Dynamical Systems — 4.22

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The stable and unstable invariant subspaces of the linearisation of a dynamical system $\dot{\underline{x}}=\underline{f}(\underline{x})$ ~~are~~ at a saddle point located at $\underline{0}$ are denoted by E^s and E^u respectively, and are spanned by the eigenvectors of the Jacobian of the linearised system ~~with the~~ with real parts of corresponding eigenvalues <0 and >0, respectively.

The stable and unstable manifold, denoted by W^s_{loc} and W^u_{loc}, respectively, have the same dimension as E^s and E^u, and are tangent to E^s and E^u at the origin, respectively, with

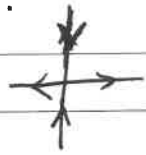
$$W^s_{loc}=\{\underline{x}:\ \phi_t(\underline{x})\to 0 \text{ as } t\to\infty\}$$
$$W^u_{loc}=\{\underline{x}:\ \phi_t(\underline{x})\to 0 \text{ as } t\to-\infty\}.$$

$$\begin{cases}\dot{x}=x+x^2+2xy+3y^2\\ \dot{y}=-y+3x^2.\end{cases}$$

~~To find W^s_{loc}:~~

Jacobian $J=\begin{pmatrix}1+2x+2y & 2x+6y\\ 6x & -1\end{pmatrix}\Bigg|_{(0,0)}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\therefore$ $(0,0)$ is a saddle point, with $\lambda_1=1$, $E^u=\{t\binom{1}{0}\}$;
$\lambda_2=-1$, $E^s=\{t\binom{0}{1}\}$.



To find W^s_{loc}: let $x=S(y)=a_2y^2+a_3y^3+\cdot\, O(y^4)$
($a_0=a_1=0$ as by tangent condition, $x'=S'(y)=0$ when $y=0$)

$\dot{x}=S'(y)\,\dot{y}$

$\Rightarrow x+x^2+2xy+3y^2=(2a_2y+3a_3y^2+4a_4y^3)(-y+3x^2)$

Sub $x=S(y)$ and only consider up to cubic order:

$a_2y^2+a_3y^3+2a_2y^3+3y^2=-2a_2y^2-3a_3y^3+\text{h.o.t.}$

Comparing coefficients:

$$\begin{cases}a_2+3=-2a_2\\ a_3+2a_2=-3a_3\end{cases}\Rightarrow\begin{cases}a_2=-1\\ a_3=\frac{1}{2}\end{cases}$$

$\therefore W^s_{loc}=\{\underline{x}:\ x=-y^2+\frac{1}{2}y^3\}$.

To find W^u_{loc}: let $y = u(x) = b_2 x^2 + b_3 x^3 + O(x^4)$.

$\Rightarrow \dot{y} = u'(x)\,\dot{x}$

$\Rightarrow -y + 3x^2 = (2b_2 x + 3b_3 x^2)(x + x^2 + 2xy + 3y^2)$

To cubic order:

$-b_2 x^2 + b_3 x^3 + 3x^2 = 2b_2 x^2 + 2b_2 x^3 + 3b_3 x^3 + h.o.t.$

$$\begin{cases} -b_2 + 3 = 2b_2 \\ b_3 = 2b_2 + 3b_3 \end{cases} \Rightarrow \begin{cases} b_2 = 1 \\ b_3 = -1 \end{cases}$$

$\therefore W^u_{loc} = \{ \underline{x} : y = x^2 - x^3 \}$.