Part II: Dynamical Systems - Revision Notes

Lectures by , notes by Zhimei Liu

1. Basic Definitions

Definition: (ω-limit set)

$\omega(\mathbf{x}) = \{\mathbf{y} : \exists$ infinite sequence $t_1, t_2, ... \to \infty$ with $\phi_{t_n}(\mathbf{x}) \to \mathbf{y}\}$

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2. Fixed Points

2.1. Linearisation

Classifying fixed points

1. **Saddle point:** $\lambda_1 < 0 < \lambda_2$
2. **Stable node:** $\lambda_1, \lambda_2 < 0$
3. **Unstable node:** $\lambda_1, \lambda_2 > 0$
4. **Stable focus:**
5. **Unstable focus:**
6. **Stella node:**
7. **Improper node:**
8. **Centre:**

Hamiltonian systems:

Definition: The system that can be written as $\dot{x} = \frac{\partial H}{\partial y}$ and $\dot{y} = -\frac{\partial H}{\partial x}$ is called the Hamiltonian system.

Hamiltonian systems are always centres or saddles.

$\dot{\mathbf{x}} \cdot \nabla H = 0 \implies$ trajectories are contours of $H(x, y)$.

Definition: (*Hyperbolic fixed point*) If none of the eigenvalues of the Jacobian at this fixed point has zero real part.

Definition: (*Hyperbolic sink*) If **all** eigenvalues have negative real parts.

Definition: (*Hyperbolic source*) If **all** eigenvalues have positive real parts.

Definition: (*Stable, unstable and centre subspaces*) The stable, unstable and centre subspaces of the linearisation of $\mathbf{f}$ at the FPs $\mathbf{x}_0$ are the 3 linear subspaces E^s, E^u and E^c spanned by the subset of (possibly generalised) eigenvectors of $\mathbf{A}$, whose eigenvalues have real parts < 0, > 0 and $= 0$, respectively.

Note: Hyperbolic points do not have a E^c

Theorem: (*Stable Manifold Theorem*) Suppose $\mathbf{0}$ is a hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and E^u and E^s are the unstable and stable subspaces of the linearisation of $\mathbf{f}$ about $\mathbf{0}$. Then $\exists$ local unstable and stable manifolds $W^u_{\text{loc}}(\mathbf{0})$ and $W^s_{\text{loc}}(\mathbf{0})$ which have the same dimension as E^u and E^s and are tangent to E^u and E^s at $\mathbf{0}$ s.t. for $\mathbf{x} \neq \mathbf{0}$ but in a sufficiently small neighborhood of $\mathbf{0}$,

$$W^u_{\text{loc}} = \{\mathbf{x} : \phi_t(\mathbf{x}) \to \mathbf{0} \text{ as } t \to -\infty\}$$

$$W^s_{\text{loc}} = \{\mathbf{x} : \phi_t(\mathbf{x}) \to \mathbf{0} \text{ as } t \to \infty\}$$

Finding Stable and unstable manifold

To find W^u_{loc}, write $y = U(x) = a_2x^2 + a_3x^3 + ...$
with $U(0) = U'(0) = 0$.

To find W^s_{loc}, write $x = S(y) = b_2y^2 + b_3y^3 + ...$
with $S(0) = S'(0) = 0$.

Then take derivatives both sides and substitute $\dot{x}$ and $\dot{y}$ and compare coefficients.

3. Stability

3.1. Stability definitions

Definition: A fixed point $\mathbf{x_0}$ is **Lyapunov stable (LS)** if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \text{s.t } |\mathbf{x} - \mathbf{x_0}| < \delta,\ |\phi_t(\mathbf{x}) - \mathbf{x_0}| < \epsilon,\ \forall t > 0$$

Informally, we can say "starts near, stays near".

Definition: A fixed point $\mathbf{x_0}$ is **Quasi-asymptotically stable (QAS)** if

$$\exists \delta > 0,\ |\mathbf{x} - \mathbf{x_0}| < 0,\ \phi_t(\mathbf{x}) \to \mathbf{x_0} \text{ as } t \to \infty$$

Informally, we can say "orbit tends to ".

Definition: A fixed point $\mathbf{x_0}$ is **asymptotically stable (AS)** if it is both LS and QAS.

3.2. Lyapunov functions

Definition: A continuous differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is a **Lyapunov function** for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on a domain D containing neighbourhood of $\mathbf{0}$ if

(i) $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ in D. (positive definite)

(ii) $\dot{V} = \mathbf{f} \cdot \nabla V \leq 0$ in D. (non-increasing)

Strict Lyapunov function: If inequality in condition (ii) is strict apart from $\mathbf{x} = \mathbf{0}$.

Lyapunov first theorem: If a Lyapunov function V exists, then $\mathbf{0}$ is Lyapunov stable.

Proof. Let ϵ be small enough, so $\{|\mathbf{x}| \leq \epsilon\} \subseteq D$.

$$m := \min\{V(\mathbf{x}) : |\mathbf{x}| = \epsilon\}$$

$$C_{m,\epsilon} := \{\mathbf{x} : V(\mathbf{x}) < m,\ |\mathbf{x}| < \epsilon\}$$

Choose $\delta > 0$ s.t. $\{|\mathbf{x}| < \delta\} \subset C_{m,\epsilon}$
Then $\{|\mathbf{x}| < \delta\} \subset C_{m,\epsilon} \subseteq \{|\mathbf{x}| < \epsilon\}$.

Lyapunov second theorem: If a strict Lyapunov function exists, then $\mathbf{0}$ is asymptotically stable.

La Salle's invariance principle: If V is a Lyapunov function on domain D which is compact (closed and bounded) and forward invariant ($\mathbf{x} \in D \implies \phi_t(\mathbf{x}) \in D\ \forall t > 0$), then

$$\omega(\mathbf{x}) \subseteq \{\mathbf{y} : V(\phi_t(\mathbf{y})) = V_0\ \forall t\} \text{ for some } V_0$$

More usefully: $\phi_t(\mathbf{x})$ tends to an invariant subset of $\{\mathbf{y} : \dot{V}(\mathbf{y}) = 0\} \cap D$.

Definition: The **domain of stability (DoS)** of an AS invariant set Λ is

$$\{\mathbf{x} : \phi_t(\mathbf{x}) \to \Lambda\}$$

If DoS is $\mathbb{R}^n$, then we say Λ is globally stable.

General method for finding DoS:

1. Find Lyapunov function V and domain D containing neighbourhood of $\mathbf{0}$ s.t.
   - $V \geq 0$ on D and $V = 0$ only at $\mathbf{x} = 0$.
   - $\dot{V} \leq 0$ on D.
2. Find k s.t. $C_k = \{\mathbf{x} : V(\mathbf{x}) \leq k\} \subseteq D$
3. Adjust k or V so that only invariant subset of $\{\dot{V} = 0\} \cap C_k$ is $\{\mathbf{0}\}$. Then La Salle's $\implies$ $C_k \subseteq$ DoS.

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4. Periodic orbit

4.1. Poincare index test

Properties of Poincare index:

1. Integral form: $I(\Gamma) = \frac{1}{2\pi} \oint d\psi = \frac{1}{2\pi} \oint d(\tan^{-1}(\frac{f_2}{f_1}))$ $= \frac{1}{2\pi} \oint \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$
2. If $I(\Gamma)$ encloses no FPs, then $I(\Gamma) = 0$.
3. Index of a closed trajectory is +1.
4. $I(\Gamma)$ is the sum of indices of all FPs enclosed by Γ.
5. Index of hyperbolic sink/source is +1, hyperbolic saddle is -1.

Poincare index test: (Test 1) Any periodic orbit must contain one or more FPs and sum of their indices is +1.

Note: POs cannot cross any invariant axes.

4.2. Dulac's criterion

Dulac's criterion: (Test 2) If there is a continuously differentiable function $\phi(x, y)$ s.t. $\nabla \cdot (\phi \mathbf{f}) \neq 0$ on a simply connected domain $D \subseteq \mathbb{R}^2$, then there are no periodic orbit that lie entirely in D.

Proof. By contradiction and divergence theorem.

Note: Often use $\phi = 1$ (called dibergence test).

Corollary: (Test 3) If $\nabla \cdot (\phi \mathbf{f}) \neq 0$ on some doubly-connected domain $D \subset \mathbb{R}^2$, then there is at most one PO entirely in D (and must enclose the hole).

Note: Can apply in damped pendulum, where there is a cylinder coordinate.

Gradient criterion: (Test 4) If $\exists$ positive function $\rho(x, y)$ s.t. $\rho \mathbf{f} = \nabla \psi$ for some single-valued function ψ on some simply connected domain D, then there are no POs entirely in D.

4.3. Poincare-Bendixson Theorem

Theorem: (Test 4) If the forward orbit $O^+(\mathbf{x})$ of some point $\mathbf{x}$ remains in a compact set (closed and bounded) $K \subseteq \mathbb{R}^2$ that contains no fixed points, then $\omega(\mathbf{x})$ is a periodic orbit.

4.4. Near-Hamiltonian flows