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Two-Time-Scale Hybrid Traffic Models for Pedestrian Crowds

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***Abstract*— This paper introduces new models to describe pedestrian crowd dynamics in a typical unidirectional environment, such as corridors, pathways, and railway platforms. Pedestrian movements are represented in a two-dimensional space that is further divided into narrow virtual lanes. Consequently, pedestrians either move in a lane following each other or change lanes, when it is desirable. Within this framework, the motions of pedestrians are modeled as a two-dimensional and two-time-scale hybrid system. A pedestrian's movement along the crowd direction is labeled as the *x* direction and modeled by a real-valued process, a solution of a differential equation in continuous time, the lane change is labeled as the *y* direction. In contrast to the *x* direction dynamics, the movements in the *y* direction only happen at some time epoch. Although the movements are still on the same time horizon as the *x* direction movements, with a slight abuse of notation and for simplicity and convenience, we use discrete time as the time indicator, and model the movements by a recursive equation taking values in a finite set. Under common assumptions of crowd movements, we prove that the crowd movements in the *x* direction will converge to a uniform distance distribution and the convergence rate is exponential. Furthermore, by using a velocity-distance function to represent the common crowd and traffic congestion scenarios, we show that all pedestrians will asymptotically move with a uniform group speed. In the *y* direction, when pedestrians naturally wish to change to faster lanes, we show that the numbers in each virtual lanes converge to a balanced distribution and hence achieves asymptotic consensus as shown typically in a crowd behavior. Stability and convergence analysis is carried out rigorously by using properties of circular matrices, stability of networked systems, and stochastic approximations. Simulation studies are used to demonstrate the main properties of our modeling approach and establish its usefulness in representing pedestrian dynamics.**

***Index Terms*— Pedestrian dynamic system, crowd behavior, hybrid system, consensus formation, convergence analysis, stochastic approximation.**

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I. Introduction

Large gathering and movements of people are very common, exemplified by political, entertainment, sports, and shopping events; public transportation; and popular tourism attractions. Crowd congestions, and even catastrophes, frequently occur in buildings and public areas, like stations, stadiums, shopping malls, sometimes causing hundreds of deaths in these accidents due to stampedes and crushes; see a list of human stampedes in [1]. As a result, crowd safety issues have drawn increased attention of authorities, event organizers, and scientists in seeking preventive and management strategies to improve crowd control.

A fundamental foundation of developing reliable crowd control methodologies is to develop representative mathematical models of crowd movements so that impact of a large variety of factors can be extensively and rigorously studied. Investigation of crowd dynamics involves many interesting and challenging issues, encompassing multi-disciplinary areas of engineering, physics, psychology, sociology, and other disciplines. The need of explaining the fascinating self-organization phenomena observed in pedestrian crowds and developing models that can reproduce crowd dynamics in a realistic way have motivated extensive scientific research [2].

This paper introduces new models to describe pedestrian crowd dynamics in a typical unidirectional environment such as corridors, pathways, and railway platforms. Crowd dynamics refer to the study of how and where crowds form and move. An understanding of crowd dynamics can be used to study critical conditions for overcrowding, critical factors that explain how, when, where, and why accidents occur, impact of space and crowd control on crowd behavior, and potential management systems to avoid accidents [3].

In this paper, pedestrian movements are represented in a two-dimensional space that is further divided into narrow virtual lanes. Consequently, pedestrians either move in a lane following each other or change lanes when it is desirable. Within this framework, the motions of pedestrians are modeled as a two-directional and two-time-scale hybrid system. A pedestrian's movement along the crowd direction is labeled as the x-direction and modeled by a real-valued process, a solution of a differential equation in continuous-time; and the lane change is labeled as the y direction and modeled by a recursive equation taking values in a finite set. Under common assumptions of crowd movements, we prove that the crowd movements in the x-direction will converge to a uniform distance distribution and the convergence rate is exponential.

Furthermore, by using a velocity-distance function to represent the common crowd and traffic congestion scenarios, we show that all pedestrians will asymptotically move with a uniform group speed. In the y-direction, when pedestrians naturally change lanes to faster lanes, we show that the numbers in each virtual lanes converge to a balanced distribution and hence achieves consensus asymptotically as shown typically in a crowd behavior. Stability and convergence analysis is carried out rigorously by using properties of circular matrices, stability of networked systems, and stochastic approximations.

To understand the relationship of this paper with the existing literature, we first note that pedestrian crowd was first empirically studied five decades ago by using direct observation, photographs, and time-lapse films [4]. Since catastrophes cannot be easily reproduced or created, simulation models have been proposed to study pedestrian and evacuation dynamics under different conditions, e.g. hydrodynamics models [5], [6], cellular automaton (CA) models [7], discrete choice models [8], [9], social force models [10], [11], nomad model [12] and optimization models [13]. Usually, these pedestrian models can be classified into macroscopic and microscopic scales based on the different choosing of the state of the system. In microscopic models, each pedestrian is represented separately and the interaction between different individuals can also be taken into account. However, in macroscopic models, the details of every pedestrian are overlooked and the crowd is described by some averaged gross quantities [14]. The macroscopic approach is based on the fact, that at high densities, the movement of pedestrian flow shows some obvious similarities with the motion of fluids and granular flows [15]. Therefore, similar with the vehicular traffic dynamics [16], the early macroscopic models of pedestrian dynamics took inspiration from hydrodynamics (Navier-Stokes type equations) and gas-kinetic theory (Boltzmann type equations) [5], [6], [17], [18].

The hydrodynamic model defines the quantities of density, mean velocity and pedestrian flow as state variables. By applying mass (corresponding to the total number of pedestrians) and momentum conservation equations, two coupled partial differential equations (PDE) are derived to determine the evolution of the system [14]. Specific models can be designed with this framework. First order models use fundamental diagrams on accelerations [19]. Second order models describe further details on acceleration functions [20]. Under this framework, Hughes' model [5] and its recent modifications [21], [22] treat the crowds as a "thinking" fluid and uses the eikonal equation to determine crowd movement velocity. In [23] and [24], the PDE framework is generalized to handle obstacles in pedestrian simulation and panic situations.

The kinetic approach approximated states of pedestrian flows by suitable probability distribution functions [14], which is valid only for large crowds. In [25], the kinetic method is expanded by modifying the laws of classical mechanics to accommodate pedestrian behaviors and strategies. Then, the model is simplified and validated in [26]. Considering the behavior of pedestrians, the kinetic model can also be combined with other models, e.g. heuristic-based models [13], [27].

With suitable calibration towards experimental data, the macroscopic models can realistically predict the time needed to evacuate the crowd [28], [29]. They are less complex than the microscopic models in model parameters and simulation. One technical issue with the models is how to include boundary conditions at exits and intersections. The macroscopic models are especially suitable for study and implementation of control laws.

The microscopic pedestrian models can be divided into discrete and continuous ones: The CA models [7] are discrete in all variables whereas the force-based models [10], [11] are fully continuous. The basic idea of CA models is that the space is partitioned into identical cells and each cell is either empty or occupied by a pedestrian or an obstacle. A pedestrian can move to the surrounding cells based on a transition probability. In the social force model, the motion of each pedestrian is described by a superposition of socio-psychological and physical forces [11]. Based on Newton's laws, the dynamics of crowd movements can be determined by some coupled ordinary differential equations (ODE). Both models can reproduce many highly realistic self-organization phenomena such as clogging, oscillations, and lane formation [2], [15]. But when the number of pedestrians is large, the prescription of all behaviors is not easy and the computational complexity is high [30]. Treating pedestrians as physical particles without anticipation and prediction abilities is another problem for some microscopic models. Besides, for the force-based models, inertia is a natural consequence which may cause oscillations around critical points and overlapping of pedestrians and it is often inappropriate for pedestrian dynamics [31]. The most recent research mainly focuses on improving the original models [13], [32], [33], studying pedestrian flows and evacuation problems under different conditions [34]–[36].

Although a lot of models have been proposed, which can capture some key aspects of the pedestrian dynamics, their properties are mostly based on simulation case studies. Theoretical results are few due to complexities of rules of motion [37]. Many self-organization phenomena are not fully understood. Some researchers have tried to provide some theoretical analysis. Helbing et al. tried to explain the lane formation in crowds of oppositely moving pedestrians [38] and "freezing by heating" phenomenon during evacuations [39]. Control theory was applied to study evacuation dynamics in [28], [29], and [40]. Optimal velocity (OV) models were used to study the instability of pedestrian flows [41] and extended to stochastic scenarios to describe one-dimentional pedestrian trajectories [42].

In contrast to above-mentioned development in the literature, this paper provides the following main contributions. It introduces virtual lanes to divide the walking space and then proposes a new pedestrian movement model. This model is described as a two-time-scale hybrid system, which combines some advantages of the macroscopic, microscopic, continuous and discrete models. Also, the two-time-scale framework makes it possible to conduct rigorous mathematical analysis to understand the typical crowd behaviors.

This paper introduces the modeling methodology to describe the pedestrian crowd behaviors. The main properties of the

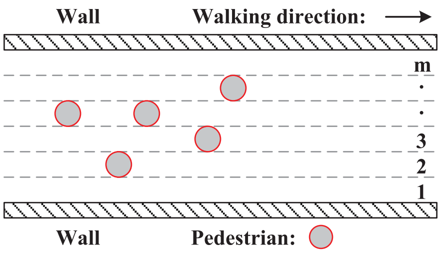

Fig. 1. The representation of the walking space.

model are demonstrated using theoretical analysis as well as simulation studies. The rest of the paper is organized into the following sections. Section II introduces the two-time-scale hybrid model. In Section III and IV, the convergence analysis in x-direction and y-direction are performed, separately. Some illustrative examples and simulation results are also showed in these sections, followed by conclusions and future work in Section V.

II. Models

A. Space Structure

The walking space is represented by the set Ω whose boundary (wall) is $\partial\Omega$; see Fig. 1. Here we aim to develop models of pedestrian movements in a unidirectional corridor, that is, all the pedestrians walk from one side towards the other side of the corridor (for example, from left to right). The length of the corridor is L and the width is W.

A cartesian coordinate system is placed on the corridor with the moving direction along the corridor labeled by the x-direction, and its orthogonal direction labeled by the y-direction. To study the dynamic behavior of pedestrians, the corridor is divided into m isometric virtual lanes in the y-direction and each lane can contain only one pedestrian in width. Within each lane, the pedestrians are viewed as a one-line group of agents moving along the positive direction of the x axis. However, pedestrians can change lanes as a movement along the y-direction. Consequently, the two-dimensional movements of pedestrians can be accommodated in this modeling framework. The walking space structure is shown in Fig. 1.

The pedestrian movements are modeled as a two-time-scale hybrid system. (1) The movement along the x-direction is in continuous time and is real-valued, governed by a differential equation; (2) the lane change movement along the y-direction is in the same time domain as the x-direction but occurs only every τ seconds and has a finite state space $\{1, 2, \dots, m\}$. Typically, a pedestrian can move towards the group speed asymptotically (a consensus-type flocking behavior) relatively fast after changing a lane. To simplify analysis, we assume that lane changes occur at the discrete time instances $k\tau$ with a constant interval τ. Within the time window $[k\tau, (k+1)\tau)$, no lane change takes place and an analysis on the lane dynamics will be performed along the x direction. This leads to a two-time-scale framework that allows dynamic system analysis on the x- and y-directions be performed individually.

This space representation permits a resemblance of pedestrian crowds to vehicle traffics on highways [43], [44]. However the number of pedestrians in each lane and the number of lanes are typically much larger than those of vehicle dynamic models.

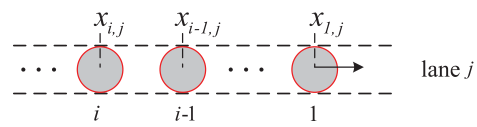

Fig. 2. The representation of pedestrians in lane j.

In our model, each pedestrian may be represented by a circle of radius r, similar to the social force model [11]. The location of a pedestrian at time t is determined by $(x(t), y(t))$. Consider a crowd of n pedestrians, indexed by subscript i. The position of each pedestrian is represented by the position of the center of the circle, $[x_i, y_i]$. x_i is a continuous variable taking values in $[0, L]$, and $y_i \in \{1, 2, \dots, m\}$.

The following assumptions form the basis of our models to capture a normal crowd behavior in congested places [11]:

- All the pedestrians and all the lanes are considered to be identical. Given the width W of the corridor and the radius r of the pedestrians, the width of a virtual lane is $W/\lfloor \mathrm{W/2r} \rfloor$.
- The pedestrians move under normal situations without panicking. As a result, the motions of pedestrians are mainly determined by heuristic-based walking behavior rather than physical interactions [13].
- Pedestrians like to walk with an individual speed which corresponds to the most comfortable one.
- Pedestrians want to reach the destination as fast as possible.
- Pedestrians keep a certain distance from other pedestrians and boundaries. The inter-person distance decreases with increasing pedestrian density in a lane.
- In the x-direction, we assume all pedestrians have the same movement rules: they observe and follow the pedestrians in front of them.
- In the y-direction, the lane changing decisions of the pedestrians are based on the current states and don't need any past information. By assigning probabilities on the decisions, they create a Markov chain whose states are the number of the pedestrians in the virtual lanes. This Markov chain is aperiodic and positive recurrent.

B. Pedestrian Dynamics in the x-Direction

First, we assume that within the time interval $[k\tau, (k+1)\tau)$, the pedestrians walk forward in the x-direction and there is no lane changing in the y-direction. Suppose that there are n_j pedestrians in lane j, $x_{i,j}(t)$ is the position of the ith pedestrian in lane j at time t. The representation of pedestrians in lane j is shown in Fig. 2. Assume that there is no overtaking in one lane (overtaking will involve a lane change to a faster lane, speeding up, and then a lane change again in a later $k\tau$). A pedestrian always tries to follow the pedestrian in front of him/her and keeps a comfortable (speed-dependent) distance [45]. In the social field concept [10], pedestrians tend to keep a certain distance with other pedestrians. Some short-range repulsive potentials can be designed to reflect this effect [11], [46].

In principle, the faster the crowd moves, the longer the distance in between the pedestrians. This distance-speed relationship is problem specific and will be generically represented by $v = f(d)$, where v is the speed and d is the front inter-person distance. $f(\cdot)$ is continuously differentiable and strictly increasing in the range of $0 < d_{min} \leq d \leq d_{max}$. We assume $L/d_{max} \leq n_j \leq L/d_{min}$. For our theoretical development, we also use an affine relationship $v = c_1 d + c_2$ to represent this function, where $c_1 > 0$ and c_2 are constants. This speed-distance relationship is quite similar to the optimal velocity model in vehicle traffic systems [43] and the pedestrian crowd model in [41]. We assume all the pedestrians have the same relationship.

The pedestrians in lane j have dynamics

$$\dot{x}_{i,j} = v_{i,j}, \quad i = 1, \ldots, n_j,$$

where $v_{i,j}$ is the "control" input. In other words, a pedestrian controls his/her walking speed based on the information he/she perceives on the front inter-person distance.

While the x-direction movement is a longitude motion, to study dynamic and asymptotic behavior of the group (flocking behavior), it is convenient and customary to view the movement in a circular track of length L. That is, the pedestrians move in a circle of length L and the $(n_j + 1)th$ pedestrian is identical to the first one [43]. This approach is common in the study of pedestrian flows [47], [48] and will be adopted in this paper for theoretical analysis.

As a result, we define the inter-person distances as follows: if $x_{i,j} - x_{i+1,j} \geq 0, d_i = x_{i,j} - x_{i+1,j}$; if $x_{i,j} - x_{i+1,j} < 0$, $d_i = L + (x_{i,j} - x_{i+1,j})$, $i = 1, 2, \ldots, n_j$. It follows that

$$\begin{aligned} \dot{d}_1 &= v_{1,j} - v_{2,j} \\ &\vdots \\ \dot{d}_{n_j} &= v_{n_j,j} - v_{1,j} \end{aligned}$$

Since the total distance is fixed at L, we have the physical constraint

$$\sum_{i=1}^{n_j} d_i = L. \tag{1}$$

By using the control action $v_{i,j} = f(d_{i-1})$, we obtain

$$\begin{aligned} \dot{d}_1 &= f(d_{n_j}) - f(d_1) \\ &\vdots \\ \dot{d}_{n_j} &= f(d_{n_j-1}) - f(d_{n_j}) \end{aligned}$$

subject to (1).

It is noted that at the steady state, the pedestrians move in a uniform lane speed $v_{1,j} = \cdots = v_{n,j} := v_j$. The lane speeds $\{v_1, \ldots, v_m\}$ will be used in the y-direction dynamic system models.

C. Pedestrian Dynamics in the y-Direction

In this subsection, we consider the lane changing of pedestrians in the y-direction. In contrast to the x-direction dynamics, the movement in the y direction, namely, the lane changes happen relatively infrequently. Different from continuous dynamics, the process values in a discrete set $\{1, \ldots, m\}$, the m virtual lanes. In fact, the y direction movements only happen at some time epoch. Although the movements still on the same time horizon as the x direction movements, with a slight abuse of notation and for simplicity and convenience, we use discrete time as the time indicator. If a pedestrian wants to walk faster and gets to his/her destination as soon as possible, he/she is likely to stay in or change to the lane with a higher speed. We assume that a pedestrian can only change one lane at a time. This implies that the pedestrian has three choices each time if there is no wall on either side: the pedestrian can change to the left, change to the right, or stay in the current lane.

The lane changing decisions of the pedestrians are based on the current states and don't need any past information. We will consider the numbers of the pedestrians in the virtual lanes which are the states of a Markov chain. n pedestrians are distributed in the m virtual lanes. For simplicity, we assume that n/m is an integer. Let $y_i(k)$ be the lane position of the ith pedestrian at time step k, $y_i(k) \in \{1, 2, \ldots, m\}$, $i = 1, 2, \ldots, n$. The number of pedestrians in the jth lane at time step k is

$$n_j(k) = \sum_{i=1}^{n} I_{\{y_i(k)=j\}} \tag{2}$$

where I_A is the indicator function of the set A: $I_A = 1$ if the condition A is satisfied; and $I_A = 0$ otherwise.

We assume the total number of pedestrians does not change with time, which implies that for all k,

$$\begin{cases} \sum_{j=1}^{m} n_j(k) = n \\ n_j(k) \in [0, n], \quad n_j(k) \text{ is integer}, \quad j = 1, 2, \ldots, m. \end{cases} \tag{3}$$

The lane speed difference is observed by pedestrians with an error: $v_j(k) - v_{j+1}(k) + \xi_{j,j+1}(k)$. Here $\xi_{j,j+1}(k)$ is the measurement/perception error which is assumed to be a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and finite variance. This reflects that fact that pedestrians are looking and guessing the speed difference with an error.

The lane change decision function is possibly defined by: $g(v_j(k) - v_{j+1}(k) + \xi_{j,j+1}(k))$. This function models how many people will likely to decide to change to faster lanes when they see a speed difference. By its physical meaning, this is a monotone function and symmetric to 0 (odd function). In other words, the larger the speed difference, the more people will try to change to the lane of higher speed. In subsequent development, we will use a simple gain function of a positive gain g in our convergence analysis.

Let $u_j(k) = -g(v_{j-1}(k) - v_j(k) + \xi_{j-1,j}(k)) + g(v_j(k) - v_{j+1}(k) + \xi_{j,j+1}(k))$. Since a lane change is a reduction of pedestrians for one lane but increase of the same number for the other lane, we can write this as the following updating algorithm

$$n_j(k+1) = n_j(k) + u_j(k), \quad j = 1, \ldots, m \tag{4}$$

subject to (3). If we look at the $(j+1)$th lane, the term $-g(v_j(k) - v_{j+1}(k) + \xi_{j,j+1}(k))$ will appear there, indicating the same value but a negative sign will be in effect. This will ensure that the condition $\sum_{j=1}^{m} x_j(k) = n$ is always satisfied. As the pedestrian numbers must be integers, we restrict $u_j(k)$ to $-\lceil g(v_{j-1}(k) - v_j(k) + \xi_{j-1,j}(k))\rceil + \lceil g(v_j(k) - v_{j+1}(k) + \xi_{j,j+1}(k))\rceil$ and $n_j(k)$ is always bounded by 0 and n, where $\lceil\cdot\rceil$ is the smallest integer above the value. The dynamic systems and lane changing decisions allow us to apply the theoretical results from the stochastic approximation method to resolve convergence issues in the following part.

III. Convergence Analysis in the x-Direction

A. Dynamic Systems

For theoretical analysis on the dynamic evolution of speed and distance in the x direction, we will assume an affine relationship $v = c_1 d + c_2$, where $c_1 > 0$ and c_2 are constants, in the range of d. As a result, the dynamics of the crowd in the x-direction become

$$\begin{aligned}\dot{d}_1 &= c_1(d_{n_j} - d_1)\\ &\vdots\\ \dot{d}_{n_j} &= c_1(d_{n_j-1} - d_{n_j})\end{aligned}$$

subject to $\sum_{i=1}^{n_j} d_i = L$. In the following derivations, we use $\mathbf{1}$ as a generic symbol of a vector of all 1s without specifying its dimension, which is always clear from the context. Namely, $\mathbf{1} = [1, 1, \ldots, 1]'$.

The system can be expressed as

$$\dot{d} = Md \tag{5}$$

where $d = [d_1, \ldots, d_{n_j}]'$, subject to $\mathbf{1}'d = L$, and

$$M = \begin{bmatrix} -c_1 & 0 & \ldots & 0 & c_1 \\ c_1 & -c_1 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & c_1 & -c_1 \end{bmatrix}.$$

B. Convergence Analysis

We first establish several lemmas.

Lemma 1: M has a simple eigenvalue at 0, and all remaining eigenvalues are in the open left half plane of the complex plane.

Proof: M is a circular matrix with its first row elements $[-c_1, 0, \ldots, 0, c_1]$, see [49] for detail. As a result, its associated polynomial is $p(x) = -c_1 + c_1 x^{n_j - 1}$. Its eigenvalues are

$$\lambda_l = -c_1 + c_1 e^{\frac{2\pi l(n_j-1)}{n_j} i}, \quad l = 0, 1, \ldots, n_j - 1$$

where i is the imaginary unit. Since

$$e^{\frac{2\pi l(n_j-1)}{n_j} i} = e^{2\pi l i} e^{-\frac{2\pi l}{n_j} i} = e^{-\frac{2\pi l}{n_j} i}$$

the eigenvalues are

$$\lambda_l = -c_1 + c_1 e^{-\frac{2\pi l}{n_j} i}, \quad l = 0, 1, \ldots, n_j - 1 \tag{6}$$

whose real parts are

$$-c_1 + c_1 \cos(\frac{2\pi l}{n_j}), \quad l = 0, 1, \ldots, n_j - 1.$$

Consequently, we have $\lambda_0 = 0$, and the real parts of the remaining eigenvalues satisfy

$$\Re(\lambda_l) = -c_1 \left(1 - \cos\left(\frac{2\pi l}{n_j}\right)\right) < 0, \quad l = 1, \ldots, n_j - 1$$

which imply that these eigenvalues are in the open left half plane of the complex plane. □

We now decompose M into

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where M_{11} is the $(n_j - 1) \times (n_j - 1)$ submatrix

$$M_{11} = \begin{bmatrix} -c_1 & 0 & \cdots & 0 \\ c_1 & -c_1 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & -c_1 \end{bmatrix}; \quad M_{12} = \begin{bmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Define

$$\widetilde{M} = M_{11} - M_{12}\mathbf{1}'.$$

Let λ_m be defined as in (6).

Lemma 2: $\widetilde{M}$ is full rank and has the eigenvalue set $\{\lambda_l, l = 1, \ldots, n_j - 1\}$.

Proof: It can be directly verified that $\widetilde{M}v = 0$ has a unique solution $v = 0$. Hence, $\widetilde{M}$ is full rank and $\lambda_0 = 0$ is not an eigenvalue of $\widetilde{M}$.

We now show that any non-zero eigenvalue of M is also an eigenvalue of $\widetilde{M}$. Suppose that λ is an eigenvalue of M with a corresponding eigenvector v. Decompose v into

$$v = \begin{bmatrix} \widetilde{v} \\ v^* \end{bmatrix}$$

where $\widetilde{v}$ is of dimension $n_j - 1$.

Since $0 = \mathbf{1}'Mv = \lambda \mathbf{1}'v$ and $\lambda \neq 0$, we have $\mathbf{1}'v = 0$ or $v^* = -\mathbf{1}'\widetilde{v}$. By considering the first $n_j - 1$ equations in $Mv = \lambda v$, we obtain

$$M_{11}\widetilde{v} + M_{12}v^* = \lambda\widetilde{v}$$

Substituting $v^* = -\mathbf{1}'\widetilde{v}$ into the equation leads to

$$M_{11}\widetilde{v} - M_{12}\mathbf{1}'\widetilde{v} = \widetilde{M}\widetilde{v} = \lambda\widetilde{v}$$

which implies that λ is also an eigenvalue of $\widetilde{M}$.

Since $\widetilde{M}$ has $n_j - 1$ eigenvalues and $\lambda_0 = 0$ is not its eigenvalue, it follows that the set of its eigenvalues must be $\{\lambda_l, l = 1, \ldots, n_j - 1\}$. □

Theorem 1: (1) System (5) has a unique equilibrium point $\frac{L}{n_j}\mathbf{1}$.

(2) From any initial condition, the solution $d(t)$ to the dynamic system (5) converges to its equilibrium point

$$\lim_{t\to\infty} d(t) = \frac{L}{n_j}\mathbf{1}.$$

(3) The convergence rate is exponential. Namely, from any given bounded initial error $e(0) = d(0) - \frac{L}{n_j}\mathbf{1}$, there exists $\sigma > 0$ and $c > 0$ such that $e(t) = d(t) - \frac{L}{n_j}\mathbf{1}$ is bounded by

$$\|e(t)\|_2 \leq ce^{-\sigma t}, \quad t \geq 0 \tag{7}$$

where $\|\cdot\|_2$ is the Euclidean norm.

Proof: The proof is divided in the following steps.

(1) Since $\lambda_0 = 0$ is a simple eigenvalue and $M\mathbf{1} = 0$, the set of null space of M is $\{a\mathbf{1}, a \in \mathbb{R}\}$. Under the constraint $\mathbf{1}'d = L$, the unique solution is $a\mathbf{1}'\mathbf{1} = an_j = L$, or $a = L/n_j$.

(2) Define $e = d - \frac{L}{n_j}\mathbf{1}$. Then

$$\dot{e} = Md = M(e + \frac{L}{n_j}\mathbf{1}) = Me$$

since $M\mathbf{1} = 0$, subject to the constraint $\mathbf{1}'e = \mathbf{1}'(d - \frac{L}{n_j}\mathbf{1}) = L - L = 0$.

Decompose e into

$$e = \begin{bmatrix} \widetilde{e} \\ e^* \end{bmatrix}$$

where $\widetilde{e}$ is of dimension $n_j - 1$. From $\mathbf{1}'e = 0$, we have $e^* = -\mathbf{1}'\widetilde{e}$. Consequently,

$$\dot{\widetilde{e}} = M_{11}\widetilde{e} + M_{12}e^* = M_{11}\widetilde{e} - M_{12}\mathbf{1}'\widetilde{e} = \widetilde{M}\widetilde{e}.$$

By Lemmas 1 and 2, $\widetilde{M}$ is Hurwitz. Hence from any initial condition $\widetilde{e}(0)$

$$\widetilde{e}(t) = e^{\widetilde{M}t}\widetilde{e}(0) \to 0, \quad t \to \infty.$$

This further implies that

$$e^*(t) = -\mathbf{1}'\widetilde{e}(t) \to 0, \quad t \to \infty.$$

Therefore,

$$e(t) \to 0, \quad t \to \infty.$$

(3) Define

$$\sigma_0 = \min_{l=1,\ldots,n_j-1} c_1 \left(1 - \cos\left(\frac{2\pi l}{n_j}\right)\right) > 0.$$

Then, for any $\sigma < \sigma_0$, there exists $c_0 > 0$ such that the matrix exponential is bounded by

$$\|e^{\widetilde{M}t}\| \leq c_0 e^{-\sigma t},$$

where the matrix norm $\|\cdot\|$ is the largest singular value. This implies

$$\begin{aligned} \|\widetilde{e}(t)\|_2 &= \|e^{\widetilde{M}t}\widetilde{e}(0)\|_2 \\ &\leq \|e^{\widetilde{M}t}\|\|\widetilde{e}(0)\|_2 \\ &\leq c_0 e^{-\sigma t}\|\widetilde{e}(0)\|_2. \end{aligned}$$

Furthermore,

$$\begin{aligned} |e^*(t)| &= |\mathbf{1}'\widetilde{e}(t)| \\ &\leq \sqrt{n_j - 1}\|\widetilde{e}(t)\|_2 \\ &\leq \sqrt{n_j - 1}c_0 e^{-\sigma t}\|\widetilde{e}(0)\|_2. \end{aligned}$$

These bounds imply (7). □

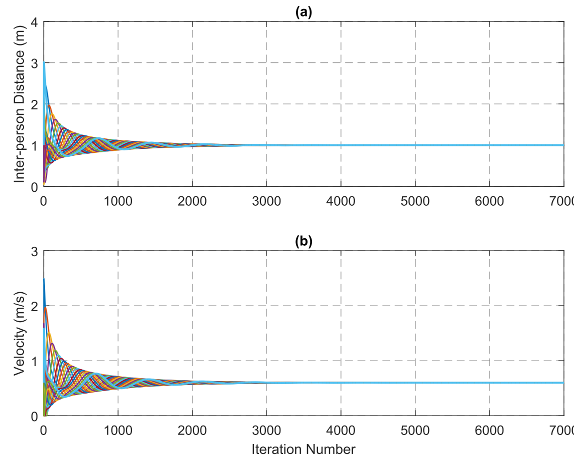

Fig. 3. Pedestrian movement simulation with $v = 0.94d - 0.34$. Different trajectories correspond to different pedestrians. (a) Inter-person distance trajectories. (b) Velocity trajectories of the pedestrians.

Remark 1:

1) Theorem 1 claims that all pedestrians in lane j will converge to a uniform distance distribution $d_1 = \cdots = d_{n_j} = L/n_j$.
2) It is noted that M in (5) has one simple eigenvalue at 0 and the remaining eigenvalues are all stable. The actual convergence rate to the final balanced distance is exponential and depends on these eigenvalues. Due to the fast exponential convergence rates, our two-time-scale framework is valid.
3) From $v_{i+1,j} = c_1 d_i + c_2$, we have that all pedestrians will asymptotically move with a group speed $v_j = c_1 L/n_j + c_2$ for lane j.

C. Illustrative Examples

We will demonstrate pedestrian behaviors in one lane by several simulation examples. We assume that the radii of the pedestrians are the same and equal to $0.2m$. The length of the lane is $L = 20m$ and the width is $0.4m$, so the lane can only contain one pedestrian in width. At first, the pedestrians are randomly distributed in the lane and their initial velocities are zero. The pedestrian number in the lane is 20. The dynamics are determined by (5). Based on these conditions, simulations are performed.

Example 1: In [45], a distance-velocity relationship is given by $d = a + bv$ with $a = 0.36m$ and $b = 1.06s$. Here we take $v = 0.94d - 0.34$ and assume all the pedestrians have the same relationship. In reality, a pedestrian will stop rather than move back when he/she is too close to the front pedestrian [46]. Also a pedestrian has an upper bound velocity due to physical limitations. Furthermore we limit the pedestrian velocities to $[0, v_{\max}]$, with $v_{\max} = 3m/s$. The simulation results are shown in Fig. 3, the subplot (a) shows how inter-person distances of different pedestrians, whose initial inter-person distances are different from each other, become gradually distributed to the consensus (they all equal $L/n = 1m$). The subplot (b) shows the convergence of the velocities of the pedestrians

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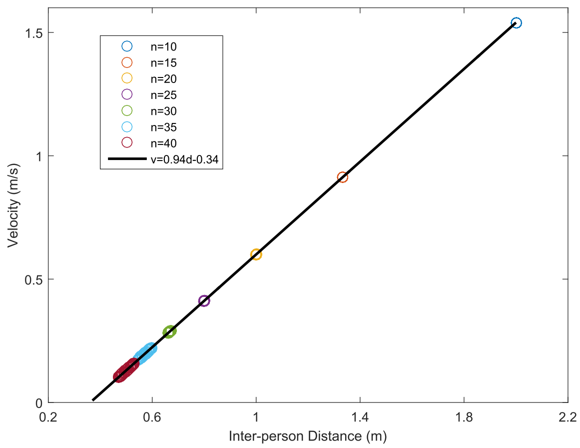

Fig. 4. The dependence between velocity and inter-person distance according to the simulation results with $n_j = 10, 15, 20, 25, 30, 35, 40$. The data lie on the line of function f.

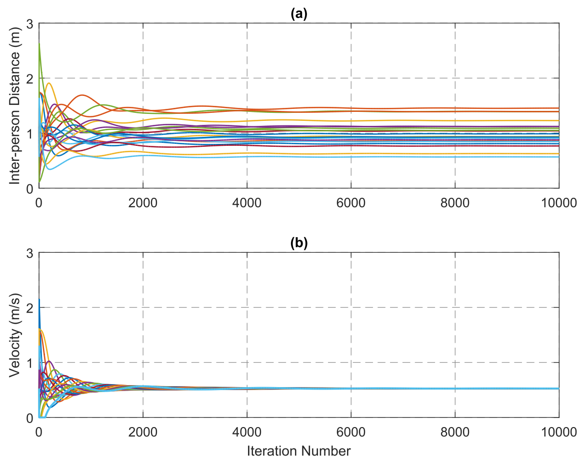

Fig. 5. Pedestrian movement simulation with different c_1 and c_2. (a) Inter-person distance trajectories. (b) Pedestrian velocity trajectories.

(equal to $0.94 * 1 - 0.34 = 0.6m/s$). Fig. 4 shows the convergent velocities of different pedestrian numbers. These results confirm that all pedestrians will asymptotically move with a group speed $v = 0.94d - 0.34 = 0.94L/n - 0.34$.

Example 2: In this case, we consider the case in which each pedestrian has slightly different c_1 and c_2 to reflect more realistic person-to-person variations in perception and speed/distance preferences. We assume that $c_1(i)$ is distributed according to a normal-distribution with $\mu_1 = 0.2$ and standard deviation $\sigma_1 = 0.94$. $c_2(i)$ is also normal distributed with mean $\mu_2 = 0.1$ and standard deviation $\sigma_2 = -0.34$. Fig. 5 shows the simulation results. From subplot (a), we can see that the inter-person distances converge to different values. This is consistent with the differences in pedestrians' speed-distance functions. Subplot (b) shows that the velocities still converge to a consensus but to a smaller velocity (about 0.52m/s) due to the variations of the functions.

Example 3: In this example, we will use a nonlinear function f rather than affine functions. We use the fundamental diagram [50] as the control input. Here we take n/L as the pedestrian density in one lane and assume that it is equal

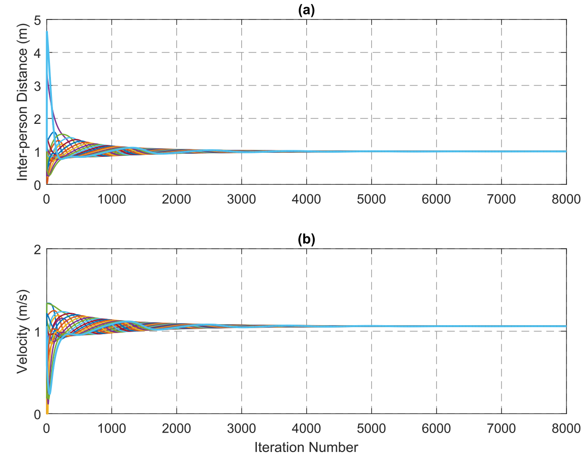

Fig. 6. Pedestrian movement simulation with nonlinear function f. (a) Inter-person distance trajectories. (b) Pedestrian velocity trajectories.

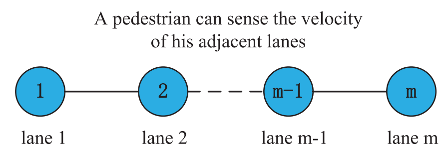

Fig. 7. The corridor is represented by a network.

to the two-dimensional density for simplicity. Hence $v = 1.34[1 - e^{-1.913(d-0.18)}]$. See [48] for the transformation from line-density to an area-density. Fig. 6 shows the convergence of the inter-person distances and velocities.

IV. Convergence Analysis in the y-Direction

A. Dynamic Systems

The y direction has a state space of m values. Each pedestrian can only observe the speeds of his/her current lane and immediate next ones. As a result, the corridor can be viewed as a network with m nodes and $m-1$ edges, as shown in Fig. 7. The nodes are linked by an information topology, represented by an undirected graph ζ whose element $(j-1, j)$ indicates an information exchange between node $j-1$ and node j. This means the pedestrians in lane $j-1$ can observe the velocity in lane j.

Denote the state vector $n(k) = [n_1(k), \ldots, n_m(k)]'$ and $u(k) = [u_1(k), \ldots, u_m(k)]'$, where the prime means the transpose of the vector. We can write (4) compactly as

$$n(k+1) = n(k) + u(k) \tag{8}$$

By assumption, the lane speed is related to the pedestrian number in that lane. For simplicity of analysis, we assume that the pedestrian makes a lane change decision based on the pedestrian number difference of the two lanes. Namely, we let $u_i(k) = h(n_{j-1}(k) - n_j(k) + \eta_{j-1,j}(k)) - h(n_j(k) - n_{j+1}(k) + \eta_{j,j+1}(k))$, where $h(\cdot)$ is the lane change decision function based on pedestrian number difference and η is the perception error. h and η have the same properties with g and ξ, respectively.

Denote $\hat{n}_{j-1,j}(k) = n_{j-1}(k) - n_j(k) + \eta_{j-1,j}(k)$ as the pedestrian number difference between lane $j-1$ and lane j. Let $\alpha(k)$ and $\eta(k)$ be the $m-1$ dimensional vectors that contain all the $\hat{n}_{j-1,j}(k)$ and $\eta_{j-1,j}(k)$ in a selected order, respectively. As a result,

$$\alpha(k) = H_1 n(k) - H_2 n(k) + \eta(k) = Hn(k) + \eta(k) \tag{9}$$

where H_1 is an $(m-1)\times m$ matrix whose rows are elementary vectors such that if the l-th element of $\alpha(k)$ is $\hat{n}_{j-1,j}(k)$ then the l-th row in H_1 is the row vector of all zeros except for a "1" at the $j-1$-th position. H_2 is an $(m-1)\times m$ matrix whose rows are elementary vectors such that if the l-th element of $\alpha(k)$ is $\hat{n}_{j-1,j}(k)$ then the l-th row in H_2 is the row vector of all zeros except for a "1" at the j-th position, and $H = H_1 - H_2$.

Due to network constraints, the information can only be used by nodes $j-1$ and j. When the changing number of pedestrians is linear, time invariant, and memoryless, we have $h(\hat{n}_{j-1,j}(k)) = \mu(k) g_{j-1j}\hat{n}_{j-1j}(k)$, where $\mu(k)$ is a global scaling factor that will be used in state updating algorithms as the recursive step size and g_{j-1j} is the gain. Let G be the $(m-1)\times(m-1)$ diagonal matrix that has g_{j-1j} as its diagonal element. In this case, $u(k) = -\mu(k)H'G\alpha(k)$.

As μ is a global variable, we may represent $u(k)$ equivalently as

$$\begin{aligned} u(k) &= -\mu(k)H'G(Hn(k) + \eta(k)) \\ &= \mu(k)Pn(k) + \mu(k)Q\eta(k) \end{aligned} \tag{10}$$

with $P = -H'GH$ and $Q = -H'G$. Then (8) becomes

$$n(k+1) = n(k) + \mu(k)(Pn(k) + Q\eta(k)) \tag{11}$$

B. Convergence Analysis

Remark 2: A square matrix $\tilde{S} = (\tilde{s}_{ij})$ is a generator of a continuous-time Markov chain if $\tilde{s}_{ij} \geq 0$ for all $i \neq j$ and $\sum_j \tilde{s}_{ij} = 0$ for each i. Also, a generator or the associated continuous-time Markov chain is irreducible if the system of equations

$$\begin{cases} \upsilon\tilde{S} = 0 \\ \upsilon\mathbf{1} = 1 \end{cases} \tag{12}$$

has a unique solution, where $\upsilon = [\upsilon_1, \ldots, \upsilon_m] \in \mathbb{R}^{1\times m}$ with $\upsilon_i > 0$ for each $i = 1, \ldots, r$ is the associated stationary distribution. Under assumption of $g_{j-1j} > 0$ and the fact that ζ is connected, we can show that (1) P has rank $m-1$ and is negative semi-define, and (2) P is a generator of a continuous-time Markov chain, and is irreducible; see [51] for a proof.

Next we will discuss the quantization effect since $n(k)$ must be integer values. We define a new variable $\zeta_j(k)$ as $\zeta_j(k) = n_j(k)/n$, $\zeta_j(k) \in [0, 1]$. Because $1/n \to 0$ as $n \to \infty$, we can consider $\zeta_j(k)$ as a continuous variable when the pedestrian number is large enough. Denote the new state vector $\zeta(k) = [\zeta_1(k), \ldots, \zeta_m(k)]'$. Then equation (11) becomes

$$\zeta(k+1) = \zeta(k) + \mu(k)(P\zeta(k) + Q\varsigma(k)) \tag{13}$$

where $\varsigma(k) = N^{-1}\eta(k)$ and N^{-1} is a diagonal matrix whose diagonal elements is $1/n$.

The original state variable is given by

$$n(k) = \lfloor n\zeta(k)\rfloor \tag{14}$$

where the floor operation is taken componentwise.

Assumption 1: The step size $\mu(k)$ satisfies the following conditions: $\mu(k) \geq 0, \mu(k) \to 0$ as $k \to \infty$, and $\sum_k \mu(k) \to \infty$.

Theorem 2: Under assumption 1, for any given initial distribution $n(0)$, if the pedestrians change lanes according to (11), then $\zeta(k) \to (1/m)\mathbf{1}$ and thus $n(k) \to (n/m)\mathbf{1}$ with probability 1 (w.p.1) as $k \to \infty$.

Proof: Following the standard techniques of the ODE (ordinary differential equation) approach [52], define $t_k = \sum_{j=0}^{k-1}\mu(j)$, the piecewise constant interpolation $\zeta^0(t) = \zeta(k)$ for $t \in [t_k, t_{k+1})$, the shift sequence $\zeta^k(t) = \zeta^0(t + t_k)$, and the process $m(t) = \max\{k : t_k \leq t\}$. Then it can be shown that $\{\zeta^k(\cdot)\}$ is equicontinuous in the extended sense; see [52, pp.101–102]. It follows from the extension of Arzeá-Ascoli theorem [52, p. 102], there is a convergent subsequence $\{\zeta^{k_\ell}(\cdot)\}$ with limit $\zeta(\cdot)$ such that the convergence is in the sense of probability one convergence and is uniform on any compact time interval. Moreover, the limit of the convergent subsequence is a solution of the equation

$$\dot{\zeta} = P\zeta. \tag{15}$$

Note that because (15) is linear, it has a unique solution for each intial condition.

The solutions of $P\zeta = 0$ give the equilibrium points of (15). Since P is a generator of a continuous-time Markov chain, the equilibrium points belong to the set $C = \{q\mathbf{1}; q \in \mathbb{R}\}$. It can be shown that the set is invariant and asymptotically stable in the sense of Lyapunov. Define $V : \mathbb{R}^m \mapsto \mathbb{R}$ by $V(\zeta) = \zeta'\zeta/2$. Then $V(0) = 0$, $V(\zeta) > 0$ for $\zeta \neq 0$, and $V(\zeta) \to \infty$ as $|\zeta| \to \infty$. Moreover, the derivative of $V(\zeta)$ along the solution of (15) is $\dot{V}(\zeta) = \zeta'P\zeta$. Similar to [51], as P is negative semi-define, we know $\zeta'P\zeta \leq 0$. By the invariant set theorem, the solution of (15) converges to C as $t \to \infty$. That is, C is a globally asymptotically stable set. Then using the techniques in [52, Ch. 5], we can show that $\zeta(k) \to C$ w.p.1 as $k \to \infty$. Furthermore, by virtue of (12), the point in $\zeta \in C$ satisfying the constraint $\zeta'\mathbf{1} = 1$ is the single point $\zeta = (1/m)\mathbf{1}$, we conclude that $\zeta(k)$ converges to $(1/m)\mathbf{1}$ w.p.1. From (14), we obtain $n(k)$ converges to $(n/m)\mathbf{1}$ w.p.1. □

Remark 3:

1) Theorem 2 shows that the number of pedestrians in each lane will converge to n/m, which means that all the pedestrians will move with the same velocity. The convergent state corresponds to the fast and smooth laminar flow state which is similar with fluid dynamics [4].
2) The convergence analysis implies that the more pedestrians are in this corridor, the slower their convergent velocity will be. This is in agreement with the fundamental diagram [48].
3) This approach can be extended to the general two-dimensional movements without lane division in a stochastic framework. Some related theoretical results are available, see [53], [54]. In contrast to the nonlinear ODEs in our model, the dynamics of some multi-agent systems are modeled in the framework of partial difference equations [55], [56]. In these works, the analyzing

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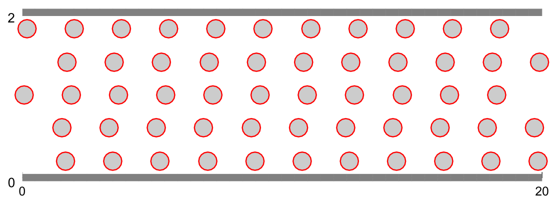

Fig. 8. The snapshot of the pedestrian movement after lane changing.

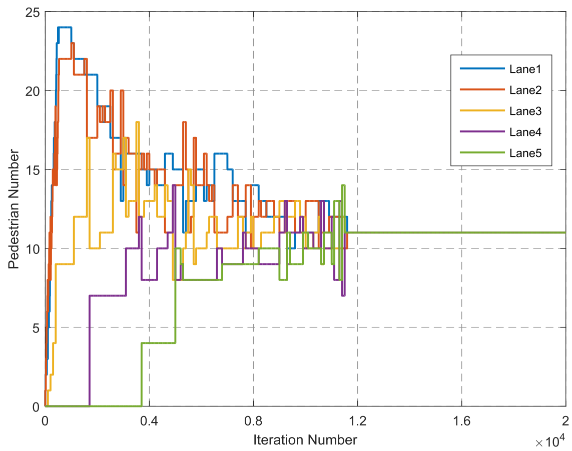

Fig. 9. The changing of pedestrian numbers in each lane.

of the systems can be realized by decoupling of spatial and temporal scales.

C. Illustrative Examples

Example 4: In this example, we consider the lane changing behavior of pedestrian crowds. Consider a corridor with length $20m$ and width $2m$. The number of virtual lanes is $m = 5$. The total number of pedestrians is 55. To reflect the lane changing behavior, we assume that initially the pedestrians enter the corridor from lane 1 and lane 2 randomly. The initial velocities of the pedestrians are assumed to be Gaussian distributed with mean $1.5m/s$ and standard deviation $0.3m/s$. $g_{j-1,j} = 0.25$. The noises are i.i.d. sequences of Gaussian noises with mean zero and variance 4. When a pedestrian changes lane, the y-coordinate of that pedestrian changes. For simplicity, if a pedestrian collides with others after changing lanes, we just place the pedestrian in the middle of the two nearest pedestrians to him/her in the destination lane. The lane changing behavior occurs every 100 simulation steps. The other conditions are the same with those in the x-directions.

Fig. 8 is a snapshot of the final state of pedestrians after lane changing and shows that the numbers reach a consensus distribution (each lane has 11 pedestrians). Fig. 9 shows the trajectories of pedestrian numbers. From this figure, we can see that the pedestrians come into the corridor from lane 1 and lane 2. As there are no pedestrians in other lanes at first, pedestrians will change to these lanes to have higher velocities. Because the pedestrian numbers should be integers, the trajectories are staircase curves. After lane changing, each lane has the same pedestrian number in the end.

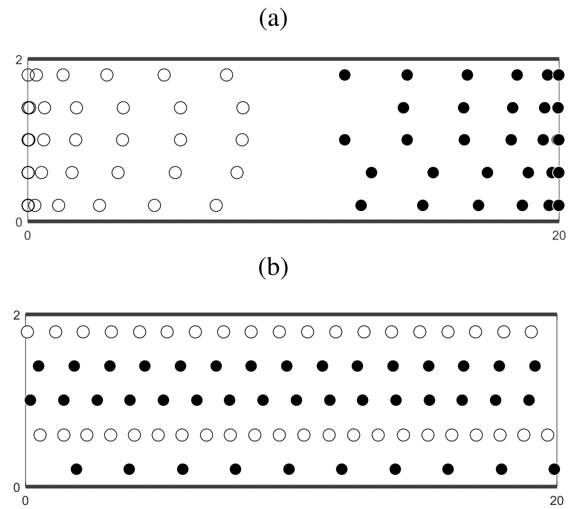

Fig. 10. The lane formation phenomenon in pedestrian counter flows. (a) The initial state. Forty pedestrians (full circles) come into the corridor from the right-hand side and walk to the left and the other forty pedestrians (hollow circles) have the opposite walking direction. (b) The final state. The pedestrians with different walking directions form into different lanes.

Example 5: In this example, we extend our model to reproduce the lane formation phenomenon observed in the real field [15]. The simulation scenario and the initial distribution of speeds of pedestrians are the same as those in the previous example. There are forty pedestrians (full circles) coming into the corridor from the right-hand side and walking toward left. The other forty pedestrians (hollow circles) come into the corridor from the left-hand side and walk toward right (as shown in Fig. 10 (a)). We still take the periodic boundary conditions. A pedestrian i will follow the front pedestrian who has the same walking direction with him/her. If he/she meets a pedestrian who has the opposite walking direction, he/she will change to the adjacent lane with a higher walking velocity with probability P_i. Here we take $P_i = 0.5$ for all pedestrians. The walking velocity of a lane is assumed to be the average velocity of all the pedestrians in that lane. From Fig. 10 (b), we can see that after lane changings, the pedestrians with different walking directions form into different lanes.

V. Conclusion

This paper introduces a new model for pedestrian crowd dynamics. The model is a two-time-scale hybrid system. Pedestrian dynamics represented by this model structure are analyzed rigorously and their convergence properties are established. The models can be used potentially to represent crowd movements, study impact of social-phycological variations, develop potential crowd control strategies, and investigate measures to avoid accidents. For instance, we can study non-steady-state states (stop-and-go and turbulent flows) and the transitions between stable and unstable states. Findings from such studies can be used to provide warning on critical crowd situations. We can study the ways to place barriers and fences to force pedestrians to walk in more organized lanes. The crowd evacuation problems on a network whose edges represent different and interconnected corridors can also be discussed.

As an introductory work for this modeling methodology, this paper has many potential extensions. Some essential practical constraints on pedestrian movements can eventually be integrated, including their positions to avoid collision, their velocities as functions of environments such as obstacles, randomness in pedestrian behavior. These constraints can be accommodated with increased complications in mathematics analysis. For example, introducing position and speed bounds will lead to iterating algorithms with projection. Stability and convergence analysis is still possible but must deal with nonlinear systems with saturation. After adding stochastic observation noises, this will become stochastic approximation algorithms with interior projection.

Within the virtual lane approach, there are rooms and flexibility to accommodate broader scenarios. It is easy to change the shape of corridors by adding, deleting and intersecting virtual lanes. Then we can study pedestrian flows at bottlenecks and crossing pedestrian flows. By subdividing the virtual lanes (letting one pedestrian occupy more than one lane), we can study the zipper effect, bypassing behavior, and group behavior. Furthermore, obstacles and traffic signs and directions can be added into the model structure to capture realistic structures of pathways and corridors. In this paper, n/m is assumed to be an integer so that convergence to a perfect consensus is possible. This condition can be relaxed and we can study quantized consensus [57] problems. Based on this basic model structure, rigorous studies of crowd control strategies become feasible, which will be discussed in our subsequent papers.

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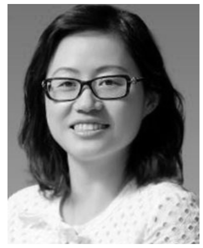

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