Paper 2, Section I

6C Mathematical Biology

Consider a model of an epidemic consisting of populations of susceptible, $S(t)$, infected, $I(t)$, and recovered, $R(t)$, individuals that obey the following differential equations

$$\begin{aligned}\frac{dS}{dt} &= aR - bSI,\\ \frac{dI}{dt} &= bSI - cI,\\ \frac{dR}{dt} &= cI - aR,\end{aligned}$$

where a, b and c are constant. Show that the sum of susceptible, infected and recovered individuals is a constant N. Find the fixed points of the dynamics and deduce the condition for an endemic state with a positive number of infected individuals. Expressing R in terms of S, I and N, reduce the system of equations to two coupled differential equations and, hence, deduce the conditions for the fixed point to be a node or a focus. How do small perturbations of the populations relax to the steady state in each case?

Math Bio

4.27

11 min

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$\dot{S} = aR - bSI$

$\dot{I} = bSI - cI$

$\dot{R} = cI - aR$

$\dot{S} + \dot{I} + \dot{R} = 0 \Rightarrow S + I + R = N$, which is the number of total population.

Find FP: $\dot{I} = 0 \Rightarrow I = 0$ or $S = \frac{c}{b}$.

— If $I = 0$, $R = 0$, $S = N - I - R = N$

— If $S = \frac{c}{b}$, then $aR = cI$, $R + I = N - \frac{c}{b}$

$\Rightarrow R = \dfrac{c(bN-c)}{b(a+c)}$, $I = \dfrac{a(bN-c)}{b(a+c)}$

$\therefore$ FPs are: $(S, I, R) = (N, 0, 0)$, $\left(\frac{c}{b}, \dfrac{a(bN-c)}{b(a+c)}, \dfrac{c(bN-c)}{b(a+c)}\right)$

Condition for endemic state:

$I(0) > 0$

$\dot{I}(t) = (bS - c)I(t)$, $\dot{I}(0) = \cancel{(bS} (bS(0) - c)I(0)$

$\Rightarrow$ need $bS(0) - c > 0$

Assume $S(0) \approx N$, then $\boxed{\frac{b}{c}N > 1}$

$R = N - S - I$

$$\begin{cases} \dot{S} = aR - bSI = a(N-S-I) - bSI \\ \dot{I} = bSI - cI. \end{cases}$$

$$J = \begin{pmatrix} -a-bI & -a-bS \\ bI & bS-c \end{pmatrix}$$

Consider the FP with $S > 0$, $I > 0$:

$(S, I) = \left(\frac{c}{b}, \dfrac{a(bN-c)}{b(a+c)}\right)$. (Note: $\dfrac{a(bN-c)}{b(a+c)} > 0$ as $\frac{b}{c}N > 1$)

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$$J = \begin{pmatrix} -a\left(\frac{a+bN}{a+c}\right) & -(a+c) \\ \frac{a(bN-c)}{a+c} & 0 \end{pmatrix} \quad \text{at FP}$$

$Tr(J) = -a\left(\frac{a+bN}{a+c}\right)$, $Det(J) = a(bN-c)$

Since $Tr(J) < 0$, $Det(J) > 0$, then the FP is stable.

For stable node:

$\lambda^2 - Tr(J)\lambda + Det(J) = 0$

$\Rightarrow$ need $Tr(J)^2 > 4Det(J)$

$\Rightarrow a^2\left(\frac{a+bN}{a+c}\right)^2 > 4a(bN-c)$

$\Rightarrow \boxed{a\left(\frac{a+bN}{a+c}\right)^2 \geq 4(bN-c)}$

For stable focus: $\boxed{a\left(\frac{a+bN}{a+c}\right)^2 < 4(bN-c)}$

Small perturbation:

For node: perturbation will decrease in magnitude and goes to FP eventually.

For focus: perturbation will oscillate around FP, and eventually will get closer in magnitude.

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