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Quantum value indefiniteness

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Abstract The indeterministic outcome of a measurement of an individual quantum is certified by the impossibility of the simultaneous, unique, definite, deterministic pre-existence of all conceivable observables from physical conditions of that quantum alone.

Keywords Quantum value indefiniteness · Quantum contextuality · Quantum oracle · Quantum randomnumber generator

1 Introduction

One of the most astounding consequences of the assumption of the validity of the quantum formalism in terms of Hilbert spaces (von Neumann 1932) is the apparent impossibility of its classical interpretation. More precisely, a classical interpretation of a quantum logical structure (Birkhoff and von Neumann 1936) is either identified with a Boolean algebra, or at least with a homomorphic embedding (structurally preserving all quantum logical relations and operations) into some Boolean algebra (Calude et al. 1999). Quantum logics are obtained by identifying (unit) vectors (associated with the one-dimensional subspaces corresponding to the linear spans of the vectors, and with the corresponding one dimensional projectors) with elementary yes–no propositions. The logical *and, or*, and *not* operations are identified with the set theoretic intersection, with the linear span of two subspaces, and with forming the orthogonal subspace, respectively. Suppose further that orthogonality among subspaces indicates mutual exclusive propositions or experimental outcomes.

Then, in at least three-dimensional Hilbert (sub)spaces, there does not exist a (classical) truth assignment on (finite sets of) elementary yes–no propositions which would

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(Rule 1—"countable additivity:") ascribe truth to exactly one observable outcome among each set of maximal commeasurable mutually exclusive outcomes, and falsity to the others, such that

(Rule 2—"noncontextuality:") for "overlapping" link observables belonging to more than one commeasurable set of observables, henceforth called *context*, the truth value remains the same, independent of the particular commeasured observables (Specker 1960; Kochen and Specker 1967; Zierler and Schlessinger 1965; Alda 1980, 1981; Kamber 1964, 1965; Peres 1991; Mermin 1993; Svozil and Tkadlec 1996; Cabello et al. 1996; Cabello 2008).

Proofs (e.g., Kochen and Specker 1967) could be finitistic and by contradiction (i.e., *via reductio ad absurdum*), so there should not be any metamathematical issues about their applicability in physics. Countable additivity (Rule 1) is the basis of a theorem (Gleason 1957; Pitowsky 1998; Richman and Bridges 1999; Dvurečenskij 1993) by Gleason which derives the Born rule $\langle \mathbf{A} \rangle = \mathtt{Tr}(\rho \mathbf{A})$, where $\langle \mathbf{A} \rangle$ and ρ stand for the expectation value of an observable $\mathbf{A}$ and for the quantum state, respectively.

Yet, there are metaphysical issues related to the impossibility of a classical interpretation of the quantum formalism; in particular the explicit and indispensible use of *counterfactuals* in the argument (Svozil 2009c). Remarkably, this has been already emphasized in the first announcement of the formal result (Specker 1960). Counterfactuals are "observables" which *could* have been measured *if* the experimenter *would have* chosen a different, i.e., complementary, measurement setup, but *actually chose another* (complementary) one. Hence, from the point of view of the quantum formalism, any proof of the impossibility of a classical interpretation of quantum mechanics uses complementary observables, which cannot possibly be simultaneously measured. Pointedly stated, from a strictly operational point of view, due to quantum complementarity (Pauli 1958, p. 7), the entities occurring in the proofs cannot physically coexist.

So, it may not be totally unjustified to ask why one should bother about nonoperational quantities and their consequences at all? There may be two affirmative apologies for the use of counterfactuals: First, although these observables could not be measured simultaneously, they are perfectly reasonable physical observables if the experimenter chooses to measure them. Secondly, through a measurement setup involving two correlated particles, two complementary observables can be measured counterfactually (Einstein et al. 1935) on two space-like separated (Weihs et al. 1998) but entangled (Schrödinger 1935a, b, 1936) particles. Because of constraints on the uniqueness of the arguments, this "indirect measurement" cannot be extended to more than two counterfactual observables (Svozil 2006).

Quantum "value (in)definiteness," sometimes also termed "counterfactual (in)definiteness" (Murata 1990), refers to the (im)possibility of the simultaneous existence of definite outcomes of conceivable measurements under certain assumptions [e.g. noncontextuality; see Rule 2 above]—that is, unperformed measurements can(not) have definite results (Peres 1978). "(In)determinacy" often (but not always) refers to the absence (presence) of causal laws—in the sense of the principle of sufficient reason stating that every phenomenon has its explanation and cause—governing a physical behavior. Thus "value (in)definiteness" relates to a static property, whereas "(in)determinacy" is often used for temporal evolutions. Sometimes, quantum value indefiniteness is considered as one of the expressions of quantum indeterminacy; another expression of quantum indeterminacy is, for instance, associated with the (radioactive) decay of some excited states (Kragh 1997, 2009).

In what follows we shall review some explicit physical consequences of the impossibility to interpret the quantum formalism classically. We shall also review consequences for the construction of quantum mechanical devices capable of generating particular indeterministic outcomes (Svozil 1990, 2009d; Rarity et al. 1994; Jennewein et al. 2000; Stefanov et al. 2000; Hai-Qiang et al. 2004; Wang et al. 2006; Fiorentino et al. 2007; Kwon et al. 2009; Pironio et al. 2010), which have been already discussed in an article by Calude and Svozil (2008).

Any particular maximal set of (mutually exclusive) observables will be called *context* (Svozil 2009a). It constitutes a "maximal collection of co-measurable observables," or, stated differently, a "classical mini-universe" located within the continuity of complementary quantum propositions. The spectral theorem suggests that a context can be formalized by a single "maximal" self-adjoint operator, such that there exist "maximal" sets of mutually compatible, co-measurable, mutually exclusive orthogonal projectors which appear in its spectral decomposition (e.g., von Neumann 1932, Sect. II.10, p. 90, English translation p. 173; Kochen and Specker 1967, Sect. 2; Neumark 1954, pp. 227–228; Halmos 1974, Sect. 84).

2 Contextual interpretation

In a "desperate" attempt to save realism (Stace 1949), Bell (Bohr 1949; Bell 1966; Heywood and Redhead 1983; Redhead 1990) proposed to abandon the noncontextuality assumption Rule 2 that the truth or falsity of an individual outcome of a measurement of some observable is independent of what other (mutually exclusive) observables are measured "alongside" of it. In Bell's own words (Bell 1966, Sect. 5), the "danger" in the implicit assumption is this[1]:

> It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously. … The result of an observation may reasonably depend not only on the state of the system … but also on the complete disposition of the apparatus.

This "contextual interpretation" of quantum mechanics will be henceforth called *contextuality*.

Notice that contextuality does not suggest that any *statistical* property is context dependent; this would be ruled out by the Born rule, which is context independent. Instead, the contextual interpretation claims that the *individual outcome*—Bell's "result of an observation"—depends on the context. This is somewhat similar to the parameter independence but outcome dependence of correlated quantum events (Shimony 1984).

The exact formalization or causes of this type of "contextual outcome dependence" remains an open question. Individual quantum events are generally *conventionalized* to happen acausally and indeterministically (Born 1926a, b); according to the prevalent quantum canon (Zeilinger 2005), "… for the individual event in quantum physics, not only do we not know the cause, there is no cause." In this belief system, indeterminism can be trivially certified by the convention of the "random outcome" of individual quantum events, a view which is further "backed" by our inability to "come up" with a causal

[1] Bell cites Bohr's remark (Bohr 1949) about "the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

model, and by the statistical analysis (Calude et al. 2010) of the assumption of stochasticity and randomness of strings generated *via* the context mismatch between preparation and measurement. Nevertheless, one should always keep in mind that this kind of indeterminism may be epistemic and not ontic. Furthermore, due to the ambiguities of a formal definition, and by reduction to the halting problem (Rogers, Jr. 1967; Davis 1965; Barwise 1978; Enderton 2001; Odifreddi 1989; Boolos et al. 2007), the incomputablity, and even more so randomness, of arbitrary (finite) sequences remains provably unprovable (Calude 2002).

2.1 Violation of probabilistic bounds

For the sake of getting a more intuitive understanding of quantum contextuality, a few examples of its consequences will be discussed next. As any violations of Boole–Bell type elements of physical reality indicate the impossibility of its classical interpretation by probabilistic constraints (Pitowsky 1986, 1989a, b, 1994), every violation of Boole–Bell type inequalities can be re-interpreted as (experimental) "proof of contextuality" (Hasegawa et al. 2006; Bartosik et al. 2009; Amselem et al. 2009; Kirchmair et al. 2009). Indeed, as expressed by Cabello (2008), "Because of the lack of spacelike separation between one observer's choice and the other observer's outcome, the immense majority of the experimental violations of Bell inequalities does not prove quantum nonlocality, but just quantum contextuality." Alas, while certainly most (with the exception of, e.g., Weihs et al. 1998)) experimental violations of Bell inequalities do not prove quantum nonlocality, these statistical violations are no direct proof of contextuality in general. Nevertheless, they may indicate counterfactual indefiniteness (Murata 1990).

Note that in a geometric framework (Froissart 1981; Cirel'son (=Tsirel'son) 1980, 1993; Pitowsky 1986, 1989a, b, 1994; Pitowsky and Svozil 2001), Boole–Bell type inequalities are just the *facet inequalities* of a classical probability (correlation) polytope obtained by (i) forming all probabilities and joint probabilities of independent events, (ii) taking all two-valued measures (interpretable as truth assignments) associated with this structure, (iii) for each of the probabilities and joint probabilities forming a vector whose components are the (encoded truth) values (either "0" or "1") of the two-valued measures (hence, the dimensionality of the problem is equal to the number of entries corresponding to probabilities and joint probabilities); every such vector is a vertex of the *correlation polytope*, (iv) applying the Minkoswki–Weyl representation theorem (e.g., Ziegler 1994, p. 29), stating that every convex polytope has a dual (equivalent) description as the intersection of a finite number of half-spaces. Such facets are given by linear inequalities, which are obtained from the set of vertices by solving the (computationally hard (Pitowsky 1990)) *hull problem*. The inequalities coincide with Boole's *"conditions of possible experience,"* and with Bell type inequalities.

Any "proof" of contextuality based on Boole–Bell type inequalities necessarily involves the *statistical* behavior of many counterfactual quantities contained in Boole–Bell type inequalities. These quantities cannot be obtained simultaneously, but merely one after another in different experimental configuration runs involving "lots of particles." Due to the statistical nature of the argument and its implicit improvable assumption that contextuality—that is, the abandonment of Rule 2—is the only possible cause for the violations of the classical probabilistic bounds, these "proofs" lack the *sufficiency* of the formal argument.

Table 1 Hypothetical counterfactual contextual outcomes of an experiment capable of violating the Boole–Bell type inequalities involving binary outcomes (denoted by "$-$, $+$") of two observables (subscripts "1, 2") on two particles (denoted by "$\mathbf{A}, \mathbf{B}$")

$\mathbf{A}_1\{\mathbf{B}_1\}$	$+$	·	·	·	·	·	·	·	·	·	$+$	·	...
$\mathbf{A}_1\{\mathbf{B}_2\}$	$-$	·	·	·	·	·	·	·	·	·	$-$	·	...
$\mathbf{A}_2\{\mathbf{B}_1\}$	·	$-$	·	·	·	·	$+$	·	·	·	$-$	·	...
$\mathbf{A}_2\{\mathbf{B}_2\}$	·	$+$	·	·	·	·	$-$	·	·	·	$+$	·	...
$\mathbf{B}_1\{\mathbf{A}_1\}$	·	·	·	·	$-$	·	·	·	·	·	$-$	$+$	...
$\mathbf{B}_1\{\mathbf{A}_2\}$	·	·	·	·	$+$	·	·	·	·	·	$+$	$-$	...
$\mathbf{B}_2\{\mathbf{A}_1\}$	·	·	·	·	$+$	·	·	$-$	·	·	$-$	·	...
$\mathbf{B}_2\{\mathbf{A}_2\}$	·	·	·	·	$-$	·	·	$+$	·	·	$+$	·	...

The expression "$\mathbf{X}\{\mathbf{Y}\}$" stands for "observable $\mathbf{X}$ measured alongside observable $\mathbf{Y}$." Time progresses from left to right; rows contain the individual conceivable, potential measurement values of the eight observables $\mathbf{A}_1\{\mathbf{B}_1\}, \mathbf{A}_1\{\mathbf{B}_2\}, \mathbf{A}_2\{\mathbf{B}_1\}, \mathbf{A}_2\{\mathbf{B}_2\}, \mathbf{B}_1\{\mathbf{A}_1\}, \mathbf{B}_1\{\mathbf{A}_2\}, \mathbf{B}_2\{\mathbf{A}_1\}$, and $\mathbf{B}_2\{\mathbf{A}_2\}$ which "simultaneously co-exist." *Dots* indicate any value in $\{-, +\}$

2.2 Tables of counterfactual "outcomes"

Previously, tables of hypothetical and counterfactual experimental outcomes have been used to argue against the noncontextual classical interpretation of the quantum probabilities (Peres 1978; Murata, 1990; Krenn and Zeilinger 1996). In what follows tables of contextual outcomes violating Rule 2 will be enumerated which could be compatible with quantum probabilities. These tables may serve as a demonstration of the kind of behavior which is required by (hypothetical and counterfactual) individual events capable of rendering the desired violations of Boole–Bell type violations of bounds on classical probabilities.

Let "$\mathbf{X}\{\mathbf{Y}\}$" stand for "observable $\mathbf{X}$ measured alongside observable (or context) $\mathbf{Y}$." Consider the hypothetical counterfactual outcomes enumerated in Table 1 for simultaneous quantum observables associated with the Clauser–Horne–Shimony–Holt inequality

$$|\mathbf{A}_1\{\mathbf{B}_1\}\mathbf{B}_1\{\mathbf{A}_1\} + \mathbf{A}_1\{\mathbf{B}_2\}\mathbf{B}_2\{\mathbf{A}_1\} + \mathbf{A}_2\{\mathbf{B}_1\}\mathbf{B}_1\{\mathbf{A}_2\} - \mathbf{A}_2\{\mathbf{B}_2\}\mathbf{B}_2\{\mathbf{A}_2\}| \leq 2. \qquad (1)$$

They are contextual, as for some cases $\mathbf{X}\{\mathbf{Y}_1\} \neq \mathbf{X}\{\mathbf{Y}_2\}$, as indicated in the enumeration. (Note that noncontextuality would imply the independence of $\mathbf{X}$ from $\mathbf{Y}$; i.e., $\mathbf{X}\{\mathbf{Y}_1\} = \mathbf{X}\{\mathbf{Y}_2\} = \mathbf{X}$.)

The difference between "truth tables" associated with configurations for the statistical arguments against value indefiniteness involving Boole–Bell type inequalities on the one hand, and for direct proofs (e.g. by the Kochen–Specker theorem) on the other hand, is that the former tables need not always contain contextual assignments—although it can be expected that the violations of noncontextuality should increase with increasing deviations from the classical Boole–Bell bounds on joint probabilities[2]—whereas the latter tables require some violation(s) of noncontextuality at every single column. For example, in the compact 18-vector configuration allowing a Kochen–Specker proof introduced in Cabello et al. (1996) and Cabello (2000) and depicted in Fig. 1, one is forced to violate the

[2] Note that for stronger-than-quantum correlations (Popescu and Rohrlich 1994; Krenn and Svozil 1998) rendering a maximal violation of the Clauser–Horne–Shimony–Holt inequality by $\mathbf{A}_1\{\mathbf{B}_1\}\mathbf{B}_1\{\mathbf{A}_1\} + \mathbf{A}_1\{\mathbf{B}_2\}\mathbf{B}_2\{\mathbf{A}_1\} + \mathbf{A}_2\{\mathbf{B}_1\}\mathbf{B}_1\{\mathbf{A}_2\} - \mathbf{A}_2\{\mathbf{B}_2\}\mathbf{B}_2\{\mathbf{A}_2\} = \pm 4$, if $\mathbf{A}_1\{\mathbf{B}_2\} = \mathbf{A}_2\{\mathbf{B}_2\}$, then $\mathbf{B}_2\{\mathbf{A}_1\} = -\mathbf{B}_2\{\mathbf{A}_2\}$, and if $\mathbf{B}_1\{\mathbf{A}_2\} = \mathbf{B}_2\{\mathbf{A}_2\}$, then $\mathbf{A}_2\{\mathbf{B}_1\} = -\mathbf{A}_2\{\mathbf{B}_2\}$.

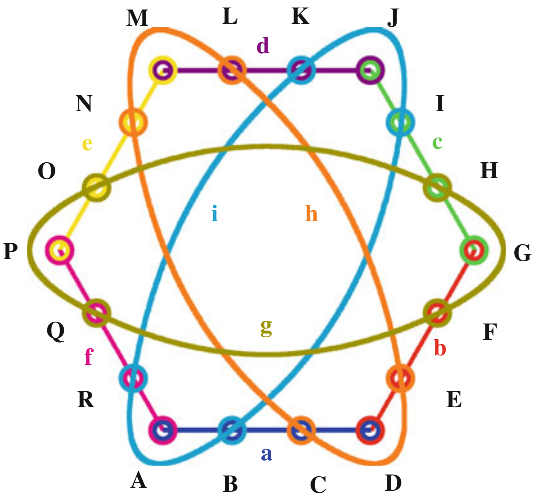

Fig. 1 (Color online) Greechie diagram of a finite subset of the continuum of blocks or contexts embeddable in four-dimensional real Hilbert space without a two-valued probability measure (Cabello et al. 1996; Cabello 2000). The proof of the Kochen–Specker theorem uses nine tightly interconnected contexts $\mathbf{a} = \{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$, $\mathbf{b} = \{\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G}\}$, $\mathbf{c} = \{\mathbf{G},\mathbf{H},\mathbf{I},\mathbf{J}\}$, $\mathbf{d} = \{\mathbf{J},\mathbf{K},\mathbf{L},\mathbf{M}\}$, $\mathbf{e} = \{\mathbf{M},\mathbf{N},\mathbf{O},\mathbf{P}\}$, $\mathbf{f} = \{\mathbf{P},\mathbf{Q},\mathbf{R},\mathbf{A}\}$, $\mathbf{g} = \{\mathbf{B},\mathbf{I},\mathbf{K},\mathbf{R}\}$, $\mathbf{h} = \{\mathbf{C},\mathbf{E},\mathbf{L},\mathbf{N}\}$, $\mathbf{i} = \{\mathbf{F},\mathbf{H},\mathbf{O},\mathbf{Q}\}$ consisting of the 18 projectors associated with the one dimensional subspaces spanned by $\mathbf{A} = (0,0,1,-1), \mathbf{B} = (1,-1,0,0), \mathbf{C} = (1,1,-1,-1), \mathbf{D} = (1,1,1,1), \mathbf{E} = (1,-1,1,-1), \mathbf{F} = (1,0,-1,0), \mathbf{G} = (0,1,0,-1)$, $\mathbf{H} = (1,0,1,0), \mathbf{I} = (1,1,-1,1), \mathbf{J} = (-1,1,1,1), \mathbf{K} = (1,1,1,-1), \mathbf{L} = (1,0,0,1)$, $\mathbf{M} = (0,1,-1,0)$, $\mathbf{N} = (0,1,1,0), \mathbf{O} = (0,0,0,1)$, $\mathbf{P} = (1,0,0,0)$, $\mathbf{Q} = (0,1,0,0)$, $\mathbf{R} = (0,0,1,1)$. Greechie diagram representing atoms by points, and contexts by maximal smooth, unbroken curves. Every observable proposition occurs in exactly two contexts. Thus, in an enumeration of the four observable propositions of each of the nine contexts, there appears to be an *even* number of true propositions. Yet, as there is an odd number of contexts, there should be an *odd* number (actually nine) of true propositions

noncontextuality assumption Rule 2 for at least one link observable. This can be readily demonstrated by considering all 36 entries per column in Table 2, Whether one violation of the noncontextuality Rule 2 is enough for consistency (i.e., the necessary extent of the violation of contextuality) with the quantum probabilities remains unknown.

Whether if such signatures of contextuality exist cannot be decided experimentally, as direct observations are operationally blocked by quantum complementarity. Thus this type of contextuality remains metaphysical.

2.3 Indirect simultaneous tests

There exist "explosion views" of counterfactual configurations involving singlet or other correlated states of two three- and more state particles which, due to the counterfactual uniqueness properties (Svozil 2006), are capable of indirectly testing the quantum contextuality assumption (Svozil 2009b) by a simultaneous measurement of two complementary contexts (Einstein et al. 1935). For the sake of explicit demonstration, consider Fig. 2 depicting three orthogonality (Greechie) diagrams of such configurations of observables. Every diagram is representable in three- or four-dimensional vector space.

Table 2 (Color online) Hypothetical counterfactual contextual outcomes of experiments associated with a compact proof of the Kochen–Specker theorem (Cabello et al. 1996; Cabello 2000) involving binary outcomes "0" or 1" of 18 observables, adding up to one within each of the nine contexts denoted by "**a**, ..., **i**"

A{**a**}	1	0	0	0	0	1	0	1	0	0	0	1	1	...
A{**f**}	0	1	0	0	0	1	0	1	0	0	0	1	1	...
B{**a**}	0	0	1	0	0	0	1	0	1	0	0	0	0	...
B{**i**}	·	0	·	0	0	·	·	0	·	·	0	0	·	...
C{**a**}	0	1	0	0	1	0	0	0	0	0	1	0	0	...
C{**h**}	·	·	·	·	·	·	·	·	·	·	·	·	·	...
D{**a**}	0	0	0	1	0	0	0	0	0	1	0	0	0	...
D{**b**}	·	·	·	·	·	·	·	·	·	·	·	·	·	...
...	·	·	·	·	·	·	·	·	·	·	·	·	·	...
P{**e**}	·	·	·	·	·	·	·	·	·	·	·	·	·	...
P{**f**}	0	0	0	1	0	0	1	0	0	1	0	0	0	...
Q{**g**}	·	·	·	·	·	·	·	·	·	·	·	·	·	...
Q{**f**}	1	0	0	0	0	0	0	0	1	0	0	0	0	...
R{**i**}	0	·	0	·	·	0	0	·	0	0	·	·	0	...
R{**f**}	0	0	1	0	1	0	0	0	0	0	1	0	0	...

The expression "**X**{**y**}" stands for "observable **X** measured alongside the context **y**." Time progresses from left to right; rows contain the individual conceivable, potential measurement values of the observables **A**{**a**}, ..., **R**{**i**} which "simultaneously co-exist." *Dots* indicate any value in {0,1} subject to at least one violation of the noncontextuality assumption, that is, **X**(**y**) $\neq$ **X**(**y**') for **y** $\neq$ **y**'

For the configuration depicted in Fig. 2a, contextuality predicts that there exist experimental outcomes with $\mathbf{A}\{\mathbf{B},\mathbf{C}\} \neq \mathbf{A}\{\mathbf{D},\mathbf{E}\}$. As detailed quantum mechanical calculations (Svozil 2009b) show, this is not predicted by quantum mechanics.

For the configuration depicted in Fig. 2b, contextuality predicts that there exist experimental outcomes with $\mathbf{A}\{\mathbf{B},\mathbf{C},\mathbf{D}\} \neq \mathbf{A}\{\mathbf{G},\mathbf{H},\mathbf{I}\}$, as well as $\mathbf{A}\{\mathbf{G},\mathbf{H},\mathbf{I}\} = \mathbf{D}\{\mathbf{E},\mathbf{F},\mathbf{G}\} = 1$, and their cyclic permutations.

For the configuration depicted in Fig. 2c, contextuality predicts that there exist experimental outcomes with $\mathbf{A}\{\mathbf{B},\mathbf{C},\mathbf{D}\} \neq \mathbf{A}\{\mathbf{B},\mathbf{E},\mathbf{F}\}$, as well as $\mathbf{B}\{\mathbf{A},\mathbf{C},\mathbf{D}\} \neq \mathbf{B}\{\mathbf{A},\mathbf{E},\mathbf{F}\}$. Again, this is not predicted quantum mechanically (Svozil 2009b).

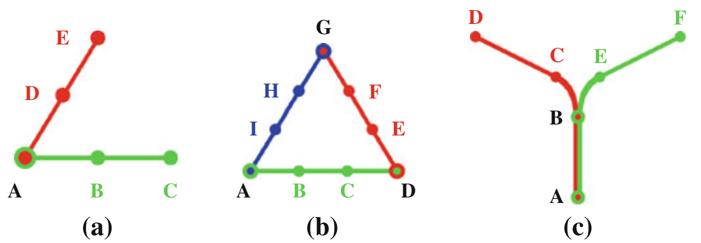

Fig. 2 (Color online) Diagrammatical representation of interlinked contexts by Greechie (orthogonality) diagrams (points stand for individual basis vectors, and entire contexts are drawn as smooth curves): **a** two tripods with a common leg; **b** three interconnected fourpods (this configuration with tripods would be irrepresentable in three-dimensional vector space (Kalmbach 1983; Pták and Pulmannová 1991)); **c** two contexts in four dimensions interconnected by two link observables

Experiment will clarify and decide the contradiction between the predictions by the contextuality assumption and quantum mechanics, but it is not too unreasonable to suspect that the quantum predictions will prevail. As a consequence, and subject to experimental falsification, any *ad hoc* "ontic" contextuality assumption might turn out to be physically unfounded.

One may argue that quantum contextuality only "appears" if measurement configurations are encountered which do not allow a set of two-valued states. The same might be said for measurement configurations allowing only a "meager" set of two-valued states which cannot be used for the construction of any homomorphic (i.e. preseving relations and operations among quantum propositions) embedding into a classical (Boolean) algebra. Alas, configurations of observables such as the one depicted in Fig. 2a are just sub-configurations of proofs of the Kochen–Specker theorem (Kochen and Specker 1967), in particular their Γ_2 and Γ_3; so it would be difficult to imagine why Fig. 2a feature context independence because of the experimenter takes into account only *two* contexts, whereas context dependence is encountered when the experimenter has in mind, say, the *entire* structure of all the 117 Kochen–Specker contexts contained in Γ_3.

3 Context translation principle

In view of the inapplicability of the quantum contextuality assumption and the fact that, although quantized systems can only be prepared in a certain single context[3] quantized systems yield measurement results when measured "along" different, nonmatching context, one may speculate that the measurement apparatus must be capable of "translating" between the preparation context and the measurement context (Svozil 2004). Variation of the capabilities of the measurement apparatus to translate nonmatching quantum contexts with its physical condition yields possibilities to detect this mechanism.

In this scenario, stochasticity is introduced *via* the context translation process; albeit not necessarily an irreversible, irreducible one, as the unitary quantum state evolution (in-between measurements) is deterministic, reversible and one-to-one (Everett 1957). Nevertheless, one may further speculate that, at least for finite experimental time series and for finite algorithmic tests, any such quasi-deterministic form of stochasticity will result in very similar statistical behaviors as is predicted for acausality.

Context translation might present an "epistemic" contextuality, since the "complete disposition of the measurement apparatus" (see Bell 1966, Sect. 5) may enter in the translation function τ formalizing the "state reduction"

$$\rho \longrightarrow \tau_{\mathtt{D}(\mathbf{X},\mathbf{Y})}(\rho) \in S_{\mathbf{X}}, \tag{2}$$

where ρ stands for the quantum state, $S_{\mathbf{X}}$ for the spectrum of the operator $\mathbf{X}, \mathtt{D}$ for the "disposition of the apparatus," $\mathbf{X}$ for the observable and $\mathbf{Y}$ for the context.

In general, even in the absence of some concrete "translation mechanism," τ is subject to some probabilistic constraints, such as Malus' law (Brukner and Zeilinger 1999). In order to be able to account for the nonlocal quantum correlation functions even at space-

[3] We would even go so far to speculate that the ignorance of state preparation resulting in mixed states is an epistemic, not ontologic, one. Thus all quantum states are "ontologically" pure.

like separations (Weihs et al. 1998) τ should also be nonlocal. Ideally, if preparation and measurement context match, and if ρ is in some eigenstate $\mathbf{E}_i$ of $\mathbf{X}$ with an associated eigenvalue x_i, then Eq. (2) reduces to its context and apparatus independent form $\mathbf{E}_i \longrightarrow \tau_{\mathrm{D}(\mathbf{X},\mathbf{Y})}(\mathbf{E}_i) = \tau_{\mathbf{X}}(\mathbf{E}_i) = x_i$ for all $\mathbf{E}_i$ in the spectral sum $\mathbf{X} = \sum_i x_i \mathbf{E}_i$. This reduction postulate appears to be the reason for an absence of contextuality in the "explosion view" type configurations discussed above.

For all the other cases, the measurement apparatus will introduce a stochastic element which, in this scenario, is the reason for the quantum indeterminism of individual events. Of course, the *degree of stochasticity* will depend on the context mismatch, and on the "disposition of the apparatus." But again, as for the *ad hoc* "ontic" type of contextuality discussed above, in no way can the measurement outcome of an individual particle be completely determined by a pre-existing element of physical reality (Einstein et al. 1935) of that particle alone. In this sense, as only observables associated with one context have a definite value and all other observables have none, one is lead to a quasi-classical "effective value indefiniteness," giving rise to a natural classical theory not requiring value definiteness.

4 Summary

We have discussed the "current state of affairs" with regard to the interpretation of quantum value indefiniteness, and the limited operationalizability of its interpretation in terms of *context dependence (contextuality)* of observables. Of course, due to complementarity, quantum counterfactuals are not directly simultaneously measurable; and thus—despite the prevalence of counterfactuals in quantum information, communication and computation theory—anyone considering their physical existence is, to paraphrase von Neumann's words (von Neumann 1951), at least empirically, "in a state of sin."

In any case, the absence of classical interpretations of the quantum formalism, and in particular the strongest expression of it—the absence of any global truth function for quantum systems of three or more mutually exclusive outcomes—presents the possibility to render a quantum random number generator by preparing a quantum state in a particular context and measuring it in another. As has been pointed out already by Calude and Svozil (2008), the resulting measurement outcomes are "quantum certified" (i.e., true with respect to the validity of quantum mechanics) and do not correspond to any pre-existing physical observable of the "isolated" individual system before the measurement process. Exactly how this kind of quantum oracle for randomness operates remains open. One may hold that, somehow, due to the lack of determinacy, this type of randomness emerges "out of nowhere" and essentially is irreducible (Zeilinger 1999, 2005). One may also put forward the idea that, at least when complementarity is involved, quantum randomness is rendered by a quasi-classical context translation which maps an incompatible preparation context into some outcome, thereby introducing stochasticity. In any case, for all practical purposes, the resulting oracles for randomness, when subjected to tests (Calude et al. 2010), might be "hardly differentiable" from each other even asymptotically.

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