MODULAR SYMBOLS AND EQUIVARIANT BIRATIONAL INVARIANTS

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ABSTRACT. We study relations between the classical modular symbols associated with congruence subgroups and Kontsevich-Pestun-Tschinkel groups $\mathcal{M}_n(G)$ associated with finite abelian groups G.

1. INTRODUCTION

Let G be a finite abelian group, acting regularly and generically freely on a smooth projective variety of dimension $n \geq 2$ over an algebraically closed field of characteristic zero. An *equivariant birational invariant* of such actions was introduced in [4]. It takes values in the abelian group

$$\mathcal{B}_n(G),$$

defined via explicit generators and relations. This group and its generalizations in [6] encode intricate geometric information, leading to new results in equivariant birational geometry, see, e.g., [3], [7], [11] and [12]. A closely related group $\mathcal{M}_n(G)$ which only differs from $\mathcal{B}_n(G)$ by torsions is also studied in [4] from a number-theoretic perspective. The simplicity of the defining relations of $\mathcal{M}_n(G)$ reveals a rich arithmetic nature: it was found that $\mathcal{M}_n(G)$ carry Hecke operators, formal (co-)multiplication maps, and are closely related to Manin's modular symbols for modular forms of weight 2, when $n = 2$.

In this note, we continue the investigation of arithmetic properties of $\mathcal{M}_n(G)$, with a particular focus on their relations with Manin symbols. Our main results are:

- We settle the algebraic structure of $\mathcal{M}_2^-(G)$, a quotient of the group $\mathcal{M}_2(G)$, for any finite abelian group G, see Proposition 3.8. The key ingredient is the construction of an isomorphism between $\mathcal{M}_2^-(G)$ and the $\mathbb{Z}$-module of classical Manin symbols for certain congruence subgroups.

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- We prove a conjecture from [4, Section 11] regarding the $\mathbb{Q}$-ranks of $\mathcal{M}_2(G)\otimes\mathbb{Q}$ when G is cyclic, and generalize the result to any finite abelian group G.

Here is the roadmap of the paper. In Section 2, we recall relevant definitions. In Section 3, we study the connections between Manin symbols and the groups $\mathcal{M}_2^-(G)$. Dimensional formulae for $\mathcal{M}_2(G)\otimes\mathbb{Q}$ are given in Section 4.

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2. Background

Let G be a finite *abelian* group, $G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$ its character group, n a positive integer and

$$\mathcal{S}_n(G)$$

the $\mathbb{Z}$-module freely generated by n-tuples of characters of G:

$$\beta = (b_1, \dots, b_n), \quad \text{such that} \quad \sum_{j=1}^{n} \mathbb{Z} b_j = G^\vee.$$

The group $\mathcal{B}_n(G)$ is defined via the quotient

$$\mathcal{S}_n(G) \to \mathcal{M}_n(G)$$

by the *reordering relation*

(O): for all $\beta = (b_1, \dots, b_n)$ and all $\sigma \in \mathfrak{S}_n$, one has

$$\beta = \beta^\sigma := (b_{\sigma(1)}, \dots, b_{\sigma(n)}),$$

and the *blowup relation*

(B): for $\beta = (b_1, b_2, \dots, b_n)$, one has

$$(b_1, b_2, \dots, b_n) = (0, b_2, \dots, b_n), \quad \text{if} \quad b_1 = b_2,$$

and otherwise

$$\beta = \beta_1 + \beta_2,$$

where

$$\beta_1 := (b_1 - b_2, b_2, \dots, b_n), \quad \beta_2 := (b_1, b_2 - b_1, \dots, b_n), \quad n \geq 2. \tag{2.1}$$

The group $\mathcal{M}_n(G)$ is defined via the quotient

$$\mathcal{S}_n(G) \to \mathcal{M}_n(G)$$

by the reordering relation **(O)**, and the *motivic blowup relation*

(M): for $\beta = (b_1, b_2, b_3, \dots, b_n)$, one has $\beta = \beta_1 + \beta_2$, where β_1 and β_2 are given as in (2.1).

The group $\mathcal{B}_n(G)$ captures equivariant birational invariants [4, Theorem 3], while the simplicity of the relation **(M)** endows $\mathcal{M}_n(G)$ with interesting arithmetic properties. It is shown in [3] that these two groups only differ by torsions, i.e.,

$$\mathcal{B}_n(G) \otimes \mathbb{Q} \simeq \mathcal{M}_n(G) \otimes \mathbb{Q}.$$

From now on, we focus on the study of $\mathcal{M}_n(G)$.

A closely related group $\mathcal{M}_n^-(G)$ is defined as the quotient of $\mathcal{S}_n(G)$ by **(O)**, **(M)** and the *anti-symmetry relation* **(A)**:

(A): $(b_1, \dots, b_n) = -(-b_1, \dots, b_n)$, for all generating symbols β.

For clarity, we distinguish symbols in $\mathcal{M}_n(G)$ and $\mathcal{M}_n^-(G)$ with the following notation:

- $\langle b_1, \dots, b_n \rangle \in \mathcal{M}_n(G)$,
- $\langle b_1, \dots, b_n \rangle^- \in \mathcal{M}_n^-(G)$.

Remark 2.1. The original definition of relations **(B)** and **(M)** in [4] is more involved, but is equivalent to the version above, by [3, Proposition 2.1].

When $n = 1$, we have

$$\mathcal{M}_1(G) = \begin{cases} \mathbb{Z}^{\phi(N)} & G = \mathbb{Z}/N, N \geq 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\phi(n)$ is Euler's totient function.

When $n = 2$, $\mathcal{M}_2(G)$ can be nontrivial for cyclic and bi-cyclic groups. Below, we present results of numerical computations of $\mathbb{Q}$-ranks of $\mathcal{M}_2(G)$ and $\mathcal{M}_2(G)^-$. Let

$$\mathcal{M}_2(G)_{\mathbb{Q}} := \mathcal{M}_2(G) \otimes \mathbb{Q}, \quad \text{and} \quad \mathcal{M}_2^-(G)_{\mathbb{Q}} := \mathcal{M}_2^-(G) \otimes \mathbb{Q}.$$

In the following tables, d and d^- denote respectively

$$\dim_{\mathbb{Q}}(\mathcal{M}_2(G)_{\mathbb{Q}}) \quad \text{and} \quad \dim_{\mathbb{Q}}(\mathcal{M}_2^-(G)_{\mathbb{Q}}).$$

When $G = C_N$ is cyclic, we have

N	2	3	4	5	7	9	11	12	13	16	17	19	23	29	31	37
d	0	1	1	2	3	5	6	7	8	10	13	16	23	36	41	58
d^-	0	0	0	0	0	1	1	2	2	3	5	7	12	22	16	40

When $G = C_{N_1} \times C_{N_2}$ is bi-cyclic, we have

N_1	2	2	2	2	2	2	3	3	3	3	4	4	4	5	6
N_2	2	4	6	8	10	16	6	3	9	27	8	16	32	25	36
d	0	2	3	6	7	21	15	7	37	235	33	105	353	702	577
d^-	0	0	0	1	1	9	7	3	19	163	17	65	257	502	433

In particular, when $G = C_p \times C_p$, for prime p, we have

p	5	7	11	13	17	19	23	29	31	37
d	46	159	855	1602	4424	6759	14047	34314	44415	88254
d^-	22	87	555	1098	3272	5139	11143	28434	37215	75942

It was discovered and proved in [4] that

$$\dim(\mathcal{M}_2^-(C_N)_{\mathbb{Q}}) = \begin{cases} 1 - \frac{\phi(N)+\phi(N/2)}{2} + \frac{N\cdot\phi(N)}{24} \cdot \prod_{p|N}(1+\frac{1}{p}) & N \text{ even}, \\ 1 - \frac{\phi(N)}{2} + \frac{N\cdot\phi(N)}{24} \cdot \prod_{p|N}(1+\frac{1}{p}) & N \text{ odd}. \end{cases}$$

The proof is based on an isomorphism between $\mathcal{M}_2^-(C_N)_{\mathbb{Q}}$ and the space of modular symbols of the congruence subgroups $\Gamma_1(N)$. From the tables above, we speculate the following identities

$$\dim(\mathcal{M}_2(C_p \times C_p)_{\mathbb{Q}}) \stackrel{?}{=} \frac{(p-1)(p^3+6p^2-p+6)}{24},$$

$$\dim(\mathcal{M}_2^-(C_p \times C_p)_{\mathbb{Q}}) \stackrel{?}{=} \frac{(p-1)(p^3-p+12)}{24},$$

also signaling a strong connection to modular forms. The remaining part of this paper is dedicated to a proof of these two identities in the general setting.

First, observe that the common factor $(p-1)$ indicates that the structure of $\mathcal{M}_2(G)$ and $\mathcal{M}_2^-(G)$ can be simplified when G is a bi-cyclic group. We explain in detail the simplification for $\mathcal{M}_2^-(G)$ below. The same argument also applies to $\mathcal{M}_2(G)$.

Bi-cyclic groups. Let $G = C_N \times C_{MN}$ be a finite bi-cyclic group. By definition, the $\mathbb{Z}$-module $\mathcal{M}_2^-(G)$ is generated by symbols

$$\beta := \langle (a_1, b_1), (a_2, b_2) \rangle^-$$

such that

$$a_1, a_2 \in C_N, \quad b_1, b_2 \in C_{MN}, \quad \mathbb{Z}(a_1, b_1) + \mathbb{Z}(a_2, b_2) = C_N \times C_{MN},$$

and subject to relations

- $\beta = \langle (a_2, b_2), (a_1, b_1) \rangle^-$,
- $\beta = \langle (a_1 - a_2, b_1 - b_2), (a_2, b_2) \rangle^- + \langle (a_1, b_1), (a_2 - a_1, b_2 - b_1) \rangle^-$,
- $\beta = -\langle (-a_1, -b_1), (a_2, b_2) \rangle^-$.

Formally, we can also denote β by a 2×2 matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

and assign a determinant:

$$\det(\beta) := a_1 b_2 - a_2 b_1 \in (\mathbb{Z}/N)^\times,$$

where the operation takes place modulo N. From the defining relations **(O)**, **(M)** and **(A)**, one can see that the linear combinations of symbols with the same determinant up to ± 1 form a submodule of $\mathcal{M}_2^-(G)$. More precisely, for $k \in (\mathbb{Z}/N)^\times$, let

$$\mathcal{S}_{2,k}(G) \tag{2.2}$$

be the *finite set* consisting of matrices/symbols

$$\beta := \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \langle (a_1, b_1), (a_2, b_2) \rangle^-$$

such that

- $(a_1, b_1), (a_2, b_2) \in (C_N \times C_{MN})^\vee$,
- $\mathbb{Z}(a_1, b_1) + \mathbb{Z}(a_2, b_2) = (C_N \times C_{MN})^\vee$,
- $\det(\beta) = k \pmod{N}$,

and

$$\mathcal{M}_{2,k}^-(G)$$

be the $\mathbb{Z}$-module freely generated by elements in the set

$$\mathcal{S}_{2,k}(G) \cup \mathcal{S}_{2,-k}(G)$$

subject to relations **(O)**, **(M)** and **(A)**. It follows that $\mathcal{M}_{2,k}^-(G)$ can be naturally identified as a submodule of $\mathcal{M}_2^-(G)$. Moreover, the algebraic structure of $\mathcal{M}^-_{2,k}(G)$ is independent of k: consider the maps

$$\mathcal{M}^-_{2,1}(G) \to \mathcal{M}^-_{2,k}(G), \quad \langle (a_1,b_1),(a_2,b_2)\rangle^- \mapsto \langle (ka_1,b_1),(ka_2,b_2)\rangle^- \ ;$$

$$\mathcal{M}^-_{2,k}(G) \to \mathcal{M}^-_{2,1}(G), \quad \langle (a_1,b_1),(a_2,b_2)\rangle^- \mapsto \langle (a_1/k,b_1),(a_2/k,b_2)\rangle^-.$$

These maps respect the defining relations and are inverse to each other. It follows that we have isomorphisms of $\mathbb{Z}$-modules, when $N \geq 3$:

$$\mathcal{M}^-_2(G) \simeq \bigoplus_{k\in(\mathbb{Z}/N)^\times/\langle\pm1\rangle} \mathcal{M}^-_{2,k}(G) \simeq \bigoplus_{\frac{\phi(N)}{2}\text{ copies}} \mathcal{M}^-_{2,1}(G).$$

Multiplication and Co-multiplication. Given an exact sequence of finite abelian groups

$$0 \to G' \to G \to G'' \to 0,$$

consider the dual sequence of their character groups

$$0 \to A'' \to A \to A' \to 0.$$

For all integers $n = n' + n'', n', n'' \geq 1$, one can define a $\mathbb{Z}$-bilinear *multiplication* map

$$\nabla : \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \to \mathcal{M}_n(G)$$

given on the generators by

$$\langle a'_1, \dots, a'_{n'}\rangle \otimes \langle a''_1, \dots, a''_{n''}\rangle \to \sum \langle a_1, \dots, a_{n'}, a''_1, \dots, a''_{n''}\rangle,$$

where the sum is over all possible lifts $a_i \in A$ of $a'_i \in A'$; and $a''_i \in A$ are understood via the embedding $A'' \hookrightarrow A$.

Dual to this construction is the $\mathbb{Z}$-linear *co-multiplication* map when G'' is non-trivial:

$$\Delta : \mathcal{M}_n(G) \to \mathcal{M}_{n'}(G') \otimes \mathcal{M}^-_{n''}(G''). \tag{2.3}$$

This map is defined on the generators by

$$\langle a_1, \cdots, a_n\rangle \mapsto \sum \langle a_{I'} \mod A''\rangle \otimes \langle a_{I''}\rangle^-,$$

where the sum is over all partition of $\{1, \dots, n\} = I' \cup I''$ such that

- $\#I' = n', \quad \#I'' = n''$;
- for all $j \in I''$, $a_j \in A'' \subset A$; and for any $i \in I'$, $a_i \mod A''$ is understood as projection of $a_i \in A$ in A/A'';
- the elements $a_j, j \in I''$, span A''.

The correctness of ∇ and Δ can be verified directly [4]; they maps also descend to well-defined $\mathbb{Z}$-module homomorphisms

$$\nabla^- : \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G'') \to \mathcal{M}^-_n(G),$$
$$\Delta^- : \mathcal{M}^-_n(G) \to \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G'').$$

3. Congruence subgroups and Modular Symbols

Congruence subgroups. Connections between $\mathcal{M}^-_2(C_N)$ and a classical congruence subgroup

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad N \geq 2,$$

were discovered in [4, Section 11]. To extend their results to bi-cyclic groups, we introduce a new family of congruence subgroups

(3.1)

$$\Gamma(N, MN) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{array}{ll} a \equiv 1 & (\mathrm{mod}\ N) \\ b \equiv 0 & (\mathrm{mod}\ N) \\ c \equiv 0 & (\mathrm{mod}\ MN) \\ d \equiv 1 & (\mathrm{mod}\ MN) \end{array} \right\}, \quad N \geq 2.$$

To see that $\Gamma(N, MN)$ is indeed a congruence subgroup, one can check that the definition (3.1) forces

$$a \equiv 1 \mod MN,$$

leading to an equivalent description of $\Gamma(N, MN)$:

(3.2)

$$\Gamma(N, MN) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{array}{ll} a \equiv 1 & (\mathrm{mod}\ MN) \\ b \equiv 0 & (\mathrm{mod}\ N) \\ c \equiv 0 & (\mathrm{mod}\ MN) \\ d \equiv 1 & (\mathrm{mod}\ MN) \end{array} \right\}, \quad N \geq 2.$$

Using (3.2), one can easily verify the following inclusion relations

$$\mathrm{SL}_2(\mathbb{Z}) \supset \Gamma_1(MN) \supset \Gamma(N, MN) \supset \Gamma(MN)$$

and conclude that $\Gamma(N, MN)$ is a congruence subgroup.

Lemma 3.1. *$[\Gamma(N, MN) : \Gamma(MN)] = M$.*

Proof. Consider the surjective group homomorphism:

$$\Gamma(N, MN) \to \mathbb{Z}/M\mathbb{Z}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{b}{N} \pmod{M}.$$

The kernel of the homomorphism is $\Gamma(MN)$. In particular,

$$\Gamma(N, MN)/\Gamma(MN) \simeq \mathbb{Z}/m\mathbb{Z}.$$

$\square$

To study the space of Manin symbols associated with $\Gamma(N, MN)$, one needs a description of the right cosets $\Gamma(N, MN) \setminus \mathrm{SL}_2(\mathbb{Z})$. Now, we show that $\Gamma(N, MN)\backslash\mathrm{SL}_2(\mathbb{Z})$ coincides with the set $\mathcal{S}_{2,1}(C_N \times C_{MN})$ introduced in (2.2). Consider a natural map:

$$\begin{gathered}\Gamma(N, MN) \setminus \mathrm{SL}_2(\mathbb{Z}) \to \mathcal{S}_{2,1}(C_N \times C_{MN}), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \mod N & b \mod N \\ c \mod MN & d \mod MN \end{pmatrix}.\end{gathered} \tag{3.3}$$

The correctness of (3.3) as a bijection between finite sets follows from elementary computations. Moreover, we have the following lemmas.

Lemma 3.2. *For* $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $i = 1, 2$, *one has*

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \equiv \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \pmod{\Gamma(N, MN)}$$

if and only if $\begin{cases} a_1 \equiv a_2 \pmod{N}, & c_1 \equiv c_2 \pmod{MN}, \\ b_1 \equiv b_2 \pmod{N}, & d_1 \equiv d_2 \pmod{MN}. \end{cases}$

Proof. Basic modular arithmetic, as in [1, Lemma 3.1]. $\square$

Lemma 3.3. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, *and* $a', b', c', d' \in \mathbb{Z}$ *such that*

$$\begin{cases} a' \equiv a \pmod{N}, & c' \equiv c \pmod{MN}, \\ b' \equiv b \pmod{N}, & d' \equiv d \pmod{MN}, \end{cases}$$

with $0 \le a', b' < N$ *and* $0 \le c', d' < MN$. *Then we have*

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{S}_{2,1}(C_N \times C_{MN}).$$

Proof. It suffices to check $\mathbb{Z}(a', c') + \mathbb{Z}(b', d') = C_N \times C_{MN}$. Indeed,

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a'd - b'c & -a'b + ab' \\ c'd - d'c & -c'b + ad' \end{pmatrix} \in \Gamma(N, MN),$$

since $ad - bc = 1$. This shows (a', c') and (b', d') generate the generators $(0, 1)$ and $(1, 0) \in C_N \times C_{MN}$. $\square$

Proposition 3.4. *The map* (3.3) *is a well-defined bijection between finite sets.*

Proof. Lemmas 3.2 and 3.3 implies (3.3) is a well-defined injection. It suffices to show it is also surjective. Let

$$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{S}_{2,1}(C_N \times C_{MN}).$$

By definition, one has $ad - bc = 1 + l_1 N$ for some l_1. The generating condition implies that $\gcd(c, d, M) = 1$. So there exists $k_1, k_2 \in C_M$ such that

$$k_1 d - k_2 c = -l_1 \pmod{M}.$$

Put

$$\gamma = \begin{pmatrix} a + k_1 N & b + k_2 N \\ c & d \end{pmatrix},$$

One computes that $\det(\gamma) \equiv 1 \pmod{MN}$, i.e., $\gamma \in \mathrm{SL}_2(\mathbb{Z}/MN)$. Let $\overline{\gamma}$ be a lift of γ in $\mathrm{SL}_2(\mathbb{Z})$ under the surjection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/MN)$. The lift $\overline{\gamma}$ is mapped to β under the map (3.3), proving surjectivity. $\square$

Modular symbols. We follow Manin's definition of modular symbols [9, Section 1.7]. Given the bijection (3.3), the space $\mathbb{M}_2(\Gamma(N, MN))$ of modular symbols of weight 2 for $\Gamma(N, MN)$ is defined via generators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{S}_{2,1}(C_N \times C_{MN})$$

subject to relations

(1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = 0,$

(2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a+b & -a \\ c+d & -c \end{pmatrix} + \begin{pmatrix} b & -a-b \\ d & -c-d \end{pmatrix} = 0,$

(3) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ or $\begin{pmatrix} a+b & -a \\ c+d & -c \end{pmatrix}$.

Relation (3) guarantees that the space of modular symbols is torsion-free. But for $\Gamma(N, MN)$, relation (3) is redundant as the condition in

(3) is never satisfied. Using relation (1), relation (2) can be rewritten:

$$\begin{aligned} 0 &\overset{(2)}{=} \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} + \begin{pmatrix} b-a & -b \\ d-c & -d \end{pmatrix} + \begin{pmatrix} -a & a-b \\ -c & c-d \end{pmatrix} \\ &\overset{(1)}{=} -\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a-b & b \\ c-d & d \end{pmatrix} + \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix}. \end{aligned}$$

Equivalently, one can rewrite defining relations of $\mathbb{M}_2(\Gamma(N, MN))$ as

(R1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$,

(R2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-b & b \\ c-d & d \end{pmatrix} + \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix}$.

Proposition 3.5. *The $\mathbb{Z}$-modules $\mathcal{M}^-_{2,1}(C_N \times C_{MN})$ and $\mathbb{M}_2(\Gamma(N, MN))$ are isomorphic when $N \in \mathbb{Z}_{>2}$ and $M \in \mathbb{Z}_{\geq 1}$.*

Proof. When $N > 2$, consider the map

$$\mathcal{M}^-_{2,1}(C_N \times C_{MN}) \to \mathbb{M}_2(\Gamma(N, MN)), \tag{3.4}$$

$$\langle (a_1, b_1), (a_2, b_2) \rangle^- \mapsto \begin{cases} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} & \text{if } a_1b_2 - a_2b_1 = 1 \pmod N, \\ \begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} & \text{if } a_1b_2 - a_2b_1 = -1 \pmod N. \end{cases}$$

The correctness of the map (3.4) can be verified directly:

- It is compatible with the relation **(O)** by construction.
- Relation **(M)** is identical to relation **(R2)** and preserves the determinants of the symbols.
- It is compatible with relation **(A)** due to the defining relation **(R1)** of $\mathbb{M}_2(\Gamma(N, MN))$.

Similarly, one can check that the map given by

$$\mathbb{M}_2(\Gamma(N, MN)) \to \mathcal{M}^-_{2,1}(C_N \times C_{MN}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \langle (a, c), (b, d) \rangle^-$$

is a well-defined inverse homomorphism to (3.4). ∎

When $N = 2$, the map (3.4) in the proof above is not well-defined as ± 1 are not distinguishable modulo 2. But in this case, the generating sets of $\mathcal{M}^-_2(C_2 \times C_{2M})$ and $\mathbb{M}_2(\Gamma(2, 2M))$ coincide: $\mathcal{S}_2(C_2 \times C_{2M})$ is simply the free $\mathbb{Z}$-module generated by elements in $\mathcal{S}_{2,1}(C_2 \times C_{2M})$. We can then consider the $\mathbb{Z}$-module

$$\mathbb{M}_2^-(\Gamma(2, 2M))$$

defined as the quotient of $\mathcal{S}_2(C_2 \times C_{2M})$ by relations **(R1)** and **(R2)**, i.e., the quotient of $\mathbb{M}_2(\Gamma(2, 2M))$ by

$$(\mathbf{O}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

Proposition 3.6. *The $\mathbb{Z}$-modules $\mathcal{M}_2^-(C_2 \times C_{2M})$ and $\mathbb{M}_2^-(\Gamma(2, 2M))$ are isomorphic for all integers $M \in \mathbb{Z}_{\geq 1}$.*

Proof. With the presence of **(O)**, the relation **(R1)** is identical to **(A)**. It follows that relations **(R1)** and **(R2)** generate the same submodule of $\mathcal{S}_2(C_2 \times C_{2M})$ as **(M)** and **(A)** does. □

It is classically known that $\mathbb{M}_2(\Gamma(N, MN))$ can be identified as

$$H_1(\overline{X(N, MN)}, \mathbb{Z}),$$

the first homology group of the complex modular curve $X(N, MN)$ compactified with respect to the cusps [9, Theorem 1.9]. We follow definitions in [10, Chapter 1.3]:

- $X(N, MN) := \Gamma(N, MN)\backslash\mathfrak{h}$, where $\mathfrak{h}$ is the upper half-plane,
- $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$, cusps are the elements of $\mathbb{P}^1(\mathbb{Q})/\Gamma(N, MN)$,
- $\mathfrak{h}^* := \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ is the extended upper half-plane,
- $\overline{X(N, MN)} := \Gamma(N, MN)\backslash\mathfrak{h}^*$.

In particular, a symbol $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to the image in $X(N, MN)$ of the geodesic path from a/c to b/d, where a, b, c and d are naturally considered as integers. Moreover, $\mathbb{M}_2^-(\Gamma(2, 2M))$ can be identified as the (-1)-eigenspace of the antiholomorphic involution on $X(2, 2M)$ given by the map $\tau \mapsto -\bar{\tau}, \tau \in \mathcal{H}$, on the universal cover. On modular symbols, ι takes the form

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \overset{\textbf{(R1)}}{=} - \begin{pmatrix} -b & -a \\ d & c \end{pmatrix} \overset{\text{mod } 2}{=} - \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

This forces a 2-torsion in $\mathbb{M}_2^-(\Gamma(2, 2M))$ each time a cusp different from ∞ is fixed by ι.

Concretely, these imply that

$$\begin{aligned}(3.5)\qquad \dim(\mathbb{M}_2(\Gamma(N,MN)_{\mathbb{Q}}) &= 2g(N,MN)+\varepsilon_\infty(N,MN)-1,\\ \dim(\mathbb{M}_2^-(\Gamma(2,2M)_{\mathbb{Q}}) &= g(2,2M)+\frac{\varepsilon_\infty(2,2M)-\varepsilon(2,2M)}{2},\\ \mathrm{Tors}(\mathbb{M}_2(\Gamma(N,MN)) = 0,\ &\mathrm{Tors}(\mathbb{M}_2^-(\Gamma(2,2M))) = (\mathbb{Z}/2)^{\varepsilon(2,2M)-1},\end{aligned}$$

where

- $g(N,MN)$ is the genus of $\overline{X(N,MN)}$ as a compact Riemann surface,
- $\varepsilon_\infty(N,MN)$ is the number of cusps, i.e., the cardinality of $\mathbb{P}^1(\mathbb{Q})/\Gamma(N,MN)$.
- $\varepsilon(2,2M)$ is the number of cusps fixed by the anti-holomorphic involution on $X(2,2M)$.
- Tors refers to the torsion subgroup.

Remark 3.7. In [5], the authors introduced birational invariants of orbifolds $\mathcal{X}$, and gave an interpretation of the invariant groups in dimension 2 in terms of quotients of the orbifold cohomology of the modular curve $X_0(p)$, see [5, Section 4 and 5]. In [8], a more refined invariant when $\mathcal{X}$ is a toric orbifold surface was introduced and related to the full orbifold cohomology of $X_0(p)$. Both of these two works established connections between the groups of birational invariants and Manin symbols, generalizing the observation in [4]. Our treatment of $\mathcal{M}_2^-(G)$ above has a similar nature.

Now we compute each term appearing in (3.5). It is well-known that

$$|\mathbb{P}^1(\mathbb{Q})/\Gamma(MN)| = \frac{M^2N^2}{2}\cdot\prod_{p|MN}(1-p^{-2}).$$

Recall from Lemma 3.1 that $[\Gamma(N,MN):\Gamma(MN)] = M$. Then

$$\varepsilon_\infty(N,MN) = \frac{MN^2}{2}\cdot\prod_{p|MN}(1-p^{-2}).$$

Using the genus formula of modular curves [2, Theorem 3.1.1], we obtain for $N\geq 3$ and $M\geq 1$:

$$g(N,MN) = 1+\frac{MN^2(MN-6)}{24}\cdot\prod_{p|MN}(1-p^{-2}).$$

To compute $\varepsilon(2, 2M)$, first observe that

$$\Gamma(2, 2M) = \bigcup_{j \in \mathbb{Z}/M} \Gamma(2M) \cdot \begin{pmatrix} 1 & 2j \\ 0 & 1 \end{pmatrix}.$$

Two reduced rational numbers a/c and a'/c' lie in the same equivalence class of cusps in $\mathbb{P}^1(\mathbb{Q})/\Gamma(2, 2M)$ if and only if

$$\frac{a}{c} \equiv \frac{a'}{c'} + 2j \pmod{\Gamma(2M)} \quad \text{for some } j \in \mathbb{Z}/M,$$

if and only if [2, Proposition 3.8.3]

$$(a', c') \equiv \pm(a + 2jc, c) \pmod{2M}, \quad \text{for some } j \in \mathbb{Z}/M.$$

A counting argument leads to

$$\varepsilon(2, 2M) = 2\phi(M) + \phi(2M), \quad M > 2.$$

We summarize the computations above and results in [4, Section 11]:

Proposition 3.8. *Let G be a finite abelian group. Then*

- *When $G = C_N$, $N \geq 5$ and N is even,*

$$\dim(\mathcal{M}_2^-(G)_{\mathbb{Q}}) = 1 - \frac{\phi(N) + \phi(N/2)}{2} + \frac{N \cdot \phi(N)}{24} \cdot \prod_{p|N} (1 + \frac{1}{p}),$$

$$\mathrm{Tors}(\mathcal{M}_2^-(G)) = (\mathbb{Z}/2)^{\phi(N)+\phi(N/2)-1}.$$

- *When $G = C_N$, $N \geq 5$ and N is odd,*

$$\dim(\mathcal{M}_2^-(G)_{\mathbb{Q}}) = 1 - \frac{\phi(N)}{2} + \frac{N \cdot \phi(N)}{24} \cdot \prod_{p|N} (1 + \frac{1}{p}),$$

$$\mathrm{Tors}(\mathcal{M}_2^-(G)) = (\mathbb{Z}/2)^{\phi(N)-1}.$$

- *When $G = C_2 \times C_{2M}$, $M \geq 3$,*

$$\dim(\mathcal{M}_2^-(G)_{\mathbb{Q}}) = 1 - \phi(M) - \frac{\phi(2M)}{2} + \frac{M^2}{3} \cdot \prod_{p|MN} (1 - p^{-2}),$$

$$\mathrm{Tors}(\mathcal{M}_2^-(G)) = (\mathbb{Z}/2)^{2\phi(M)+\phi(2M)-1}.$$

- *When $G = C_N \times C_{MN}$, $N \geq 3$, $M \geq 1$,*

$$\dim(\mathcal{M}_2^-(G)_{\mathbb{Q}}) = \frac{\phi(N)}{2} \left(1 + \frac{M^2 N^3}{12} \cdot \prod_{p|MN} (1 - p^{-2}) \right),$$

$$\mathrm{Tors}(\mathcal{M}_2^-(G)) = 0.$$

- $\mathcal{M}_2^-(C_2) = \mathcal{M}_2^-(C_3) = \mathbb{Z}/2, \quad \mathcal{M}_2^-(C_4) = \mathcal{M}_2^-(C_2^2) = (\mathbb{Z}/2)^2.$

- $\mathcal{M}_2^-(G) = 0$ *if* G *is not in any of the cases above.*

4. Dimensional Formulae

Consider the natural quotient map of $\mathcal{M}_2(G)$ by relation **(A)**

$$\mu^- : \mathcal{M}_2(G) \to \mathcal{M}_2^-(G).$$

In this section, we determine the $\mathbb{Q}$-rank of the kernel of μ^-. First, we introduce an auxiliary group

$$\mathcal{M}_1^+(G)$$

defined as the quotient of $\mathcal{M}_1(G) = \mathcal{S}_1(G)$ by the relation

$$(\mathbf{P}) : \langle a_1 \rangle = \langle -a_1 \rangle,$$

and denote by $\langle a_1 \rangle^+ \in \mathcal{M}_1^+(G)$ the image of $\langle a \rangle \in \mathcal{M}_1(G)$ under the natural projection

$$\mu^+ : \mathcal{M}_1(G) \to \mathcal{M}_1^+(G).$$

We have

$$\mathcal{M}_1^+(G) = \begin{cases} \mathbb{Z}^{\frac{\phi(N)}{2}} & G = C_N, N > 2, \\ \mathbb{Z} & G = C_N, N = 1, 2, \\ 0 & \textit{otherwise}. \end{cases}$$

Given a finite abelian group G and a subgroup $G' \subsetneq G$ such that $G' = C_d$ for some $d \in \mathbb{Z}_{\geq 1}$, there is a map

$$\nu_{G'} : \mathcal{M}_n(G) \to \mathcal{M}_1^+(G') \otimes \mathcal{M}_{n-1}^-(G''), \tag{4.1}$$

obatined as the composition of the co-multiplication map and μ^+. Notice that $\nu_{G'}$ is non-trivial only when G' is cyclic. Put

$$\nu := \bigoplus_{G' \subsetneq G} \nu_{G'},$$

where the sum runs through all proper cyclic subgroups (including the trivial one) $G' \subsetneq G$. We will show that the restriction of ν to

$$\mathcal{K}_n(G) := \ker\big(\mathcal{M}_n(G) \to \mathcal{M}_n^-(G)\big)$$

is an isomorphism over $\mathbb{Q}$. Formally, consider the map

$$\nu_{\mathcal{K}_n(G)} : \mathcal{K}_n(G) \to \bigoplus_{G' \subsetneq G} \mathcal{M}_1^+(G') \otimes \mathcal{M}_{n-1}^-(G/G'). \tag{4.2}$$

We construct an inverse of $\nu_{\mathcal{K}_n(G)}$ over $\mathbb{Q}$:

$$\psi : \bigoplus_{G' \subsetneq G} \mathcal{M}_1^+(G') \otimes \mathcal{M}_{n-1}^-(G/G') \to \mathcal{K}_n(G) \tag{4.3}$$

in the following way:

Let $G' = C_d \subsetneq G$ be a cyclic subgroup of G. We denote by

$$A, A', \text{and } A''$$

the character group of

$$G, G', \text{and } G/G'$$

respectively. For any

$$\langle a \rangle^+ \in \mathcal{M}_1^+(C_{d_i})$$

and

$$\langle b_1, b_2, \ldots, b_{n-1} \rangle^- \in \mathcal{M}_{n-1}^-(G/G'),$$

we set

$$\boldsymbol{b} := \{b_1, b_2, \ldots, b_{n-1}\},$$

and

$$\boldsymbol{\omega}(a, \boldsymbol{b}) := \langle a \rangle^+ \otimes \langle b_1, \ldots, b_{n-1} \rangle^- \in \mathcal{M}_1^+(G') \otimes \mathcal{M}_{n-1}^-(G/G').$$

Find an arbitrary lift $a' \in A$ of $a \in A'$ and put

$$\boldsymbol{\gamma}(a, \boldsymbol{b}) := \langle a', b_1, \ldots, b_{n-1} \rangle + \langle -a', b_1, \ldots, b_{n-1} \rangle \in \mathcal{K}_n(G),$$

where b_i are understood via the embedding $A'' \subset A$. Then we define

$$\psi(\boldsymbol{\omega}(a, \boldsymbol{b})) := \frac{1}{2}\, \boldsymbol{\gamma}(a, \boldsymbol{b}). \tag{4.4}$$

Notice that ψ is defined over $\mathbb{Q}$. It is not hard to see that

$$\nu_{G'}\big(\frac{1}{2}\, \boldsymbol{\gamma}(a, \boldsymbol{b}) \big) = \boldsymbol{\omega}(a, \boldsymbol{b}) \tag{4.5}$$

and the map ψ is compatible with relations **(O)** and **(M)**. It remains to check that the construction is independent of the lift a' and ψ is also compatible with relations **(P)** and **(A)** as a homomorphism between $\mathbb{Q}$-vector spaces.

Lemma 4.1. *With the notation above, the definition of ψ is independent of the choice of the lift a' of a.*

Proof. Let $a_1, a_2 \in A$ be two lifts of $a \in A'$, i.e., there exists $g \in A''$ such that $a_2 = a_1 + g$. Relations **(S)** and **(M)** imply that

$$\langle a_1, b_1, \ldots \rangle = \langle a_1 - b_1, b_1, \ldots \rangle + \langle a_1, b_1 - a_1, \ldots \rangle,$$
$$\langle b_1 - a_1, b_1, \ldots \rangle = \langle -a_1, b_1, \ldots \rangle + \langle a_1, b_1 - a_1, \ldots \rangle.$$

Taking the difference between the two lines above, one has

$$\langle a_1, b_1, \ldots \rangle + \langle -a_1, b_1, \ldots \rangle = \langle a_1 - b_1, b_1, \ldots \rangle + \langle b_1 - a_1, b_1, \ldots \rangle.$$

Iterating this process with b_i, we obtain

$$\langle a_1, b_1, \dots\rangle + \langle -a_1, b_1, \dots\rangle = \langle a_1 - \sum_{i=1}^{n-1} m_i b_i, b_1, \dots\rangle + \langle \sum_{i=1}^{n-1} m_i b_i - a_1, b_1, \dots\rangle$$

where $m_i \in \mathbb{Z}_{\geq 0}$ for all i. Since b_i generate A'', we conclude that

$$\langle a_1, b_1, \dots\rangle + \langle -a_1, b_1, \dots\rangle = \langle a_2, b_1, \dots\rangle + \langle -a_2, b_1, \dots\rangle.$$

$\square$

Notice that Lemma 4.1 also implies that ψ is compatible with the relation **(P)**. Indeed, let a' be a lift of $a \in A'$ in A and a'' a lift of $-a \in A'$ in A. Then $a'' = -a' + g$ for some $g \in A''$ and thus $\boldsymbol{\gamma}(a, \boldsymbol{b}) = \boldsymbol{\gamma}(-a, \boldsymbol{b})$. The compatibility of ψ with the relation **(A)** is reduced to the following lemma.

Lemma 4.2. *Let $n \geq 2$ be an integer, G be a finite abelian group and $\langle a_1, \dots, a_n\rangle$ be any generating symbol of $\mathcal{M}_n(G)$, one has*

$$\sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \langle \varepsilon_1 a_1, \varepsilon_2 a_2, a_3, \dots, a_n\rangle = 0 \in \mathcal{M}_n(G) \otimes \mathbb{Q}.$$

Proof. For simplicity, we denote the sum in the assertion by

$$\delta(\langle a_1, \dots, a_n\rangle) := \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \langle \varepsilon_1 a_1, \varepsilon_2 a_2, a_3, \dots, a_n\rangle.$$

Consider a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\delta(\langle a_1, \dots, a_n\rangle)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \delta(\langle a_1, a_2, a_3, \dots, a_n\rangle) = \delta(\langle aa_1 + ba_2, ca_1 + da_2, a_3, \dots, a_n\rangle).$$

Equivalently, we can view this as an action of $\mathrm{SL}_2(\mathbb{Z})$ on $(G^\vee)^2$. The action is in fact trivial in $\mathcal{M}_n(G)$. It suffices to check this on generators of $\mathrm{SL}_2(\mathbb{Z})$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

By symmetry, it is clear that

$$\delta(\langle a_1, a_2, \dots, a_n\rangle) = \delta(\langle a_2, -a_1, \dots, a_n\rangle).$$

On the other hand, one has

$$\begin{aligned}
&\delta(\langle a_1+a_2, a_1, a_3, \ldots, a_n\rangle)\\
=\ &\langle a_1+a_2, a_1, \ldots\rangle + \langle -a_1-a_2, -a_1, \ldots\rangle + \langle a_1+a_2, -a_1, \ldots\rangle +\\
&\langle -a_1-a_2, a_1, \ldots\rangle\\
&\text{applying } \mathbf{(M)} \text{ to the first two terms above}\\
=\ &\langle a_1, a_2, \ldots\rangle + \langle -a_1, -a_2, \ldots\rangle + \langle a_1+a_2, -a_2, \ldots\rangle +\\
&\langle -a_1-a_2, a_1, \ldots\rangle + \langle -a_1-a_2, a_2, \ldots\rangle + \langle a_1+a_2, -a_1, \ldots\rangle\\
&\text{applying } (\mathbf{M}) \text{ to the last four terms above}\\
=\ &\langle a_1, a_2, \ldots\rangle + \langle -a_1, -a_2, \ldots\rangle + \langle a_1, -a_2, \ldots\rangle + \langle -a_1, a_2, \ldots\rangle\\
=\ &\delta(\langle a_1, a_2, \ldots, a_n\rangle).
\end{aligned}$$

Consider

$$S := \sum_{a,b} \langle a, b, a_3, \ldots, a_n\rangle, \tag{4.6}$$

where the sum runs over the $\mathrm{SL}_2(\mathbb{Z})$-orbit of (a_1, a_2) in $(G^\vee)^2$. Observe that the orbit is finite as G is a finite group. Applying relation **(M)** to each term in the sum, one finds that

$$\begin{aligned}
S &= \sum_{a,b} \langle a-b, b, a_3, \ldots, a_n\rangle + \langle a, b-a, a_3, \ldots, a_n\rangle\\
&= 2\sum_{a,b} \langle a, b, a_3, \ldots, a_n\rangle
\end{aligned}$$

since

$$\begin{pmatrix} a-b \\ b \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a \\ b-a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

Similarly, averaging δ over this orbit leads to

$$\begin{aligned}
&\sum_{a,b} \delta(\langle a, b, a_3, \ldots, a_n\rangle)\\
=\ &\sum_{a,b} \langle a, b, \ldots\rangle + \langle -a, b, \ldots\rangle + \langle a, -b, \ldots\rangle + \langle -a, b, \ldots\rangle\\
&\text{applying (4.6) to each term}\\
=\ &2 \cdot \sum_{a,b} \delta(\langle a, b, a_3, \ldots, a_n\rangle).
\end{aligned}$$

Recall that δ is invariant under the $\mathrm{SL}_2(\mathbb{Z})$-action. We conclude that

$$\delta(\langle a_1, \ldots, a_n\rangle) = 0 \in \mathcal{M}_n(G) \otimes \mathbb{Q}.$$

□

Proposition 4.3. *The map ψ is well-defined over $\mathbb{Q}$. In addition, $\nu_{\mathcal{K}_n(G)}$ and ψ are inverse to each other over $\mathbb{Q}$.*

Proof. The correctness of ψ is due to Lemma 4.1 and 4.2. By definition, $\mathcal{K}_n(G)$ is generated by

$$\boldsymbol{\gamma}(a, \boldsymbol{b}) = \langle a, b_1, \ldots, b_{n-1}\rangle + \langle -a, b_1, \ldots, b_{n-1}\rangle.$$

Let G' be the subgroup of G such that

$$\sum_{i=1}^{n-1} \mathbb{Z} b_i = (G/G')^\vee.$$

The definition of the co-multiplication map ensures that

$$\nu_{\mathcal{K}_n(G)}(\boldsymbol{\gamma}(a, \boldsymbol{b})) = \nu_{G'}(\boldsymbol{\gamma}(a, \boldsymbol{b}))$$

and one can deduce from (4.5) that

$$\psi \circ \nu_{\mathcal{K}_n(G)}(\boldsymbol{\gamma}(a, \boldsymbol{b})) = \psi(2\,\boldsymbol{\omega}(a, \boldsymbol{b})) = \boldsymbol{\gamma}(a, \boldsymbol{b}),$$

where the last equality holds by Lemma 4.1. Similarly, for any

$$\boldsymbol{\omega}(a, \boldsymbol{b}) = \langle a\rangle^+ \otimes \langle b_1, \ldots, b_{n-1}\rangle^- \in \mathcal{M}_1^+(G') \otimes \mathcal{M}_{n-1}^-(G/G'),$$

one has

$$\nu_{\mathcal{K}_n(G)} \circ \psi(\boldsymbol{\omega}(a, \boldsymbol{b})) = \nu_{\mathcal{K}_n(G)}(\frac{1}{2}\boldsymbol{\gamma}(a, \boldsymbol{b})) = \boldsymbol{\omega}(a, \boldsymbol{b}).$$

It follows that ψ and $\nu_{\mathcal{K}_n(G)}$ are inverse to each other as homomorphisms between $\mathbb{Q}$-vector spaces. □

Dimensional Formulae. Proposition 4.3 provides an effective computation for

$$\dim(\mathcal{M}_n(G)_{\mathbb{Q}}) - \dim(\mathcal{M}_n^-(G)_{\mathbb{Q}}).$$

In particular, it implies the hypothetical formula (note that the original formula in [4, Section 11] is wrong)

$$\begin{aligned}
&\dim(\mathcal{M}_2(C_N)_{\mathbb{Q}}) - \dim(\mathcal{M}_2^-(C_N)_{\mathbb{Q}}) \\
&\overset{N\geq 5}{=} \begin{cases} \dfrac{\phi(N)}{2} + \dfrac{1}{4} \displaystyle\sum_{d|N, 3\leq d\leq N/3} \phi(d)\phi(N/d) & N \text{ odd}, \\ \dfrac{\phi(N) + \phi(\frac{N}{2})}{2} + \dfrac{1}{4} \displaystyle\sum_{d|N, 3\leq d\leq N/3} \phi(d)\phi(N/d) & N \text{ even}. \end{cases}
\end{aligned}$$

Combining this with Proposition 3.8, we obtain an effective computation for

$$\dim(\mathcal{M}_2(G)_{\mathbb{Q}}).$$

For example, when $G = C_p \times C_p$, p an odd prime, one has

$$\dim(\mathcal{M}_2(C_p \times C_p) \otimes \mathbb{Q}) - \dim(\mathbb{Q} \otimes \mathcal{M}_2^-(C_p \times C_p)) = \frac{(p+1)(p-1)^2}{4}$$

and thus

$$\dim(\mathcal{M}_2(C_p \times C_p) \otimes \mathbb{Q}) = \frac{(p-1)(p^3+6p^2-p+6)}{24},$$

which is consistent with results of computer experiments recorded in Section 2.

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