DS

4.29.

10 min

2007/3/I/7E

Poincaré–Bendixon Theorem for a system $\dot{x} = f(x)$ in $\mathbb{R}^2$ states that: If the forward orbit of the system is trapped in a closed and bounded domain with no fixed point in this domain, then $\omega(x)$ is a periodic orbit.

$$\begin{cases} \dot{x} = x - y - x^3 - xy^2 - k^2xy^2 \\ \dot{y} = y + x - x^2y - y^3 - k^2x^2y \end{cases}$$

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y})$$
$$= \frac{1}{r}(x^2 - \cancel{xy} - x^4 - x^2y^2 - k^2x^2y^2 + y^2 + \cancel{xy} - x^2y^2 - y^4 - k^2x^2y^2)$$
$$= \frac{1}{r}((x^2+y^2) - (x^4+y^4+2x^2y^2) - 2k^2x^2y^2)$$
$$= \frac{1}{r}[r^2 - r^4 - 2k^2r^4\cos^2\theta\sin^2\theta]$$
$$= r - r^3 - 2k^2r^3\cos^2\theta\sin^2\theta$$
$$= r - r^3 - \frac{1}{2}k^2r^3\sin^2 2\theta$$

Extremise over $\theta$: $\sin^2 2\theta \in [0,1]$.


$$r - r^3 - \frac{1}{2}k^2r^3 \le \dot{r} \le r - r^3$$
$$\parallel$$
$$(r - (1+\tfrac{1}{2}k^2)r^3)$$

So $\dot{r} \ge 0$ if $r - (1+\frac{1}{2}k^2)r^3 \ge 0$, i.e. $0 \le r \le \sqrt{\frac{2}{2+k^2}}$

$\dot{r} \le 0$ if ~~r≥1~~ $r - r^3 \le 0$, i.e. $r \ge 1$

$\therefore$ $D := \{x : \sqrt{\frac{2}{2+k^2}} \le r \le 1\}$.

Then once orbit enters the annulus, it stays within it. And $D$ is closed and bounded.

Hence, by Poincaré–Bendixson Theorem, $\exists$ a periodic orbit in the region $\frac{2}{2+k^2} \le x^2 + y^2 \le 1$