ütt 2

Aufgabe 1. (a) ① $A \rightarrow B$

P(A)

T	F
0.3	0.7

P(B)

P(A)	T	F
T	0.2	0.8
F	0.4	0.6

mit $P(A,B) = P(A) \times P(B|A)$

Dann $P(B) = P(B=T) = \sum_{A\in\{T,F\}} P(B=T, A)$

$$= P(B_T, A_T) + P(B_T, A_F)$$
$$= 0.3 \times 0.2 + 0.7 \times 0.4$$
$$= 0.34$$

② $B \rightarrow C$

P(B)

T	F
0.34	0.66

P(C)

P(B)	T	F
T	0.7	0.3
F	0.3	0.5

mit $P(B,C) = P(B) \times P(C|B)$

Dann $P(C) = P(C=T) = \sum_{B\in\{T,F\}} P(C=T, B)$

$$= P(C_T, B_T) + P(C_T, B_F)$$
$$= 0.34 \times 0.7 + 0.66 \times 0.3$$
$$= 0.436$$

(b) ① $A \rightarrow B$ same as (a)① $\Leftrightarrow P(B) = 0.34$

② $A \rightarrow C$

P(A)

T	F
0.3	0.7

P(C)

P(A)	T	F
T	0.7	0.3
F	0.6	0.4

$P(C) = \sum_{A\in\{T,F\}} P(C=T, A) = 0.3 \times 0.7 + 0.7 \times 0.6 = 0.63$

P(D)

P(C)	P(B)	T	F
T	T	0.9	0.1
T	F	0.3	0.7
F	T	0.5	0.5
F	F	0.3	0.7

③ $C \rightarrow D \leftarrow B$ mit head to head haben wir

$P(B, C, D) = P(B) \times P(C) \times P(D|B, C)$

$P(B, C) = P(B) \times P(C)$

Dann $P(D) = P(D=T) = \sum_{B,C\in\{T,F\}} P(D=T, B, C)$

$$= (0.9 \times 0.63 \times 0.34)_{TTT} + (0.3 \times 0.63 \times 0.66)_{TTF}$$
$$+ (0.5 \times 0.37 \times 0.34)_{TFT} + (0.3 \times 0.37 \times 0.66)_{TFF}$$

$$= 0.19278 + 0.12474 + 0.0629 + 0.07326$$
$$= 0.45368$$

Aufgabe 2.

(a) $p(A,B,C,D,E,F) = P(A) \times P(B|A) \times P(C|B) \times P(D|B) \times P(E|D,C) \times P(F|E)$

(b) $p(A,B,C,D,E) = \Sigma_{F \in \{T,F\}} P(A,B,C,D,E,F)$

$$= P(A,B,C,D,E,F=T) + P(A,B,C,D,E,F=F)$$

$$\overset{(a)}{=} P(A) \times P(B|A) \times P(C|B) \times P(D|B) \times P(E|D,C) \times \big(\underbrace{P(F_T|E) + P(F_F|E)}_{=1}\big)$$

$$= P(A) \times P(B|A) \times P(C|B) \times P(D|B) \times P(E|D,C)$$

(c) sei min. set $S_1 = \{B\}$ oder $S_2 = \{E\}$

Dann A and F d-separates

bew: mit Def 4.8 1. haben wir:

$i_k \in S$ and $i_{k-1} \to i_k \to i_{k+1}$ gibt $A \to \underset{i_k}{B}$, $B \to D$, $B \to C$

and $D \to E$, $C \to E \to F$

Dann sei $S_{i_{k-1}} = \{A\}$, $S_{i_{k+1}} = \{F\}$ $\Rightarrow \boxed{S_{i_k} = \{B,C,D,E\}}$

S_{i_k} blocked nodes A and F

And every path between in A and F is blocked by S_{i_k}

so S_{i_k} is <u>d-separated</u> node A and F.

Then find we the <u>min. set</u> in S_{i_k}

sei $S_1 = \{B\}$ oder $S_2 = \{E\}$ oder $S_3 = \{C,D\}$ Kann auch d-Separated node A and F.

so <u>min set</u> is $S_1 = \{B\}$ oder $S_2 = \{E\}$.

(d) Prove $C \perp\!\!\!\perp_G D \perp B \iff$ node C and D is blocked by B

weil $C, D, B \to$ $B \to C$, $B \to D$ $\iff C \leftarrow B \rightarrow D$

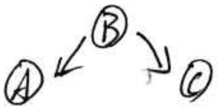
if is not blocked by B, so we Kann find another way connect node C and D. But we can't.

so all paths between them are d-separated $\iff C \perp\!\!\!\perp_G D \perp B$

$((D \to E \leftarrow C)$ is also d-separated$)$.

Aufgabe 3.

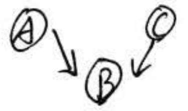
(c) $A \leftarrow B \rightarrow C$ (tail to tail)



$$P(A,B,C) = P(B) \times P(A|B) \times P(C|B) \quad \text{mit } P(A,C|B) = P(A,B,C)/P(B)$$

Dann haben wir $P(A,C|B) = P(A|B) \times P(C|B) \Leftrightarrow A \perp\!\!\!\perp C \mid B$

(d) $A \rightarrow B \leftarrow C$ (head to head)



$$P(A,B,C) = P(A) \times P(C) \times P(B|A,C) \quad \text{mit } P(B|A,C) = \frac{P(A,B,C)}{P(A,C)}$$

Dann $P(A,C) = P(A) \times P(C) \Leftrightarrow A \perp\!\!\!\perp C$

(a) $A \rightarrow B \rightarrow C$ (head to tail)

$$P(A,B,C) = P(A) \times P(B|A) \times P(C|B) \quad \text{mit } \begin{cases} P(B|A) = \dfrac{P(A,B)}{P(A)} = \dfrac{P(B)P(A|B)}{P(A)} \\ P(C|B) = \dfrac{P(C,B)}{P(B)} \end{cases}$$

Dann $P(A,B,C) = P(B) \times P(A|B) \times P(C|B) \Leftrightarrow A \perp\!\!\!\perp C \mid B$.

(b) $A \leftarrow B \leftarrow C$ (head to tail)

$$P(A,B,C) = P(C) \times P(B|C) \times P(A|B) \quad \text{mit } P(B|C) = \frac{P(B,C)}{P(C)} = \frac{P(B)P(C|B)}{P(C)}$$

$$= P(B) \times P(C|B) \times P(A|B) \Leftrightarrow A \perp\!\!\!\perp C \mid B.$$

Aufgabe 4. (discrete)

(a) weil $E(X) = \sum_{x \in X} x \cdot p_X(x) = x_1 p_X(x_1) + \cdots + x_K p_X(x_K)$

Dann (a) $E(aX+Y) = aE(X) + E(Y)$ Können wir $\Rightarrow \begin{cases} E(aX) = aE(X) & (1) \\ E(X+Y) = E(X) + E(Y) & (2) \end{cases}$

In (1): $E(aX) = \sum_{x \in X} ax \cdot p_X(x) = ax_1 p_X(x_1) + \cdots + ax_K p_X(x_K)$

$$= a(x_1 p_X(x_1) + \cdots + x_K p_X(x_K))$$

$$= aE(X)$$

In (2): $E(X+Y)=E(X)+E(Y)$

$$=\sum_{1\le i,j\le n}(x_i+y_j)P(x=x_i,y=y_j)=\sum_{1\le i,j\le n}(x_i+y_j)P(x=x_i)P(y=y_i)$$

$$=\sum_{1\le i\le n}x_iP(x=x_i)+\sum_{1\le j\le n}y_jP(y=y_j)=\sum_{1\le i\le n}x_iP(x=x_i)+E(Y)=E(X)+E(Y)$$

(b) $Var(aX)=a^2Var(X)$

weil $Var(X)=E((X-\mu)^2)=E((X-E(X))^2)$ mit $E(aX)=aE(X)$

haben wir $Var(ax)=E((ax-E(ax))^2)$

$$=E((ax-aE(x))^2)$$

$$=E(a^2(x-E(x))^2)$$

$$=a^2E((x-E(x))^2)$$

$$=a^2Var(x)$$

Aufgabe 5.

$$N(x|\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

(1) sei $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

Dann $\log f(x)=\log\frac{1}{\sigma\sqrt{2\pi}}-\frac{1}{2\sigma^2}\cdot(x-\mu)^2$ (mit $\log e^K=K$)

Dann $\frac{f'(x)}{f(x)}=0-\frac{1}{\sigma^2}(x-\mu)=0$ $f'(x)=0$, $x=\mu$

$$f'(x)=-\frac{1}{\sigma^2}(x-\mu)\cdot f(x) \quad (1)$$

$$f''(x)=-\frac{1}{\sigma^2}[1\cdot f(x)+(x-\mu)f'(x)]$$

$$=-\frac{f(x)}{\sigma^2}(1-(x-\mu)\cdot\frac{(x-\mu)^2}{\sigma^2}\cdot f(x)]$$

$$f''(x)=-\frac{f(x)}{\sigma^2}[1-\frac{(x-\mu)^2}{\sigma^2}]$$

$\cancel{f''(x)}$ $f''(\mu)=-\frac{f(\mu)}{\sigma^2}=\frac{1}{\sigma^2}\cdot\frac{1}{\sigma\sqrt{2x}}<0$

and $x=\mu$ $\boxed{(\cdot\text{Avg}=\text{mode})}$

so $\rightarrow$

μ is also the mode of Gaussian normal distribution