Blatt 03

Aufgabe 1. (b) $f(x|\lambda) = \lambda e^{-\lambda x}$ for $\lambda > 0$, $x_i \geq 0$

$\Rightarrow L(\lambda|x_i) = \lambda e^{-\lambda x_i}$

① $\frac{d}{d\lambda} L(\lambda|x_1 \cdots x_n) = \frac{d}{d\lambda} \lambda^n [e^{-\lambda(x_1 + \cdots + x_n)}]$

$= \frac{d}{d\lambda} \log(\lambda^n [e^{-\lambda(x_1 + \cdots + x_n)}])$

$= \frac{d}{d\lambda} \log(\lambda^n) + \log[e^{-\lambda(x_1 + \cdots + x_n)}]$

$= \frac{d}{d\lambda} n\log(\lambda) - \lambda(x_1 + \cdots + x_n)$

$= n \cdot \frac{1}{\lambda} - (x_1 + \cdots + x_n)$

② sei $0 = n \cdot \frac{1}{\lambda} - (x_1 + \cdots + x_n)$

$\Longleftrightarrow \lambda = n / (x_1 + \cdots + x_n)$

(a) $f(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$

$\Rightarrow L(\mu, \sigma^2) = \prod_{i=1}^{n} f(y|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma^2}\right)$

$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\sigma^2}\right)$

$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\sigma^2}\right)$

Dann ①

$\ln L(\mu, \sigma^2) = \ln(2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\sigma^2}\right)$

$= \ln(2\pi\sigma^2)^{-\frac{n}{2}} + \ln\exp\left(-\sum_{i=1}^{n} \left(\frac{y_i-\mu)^2}{2\sigma^2}\right)\right)$

$= -\frac{n}{2}\ln 2\pi\sigma^2 - \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\sigma^2}$

$= -\frac{n}{2}(\ln 2\pi + \ln\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} (x_i-\mu)^2$

Dann ② $\begin{cases} \frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu) = 0 \\ \frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i-\mu)^2 = 0 \end{cases} \Longrightarrow \begin{cases} \sum_{i=1}^{n}(y_i-\mu) = 0 \quad (1) \Rightarrow \mu = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x} \\ \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2 \end{cases}$

D.h. für μ gilt: $\mu = \overline{x}$; $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2$

(c) $g(x|\alpha, \lambda) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x_i^{(\alpha-1)} e^{-\lambda x_i}$ for λ.

$\log \lambda(x_i) = \alpha\log\lambda - \log\Gamma(\alpha) + (\alpha-1)\log x_n - \lambda x_n$

① $\Rightarrow L(x|\alpha, \lambda) = \sum_{n=0}^{N-1} \log p(x_n)$

$= N \cdot \alpha\log\lambda - N\log\Gamma(\alpha) + (\alpha-1)\sum_{n=0}^{N-1}\log x_n - \lambda\sum_{n=0}^{N-1} x_n$

② $\frac{d}{d\lambda}(N\alpha\log\lambda - N\log\Gamma(\alpha) + (\alpha-1)\sum_{n=0}^{N-1}\log x_n - \lambda\sum_{n=0}^{N-1} x_n] = 0$

$\Longleftrightarrow \frac{N\alpha}{\lambda} - \sum_{n=0}^{N-1} x_n = 0$

$\hat{\lambda} = \frac{\alpha}{\frac{1}{N}\sum_{n=0}^{N-1} x_n} \Rightarrow \hat{\lambda} = \frac{\alpha}{\overline{x}}$. (weil ↓)

$\frac{d}{d\alpha}(N\alpha\log\lambda - N\log\Gamma(\alpha) + (\alpha-1)\sum_{n=0}^{N-1}\log x_n - \lambda\sum_{n=0}^{N-1} x_n] = 0$

$\Longleftrightarrow N\log\lambda - N\frac{1}{\Gamma(\alpha)}\Gamma'(\alpha) + \sum_{n=0}^{N-1}\log x_n = 0$

$\alpha = \log\lambda + \frac{1}{N}\sum_{n=0}^{N-1}\log x_n$

$\Rightarrow$ zusammen $\lambda = \frac{\alpha}{\overline{x}}$.

Aufgabe 2.

$$p(\theta) = Dir(\theta|\alpha) = \frac{1}{B(\alpha)} \prod_{k=1}^{K} \theta_k^{\alpha_k - 1} \cdot 1$$

mit $B(\alpha) = \frac{\prod_{i=1}^{K} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{K} \alpha_i)}$ $\alpha = (\alpha_1 \cdots \alpha_k)$

$$p(D|\theta) = Mu(x|n,\theta) = \prod_{k=1}^{K} \theta_k^{N_k}$$

Dann $p(\theta|D) = p(D|\theta) \cdot p(\theta) = \prod_{k=1}^{K} \theta_k^{N_k} \cdot \theta_k^{\alpha_k - 1} = \prod_{k=1}^{K} \theta_k^{\alpha_k + N_k - 1}$

$$= Dir(\theta | \alpha_1 + N_1 + \cdots + \alpha_k + N_k)$$

$$= Dir(\theta | \alpha + x)$$