n : \overline{\mathcal{M}}_{0,n} \to \overline{L}_n \to \mathbb{P}^{n-3},$$

where we blow up linear subspaces spanned by just $(n-2)$ points in linear general position. (Note that our indexing of $\overline{L}_n$ differs from [LM00].) The first arrow contracts the boundary divisors

$$D_{i_1,\dots,i_k,(n-1),n}, \{i_1,\dots,i_k\} \subset \{1,\dots,n-2\}, \quad k \leq n-5,$$

by allowing points indexed by

$$\{1, \dots, n-2\} \setminus \{i_1, \dots, i_k\}$$

to coincide.

We record some properties:

- $\overline{L}_n$ is toric [LM00, Section 2.6];
- the Losev-Manin construction is equivariant under $\mathfrak{S}_{n-2} \times \mathfrak{S}_2 \subset \mathfrak{S}_n$, realized as permutations of $\{1, \dots, n-2\}$ and $\{n-1, n\}$ [LM00, Theorem 2.5(b)].

The constructions of Losev-Manin give an explicit realization of the torus T and its character module $\mathfrak{X}^*(\mathsf{T})$. Let P denote the permutation module for $\mathfrak{S}_{n-2}$ associated with the first $n-2$ letters and L the non-trivial rank-one module for $\mathfrak{S}_2$ corresponding to $n-1$ and n. We regard these as modules for $\mathfrak{S}_{n-2} \times \mathfrak{S}_2$. Consider the short exact sequence

$$0 \to P_0 \to P \to \mathbb{Z} \to 0$$

associated with summing over the $n-2$ letters. Then we have

$$\mathfrak{X}^*(\mathsf{T}) = L \otimes P_0. \tag{2.1}$$

Indeed, we may describe the open torus orbit in $\overline{L}_n$ in geometric terms: We identify the points $n-1$ and n as 0 and ∞ and the first $n-2$ points as elements of

$$\mathrm{Hom}(P, \mathbb{P}^1 \setminus \{0, \infty\}) = \mathrm{Hom}(P, \mathsf{T}_L),$$

where T_L is the rank-one torus associated with L. To get moduli, we quotient out by the diagonal action of T_L.

We record one last observation: Consider the Kapranov blowups associated with points $n-1$ and n:

$$\beta_n[n-1], \beta_n[n] : \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}.$$

These two maps are related by an elementary Cremona transformation

$$\mathrm{Cr} : \mathbb{P}^{n-3} \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-3}$$

associated with the points indexed by $\{1, \dots, n-2\}$. This is equivariant for the T-actions and we obtain a birational contraction

$$\overline{L}_n \to \mathrm{Graph}(\mathrm{Cr}).$$

We summarize this as follows:

Proposition 6. *Consider a twist of $\overline{\mathcal{M}}_{0,n}$ associated with a subgroup of $\mathfrak{S}_n$ leaving a pair of points invariant. This variety is necessarily toric, realized as a twist of the Losev-Manin space.*

This applies in both equivariant and Galois-theoretic situations.

The Gelfand-MacPherson correspondence. Our main source is Kapranov [Kap93].

Let $\mathrm{Mat}(2,n)$ denote the $2\times n$ matrices. The group GL_2 acts via multiplication from the left

$$\mathrm{A}\cdot\mathrm{M}\mapsto\mathrm{AM}$$

and the torus $\mathsf{T}=\mathbb{G}_m^n$ acts via multiplication from the right

$$\mathrm{M}\cdot\mathsf{T}\mapsto\mathrm{M}\mathsf{T},\quad \mathsf{T}=\mathrm{diag}(t_1,\dots,t_n).$$

Considering the action by the product $\mathrm{GL}_2\times\mathbb{G}_m^n$, with the elements

$$\left(t^{-1}\,\mathrm{I}_2,\mathrm{diag}(t,t,\dots,t)\right)$$

in the kernel, we obtain a faithful action of the quotient group

$$(\mathrm{GL}_2\times\mathbb{G}_m^n)/\mathbb{G}_m.$$

We have an exact sequence

$$1\to\mu_2\to\mathrm{SL}_2\times\mathbb{G}_m^n\to(\mathrm{GL}_2\times\mathbb{G}_m^n)/\mathbb{G}_m\to 1,$$

where

$$\mu_2=(-\mathrm{I}_2,\mathrm{diag}(-1,-1,\dots,-1)).$$

The invariant theory quotient is

$$\mathrm{SL}_2\backslash\mathrm{Mat}(2,n)=\mathrm{CGr}(2,n),$$

the cone over the Grassmannian $\mathrm{Gr}(2,n)$ in its Plücker imbedding. The residual action of $\mathbb{G}_m^n$ on this cone has generic stabilizer μ_2; the action on the Grassmannian has generic stabilizer $\mathbb{G}_m=\mathrm{diag}(t,t,\dots,t)$. On the other hand, the geometric invariant theory quotient

$$\mathrm{Mat}(2,n)/\!/\mathbb{G}_m,\quad \mathbb{G}_m=\mathrm{diag}(t,t,\dots,t)$$

yields $(\mathbb{P}^1)^n$ with factors induced by the columns of the matrix. The residual SL_2 acts on this product with the distinguished linearization introduced above, which is $\mathfrak{S}_n$-symmetric. Again, this action fails to be faithful, as $\mu_2\subset\mathrm{SL}_2$ acts trivially.

The Gelfand-MacPherson construction yields isomorphisms

$$(\mathrm{CGr}(2,n)\setminus\{0\})\,/\mathbb{G}_m^n\xrightarrow{\sim}\mathrm{SL}_2\backslash\!\backslash(\mathbb{P}^1)^n, \tag{2.2}$$

where both sides are interpreted as GIT quotients [Kap93, 2.4.7]. Note that we have numerous choices for how to linearize the actions on the left- and right-hand sides, reflecting linearizations of the torus action and ample line bundles on the product; Kapranov's result makes clear how to identify these choices. Let X_n denote the quotient arising from the $\mathfrak{S}_n$-symmetric linearization.

Recall that the stable and strictly semistable loci on $(\mathbb{P}^1)^n$ are easily identified

$$(p_1, \dots, p_n) \text{ stable if there is no point with multiplicity } \geq \frac{n}{2}. \tag{2.3}$$

It is semistable if all points have multiplicity $\leq \frac{n}{2}$. For odd n, stable and semistable coincide; for even $n = 2m$, collections of points where m indices coincide are strictly stable, with closed orbits consisting of collections where

$$p_{i_1} = \cdots = p_{i_m}, \quad p_{i_{m+1}} = \cdots = p_{i_{2m}}, \quad \{i_1, \dots, i_{2m}\} = \{1, \dots, 2m\}.$$

In particular, $X_{2m}, m \geq 3$ has $\frac{1}{2}\binom{2m}{m}$ distinguished singular points over which the orbits are identified.

The stable loci on the Grassmannian $\mathrm{Gr}(2, n)$ for the action of $\mathbb{G}_m^n \cap \mathrm{SL}_n$ may be described as well: Choose a basis diagonalizing the torus action and let $(A_{ij}), 1 \leq i < j \leq n$ denote the associated Plücker coordinates. The point (A_{ij}) is stable if there are

1. no index i with $A_{ij} = 0$ for every j; and
2. no subset $I \subset \{1, \dots, n\}$ with $|I| \geq \frac{n}{2}$ and $A_{ij} = 0$ for all $i, j \in I$.

These descriptions yield an $\mathfrak{S}_n$-equivariant stratified blowup [Kap93, 0.4.3,4.1.8]

$$\beta : \overline{\mathcal{M}}_{0,n} \to X_n.$$

This blows down all the boundary divisors D_I except those where $|I|$ or $|I^c| = 2$. The divisors D_I with $2|I| = n$ are collapsed to the distinguished singular points $\Sigma \subset X_{2m}$ where $m = |I|$ and $n = 2m$.

The Gelfand-MacPherson construction is a powerful tool for computing class groups. The induced homomorphism

$$\beta_* : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}) = \mathrm{Cl}(\overline{\mathcal{M}}_{0,n}) \to \mathrm{Cl}(X_n) \tag{2.4}$$

is surjective because β is a fibration away from the distinguished singular points. Thus we get an exact sequence

$$0 \to N \to M \to Q \to 0, \tag{2.5}$$

where

$$N = \ker(\beta_*), \quad M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}).$$

In particular, N is generated by the D_I where $|I|, |I^c| \neq 2$. We can easily compute Q is well. Write

$$\mathfrak{X}^*(\mathbb{G}_m^n) = \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n,$$

so the quotient acting faithfully on the $\mathrm{CGr}(2,n)$ has characters

$$\{\sum a_i g_i : a_i \in \mathbb{Z}, \sum a_i \equiv 0 \pmod 2\}.$$

These give rise to line bundles on $X_n \setminus \Sigma$ and divisor classes on the full space. Thus we deduce that

$$Q \subset \mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$$

as an index-two subgroup. Note that the element $g_{i_1} + g_{i_2}, i_1 \neq i_2$ corresponds to the boundary divisor $D_{i_1 i_2}$; indeed, this locus is cut out by the 2×2 determinant on $\mathbb{P}^1_{i_1} \times \mathbb{P}^1_{i_2}$. Since Q is an index-two subgroup of a permutation module, we have

$$\mathrm{H}^1(G,Q) = 0 \text{ or } \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{H}^1(G,M) = 0 \text{ or } \mathbb{Z}/2\mathbb{Z}. \tag{2.6}$$

When n is odd, i.e., $n = 2m+1$, then X_{2n+1} is nonsingular,

$$\mathrm{Pic}(X_{2m+1}) = \mathrm{Cl}(X_{2m+1}),$$

and β is the iteration of a sequence of blowups along smooth disjoint centers. Precisely, we blow up the strata where m points coincide, then where $m-1$ points coincide, etc. (see [Has03, §8]); this is naturally equivariant under the $\mathfrak{S}_{2m+1}$-action. By the blowup formula [Ful98, Prop. 6.7], we have

$$\mathrm{Pic}(\overline{\mathcal{M}}_{0,2m+1}) = \mathrm{Pic}(X_{2m+1}) \oplus \{\text{free group on the exceptional divisors}\}.$$

We summarize this in algebraic terms:

Proposition 7. *For odd $n = 2m+1$, the exact sequence (2.5) splits $\mathfrak{S}_{2m+1}$-equivariantly:*

$$M \simeq N \oplus Q.$$

On the other hand, for n even, e.g., $n = 6$, there are examples of $G \subset \mathfrak{S}_n$ such that the sequence does not split equivariantly, since in those cases $\mathrm{H}^1(G,Q) \neq 0$ while $\mathrm{H}^1(G,M) = 0$ (see Example 26).

We return to the isomorphism (2.2) over nonclosed fields. Up to this point, we have been working with schemes but this is compatible with the μ_2-gerbe structure over the dense open subset where this is the full stabilizer. When $n = 2m$ the stabilizers may be larger, e.g., where the sequence in $(\mathbb{P}^1)^{2m}$ consists of m copies of a pair of points conjugate over a quadratic extension. In the cone over the Grassmannian, $2\binom{m}{2} = m^2 - m$ coordinates vanish and the m^2 remaining coordinates are equal to the determinant of the conjugate pair.

We can apply the same analysis to nonsplit actions. This includes working over nonclosed fields, where the n points are a Galois orbit, or in the equivariant context, where the n points are invariant under the action of a finite group. In the former situation, over a ground field F of characteristic zero, let E/F be an étale algebra of degree n classified by a representation of the Galois group $\Gamma_F \to \mathfrak{S}_n$. We replace the group $(\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m$ with $(\mathrm{GL}_2 \times R_{E/F}\mathbb{G}_m)/\mathbb{G}_m$ and $(\mathbb{P}^1)^n$ with $R_{E/F}\mathbb{P}^1$ (see [FR18, §4]). Note however that twisting $\mathrm{Mat}(2, n) = \mathbb{A}^{2n}$ yields a variety isomorphic to $\mathbb{A}^{2n}$, albeit with an action of a nonsplit torus.

The μ_2-gerbe has an explicit geometric interpretation along $\mathcal{M}_{0,n}$: It is encoded by the universal family

$$\phi : \mathcal{C}_{0,n} \to \mathcal{M}_{0,n},$$

a conic fibration, in general.

3. Rationality constructions

In this section, we work over an arbitrary field F, and we let Γ be the absolute Galois group of F.

Schubert calculus background. Our reference is [Kly85].

Consider the Grassmannian $\mathrm{Gr} = \mathrm{Gr}(p, p+q)$ of p-dimensional subspaces of a vector space of dimension $p + q$. The maximal torus $\mathsf{T} = \mathbb{G}_m^{p+q}$ acts diagonally on the vector space. Let X be a generic orbit in Gr.

We set combinatorial notation: Consider shuffles of $\{1, \ldots, p+q\}$

$$I = \{i_1 < \cdots < i_p\}, \quad J = \{j_1 < \cdots < j_q\}.$$

For each such shuffle, record the pairs $(k, \ell), k = 1, \ldots, p, \ell = 1, \ldots, q$, such that $i_k > j_\ell$. Write

$$\lambda_{p+1-k} = \#\{\ell : j_\ell < i_k\}$$

and note that

$$q \geq \lambda_1 \geq \cdots \geq \lambda_p.$$

Write $\lambda = (\lambda_1, \ldots, \lambda_p)$ and use the same notation for the associated Young diagram, which fits into a $p \times q$ rectangle. The *height* $\mathrm{ht}(\lambda)$ is the number of indices i with $\lambda_i > 0$. Set $|\lambda| = \lambda_1 + \cdots + \lambda_p$ and let σ_λ denote the associated Schubert cycle on Gr, a class in $\mathrm{H}^{2|\lambda|}(\mathrm{Gr}, \mathbb{Z})$.

We recall dimension formulae for representations. Let V be an n-dimensional vector space and $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition of $|\lambda|$ as above; in particular, $n \geq \mathrm{ht}(\lambda)$. The Schur functor $\mathbb{S}_\lambda(V)$ is a representation of $\mathrm{SL}(V)$ with dimension [FH91, Theorem 6.3, Exercise 6.4]:

$$\begin{aligned} d_n(\lambda) := \dim \mathbb{S}_\lambda(V) &= \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \\ &= \prod_{(a,b)} \frac{n - a + b}{h_{ab}}, \end{aligned}$$

where $a = 1, \dots, n$ labels the rows of λ (from top to bottom), b labels the columns (from left to right), and h_{ab} labels the "hook length". This is defined as the number of boxes immediately below and to the right of a given box, including the box. For $n < \mathrm{ht}(\lambda)$ we set $d_n(\lambda) = 0$.

For example, when $\lambda = (\lambda_1, \lambda_2, 0, \dots)$ and $n \geq 2$,

$$\begin{aligned} &d_n(\lambda_1, \lambda_2) = \\ &\frac{(n-1+1)\cdots(n-1+\lambda_1)}{1\cdots(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+2)\cdots(\lambda_1+1)} \frac{(n-2+1)\cdots(n-2+\lambda_2)}{1\cdots\lambda_2} \\ &\qquad = \binom{n-1+\lambda_1}{\lambda_1}\binom{n-2+\lambda_2}{\lambda_2}\frac{\lambda_1-\lambda_2+1}{\lambda_1+1}. \end{aligned}$$

For instance,

$$d_n(2,1) = \frac{(n+1)n(n-1)}{3}, \quad n \geq 1.$$

Another combinatorial quantity is

$$m_k(\lambda) := \sum_{i=0}^{k} (-1)^i \binom{|\lambda|+1}{i} d_{k-i}(\lambda).$$

If λ has height k then $m_k(\lambda) = d_k(\lambda)$, as the terms in the sum with $i > 0$ are zero.

We record a fact that we will use repeatedly in examples:

Proposition 8. *Fix an integer $d \geq 0$. If $f(x)$ is a polynomial of degree $\leq d$ then the $(d+1)$th iterated difference*

$$\sum_{i=0}^{d+1} (-1)^i \binom{d+1}{i} f(x-i) = 0.$$

When $\lambda = (\lambda_1, \lambda_2, 0, \ldots)$ we have:

$$m_k(\lambda_1, \lambda_2) =$$
$$\sum_{i=0}^{k} (-1)^i \binom{\lambda_1 + \lambda_2 + 1}{i} \binom{k - i - 1 + \lambda_1}{\lambda_1} \binom{k - i - 2 + \lambda_2}{\lambda_2} \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1}.$$

For instance, when $\lambda_1 = 2$ and $\lambda_2 = 1$ we have

$$\begin{aligned} m_k(2,1) &= \sum_{i=0}^{k} (-1)^i \binom{4}{i} \frac{(k-i+1)(k-i)(k-i-1)}{3} \\ &= 2\left(\binom{k+1}{3} - 4\binom{k}{3} + 6\binom{k-1}{3} - 4\binom{k-2}{3} + \binom{k-3}{3}\right) \\ &= \begin{cases} 2 & \text{if } k = 2, \\ 0 & \text{if } k \geq 3. \end{cases} \end{aligned}$$

For general λ_1 and λ_2,

$$m_2(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2 + 1$$

and

$$m_3(\lambda_1, \lambda_2) = \frac{\lambda_1(\lambda_2 - 1)(\lambda_1 - \lambda_2 + 1)}{2}.$$

Theorem 9. [Kly85, Theorem 5] *If X is the generic torus orbit in $\mathrm{Gr} = \mathrm{Gr}(p, p+q)$ and λ is a partition with $|\lambda| = p + q - 1$ then*

$$[X] \cdot \sigma_\lambda = m_p(\lambda).$$

For example, take $p = 2$. For $q = 2$

$$[X] \cdot \sigma_{21} = 2$$

and when $q = 3$ we have

$$[X] \cdot \sigma_{22} = 1, \quad [X] \cdot \sigma_{31} = 3.$$

For general q, we have $\lambda_1 \geq \lambda_2 = q + 1 - \lambda_1 \geq 0$, i.e.,

$$\frac{q+1}{2} \leq \lambda_1 \leq q + 1.$$

Here we have

$$[X] \cdot \sigma_{\lambda_1\, q+1-\lambda_1} = 2\lambda_1 - q;$$

in particular, when $q = 2m - 1$ and $\lambda_1 = m$ we find

$$[X] \cdot \sigma_{m\, m} = 1.$$

Remark 10. The signs in the formula for $m_k(\lambda)$ obscure the positivity of the result. An alternate formula [BF17, Theorem 5.1] makes this clearer:

$$[X] = \sum_{\lambda \subset (q-1)^{p-1}} \sigma_\lambda \sigma_{\widetilde{\lambda}},$$

where $\widetilde{\lambda}$ is the complement to λ in the rectangle $(q-1)^{p-1}$:

$$\lambda = (\lambda_1, \dots, \lambda_{p-1}), \quad \widetilde{\lambda} = (q-1-\lambda_{p-1}, \dots, q-1-\lambda_1).$$

We refer the reader to [Lia23] for the combinatorics directly relating these formulas.

This extends to general $p \in \mathbb{N}$:

Proposition 11. *Let V be a vector space with $\dim(V) = mp + 1$ so that*

$$q = (m-1)p + 1 \quad \textit{and} \quad (p-1)(q-1) = (m-1)(p-1)p.$$

Consider the coefficient of

$$\sigma_{\underbrace{(m-1)(p-1)\dots(m-1)(p-1)}_{p \text{ times}}}$$

in the expansion of $[X]$ in $\mathrm{H}^{2(p-1)(q-1)}(\mathrm{Gr}(p, p+q))$. This equals 1, i.e.,

$$[X] \cdot \sigma_{\underbrace{m \dots m}_{p \text{ times}}} = 1.$$

Indeed, this follows from Klyachko's formula (Theorem 9) and

$$m_p(\underbrace{m, \dots, m}_{p \text{ times}}) = d_p(\underbrace{m, \dots, m}_{p \text{ times}}) = 1.$$

Example 12. When $\dim(V) = 3m+1$ the generic orbit X for the action of T on $\mathrm{Gr}(3, V)$ has codimension $3(3m-2) - 3m = 6(m-1)$ and

$$[X] \cdot \sigma_{m\,m\,m} = m_3(m, m, m) = d_3(m, m, m) = 1.$$

This is not the case when $\dim(V) = 3m+2, m > 1$, e.g.,

$$[X] = 10\sigma_{5,3} + 8\sigma_{5,2,1} + 15\sigma_{4,4} + 15\sigma_{4,3,1} + 6\sigma_{4,2,2} + 3\sigma_{3,3,2}.$$

Grassmann geometry and rationality.

Theorem 13. *Let T be a maximal torus – possibly nonsplit - for SL_{pm+1} over a field F. Take $\mathrm{Gr}(p,V)$ for $\dim_F(V) = pm+1$ with the resulting T-action. Choose a subspace $W \subset V$ with*

$$\dim_F(W) = (p-1)m + 1$$

and transverse to T in the sense that $\mathrm{Gr}(p,W) \subset \mathrm{Gr}(p,V)$ meets some stable T-orbit properly. Then $\mathrm{Gr}(p,W)$ is a rational section of the quotient

$$\mathrm{Gr}(p,V) \stackrel{\sim}{\dashrightarrow} \mathrm{Gr}(p,V)/\mathsf{T}.$$

Thus if $\mathrm{Gr}(p,W)$ is rational, linearizable, or stably linearizable then the same holds true of the quotient.

Proof. The stability assumption guarantees that the quotient map is defined over a non-empty open subset of $\mathrm{Gr}(p,W)$. Properness of the intersection – which has degree one by Proposition 11! – implies $\mathrm{Gr}(p,W)$ is mapped birationally to the quotient. $\square$

Proposition 14. *Retain the notation of Theorem 13.*

If F is infinite then $\mathrm{Gr}(p,V)$ admits a codimension-m subspace $W \subset V$ satisfying the transversality condition.

If F is finite and $p = 2$ then $\mathrm{Gr}(2,V)$ admits a stable F-rational point.

If F is arbitrary and $p = 2$ then for each stable point there exists a subspace W satisfying the transversality assumption.

Combining with Theorem 13 gives a generalization of the results of [FR18]:

Corollary 15. *Let F be a finite field and ρ a representation of its Galois group in $\mathfrak{S}_{2m+1}$. Then ${}^{\rho}\overline{\mathcal{M}}_{0,2m+1}$ is rational over F.*

We also obtain analogs in higher dimensions:

Corollary 16. *Let $m \geq 1$ and $p \geq 2$ be integers. Consider the moduli space of $pm+1$ points in $\mathbb{P}^{p-1}$ up to projective equivalence. Let X be a variety obtained by twisting via permutations of the points, over an infinite field F. Then X is rational.*

Proof of Proposition 14. Assume F is infinite; here we use [Kap93, §1.2]. While Kapranov assumes the ground field has characteristic zero, the toric constructions and interpretation of $\overline{\mathcal{M}}_{0,n}$ as a Chow quotient for the PGL_2-action are valid in positive characteristic [GG14].

The Grassmannian is rational over F so its F-rational points are Zariski dense. We note that the torus action determines a collection of $\overline{F}$-subspaces

$$V_I \subset V, \quad \emptyset \neq I = \{i_1, \dots, i_r\} \subset \{0, \dots, mp\},$$

spanned by eigenvectors of the torus. Consider the

$$W \in \mathrm{Gr}(mp + 1 - m, mp + 1)$$

meeting some of these improperly, i.e.,

$$\dim(W \cap V_I) > \dim(W) + \dim(V) - \dim(V_I).$$

This is a Zariski closed proper subset of the Grassmannian, defined over F; its complement has F-rational points. Given such a subspace $W \subset V$, choose

$$w \in \Lambda \subset W, \quad \dim(\Lambda) = p,$$

defined over F, with w not contained in any of the $V_I \subsetneq V$ and Λ meeting all the V_I properly. Thus Λ is stable for the torus action and the torus orbit of Λ meets $\mathrm{Gr}(p, W)$ transversally there.

Now assume that F is finite and $p = 2$. We use the stability criterion (2.3) for points on $\mathbb{P}^1$ and Kapranov's analysis of the Gelfand-MacPherson correspondence. Here the Galois action ρ on the $2m + 1$ points is encoded by a single element $\sigma \in \mathfrak{S}_{2m+1}$. Express σ as a product of r disjoint cycles of lengths ℓ_i with

$$\ell_1 + \cdots + \ell_r = 2m + 1, \quad \ell_1 \geq \ell_2 \geq \cdots \geq \ell_r.$$

Only ℓ_1 can possibly be greater than m; if $\ell_1 \leq m$ then we have $r \geq 3$. When $\ell_1 > m$, choose a configuration of ℓ_1 points defined over a degree-ℓ_1 extension of F. Allow the remaining points to all coincide. We turn to the situation where $\ell_1 \leq m$. If $r = 3$ then we allow ℓ_1 points to coincide with $[0, 1]$, ℓ_2 points to coincide with $[1, 0]$, and ℓ_3 points to coincide with $[1, 1]$. We may therefore assume that $r \geq 4$ and work inductively on r. There exists two indices, say ℓ_3 and ℓ_4, whose sum is less than m. Use this to "degenerate" to a new partition of $2m + 1$, refined by $(\ell_1, \dots, \ell_r)$ but of length $r - 1$, all of whose entries are less than m. For example, we could take

$$(\ell_1, \ell_2, \ell_3 + \ell_4, \ell_5, \dots, \ell_r).$$

Continuing in this way, we generate a partition

$$\{1, 2, \dots, r\} = A \sqcup B \sqcup C$$

such that

$$\sum_{a\in A} \ell_a, \sum_{b\in B} \ell_b, \sum_{c\in C} \ell_c \leq m.$$

Let points coincide in three groups according to this coarsening of our original partition, the first group to $[0,1]$, the second to $[1,0]$, and the third to $[1,1]$.

Assume $p=2$ and F is arbitrary. We continue to assume that $\Lambda \subset V$ is a two-dimensional subspace that is stable in the sense of Geometric Invariant Theory. Let $\mathbf{T}_{2m}$ denote the tangent space to the torus orbit at Λ

$$\mathbf{T}_{2m} \subset \mathrm{Hom}(\Lambda, V/\Lambda),$$

an $2m$-dimensional subspace of the tangent space to $\mathrm{Gr}(2,V)$ at Λ. We claim there exists a subspace

$$\Lambda \subset W \subset V,$$

where W has codimension m in V, such that the composition

$$\mathbf{T}_{2m} \subset \mathrm{Hom}(\Lambda, V/\Lambda) \twoheadrightarrow \mathrm{Hom}(\Lambda, V/W)$$

has full rank $2m$. Since the latter space is the normal directions to $\mathrm{Gr}(2,W)$ at Λ, this will yield transversality.

We record some basic geometry:

Lemma 17. *There is a distinguished orbit*

$$\mathbb{P}^1 \times \mathbb{P}^{m-2} \simeq \mathbb{P}(\Lambda^*) \times \mathbb{P}(V/\Lambda) \subset \mathbb{P}(\mathrm{Hom}(\Lambda, V/\Lambda))$$

invariant under automorphisms of $\mathrm{Gr}(2,V)$ *fixing* $[\Lambda]$.

The subspace $\mathbb{P}^{2m-1} \simeq \mathbb{P}(\mathbf{T}_{2m})$ *cuts out the graph of a rational normal curve*

$$\begin{aligned} \varrho : \mathbb{P}^1_{s_0,s_1} &\hookrightarrow \mathbb{P}^{2m-2}_{x_0,\dots,x_{2m-2}} \\ [s_0,s_1] &\mapsto [s_0^{2m-2},\dots,s_1^{2m-2}]. \end{aligned}$$

In these coordinates, the rational normal curve has equations

$$s_0x_{i+1} = s_1x_i, \quad i = 0,\dots,2m-1.$$

Let $\Gamma \subset \mathbb{P}^1$ *denote the length-*$(2m+1)$ *subscheme that is the image of the eigenvectors for* $\mathbf{T}_{2m}$ *under* $V^* \twoheadrightarrow \Lambda^*$. *Then* ϱ *realizes the Gale transform for* $\Gamma \subset \mathbb{P}^1$ *as a subscheme of* $\mathbb{P}^{2m-2}$ *contained in a rational normal curve.*

The first assertion reflects the fact that the parabolic subgroup of PGL_{2m+1} fixing $[\Lambda]$ has semisimple part $(\mathrm{GL}_2 \times \mathrm{GL}_{2m-1})/\mathbb{G}_m$. Note that the unipotent part acts trivially on the tangent space. The second assertion is true for the generic codimension-$(2m-2)$ linear slice of $\mathbb{P}^1 \times \mathbb{P}^{2m-1}$. Of course, one has to show that this applies in our siutation! This follows from the third assertion, a special case of [EP00, Corollary 3.2] – the first application following the statement. This completes the proof of the lemma.

Returning to the proof of the Proposition, we may take W as the subspace given by

$$\{x_{2j} = 0, j = 0, \ldots, m-1\},$$

where we interpret $x_j \in (V/\Lambda)^* a$. It is clear that the products

$$\{s_i x_{2j}, i = 0, 1, j = 0, \ldots, m-1\}$$

have the desired spanning property; the elements

$$s_0^{2m-1}, \ldots, s_1^{2m-1}$$

are a basis for bilinear forms of degree $2m-1$. □

Partitioning the points. We start with a general construction: Let $n \geq 3$ be an integer and $n = \ell m$ a factorization in integers $\ell, m > 1$. Suppose that $H \subset \mathfrak{S}_\ell, A \subset \mathfrak{S}_m$ are subgroups. The *wreath product*

$$A \wr H = A \wr_{1,\ldots,\ell} H$$

is the semidirect product $A^\ell \rtimes H$ where

$$(a_1, \ldots, a_\ell) \cdot h = (a_{h^{-1}(1)}, \ldots, a_{h^{-1}(\ell)}).$$

This comes with a natural embedding

$$\rho : A \wr H \hookrightarrow \mathfrak{S}_{\ell m}$$

as permutations of pairs

$$(i, j), \quad i \in \{1, \ldots, m\}, j \in \{1, \ldots, \ell\}.$$

Now assume that $m \geq 3$. Forgetting maps yield an equivariant morphism

$$\phi : {}^\rho\overline{\mathcal{M}}_{0,\ell m} \to \prod_H {}^\alpha\overline{\mathcal{M}}_{0,m},$$

where $\alpha : A \hookrightarrow \mathfrak{S}_m$ and the twisted product denotes ℓ copies of the moduli space with the associated H-action. The generic fiber of this morphism is irreducible of dimension

$$(\ell m - 3) - \ell(m-3) = 3\ell - 3.$$

It is birational to the Hilbert scheme of multidegree-$(1,\dots,1)$ curves in the H-twisted product $\prod_H C_j$ of ℓ genus-zero curves. Geometrically, this is a compactification of the homogeneous space

$$\underbrace{\mathrm{PGL}_2 \times \cdots \times \mathrm{PGL}_2}_{\ell \text{ times}} / \mathrm{PGL}_2$$

with the last PGL_2 embedded diagonally.

We record some observations on the generic fiber of ϕ:

- Suppose $\ell = 2$. Geometrically, $(1,1)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are parametrized by $\mathbb{P}^3$ – the dual to the projective space containing the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. Over an arbitrary field the fiber is a Brauer-Severi threefold.
- Suppose that m is odd. Then the genus-zero curves C_j appearing in the twisted product are split and – over the extension/-subgroup associated with $A^\ell \subset A \wr H$ – isomorphic to $\mathbb{P}^1$'s. Here the twisted product $\prod_H C_j$ is rational, as it is isomorphic to the restriction of scalars of $\mathbb{P}^1$.
- Now assume $\ell = 2$ and m odd. Here the generic fiber of ϕ is isomorphic to $\mathbb{P}^3$ over the function field/linearizable for the full wreath product.

Example 18. Suppose $n = 6$ and consider $G = \mathfrak{S}_3 \wr \mathfrak{S}_2 \subset \mathfrak{S}_6$, a subgroup of index 10 preserving an unordered partition

$$\{1,2,3,4,5,6\} = \{i,j,k\} \sqcup \{a,b,c\}.$$

Then the associated ${}^\rho\overline{\mathcal{M}}_{0,6}$ is rational/linearizable. These actions correspond to situations where the associated Segre threefold admits an invariant node (cf. Theorem 33 below).

Theorem 19. *Let $n = 2m$, with $m \geq 3$ odd. Fix a subgroup $A \subset \mathfrak{S}_m$ and the diagonal subgroup*

$$G := A \times \mathfrak{S}_2 \subset A \wr \mathfrak{S}_2 \subset \mathfrak{S}_{2m}.$$

- *For each Galois representation $\rho : \Gamma \to G$ the twist ${}^\rho\overline{\mathcal{M}}_{0,n}$ is rational over F.*
- *The G action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.*

Proof. We assume $\mathfrak{S}_m$ permutes the points with odd and even indices respectively.

We focus first on the arithmetic case. Let L/F be the quadratic extension associated with A. Over L, the generic point of the twisted moduli space corresponds to $\mathbb{P}^1$ equipped with reduced and disjoint zero-cycles $Z_{odd}, Z_{even} \subset \mathbb{P}^1$ of length m. The parity of m ensures that the underlying curve is $\mathbb{P}^1$.

Note that the variety ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ is already stably rational over L by Proposition 4.

Consider forgetting the even and odd points

$$(\pi_{odd}, \pi_{even}) : ({}^{\rho}\overline{\mathcal{M}}_{0,n})_L \to {}^{\varpi_{odd}}\overline{\mathcal{M}}_{0,m} \times {}^{\varpi_{even}}\overline{\mathcal{M}}_{0,m}$$

where the Galois actions come via restriction to the even and odd points. These actions are conjugate for the quadratic extension L/F. Descent therefore gives a morphism over F

$$\phi : {}^{\rho}\overline{\mathcal{M}}_{0,n} \to R_{L/F}({}^{\varpi_{odd}}\overline{\mathcal{M}}_{0,m}),$$

where the target is the restriction of scalars. The twists of $\overline{\mathcal{M}}_{0,m}$ are rational over L by [FR18] and Corollary 15. The restriction of scalars of a rational variety is rational.

We claim that the generic fiber of ϕ is rational over the function field of the base, which implies rationality for ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ over F. This follows from the analysis above for $\ell = 2$ and odd m.

For the equivariant case, our geometric argument shows that the G-variety $\overline{\mathcal{M}}_{0,n}$ is birationally the projectivization of an equivariant vector bundle over a stably linearizable variety (by Proposition 4). Note that restriction of scalars in the arithmetic situation corresponds to passing to an induced representation in the equivariant context; thus stable linearizability is clearly preserved. We conclude then that $\overline{\mathcal{M}}_{0,n}$ is stably linearizable. $\square$

Corollary 20. *Let C_{2m}, with m odd, be a cyclic group. Then twists of $\overline{\mathcal{M}}_{0,n}$ by this group are rational (in the Galois case) and stably linearizable (in the equivariant situation).*

Proof. If the action has an odd orbit then this follows from Propositions 2 and 4. Otherwise, all the orbits are even and we may apply Theorem 19. $\square$

Remark 21. Similar reasoning applies for a Galois action

$$\rho : \Gamma \to \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2} \subset \mathfrak{S}_{m_1+m_2}, \quad m_1, m_2 \geq 3 \text{ odd},$$

with restricted actions ϖ_1 and ϖ_2 on the first m_1 points and last m_2 points respectively. Proposition 4 already gives stable rationality in this case. The forgetting morphism

$$\phi : {}^{\rho}\overline{\mathcal{M}}_{0,m_1+m_2} \to {}^{\varpi_1}\overline{\mathcal{M}}_{0,m_1} \times {}^{\varpi_2}\overline{\mathcal{M}}_{0,m_2}$$

has generic fiber birational to $\mathbb{P}^3$ by the reasoning above. Since the factors ${}^{\varpi_i}\overline{\mathcal{M}}_{0,m_i}$ are rational, ${}^{\rho}\overline{\mathcal{M}}_{0,m_1+m_2}$ is rational as well.

4. Stable linearizability via torsors

Let G be a finite group and T a G-torus, i.e., a torus equipped with a representation of G on its character module $\mathfrak{X}^*(\mathsf{T})$. Recall that T is stably linearizable if $\mathfrak{X}^*(\mathsf{T})$ is stably permutation, see, e.g., [HT23, Proposition 2].

Proposition 22. *Let U be a smooth quasi-projective variety with G-action. Assume that we have a T-torsor*

$$\mathcal{P} \to U,$$

i.e., a T-principal homogeneous space over U, in the category of G-varieties. Assume that

- *the G-action on U is generically free,*
- *the characters $\mathfrak{X}^*(\mathsf{T})$ are a stably permutation G-module,*
- *the G-action on $\mathcal{P}$ is stably linearizable.*

Then the G-action on U is stably linearizable.

Proof. We claim there is a G-equivariant birational map,

$$\begin{tikzcd} \mathcal{P} \arrow[rr, dashed, "\sim"] \arrow[dr] & & \mathsf{T} \times U \arrow[dl] \\ & U & \end{tikzcd}$$

which would follow if $\mathcal{P} \to U$ admits a G-equivariant rational section. We clearly have such a section after discarding the G-action, by Hilbert's Theorem 90.

Since T is stably permutation, a product $\mathsf{T} \times \mathsf{T}_1$, where T_1 is a permutation torus, is isomorphic to a permutation torus and may be realized as a dense open subset of affine space. It follows that we have an open embedding

$$\begin{tikzcd} \mathcal{P} \times_U \mathsf{T}_1 \arrow[rr, hook] \arrow[dr] & & \mathcal{V} \arrow[dl] \\ & U & \end{tikzcd}$$

where $\mathcal{V} \to U$ is a vector bundle with G-action. The vector bundle admits a rational section (by the No-Name Lemma) thus $\mathcal{P}$ does as well.

We assumed that $\mathcal{P}$ is stably linearizable, i.e. $\mathcal{P} \times \mathbb{G}_m^r$ is linearizable for some r. Thus $U \times \mathsf{T} \times \mathbb{G}_m^r$ is as well. We observed that T is stably linearizable because its character module is stably permutation, i.e. $\mathsf{T} \times \mathsf{T}_1$ is a permutation torus. Another application of the No-Name Lemma, using the assumption that the action on U is generically free, gives that U is stably linearizable. $\square$

We recall the exact sequence (2.5)

$$0 \to N \to M \to Q \to 0,$$

with $M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$, N an $\mathfrak{S}_n$-permutation module, and Q is an index-2 submodule of the permutation module $\mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$. We record:

- if $\mathrm{H}^1(G, Q) = 0$ for some $G \subset \mathfrak{S}_n$, then also $\mathrm{H}^1(G, M) = 0$, by the long exact sequence in cohomology,
- if Q is a stably permutation G-module, then the sequence splits and $\mathrm{Pic}(\overline{\mathcal{M}}_{n,0})$ is a stably permutation module, by [CTS77, Lemma 1].

Theorem 23. *Let $G \subseteq \mathfrak{S}_n$ be a subgroup such that Q is a stably permutation module. Then the G-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.*

Let X be a form of $\overline{\mathcal{M}}_{0,n}$ over F such that the action of the absolute Galois group on Q gives rise to a stable permutation module. Then X is stably rational over F.

Proof. For the equivariant statement, we apply Proposition 22. Here T, with character module Q acts on $\mathrm{CGr}(2, n)$ (see Section 2). Let $V \subset \mathrm{CGr}(2, n)$ the open subset over which T acts freely and $U \subset X_n$ the corresponding locus in the quotient, i.e., remove all the strictly semistable points. We have a torsor

$$V \xrightarrow{\mathsf{T}} U.$$

By [HT23, Proposition 19], the $\mathfrak{S}_n$-action on $\mathrm{Gr}(2, n)$ (and its cone) is stably linearizable. Assuming that $Q = \mathfrak{X}^*(\mathsf{T})$ is a stable permutation module for $G \subset \mathfrak{S}_n$, and applying Proposition 22, we conclude that the G-action on U, and thus $\overline{\mathcal{M}}_{0,n}$, is stably linearizable as well.

The Galois-theoretic result is proven analogously, with [BCTSSD85, Prop. 3] playing the role of Proposition 22. This is an application of the torsor formalism of [CTS87]. $\square$

Remark 24. There exist linearizable G-actions on $\overline{\mathcal{M}}_{0,n}$ such that the induced action on Q is not stably permutation. Consider n even and $G = C_2$ generated by $\sigma := (1, 2) \cdots (n-1, n)$; we have $\mathrm{H}^1(C_2, Q) \neq 0$ (see Remark 31) so Q is not stably permutation. This action is equivariantly birational – by Proposition 6 – to an action on a torus $\mathsf{T} = \mathbb{G}_m^{n-3}$.

Its character module consists of the elements of $\mathbb{Z}^{n-2}$ – the twisted permutation module on $\{1,\dots,n-2\}$ – whose coordinates sum to zero (see Equation 2.1). The action of C_2 on the twisted permutation module consists of $(n-2)/2$ copies of $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Hence $\mathfrak{X}^*(\mathsf{T})$ decomposes as a sum of $\frac{n}{2}-2$ permutation modules and one invariant, a permutation module. We conclude T is linearizable.

Remark 25. By [FR18, Remark 5.5], for *odd* n, every form of $\overline{\mathcal{M}}_{0,n}$ over a nonclosed field F is an F-rational variety. *A priori*, this does *not* imply that $\overline{\mathcal{M}}_{0,n}$ is (stably) linearizable for $\mathfrak{S}_n$. However, this does imply that M is a stable permutation module, for the $\mathfrak{S}_n$-action.

For n *odd*, we have

$$M \simeq N \oplus Q, \tag{4.1}$$

as $\mathfrak{S}_n$-modules, by Proposition 7. Since N is a permutation module, for all n, and M a stably permutation module, for odd n, we see that Q is also stably permutation, for odd n. Thus, the $\mathfrak{S}_n$-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable, by Theorem 23.

The splitting (4.1) can also be seen explicitly: Recall that under the Kapranov basis, $Q = M/N$ is generated by the image of the classes

$$H, \quad E_i, \quad i = 1,\dots,n-1$$

in M under the projection modulo N. The $\mathbb{Z}$-linear map

$$s : Q \to M,$$

given on these generators by

$$H \mapsto H + \sum_{\substack{I\subset\{1,\dots,n-1\},\\ |I|=\frac{n-1}{2},\dots,n-4.}} (|I|-1)\cdot E_I, \qquad E_i \mapsto E_i + \sum_{\substack{I\subset\{1,\dots,n-1\}, i\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I.$$

is a section of the exact sequence (2.5). We check that it is $\mathfrak{S}_n$-equivariant. Let $\tau = (1,2)$ and $\sigma = (1,\dots,n)$. In Q, one has

$$H = D_{12} + \sum_{i=3}^{n} E_i$$

and $\tau(H) = H$, $\tau(E_1) = E_2$, $\tau(E_2) = E_1$ and $\tau(E_i) = E_i$. Note that s is τ-equivariant by construction. Next, observe

$$\begin{aligned}
s\sigma(H) &= s\left(\sigma\left(D_{12} + \sum_{i=3}^{n} E_i\right)\right) = s\left((n-3)H - (n-4)\sum_{i=2}^{n-1} E_i\right) \\
&= (n-3)H - (n-4)\sum_{i=2}^{n-1} E_i - \sum_{\substack{I\subset\{1,\dots,n-1\},1\notin I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} (n-|I|-3)\cdot E_I \\
&\quad + \sum_{\substack{I\subset\{1,\dots,n-1\},1\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} (|I|-1)\cdot E_I.
\end{aligned}$$

$$\begin{aligned}
\sigma s(H) &= \sigma\left(H + \sum_{|I|=\frac{n-1}{2},\dots,n-4} (|I|-1)\cdot E_I\right) \\
&= \sigma\left(D_{n-2,n-1} + \sum_{\substack{I\subset\{1,\dots,n-3\},\\ |I|=1,\dots,n-4.}} E_I + \sum_{|I|=\frac{n-1}{2},\dots,n-4.} (|I|-1)\cdot E_I\right) \\
&= E_{n-1} + \sum_{\substack{I\subset\{2,\dots,n-2\}\\ |I|=1,\dots,n-5}} D_{I\cup\{n-1,n\}} + \sum_{i=2}^{n-2} D_{1,i} \\
&\quad + \underbrace{\sum_{\substack{I\subset\{2,\dots,n-1\},\\ |I|=\frac{n-1}{2}-1,\dots,n-5.}} E_{\{1\}\cup I} + \sum_{\substack{I\subset\{2,\dots,n-1\},\\ |I|=2,\dots,\frac{n-1}{2}-1.}} (n-3-|I|)E_I}_{:=A} \\
&= (n-3)H - (n-4)\sum_{i=2}^{n-1} E_i - \sum_{\substack{I\subset\{2,\dots,n-1\},i\notin I,\\ |I|=1,\dots,n-4.}} E_I \\
&\quad + \sum_{\substack{I\subset\{2,\dots,n-2\},\\ |I|=1,\dots,n-5.}} D_{I\cup\{n-1,n\}} + A.
\end{aligned}$$

One can then verify $\sigma s(H) = s\sigma(H)$ by comparing the coefficients of each generator E_I. To check actions on E_i, for $i = 1,\dots,n-2$, one has

$$
\begin{aligned}
s\sigma(E_i) &= s(H - \sum_{k=2, k\neq i+1}^{n-1} E_k) \\
&= H - \sum_{k=2, k\neq i+1}^{n-1} E_k - \sum_{\substack{I\subset\{1,\dots,n-1\},\\ 1, i+1\notin I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I \quad + \sum_{\substack{I\subset\{1,\dots,n-1\},\\ 1, i+1\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sigma s(E_i) &= \sigma(E_i + \sum_{\substack{I\subset\{1,\dots,n-1\}, i\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I) \\
&= H - \sum_{\substack{I\subset\{2,\dots,n-1\}, i+1\notin I,\\ |I|=1,\dots,n-4}} D_{I\cup\{n\}} + \sum_{\substack{I\subset\{2,\dots,n-1\},\\ |I|=\frac{n-1}{2}-1,\dots,n-5.}} D_{I\cup\{1,n\}} \\
&\quad + \sum_{\substack{I\subset\{2,\dots,n-1\}, i+1,\notin I\\ |I|=2,\dots,\frac{n-1}{2}-1.}} E_I.
\end{aligned}
$$

Similarly, one can check $\sigma s(E_i) = s\sigma(E_i)$ for $i \neq n-1$ by comparing the coefficients. Finally, one can verify

$$s(\sigma(E_{n-1})) = s(E_1) = \sigma(s(E_{n-1})).$$

5. Computing cohomology

In this section, we study the G-module

$$M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}),$$

and the quotient $Q = M/N$, from (2.5), for various $G \subset \mathfrak{S}_n$.

Cohomological criteria. We focus on two properties, which are necessary for linearizability of a regular G-action on a smooth projective rational variety X, see, e.g., [BP13, Proposition 2.5]:

(H1) For all subgroups $G' \subset G$ one has

$$\mathrm{H}^1(G', \mathrm{Pic}(X)) = \mathrm{H}^1(G', \mathrm{Pic}(X)^*) = 0.$$

(SP) The G-module $\mathrm{Pic}(X)$ is stably permutation.

Since H^1 vanishes on permutation modules, (**SP**) implies **(H1)**, but the converse does not hold, in general. Computationally, it is easier to check **(H1)**.

Example 26. For $n = 6$ and $G \subseteq \mathfrak{S}_6$, property **(H1)** for the action on $M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$ does not imply **(SP)**, e.g., for the action of

$$G \simeq C_2 \times C_4 := \langle (3,4), (1,2,5,6) \rangle,$$

and

$$G \simeq (C_2)^3 := \langle (1,5)(2,6), (3,4), (1,2)(5,6) \rangle,$$

see the analysis in [CTZ23, Section 6], as well as [Kun87, Section 4]. Furthermore, there are $G \subset \mathfrak{S}_6$ such that

- Q fails **(H1)** but M satisfies it, e.g., for $G = \langle (1,2)(3,4)(5,6) \rangle$, one has

  $$\mathrm{H}^1(G, M) = 0, \quad \mathrm{H}^1(G, Q) = \mathbb{Z}/2.$$

  Actually, M is a permutation module while Q is not. Indeed, under appropriate choices of basis, M is of the form

  $$\mathbb{Z}^4 \oplus \mathbb{Z}[C_2]^6,$$

  and Q is of the form

  $$\mathbb{Z} \oplus \mathbb{Z}[C_2]^2 \oplus \mathbb{Z}[e],$$

  where G acts on e via -1.
- Both Q and M fail **(H1)**: all groups containing $G = C_2^2$ from Proposition 27, in these cases we have

  $$\mathrm{H}^1(G, M) = \mathrm{H}^1(G, Q) = \mathbb{Z}/2.$$

Statement of results.

Proposition 27. *For $n_1, n_2, n_3 \in \mathbb{N}$ with $2(n_1 + n_2 + n_3) = n$ let*

$$\iota_1 = (1,2)\dots(2n_1 - 1, 2n_1)(2(n_1 + n_2) + 1, 2(n_1 + n_2) + 2)\dots(n-1, n),$$

$$\iota_2 = (2n_1 + 1, 2n_1 + 2), \dots, (2(n_1 + n_2) - 1, 2(n_1 + n_2))\dots(n-1, n),$$

and put $G := \langle \iota_1, \iota_2 \rangle$. Then

$$\mathrm{H}^1(G, M) = \mathbb{Z}/2.$$

The first part of Theorem 1 follows:

Corollary 28. *For every even $n > 5$ and every subgroup of $\mathfrak{S}_n$ containing G, the induced action on $\overline{\mathcal{M}}_{0,n}$ is not stably linearizable.*

For example, when $n_1 = n_2 = n_3 = 1$

$$\iota_1 = (12)(56), \quad \iota_2 = (34)(56),$$

and the corresponding action on $\overline{\mathcal{M}}_{0,6}$, which is $\mathfrak{S}_6$-equivariantly birational to the Segre cubic, is not stably linearizable.

We apply the results above to rationality questions over nonclosed fields, completing the proof of Theorem 1:

Theorem 29. *Let F be a field admitting a biquadratic extension. Then, for all even $n \geq 6$ there exist forms of $\overline{\mathcal{M}}_{0,n}$ over F that are not retract rational, and thus not stably rational, over F.*

In particular, this yields nonrational forms over $F = \mathbb{C}(t)$, a field with trivial Brauer group.

Proof. Indeed, let $G \simeq C_2^2$ be the group identified in Proposition 27, with $\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})) = \mathbb{Z}/2$. Let $\Gamma = \mathrm{Gal}(F'/F)$ be the Galois group of the biquadratic extension F'/F. We construct a form X of $\overline{\mathcal{M}}_{0,n}$ over F such that Γ acts on $\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ via G. This gives an **(H1)**-obstruction to retract rationality. □

Proof of Proposition 27. Put

$$\sigma := \iota_1\iota_2 = (1,2)\cdots(2(n_1+n_2)-1, 2(n_1+n_2)),$$
$$\tau := \iota_2 = (2n_1+1, 2n_1+2)\cdots(n-1,n),$$

so that $G = \langle\sigma,\tau\rangle$. We will repeatedly use the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\langle\tau\rangle, A^\sigma) \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(\langle\sigma\rangle, A)^\tau, \tag{5.1}$$

with the usual notation for invariants under the actions of σ, τ.

Step 1. Observe that M admits a decomposition, as a G-module,

$$M = L \oplus P,$$

where L consists of $\mathbb{Z}$-linear combinations of H and E_I, with $n-1 \notin I$, and P is generated, over $\mathbb{Z}$, by E_I with $n-1 \in I$. We have

$$\mathrm{H}^1(G, M) = \mathrm{H}^1(G, L) \oplus \mathrm{H}^1(G, P).$$

Step 2. The involution σ is contained in $\mathfrak{S}_{n-1}$, permuting $(n-1)$ points and therefore linearizable. Thus

$$\mathrm{H}^1(\langle\sigma\rangle, M) = \mathrm{H}^1(\langle\sigma\rangle, L) = \mathrm{H}^1(\langle\sigma\rangle, P) = 0.$$

Moreover, P is a G-permutation module. Indeed, for I with $n-1 \in I$, $\sigma E_I = E_{\sigma(I)} \in P$, and $\tau E_I = E_{(\tau\cdot(n-1,n))(I)} \in P$. It follows that

$$\mathrm{H}^1(G, P) = 0,$$

and

$$\mathrm{H}^1(G, M) = \mathrm{H}^1(G, L) = \mathrm{H}^1(\langle\tau\rangle, L^\sigma).$$

Remark 30. Geometrically, cohomology is already contributed on the *toric model* $\overline{L}_n$, obtained by blowing up $(n-2)$ general points on $\mathbb{P}^{n-3}$.

Step 3. Let $N \subset L$ be the submodule of $\mathbb{Z}$-linear combinations of E_I with $|I| \geq 2$ and $n-1 \notin I$. We have a short exact sequence

$$0 \to N \to L \to Q \to 0,$$

of G-modules, with Q generated by $H, E_1, \ldots, E_{n-2}$, modulo N, and the corresponding long exact sequence of $\langle \tau \rangle$-modules:

$$0 \to N^\sigma \to L^\sigma \to Q^\sigma \to \mathrm{H}^1(\langle \sigma \rangle, N) \to \ldots$$

Since $\sigma(E_I) = E_{\sigma(I)}$, the σ-action on N yields naturally a permutation module, realized via permutation of indices of E_I. So

$$\mathrm{H}^1(\langle \sigma \rangle, N) = 0.$$

The short exact sequence

$$0 \to N^\sigma \to L^\sigma \to Q^\sigma \to 0$$

gives rise to the long exact sequence

$$\mathrm{H}^1(\langle \tau \rangle, N^\sigma) \to \mathrm{H}^1(\langle \tau \rangle, L^\sigma) \to \mathrm{H}^1(\langle \tau \rangle, Q^\sigma) \to \mathrm{H}^2(\langle \tau \rangle, N^\sigma). \tag{5.2}$$

Step 4. The $\langle \tau \rangle$-module N^σ has the form:

$$N^\sigma = \mathbb{Z}[\langle \tau \rangle] \oplus \cdots \oplus \mathbb{Z}[\langle \tau \rangle].$$

In particular,

$$\mathrm{H}^1(\langle \tau \rangle, N^\sigma) = \mathrm{H}^2(\langle \tau \rangle, N^\sigma) = 0.$$

Indeed, a $\mathbb{Z}$-basis of N^σ is given by

$$e_I := \begin{cases} E_I + E_{\sigma(I)} & \text{if } \ \sigma(I) \neq I, \\ E_I & \text{if } \ \sigma(I) = I, \end{cases}$$

for

$$I \subset \{1, 2, \ldots, n-2\}, \quad 2 \leq |I| \leq n-4.$$

To show that N^σ is a direct sum of copies of $\mathbb{Z}[\langle \tau \rangle]$, it suffices to show that $\tau(e_I) = e_{I'}$, for some $I' \neq I$ and $e_I \neq e_{I'}$. Observe that

$$\sigma(I)^c = \sigma(I^c), \quad I^c := \{1, \ldots, n-2\} \setminus I.$$

There are three cases:

- If $\sigma(I) = \tau(I) = I$, then
$$\tau(e_I) = \tau(E_I) = D_{I\cup\{n-1\}} = E_{I^c} = e_{I^c}$$
and thus $e_I \neq e_{I^c}$.
- If $\sigma(I) \neq I$ and $\tau(I) = I$, then
$$\begin{aligned}\tau(e_I) &= \tau(E_I) + \tau(E_{\sigma(I)}) = D_{I\cup\{n-1\}} + D_{\sigma(I)\cup\{n-1\}}\\ &= E_{I^c} + E_{\sigma(I)^c} = E_{I^c} + E_{\sigma(I^c)} = e_{I^c}.\end{aligned}$$
Since $I^c \neq I$ and $I^c \neq \sigma(I^c)$, we know that $e_I \neq e_{I^c}$.
- If $\tau(I) \neq I$, then $\sigma(I) \neq I$, and
$$\begin{aligned}\tau(e_I) &= E_{\tau(I)^c} + E_{(\tau\sigma(I))^c} = E_{\tau(I)^c} + E_{(\sigma\tau(I))^c}\\ &= E_{\tau(I)^c} + E_{\sigma(\tau(I)^c)} = e_{\tau(I)^c}.\end{aligned}$$
To be concrete, assume that $1 \in I$ and $2 \notin I$. Then $1 \in \tau(I)^c$ and $1 \notin \sigma(I)$, so that $\tau(I)^c \neq \sigma(I)$. Since $|I| \geq 2$, one can see that $\tau(I)^c \neq I$ and thus $e_{\tau(I)^c} \neq e_I$.

In conclusion, $\tau(e_I) \neq e_I$, in all cases, and N^σ is as claimed, and thus has vanishing first and second cohomology. It follows that
$$\mathrm{H}^1(\langle\tau\rangle, M^\sigma) = \mathrm{H}^1(\langle\tau\rangle, L^\sigma) = \mathrm{H}^1(\langle\tau\rangle, Q^\sigma).$$

Step 5. To show that $\mathrm{H}^1(\langle\tau\rangle, Q^\sigma) = \mathbb{Z}/2$, let
$$\Sigma_i := \sum_{|I|=i} E_I,$$
where the sum is over $I \subseteq \{1, 2, \dots, n-2\}$ with $|I| = i$. Put $\Sigma := \Sigma_1$ and set
$$\begin{aligned}&e_0 := H - \Sigma,\\ &e_i := H - \Sigma + (E_{2i-1} + E_{2i}), && 1 \leq i \leq n_1 + n_2,\\ &w_j := E_{2j-1}, && n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2},\\ &v_j := H - \Sigma + E_{2j}, && n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}.\end{aligned}$$
Then
$$\{e_i, w_j, v_j\}$$
for $0 \leq i \leq n_1 + n_2$ and $n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}$ gives a $\mathbb{Z}$-basis of Q^σ. Moreover, for $1 \leq i \leq n_1 + n_2$ and $n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}$, one has
$$\tau(e_0) = -e_0, \quad \tau(e_i) = e_i, \quad \text{and} \quad \tau(w_j) = v_j.$$

Indeed, Q^σ is generated, over $\mathbb{Z}$, by

$$H, (E_1+E_2), \ldots, (E_{2(n_1+n_2)-1}+E_{2(n_1+n_2)}), E_{2(n_1+n_2)+1}, \ldots, E_{n-2}.$$

We now show that $\{e_i, w_j, v_j\}$ gives another basis. First, observe that

$$H - \Sigma = D_{34\ldots n} - (E_1+E_2) + \underbrace{\sum_{1,2\notin I, E_I\in N} E_I}_{\in N^\sigma} .$$

Indeed, if $1,2 \notin I$ and $E_I \in N$, $1,2\notin \sigma(I)$ and $E_{\sigma(I)}$ will also appear in the summand. Then $\sigma(H-\Sigma) = H - \Sigma \pmod{N^\sigma}$ and

$$e_j, w_j, v_j \in Q^\sigma.$$

Moreover, $\{e_j, w_j, v_j\}$ generates Q^σ since

$$E_{2i-1} + E_{2i} = e_i - e_0, \quad E_{2j} = v_j - e_0$$

and

$$H = (\frac{4-n}{2})e_0 + \sum_{i=1}^{n_1+n_2} e_i + \sum_{j=n_1+n_2+1}^{\frac{n-2}{2}} (w_j+v_j)\,.$$

To compute the τ-action on this basis, one can first compute

$$\begin{aligned}
H - \Sigma =& D_{34\ldots n} - (E_1+E_2) \pmod{N^\sigma}\\
\overset{\tau}{\longmapsto}& D_{34\ldots n} - D_{1,n-1} - D_{2,n-1}\\
=& D_{34\ldots n} - 2H + 2\Sigma - (E_1+E_2) \pmod{N^\sigma}\\
=& H - \Sigma + (E_1+E_2) - 2H + 2\Sigma + (E_1+E_2) \pmod{N^\sigma}\\
=& -H + \Sigma,
\end{aligned}$$

i.e.,

$$\tau(e_0) = -e_0.$$

Then we have

$$\begin{aligned}
H - \Sigma + E_{2i-1} + E_{2i} \overset{\tau}{\longmapsto}& -H + \Sigma + D_{2i-1,n-1} + D_{2i,n-1}\\
=& -H + \Sigma + 2H - 2\Sigma + (E_{2i-1}+E_{2i}) \pmod{N^\sigma}\\
=& \quad H - \Sigma + (E_{2i-1}+E_{2i}) \pmod{N^\sigma}.
\end{aligned}$$

Note that the equalities hold for all $1 \le i \le \frac{n}{2}$. In particular,

$$\tau(e_i) = e_i, \quad \text{for} \quad 1 \le i \le n_1+n_2.$$

Finally,

$$\begin{aligned}\tau(w_j) = D_{2_j,n-1} &= H - \Sigma + E_{2j} - \sum_{\substack{2j\notin I\\ E_I\in N}} E_I \\ &= H - \Sigma + E_{2j} \pmod{N^\sigma},\end{aligned}$$

i.e.,

$$\tau(w_j) = v_j, \quad \text{for} \quad n_1+n_2+1 \leq j \leq \frac{n-2}{2}.$$

In conclusion,

$$Q^\sigma = \mathbb{Z}[e_0] + \sum_{i=1}^{n_1+n_2} \mathbb{Z}[e_i] + \sum_{j=n_1+n_2+1}^{\frac{n-2}{2}} \mathbb{Z}[w_j, v_j],$$

where τ acts trivially on e_i, permutes w_j and v_j, and the unique (-1)-eigenvector e_0 contributes to

$$\mathrm{H}^1(\langle\tau\rangle, Q^\sigma) = \mathbb{Z}/2.$$

This completes the proof of Proposition 27.

Remark 31. Notice that when $n_1 = n_2 = 0$, the argument above shows

$$\mathrm{H}^1(C_2, Q) = \mathbb{Z}/2,$$

where the C_2 is generated by $(1,2)(3,4)\ldots(n-1,n)$. Computational experiments suggest that

$$\mathrm{H}^1(H, M) = 0,$$

for all cyclic subgroups $H \subset \mathfrak{S}_n$.

Small dimensional examples.
$\mathbf{n = 6}$: By Theorem 1 and the analysis in Section 6 of [CTZ23], we know that the G-action on $\mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$ satisfies **(SP)** iff the G-action is linearizable, thus, nonlinearizable actions are not stably linearizable, as they fail **(SP)**.

Remark 32. This indicates an error in the application in [HT23, p. 295]: Proposition 21 there asserts that the standard and non-standard actions of $\mathfrak{A}_5$ are stably birational, contradicting our cohomology computation. The gap occurs in the sentence: "However, for any finite group G and automorphism $a : G \to G$, precomposing by a yields an action on G-modules; this respects permutation and stably permutation modules."

n = 8: There is a unique (conjugacy class of) $G' = C_2^2 \subset \mathfrak{S}_8$ such that

$$\mathrm{H}^1(G', \mathrm{Pic}(\overline{\mathcal{M}}_{0,8})) = \mathbb{Z}/2,$$

and all $G \subseteq \mathfrak{S}_8$ failing **(H1)** on M contain G'. With `magma`, we find:

- There are 66 (conjugacy classes of) groups containing this G'.
- Of the remaining 230 classes, 96 are contained in the (unique) $\mathfrak{S}_7 \subset \mathfrak{S}_8$, the corresponding actions are linearizable.
- After that, there are 56 contained in the (unique) $\mathfrak{S}_6 \times C_2$ – these actions are birational to an action on a 5-dimensional torus; such actions have been analyzed, over nonclosed fields, in [HY17].
- We are left with 78 classes. Applying [HY17, Algorithm F4] to these classes, we found at least 37 classes of groups $G \subset \mathfrak{S}_8$ having vanishing cohomology but with $\mathrm{Pic}(\overline{\mathcal{M}}_{0,8})$ failing the **(SP)** condition.
- Among the 41 remaining classes, 13 leave invariant an odd cycle. These actions are stably linearizable by Proposition 2.
- There are 28 remaining classes, including a minimal

  $$C_2^2 = \langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,7)(6,8) \rangle,$$

  which (up to conjugation) is contained in every remaining class. The action of this C_2^2 on M yields a permutation module:

  $$\mathbb{Z}[C_2^2]^{19} \oplus \mathbb{Z}[C_2^2/C_2]^3 \oplus \mathbb{Z}[C_2^2/C_2']^3 \oplus \mathbb{Z}[C_2^2/C_2'']^3 \oplus \mathbb{Z}^5.$$

  However, on Q, this action fails **(H1)**, and Theorem 23 is not applicable to any of these cases.

n = 10: We find more minimal groups contributing cohomology:

$$\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,10})) = \mathbb{Z}/2$$

when

- $G = C_2^2 = \langle (1,2)(3,4)(5,6)(7,8), (1,2)(9,10) \rangle$,
- $G = C_2^2 = \langle (1,2)(3,4)(5,6), (5,6)(7,8)(9,10) \rangle$,
- $G = C_2 \times C_4 = \langle (3,6)(8,10), (1,2)(5,9), (1,2)(3,10,6,8)(4,7) \rangle$,
- $G = \mathfrak{D}_4 = \langle (3,6)(8,10), (1,2)(5,9)(8,10), (1,2)(3,10,6,8)(4,7) \rangle$.

6. Three-dimensional case

Next, we give a criterion for rationality of the Segre cubic, exhibit forms failing stable rationality over arbitrary fields admitting a biquadratic extension, and establish stable rationality, provided Q is stably permutation, for the action of the absolute Galois group.

Recall that X_6 denotes the symmetrically linearized GIT quotient with equivalent presentations:

- $(\mathbb{P}^1)^6$ under the diagonal action of SL_2; or
- $\mathrm{Gr}(2,6)$ under the diagonal action of the torus $\mathsf{T} \simeq \mathbb{G}_m^5$.

These have ten isolated nodes, the images of the $D_I, |I| = 3$ under the blow down $\beta : \overline{\mathcal{M}}_{0,6} \to X_6$. These are classically embedded $X_6 \subset \mathbb{P}^4$ as cubic threefolds, known as Segre cubic threefolds [CTZ23]. The remaining boundary divisors $D_I, |I| = 2$ correspond to planes passing through four nodes.

Theorem 33. *Let X be a form of the Segre cubic threefold over a nonclosed field F of characteristic zero, and $\tilde{X}$ its standard resolution of singularities, a form of $\overline{\mathcal{M}}_{0,6}$. Then X is rational over F if and only if the Galois-module* $\mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$ *satisfies* (**SP**).

Proof. This is closely related to the linearizability result [CTZ23, Theorem 1]. The group-theoretic analysis there shows that the only cases where the Galois action on the Picard group is stably permutation are:

- when one of the ten nodes is Galois invariant;
- the Galois action is contained in an $\mathfrak{S}_5$-action associated with permutations of *five* of the marked points;
- the Galois group acts via C_2^2, leaving three planes invariant, and the set of nodes splits into a union of five Galois orbits of length two.

Note that the first two cases are easily shown to be rational: Projecting from a node gives a birational map to $\mathbb{P}^3$, cf. Example 18. And when the action factors through $\mathfrak{S}_5$, the moduli space arises via the Kapranov construction, i.e., is a blow-up of $\mathbb{P}^3$.

Recall that in the third case, the Galois action factors through $\mathfrak{S}_2 \times \mathfrak{S}_4 \subset \mathfrak{S}_6$ corresponding to a partition of the six points conjugate to

$$\{1,2,3,4,5,6\} = \{3,4\} \cup \{1,2,5,6\}.$$

Our $C_2 \times C_2$ action is conjugate to

$$\langle (34), (15)(26) \rangle \subset \mathfrak{S}_6$$

This leaves the boundary divisors D_{34}, D_{15}, and D_{26} invariant. Identifying singular points with the boundary divisors in $\overline{\mathcal{M}}_{0,6}$, the orbits are

$$\begin{aligned}&\{D_{123} = D_{456}, D_{124} = D_{356}\}, \quad \{D_{125} = D_{346}, D_{156} = D_{234}\},\\&\{D_{126} = D_{345}, D_{256} = D_{134}\}, \quad \{D_{135} = D_{246}, D_{145} = D_{236}\},\\&\{D_{136} = D_{245}, D_{146} = D_{235}\}.\end{aligned}$$

We emphasize that the invariant divisor classes reflect boundary divisors defined over F. Indeed, our moduli space has F-rational smooth points so there is no obstruction to descending Galois-invariant divisors.

We claim this moduli space is birational over F to a toric threefold, i.e., an equivariant compactification of a nonsplit torus over F.

Consider the Losev-Manin moduli space associated to the partition above. Specifically, points 3 and 4 are not permitted to collide with other points but points from $\{1, 2, 5, 6\}$ may collide with one another. This is toric by Proposition 6, i.e., the orbits of the homogeneous quartic forms vanishing along $\{1, 2, 5, 6\}$ modulo the torus fixing $\{3, 4\}$. This geometric description is compatible with the Galois action.

Rationality of three-dimensional toric varieties has been settled in [Kun87, Theorem 2]: The variety is rational over F iff the Picard module is stably permutation for the Galois action.

Here is an alternative rationality construction: Pick one of the boundary divisors $D_I, |I| = 2$ invariant under the Galois action. With our choice of indexing this could be D_{34}, D_{15}, or D_{26}; we take D_{34}. This corresponds to a plane $P \subset X$ containing four ordinary singularities, i.e., the images of $D_{34j}, j = 1, 2, 5, 6$. We blow this plane up – inducing a small resolution of the four singularities – and then blow down the proper transform of the plane. This yields a complete intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$ with six singularities, the images of the singularities of X *not* contained in P. Under the $C_2 \times C_2$ action, we have three orbits each with two singular points. For each orbit, the line joining the singularities is contained in $X_{2,2}$. Projecting from that line gives

$$X_{2,2} \overset{\sim}{\dashrightarrow} \mathbb{P}^3;$$

the birationality is classical cf. [CTSSD87, Proposition 2.2]. □

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Department of Mathematics, Brown University Box 1917, 151 Thayer Street, Providence, RI 02912, USA

Email address: `brendan_hassett@brown.edu`

Courant Institute, 251 Mercer Street, New York, NY 10012, USA

Email address: `tschinkel@cims.nyu.edu`

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA

Email address: `zhijia.zhang@cims.nyu.edu`

Courant Institute, 251 Mercer Street, New York, NY 10012, USA