Machine Learning

Section 2: Plausible Reasoning and Bayes Rule

合理推理和贝叶斯法则

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Life's most important questions are, for the most part, nothing but probability problems.

Pierre-Simon Laplace

生活中最重要的问题，在大多数情况下，不过是概率问题。
皮埃尔-西蒙-拉普拉斯

Sophie Feytaud (fl.1841), Pierre-Simon Laplace, public domain

Probability theory as an extension of logic

作为逻辑学延伸的概率论

Deductive and plausible reasoning (1)

Jaynes, 2003, Sec. 1

演绎性和合理性推理

C. Löser, Otterndorf Regenwolke April-2017 DSC 1499, CC BY 3.0 DE

A = 到下午6点会开始下雨
B = 天空将在下午6点前变得多云

A = it will start to rain by 6pm

B = the sky will become cloudy before 6pm

Deductive reasoning: 演绎性推理

if A is true, then B is true 大前提
A is true 小前提

B is true 结论

"modus ponens" 辩证法

if A is true, then B is true
B is false

A is false

"modus tollens"

Deductive and plausible reasoning (2)

Jaynes, 2003, Sec. 1

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A = it will start to rain by 6pm

B = the sky will become cloudy before 6pm

Plausible reasoning: 归纳推理

if A is true, then B is true
B is true

A becomes more plausible

A变得更有说服力

if A is true, then B is true
A is false

B becomes less plausible

Deductive and plausible reasoning (3)

Jaynes, 2003, Sec. 1

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A是真的，意味着B是真的。
A是假的，意味着B变得不太可信。
B是真的，意味着A变得更加可信。
B是假的，意味着A是假的。

A = it will start to rain by 6pm

B = the sky will become cloudy before 6pm

Overview:

- Assume: if A is true, then B is true.
  - A is true, implies that B is true.
  - A is false, implies that B becomes less plausible.
  - B is true, implies that A becomes more plausible.
  - B is false, implies that A is false.

Deductive and plausible reasoning (4)

Jaynes, 2003, Sec. 1

A = it will start to rain by 6pm

B = the sky will become cloudy before 6pm

Plausible reasoning:

if A is true, then B more plausible
B is true

A becomes more plausible

if A is true, then B more plausible
A is false

B becomes less plausible

Deductive and plausible reasoning (5)

Jaynes, 2003, Sec. 1

A = it will start to rain by 6pm

B = the sky will become cloudy before 6pm

Plausible reasoning:

if A is true, then B more plausible
A is true

B becomes more plausible

if A is true, then B more plausible
B is false

A becomes less plausible

Deductive and plausible reasoning (6)

Jaynes, 2003, Sec. 1

Overview:

- Assume: if *A* is true, then *B* is true.

- *A* is true, implies that *B* is true.
- *A* is false, implies that *B* becomes less plausible.
- *B* is true, implies that *A* becomes more plausible.
- *B* is false, implies that *A* is false.

A是真的，意味着B是真的。
A是假的，意味着B变得不太可信。
B是真的，意味着A变得更加可信。
B是假的，意味着A是假的。

- Assume: if *A* is true, then *B* becomes more plausible.
  - *A* is true, implies that *B* becomes more plausible.
  - *A* is false, implies that *B* becomes less plausible.
  - *B* is true, implies that *A* becomes more plausible.
  - *B* is false, implies that *A* becomes less plausible.

假设：如果A是真的，那么B就会变得更加可信
A是真的，意味着B变得更加可信。
A是假的，意味着B变得不那么可信。
B是真的，意味着A变得更加可信。
B是假的，意味着A变得不那么可信。

How can we formalize plausibility?

我们怎样才能使可信性正规化？

Formalizing plausibility (1) — Cox's axioms

Jaynes, 2003, Sec. 1-2

可信性的形式化（1）--考克斯的公理

Cox's axioms (formalizing common sense)

考克斯公理(常识的形式化)

- plausibility of B assuming that A is true is a real number $p(B|A)$
- plausibility $p(B|A)$ complies with common sense
- plausibility $p(B|A)$ is consistent

考克斯定理（警告：证明不是完全严格的）

Cox's theorem (WARNING: proof is not completely rigorous)

- product rule: $p(A, B|C) = p(A|B, C)p(B|C) = p(B|A, C)p(A|C)$
- sum rule: $p(A|C) + p(\neg A|C) = 1$

假设A是真的 B的可信度是一个实数p(B|A)

可信度p(B|A)符合常识

可信度p(B|A)是一致的

Notes

- the product rule implies Bayes' rule

Formalizing plausibility (2) — Kolmogorov's axioms

Pearl, 1988; `http://en.wikipedia.org/wiki/Probability_axioms`

可信性的形式化（2）--科尔莫戈罗夫的公理

Kolmogorov's axioms (plausibility as a measure) 作为衡量标准的可信性

- $0 \leq p(A) \leq 1$
- $p(\Omega) = 1$
- $p(A \text{ or } B) = p(A) + p(B)$ if A and B are mutually exclusive

如果A和B是相互排斥的

Theorem

- $p(A) = p(A, B) + p(A, \neg B)$
- $p(A) + p(\neg A) = 1$

Definition of conditional probability

- for $p(A) > 0$ define $p(B|A) = p(A, B)/p(A)$

Theorem

- Bayes' rule: $p(B|A) = p(A|B)p(B)/p(A)$

Formalizing plausibility (3) — Cox vs. Kolmogorov

Richard Threlkeld Cox

似是而非应该符合常理

Jack Engeman, "Richard T. Cox", public domain

- plausibilities should comply with common sense
- defines $p(B|A)$
- Bayes' rule is derived, from basic desiderata, thus very well justified

贝叶斯规则是由基本的考虑因素推导出来的，因此有很好的合理性

Andrey Nikolajevich Kolmogorov

Konrad Jacobs, "Kolmogorov", cropped, CC BY-SA 2.0 DE

似是而非应该有一个关于样本空间的衡量标准

- plausibilities should be a measure on sample space
- defines $p(A)$, defines $p(B|A)$
- Bayes' rule is "defined", i.e. follows directly from definition of $p(B|A)$
- Kolmogorov's approach is more rigorous

科尔莫戈罗夫的方法更加严格

贝叶斯规则是"定义的"，即直接从p(B|A)的定义中得出。

Note:
- Debate: is Bayes' rule an axiom?
- Plausibilities are just probabilities.

归纳性只是概率

The theory of probability is basically just common sense reduced to calculus (1814).
Pierre-Simon Laplace

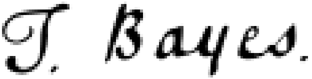
概率论基本上只是将常识简化为微积分（1814）。

Sophie Feytaud (fl.1841), Pierre-Simon Laplace, public domain

Bayes’ rule (1) — inverting probabilities

贝叶斯法则（1）--倒置概率

Unknown



Thomas Bayes, Bayes’ signature, public domain

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Bayes’ rule inverts probabilistic relationships, it translates between $p(B|A)$ and $p(A|B)$.
adapted from Barber, 1.1, 2012

贝叶斯规则颠倒了概率关系，它在p（B|A）和p（A|B）之间进行转换。
改编自Barber, 1.1, 2012

Bayes' rule (2) — updating beliefs 更新信念

Plug-in data and some hypothesis: 插入数据和一些假设

$$p(H|\text{data}) = \frac{p(\text{data}|H)p(H)}{p(\text{data})}$$

Before you see the data: (right-hand-side of Bayes' rule)

- value your beliefs as probabilities
- without seeing data your prior belief in the hypothesis H is $p(H)$
- knowing that H is true, you believe that the data has a certain likelihood $p(\text{data}|H)$; knowing that it is wrong, its likelihood is $p(\text{data}|\neg H)$
- the evidence $p(\text{data})$ can be calculated from known quantities using the sum rule, $p(\text{data}) = p(\text{data}|H)p(H) + p(\text{data}|\neg H)p(\neg H)$

After you have seen the data: (left-hand-side of Bayes' rule)

- Bayes' rule calculates your posterior belief $p(H|\text{data})$ about the hypothesis after seeing the data

Bayes' rule tells you how to update your beliefs!

Bayes' rule (3) — monster vs. mouse

当你想睡觉的时候，你听到床下有一些声音。.
While you tried to sleep, you hear some noise under the bed. . .

$$n = \text{some noise under your bed}$$
$$M = \text{a monster under your bed}$$
$$m = \text{a mouse under your bed}$$
$$e = \text{something else (e.g. only air) under your bed}$$

Question:

What is the type of n, M, m, e?

Answer:

Boolean, i.e. possible values are true/false.

布尔型，即可能的值是真/假。

Bayes' rule (3) — monster vs. mouse

While you tried to sleep, you hear some noise under the bed...

$$
\begin{aligned}
n &= \text{some noise under your bed} \\
M &= \text{a monster under your bed} \\
m &= \text{a mouse under your bed} \\
e &= \text{something else (e.g. only air) under your bed}
\end{aligned}
$$

Note:

变量M、m和e是相互排斥的。
即只有其中一个是真的。

The variables M, m and e are mutually exclusive,
i.e. only one of them can be true.

使用你的世界知识来分配概率。

Use your world knowledge to assign probabilities:

- for $p(M)$, $p(m)$, $p(e)$, with $p(M) + p(m) + p(e) = 1$.
- for $p(n|M)$, $p(n|m)$, $p(n|e)$, with values between zero and one.

Bayes' rule (3) — monster vs. mouse

While you tried to sleep, you hear some noise under the bed...

n = some noise under your bed

M = a monster under your bed

m = a mouse under your bed

e = something else (e.g. only air) under your bed

Joint probabilities:

联合概率

$p(n, M) = p(n \mid M)p(M)$	noisy monster	嘈杂的怪物
$p(n, m) = p(n \mid m)p(m)$	noisy mouse	嘈杂的老鼠
$p(n, e) = p(n \mid e)p(e)$	noisy something else	嘈杂的其他东西

Given the noise, was it a monster?

$$p(M \mid n) = \frac{p(n \mid M)p(M)}{p(n)} = \frac{p(n, M)}{p(n, M) + p(n, m) + p(n, e)}$$

Bayes' rule (4) — monster vs. mouse

Given the noise, was it a monster, a mouse or something else?

$$p(M|n) = \frac{p(n|M)p(M)}{p(n)} = \frac{p(n|M)p(M)}{p(n,M) + p(n,m) + p(n,e)}$$

$$p(m|n) = \frac{p(n|m)p(m)}{p(n)} = \frac{p(n|m)p(m)}{p(n,M) + p(n,m) + p(n,e)}$$

$$p(e|n) = \frac{p(n|e)p(e)}{p(n)} = \frac{p(n|e)p(e)}{p(n,M) + p(n,m) + p(n,e)}$$

Bayes' rule in words:

The probability of some hypothesis H after seeing the data is the ratio between "how well does H describe the data" compared to all possible explanations.

所有可能的解释的概率之和为1。

Probabilities of all possible explanations sum to one:

$$p(M) + p(m) + p(e) = 1 \qquad \text{before hearing the noise}$$

$$p(M|n) + p(m|n) + p(e|n) = 1 \qquad \text{after hearing the noise}$$

Deductive and plausible reasoning — revisited (1)

Jaynes, 2003, Sec. 1

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A = it will start to rain by 6pm

B = the sky will become cloudy before 6pm

Deductive reasoning:

if A is true, then B is true
A is true

B is true

"modus ponens"

if A is true, then B is true
B is false

A is false

"modus tollens"

Deductive and plausible reasoning — revisited (2)

Jaynes, 2003, Sec. 1

Overview:

- Assume: if A is true, then B is true.
  - A is true, implies that B is true.
  - A is false, implies that B becomes less plausible.
  - B is true, implies that A becomes more plausible.
  - B is false, implies that A is false.

- Assume: if A is true, then B becomes more plausible.
  - A is true, implies that B becomes more plausible.
  - A is false, implies that B becomes less plausible.
  - B is true, implies that A becomes more plausible.
  - B is false, implies that A becomes less plausible.

Deductive and plausible reasoning — revisited (3)

Semanitics:

Plausibility is measured as probability.

The value $p(B|A)$ is the probability of B assuming that A is true.

The value $p(B)$ is the probability of B assuming nothing.

$p(A) = 1$ is the statement that A is true assuming nothing.

p(A)=1是指假设没有任何事情发生，A是真的

$p(A) > p(B)$ is the statement that A is more probable than B.

Basic laws of probability:

$p(A, B) = p(A\|B)p(B) = p(B\|A)p(A)$	product rule
$p(A, B\|C) = p(A\|B, C)p(B\|C) = p(B\|A, C)p(A\|C)$	product rule
$p(A) + p(\neg A) = 1$	sum rule
$p(A\|C) + p(\neg A\|C) = 1$	sum rule
$p(A, B) + p(A, \neg B) = p(A)$	sum rule

Deductive and plausible reasoning — revisited (4)

Deductive reasoning:

$$\frac{\text{if } A \text{ is true, then } B \text{ is true} \quad A \text{ is true}}{B \text{ is true}}$$

$$\frac{\text{if } A \text{ is true, then } B \text{ is true} \quad B \text{ is false}}{A \text{ is false}}$$

Probabilistic reasoning:

概率推理

- assume $p(B|A) = 1$
- show $p(B|A) = 1$
- proof: by assumption

- assume $p(B|A) = 1$
- show $p(\neg A|\neg B) = 1$
- proof: apply Bayes' rule

Deductive and plausible reasoning — revisited (5)

Plausible reasoning:

$$\frac{\text{if } A \text{ is true, then } B \text{ is true} \quad B \text{ is true}}{A \text{ becomes more plausible}}$$

$$\frac{\text{if } A \text{ is true, then } B \text{ is true} \quad A \text{ is false}}{B \text{ becomes less plausible}}$$

Probabilistic reasoning:

- assume $p(B|A) = 1$
- show $p(A|B) \geq p(A)$
- proof: apply Bayes' rule

- assume $p(B|A) = 1$
- show $p(B|\neg A) \leq p(B)$
- proof: apply Bayes' rule

Deductive and plausible reasoning — revisited (6)

Plausible reasoning:

if A is true, then B more plausible
B is true

A becomes more plausible

if A is true, then B more plausible
A is false

B becomes less plausible

Probabilistic reasoning:

- assume $p(B|A) \geq p(B)$
- show $p(A|B) \geq p(A)$
- proof: apply Bayes' rule

- assume $p(B|A) \geq p(B)$
- show $p(B|\neg A) \leq p(B)$
- proof: apply Bayes' rule

Probabilistic reasoning — overview

Lemma 2.1

$p(B|A) = 1$ *implies*意味着

- $p(B|A) = 1$ *"modus ponens"*
- $p(B|\neg A) \leq p(B)$
- $p(A|B) \geq p(A)$
- $p(\neg A|\neg B) = 1$ *"modus tollens", alternative* $p(A|\neg B) = 0$

Lemma 2.2

$p(B|A) \geq p(B)$ *implies*

- $p(B|A) \geq p(B)$
- $p(B|\neg A) \leq p(B)$
- $p(A|B) \geq p(A)$
- $p(\neg A|\neg B) \geq p(\neg A)$

The rules of probability, combined with Bayes' rule make for a complete reasoning system, one which includes traditional deductive logic as a special case (see Jaynes, 2004).

adapted from Barber, 1.2, 2012

概率规则与贝叶斯规则相结合，构成了一个完整的推理系统，其中包括作为特例的传统演绎逻辑（见杰恩斯，2004）。

改编自Barber, 1.2, 2012

The actual science of logic is conversant at present only with things either certain, impossible, or entirely doubtful, none of which (fortunately) we have to reason on. Therefore the true logic for this world is the calculus of Probabilities, which takes account of the magnitude of the probability which is, or ought to be, in a reasonable man's mind (1850).

James Clerk Maxwell

目前，实际的逻辑学只涉及确定的、不可能的或完全可疑的事物，（幸运的是）我们都不需要对这些事物进行推理。因此，这个世界上真正的逻辑是概率的计算，它考虑到一个有理智的人心中的或应该有的概率的大小（1850）。

詹姆斯-克拉克-麦克斯韦

George J. Stodart, Engraving of James Clerk Maxwell, public domain

Interpretations of probability

- http://plato.stanford.edu/archives/sum2012/entries/probability-interpret/
- http://en.wikipedia.org/wiki/Probability_interpretations

Approaches to reasoning under uncertainty in AI

Pearl, 1988, 1.1.3人工智能中不确定情况下的推理方法

Three schools (according to Pearl):

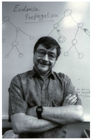
1. **Logicists:** non-numerical techniques, e.g. non-monotonic logic, circumscription

John McCarthy

2. **Neo-calculists:** numerical techniques, but not probabilities, instead new calculi, e.g. Dempster-Shafer calculus, fuzzy logic, certainty factors

Lotfi Zadeh

3. **Neo-probabilists:** probability theory together with clever computations



Judea Pearl

Summary

From Logic to Probability theory

- Deductive vs. plausible reasoning
- Cox's and Kolmogorov's approaches
- Rules of probability
- Bayes rule

演绎性推理与合理性推理
考克斯和科尔莫戈罗夫的方法
概率的规则
贝叶斯规则

Next section:

What exactly do we mean by $p(A)$ or $p(A, B)$, etc?

我们所说的p(A)p(A, B)等究竟是什么意思？