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Three-Dimensional Algebraic Tori*

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§1. Statement of the problem and fundamental results

Any algebraic torus T, whose dimension does not exceed 2, is rational over the field of definition [**14**, Theorems 4.73 and 4.74]. This means that the field of rational functions $k(T)$ is a purely transcendental extension of the ground field k. On the other hand, an example is known of a three-dimensional torus which is not rational [**14**, chap. 4, §9]. This fact gives rise to the problem of the birational classification of the three-dimensional algebraic tori.

Let k be an arbitrary field, T an algebraic k-torus (that is, a linear algebraic k-group, which for some extension of the ground field $L \supset k$ becomes isomorphic to the direct product of n copies of the multiplicative group $G_{m,L}$; the number n is called the dimension of T, and the field L is the splitting field). Tori defined over k and split by L, are called L/k-tori. The simplest example of a nonrational torus is constructed in the following manner. Let L be a biquadratic extension of k, and $N_{L/k}\colon R_{L/k}(G_m) \to G_m$ be the norm map ($R_{L/k}$ denotes the A. Weil functor of the restricting of the ground field), and $R^{(1)}_{L/k}(G_m)$ the kernel of $N_{L/k}$ (what is called the norm hypersurface). The torus $R^{(1)}_{L/k}(G_m)$ has dimension three and is not rational.

Let us sharpen the statement of the problem regarding the birational classification of three-dimensional tori. Let k_s be the separable closure of k, and G_s be the Galois group of the extension k_s/k. Every k-torus T is uniquely determined by the module of rational characters $\hat{T} = \mathrm{Hom}_{k_s}(T \otimes_k k_s, G_{m,k_s})$. Then $\hat{T}$ is a torsion-free G_s-module of finite $\mathbb{Z}$-rank; in G_s there exists a closed subgroup G_0 of finite index acting on $\hat{T}$ trivially. Let us denote by L the fixed subfield in k_s

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corresponding to the subgroup G_0; L/k is a finite extension with Galois group $G = G_s/G_0$ and $\hat{T}$ is a faithful G-module.

The torus T is uniquely determined by the extension L/k and the G-module $\hat{T}$. The following remark is important [**15**]: every G-module $\hat{T}$ determines a whole series of k-tori (it is sufficient to construct the series of extensions L_i/k with the group G), and these tori can be nonisomorphic over k (since their modules of characters are nonisomorphic as G_s-modules). Moreover, it was proved in [**3**] that if L and F are nonisomorphic biquadratic extensions of the field k, then the tori $R^{(1)}_{L/k}(G_m)$ and $R^{(1)}_{F/k}(G_m)$ are not birationally equivalent over k. Thus, for $k = \mathbf{Q}$, the set of the birational equivalence classes of three-dimensional k-tori is infinite.

Now let T be a three-dimensional k-torus; $h_T\colon G_s \to GL(3, \mathbf{Z})$; the representation corresponding to the module $\hat{T}$; $W_T = h_T(G_s) = h_T(G)$ a finite subgroup in $GL(3, \mathbf{Z})$, isomorphic to the group G as an abstract group. Under the correspondence $T \to W_T$ between the three-dimensional tori and the finite subgroups in $GL(3, \mathbf{Z})$, conjugate subgroups correspond to isomorphic tori.

Let us recall that the stable equivalence of tori T_1 and T_2 of equal dimension means the birational equivalence of the varieties $T_1 \times G_m^d$ and $T_2 \times G_m^d$, and the stable rationality of the torus T means the rationality of $T \times G_m^d$ for some $d \geq 0$. One of the basic problems of the theory of algebraic tori consists in obtaining an answer to the Zariski conjecture that stably equivalent tori of equal dimension are birationally equivalent. In particular, it is not known if each stably rational torus is rational (for more detail see [**14**, chap. 5]).

The fundamental results of this article are the following.

Theorem 1. *Let U_i and U_j be two distinct subgroups in $GL(3, \mathbf{Z})$ from the following list*:

$$U_1 = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$$

$$U_2 = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$\cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$$

$$U_3 = \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \cong D_4$$

$$U_4 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \cong D_4$$

$$U_5 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \right\} \cong A_4$$

$$U_6 = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong D_4 \times \mathbf{Z}_2$$

$$U_7 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong A_4 \times \mathbf{Z}_2$$

$$U_8 = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong S_4$$

$$U_9 = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\} \cong S_4$$

$$U_{10} = \left\{ \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong S_4$$

$$U_{11} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong S_4 \times \mathbf{Z}_2$$

$$U_{12} = \left\{ \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong S_4 \times \mathbf{Z}_2$$

$$W_1 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cong \mathbf{Z}_4 \times \mathbf{Z}_2$$

$$W_2 = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

$$\cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$$

$$W_3 = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong S_4 \times \mathbf{Z}_2$$

(here and subsequently, $\mathbf{Z}_n$, D_n, A_n and S_n denote, as usual, the cyclic, dihedral, alternating and symmetric groups). Let T_i and T_j be the three-dimensional tori to which the subgroups U_i and U_j correspond. Then T_i and T_j are not stably rational and are not mutually stably equivalent in the case when U_i and U_j are not isomorphic as abstract groups.

Theorem 2. *All three-dimensional tori, except the ones described in Theorem 1, are rational.*

Remark. Theorems 1 and 2 give a birational classification of the three-dimensional algebraic tori. In particular, from them we obtain a positive answer to the Zariski conjecture for this class of algebraic varieties: every stably rational three-dimensional torus is rational.

§2. Projective models of three-dimensional tori

In analyzing the birational properties of algebraic tori, the embedding of a torus into a projective variety (nonsingular or even normal) often proves useful. The properties of such models, called toric varieties (or toroidal embeddings) were studied in the papers [**4**], [**5**], [**7**], basically considering the case of an algebraically closed field k. The case of the nonclosed fields that we need was investigated in [**16**].

First of all, let us construct models for the tori which correspond to the maximal subgroups in $GL(3, \mathbf{Z})$. There are four such subgroups: they are the full integral automorphism groups of the quadratic forms [**12**]:

$$F = l(x^2 - xy + y^2) + mz^2 \quad (l, m > 0), \qquad C = x^2 + y^2 + x^2,$$

$$S = x^2 + y^2 + x^2 + xy + yx + zx, \quad P = 3(x^2 + y^2 + z^2) - 2(xy + yz + zx).$$

The first subgroup is isomorphic to $D_6 \times \mathbf{Z}_2$, and the remaining ones to $S_4 \times \mathbf{Z}_2$ (as abstract groups).

To construct a projective model of a three-dimensional L/k-torus T, it is necessary, in accordance with the general theory [**4**], [**5**], [**7**], [**16**], to obtain a partition of $\mathbf{R}^3$ into polyhedral cones, stable under the action of G in $\mathbf{Z}_3$, corresponding to the module $\hat{T}^* = \mathrm{Hom}(\hat{T}, \mathbf{Z})$. To each element s of the part one can associate the affine variety $V_s = \mathrm{Spec}\, \mathbf{Z}[s \cap \mathbf{Z}^3]$, and the variety V is obtained by gluing the affine pieces V_s. In our case, the varieties V have the following form.

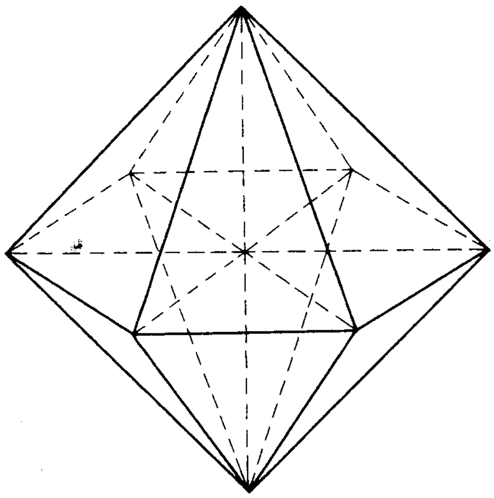

Figure 1.

Case F. The part is depicted in Fig. 1. The corresponding lattice in three-dimensional space is the direct product of the regular hexagonal lattice in the plane (x, y) and the lattices on the line z. The variety V_F is the direct product $X \times \mathbf{P}^1$, where X is a Del Pezzo surface of degree 6.

Case C. In this case, the part corresponds to the cubic lattice in three-dimensional space (Fig. 2). The projective model has the form $V_C = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

Case S. The simplest invariant part is obtained in the following manner. The vectors $f_1 = (-1, 1, 1)$, $f_2 = (1, -1, 1)$, $f_3 = (1, 1, -1)$, $f_4 = -f_1 - f_2 - f_3$, $\hat{f}_i = -f_i$ $(1 \le i \le 4)$ serve as the edges.

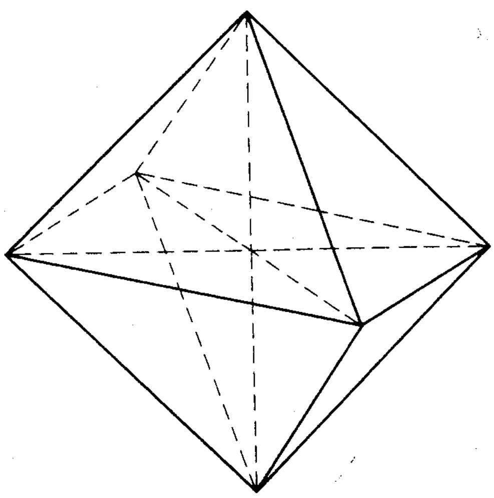

Figure 2.

(Let us recall that the action of G corresponds not to the module $\hat{T}$, but to the dual module $\hat{T}^*$), and the invariant cones are $s_{ijkl} = \langle \hat{f}_i, \hat{f}_j, \hat{f}_k, \hat{f}_l \rangle$, ($i, j, k, l$ are distinct numbers from 1 to 4). To each of the six cones s_{ijkl} corresponds an affine variety, which can be written as the equation $xy - zt = 0$ in $\mathbf{A}^4$. The variety obtained by gluing these pieces together is singular (let us call it the partial model of type S). To construct a nonsingular model, one needs to add to the partition six more vectors $f_{ij} = f_i + f_j$ ($1 \le i, j \le 4$, $i \ne j$) and barycentrically subdivide the cones s_{ijkl}. This model obtained can be described in the following way. It is obtained from $\mathbf{P}^3$ by a sequence of monoidal transformations: let us blow up the vertices and edges of the coordinate "tetrahedron" (Fig. 3).

Case P. It is possible to span the invariant partition by the vectors $g_1 = (0, 1, 1)$, $g_2 = (1, 0, 1)$, $g_3 = (1, 1, 0)$, $\hat{g}_i = -g_i$, $g_{ij} = g_i - g_j$ ($1 \le i, j \le 3$, $i \ne j$).

The cones of the partition are

$$S_1 = \langle g_1, g_2, g_3 \rangle, \quad S_2 = \langle \hat{g}_1, g_{21}, g_{31} \rangle, \quad S_3 = \langle \hat{g}_{-}, g_{12}, g_{32} \rangle,$$

$$S_4 = \langle \hat{g}_3, g_{13}, g_{23} \rangle, \quad S_5 = -S_1, \quad S_6 = -S_2, \quad S_7 = -S_3, \quad S_8 = -S_4,$$

$$S_9 = \langle g_1, g_{-}, g_{13}, g_{23} \rangle, \quad S_{10} = \langle g_1, g_3, g_{12}, g_{32} \rangle,$$

$$S_{11} = \langle g_2, g_3, g_{21}, g_{31} \rangle, \quad S_{12} = -S_9, \quad S_{13} = -S_{10}, \quad S_{14} = -S_{11}.$$

To each of the first eight cones there correspond the affine space $\mathbf{A}^4$, and to each of the remaining six, a hypersurface $xy - zt = 0$ in $\mathbf{A}^4$. One can write the variety obtained after gluing these affine pieces by the equation

$$x_0 y_0 z_0 t_0 = x_1 y_1 z_1 t_1 \quad \text{in} \quad \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1.$$

This variety (the partial model of type P) is singular. Adding to the partition the

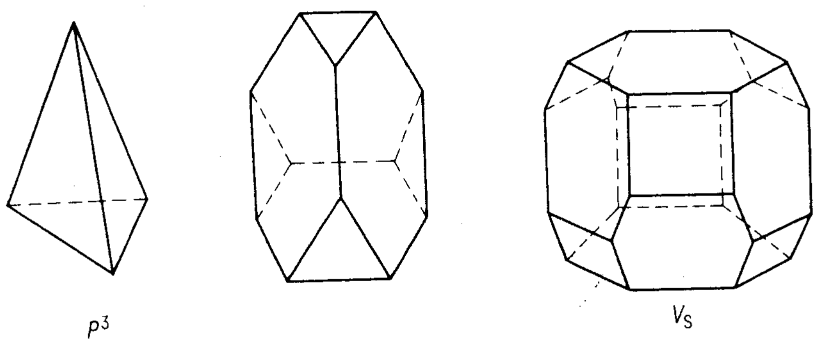

Figure 3.

vectors $g_{ijkl} = g_i + g_j - g_k$, $\hat{g}_{ijk} = -g_{ijk}$ (i, j, k are different numbers from 1 to 3) and barycentrically subdividing the cones s_i ($9 \leq i \leq 14$), we obtain a nonsingular model V_P. This process corresponds to blowing up the six singular points of the partial model.

Let us remark that for many subgroups in $GL(3, \mathbf{Z})$, these models are not minimal. In particular, let us consider subgroups of type S of the following form. To the group $G = \mathrm{Aut}(S) \cong S_4 \times \mathbf{Z}_2$ there corresponds the representation

$$a \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \quad b \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a, b and c are the generators of the group $S_4 \times \mathbf{Z}_2$: $a^4 = b^2 = (ab)^3 = c^2 = 1$, $ac = ca$, $bc = cb$).

The following assertion holds.

Lemma 1. *For the tori to which the subgroups $H \subseteq S_4 = \{a, b\}$ correspond, the minimal projective model is $V_H \cong \mathbf{P}^3$.*

Proof. For the subgroups of the form described above, the partition spanned by the vectors f_1, f_2, f_3, f_4 and consisting of the cones

$$S_1 = \langle f_2, f_3, f_4 \rangle, \quad S_2 = \langle f_1, f_3, f_4 \rangle, \quad S_3 = \langle f_1, f_2, f_4 \rangle, \quad S_4 = \langle f_1, f_2, f_3 \rangle,$$

is H-invariant. To each of the four cones there corresponds an affine space $\mathbf{A}^3$; gluing them together, we obtain the model $V_H = \mathbf{P}^3$.

This fact will be used in §3.

To each projective nonsingular model V of an L/k-torus T, there corresponds the canonical exact sequence of G-modules $0 \to \hat{T} \to S \to \hat{N} \to 0$ [**14**], where S is a permutation module (that is, a module possessing a finite $\mathbf{Z}$-basis, on which G acts by permutation: in the given case, the generators are the divisors of V, with supports outside T) and $\hat{N} \cong \mathrm{Pic}\, V_L$ is a flasque module (that is, $H^{-1}(G', \hat{N}) = 0$ for all subgroups G' of G). The similarity class $[\hat{N}]$ of the module $\hat{N}$ (it is also denoted by $p(T)$ or $p(\hat{T})$) is called the Picard class (modules A_1 and A_2 are called similar if $A_1 \oplus S_1 \cong A_2 \oplus S_2$ for some permutation modules S_1 and S_2). The class $p(T)$ is an important birational invariant of the torus T; in particular, the tori T_1 and T_2 are stably equivalent if and only if $p(T_1) = p(T_2)$ (for more detail see [**14**]).

For the three-dimensional tori, to which the subgroups of type F, C, S and P correspond, the Picard modules of the projective models have ranks 5, 3, 11 and 15, respectively. We have $\mathrm{Pic}\, V_F \cong \mathbf{Z} \oplus M$, where the module M is similar to a permutation module and Pic V_C is a permutation module. The modules Pic V_S and Pic V_P will be analyzed in detail in §§4 and 5. For the subgroups H from Lemma 1, Pic $V_H \cong \mathbf{Z}$ (a module of rank 1 with the trivial action of H).

Remark. The results formulated in §1 can be reformulated in the language of Picard classes in the following way: if T runs over the set of three-dimensional k-tori, then the set $\{p(T)\}$ of Picard classes contains 15 nonzero elements.

§3. Rational three-dimensional tori

In the group $GL(3, \mathbf{Z})$ there are 73 finite subgroups (up to conjugacy). The matrix method of writing these subgroups was given in [**13**] (in Tahara's list there are 74 subgroups; one subgroup was enumerated twice; for a correction see [**1**]). In this section we eliminate the subgroups corresponding to the rational tori—the rational subgroups. Thereby, we prove Theorem 2.

Proposition 1. *The following subgroups of $GL(3, \mathbf{Z})$ are rational*:

(a) *the subgroups corresponding to split representations*;

(b) *the subgroups for which the corresponding module contains nonzero invariant elements*;

(c) *the subgroup preserving the quadratic form $\delta = x^2 + y^2 + z^2$*;

(d) *the subgroups described in Lemma* 1;

(e) *the subgroups isomorphic to $\mathbf{Z}_4$*;

(f) *the subgroups isomorphic to S_3*.

Proof. The subgroups of type (a) correspond to tori *representable in the form of a direct product of tori of dimension no more than* 2, *and consequently, are rational.*

Let T be a torus to which subgroup (b) corresponds. Since the module of the characters $\hat{T}$ contains nonzero invariant elements, one can construct the exact sequence $0 \to \hat{T}_a \to \hat{T} \to \mathbf{Z}^d \to 0$, where $d > 0$. By Proposition 4.22 of [**14**] there is a birational equivalence $T \simeq T_a \times G_m^d$. Since the dimension of T_a does not exceed 2, it is rational; this means that the torus T is rational also.

Assertion (c) was proved in [**17**] in the more general case of the quadratic form $x_1^2 + \cdots + x_n^2$.

Tori of type (d) are rational, since by Lemma 1 they admit an open embedding into $\mathbf{P}^3$.

Tori of any dimension, split over a Galois extension with the group $G = \mathbf{Z}_4$, are rational [**14**, Theorem 4.76].

The same is true in the case $G = S_3$ [**8**].

Remark. Proposition 1 (assertions (a) and (c)) shows that all the tori of type F and type C are rational.

The majority of the subgroups from Tahara's list are eliminated after applying Proposition 1.

All five groups of order 2 belong to type (a).

Groups of order 3: W_1 belongs to type (a), and W_2 to type (b).

Groups of order 4: W_1-W_4 belong to type (e), W_5-W_{11} to type (a), W_{13} and W_{15} to type (b), and W_{12} to type (d).

Groups of order 6: W_1-W_3 belong to type (a), W_4 to type (c), and W_5-W_{10} to type (f).

Groups of order 8: W_1, W_3, W_4, W_7-W_{10} belong to type (a), W_{12} to type (b), and W_{14} to type (d).

Groups of order 12: W_1-W_7 belong to type (a), W_8 and W_9 to type (c), and W_{11} to type (d).

Groups of order 16: W_1 belongs to type (a).

Groups of order 24: W_4 and W_5 belongs to type (a) W_6 and W_7 to type (c), and W_{11} to type (d).

Groups of order 48: W_1 belongs to type (c).

Thus, it remains to investigate 15 subgroups of $GL(3, \mathbf{Z})$. The group of order 4:

$$U_1 = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \quad (W_{14}).$$

The groups of order 8:

$$W_1 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong \mathbf{Z}_4 \times \mathbf{Z}_2 \quad (W_2).$$

$$W_2 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$\cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \quad (W_5).$$

$$U_2 = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$\cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \quad (W_6).$$

$$U_3=\left\{\begin{pmatrix}0&-1&0\\0&0&-1\\1&1&1\end{pmatrix},\begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix}\right\}\cong D_4\quad (W_{11}).$$

$$U_4=\left\{\begin{pmatrix}0&1&0\\0&0&1\\-1&-1&-1\end{pmatrix},\begin{pmatrix}1&1&1\\0&0&-1\\0&-1&0\end{pmatrix}\right\}\cong D_4\quad (W_{13}).$$

The group of order 12:

$$U_5=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&0&0\\-1&0&1\\-1&1&0\end{pmatrix}\right\}\cong A_4\quad (W_{10}).$$

The group of order 16:

$$U_6=\left\{\begin{pmatrix}0&-1&0\\0&0&-1\\1&1&1\end{pmatrix},\begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right\}$$
$$\cong D_4\times \mathbf{Z}_2\quad (W_2).$$

The groups of order 24:

$$U_7=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&0&0\\-1&0&1\\-1&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right\}\cong A_4\times \mathbf{Z}_2\quad (W_2).$$

$$W_3=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\-1&-1&-1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right\}$$
$$\cong A_4\times \mathbf{Z}_2\quad (W_3).$$

$$U_8=\left\{\begin{pmatrix}0&-1&0\\0&0&-1\\1&1&1\end{pmatrix},\begin{pmatrix}1&1&1\\0&-1&0\\0&0&-1\end{pmatrix}\right\}\cong S_4\quad (W_{10}).$$

$$U_9=\left\{\begin{pmatrix}0&0&1\\-1&0&1\\0&-1&1\end{pmatrix},\begin{pmatrix}-1&0&1\\0&-1&1\\0&0&1\end{pmatrix}\right\}\cong S_4\quad (W_8).$$

$$U_{10}=\left\{\begin{pmatrix}0&0&-1\\1&0&-1\\0&1&-1\end{pmatrix},\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&-1\end{pmatrix}\right\}\cong S_4\quad (W_9).$$

The groups of order 48:

$$U_{11} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$\cong S_4 \times \mathbf{Z}_8 \quad (W_3).$$

$$U_{12} = \left\{ \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong S_4 \times \mathbf{Z}_2 \quad (W_2).$$

For convenience, we have changed the matrix method of writing some groups introduced in [**13**]; on the right we present Tahara's notation in parentheses. Theorem 2 is proved.

To analyze the remaining subgroups, we need a more thorough study of the Picard modules of the projective models of the corresponding tori. All the tori to which the subgroups listed above (except W_1, W_2 and W_3) correspond are not even stably rational, since these groups contain the subgroup U_1 (it corresponds to the torus $R^{(1)}_{L/k}(G_m)$, where L/k is a biquadratic extension). This subgroup enjoys the property that $H^1(U_1, \operatorname{Pic} V) = \mathbf{Z}_2$, while the vanishing of the first cohomology of the Picard module is a necessary condition for stable rationality.

§4. Picard classes of three-dimensional tori of type S and type P

Let us place a subgroup in $GL(3, \mathbf{Z})$ and the corresponding torus in type S or type P, depending on whether this subgroup preserves the quadratic form

$$S = x^2 + y^2 + x^2 + xy + yz + zx$$

or the form

$$P = 3(x^2 + y^2 + z^2) - 2(xy + yz + zx).$$

Among the 15 subgroups under investigation, three belong to the type $S(W_3, U_8, U_{11})$, five to the type $P(U_5, U_7, U_9, U_{10}, U_{12})$, and one can place the remaining seven in any of the types $(U_1, U_2, W_1, W_2, U_3, U_4, U_6)$.

First let us write down the explicit formulas that define modules $\hat{N}_S = \operatorname{Pic} V_S$ and $\hat{N}_P = \operatorname{Pic} V_P$ of the projective models of the tori to which the maximal subgroups of types S and P (U_{11} and U_{12}) correspond.

Proposition 2. *The modules $\hat{N}_S$ and $\hat{N}_P$ have ranks* 11 *and* 15 *respectively. The action of the group $G = S_4 + \mathbf{Z}_2$ has the following form.*

For the module $\hat{N}_S$:

$$\begin{array}{lll}
ae_1 = e_1 & be_1 = e_1 & ce = -e_1 - e_6 - e_7 - e_8 - e_9 - e_{10} - e_{11} \\
ae_2 = e_3 & be_2 = e_3 & ce_2 = e_1 + e_2 + e_6 + e_7 + e_8 \\
ae_3 = e_4 & be_3 = e_2 & ce_3 = e_1 + e_3 + e_6 + e_9 + e_{10} \\
ae_4 = e_5 & be_4 = e_4 & ce_4 = e_1 + e_4 + e_7 + e_9 + e_{11} \\
ae_5 = e_2 & be_5 = e_5 & ce_5 = e_1 + e_5 + e_8 + e_{10} + e_{11} \\
ae_6 = e_9 & be_6 = e_6 & ce_6 = e_{11} \\
ae_7 = e_{10} & be_7 = e_9 & ce_7 = e_{10} \\
ae_8 = e_6 & be_8 = e_{10} & ce_8 = e_9 \\
ae_9 = e_{11} & be_9 = e_7 & ce_9 = e_8 \\
ae_{10} = e_7 & be_{10} = e_8 & ce_{10} = e_7 \\
ae_{11} = e_8 & be_{11} = e_{11} & ce_{11} = e_6.
\end{array} \tag{2}$$

For the module $\hat{N}_P$:

$$\begin{array}{lll}
ae_1 = \hat{e}_3 & be_1 = e_2 & ce_1 = \hat{e}_1 \\
ae_2 = e_6 & be_2 = e_1 & ce_2 = \hat{e}_2 \\
ae_3 = e_8 & be_3 = e_3 & ce_3 = \hat{e}_3 \\
ae_4 = \hat{e}_1 & be_4 = e_5 & ce_4 = e_5 \\
ae_5 = e_1 & be_5 = e_4 & ce_5 = e_4 \\
ae_6 = \hat{e}_2 & be_6 = e_8 & ce_6 = e_7 \\
ae_7 = e_2 & be_7 = e_9 & ce_7 = e_6 \\
ae_8 = e_4 & be_8 = e_6 & ce_8 = e_9 \\
ae_9 = e_5 & be_9 = e_7 & ce_9 = e_8 \\
ae_{10} = e_{15} & be_{10} = e_{10} & ce_{10} = e_{13} \\
ae_{11} = e_{14} & be_{11} = e_{12} & ce_{11} = e_{14} \\
ae_{12} = e_{10} & be_{12} = e_{11} & ce_{12} = e_{15} \\
ae_{13} = e_{12} & be_{13} = e_{13} & ce_{13} = e_{10} \\
ae_{14} = e_{11} & be_{14} = e_{15} & ce_{14} = e_{11} \\
ae_{15} = e_{13} & be_{15} = e_{14} & ce_{15} = e_{12}.
\end{array} \tag{3}$$

For brevity:

$$\hat{e}_1 = e_2 + e_4 - e_5 + e_6 - e_7 + e_{10} + e_{11} - e_{12} - e_{13} - e_{14} + e_{15},$$

$$\hat{e}_2 = e_2 - e_4 + e_5 + e_8 - e_9 + e_{10} - e_{11} + e_{12} - e_{13} + e_{14} - e_{15},$$

$$\hat{e}_3 = e_3 - e_6 + e_7 = e_8 + e_9 - e_{10} + e_{11} + e_{12} + e_{13} - e_{14} - e_{15}.$$

Proof. The vectors $f_1 = (-1, 1, 1)$, $f_2 = (1, -1, 1)$, $f_3 = (1, 1, -1)$, $f_4 = -f_1 - f_2 - f_2$, $\hat{f}_i = -f_i$, $f_{ij} = f_i + f_j$ $(i, j = 1, \ldots, 4,\ i \neq j)$ serve as the edges of the partition S (see §2). Their images in Pic V_S satisfy the relationships:

$$f_1 - \hat{f}_1 - f_4 + \hat{f}_4 + f_{12} + f_{13} - f_{24} - f_{34} = 0,$$

$$f_2 - \hat{f}_2 - f_4 + \hat{f}_4 + f_{12} + f_{23} - f_{14} - f_{34} = 0,$$

$$f_3 - \hat{f}_3 - f_4 + \hat{f}_4 + f_{13} + f_{23} - f_{14} - f_{24} = 0.$$

From these relations it follows that the algebraic sum

$$\hat{f}_i - f_i - \sum_{\substack{j=1 \\ j \neq 1}}^{4} f_{ij}$$

is constant for $i = 1, \ldots, 4$. Denoting this sum by e_1 and letting $e_2 = f_1$, $e_3 = f_2$, $e_4 = f_3$, $e_5 = f_4$, $e_6 = f_{12}$, $e_7 = f_{13}$, $e_8 = f_{14}$, $e_9 = f_{23}$, $e_{10} = f_{24}$, $e_{11} = f_{34}$, we obtain formulas (2).

In the case of the partition P, the generators and relations of Pic V_P have the form:

$$g_1 = (0, 1, 1), \quad g_2 = (1, 0, 1), \quad g_3 = (1, 1, 0), \quad \hat{g}_i = -g_i,$$

$$g_{ij} = g_i - g_j, \quad g_{ijk} = g_i + g_j - g_k, \quad \hat{g}_{ijk} = -g_{ijk} \quad (1 \leq i, j, k \leq 3;\ i \neq j \neq k);$$

$$g_1 - \hat{g}_1 + g_{12} - g_{21} + g_{13} - g_{31} + g_{123} + g_{132} - g_{231} - \hat{g}_{123} - \hat{g}_{132} + \hat{g}_{231} = 0,$$

$$g_2 - \hat{g}_2 + g_{21} - g_{12} + g_{23} - g_{32} + g_{123} + g_{321} - g_{132} - \hat{g}_{123} - \hat{g}_{321} + \hat{g}_{132} = 0,$$

$$g_3 - \hat{g}_3 + g_{31} + g_{32} - g_{23} + g_{132} + g_{231} - g_{123} - \hat{g}_{132} - \hat{g}_{231} + \hat{g}_{123} = 0.$$

Letting $e_1 = g_1$, $e_2 = g_2$, $e_3 = g_3$, $e_4 = g_{12}$, $e_5 = g_{21}$, $e_6 = g_{13}$, $e_7 = g_{31}$, $e_8 = g_{23}$, $e_9 = g_{32}$, $e_{10} = g_{123}$, $e_{11} = g_{132}$, $e_{12} = g_{231}$, $e_{13} = \hat{g}_{123}$, $e_{14} = \hat{g}_{132}$, $e_{15} = \hat{g}_{231}$, we obtain formulas (3).

Let us show that the subgroups $W_1 - W_3$ correspond to nonrational tori. Let $G = W_1$, and

$$0 \to \hat{T} \to S \to N \to 0 \tag{4}$$

be the canonical resolution. Let us assume that N is a direct summand of a permutation module. Let us use the results of Endo and Miyata [**14**].

Lemma 2. *If G is a finite P-group, and M is a direct summand of a permutation G-module, then $\tilde{M} = M \otimes_{\mathbf{Z}} Z_P$ is a permutation $Z_P[G]$-module.*

By Lemma 2, $\tilde{N} = N \otimes_{\mathbf{Z}} Z_2$ is a permutation $Z_2[G]$-module. From (2) we have $\operatorname{rk} N = 11$ and $\operatorname{rk} N^G = 3$, meaning that

$$\tilde{N} = Z_2[G] \oplus Z_2[G/H] \oplus Z_2, \tag{5}$$

where $\# H = 4$. It follows from (5) that $\hat{H}^0(G, N) = \mathbf{Z}_4 \times \mathbf{Z}_8$ ($\hat{H}^i(G, N) \cong \hat{H}^i(G, \tilde{N})$, since G is a 2-group). On the other hand, in (4) we have $\hat{H}^0(G, \hat{T}) = H^1(G, S) = 0$, since $\hat{T}^G = 0$, and S is a permutation module; $S = \mathbf{Z}[G] \otimes \mathbf{Z}[G/U] \oplus \mathbf{Z}[G/H]$, where $\# U = 2$ and $\# H = 4$, meaning that $\hat{H}^0(G, S) = \mathbf{Z}_2 \times \mathbf{Z}_4$; a direct computation yields $H^1(G, \hat{T}) = \mathbf{Z}_2$. We obtain the exact sequence $0 \to \mathbf{Z}_2 \times \mathbf{Z}_4 \to \mathbf{Z}_4 \times \mathbf{Z}_8 \to \mathbf{Z}_2 \to 0$, a contradiction.

The case $G = W_2$ is analyzed similarly. We have $\operatorname{rk} N = 11$ and $\operatorname{rk} N^G = 4$; for $\tilde{N}$ there are two possibilities:

$$\tilde{N} \cong Z_2[G] \oplus Z_2 \oplus Z_2 \oplus Z_2, \tag{6}$$

$$\tilde{N} \cong Z_2[G/V_1] \oplus Z_2[G/V_2] \oplus Z_2[G/H] \oplus Z_2, \tag{7}$$

where $\# V_i = 2$, $\# H = 4$. In (4), $S = \mathbf{Z}[G] \oplus \bigoplus_{i=1}^{3} \mathbf{Z}[G/H_i]$, where $\# H_i = 4$; whence $\hat{H}^0(G, S) = (\mathbf{Z}_4)^3$; as in the previous case, computation yields $H^1(G, \hat{T}) = \mathbf{Z}_2$. From (6) and (7) we obtain the exact sequences:

$$0 \to (\mathbf{Z}_4)^3 \to (\mathbf{Z}_8)^3 \to \mathbf{Z}_2 \to 0,$$

$$0 \to (\mathbf{Z}_4)^3 \to (\mathbf{Z}_2)^2 \times \mathbf{Z}_4 \times \mathbf{Z}_8 \to \mathbf{Z}_2 \to 0.$$

The first of these sequences is impossible because of the noncoincidence of the orders of the groups; and the second, because it is impossible to embed $(\mathbf{Z}_4)^3$ into $(\mathbf{Z}_2)^2 \times \mathbf{Z}_4 \times \mathbf{Z}_8$.

The non-rationality of the torus corresponding to W_3 follows from the non-rationality of the torus corresponding to W_2.

Thus the tori corresponding to the groups U_1–U_{12} and W_1–W_3 are not stably rational. To complete the proof of Theorem 1, it remains to prove that among these tori there are not any stably equivalent ones. In the next section, this will be proved in detail for the groups U_1–U_{12}; the proof is similar for the groups W_1–W_3.

§5. Analysis of the remaining subgroups of $GL(3, \mathbf{Z})$

For convenience let us represent the remaining unanalyzed 12 subgroups of $GL(3, \mathbf{Z})$ as vertices of a graph (Fig. 4).

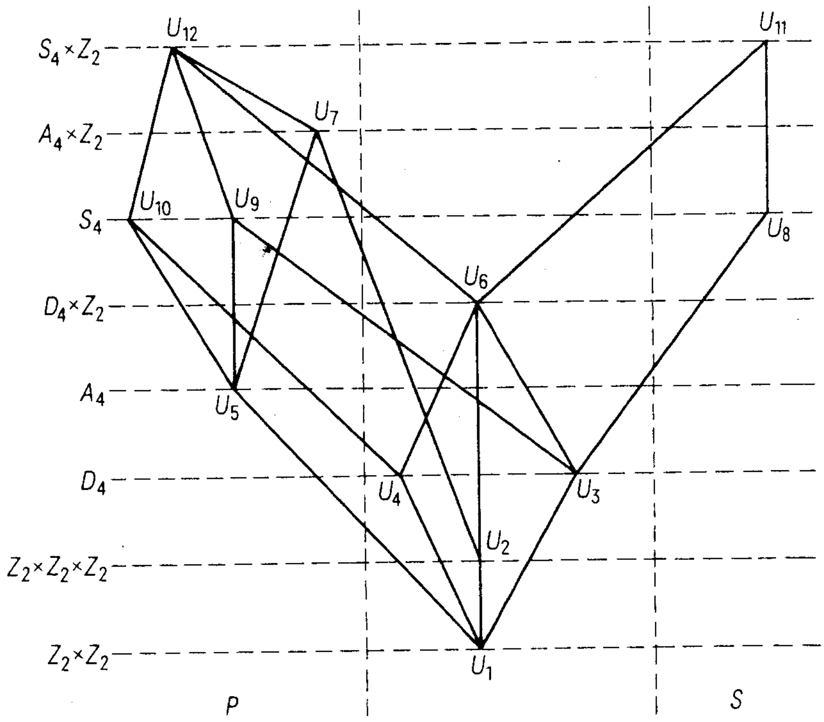

Figure 4.

The vertical dotted lines separate the subgroups of type P and type S into the left and right parts of the figure. The subgroups belonging to both types are placed in the central one. If vertices of the graph are joined by an edge, then an upper group contains a lower group as a subgroup.

Let U_i be one of the above 12 subgroups, let T_i be the torus to which the subgroup U_i corresponds, and let $[N_i]$ be the Picard class of T_i. As was noted in §3, none of the tori T_i is stably rational, since $H^1(U_1, N_1) = \mathbf{Z}_2$.

Let us call a module M almost indecomposable if $\tilde{M} = M \otimes_{\mathbf{Z}} Z_2$ does not contain a permutation direct summand.

Proposition 3. *In the similarity class* $[N_i]$ $(i \neq 5, 9, 10)$ *there exists an almost indecomposable module.*

Proof. We will use the following facts from the theory of integral representations:

(1) if G is a p-group, then for a G-module, integral indecomposability is equivalent to p-adic indecomposability [**6**];

(2) if G is a finite group, and Z_p denotes the ring of p-adic integers, then for Z_pG-modules, the Krull–Schmidt unique decomposition theorem holds [**2**].

Furthermore, if G is a finite group, we will denote by $S(G)$ the class of permutation G-modules, and by $D(G)$ the class of G-modules which are direct summands of permutation modules; set

$$H^i(G) = \{M \mid H^i(G', M) = 0 \ \forall G' \subseteq G\}, \qquad H(G) = H^1(G) \cap H^{-1}(G).$$

It is evident that $S(G) \subseteq D(G) \subseteq H(G)$.

First, let us analyze the case of the group U_1. An indecomposable module N_1 in the Picard class is obtained in the following way. By formula (2) there exists an exact sequence

$$0 \to S_0 \to \hat{N}_S \to N_0 \to 0, \tag{8}$$

where $S_0 = \{e_6, \ldots, e_{11}\}$ is a permutation module. A direct calculation shows that N_0 is a flasque U_1-module, that is, $N_0 \in H^{-1}(U_1)$. This means that the sequence (8), considered as a sequence of U_1-modules, splits. Hence one can choose as N_1 the restriction of the module N_0 to the subgroup U_1. The indecomposability of N_1 follows from the classification of the integral representations of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ [**10**]. It is not hard to obtain a purely algebraic description of the module N_1 without using projective models (see, e.g. [**9**]).

Remark. The module N_0 corresponds to the partial model of type S (see §2).

The case of the group U_3 is more complicated, since for the cyclic subgroup H_0 of order 4, generated by the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix},$$

we have $H^{-1}(H_0, N_0) = \mathbf{Z}_2$.

Let $S_1 = \{e_6, e_8, e_9, e_{11}\}$, and let N_3 (N_4) denote the restriction of $\hat{N}_S/S_1$ to the subgroup U_3 (U_4).

Lemma 3. *The U_3-module N_3 is almost indecomposable.*

Proof. Let us assume that $\tilde{N}_3 = \tilde{N} \oplus \tilde{S}$, where $\tilde{S}$ is a permutation module. From formulas (2) we have

$$0 \to S_2 \to N_3 \to M \to 0, \tag{9}$$

where $S_2 = \{e_7, e_{10}\}$ is a permutation U_3-module. Passing to the restriction to the subgroup $U_1 = \langle a^2, ab \rangle$ (we denote it by a bar) and considering that $\bar{M} \cong N_1$ is a flasque U_1-module, we obtain $N_3 \cong \bar{S}_2 \oplus N_1$. This means that rk $S = 2$ and for S there are three possibilities: (1) $S \cong \mathbf{Z}[G/U_1]$; (2) $S \cong \mathbf{Z}[G/H_0]$; and (3) $S = \mathbf{Z}[G/U_0]$, where $U_0 = \langle a^2, b\rangle$. The first case is impossible, since S_2 is not a trivial U_1-module. The restrictions of the module N_3 to H_0 and U_0 are direct sums of modules of ranks 4, 2 and 1; they do not contain direct summands of the form $\mathbf{Z} \oplus \mathbf{Z}$, showing the impossibility of cases 2 and 3.

The following lemma is proved similarly to Lemma 3.

Lemma 4. *The U_4-module N_4 is almost indecomposable.*

Corollary. *Let N_8 be the restriction of $\hat{N}_S/S_1$ to the subgroup U_8. The module N_8 is almost indecomposable.*

Let us denote by N_6 the restriction of $\hat{N}_S$ to the subgroup U_6. The following fact is proved similarly to Lemma 3.

Lemma 5. *The U_6-module N_6 is almost indecomposable.*

Corollary. *The module $\hat{N}_S$ is almost indecomposable.*

Let us now consider the groups of type P. From formulas (3) we deduce the exact sequence

$$0 \to S \to \hat{N}_P \to M_0 \to 0, \tag{10}$$

where $S = \{e_{10}, \ldots, e_{16}\}$ is a permutation module. By computing the cohomology, we obtain that M_0 is a flasque $U_2(U_5, U_7)$-module of rank 9. It corresponds to the partial model of type P. Let us denote by N_i the restriction of M_0 to U_i $(i = 2, 7)$.

The following fact is established similarly to Lemma 3.

Lemma 6. *The U_2-module N_2 is almost indecomposable.*

Corollary. *The U_7-module N_7 is almost indecomposable.*

In the case of the group U_5 the situation is somewhat different. Let us denote by M_5 the restriction of M_0 to U_5. From formulas (3) we obtain the exact sequence

$$0 \to S' \to M_5 \to N_5 \to 0, \tag{11}$$

where $S' = \{e_1 + e_2 + e_3, e_1 + e_4 + e_6, e_2 + e_5 + e_8, e_3 + e_4 + e_5\} \cong \mathbf{Z}[G/H]$ is a permutation module (H being the cyclic subgroup of order 3), and the restriction of N_5 to U_1 is isomorphic to N_1. Since $\mathrm{Ext}_{U_5}(N_5, S') \cong H^1(H, N_5) = \mathbf{Z}_3$, we have the decomposition over Z_2:

$$\tilde{M}_5 \cong \tilde{N}_5 \oplus \tilde{S}', \tag{12}$$

where $\tilde{N}_5$ is indecomposable because of the indecomposability of N_1.

Let us denote by M_9 (M_{10}) the restriction of $\hat{N}_P$ to U_9 (U_{10}). From formulas (3) we obtain the exact sequences:

$$\begin{gathered} 0 \to S_9 \to M_9^0 \to N_9 \to 0, \\ 0 \to S_{10} \to M_{10} \to N_{10} \to 0, \end{gathered} \tag{13}$$

where

$$\begin{aligned} S_9 = \{&e_1 + e_2 + e_3 + e_{10} + e_{11} + e_{12} + e_{12},\ e_1 + e_4 + e_6 + r_{10} + e_{11} + e_{15}, \\ &e_3 + e_7 + e_9 + e_{11} + e_{12} + e_{13},\ e_2 + e_5 + e_8 + e_{10} + e_{12} + e_{14}\}, \\ S_{10} = \{&e_1 + e_2 + e_3 + e_{10} + e_{11} + e_{12},\ e_1 + e_4 + e_6 + e_{10} + e_{11} + e_{12} \\ &- e_{13} - e_{14} - e_{15},\ e_1 + e_4 + e_6,\ e_2 + e_5 + e_1,\ e_3 + e_4 + e_9\} \end{aligned}$$

are permutation modules. As in the case of the group U_5, we have the 2-adic decompositions:

$$\tilde{M}_i \cong \tilde{N}_i \oplus \tilde{S}_i \qquad (i = 9, 10). \tag{14}$$

Lemma 7. *The modules $\tilde{N}_9$ and $\tilde{N}_{10}$ do not contain permutation direct summands.*

Proof. Let us consider the module $\tilde{N}_9$. Let us denote its restrictions to U_5 and U_3 by $\tilde{A}_9$ and $\tilde{B}_9$. The module M_5 being flasque, from the exact sequence (10), and the decompositions (12) and (14), we obtain the decomposition $\tilde{A}_9 \cong \tilde{N}_5 \oplus \tilde{S}_5$. Here S_5 is the restriction of $S = \{e_{10}, \dots, e_{15}\}$ to U_5; it is a 2-adically indecomposable according to the classification of Nazarova **[11]**; in the notation of this paper, S_5 is a module of type

$$\begin{pmatrix} 1 & x \\ 0 & D_9 \end{pmatrix}.$$

From formulas (3) we obtain the decomposition of the module $\tilde{B}_9$: $\tilde{B}_9 \cong \tilde{N}_3 \oplus \tilde{S}_3$, where $S_3 = \{e_{10}, e_{12}, e_{13}, e_{15}\}$. The ranks of the modules $\tilde{S}_5$ and $\tilde{S}_3$ are equal to 6 and 4, respectively. The assertion of the lemma follows.

The case of the module $\tilde{N}_{10}$ is analyzed similarly, using the restrictions to the subgroups U_5 and U_4.

Lemma 8. *The module $\hat{N}_P$ is almost indecomposable.*

Proof. Let us denote by A_{12} and B_{12} the restrictions of $\hat{N}_P$ to the subgroups U_7 and U_6. As in Lemma 7, we obtain the decompositions: $\tilde{A}_{12} \cong \tilde{N}_7 \oplus \tilde{S}_7$, $\tilde{B}_{12} \cong N_6 \oplus S_6$, where S_7 is the restriction of S to U_7 and $S_6 = \{e_2, \hat{e}_2, e_6, e_7\}$. The noncoincidence of the ranks of $\tilde{S}_7$ and $\tilde{S}_6$ proves the lemma.

Lemma 8 completes the proof of Proposition 3.

Let us now complete the proof of Theorem 1.

Let T_i (T_j) be the torus to which the subgroup U_i (U_j) corresponds, and let L_i (L_j) be its splitting field. Let us observe that there exists a common splitting field L of the tori T_i and T_j; the simplest example of such a field is the composite of the fields L_i and L_j for some embedding of the field k into its algebraic closure. Let us write $G = \mathrm{Gal}(L/k)$; one can consider $\hat{T}_i$ and $\hat{T}_j$, as well as the representatives of their Picard classes as G-modules (in general, they are not faithful). Let us denote by N_i, $i \neq 5, 9, 10$, the representatives of the Picard classes constructed in Proposition 3; and by N_5, N_9 and N_{10}, the almost indecomposable modules from (11) and (13). Let us assume that the tori T_i and T_j are stably equivalent. Since the Picard modules are similar, from Proposition 3, and Lemmas 7 and 8, there follows the isomorphism

$$\tilde{N}_i \cong \tilde{N}_j, \tag{15}$$

since $\tilde{N}_i$ and $\tilde{N}_j$ cannot be direct summands of permutation modules, because of the equality $H^1(U_i, N_i) = \mathbf{Z}_2$. But if U_i and U_j are not isomorphic as abstract groups, then $L_i \neq L_j$, and the isomorphism (15) is impossible, because G contains a subgroup acting trivially on N_i and nontrivially on N_j.

Theorem 1 is proved.

Remarks.

1. Let U_i and U_j be isomorphic as abstract groups. Let us show that for $L_3 = L_4$, the tori T_3 and T_4 are stably equivalent. It follows from formulas (2) that the group $U_3 = \langle a, b\rangle$ acts on $N_3 = \{E_1, \dots, E_7\}$ as follows

$$\begin{array}{ll}
aE_1 = -E_1 - E_6 - E_7 & bE_1 = E_1. \\
aE_2 = E_1 + E_3 + E_7 & bE_2 = E_5 \\
aE_3 = E_1 + E^4 + E^6 & bE_3 = E_4 \\
aE_4 = E_1 + E_5 + E_7 & bE_4 = E_3 \\
aE_5 = E_1 + E_2 + E_6 & bE_5 = E_2 \\
aE_6 = E_6 & bE_6 = E_7 \\
aE_7 = E_7 & bE_7 = E_6.
\end{array}$$

The action of the group U_4 on $N_4 = \{F_1, \dots, F_7\}$ has the form:

$$\begin{aligned} aF_1 &= F_1 & bF_1 &= -F_1 - F_6 - F_7 \\ aF_2 &= F_3 & bF_2 &= F_1 + F_5 + F_7 \\ aF_3 &= F_4 & bF_3 &= G_1 + F_4 + F_6 \\ aF_4 &= F_5 & bF_4 &= F_1 + F_3 + F_7 \\ aF_5 &= F_2 & bF_5 &= F_1 + F_2 + F_6 \\ aF_6 &= F_7 & bF_6 &= F_6 \\ aF_7 &= F_6 & bF_7 &= F_7. \end{aligned}$$

Changing the basis $F_1 = E_6$, $F_2 = E_1 + E_2$, $F_3 = E_3 - E_6$, $F_4 = E_1 + E_4$, $F_5 = E_5 - E_6$, $F_6 = -E_1 - E_4 - E_7$, $F_7 = E_1$, we establish the isomorphism of the modules N_3 and N_4.

2. In all the remaining cases, there are no stably equivalent tori among T_i and T_j. The torus T_8 is stably equivalent neither to T_9 nor to T_{10}, because of the noncoincidence of the groups $H^1(U, N)$ for $U = A_4$; we have $H^1(U, \hat{N}_S) = 0$, whereas $H^1(U, \hat{N}_P) = \mathbf{Z}_2$. For the same reason, stable equivalence of the tori T_{11} and T_{12} is impossible. The tori T_9 and T_{10} cannot be stably equivalent because the modules N_9 and N_{10} are almost indecomposable, and their restrictions to the cyclic subgroup of the order 4 are not isomorphic.

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