Ex.1. $H = I_n - \frac{1}{n} 1_n 1_n^T$

1. H is symmetric $\Rightarrow$ weil

$$H^T = (I_n - \tfrac{1}{n} 1_n 1_n^T)^T$$
$$= I_n^T - \tfrac{1}{n}(1_n 1_n^T)^T$$
$$= I_n - \tfrac{1}{n}(1_n^T)^T 1_n^T$$
$$= I_n - \tfrac{1}{n} 1_n 1_n^T$$
$$= H. \quad \square$$

D.h. $H = H^T$

2. H is idempotent $\Rightarrow$ weil

$$HH = H\cdot(I - \tfrac{1}{n} 1_n 1_n^T)$$
$$= HI - H\cdot\tfrac{1}{n} 1_n 1_n^T \quad (\text{weil } H\cdot I = H)$$
$$= H - (I - \tfrac{1}{n} 1_n 1_n^T)\cdot \tfrac{1}{n} 1_n 1_n^T$$
$$= H - \tfrac{1}{n} 1_n 1_n^T + \tfrac{1}{n^2} 1_n 1_n^T \cdot 1_n 1_n^T$$
$$= H - \tfrac{1}{n} 1_n 1_n^T + \tfrac{1}{n} 1_n 1_n^T \quad (\text{weil } 11^T11^T = n11^T)$$
$$= H$$

D.h. $HH = H$ $\quad\square$

$1_n 1_n^T = \sum_{i=1}^{n} |x|$

3. ~~H~~ weil 1_n is an eigenvector with eigenvalue 0

Dann $H\cdot 1 = [I - \frac{1}{n} 1_n 1_n^T] 1_n = I - \frac{1}{n} 1_n (1_n^T 1_n) = 0$. $\square$

4. $H = I_n - \frac{1}{n} 1_n 1_n^T$ für $1_n 1_n^T$ ~~müssen~~ Assume eigenvalue be λ, eigenvetor x.

Then $1_n 1_n^T x = \lambda x \rightsquigarrow (1_n 1_n^T)^2 x = \lambda^2 x$ and $(1_n 1_n^T)^2 = n(1_n 1_n^T)$ gilt $\lambda^2 - n\lambda = 0$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 = n$

① mit $\lambda_1 = 0$ $R(0\cdot I - 1_n 1_n^T) = n - k_1 \rightsquigarrow k_1 = n-1$ ~~$\rightarrow n = 1$~~ ✓.

② mit $\lambda_2 = n$. because there must be n eigenvalues ~~0~~ for matrix oder n,
so n is 1!

$\Rightarrow$ Together $\rightsquigarrow$ H have $\ell - \frac{1}{n}\lambda + 1$ mit $\lambda_1 = 0$ or $\lambda_2 = n$

so the eigenvalues of $\ell \cdot H$ are 0 of multiplicity 1 and 1 of multiplicity $n-1$ $\square$

5. XH has mean zero $\Rightarrow$

weil mit 3. ($H\mathbb{1}_n = 0$) haben wir.

$$\frac{1}{n} X H \mathbb{1}_n = \frac{1}{n} X \cdot 0_n = 0_d \quad \square$$

Ex4. 1. $\phi(a) = (a_1^2, a_2^2, \sqrt{2}a_1a_2, \sqrt{2}a_1, \sqrt{2}a_2, 1)^T$

Then: $$\phi(a)^T\phi(b) = a_1^2b_1^2 + a_2^2b_2^2 + 2a_1a_2b_1b_2 + 2a_1b_1 + 2a_2b_2 + 1$$
$$= (a_1b_1 + a_2b_2)^2 + 2a_1a_2b_1b_2 + 2a_1b_1 + 2a_2b_2 + 1$$
$$= (a^Tb)^2 + 2a^Tb + 1$$
$$= (a^Tb+1)^2$$
$$= K(a,b)$$

$\square$

2. $$K(a,b) = (a^Tb+1)^3 = (a^Tb+1)(a^Tb+1)(a^Tb+1)$$
$$= ((a^Tb)^2 + 2a^Tb + 1)(a^Tb+1)$$
$$= (a^Tb)^3 + (a^Tb)^2 + 2(a^Tb)^2 + 2a^Tb + a^Tb + 1$$
$$= (a^Tb)^3 + 3(a^Tb)^2 + 3a^Tb + 1$$
$$= (a_1b_1 + a_2b_2)^3 + 3(a_1b_1 + a_2b_2)^2 + 3(a_1a_2b_1b_2 + a_1b_1 + a_2b_2) + 1$$
$$= a_1^3b_1^3 + 3a_1^2b_1^2a_2b_2 + 3a_2^2b_2^2a_1b_1 + a_2^3b_2^3 + 3a_1^2b_1^2 + 6a_1a_2b_1b_2 + 3a_2^2b_2^2 + 3a_1a_2b_1b_2 + 3a_1b_1 + 3a_2b_2 + 1$$
$$= a_1^3b_1^3 + a_2^3b_2^3 + 3a_1^2b_1^2a_2b_2 + 3a_2^2b_2^2a_1b_1 + 9a_1a_2b_1b_2 + 3a_1^2b_1^2 + 3a_2^2b_2^2 + 1 + 3a_1b_1 + 3a_2b_2$$

D.h. $\phi(a) = (a_1^3, a_2^3, \sqrt{3}a_1^2b_1^2, \sqrt{3}a_2^2b_2^2, 3a_1a_2, \sqrt{3}a_1^2, \sqrt{3}a_2^2, \sqrt{3}a_1, \sqrt{3}a_2, 1) \rightsquigarrow \mathbb{R}^{10}$

3.

3. mit Binomial theorem wissen wir

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \dots + \binom{n}{n}x^0 y^n$$

$$\Rightarrow (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

und mit

$$(1+x)^n = \binom{n}{0}x^0 + \dots + \binom{n}{n}x^n$$

$$\Rightarrow (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

zusammen: haben wir

$$\text{dimenstion} = \text{höchsten } (p+1) + p + + p-1 + \dots + 1$$

$$= \cancel{(p+1+1)\cdot p/2} \quad 1+2+\dots+p+1$$

$$= \cancel{\tfrac{1}{2}p^2+p} \quad \sum_{n=1}^{p+1} p$$

$$= (1+p)(p+2)/2$$

$$= \tfrac{1}{2}(p^2+3p+2)$$

Bsp: $p=2 \rightsquigarrow d = \frac{1}{2}(4+6+2) = 6$ ✓

$p=3 \rightsquigarrow d = \frac{1}{2}(9+\cancel{12}^{9}+2) = 10$ ✓.

Ω