

Exercise set #3

Please submit your solutions in teams of two using the sciebo file-drop folder. The link is available in ILIAS. For the formatting please stick to the `submission_guideline.pdf` that you can find on sciebo. In the case of multiple uploads we will consider the latest. Uploads after the deadline will be deleted without further notice.

1. MLEs

Derive the maximum likelihood estimators for the following distributions. For this write down the log-likelihood function given n observations x_1, \dots, x_n and determine the maximum with respect to the parameter.

- (a) Gaussian normal distribution $\mathcal{N}(y|\mu, \sigma^2)$ for μ .
- (b) Exponential distribution with probability density function $f(x|\lambda) = \lambda e^{-\lambda x}$ for $\lambda > 0$ given observations $x_i \geq 0$.
- (c) Gamma distribution with probability density function $g(x|\alpha, \lambda) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x_i^{(\alpha-1)} e^{-\lambda x_i}$ for parameter λ given observations $x_i \geq 0$ and a known $\alpha > 0$.

Hint: The derivative of $\log(y)$ is $\frac{1}{y}$.

45 points

2. Dirichlet-multinomial model

Throwing a (not necessarily fair) K -sided die n times allows us to infer posteriors for the unknown probabilities. The data is $\mathcal{D} = (x_1, \dots, x_K)$ with x_j being the number of times you have seen side j . Assume a Dirichlet prior (with (hyper-)parameter vector α) for the parameter vector $\theta = (\theta_1, \dots, \theta_K)$ with $0 \leq \theta_j \leq 1$ and $\sum_j \theta_j = 1$ and a multinomial likelihood for your data, i.e.,

$$p(\theta) = \text{Dir}(\theta|\alpha) \qquad p(\mathcal{D}|\theta) = \text{Mu}(x|n, \theta)$$

Show that the posterior is also Dirichlet, i.e., show

$$p(\theta|\mathcal{D}) = \text{Dir}(\theta|\alpha + x)$$

Hint: You do not have to calculate the normalization constant, i.e., prove that the posterior is proportional to a Dirichlet distribution with parameter $\alpha + x$.

30 points

3. Inference for a difference in proportions (programming task)

Recall the story from the lecture “Two sellers at Amazon have the same price. One has 90 positive and 10 negative reviews. The other one 2 positive and 0 negative. Who should you buy from?” Write down the posterior probabilities about the reliability (as in the lecture).

- (a) Calculate $p(\theta_1 > \theta_2|\mathcal{D}_1, \mathcal{D}_2)$ using quadrature, e.g., by using the function `dblquad` from `scipy.integrate`.
- (b) Calculate $p(\theta_1 > \theta_2|\mathcal{D}_1, \mathcal{D}_2)$ using Monte Carlo integration¹. You can generate Beta distributed samples with the function `scipy.stats.beta.rvs(a,b,size)`.

25 points

¹https://en.wikipedia.org/wiki/Monte_Carlo_integration