homework2

February 5, 2019

1 Homework 2 - Berkeley STAT 157

Handout 1/29/2019, due 2/5/2019 by 4pm in Git by committing to your repository.

```
In [17]: from mxnet import nd, autograd, gluon
```

2 1. Multinomial Sampling

Implement a sampler from a discrete distribution from scratch, mimicking the function `mxnet.ndarray.random.multinomial`. Its arguments should be a vector of probabilities p. You can assume that the probabilities are normalized, i.e. tha they sum up to 1. Make the call signature as follows:

```
samples = sampler(probs, shape)

probs   : An ndarray vector of size n of nonnegative numbers summing up to 1
shape   : A list of dimensions for the output
samples : Samples from probs with shape matching shape
```

Hints:

1. Use `mxnet.ndarray.random.uniform` to get a sample from $U[0, 1]$.
2. You can simplify things for `probs` by computing the cumulative sum over `probs`.

```
In [18]: def sampler(probs, shape):
             ## Add your codes here
             ret=nd.zeros(shape=shape)
             m, n=shape
             num=1000 # time of sampling from U[0, 1]
             x=nd.zeros(shape=n)
             y=nd.random.uniform(shape=shape)
             for i in range(m):
                 for j in range(n):
                     x1=y[i]<probs[:j+1].sum()
                     x2=y[i]>probs[:j].sum()
                     for k in range(n):
                         if x1[k]==1 and x2[k]==1:
```

```
                    ret[i, k]=j
    return ret

# a simple test
sampler(nd.array([0.2, 0.3, 0.5]), (1000,3))
```

Out[18]:

```
[[2. 2. 2.]
 [2. 2. 2.]
 [2. 2. 1.]
 ...
 [0. 1. 1.]
 [2. 2. 1.]
 [2. 1. 0.]]
<NDArray 1000x3 @cpu(0)>
```

3 2. Central Limit Theorem

Let's explore the Central Limit Theorem when applied to text processing.

- Download https://www.gutenberg.org/ebooks/84 from Project Gutenberg
- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

$$n_{\text{the}}[i] = \sum_{j=1}^{i} \{w_j = \text{the}\}$$

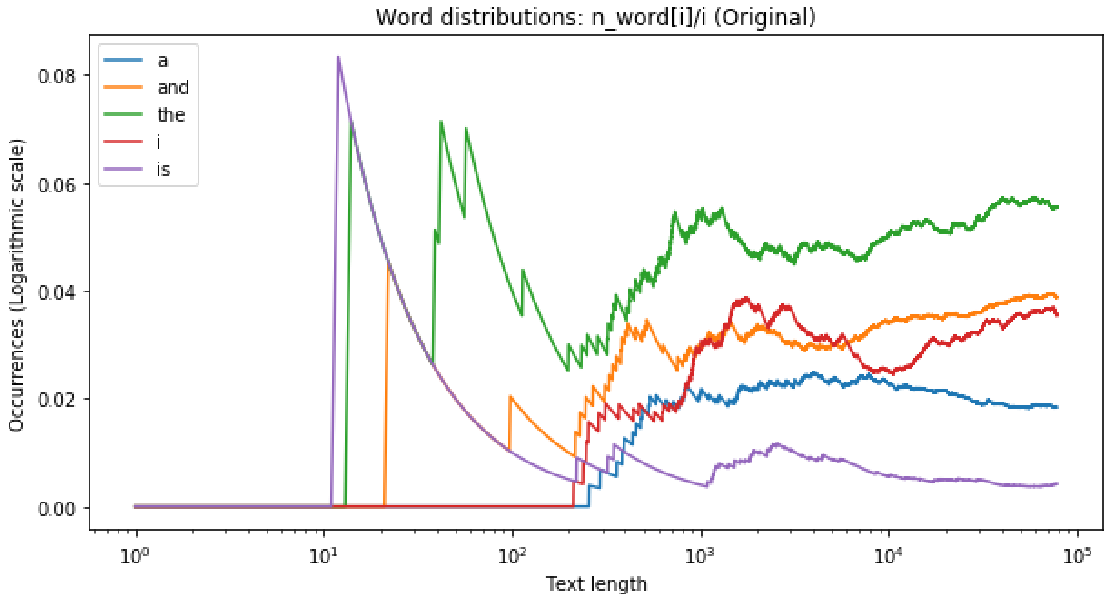
- Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.
- Find an envelope of the shape $O(1/\sqrt{i})$ for each of these five words.
- Why can we **not** apply the Central Limit Theorem directly?
  - Because in sentences, words are not uniformly distributed among different positions.
- How would we have to change the text for it to apply?
  - We can shuffle all words randomly.
- Why does it still work quite well?
  - Because after shuffle, all words are uniformly and independently distributed among different locations, which meets the requirements of Central Limit Theorem.

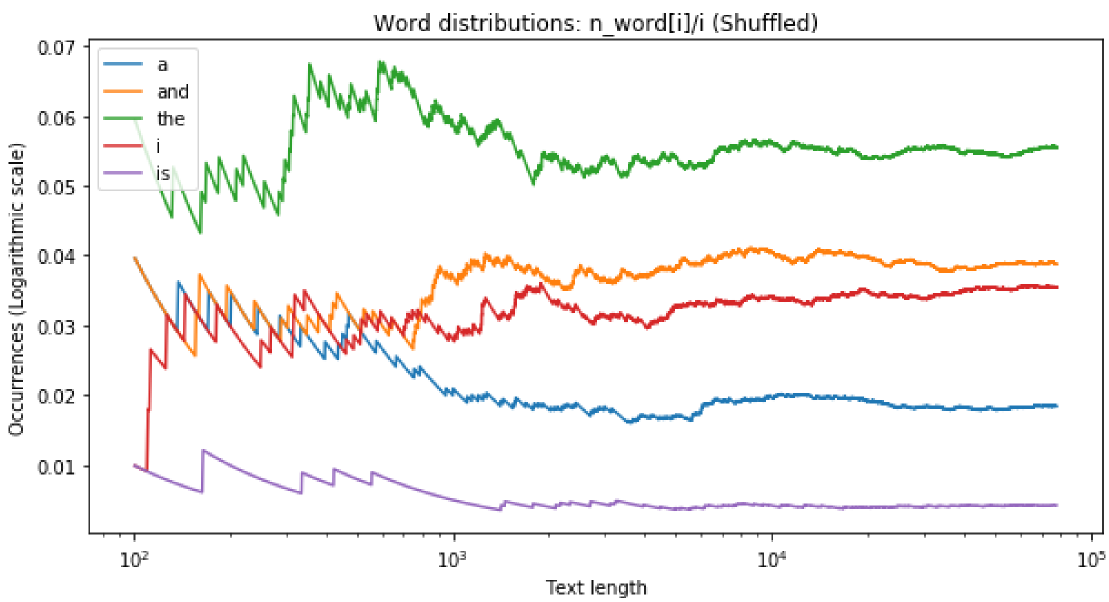
In [19]:

```
import re
import string
import matplotlib.pyplot as plt
import numpy as np
from random import shuffle
filename = gluon.utils.download('https://www.gutenberg.org/files/84/84-0.txt')
with open(filename, encoding='utf-8') as f:
```

```
    book = f.read()
tokens=['a', 'and', 'the', 'i', 'is']
book=book.lower()
exclude = set(string.punctuation)
book=''.join(ch for ch in book if ch not in exclude)
word_list=book.split()
list_len=len(word_list)
word_count=np.zeros(shape=(list_len, len(tokens)))
for i in range(list_len):
    for j in range(len(tokens)):
        if tokens[j]==word_list[i]:
            word_count[i, j]=word_count[i-1, j]+1
        else:
            word_count[i, j]=word_count[i-1, j]
# print(word_count[0:100])
x=np.arange(1, list_len+1)
plt.figure(figsize=(10, 5))
for i in range(len(tokens)):
    plt.semilogx(x, word_count[:,i]/x, label=tokens[i])
plt.title('Word distributions: n_word[i]/i (Original)')
plt.xlabel('Text length')
plt.ylabel('Occurrences (Logarithmic scale)')
plt.legend(loc='upper left')
plt.show()

# shuffle words in word_list
shuffle(word_list)
word_count=np.zeros(shape=(list_len, len(tokens)))
for i in range(list_len):
    for j in range(len(tokens)):
        if tokens[j]==word_list[i]:
            word_count[i, j]=word_count[i-1, j]+1
        else:
            word_count[i, j]=word_count[i-1, j]
plt.figure(figsize=(10, 5))
for i in range(len(tokens)):
    plt.semilogx(x[100:], word_count[100:,i]/x[100:], label=tokens[i])
plt.title('Word distributions: n_word[i]/i (Shuffled)')
plt.xlabel('Text length')
plt.ylabel('Occurrences (Logarithmic scale)')
plt.legend(loc='upper left')
plt.show()
```





3.1 3. Denominator-layout notation

We used the numerator-layout notation for matrix calculus in class, now let's examine the denominator-layout notation.

Given $x, y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we have

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \quad \frac{\partial \mathbf{y}}{\partial x} = \left[\frac{\partial y_1}{\partial x}, \frac{\partial y_2}{\partial x}, \ldots, \frac{\partial y_m}{\partial x}\right]$$

and

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_m}{\partial x_2} \\ \vdots \\ \frac{\partial y_1}{\partial x_n}, \frac{\partial y_2}{\partial x_n}, \ldots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Questions:

1. Assume $\mathbf{y} = f(\mathbf{u})$ and $\mathbf{u} = g(\mathbf{x})$, write down the chain rule for $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$
2. Given $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, assume $z = \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$, compute $\frac{\partial z}{\partial \mathbf{w}}$.

Answers:

1. $\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial \mathbf{y}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$
2. Suppose $\mathbf{u} = \mathbf{X}\mathbf{w} - \mathbf{y}$, $\mathbf{v} = \mathbf{X}\mathbf{w}$, then

$$\frac{\partial z}{\partial \mathbf{w}} = \frac{\partial z}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{w}}$$

where

$$z = \|\mathbf{u}\|^2, \mathbf{u} = \mathbf{v} - \mathbf{y}, \mathbf{v} = \mathbf{X}\mathbf{w} \frac{\partial z}{\partial \mathbf{u}} = 2\mathbf{u}^\top \frac{\partial \mathbf{u}}{\partial \mathbf{v}} = \mathbf{I} - \frac{\partial \mathbf{y}}{\partial \mathbf{v}} = \mathbf{I} \frac{\partial \mathbf{v}}{\partial \mathbf{w}} = \mathbf{X}$$

Therefore

$$\frac{\partial z}{\partial \mathbf{w}} = 2(\mathbf{X}\mathbf{w} - \mathbf{y})^\top \mathbf{X}$$

3.2 4. Numerical Precision

Given scalars x and y, implement the following `log_exp` function such that it returns a numerically stable version of

$$-\log\left(\frac{e^x}{e^x + e^y}\right)$$

```
In [20]: def log_exp(x, y):
             ## add your solution here
             return -nd.log(nd.divide(nd.exp(x), nd.exp(x)+nd.exp(y)))
```

Test your codes with normal inputs:

```
In [21]: x, y = nd.array([2]), nd.array([3])
         z = log_exp(x, y)
         z
```

Out[21]:
```
[1.3132617]
<NDArray 1 @cpu(0)>
```

Now implement a function to compute $\partial z/\partial x$ and $\partial z/\partial y$ with `autograd`

In [22]:
```
def grad(forward_func, x, y):
    x.attach_grad()
    y.attach_grad()
    with autograd.record():
        z=forward_func(x, y)
    z.backward()
    print('x.grad =', x.grad)
    print('y.grad =', y.grad)
```

Test your codes, it should print the results nicely.

In [23]:
```
grad(log_exp, x, y)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

But now let's try some "hard" inputs

In [24]:
```
x, y = nd.array([50]), nd.array([100])
grad(log_exp, x, y)
```

```
x.grad =
[nan]
<NDArray 1 @cpu(0)>
y.grad =
[nan]
<NDArray 1 @cpu(0)>
```

Does your code return correct results? If not, try to understand the reason. (Hint, evaluate `exp(100)`). Now develop a new function `stable_log_exp` that is identical to `log_exp` in math, but returns a more numerical stable result.

In [25]:
```
def stable_log_exp(x, y):
    ## Add your codes here
    if x>y:
        return -nd.log(1/(1+nd.exp(y-x)))
    else:
        return -nd.log(nd.exp(x-y)/(1+nd.exp(x-y)))
grad(stable_log_exp, x, y)
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[1.]
<NDArray 1 @cpu(0)>
```