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Computation in Sofic Quantum Dynamical Systems

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Abstract We analyze how measured quantum dynamical systems store and process information, introducing sofic quantum dynamical systems. Using recently introduced information-theoretic measures for quantum processes, we quantify their information storage and processing in terms of *entropy rate* and *excess entropy*, giving closed-form expressions where possible. To illustrate the impact of measurement on information storage in quantum processes, we analyze two spin-1 sofic quantum systems that differ only in how they are measured.

1 Introduction

Symbolic dynamics originated as a method to study general dynamical systems when, nearly 100 years ago, Hadamard used infinite sequences of symbols to analyze the distribution of geodesics; see Ref. (Kitchens 1998) and references therein. In the 1930's and 1940's Hedlund and Morse coined the term *symbolic dynamics* (Hedlund and Morse 1938, 1940). In the 1940's Shannon used sequence spaces to describe information channels (Shannon and Weaver 1962). Subsequently, the techniques and ideas have found significant applications beyond dynamical systems, in data storage and transmission, as well as linear algebra (Lind and Marcus 1995).

On the other side of the same coin one finds finite-state automata that are used as codes for sequences. Sequences that can be coded this way define the *regular languages*. Dynamical systems can also be coded with finite-state automata. Transforming a dynamical system into a finite-state code is done using the tools from symbolic dynamics

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(Lind and Marcus 1995). *Sofic systems* are the particular class of dynamical systems that are the analogs of regular languages in automata theory.

Yet another thread in the area of dynamical systems—the study of the quantum behavior of classically chaotic systems—gained much interest (Gutzwiller 1990), most recently including the role of measurement. It turns out that measurement interaction leads to genuinely chaotic behavior in quantum systems, even far from the semi-classical limit (Habib et al. 2006). Classical dynamical systems can be embedded in quantum dynamical systems as the special class of commutative operators, which allow for unambiguous assignment of joint probabilities to two observations (Alicki and Fannes 2001).

An attempt to construct symbolic dynamics for quantum dynamical systems was made by Beck and Graudenz (1992) and Alicki and Fannes (2001). Definitions of entropy followed from this. However, no connection to quantum finite-state automata was established there.

Here we forge a link between quantum dynamical systems and quantum computation in general, and with quantum automata in particular, by extending concepts from symbolic dynamics to the quantum setting. The recently introduced computational model class of *quantum finite-state generators* provides the required link between quantum dynamical systems and the theory of automata and formal languages (Wiesner and Crutchfield 2008). It gives access to an analysis of quantum dynamical systems in terms of symbolic dynamics. Here, we strengthen that link by studying explicit examples of quantum dynamical systems. We construct their quantum finite-state generators and establish their *sofic* nature. In addition we review tools that give an information-theoretic analysis for quantifying the information storage and processing of these systems. It turns out that both the sofic nature and information processing capacity depend on the way a quantum system is measured.

2 Quantum finite-state generators

To start, we recall the *quantum finite-state generators* (QFGs) defined in Ref. (Wiesner and Crutchfield 2008). They consist of a finite set of *internal states* $Q = \{q_i : i = 1, \ldots, |Q|\}$. The *state vector* is an element of a $|Q|$-dimensional Hilbert space: $\langle\psi| \in \mathcal{H}$. At each time step a quantum generator outputs a symbol $s \in \mathcal{A}$ and updates its state vector.

The temporal dynamics is governed by a set of $|Q|$-dimensional *transition matrices* $\{T(s) = U \cdot P(s), s \in \mathcal{A}\}$, whose components are elements of the complex unit disk and where each is a product of a unitary matrix U and a projection operator $P(s)$. U is a $|Q|$-dimensional unitary *evolution operator* that governs the evolution of the state vector $\langle\psi|$. $\mathbf{P} = \{P(s) : s \in \mathcal{A}\}$ is a set of *projection operators*—$|Q|$-dimensional Hermitian matrices—that determines how the state vector is measured. The operators are mutually orthogonal and span the Hilbert space: $\sum_s P(s) = \mathbb{1}$.

Each output symbol s is identified with the measurement outcome and labels one of the system's eigenvalues. The projection operators determine how output symbols are generated from the internal, hidden unitary dynamics. They are the only way to observe a quantum process's current internal state.

A quantum generator operates as follows. U_{ij} gives the transition amplitude from internal state q_i to internal state q_j. Starting in state vector $\langle\psi_0|$ the generator updates its state by applying the unitary matrix U. Then the state vector is projected using $P(s)$ and renormalized. Finally, symbol $s \in \mathcal{A}$ is emitted. In other words, starting with state vector

$\langle \psi_0 |$, a single time-step yields $\langle \psi(s) | = \langle \psi_0 | U \cdot P(s)$, with the observer receiving measurement outcome s.

2.1 Process languages

The only physically consistent way to describe a quantum system under iterated observation is in terms of the observed sequence $\overleftrightarrow{S} \equiv \ldots S_{-2} S_{-1} S_0 S_1 \ldots$ of discrete random variables S_t. We consider the family of *word distributions*, $\{\Pr(s_{t+1}, \ldots, s_{t+L}) : s_t \in \mathcal{A}\}$, where $\Pr(s_t)$ denotes the probability that at time t the random variable S_t takes on the particular value $s_t \in \mathcal{A}$ and $\Pr(s_{t+1}, \ldots, s_{t+L})$ denotes the joint probability over sequences of L consecutive measurement outcomes. We assume that the distribution is stationary:

$$\Pr(S_{t+1}, \ldots, S_{t+L}) = \Pr(S_1, \ldots, S_L). \tag{1}$$

We denote a block of L consecutive variables by $S^L \equiv S_1 \ldots S_L$ and the lowercase $s^L = s_1 s_2 \cdots s_L$ denotes a particular measurement sequence of length L. We use the term *quantum process* to refer to the joint distribution $\Pr(\overleftrightarrow{S})$ over the infinite chain of random variables. A quantum process, defined in this way, is the quantum analog of what Shannon referred to as an *information source* (Cover and Thomas 1991).

Such a quantum process can be described as a *stochastic language* $\mathcal{L}$, which is a *formal language* with a probability assigned to each word. A stochastic language's word distribution is normalized at each word length:

$$\sum_{\{s^L \in \mathcal{L}\}} \Pr\left(s^L\right) = 1, \;\; L = 1, 2, 3, \ldots \tag{2}$$

with $0 \leq \Pr(s^L) \leq 1$ and the consistency condition $\Pr(s^L) \geq \Pr(s^L s)$.

A *process language* is a stochastic language that is *subword closed*: all subwords of a word are in the language.

We can now determine word probabilities produced by a QFG. Starting the generator in $\langle \psi_0 |$, the probability of output symbol s is given by the state vector without renormalization:

$$\Pr(s) = \langle \psi(s) | \psi(s) \rangle. \tag{3}$$

While the probability of outcome s^L from a measurement sequence is

$$\Pr(s^L) = \langle \psi(s^L) | \psi(s^L) \rangle. \tag{4}$$

In (Wiesner and Crutchfield 2008) the authors established a hierarchy of process languages and the corresponding quantum and classical computation-theoretic models that can recognize and generate them.

2.2 Alternative quantum finite-state machines

Said most prosaically, we view quantum generators as representations of the word distributions of quantum process languages. Despite similarities, this is a rather different emphasis than that used before. The first mention of *quantum automata* as an empirical description of physical properties was made by Albert in 1983. Albert's results were subsequently criticized by Peres for using an inadequate notion of measurement (Peres 1984). In a computation-theoretic context, quantum finite-state automata were introduced by several authors and in varying ways, but all as devices for recognizing word membership in a language. For the most widely discussed quantum automata, see Refs. (Moore and Crutchfield 2000; Kondacs and Watrous 1997; Aharonov et al. 1998). Ref. (Ambainis and Watrous 2002) summarizes the different classes of languages which they can recognize. Quantum transducers were introduced by Freivalds and Winter (2001). Their definition, however, lacks a physical notion of measurement. We, then, introduced quantum finite-state machines, as a type of transducer, as the general object that can be reduced to the special cases of quantum recognizers and quantum generators of process languages (Wiesner and Crutchfield 2008).

3 Information processing in a spin-1 dynamical system

We will now investigate concrete examples of quantum processes. Consider a spin-1 particle subject to a magnetic field which rotates the spin. The state evolution can be described by the following unitary matrix:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{5}$$

Since all entries are real, U defines a rotation in $\mathbb{R}^3$ around the y-axis by angle $\frac{\pi}{4}$ followed by a rotation around the x-axis by an angle $\frac{\pi}{2}$.

Using a suitable representation of the spin operators J_k (Peres 1993, p. 199):

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0, \end{pmatrix}, \tag{6}$$

the relation $P_k = 1 - J_k^2$ defines a one-to-one correspondence between the projector P_k and the square of the spin component along the k-axis. The resulting measurement answers the yes-no question, Is the square of the spin component along the k-axis zero?

Consider the observable J_y^2. Then the following projection operators together with U in Eq. 4 define a quantum finite-state generator:[1]

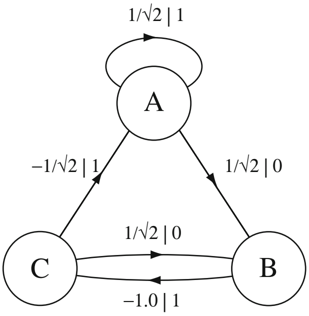
$$\begin{aligned} & P(0) = |010\rangle\langle 010| \\ \text{and} \quad & P(1) = |100\rangle\langle 100| + |001\rangle\langle 001|. \end{aligned} \tag{7}$$

A graphical representation of the automaton is shown in Fig. 1. States A, B, and C correspond to the eigenstates $\langle 100|$, $\langle 010|$, and $\langle 001|$, respectively, of the observable J_y^2. The amplitudes attached to the transitions are given by the unitary matrix in Eq. 4. Since eigenstates A and C have degenerate eigenvalues the output symbol is the same when the machine enters either of these states.

The process language generated by this QFG is the so-called *Golden-Mean Process* language (Kitchens 1998). The word distribution is shown in Fig. 2. It is characterized by the set of *irreducible forbidden words* $\mathcal{F} = \{00\}$: no consecutive zeros occur. In other words, for the spin-1 particle the spin is never measured twice in a row along the y-axis. This restriction—the dominant structure in the process—is a *short-range correlation* since the measurement outcome at time t only depends on the immediately preceding one at time

[1] These row and column vectors should not be confused with the qubit notation used in quantum computation.

Fig. 1 The Golden Mean quantum generator



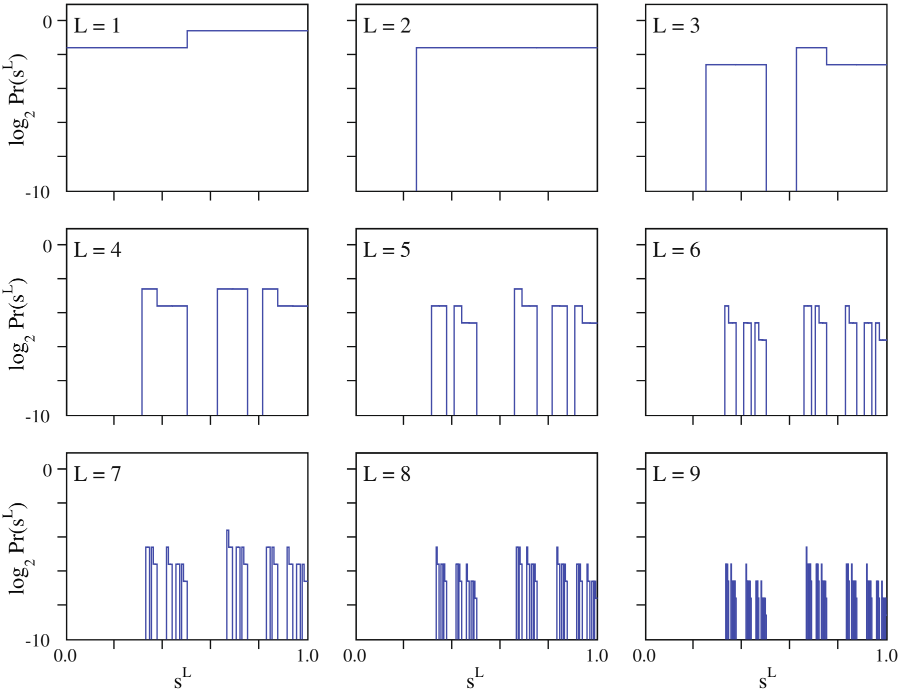


Fig. 2 Golden Mean process language: Word {00} has zero probability; all others have nonzero probability. The logarithm base 2 of the word probabilities is plotted versus the binary string s^L, represented as base-2 real number "0.s^L". To allow word probabilities to be compared at different lengths, the distribution is normalized on [0,1]—that is, the probabilities are calculated as densities

$t-1$. If the outcome is 0, the next outcome will be 1 with certainty. If the outcome is 1, the next measurement is maximally uncertain: outcomes 0 and 1 occur with equal probability.

Consider the same Hamiltonian, but now use instead the observable J_x^2. The corresponding projection operators are:

$$\begin{aligned} & P(0) = |100\rangle\langle 100| \\ \text{and} \quad & P(1) = |010\rangle\langle 010| + |001\rangle\langle 001|. \end{aligned} \tag{8}$$

The QFG defined by U and these projection operators is shown in Fig. 3. States A, B, and C correspond to the eigenstates of the observable J_x^2. Since J_x^2 and J_y^2 commute they have the same eigenstates. The process language generated by this QFG is the so-called *Even Process* language (Hirsch et al. 1970; Kitchens 1998). The word distribution is shown in Fig. 4. It is defined by the infinite set of irreducible forbidden words $\mathcal{F} = \{01^{2k-1}0\}$, $k = 1, 2, 3, \ldots$. That is, if the spin component equals 0 along the x-axis it will be zero an even number of consecutive measurements before being observed to be nonzero. This is a type of *infinite correlation*: For a possibly infinite number of time steps the system tracks the evenness or oddness of the number of consecutive measurements of "spin component equals 0 along the x-axis".

Note that changing the measurement, specifically choosing J_z^2 as the observable, yields a QFG that generates the *Golden Mean* process language again.

The two processes produced by these quantum dynamical systems are well known in the context of symbolic dynamics (Lind and Marcus 1995)—a connection we will return to shortly. Let us first turn to another important property of finite-state machines and explore its role in computational capacity and dynamics.

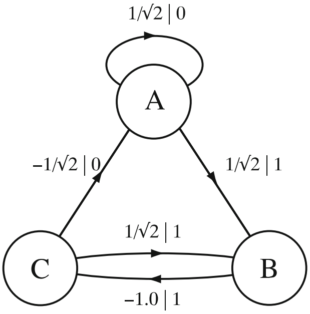
4 Determinism

The label *determinism* is used in a variety of senses, some of which are seemingly contradictory. Here, we adopt the notion, familiar from automata theory (Hopcroft et al. 2001), which differs from that in physics, say, of non-stochasticity. One calls a finite-state machine (classical or quantum) *deterministic* whenever the transition from one state to the next is uniquely determined by the output symbol (or input symbol for recognizers). It is important to realize that a *deterministic* finite-state machine can still behave stochastically—stochasticity here referring to the positive probability of generating symbols. Once the symbol is determined, though, the transition taken by the machine to the next state is unique. Thus, what is called a *stochastic process* in dynamical systems theory can be described by a deterministic finite-state generator without contradiction.

We can easily check the two quantum finite-state machines in Figs. 1 and 3 for determinism by inspecting each state and its outgoing transitions. One quickly sees that both generators are deterministic. In contrast, the third QFG mentioned above, defined by U in Eq. 4 and J_z^2, is nondeterministic.

Determinism is a desirable property for various reasons. One is the simplicity of the mapping between observed symbols and internal states. Once the observer synchronizes to the internal state dynamics, the output symbols map one-to-one *onto* the internal states. In

Fig. 3 The even process quantum generator



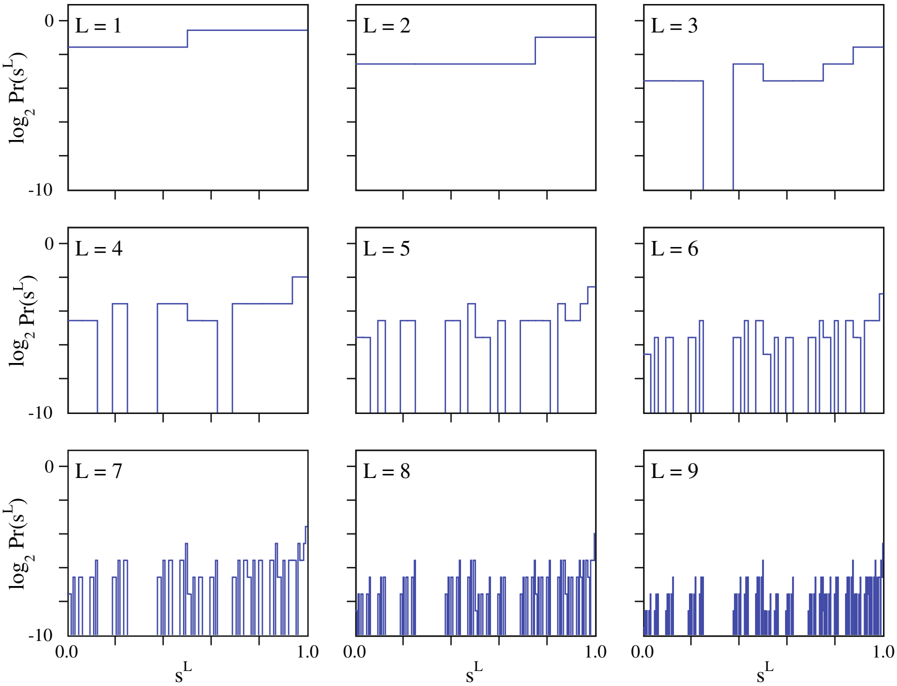


Fig. 4 Even process language: Words $\{01^{2k-1}0\}$, $k = 1,2,3,...$ have zero probability; all others have nonzero probability

general, though, the observed symbol sequences do not track the internal state dynamics (orbit) directly. [This brings one to the topic of hidden Markov models (Bishop 2006).]

For optimal prediction, however, access to the internal state dynamics is key. Thus, when one has a deterministic model, the observed sequences reveal the internal dynamics. Once they are known and one is synchronized, the process becomes optimally predictable. A final, related reason why determinism is desirable is that closed-form expressions can be given for various information processing measures, as we will discuss in Sect. 6.

5 Sofic systems

In symbolic dynamics, sofic systems are used as tractable representations with which to analyze continuous-state dynamical systems, as given in (Kitchens 1998; Lind and Marcus 1995). Let the alphabet $\mathcal{A}$ together with an $n \times n$ adjacency matrix (with entries 0 or 1) define a directed graph $G = (V,E)$ with V the set of vertices and E the set of edges. Let X be the set of all infinite admissible sequences of edges, where *admissible* means that the sequence corresponds to a path through the graph. Let T be the shift operator on this sequence; it plays the role of the time-evolution operator of the dynamical system. A *sofic system* is then defined as the pair (X, T) (Weiss 1973). The Golden Mean and the Even process are standard examples of *sofic systems*. The Even system, in particular, was introduced by Hirsch et al. (1970).

Whenever the rule set for admissible sequences is finite one speaks of a *subshift of finite type*. The Golden Mean process is a subshift of finite type. Words in the language are defined by the finite (single) rule of not containing the subword 00. The Even Process, on the other hand, is not of finite type, since the number of rules is infinite: The forbidden words $\{01^{2k+1}0\}$ cannot be reduced to a finite set. As we noted, the rule set, which determines allowable words, implies the process has a kind of infinite memory. One refers, in this case, to a *strictly sofic* system.

The spin-1 example above appears to be the first time a strictly sofic system has been identified in quantum dynamics. This ties quantum dynamics to languages and quantum automata theory in a way similar to that found in classical dynamical systems theory. In the latter setting, words in the sequences generated by sofic systems correspond to *regular languages*—languages recognized by some finite-state machine. We now have a similar construction for quantum dynamics. For any (finite-dimensional) quantum dynamical system under observation we can construct a QFG, using a unitary operator and a set of projection operators. The language it generates can then be analyzed in terms of the rule set of admissible sequences. One interesting open problem becomes the question whether the words produced by sofic quantum dynamical systems correspond to the regular languages.

6 Information-theoretic analysis

The process languages generated by the spin-1 particle under a particular observation scheme can be analyzed using well known information-theoretic quantities such as *Shannon block entropy* and *entropy rate* (Cover and Thomas 1991) and others introduced in Ref. (Crutchfield and Feldman 2003). Here, we will limit ourselves to the *excess entropy*. The applicability of this analysis to quantum dynamical systems has been shown in Ref. (Crutchfield and Wiesner 2007), where closed-form expressions are given for some of these quantities when the generator is known.

We can use the observed behavior, as reflected in the word distribution, to come to a number of conclusions about how a quantum process generates randomness and stores and transforms historical information. The *Shannon entropy* of length-L sequences is defined

$$H(L) \equiv - \sum_{s^L \in \mathcal{A}^L} \Pr(s^L) \log_2 \Pr(s^L). \tag{9}$$

It measures the average surprise in observing the "event" s^L. The reference (Crutchfield and Feldman 2003) showed that a stochastic process's informational properties can be derived systematically by taking derivatives and then integrals of $H(L)$, as a function of L. For example, the *source entropy rate* h_μ is the rate of increase with respect to L of the Shannon entropy in the large-L limit:

$$h_\mu \equiv \lim_{L \to \infty} [H(L) - H(L-1)], \tag{10}$$

where the units are *bits/measurement* (Cover and Thomas 1991).

The reference (Crutchfield and Wiesner 2007) showed that the entropy rate of a quantum process can be calculated directly from its QFG, when the latter is deterministic. A closed-form expression for the entropy rate in this case is given by:

Table 1 Information storage and generation for example quantum processes: entropy rate h_μ and excess entropy **E**

Quantum dynamical system spin-1 particle		
Observable	J_y^2	J_x^2
h_μ [*bits/measurement*]	0.666	0.666
E [*bits*]	0.252	0.902

$$h_\mu = -|Q|^{-1} \sum_{i=0}^{|Q|-1} \sum_{j=0}^{|Q|-1} |U_{ij}|^2 \log_2 |U_{ij}|^2, \tag{11}$$

The entropy rate h_μ quantifies the irreducible randomness in processes: the randomness that remains after the correlations and structures in longer and longer sequences are taken into account.

The latter, in turn, is measured by a complementary quantity. The amount $I(\overleftarrow{S}; \overrightarrow{S})$ of mutual information (Cover and Thomas 1991) shared between a process's past $\overleftarrow{S}$ and its future $\overrightarrow{S}$ is given by the *excess entropy* **E** (Crutchfield and Feldman 2003). It is the subextensive part of $H(L)$:

$$\mathbf{E} = \lim_{L \to \infty} [H(L) - h_\mu L]. \tag{12}$$

Note that the units here are *bits*.

Ref. (Crutchfield and Feldman 2003) gives a closed-form expression for **E** for *order-R* Markov processes—those in which the measurement symbol probabilities depend only on the previous $R-1$ symbols. In this case, Eq. 12 reduces to:

$$E = H(R) - R \cdot h_\mu, \tag{13}$$

where $H(R)$ is a sum over $|\mathcal{A}|^R$ terms. Given that the quantum generator is deterministic we can simply employ the above formula for h_μ and compute the block entropy at length R to obtain the excess entropy for the order-R quantum process.

Ref. (Crutchfield and Wiesner 2007) computes these entropy measures for various example systems, including the spin-1 particle. The results are summarized in Table 1. The value for the excess entropy of the Golden Mean process obtained by using Eq. 13 agrees with the value obtained from simulation data, shown in Table 1. The entropy, $h_\mu = 2/3$ bits per measurement, is the same for both processes, and thus they have the same amount of irreducible randomness. The excess entropy, though, differs markedly. The Golden Mean process (J_y^2 measured) stores, on average, **E** $\approx$ 0.25 bits at any given time step. The Even Process (J_x^2 measured) stores, on average, **E** $\approx$ 0.90 bits, which reflects its longer memory of previous measurements.

7 Conclusion

We have shown that quantum dynamical systems store information in their dynamics. The information is accessed via measurement. Closer inspection would suggest even that information is *created* through measurement. The key conclusion is that, since both processes are represented by a 3-state QFG constructed from the same internal quantum dynamics, it is the means of observation alone that affects the amount of memory. This was illustrated with the particular examples of the spin-1 particle in a magnetic field.

Depending on the choice of observable the spin-1 particle generates different process languages. We showed that these could be analyzed in terms of the block entropy—a measure of uncertainty, the entropy rate—a measure of irreducible randomness, and the excess entropy—a measure of structure. Knowing the (deterministic) QFG representation, these quantities can be calculated in closed form.

We established a connection between quantum automata theory and quantum dynamics, similar to the way *symbolic dynamics* connects classical dynamics and automata. By considering the output sequence of a repeatedly measured quantum system as a shift system we found quantum processes that are sofic systems. Taking one quantum system and observing it in one way yields a subshift of finite type. Observing it in a different way yields a (strictly sofic) subshift of infinite type. Consequently, not only the amount of memory but also the soficity of a quantum process depend on the means of observation.

This can be compared to the fact that, classically the Golden Mean and the Even sofic systems can be transformed into each other by a two-block map. The adjacency matrix of the graphs is the same. A similar situation arises here. The unitary matrix, which is the corresponding adjacency matrix of the quantum graph, is the same for both processes.

The preceding attempted to forge a link between quantum dynamical systems and quantum computation by extending concepts from symbolic dynamics to the quantum setting. We believe the results suggest further study of the properties of quantum finite-state generators and the processes they generate is necessary and will shed light on a number of questions in quantum information processing. One open technical question is whether sofic quantum systems are the closure of quantum subshifts of finite-type, as they are for classical systems (Weiss 1973).

References

Aharonov D, Kitaev A, Nisan N (1998) Quantum circuits with mixed states. In: 30th annual ACM symposium on the theory of computing, Dallas, Texas, pp 20–30

Albert DZ (1983) On quantum-mechanical automata. Phys Lett 98A:249–251

Alicki R, Fannes M (2001) Quantum dynamical systems. Oxford University Press, Oxford

Ambainis A, Watrous J (2002) Two-way finite automata with quantum and classical states. Theor Comput Sci 287:299–311

Beck C, Graudenz D (1992) Symbolic dynamics of successive quantum-mechanical measurements. Phys Rev A 46(10):6265–6276 doi:10.1103/PhysRevA.46.6265

Bishop CM (2006) Pattern recognition and machine learning. Springer Verlag, Singapore

Cover T, Thomas J (1991) Elements of information theory. Wiley-Interscience, New York

Crutchfield JP, Feldman DP (2003) Regularities unseen, randomness observed: Levels of entropy convergence. Chaos 13:25–54

Crutchfield JP, Wiesner K (2007) Intrinsic quantum computation. e-print arxiv/quant-ph/0611202 (in press)

Freivalds R, Winter A (2001) Quantum finite state transducers. Lect Notes Comput Sci 2234:233–242

Gutzwiller MC (1990) Chaos in classical and quantum mechanics. Springer Verlag, New York

Habib S, Jacobs K, Shizume K (2006) Emergence of chaos in quantum systems far from the classical limit. Phys Rev Lett 96:010403–06

Hedlund GH, Morse M (1938) Symbolic dynamics I. Amer J Math 60:815–866

Hedlund GH, Morse M (1940) Symbolic dynamics II. Amer J Math 62:1–42

Hirsch M, Palis J, Pugh C, Shu M (1970) Neighborhoods of hyperbolic sets. Invent Math 9:121–134

Hopcroft JE, Motwani R, Ullman JD (2001) Introduction to automata theory, languages, and computation. Addison-Wesley, Reading, MA

Kitchens BP (1998) Symbolic dynamics: one-sides, two-sided, and countable state Markov shifts. Springer Verlag, Berlin, Heidelberg

Kondacs A, Watrous J (1997) On the power of quantum finite state automata. In: 38th IEEE conference on foundations of computer science, pp 66–75

Lind D, Marcus B (1995) An introduction to symbolic dynamics and coding. Cambridge University Press, Cambridge

Moore C, Crutchfield JP (2000) Quantum automata and quantum grammars. Theor Comput Sci 237:275–306

Peres A (1984) On quantum-mechanical automata. Phys Lett 101A:249–250

Peres A (1993). Quantum theory: concepts and methods. Kluwer Academic Publishers, Dordrecht

Shannon CE, Weaver W (1962) The mathematical theory of communication. University of Illinois Press, Champaign-Urbana

Weiss (1973) Subshifts of finite type and sofic systems. Monatshefte für Mathematik 77:462–474

Wiesner K, Crutchfield JP (2008) Computation in finitary stochastic and quantum processes. Physica D 237:1173–1195