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A day-to-day route flow evolution process towards the mixed equilibria

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ABSTRACT

This study investigates a travelers' day-to-day route flow evolution process under a predefined market penetration of advanced traveler information system (ATIS). It is assumed that some travelers equipped with ATIS will follow the deterministic user equilibrium route choice behavior due to the complete traffic information provided by ATIS, while the other travelers unequipped with ATIS will follow the stochastic user equilibrium route choice behavior. The interaction between these two groups of travelers will result in a mixed equilibrium state. We first propose a discrete day-to-day route flow adjustment process for this mixed equilibrium behavior by specifying the travelers' route adjustment principle and adjustment ratio. The convergence of the proposed day-to-day flow dynamic model to the mixed equilibrium state is then rigorously demonstrated under certain assumptions upon route adjustment principle and adjustment ratio. In addition, without affecting the convergence of the proposed day-to-day flow dynamic model, the assumption concerning the adjustment ratio is further relaxed, thus making the proposed model more appealing in practice. Finally, numerical experiments are conducted to illustrate and evaluate the performance of the proposed day-to-day flow dynamic model.

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1. Introduction

Traffic assignment models are used to predict link or route flows in the transportation networks. Traditionally, these models were formulated under the user equilibrium (UE) or stochastic user equilibrium (SUE) principle, in which no traveler can reduce his/her actual or perceived travel time by unilaterally changing routes (Sheffi, 1985). Since the UE or SUE traffic assignment models focus on the final equilibrium state, they are unable to describe driver's learning behavior and route adjustment process. As an important supplement to the traditional UE and SUE models, extensive studies have focused on the learning behavior modelling of commuters over the past decades (Cantarella, 2013; Hu and Mahmassani, 1997; Iida et al., 1992; Jha et al., 1998; Nakayama and Kitamura, 2000; Wu et al., 2013; Zhang et al., 2001) and a number of day-to-day flow dynamic models have been proposed to depict the evolution process of traffic flows before achieving the UE or SUE state (Guo and Liu, 2011; Horowitz, 1984; Smith, 1983, 1984).

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1.1. Literature review

The day-to-day flow dynamic models in the existing literature can be divided into the continuous or discrete day-to-day flow dynamic models based on their degree of time dispersion. The continuous day-to-day flow dynamic models use sufficiently short time steps, and differential equations are generally employed to formulate these models (Friesz et al., 1994; Smith and Wisten, 1995; Smith, 1983, 1984). Although they have elegant mathematical properties, they are not applicable in practice since the travel behaviors of travelers are generally adjusted in a discrete manner, e.g., daily. On the contrary, the discrete day-to-day flow dynamic models permit relatively long time steps, such as days or weekdays, and they are more reasonable from a practical point of view (Horowitz, 1984; Zhang et al., 2001). However, in order to guarantee that the discrete day-to-day flow dynamic models would evolve to an equilibrium state, some additional assumptions on the travelers' adjustment ratio are required, which will be discussed subsequently.

The day-to-day flow dynamic models can also be sorted into the route-based or link-based models. The route-based models aim to simulate the evolution process of route flows (Horowitz, 1984; Smith, 1983, 1984). They can directly reflect the adjustment process of travelers' route choice behavior. An important input parameter for the route-based models is the initial route flow pattern, and different initial route flow patterns may lead to different evolution processes. However, since the initial route flows are non-unique and unobservable in practice, the route-based models are merely limited to the theoretical studies. In order to overcome the essential shortcomings of the route-based models, the link-based models built on link flow are proposed (He et al., 2010). With the observable initial link flow pattern, the evolution process of link flows can be easily determined. However, travelers' heterogeneous route choice behavior cannot be observed in the link-based models because these models are built on the aggregate link flows.

In addition, the day-to-day flow dynamic models can be classified as deterministic or stochastic models according to the random nature of the problem being studied. Deterministic models assume that the travelers' route choice mechanism in each day is determined in advance. So the flow evolution trajectory can be explicitly predicted. On the contrary, stochastic models consider uncertainty in travelers' route choice decision-making process. They can provide the probability distribution of flow states. Obviously, stochastic models are more general than deterministic models. However, the computational burden of stochastic models prohibits their implementations in the large-scale transportation networks.

Furthermore, the day-to-day flow dynamic models can be categorized by various travelers' route choice behaviors. Prominent examples include the day-to-day flow dynamic UE models and day-to-day flow dynamic SUE models, in which travelers follow the UE or SUE behavior, respectively. We summarize the typical day-to-day flow dynamic models in Table 1.

1.2. Objectives and contributions

With the development of advanced traveler information system (ATIS), the accurate information of the road traffic condition can be provided to travelers with ease and thereby helps them make informed route choices. Hence it is reasonable to assume that travelers equipped with ATIS will follow the UE route choice behavior, whereas travelers without ATIS will choose their routes in a SUE manner. The resultant equilibria of travelers under a predefined market penetration of ATIS is referred to as the mixed equilibria of UE and SUE. From the column 5 in Table 1, it is apparent that all the existing models in the literature were formulated based on a single equilibrium concept, i.e., either UE or SUE. To the best of our knowledge, the influence of ATIS on the day-to-day flow evolution process towards the mixed equilibria has not been investigated so far. Therefore, in this paper we aim to fill this gap by proposing a discrete dynamic model to simulate the day-to-day route flow adjustment process under a predefined market penetration of ATIS.

Unlike the existing studies for a single equilibrium state (i.e., UE or SUE) in which all the travelers were assumed to follow the same route adjustment behavior, different route adjustment principles are proposed in this study for travelers with or without ATIS. Specifically, travelers with ATIS are likely to choose the shortest route under the current traffic condition for their trips in the next day, while travelers without ATIS are supposed to follow the logit-based SUE principle based on the current traffic condition. In addition, the adjustment ratio is assumed to satisfy certain conditions to guarantee the convergence of the proposed day-to-day flow dynamic model towards the mixed equilibria. Reasonable interpretations of these assumptions are presented to demonstrate that the route adjustment principle and adjustment ratio proposed in this study have rich behavioral implications other than be a mathematical expression.

The contributions of this study are fourfold: First, to simulate the evolution process of travelers' route flows with and without ATIS, we propose a discrete mixed behavior route flow dynamic model, which has not been investigated so far. Second, we specify the target flow and flow adjustment ratio, and elaborate their behavioral implications. The necessity to impose such a requirement on the flow adjustment ratio is illustrated by a counterexample in which the well-known rational behavior principle used in the continuous models is not sufficient for convergence for our discrete route flow dynamic model. Third, we show that our model may result in multiple possible route flow evolution trajectories, and present thee specific cases to refine them. Fourth, we demonstrate the convergence of our day-to-day flow dynamic model to the mixed equilibrium state even with a relaxation of the assumption on the flow adjustment ratio. Our proof essentially establishes the convergence of the partial linearization method under the (relaxation of) Goldstein rule.

The remainder of the paper is organized as follows. The mixed equilibrium behavior model and its properties are first reviewed in Section 2. We then propose a discrete day-to-day flow dynamic model in Section 3 for which the travelers' route adjustment principle and adjustment ratio are elaborated. The convergence of the day-to-day flow dynamic model is

Table 1
Typical day-to-day flow dynamic models.

Authors	Degree of time dispersion	Link/path-based	Random nature	Route choice behavior	Remarks
Huang et al. (2008)	Discrete	Path-based	Deterministic	SUE	A model to investigate the evolutions of daily path travel time, daily ATIS compliance rate and yearly ATIS adoption was presented
Horowitz (1984)	Discrete	Path-based	Deterministic	SUE	An investigation of the stability of stochastic equilibrium in a two-link network was presented
Smith (1983)	Continuous	Path-based	Deterministic	UE	The simplex gravity flow dynamic model was proposed to calculate user equilibrium
Smith (1984)	Continuous	Path-based	Deterministic	UE	The proportional flow adjustment model was proposed
Smith and Wisten (1995)	Continuous	Path-based	Deterministic	UE	A day-to-day flow dynamic model in a continuous setting was presented and the existence of a dynamic user-equilibrium was demonstrated
Friesz et al. (1994)	Continuous	Path-based	Deterministic	UE	A network tatonnement model was proposed to depict day-to-day adjustment processes of network flows
Zhang et al. (2001)	Continuous/discrete	Path-based	Deterministic	UE	The equivalence between stationery link flow pattern and traffic network equilibria was established under three different route choice mechanisms
He et al. (2010)	Continuous	Link-based	Deterministic	UE	Two shortcomings of the path-based day-to-day model were showed
Guo et al. (2013)	Discrete	Link-based	Deterministic	UE	A discrete rational adjustment process was formulated
Zhang and Nagurney (1996)	Continuous	Path-based	Deterministic	UE	The theory of project dynamic system was used to study the stability of a route choice adjustment process
Author	Degree of time dispersion	Link/path-based	Random nature	Route choice behavior	Remarks
Yang and Zhang (2009)	Continuous	Path-based	Deterministic	UE	A rational behavior adjustment process was introduced.
Han and Du (2012)	Continuous	Link-based	Deterministic	UE	A rigorous analysis of the model by He et al. (2010) was performed
Guo et al. (2015)	Continuous	Link-based	Deterministic	UE	A general form of the link-based dynamical system model was proposed
Meneguzzer (2012)	Discrete	Path-based	Deterministic	SUE	Two alternative dynamic process models of combined traffic assignment and control were proposed
Xiao et al.(2016)	Continuous	Path-based	Deterministic	UE	A second-order day-to-day dynamic model considering travelers' learning process and route swapping behavior was developed
Tan et al. (2015)	Continuous	Path-based	Deterministic	UE	A dynamic congestion pricing schemes was investigated to drive traffic flow pattern towards system optimum measured in monetary and/or time units, in a network with heterogeneous users
Xu et al. (2016)	Discrete	Path-based	Deterministic	UE	A trial-and-error method for implementing the traffic-restraint congestion-pricing scheme with day-to-day flow dynamics was proposed to maintain traffic flow within a desirable threshold for some target links
Cascetta (1989)	Discrete	Path-based	Stochastic	SUE	A stochastic process approach was proposed to analyze temporal dynamics in transportation networks
Cantarella and Cascetta (1995)	Discrete	Link-based	Deterministic/stochastic	SUE	A fixed point approach for the day-to-day flow dynamic model was proposed
Watling (1999)	Discrete/continuous	Path-based	Deterministic/stochastic	SUE	Stability of the day-to-day flow dynamic model was investigated
Hazelton and Watling (2004)	Discrete	Path-based	Stochastic	SUE	An approximation method to estimate the equilibrium covariance matrix for the class of Markov assignment models with linear exponential learning filters was devised
Yang (2005)	Discrete/continuous	Path-based	Stochastic	UE/SUE	The evolutionary game theory approach was adopted to study the day-to-day traffic flow dynamics
Liu et al. (2017)	Discrete	Path-based	Stochastic	SUE	A nonlinear distance-based congestion pricing scheme was presented in a network considering stochastic day-to-day dynamics

rigorously demonstrated in Section 4. Furthermore, Section 5 presents the numerical examples to evaluate the performance of the proposed day-to-day route flow dynamic model. Finally, we end with conclusions and future research directions in Section 6.

2. Notations, assumptions and problem statement

Consider a connected transportation network denoted by $G = (N, A)$ where N is the set of nodes and A is the set of directed links. Travelers in the network are divided into two groups according to whether they are equipped with ATIS or not. As discussed by Yang (1998), an ATIS can provide the sufficient traffic information to travelers. Such information will help them to reduce their travel time uncertainty. Therefore, it is reasonable to assume that travelers from the first group who are equipped with an ATIS have complete information about the network traffic conditions and their route choice behavior follows the UE principle, while travelers from the second group who are not equipped an ATIS have incomplete information of network traffic conditions and their route choice behavior obey the logit-based SUE principle.

For the origin-destination (OD) pair $(p, q) \in C \subset N \times N$, the travel demand of travelers in the first and second groups are denoted as d_{pq} and $\hat{d}_{pq}$ respectively. The set of routes from origin p to destination q is denoted by R_{pq}, and the flows of travelers on the route $r \in R_{pq}$ in the first and second groups are represented by h_{pqr} and $\hat{h}_{pqr}$. These route flows are grouped into two vectors: $\mathbf{h} = (h_{pqr} : (p, q) \in C, r \in R_{pq})$ and $\hat{\mathbf{h}} = (\hat{h}_{pqr} : (p, q) \in C, r \in R_{pq})$. Traffic flow on link $a \in A$, denoted by v_a, is the aggregated link flow from both groups and can be expressed by

$$v_a = \sum_{(p,q)\in C}\sum_{r\in R_{pq}} \delta_{pqra}(h_{pqr} + \hat{h}_{pqr}) \tag{1}$$

where $\delta_{pqra} = 1$ if route $r \in R_{pq}$ contains link a, and 0 otherwise. We assume that the travel time function on link $a \in A$, denoted by $t_a(v_a)$, is positive, differentiable and strictly increasing with respect to link flow v_a. The travel time function on route $r \in R_{pq}$,$(p, q) \in C$ is thus given by

$$c_{pqr}(\mathbf{h}, \hat{\mathbf{h}}) = \sum_{a\in A} \delta_{pqra} t_a(v_a) \tag{2}$$

Let π_{pq} denote the minimal travel time between OD pair $(p, q) \in C$ at equilibria. The mixed behavior equilibrium problem is to find a route-flow pattern $\begin{pmatrix} \mathbf{h}^* \\ \hat{\mathbf{h}}^* \end{pmatrix}$, such that, for each route $r \in R_{pq}$ between every OD pair (p, q), the following conditions hold:

$$c_{pqr}(\mathbf{h}^*, \hat{\mathbf{h}}^*) \begin{cases} = \pi_{pq}, & \text{if } h^*_{pqr} > 0, \quad (p, q) \in C, \quad r \in R_{pq} \\ \geqslant \pi_{pq}, & \text{if } h^*_{pqr} = 0, \quad (p, q) \in C, \quad r \in R_{pq} \end{cases} \tag{3}$$

$$\hat{h}^*_{pqr} = \frac{\exp(-\theta c_{pqr})}{\sum_{r\in R_{pq}} \exp(-\theta c_{pqr})} \hat{d}_{pq}, \quad (p, q) \in C, \quad r \in R_{pq} \tag{4}$$

where the parameter θ is an aggregate indicator for travelers' perception of travel time. A larger θ implies that the route choices of travelers are more deterministic, while a smaller value means travelers are more uncertain about their travel times such that their route choice behavior is more random.

It can be seen that Eq. (3) is the UE conditions for travelers in the first group and Eq. (4) is the logit-based SUE conditions for travelers in the second group. These conditions imply that at the mixed equilibria, travelers in the first group who are equipped with ATIS will choose the minimum time routes, while travelers from the second group who are unequipped with ATIS will choose their routes in accordance with the logit-based route choice probability.

Yang (1998) has shown that the mixed equilibrium conditions are rightly the optimal conditions for the following mixed behavior equilibrium problem (MBEP):

[MBEP]

$$\min_{(\mathbf{h},\hat{\mathbf{h}})\in\Omega_1\times\Omega_2} \quad Z(\mathbf{h}, \hat{\mathbf{h}}) = f(\mathbf{h}, \hat{\mathbf{h}}) + g(\hat{\mathbf{h}}) \tag{5}$$

where

$$f(\mathbf{h}, \hat{\mathbf{h}}) = \sum_{a\in A} \int_0^{v_a} t_a(s) ds \tag{6}$$

$$g(\hat{\mathbf{h}}) = \frac{1}{\theta} \sum_{(p,q)\in C}\sum_{r\in R_{pq}} \hat{h}_{pqr} \ln \hat{h}_{pqr} \tag{7}$$

$$\Omega_1 = \left\{ \mathbf{h} \middle| \sum_{r\in R_{pq}} h_{pqr} = d_{pq}, h_{pqr} \geqslant 0 \ (p, q) \in C, r \in R_{pq} \right\} \tag{8}$$

$$\Omega_2 = \left\{ \hat{\mathbf{h}} \middle| \sum_{r \in R_{pq}} \hat{h}_{pqr} = \hat{d}_{pq}, \hat{h}_{pqr} > 0 \ (p,q) \in C, r \in R_{pq} \right\} \tag{9}$$

It is easy to show that the objective function of [MBEP] is not strictly convex, so the solution of path flow to this problem is not unique. Thus, [MBEP] inherits the initial-flow nonuniqueness problem and path-overlapping problem of previous path-based models. Readers may refer to He et al. (2010) for a detailed discussion of these problems.

Moreover, the mixed behavior equilibrium conditions 3 and 4 can also be reformulated in a variational inequality (VI) form and the equivalence between [MBEP] and the VI model is established in the following proposition:

Proposition 1. *A route flow pattern* $\begin{pmatrix} \mathbf{h}^* \\ \hat{\mathbf{h}}^* \end{pmatrix}$ *is at the mixed equilibrium state if and only if it solves the VI problem*[1]:

$$\begin{bmatrix} \mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*) \\ \mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*) + \frac{1}{\theta}(\ln(\hat{\mathbf{h}}^*) + \mathbf{I}) \end{bmatrix}^T \begin{pmatrix} \mathbf{h} - \mathbf{h}^* \\ \hat{\mathbf{h}} - \hat{\mathbf{h}}^* \end{pmatrix} \geqslant 0, \quad \forall (\mathbf{h}, \hat{\mathbf{h}}) \in \Omega_1 \times \Omega_2 \tag{10}$$

where $\mathbf{I}$ *is a vector whose elements are all* 1. *In the above equation,* $\mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*)$ *is the vector of route travel time for travelers in the first group, and* $\mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*) + \frac{1}{\theta}(\ln(\hat{\mathbf{h}}^*) + \mathbf{I})$ *can be regarded as the vector of generalized route travel time for travelers in the second group.*

Let us define

$$\hat{\mathbf{c}}(\mathbf{h}^*, \hat{\mathbf{h}}^*) = \mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*) + \frac{1}{\theta}(\ln(\hat{\mathbf{h}}^*) + \mathbf{I}) \tag{11}$$

then Eq. (10) can be viewed as the VI formulation for the conventional UE conditions in terms of the generalized route travel costs $\begin{bmatrix} \mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*) \\ \hat{\mathbf{c}}(\mathbf{h}^*, \hat{\mathbf{h}}^*) \end{bmatrix}$ for travelers in the two groups.

In addition, there exists another interpretation of Eq. (10). It implies that the total generalized network travel time cannot be further reduced at the mixed equilibrium state. In other words, if the mixed equilibrium state has not yet achieved on the current day k, based on the current day's path travel time, the total generalized travel time in the network could be potentially reduced on the next day $k+1$. This study aims to examine how the route flows evolve on a day-to-day basis to the mixed equilibria. More specifically, given the current route flow pattern $\begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix}$ on the day k, we seek for a behavioral route flow adjustment process from the current route flow pattern $\begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix}$ to the route flow pattern $\begin{pmatrix} \mathbf{h}^{(k+1)} \\ \hat{\mathbf{h}}^{(k+1)} \end{pmatrix}$ on the next day $k+1$, such that the series of route flow patterns will converge to the mixed equilibrium state when time approaches infinity.

3. A discrete day-to-day route flow dynamic model for the mixed equilibrium behavior

Throughout this study, we assume that the discrete day-to-day flow dynamic model will finally evolve to a mixed equilibrium state, and ignore the cases in which the model lead to oscillatory route flows (disequilibrium state). Such an assumption is reasonable because from a practical point of view, the equilibrium route flow state will finally be achieved. (The route choice behavior of travelers will become stable after a period of time.) This viewpoint is widely recognized by researchers focusing on topics related to UE or SUE problems.

3.1. General framework of the discrete day-to-day route flow dynamic model

To investigate the evolution process of path flow to the mixed equilibrium state, a generic discrete day-to-day route flow dynamic model is presented as follows:

$$\begin{pmatrix} \mathbf{h}^{(k+1)} \\ \hat{\mathbf{h}}^{(k+1)} \end{pmatrix} = \left(1 - \alpha^{(k)}\right) \begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix} + \alpha^{(k)} \begin{pmatrix} \mathbf{y}^{(k)} \\ \hat{\mathbf{y}}^{(k)} \end{pmatrix} \tag{12}$$

where $\begin{pmatrix} \mathbf{h}^{(k+1)} \\ \hat{\mathbf{h}}^{(k+1)} \end{pmatrix}$ is the route flow pattern in the next day $k+1$; $\begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix}$ is the route flow pattern in day k; $\begin{pmatrix} \mathbf{y}^{(k)} \\ \hat{\mathbf{y}}^{(k)} \end{pmatrix}$ is the target route flow in the next day $k+1$ after the travelers finish their trips on the day k; $0 < \alpha^{(k)} \leqslant 1$ is the route flow adjustment ratio on the day k. According to Eq. (12), we can see that the route flows on the day $k+1$ consist of two components. One component includes the travelers who do not change their routes taken on the previous day while the other component consists of the rest travelers who take different routes. The proportion of these two components is determined by the adjustment ratio $\alpha^{(k)}$.

[1] The detailed proofs for all the propositions in this study can be found in the appendix.

The discrete day-to-day route flow dynamic model shown by Eq. (12) is a general framework that accommodates various specific discrete day-to-day flow adjustment processes (Guo et al., 2013). These models differ in their choices of target flows and adjustment ratios. No matter what types of the target flow and adjustment ratio are adopted, it is essential that they should well reflect the meaningful route adjustment behavior of travelers. Bearing in mind this fundamental principle, we next discuss how to choose the target flow pattern $\begin{pmatrix}\mathbf{y}^{(k)}\\ \hat{\mathbf{y}}^{(k)}\end{pmatrix}$ and the flow adjustment ratio $\alpha^{(k)}$ for the proposed mixed equilibrium behavior.

3.2. Determination of the target flow

Psychologically, the motivation of travelers to adjust their route choice comes from their recognition of an alternative route with less time. As a result, based on todays' network traffic conditions, travelers with ATIS tend to choose the route whose actual time is minimum on the next day, whereas travelers without ATIS tend to choose the perceived minimum time route in their own mind. In order to capture the cost-minimization behavior of travelers, we assume that target flows of the two groups of travelers are given by the optimal solution to the following two minimization subproblems, respectively:

[RTFA]:

$$\min_{\mathbf{h}\in\Omega_1} \sum_{(p,q)\in C}\sum_{r\in R_{pq}} c_{pqr}^{(k)} h_{pqr} \tag{13}$$

$$\min_{\hat{\mathbf{h}}\in\Omega_2} \sum_{(p,q)\in C}\sum_{r\in R_{pq}} c_{pqr}^{(k)} \hat{h}_{pqr} + \frac{1}{\theta}\sum_{(p,q)\in C}\sum_{r\in R_{pq}} \hat{h}_{pqr}\ln\hat{h}_{pqr} \tag{14}$$

Eqs. (13) and (14) are called rational target flow assumption (RTFA) in this study. The interpretation of RTFA is as follows: The first sub-model expressed by Eq. (13) is the all-or-nothing traffic assignment problem given the fixed route travel time $\mathbf{c}^{(k)}$. Hence, the target route flow $\mathbf{y}^{(k)}$, an optimal solution to the first sub-model, is the flow of travelers who are from the first group and tend to use the shortest routes experienced on the day k for their trips on the next day $k+1$. The second sub-model in Eq. (14) is the conventional logit-based stochastic loading problem given the fixed route travel time $\mathbf{c}^{(k)}$, whose optimal solution can be expressed in the following closed form:

$$\hat{y}_{pqr}^{(k)} = \hat{d}_{pq}\frac{\exp(-\theta c_{pqr}^{(k)})}{\sum_{l\in R_{pq}}\exp(-\theta c_{pql}^{(k)})}, \quad \forall r\in R_{pq}, \quad \forall (p,q)\in C \tag{15}$$

Therefore, the target flow $\hat{\mathbf{y}}^{(k)} = (\hat{y}_{pqr}^{(k)} : r\in R_{pq}(p,q)\in C)$ is the flow of travelers in the second group who follow the logit-based SUE principle for their trips on the next day $k+1$. By the theory of logit model, travelers in the second group tend to choose the perceived minimum time route in their own mind. Since different traveler have different perception errors, all routes between each OD pair will be used on the next day and these routes have the same minimum perceived travel cost.

We would like to point out that the day-to-day route flow dynamic model proposed in this paper may simulate multiple possible evolution trajectories. Since the objective function of [RTFA] is not strictly convex (c.f. Eq. (13)), its solution is not unique. That is, on each day, multiple target route flows may be obtained. Therefore, if an evolution process is repeated from the same starting point, multiple candidate trajectories may be expected. From a practical point of view, the multiple-evolution-trajectory problem is not a good feature because it makes the model unable to give meaningful predictions. In order for the evolution trajectory of the proposed model to be predictable, some specific circumstances should be presumed, which are discussed as follows:

Case 1: Travelers have average propensity in route choice.

When facing multiple shortest paths, it is natural to assume that each of them is equally likely to be chosen. As a result, travelers in the first group will equally be assigned to all the shortest paths between each OD pair. Such an assumption captures travelers' average propensity to choose their route. It also leads to a unique target route flow pattern for subproblem (13).

Case 2: Government inducement is conducted.

Government inducement is also a good way to obtain a unique flow pattern. It can be implemented through dynamic pricing (or subsidy). For example, if on the day k, there exist two routes with the same travel cost, then the government can broadcast via ATIS, notifying that traveling through one route will be charged (or subsided). Such a route based toll pricing (or subsidy) can feasibly be implemented, with the help of the GPS and ATIS in vehicles. As a result, travelers equipped with ATIS will choose the unique route that is induced by the government. In this sense, the model proposed in this paper also provides a way to induce and control the traffic flow.

Case 3: People have some kind of travel inertia.

Following the idea in He et al. (2010), it is reasonable to assume travelers in the first group have some kind of inertia, i.e., they are reluctant to change their route. Such an assumption can be reflected by the following minimization problems.

$$\min \lambda_{\mathbf{h}\in\Omega_1} \quad \lambda \sum_{(p,q)\in C} \sum_{r\in R_{pq}} c_{pqr}^{(k)} h_{pqr} + (1-\lambda)\delta D(\mathbf{h}, \mathbf{h}^{(k)}) \tag{16}$$

where $D(\mathbf{h}, \mathbf{h}^{(k)})$ is a strictly convex function that measures the distance between the target flow $\mathbf{y} = \mathbf{h}$ and the current flow $\mathbf{h}^{(k)}$, δ is a scaling factor, and λ is a positive scalar such that $0 < \lambda < 1$. Typical specification of $D\left(\mathbf{h}, \mathbf{h}^{(k)}\right)$ could take the form of the Euclidean distance, i.e. $D(\mathbf{h}, \mathbf{h}^{(k)}) = \sum_{(p,q)\in C} \sum_{r\in R_{pq}} \left(h_{pqr} - h_{pqr}^{(k)}\right)^2$.

To obtain a better understanding of travel inertia, let's look into problem (16). If $\lambda \to 0$, it tends to $\min_{\mathbf{h}\in\Omega_1} D(h_{pqr}, h_{pqr}^{(k)})$. The solution of this model is $\mathbf{y}^{(k)} = \mathbf{h}^{(k)}$, i.e., the target flow for the next day is the same as the flow on the current day, which means travelers are unwilling to change their route. If $\lambda \to 1$, it results in the shortest path problem. The target flow between each OD pair will be along the minimum cost routes, which implies travelers are active to change their routes according to current day's route costs. By choosing λ within the interval (0, 1), Eq. (16) corresponds to a route choice problem where the degree of inertia of the travelers varies from no inertia to full inertia. Therefore, problem (16) captures travelers' cost minimization behavior as well as their inertia behavior, and the value of λ may be viewed as a parameter that balances them.

Obviously, the objective function of problem (16) is strictly convex, and this guarantees the uniqueness of the target route flow. It is worth to note that, due to the quadratic term in the problem (16), the target flow may be assigned to non-shortest paths. This is quite different from the Case 1 and Case 2, in which target flows are only assigned to the shortest paths.

In view of the discussions above, under the specific cases 1–3, the target route flow is uniquely determined. Therefore, the day-to-day evolution trajectory can be predicted in advance, which allows the implementation of the model in reality.

Next, we discuss the relationship between the RTFA and the well-known rational behavior principle proposed for the continuous day-to-day flow dynamic models (Yang and Zhang, 2009; Zhang et al., 2001) as revealed by Proposition 2.

Proposition 2. *If the current route flow pattern* $\begin{pmatrix}\mathbf{h}^{(k)}\\ \hat{\mathbf{h}}^{(k)}\end{pmatrix}$ *is an optimal solution to [RTFA], then it is exactly the route flow pattern at the mixed equilibrium state. Otherwise, based on the current day's path travel costs, the total generalized travel time of the network will decrease on the next day* $k+1$ *if the route flows are adjusted in accordance to Eq. (12), namely:*

$$\mathbf{c}(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\mathbf{h}^{(k+1)} - \mathbf{h}^{(k)}) + \hat{\mathbf{c}}(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\hat{\mathbf{h}}^{(k+1)} - \hat{\mathbf{h}}^{(k)}) < 0 \tag{17}$$

It can be seen that Eq. (17) is a variation of the rational behavior principle. Therefore we can infer that the proposed day-to-day flow dynamic model under the RTFA follows the rational behavior principle.

3.3. Determination of the flow adjustment ratio

Before we explore how to determine the flow adjustment ratio in our discrete day-to-day flow dynamic model, the necessity to specify such a flow adjustment ratio will be first discussed by comparing the discrete model with the continuous one.

As we know, the flow adjustment ratio does not appear explicitly in the framework of a continuous flow dynamic model, whereas for a discrete model, one needs to specify the value of flow adjustment ratio. For the convergence of the continuous model, it suffices to say that if the rational behavior principle holds, the stationary flow pattern will coincide with the flow pattern at the equilibrium state (Yang and Zhang, 2009). However, for the discrete day-to-day flow dynamic model proposed in this study, the establishment of the rational behavior principle only is not sufficient to ensure such a coincidence. Moreover, although the continuous models can be roughly regarded as a discrete model with an arbitrarily small flow adjustment ratio (Iserles, 2009), how to quantitatively identify this arbitrarily small flow adjustment ratio in our discrete day-to-day dynamic model to guarantee its convergence to the mixed equilibrium state remains to be known. Next, an example will be used to demonstrate that an arbitrarily small flow adjustment ratio in a discrete day-to-day dynamic model cannot ensure its convergence to an equilibrium state.

Example 1. Consider the simple network which consists of one OD pair connected by two parallel links, i.e., link 1 and link 2. The OD demand is 200, with 80% of travelers in the first group, and 20% of travelers in the second group. The dispersion parameter θ in the logit-based SUE model is set to be 1, and the link time functions for both links are defined respectively as follows:

$$t_1(v_1) = 12 \cdot \left[1 + 0.15\left(\frac{v_1}{200}\right)^4\right], \quad t_2(v_2) = 10 \cdot \left[1 + 0.15\left(\frac{v_2}{150}\right)^4\right] \tag{18}$$

The target flow for this network is obtained by solving Eqs. (13) and (14). According to Proposition 2, it satisfies the rational behavior principle.

We test the performance of our discrete day-to-day flow dynamic model under different small constant flow adjustment ratios. Fig. 1 depicts the evolution processes of flows on link 1 when the adjustment ratio α is set to be 0.1, 0.03, 0.07, and 0.01. In order to make a clear illustration, we only show trajectories of link flows that are near the equilibria. From Fig. 1, we can observe that a smaller flow adjustment ratio will make the corresponding trajectory more "close" to the equilibria. However, "close" does not mean "converge". In fact, no matter how small the adjustment ratio is, irregular oscillation still occurs. The deviation of oscillation does not decrease as the number of days tends to infinity, which suggests that the evolution process does not converge to an equilibrium state.

The rationality behind the non-convergence of the above example is as follows. By the cost approximation theory (Patriksson, 1998), only if both of the two subproblems are strongly convex, it suffices to prove the convergence of the discrete day-to-day flow dynamic model under small constant flow adjustment ratio. However, from Eqs. (13) and (14), it can be seen that neither of the two subproblems in [RTFA] is strongly convex. Therefore, the constant adjustment ratio may not yield sufficient decrease of the potential function (Eq. (5)) and the discrete evolution process may oscillate rather than converge to an equilibrium state. Furthermore, a practical drawback of constant flow adjustment ratio is that, if the current flow pattern is far from the equilibria and the adjustment ratio is very small (as is the case for the continuous model), then it will take a very long time (more than 500 days) for travelers to reach near-equilibrium state. This is not realistic at all.

To guarantee that the discrete route flow dynamic model converges in a moderate manner, the flow adjustment ratio $\alpha^{(k)}$ needs to satisfy some sort of line search rules, such as Armijo rule, Goldstein rule or MSA step size rule. From a behavioral point of view, the Goldstein rule is more appropriate than the other two rules. The reasons are elaborated as follows.

As stated above, $\alpha^{(k)}$ represents the proportion of travelers who reconsider their routes in the next day. In the MSA rule, $\alpha^{(k)}$ is a predetermined sequence that converges to zero (e.g., $\alpha^{(k)} = 1/k$). This means that during the evolution process, the proportion of travelers who reconsider their routes is pre-arranged without using any online information about the road traffic condition. Obviously, $\alpha^{(k)}$ in the MSA rule cannot provide a convincing behavioral interpretation. The Armijo rule is also unrealistic. By the definition of this rule, at each evolution step, the set of $\alpha^{(k)}$ that satisfies the Armijo inequality is a series of discrete points. Since $\alpha^{(k)}$ reflects travelers' aggregate behavior, It is hard to imagine that $\alpha^{(k)}$ is chosen from a set of discrete points, rather than a continuous interval.

In contrast, the Goldstein rule overcomes the drawbacks of the above two rules. On the one hand, it is determined by the network flow information of recent days at each evolution step. On the other hand, the set of $\alpha^{(k)}$ that satisfies this rule is an interval. As a result, Goldstein rule provides sufficient behavioral justification of $\alpha^{(k)}$. The requirement of the Goldstein rule is that $\alpha^{(k)}$ satisfies the following two inequalities, which are called rational flow adjustment assumption (RFAA) in this paper. We caution that this assumption may need to be verified by empirical studies in the future.

[RFAA]:

$$Z(\mathbf{h}^{(k+1)}, \hat{\mathbf{h}}^{(k+1)}) - Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \leqslant \sigma\alpha^{(k)}\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} \tag{19}$$

$$Z(\mathbf{h}^{(k+1)}, \hat{\mathbf{h}}^{(k+1)}) - Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \geqslant (1-\sigma)\alpha^{(k)}\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} \tag{20}$$

where σ is a predefined parameter satisfying $0 < \sigma < 1/2$, and $\mathbf{d}^{(k)} = \begin{pmatrix} \mathbf{y}^{(k)} - \mathbf{h}^{(k)} \\ \hat{\mathbf{y}}^{(k)} - \hat{\mathbf{h}}^{(k)} \end{pmatrix}$ is the difference between the target route flows and actual route flows on the day k. It can be seen the RFAA is essentially the Goldstein line search rule for the unconstraint minimization problem. Other than its mathematical expression, the behavioral implications of the RFAA are elaborated below.

On the one hand, the flow adjustment ratio $\alpha^{(k)}$ shown in Eqs. (19) and (20) is determined by the network flow information of the current days at each evolution step, which is reasonable considering the fact that travelers tend to make use of the information collected from the recent days. On the other hand, it holds that

$$\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} = \begin{pmatrix} \mathbf{c}(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \\ \hat{\mathbf{c}}(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \end{pmatrix}^T \begin{pmatrix} \mathbf{y}^{(k)} \\ \hat{\mathbf{y}}^{(k)} \end{pmatrix} - \begin{pmatrix} \mathbf{c}(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \\ \hat{\mathbf{c}}(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \end{pmatrix}^T \begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix} \tag{21}$$

where $\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)}$ is the difference between the total generalized travel time in terms of the target flow $\begin{pmatrix} \mathbf{y}^{(k)} \\ \hat{\mathbf{y}}^{(k)} \end{pmatrix}$ and route flow $\begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix}$. Since the target flow is the aim of all the travelers in the network, $\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)}$ can be interpreted as the expected reduction of the total travel time based on the route time on the day k. In reality, only $\alpha^{(k)}$ fraction of travelers will reconsider their route in the next day, and the other $1 - \alpha^{(k)}$ fraction of travelers does not change their route due to their inertia. As a result, $\alpha^{(k)}\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)}$ can be interpreted as the actual reduction of the total travel time, which is induced by the active travelers. Therefore, the conditions given by the RFAA imply that the reduction of potential function $Z(\mathbf{h}, \hat{\mathbf{h}})$ from the current day k to the next day $k+1$ is less than or equal to a given proportion of the actual reduction of total generalized travel time (see Eq. (19)), and at the same time is larger than or equal to another given proportion of the actual reduction of total generalized travel cost (see Eq. (20)). Based on the above elaborations, we can see that the RFAA provides

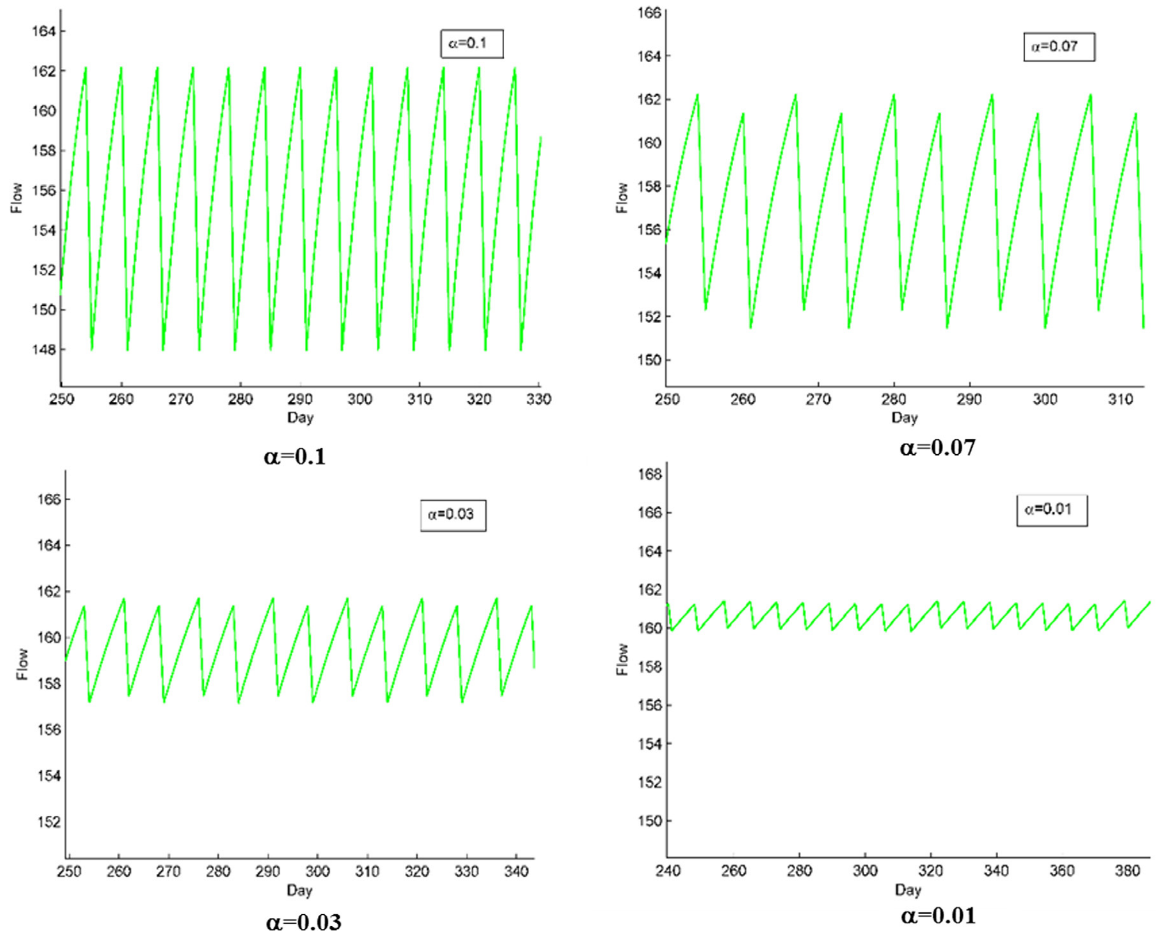

Fig. 1. Flow evolution of link 1 with different adjustment ratios.

sufficient behavior justification for the choice of $\alpha^{(k)}$. The following proposition demonstrates the well-definedness of the RFAA.

Proposition 3. *The RFAA is reasonable, i.e., we can find an appropriate $\alpha^{(k)}$ that satisfies this assumption.*

Note that in reality the flow adjustment ratio $\alpha^{(k)}$ represents the proportion of travelers who reconsider their route. It should belong to the interval $[0,1]$. However, there exist some rare cases that on some days, $\alpha^{(k)} \in [0,1]$ does not satisfy the RFAA, i.e., all elements in the feasible region of the RFAA are larger than 1. In such cases when reality does not coincide with the RFAA, we will follow the reality, and choose $\alpha^{(k)}$ to be any value between [0, 1]. Doing so does not affect the convergence of the dynamic model which is justified by Proposition 6 in the next section.

4. Global convergence of the discrete day-to-day route flow dynamic model

In this section, we will rigorously prove a critical proposition that the discrete day-to-day route flow dynamic model for the mixed equilibrium behavior will eventually evolve to the mixed equilibrium state. Note that our proof essentially establishes the convergence of the partial linearization method with Goldstein rule. Before we proceed to draw this important conclusion, the following proposition that the expected reduction of total travel time will tend to zero as time approaches infinity, will be demonstrated. It implies that as times goes on, travelers will receive less and less benefit if they reconsider their routes.

Proposition 4. *If the target flow is defined as the optimal solution to [RTFA], and the flow adjustment ratio satisfies the RFAA, then* $\lim_{k\to\infty} \nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} = 0$.

Based on Proposition 4, we can prove the convergence of our route flow dynamic model. The proof and derivation is a simple modification of convergence demonstration for the partial linearization method provided by Patriksson [see P231 in Patriksson (1993)]. In other words, we have the following proposition.

Proposition 5. *The proposed discrete day-to-day route flow dynamic model converges to the mixed equilibrium state.*

The last proposition relaxes the restriction of RFAA on the flow adjustment ratio $\alpha^{(k)}$. It states that the convergence of the route flow dynamic model can still be guaranteed, even if there exist an infinite subset of days on which $\alpha^{(k)}$ satisfies the RFAA, namely:

Proposition 6. *Assume that the potential function of the mixed equilibrium behavior model (5) satisfies $Z(\mathbf{h}^{(k+1)}, \hat{\mathbf{h}}^{(k+1)}) \leqslant Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})$ for any day $k \in N$ that is not an equilibria. If there is an infinite subset $K \subseteq N$ such that $\alpha^{(k)}$ satisfies the RFAA for all the $k \in K$, then the discrete day-to-day route flow adjustment process converges to a mixed equilibrium state.*

5. Numerical experiments

In this section, the network shown in Fig. 2 from Nguyen and Dupuis (1984) is used to illustrate the convergent performance of the proposed discrete day-to-day route flow dynamic model. It consists of 13 nodes, 19 links, 4 OD pairs and 25 routes. The traffic demand pattern between the four OD pairs is assumed to be $(d_{12}, d_{13}, d_{42}, d_{43}) = (200, 200, 200, 200)$. Without loss of generality, we assume that for each OD pair, 80% of the travelers are equipped with ATIS, while the rest 20% are unequipped with ATIS. The dispersion parameter θ for unequipped travelers is set to be 1.

The incidence relationship of routes and links for the network is tabulated in Table 2.

The link travel time functions follow the BPR form below, with free flow travel time t_a^0, and link capacity C_a given in Table 3.

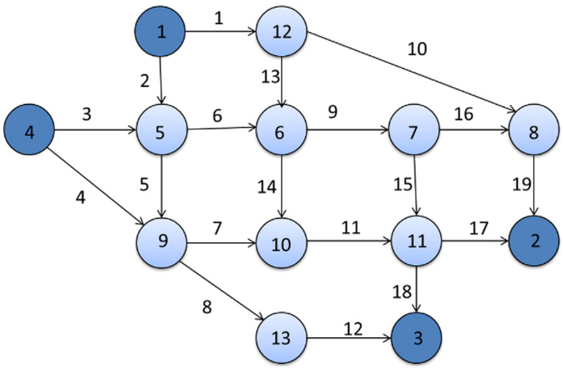

Fig. 2. Nguyen–Dupuis network.

Table 2
Link–route incidence relationship.

OD pair	Route no.	Links sequence	OD pair	Route no.	Link sequence
(1, 2)	1	1, 10, 19	(1, 3)	9	2, 5, 8, 12
	2	2, 6, 9, 16, 19		10	2, 6, 9, 15, 18
	3	2, 6, 9, 15, 17		11	2, 6, 14, 11, 18
	4	2, 6, 14, 11, 17		12	2, 5, 7, 11, 18
	5	2, 5, 7, 11, 17		13	1, 13, 9, 15, 18
	6	1, 13, 9, 16, 19		14	1, 13, 14, 11, 18
	7	1, 13, 9, 15, 17	(4, 3)	20	4, 8, 12
	8	1, 13, 14, 11, 17		21	4, 7, 11, 18
(4, 2)	15	4, 7, 11, 17		22	3, 5, 8, 12
	16	3, 6, 9, 16, 19		23	3, 6, 9, 15, 18
	17	3, 6, 9, 15, 17		24	3, 6, 14, 11, 18
	18	3, 6, 14, 11, 17		25	3, 5, 7, 11, 18
	19	3, 5, 7, 11, 17			

Table 3
The parameters in the link travel time functions.

Link no.	1	2	3	4	5	6	7	8	9	10
t_a	6	5	6	7	6	8	5	10	11	11
C_a	200	200	150	200	100	100	150	150	200	100
Link no.	11	12	13	14	15	16	17	18	19	
t_a	15	8	6	7	5	9	10	8	5	
C_a	200	150	150	100	250	200	150	200	100	

$$t_a(v_a) = t_a^0\left[1 + 0.15 \times \left(\frac{v_a}{C_a}\right)^4\right] \tag{22}$$

The initial flow is chosen as a route flow pattern at the mixed equilibria. As is discussed in Section 2, the route flow patterns at the mixed equilibria are not unique. We use the most-likely path flow method proposed by Rossi et al. (1989) to identify a reasonable path flow from the set of route flows at the equilibria. The initial network conditions are given in Tables 4–6.

Assume that on the day 1, a 50% capacity reduction happens on Link 4. Then travelers may have to reconsider their routes based on the modified link cost on the next day. For the travelers in the first group between OD pair (1, 3), three routes (route 9, 10 or 13) can be potentially used, all of which have equal and minimal costs at the end of day 1 (see row 2 in Table 6). For simplicity, we only consider two cases below:

Case 1: Travelers in the first group between OD pair (1, 3) tend to choose route 9.
Case 2: Travelers in the first group between OD pair (1, 3) tend to choose route 13.

Table 4
Initial route flows for travelers without ATIS.

OD pair	Route flow							
(1, 2)	139.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(1, 3)	35.9	25.5	0.0	0.0	80.0	0.0		
(4, 2)	111.8	0.0	30.9	0.0	0.0			
(4, 3)	143.5	0.6	0.0	0.0	0.0	0.0		

Table 5
Initial route flows for travelers with ATIS.

OD pair	Route flow							
(1, 2)	34.8	2.5	7.4	0.1	5.1	2.5	7.5	0.1
(1, 3)	16.3	16.3	0.2	11.1	16.5	0.2		
(4, 2)	20.6	6.8	20.0	0.2	13.6			
(4, 3)	17.8	12.2	11.8	11.8	0.1	8.1		

Table 6
Initial route travel times.

OD pair	Route travel times							
(1, 2)	50.0	52.7	51.6	56.0	52.0	52.7	51.6	56.0
(1, 3)	43.8	43.8	48.2	44.2	43.8	48.2		
(4, 2)	52.7	53.8	52.7	57.1	53.1			
(4, 3)	44.5	44.9	44.9	44.9	49.3	45.3		

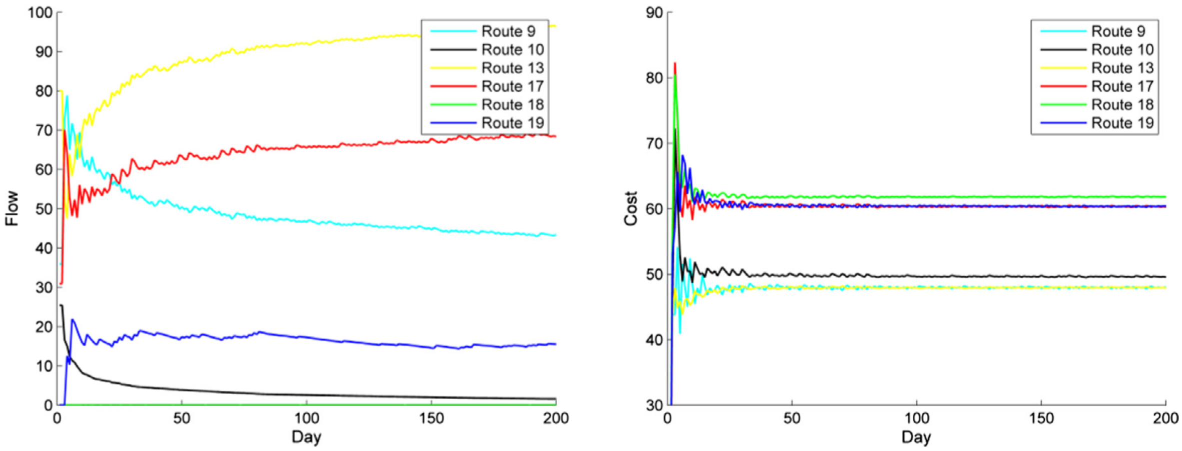

Fig. 3. Evolutionary trajectories of route flows (left) and route travel costs (right) for travelers with ATIS in Case 1.

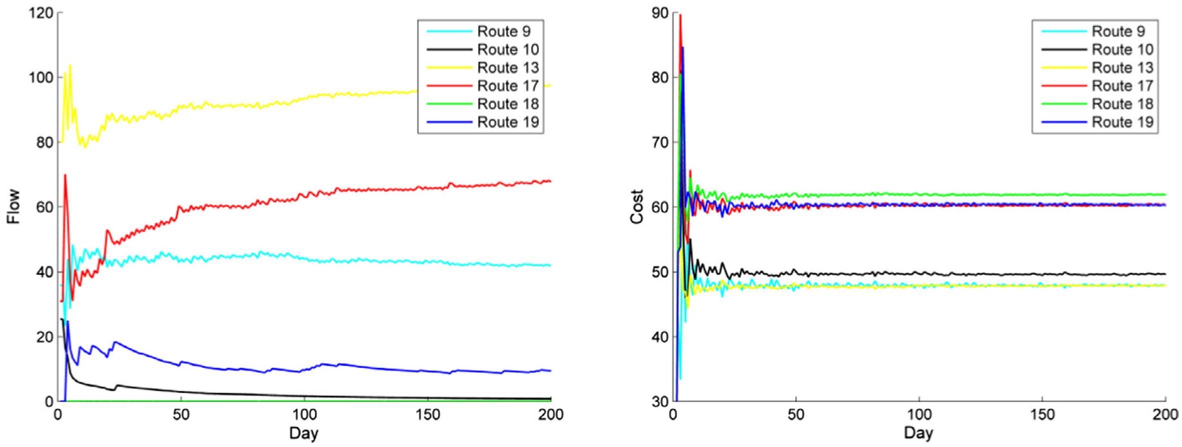

Fig. 4. Evolutionary trajectories of route flows (left) and route travel costs (right) for travelers with ATIS in Case 2.

We choose six typical routes in the aforementioned network, namely, routes 9, 10 and 13 for OD pair (1, 3), routes 17–19 for OD pair (4, 2). Only the flows on these selected routes will be examined. To find a series of suitable $\alpha^{(k)}$ that satisfy the RFAA, we use the inexact line search procedure proposed in Algorithm 2.5.1 by Sun and Yuan (2006).

Fig. 3 shows the evolution process of route flows and correspondent route travel time for travelers in the first group in Case 1, while Fig. 4 illustrates the same data in Case 2. By comparing the two figures above, we can see that although the initial flow patterns are the same, the flow evolution trajectories in Case 1 and Case 2 are somewhat different. This illustrates a drawback of our model that multiple candidate trajectories may be expected.

Furthermore, it is observed that in both cases, the flow and travel time trajectories of all routes evolve to a stationary state. At the stationary state, for OD pair (1, 3), route 9 and 13 have non-zero flows while route 10 has a zero flow. The corresponding travel times for route 9 and 13 are equal, both of which are smaller than that for route 10. For OD pair (4, 2), the flows on route 17 and 19 are positive while the flow on route 18 is zero. Correspondingly, the travel times for route 17 and 19 are less than that for route 18. The above observation indicates that travelers with ATIS will evolve to a UE state, in which the travel costs of all the used routes between the same OD pair are equal and minimal.

Fig. 5 depicts the evolution process of route flows and generalized route travel times for travelers in the second group in case 1. Fig. 6 shows the same report data in case 2. Although travelers without ATIS will follow SUE route choice principle and hence do not suffer from the problem of nonunique target flow, they are influenced by travelers that are equipped with ATIS. As a result, the evolution trajectories for travelers without ATIS in Case 1 and Case 2 are also different. In addition, we can observe that in both cases, all the routes have strictly positive stationary flows and the generalized route travel costs are the same for all the routes between each OD pair. This indicates that travelers without ATIS will evolve to the SUE state where no travelers can unilaterally change route to reduce his/her perceived travel cost.

Next, the convergent performance of the proposed day-to-day route flow dynamic model will be further evaluated given the relaxation on the choice of adjustment ratio $\alpha^{(k)}$ as stated in Proposition 6. For simplicity, we only test Case 1 and assume that on odd days, $\alpha^{(k)}$ is set to make the value of the potential function decrease merely, while on even days, $\alpha^{(k)}$ follows the

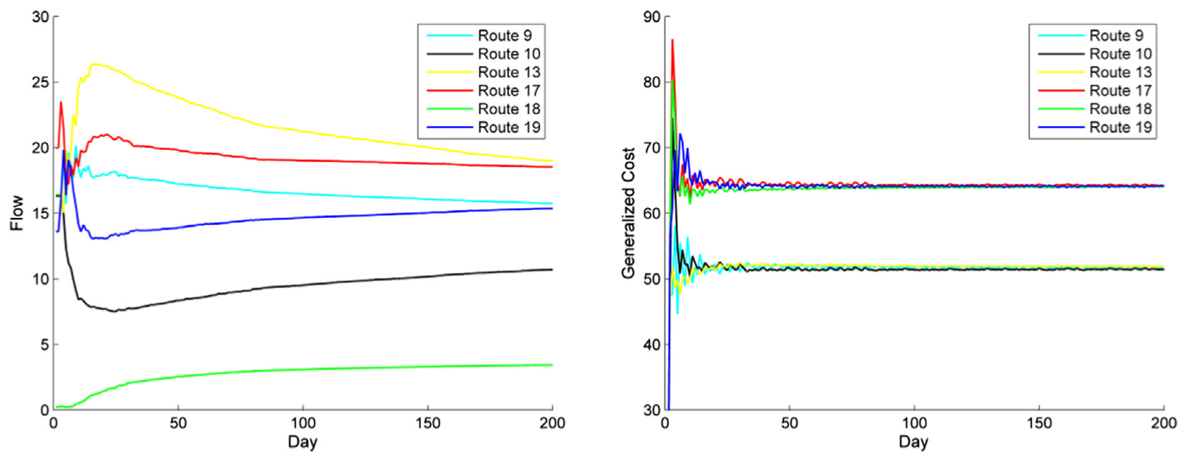

Fig. 5. Evolutionary trajectories of route flows (left) and route travel costs (right) for travelers without ATIS in Case 1.

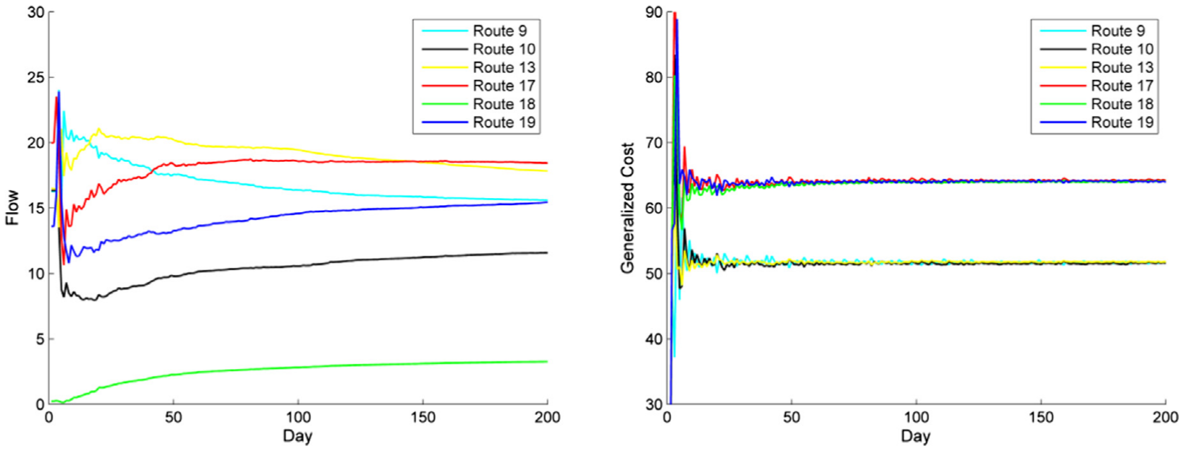

Fig. 6. Evolutionary trajectories of route flows (left) and route travel costs (right) for travelers without ATIS in Case 2.

RFAA. To guarantee that $\alpha^{(k)}$ makes the objective decrease on odd days, we use the bisection method. Namely, first let $\alpha^{(k)} = 1$, and then take $\alpha^{(k)} \leftarrow \alpha^{(k)}/2$, until $\alpha^{(k)}$ decrease the value of the potential function.

Fig. 7 shows the flow and cost evolutions of different routes for travelers in the first group when the RFAA is relaxed. Similar to the results obtained in Fig. 3, it is clear that, although the conditions of RFAA cannot be ensured to hold every day, the day-to-day route flows of travelers with ATIS still converge to a stationary state, which coincides with the UE state.

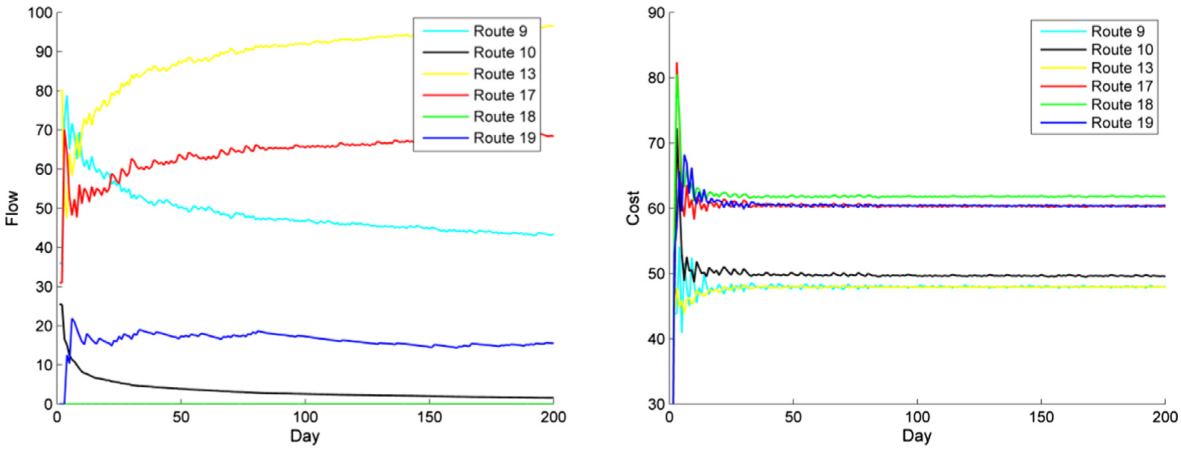

Fig. 7. Evolutionary trajectories of route flows (left) and route travel costs (right) for travelers without ATIS under a relaxation of RFAA.

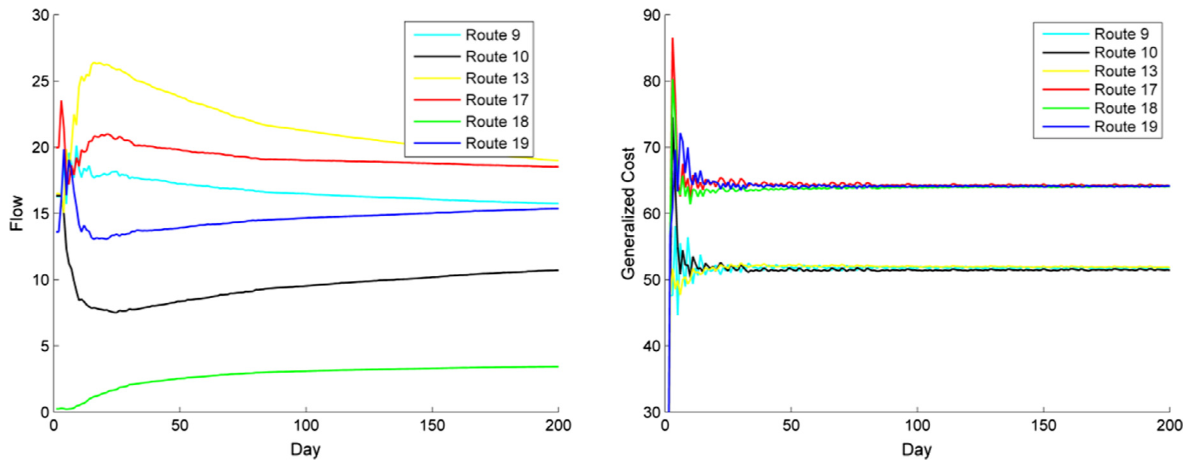

Fig. 8. Evolutionary trajectories of route flows (left) and route travel costs (right) for travelers with ATIS under a relaxation of RFAA.

Fig. 8 depicts the flow and generalized cost evolutions of different routes for travelers in the second group under the conditions in Proposition 6. Clearly, we can observe that when the RFAA is relaxed, the route flows of travelers without ATIS still converge to the SUE state. So Fig. 8 illustrates the effectiveness of Proposition 6.

6. Conclusions

With the development of ATIS, travelers have access to accurate information of the road traffic condition with ease and thereby can make a more informed route choice. However, the introduction of ATIS will inevitably affect the flow evolution process of travelers on a day-to-day basis. In this paper, a discrete day-to-day flow dynamic model is proposed to model the day-to-day evolution process of route flows towards a mixed equilibrium state under a predefined market penetration of ATIS. This problem has not be examined by previous studies, which validates the novelty of our study. The travelers' route adjustment principle and adjustment ratio are first specified for the proposed model and reasonable behavioral interpretations of these assumptions are elaborated to facilitate a good understanding. Then the convergence of the proposed day-to-day flow dynamic model to the mixed equilibrium state is rigorously demonstrated under certain assumptions upon route adjustment principle and adjustment ratio. Furthermore, a relaxation of the assumption upon the adjustment ratio is also investigated which makes our model more appealing to be implemented in practice. At last, numerical experiments are conducted to illustrate the convergence of the proposed day-to-day route flow dynamic model. This study provides insights into the influence of the adoption of ATRS on the day-to-day flow dynamics.

For future research, several extensions can be made in the following directions. First, as mentioned in Section 3, empirical analyses are necessary to validate our model as well as the rationality of RFAA. In addition, the current research focuses on a discrete dynamic model. Analogously, it is interesting to develop a continuous time model for mixed equilibrium behavior, where Lyapunov theorem may be applied in the analysis of system stability. Moreover, our flow adjustment process is modelled based on route flows. Correspondingly, it is interesting to investigate the link based day-to-day flow dynamic models. Finally, this study assumes that the link travel time function is separable. Extending the study to the cases where link travel times are asymmetric and non-separable is also a challenge for future research.

Acknowledgments

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Appendix A. Proofs for Propositions 1–6

A.1. Proof of Proposition 1

It is clear that a route flow pattern $\begin{pmatrix} \mathbf{h}^* \\ \hat{\mathbf{h}}^* \end{pmatrix}$ is a solution to the VI problem (10) if and only if it solves the following linear programming problem:

$$\min_{(\mathbf{h},\hat{\mathbf{h}})\in\Omega_1\times\Omega_2} \begin{bmatrix} \mathbf{c}(\mathbf{h}^*,\hat{\mathbf{h}}^*) \\ \hat{\mathbf{c}}(\mathbf{h}^*,\hat{\mathbf{h}}^*) \end{bmatrix}^T \begin{pmatrix} \mathbf{h} \\ \hat{\mathbf{h}} \end{pmatrix} \tag{A1}$$

where $\mathbf{c}(\mathbf{h}^*,\hat{\mathbf{h}}^*)$ is the route travel time for travelers in the first group and $\hat{\mathbf{c}}(\mathbf{h}^*,\hat{\mathbf{h}}^*)$ is the generalized route travel time for travelers in the second group.

By the KKT optimality conditions, the following necessary and sufficient optimality conditions for problem (A1) can be obtained.

$$\begin{bmatrix} \mathbf{c}(\mathbf{h}^*,\hat{\mathbf{h}}^*) - \Lambda\pi^* \\ \hat{\mathbf{c}}(\mathbf{h}^*,\hat{\mathbf{h}}^*) - \Lambda\hat{\pi}^* \end{bmatrix}^T \begin{pmatrix} \mathbf{h} \\ \hat{\mathbf{h}} \end{pmatrix} = 0 \tag{A2}$$

$$\mathbf{c}(\mathbf{h}^*,\hat{\mathbf{h}}^*) - \Lambda\pi^* \geqslant 0 \tag{A3}$$

$$\hat{\mathbf{c}}(\mathbf{h}^*,\hat{\mathbf{h}}^*) - \Lambda\hat{\pi}^* \geqslant 0 \tag{A4}$$

$$\Lambda\mathbf{h}^* - \mathbf{d} = 0 \tag{A5}$$

$$\Lambda\hat{\mathbf{h}}^* - \hat{\mathbf{d}} = 0 \tag{A6}$$

$$\mathbf{h}^* \geqslant 0 \tag{A7}$$

$$\hat{\mathbf{h}}^* \geqslant 0 \tag{A8}$$

where π^* and $\hat{\pi}^*$ are the optimal Lagrange multipliers corresponding to the flow conservation constraints for travelers in the first and second group, and Λ is the OD-route incidence matrix.

It is well known that logit model assigns strict positive flows to all the routes between each OD pair. Thus, for the travelers in the second group, $\hat{h}_{pqr} > 0\ \forall\ (p,q) \in C,\ r \in R_{pq}$, and Eq. (A2) thus becomes

$$(\mathbf{c}(\mathbf{h}^*, \hat{\mathbf{h}}^*) - \Lambda\mathbf{c}^*)^T \mathbf{h} = 0 \tag{A9}$$

$$\hat{\mathbf{c}}(\mathbf{h}^*, \hat{\mathbf{h}}^*) - \Lambda\hat{\mathbf{c}}^* = 0 \tag{A10}$$

It can be seen that Eq. (A9) and Eq. (A3) are actually the UE conditions for travelers in the first group (see Eq. (3)). To derive the logit-based SUE conditions for travelers in the second group, we rearrange Eq. (A10), and thus obtain

$$\hat{h}_{pqr} = \exp(\theta\pi_{pq} - \theta c_{pqr} - 1) \tag{A11}$$

Combining Eqs. (A11) and (A6), we have the logit route choice probability expression as follows,

$$\hat{h}^*_{pqr} = \frac{\exp(-\theta c_{pqr})}{\sum_{r \in R_{pq}} \exp(-\theta c_{pqr})} \hat{d}_{pq}, \quad (p,q) \in C, \quad r \in R_{pq} \tag{A12}$$

Therefore, it follows from Eq. (A9), Eq. (A3) and Eq. (A12) that the VI problem (10) is equivalent to the mixed behavior equilibrium conditions (3) and (4).

A.2. Proof of Proposition 2

We first show that the RTFA can be derived by partially linearizing the objective function $Z(\mathbf{h}, \hat{\mathbf{h}})$. Specially, by using a first-order approximation to the first term in Eq. (5), we can obtain the following partial linearization subproblem.

[Partial-L]

$$\min_{(\mathbf{h},\ \hat{\mathbf{h}}) \in \Omega_1 \times \Omega_2} \quad \nabla_{\mathbf{h}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\mathbf{h} - \mathbf{h}^{(k)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\hat{\mathbf{h}} - \hat{\mathbf{h}}^{(k)}) + g(\hat{\mathbf{h}}) \tag{A13}$$

It can be easily seen that the model [Partial-L] has at least one optimal solution because it is a convex programming problem with non-empty feasible solution set. Furthermore, it can be decomposed into two independent sub-models:

$$\min_{\mathbf{h} \in \Omega_1} \quad \nabla_{\mathbf{h}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\mathbf{h} - \mathbf{h}^{(k)}) \tag{A14}$$

$$\min_{\hat{\mathbf{h}} \in \Omega_2} \quad \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\hat{\mathbf{h}} - \hat{\mathbf{h}}^{(k)}) + g(\hat{\mathbf{h}}) \tag{A15}$$

which can be rewritten as follows respectively:

$$\min_{\mathbf{h} \in \Omega_1} \quad \sum_{(p,q) \in C} \sum_{r \in R_{pq}} c^{(k)}_{pqr} h_{pqr} \tag{A16}$$

$$\min_{\hat{\mathbf{h}} \in \Omega_2} \quad \sum_{(p,q) \in C} \sum_{r \in R_{pq}} c^{(k)}_{pqr} \hat{h}_{pqr} + \frac{1}{\theta} \sum_{(p,q) \in C} \sum_{r \in R_{pq}} \hat{h}_{pqr} \ln \hat{h}_{pqr} \tag{A17}$$

It thus can be seen that Eqs. (A16) and (A17) are exactly the RTFA.

Based on the above discussions, we can readily prove Proposition 2 next. It can be readily seen that the objective function of [Partial-L] is convex. By Theorem 3.4.3 in Bazaraa et al. (2006), if $\begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix}$ is a minimizer to [Partial-L], the following VI holds, i.e.,

$$\begin{bmatrix} \nabla_{\mathbf{h}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \\ \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T + \nabla g(\hat{\mathbf{h}}^{(k)}) \end{bmatrix}^T \begin{pmatrix} \mathbf{h} - \mathbf{h}^{(k)} \\ \hat{\mathbf{h}} - \hat{\mathbf{h}}^{(k)} \end{pmatrix} \geqslant 0 \tag{A18}$$

By simple calculations, it can be verified that Eq. (A18) is equivalent to the VI formulation of the [MBEP] (see Eq. (10)). Therefore, $\begin{pmatrix}\mathbf{h}^{(k)}\\ \hat{\mathbf{h}}^{(k)}\end{pmatrix}$ is rightly the mixed equilibrium route flow pattern. Clearly, the converse of this proposition is also true.

If $(\begin{smallmatrix}\mathbf{h}^{(k)}\\ \hat{\mathbf{h}}^{(k)}\end{smallmatrix})$ does not solve [Partial-L], since $(\begin{smallmatrix}\mathbf{y}^{(k)}\\ \hat{\mathbf{y}}^{(k)}\end{smallmatrix})$ is a solution to this problem, then

$$\begin{aligned}&\left[\nabla_{\mathbf{h}} f(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\mathbf{y}^{(k)}-\mathbf{h}^{(k)})+\nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)})+g(\hat{\mathbf{y}}^{(k)})\right]\\ &<\left[\nabla_{\mathbf{h}} f(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\mathbf{h}^{(k)}-\mathbf{h}^{(k)})+\nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\hat{\mathbf{h}}^{(k)}-\hat{\mathbf{h}}^{(k)})+g(\hat{\mathbf{h}}^{(k)})\right]=g(\hat{\mathbf{h}}^{(k)})\end{aligned} \tag{A19}$$

By the convexity of g, we have

$$g(\hat{\mathbf{y}}^{(k)})-g(\hat{\mathbf{h}}^{(k)}) \geqslant \nabla g(\hat{\mathbf{h}}^{(k)})^T(\hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)}) \tag{A20}$$

It follows from Eq. (A19) and Eq. (A20) that

$$\begin{bmatrix}\nabla_{\mathbf{h}} f(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\\ \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T+\nabla g(\hat{\mathbf{h}}^{(k)})\end{bmatrix}^T\begin{pmatrix}\mathbf{y}^{(k)}-\mathbf{h}^{(k)}\\ \hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)}\end{pmatrix}<0 \tag{A21}$$

which is equivalent to

$$\mathbf{c}(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\mathbf{y}^{(k)}-\mathbf{h}^{(k)})+\hat{\mathbf{c}}(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)})<0 \tag{A22}$$

where $\hat{\mathbf{c}}(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})=\mathbf{c}(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})+\frac{1}{\theta}(\hat{\mathbf{h}}^{(k)}+\mathbf{I})$ is the generalized route travel cost for travelers in the second group. Substituting Eq. (12) into Eq. (A22), we have

$$\mathbf{c}(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\mathbf{h}^{(k+1)}-\mathbf{h}^{(k)})+\hat{\mathbf{c}}(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T(\hat{\mathbf{h}}^{(k+1)}-\hat{\mathbf{h}}^{(k)})<0 \tag{A23}$$

which is the desired result.

A.3. Proof of Proposition 3

Define

$$\varphi_1(\alpha)=Z(\mathbf{h}^{(k)}+\alpha(\mathbf{y}^{(k)}-\mathbf{h}^{(k)}),\hat{\mathbf{h}}^{(k)}+\alpha(\hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)}))-[Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})+\alpha\sigma\nabla Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\mathbf{d}^{(k)}] \tag{A24}$$

By simple calculations, we have

$$\varphi_1(0)=0,\varphi_1'(0)=(1-\sigma)\nabla Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\mathbf{d}^{(k)}<0. \tag{A25}$$

Therefore $\varphi_1(\alpha)<0$ for sufficiently small $\alpha>0$.

On the other hand, since $\nabla Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\mathbf{d}^{(k)}<0$, we have

$$\varphi_1(\alpha)\to+\infty \text{ as } \alpha\to+\infty \tag{A26}$$

Hence, by the continuity of φ_1, there exists $\alpha'>0$ such that $\varphi_1(\alpha')=0$.

Let α_1^* be the smallest α' with $\varphi_1(\alpha')=0$. Then $\varphi_1(\alpha)<0$ for all $\alpha\in(0,\alpha_1^*)$. That is,

$$Z(\mathbf{h}^{(k)}+\alpha(\mathbf{y}^{(k)}-\mathbf{h}^{(k)}),\hat{\mathbf{h}}^{(k)}+\alpha(\hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)}))-Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})<\alpha\sigma\nabla Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\mathbf{d}^{(k)} \tag{A27}$$

for all $\alpha\in(0,\alpha_1^*)$. Hence Eq. (18) in the RFAA holds.

Define

$$\varphi_2(\alpha)=Z(\mathbf{h}^{(k)}+\alpha(\mathbf{y}^{(k)}-\mathbf{h}^{(k)}),\hat{\mathbf{h}}^{(k)}+\alpha(\hat{\mathbf{y}}^{(k)}-\hat{\mathbf{h}}^{(k)}))-[Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})+\alpha(1-\sigma)\nabla Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\mathbf{d}^{(k)}] \tag{A28}$$

Clearly,

$$\varphi_2(0)=0,\varphi_2'(0)=\sigma\nabla Z(\mathbf{h}^{(k)},\hat{\mathbf{h}}^{(k)})^T\mathbf{d}^{(k)}<0. \tag{A29}$$

So $\varphi_2(\alpha)<0$ for sufficiently small $\alpha>0$. Since $0<\sigma<1/2$, from Eq. (A24) and Eq. (A28), we have

$$\varphi_2(\alpha_1^*)>\varphi_1(\alpha_1^*)=0 \tag{A30}$$

By the continuity of φ_2, there exists $\alpha_2^*\in(0,\alpha_1^*)$ for which $\varphi_2(\alpha_2^*)=0$. It is obvious that $\varphi_2(\alpha)$ is a one-dimensional convex function. $(0,\phi(0))$ and $(\alpha_2^*,\varphi_2(\alpha_2^*))$ are two zero points of φ_2. Since $\varphi_2'(0)<0$, it must be true that

$$\varphi_2'(\alpha_2^*)>0 \tag{A31}$$

Hence, there exists an $\bar{\alpha}$ that close to the right of α_2^* such that $\varphi_2(\bar{\alpha})>0$.

From the statements above, we have found an $\bar{\alpha} \in (\alpha_2^*, \alpha_1^*)$, that satisfies Eq. (18) and Eq. (19). By the smoothness of Z and ∇Z, there is an interval around $\bar{\alpha}$ for which the RFAA holds.

A.4. Proof of Proposition 4

Since $Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})$ is monotonically decreasing, and $Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})$ is bounded on $\Omega_1 \times \Omega_2$ it follows that

$$Z(\mathbf{h}^{(k+1)}, \hat{\mathbf{h}}^{(k+1)}) - Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \to 0 \tag{A32}$$

If Eq. (18) in the RFAA holds, i.e.,

$$Z(\mathbf{h}^{(k+1)}, \hat{\mathbf{h}}^{(k+1)}) - Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) \leqslant \sigma\alpha^{(k)}\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} \tag{A33}$$

by the proof of Eq. (A27) in Proposition 3, we can find an $\alpha^{(k)}$ such that $0 < \alpha^{(k)} < 1$. Then it follows from Eq. (A32) and Eq. (A33) that

$$\alpha^{(k)}\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} \to 0 \tag{A34}$$

Assume that $\lim_{k\to\infty} \nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} = 0$ does not hold. By Eq. (A21), $\nabla Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T \mathbf{d}^{(k)} < 0$ for all k. Then there must exist $\varepsilon > 0$ and a subsequence $k_j \in K$ such that

$$\nabla Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)})^T \mathbf{d}^{(k_j)} < -\varepsilon \text{ and } \alpha^{(k_j)} \to 0 \tag{A35}$$

Therefore, from Eq. (19) in the RFAA, there must be some iteration $\bar{k}_j \geqslant 0$ after which the following inequality holds:

$$\frac{Z(\mathbf{h}^{(k_j)} + \alpha^{(k_j)}(\mathbf{y}^{(k_j)} - \mathbf{h}^{(k_j)}), \hat{\mathbf{h}}^{(k_j)} + \alpha^{(k_j)}(\hat{\mathbf{y}}^{(k_j)} - \hat{\mathbf{h}}^{(k_j)})) - Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)})}{\alpha^{(k_j)}} \geqslant (1-\sigma)\nabla Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)})^T \mathbf{d}^{(k_j)}, k_j \in K, k_j \geqslant \bar{k}_j \tag{A36}$$

By the mean value theorem, this inequality can be written as

$$\nabla Z(\mathbf{h}^{(k_j)} + \tilde{\alpha}^{(k_j)}(\mathbf{y}^{(k_j)} - \mathbf{h}^{(k_j)}), \hat{\mathbf{h}}^{(k_j)} + \tilde{\alpha}^{(k_j)}(\hat{\mathbf{y}}^{(k_j)} - \hat{\mathbf{h}}^{(k_j)}))^T \mathbf{d}^{(k_j)} \geqslant (1-\sigma)\nabla Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)})^T \mathbf{d}^{(k_j)} \tag{A37}$$

where $\tilde{\alpha}^{(k)} \in [0, \alpha^{(k)}]$.

Taking limits on both sides of the above inequality, we have

$$\nabla Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T \mathbf{d}^{(\infty)} \geqslant (1-\sigma)\nabla Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T \mathbf{d}^{(\infty)} \tag{A38}$$

or

$$\sigma\nabla Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T \mathbf{d}^{(\infty)} \geqslant 0 \tag{A39}$$

Since $0 < \sigma < 1/2$, it follows that

$$\nabla Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T \mathbf{d}^{(\infty)} \geqslant 0 \tag{A40}$$

which contradicts Eq. (A35).

Therefore, we have

$$\nabla Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T \mathbf{d}^{(\infty)} \geqslant 0 \tag{A41}$$

In other words,

$$\nabla_{\mathbf{h}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\mathbf{y}^{(\infty)} - \mathbf{h}^{(\infty)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) + \nabla g(\hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) = 0 \tag{A42}$$

A.5. Proof of Proposition 5

To prove Proposition 5, it is equivalent to show that $\begin{pmatrix} \mathbf{h}^{(\infty)} \\ \hat{\mathbf{h}}^{(\infty)} \end{pmatrix}$ is a mixed equilibrium route flow pattern.

Since $\begin{pmatrix} \mathbf{y}^{(k)} \\ \hat{\mathbf{y}}^{(k)} \end{pmatrix}$ solves [Partial-L], we have

$$\nabla_{\mathbf{h}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\mathbf{y}^{(k)} - \mathbf{h}^{(k)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\hat{\mathbf{y}}^{(k)} - \hat{\mathbf{h}}^{(k)}) + g(\hat{\mathbf{y}}^{(k)}) \leqslant \nabla_{\mathbf{h}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\mathbf{h} - \mathbf{h}^{(k)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})^T (\hat{\mathbf{h}} - \hat{\mathbf{h}}^{(k)}) + g(\hat{\mathbf{h}}) \tag{A43}$$

for all the $(\mathbf{h}, \hat{\mathbf{h}}) \in \Omega_1 \times \Omega_2$. Taking limits on both sides of the above inequality, it follows that

$$\nabla_{\mathbf{h}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\mathbf{y}^{(\infty)} - \mathbf{h}^{(\infty)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) + g(\hat{\mathbf{y}}^{(\infty)}) \leqslant \nabla_{\mathbf{h}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\mathbf{h} - \mathbf{h}^{(\infty)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{h}} - \hat{\mathbf{h}}^{(\infty)}) + g(\hat{\mathbf{h}}) \tag{A44}$$

From Eq. (A42), we have

$$\nabla_{\mathbf{h}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\mathbf{y}^{(\infty)} - \mathbf{h}^{(\infty)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) = -\nabla g(\hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) \tag{A45}$$

Substituting Eq. (A45) into Eq. (A44), we obtain

$$g(\hat{\mathbf{y}}^{(\infty)}) - \nabla g(\hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) \leqslant \nabla_{\mathbf{h}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\mathbf{h} - \mathbf{h}^{(\infty)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{h}} - \hat{\mathbf{h}}^{(\infty)}) + g(\hat{\mathbf{h}}) \tag{A46}$$

By the convexity of g, we have

$$g(\hat{\mathbf{y}}^{(\infty)}) - g(\hat{\mathbf{h}}^{(\infty)}) \geqslant \nabla g(\hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{y}}^{(\infty)} - \hat{\mathbf{h}}^{(\infty)}) \tag{A47}$$

It follows from Eq. (A46) and Eq. (A47) that

$$g(\hat{\mathbf{h}}^{(\infty)}) \leqslant \nabla_{\mathbf{h}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\mathbf{h} - \mathbf{h}^{(\infty)}) + \nabla_{\hat{\mathbf{h}}} f(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)})^T (\hat{\mathbf{h}} - \hat{\mathbf{h}}^{(\infty)}) + g(\hat{\mathbf{h}}) \tag{A48}$$

for all the $(\mathbf{h}, \hat{\mathbf{h}}) \in \Omega_1 \times \Omega_2$.

In view of Eq. (A48), $\begin{pmatrix}\mathbf{h}^{(\infty)} \\ \hat{\mathbf{h}}^{(\infty)}\end{pmatrix}$ is an optimal solution to [Partial-L]. By Proposition 2, $\begin{pmatrix}\mathbf{h}^{(\infty)} \\ \hat{\mathbf{h}}^{(\infty)}\end{pmatrix}$ is a mixed equilibrium route flow pattern, which implies the route flow dynamic model converges to the mixed equilibrium state.

A.6. Proof of Proposition 6

Let $k_j \in K$ be the infinite subsequence of days that satisfy the RFAA. Using a similar proof as in Proposition 4, we have

$$\lim_{k_j \to \infty} \nabla Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)})^T \mathbf{d}^{(k_j)} = 0 \tag{A49}$$

Then by Proposition 5, every limit point of the subsequence $k_j \in K$ is a mixed equilibrium state. Without loss of generality, we assume that $\begin{pmatrix}\mathbf{h}^{(k_j)} \\ \hat{\mathbf{h}}^{(k_j)}\end{pmatrix}$ converges to some equilibrium point $\begin{pmatrix}\mathbf{h}^{(\infty)} \\ \hat{\mathbf{h}}^{(\infty)}\end{pmatrix}$ satisfying Eq. (10). Since Z is convex, it follows that $\begin{pmatrix}\mathbf{h}^{(\infty)} \\ \hat{\mathbf{h}}^{(\infty)}\end{pmatrix}$ is an optimal solution to problem (5), i.e.,

$$Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) = Z^* \tag{A50}$$

where Z^* is a minimizer of the potential function Z on $\Omega_1 \times \Omega_2$.

Next we prove that for any day $k \in N$, every limit point of the sequence $\begin{pmatrix}\mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)}\end{pmatrix} k \in N$ also minimizes the problem (5). By the continuity of Z,

$$\lim_{k_j \to \infty} Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)}) = Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) \tag{A51}$$

In other words, for any $\varepsilon > 0$, there exist nonnegative integer N_1 such that

$$Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)}) - Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) < \varepsilon \tag{A52}$$

for all $k_j \geqslant N_1$.

Since the sequence $Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)})$ is monotonically decreasing, for any $k > N_1$ we have

$$Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) < Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) < Z(\mathbf{h}^{(N_1)}, \hat{\mathbf{h}}^{(N_1)}) \tag{A53}$$

Therefore, if follows from Eq. (A52) and Eq. (A53) that

$$Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) - Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) < Z(\mathbf{h}^{(N_1)}, \hat{\mathbf{h}}^{(N_1)}) - Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) < \varepsilon \tag{A54}$$

In other words,

$$\lim_{k \to \infty} Z(\mathbf{h}^{(k)}, \hat{\mathbf{h}}^{(k)}) = \lim_{k_j \to \infty} Z(\mathbf{h}^{(k_j)}, \hat{\mathbf{h}}^{(k_j)}) = Z(\mathbf{h}^{(\infty)}, \hat{\mathbf{h}}^{(\infty)}) = Z^* \tag{A55}$$

from which we can see that any limit point of $\begin{pmatrix} \mathbf{h}^{(k)} \\ \hat{\mathbf{h}}^{(k)} \end{pmatrix}$ is also an optimal solution to problem (5). Therefore, it can be concluded that the route flow dynamic model converges to the mixed equilibrium state.

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