Dynamical Systems 4.22

2011/1/I/7C 15min

$$\begin{cases} \dot{x} = \mu^2 x - xy \\ \dot{y} = -y + x^2 \end{cases}$$

To find FPs : $\dot{x} = 0 \Rightarrow x = 0$ or $y = \mu^2$

$\dot{y} = 0 \Rightarrow y = x^2$

$\therefore$ FPs are : $(x, y) = (0, 0)$, $(\pm\mu, \mu^2)$

Linearised system has Jacobian:

$$J = \begin{pmatrix} \mu^2 - y & -x \\ 2x & -1 \end{pmatrix}$$

— $(0,0)$: $J = \begin{pmatrix} \mu^2 & 0 \\ 0 & -1 \end{pmatrix}$, $\lambda_1 = \mu^2 > 0$, $\lambda_2 = -1 < 0 \Rightarrow$ saddle

— (μ, μ^2) : $J = \begin{pmatrix} 0 & -\mu \\ 2\mu & -1 \end{pmatrix}$, $\mathrm{Det}(J - \lambda I) = \begin{vmatrix} -\lambda & -\mu \\ 2\mu & -1-\lambda \end{vmatrix}$

$$= \lambda(\lambda+1) + 2\mu^2$$
$$= \lambda^2 + \lambda + \tfrac{1}{4} + (2\mu^2 - \tfrac{1}{4}) = 0$$

$\Rightarrow (\lambda + \frac{1}{2})^2 = \frac{1}{4} - 2\mu^2$

① If $\frac{1}{4} - 2\mu^2 > 0$, i.e. $-\frac{1}{2\sqrt{2}} < \mu < \frac{1}{2\sqrt{2}}$

$\lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2\mu^2}$, both eigenvalues are negative

$\Rightarrow$ stable nodes

☆ ② If $\frac{1}{4} - 2\mu^2 = 0$, i.e. $\mu^2 = \frac{1}{8}$

$\lambda_1 = \lambda_2 = -\frac{1}{2} \Rightarrow$ stable stella node

③ If $\frac{1}{4} - 2\mu^2 < 0$, i.e. $\mu > \frac{1}{2\sqrt{2}}$ or $\mu < -\frac{1}{2\sqrt{2}}$.

$\lambda = -\frac{1}{2} \pm \sqrt{2\mu^2 - \frac{1}{4}}\, i$, stable focus

— $(-\mu, \mu^2)$: Same types of FPs as (μ, μ^2).

At $(0,0)$, the FP is a saddle. $\lambda_1 = \mu^2$ corresponds to unstable invariant subspace $E^u = \{t\binom{1}{0}\}$;

$\lambda_2 = -1$ corresponds to stable invariant subspace $E^s = \{t\binom{0}{1}\}$.

— To find W^u_{loc}, let $y = S(x) = a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5)$

$(a_0 = a_1 = 0$ as W^u_{loc} should be tangent to E^u and $\underline{0} \in W^u_{loc}$

$\Rightarrow \dot{y} = S'(x)\dot{x}$

$-y + x^2 = (2a_2 x + 3a_3 x^2 + 4a_4 x^3)(\mu^2 x - xy)$

To order 4 :

$-a_2 x^2 - a_3 x^3 - a_4 x^4 + x^2 = 2\mu^2 a_2 x^2 + 3\mu^2 a_3 x^3 + 4a_4 \mu^2 x^4 - 2a_2^2 x^4$

Comparing coefficient: $\begin{cases} -a_2+1=2\mu^2 a_2 \\ -a_3=3\mu^2 a_3 \\ -a_4=4a_4\mu^2-2a_2^2 \end{cases} \Rightarrow \begin{cases} a_2=\frac{1}{2\mu^2+1} \\ a_3=0 \\ a_4=\frac{2}{(2\mu^2+1)^2(4\mu^2+1)} \end{cases}$

— To find W^s_{loc}, let $x=U(y)=b_2y^2+b_3y^3+b_4y^4+O(y^5)$

$\dot{x}=U'(y)\,\dot{y}$

$\Rightarrow \mu^2x-xy=(2b_2y+3b_3y^2+4b_4y^3)(-y+x^2)$

$\Rightarrow \mu^2b_2y^2+\mu^2b_3y^3+\mu^2b_4y^4-b_2y^3-b_3y^4$

$= -2b_2y^2-3b_3y^3-4b_4y^4$

$\begin{cases} \mu^2b_2=-2b_2 \\ \mu^2b_3-b_2=-3b_3 \\ \mu^2b_4-b_3=-4b_4 \end{cases} \Rightarrow \begin{cases} b_2=0 \\ b_3=0 \\ b_4=0 \end{cases}$

$\therefore W^s_{loc}=\{x=0\}$