AM115 In-class exercise
Models of competitive running

In this exercise, we examine models of competitive running and use them to illustrate a few points on parameter fitting[1].

Task 1: Discuss how you would model a sprint.

The answer of course depends on what you want to use the model for. We shall model running as a one-dimensional movement of the runner and consider the balance of forces. The model we will use is the Hill-Keller model[2]. For a sprint race, the model assumes the runner makes their best effort and applies the maximum propulsive force (treated as proportional to the runner's mass) throughout the race[3]. It further assumes that the resistive force is proportional to the velocity and the mass of the runner, on the grounds that it is mainly associated with dissipative effects of muscles and joints, which resist the fast motions. Wind resistance certainly has an effect[4], but is considered secondary. These considerations lead us to the following equation:

$$\frac{dv}{dt} = P_{\max} - kv \tag{1}$$

where v is the velocity and dv/dt is the acceleration, P_{max} is the maximum acceleration, and k characterizes how quickly the resistance acceleration increases with velocity. We also have the equation that the time derivative of position, x, is the velocity:

$$\frac{dx}{dt} = v \tag{2}$$

The time it takes for the sprinter to respond to the starting signal of the race, i.e. the reaction time, is not part of the present model, so we will take time t=0 to be the time when the sprinter starts running (i.e. time of the starting signal plus the reaction time), at which time the speed and position are both zero but the sprinter has started to accelerate. So the initial conditions are:

$$x(t=0) = 0$$
$$v(t=0) = 0$$

[1] This activity is based on Kurt Bryan's book "Differential equations: A toolbox for modeling the world", but with substantial modifications.

[2] A.V. Hill. The physiological basis of athletic records. The Scientific Monthly, 21(4):409–428, 1925; J.B. Keller. A theory of competitive running. Physics Today, 26(9):43, 1973.

[3] This assumption may not be valid for Usain Bolt during the last 20 meters at the 2008 Beijing Olympics finals, during which he visibly slowed down. Data from that event are also included in the csv file.

[4] Part of the reasons that the runners did better in the men's 100m sprint finals of the 2009 Berlin World Championship is that there was tailwind of 0.9 meters per second, compared to 0 wind in the 2008 Beijing Olympics. The maximum allowed tailwind is 2 meters per second. For the same reason, no records are valid when the altitude is greater than 1000 meters above the sea level as air gets thinner with altitude, reducing the resistance, although there could be other physiological advantages associated with higher altitudes. For more discussions, see Pritchard, Mathematical models of running. SIAM Review, 35(3):359–379, 1993

With parameters P_{max} and k and the initial conditions, we can integrate equations (1) and (2) to get the position of the runner as a function of time. You can use the analytical solution, derived in Appendix 1 or use an ODE solver to obtain a numerical solution.

Task 2: Fit the model to the data collected for the sprinter Usain Bolt during the 100 meters sprint finals at the 2009 World Championships (included in UsainBolt.csv) to obtain the parameters P_{max} and k. We make two notes here:

a) The reaction time is not included in the model and our t=0 is time of the starting signal plus the reaction time. You need to shift the time data to account for that.
b) The time data for the given positions were collected using infrared laser aimed at the lower back of the athletes and were given an error estimate of about 0.01 seconds[5], with the positions considered to be precisely known. Since the solution of the sprinter model is given in terms of positions as a function of time, one would need to interpolate the solution to the positions given in the data. For this exercise, to simply things, we shall consider the time to be precisely known and the positions have errors of ~0.1 meters. Moreover, while, in general, a numerical solver works well, here it is simpler to use the analytical solution shown in Eq. 7 in Appendix 1, because it gives an explicit expression for x given t.

After fitting the model, don't forget the residual analysis and the estimation of parameter uncertainties.

Task 3: Use the model and your estimated parameters to compute the time it would take a world champion runner to run 200 meters, 400 meters, 1000 meters, 1500 meters, 2000 meters, 3000 meters, etc. Compare your results with the actual world records and discuss reasons for agreements and/or disagreements. You can find the world records at https://www.worldathletics.org/records/by-category/world-records or other sites.

Task 4 (optional, extra credit): Consider a model that includes both the linear resistance and a quadratic resistance (for example from the wind), i.e.

$$\frac{dv}{dt} = P_{\max} - kv - \sigma v^2$$

Try to fit this model to the data and evaluate the residual, the parameters, and the parameter uncertainties. What conclusions do you draw? To fit the model, you can either use an ODE solver to obtain a numerical solution or use the analytical solution derived in Appendix 2 (if you are only interested in the solution itself, it's Eq. (13)).

Additional notes: In task 3, you would have found that the model does not predict the world records well beyond 200 meters. The reason is that after some critical distance, the athlete's initial reserve energy E_0 (per unit mass) will be exhausted and continued energy expenditure will need to be supplied through the oxygen uptake, which is assumed to be constant at the maximum oxygen uptake again on the ground that the athlete is making their maximum effort. This leads to the following equations:

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[5] Graubner, R., and E. Nixdorf. "Biomechanical Analysis of the Sprint and Hurdles Events At the 2009 IAAF World Championships in Athletics." *New Studies in Athletics* 26.1+2 (2011): 19-53.

$$\frac{dv}{dt} = P(t) - kv$$
$$\frac{dE}{dt} = \sigma - P(t)v(t) \tag{3}$$

Note that the energy supply through oxygen uptake is assumed to be constant, while P(t) is now recognized as something the athlete can control and is a function of time (but must be less than P_{max}). For longer distance (longer than 200 meters), one can then solve an optimal control problem to determine P(t) in order to finish the distance in minimum time. This was done by Keller (1973) (see reference in footnote 2) and gave good match for world records up to 10 kilometers, and we will approach this problem using dynamic programming later in the course. For even longer distances, efficient heat removal from the body is thought to be important. Some argue that efficient heat removal by our ancestors allowed for persistence hunting, in which animals are chased into the state of hyperthermia.

Appendix 1: Analytical solution to the sprint model with linear resistance (for reference)

Start with

$$\frac{dv}{dt} = P_{\max} - kv \tag{4}$$

Rearrange and multiple on both sides the integration factor:

$$e^{kt}\left(\frac{dv}{dt} + kv\right) = P_{\max} e^{kt}$$

we have

$$\frac{d}{dt}\left(e^{kt}v\right) = P_{\max} e^{kt} \tag{5}$$

Integrate and apply the initial condition v(t=0)=0, we have:

$$v(t) = \frac{P_{\max}}{k}\left(1 - e^{-kt}\right) \tag{6}$$

With the initial condition x(t=0)=0, the sprinter's position at time t is:

$$\begin{aligned} x(t) &= \int_0^t v(t')\,dt' \\ &= \int_0^t \frac{P_{\max}}{k}\left(1 - e^{-kt'}\right)dt' \\ &= \frac{P_{\max}}{k^2}\left(kt + e^{-kt} - 1\right) \end{aligned} \tag{7}$$

Appendix 2: Analytical solution to the sprint model with linear and quadratic resistance (for reference)

We shall only sketch out the steps and write down the final result. Start with

$$\frac{dv}{dt} = P_{\max} - kv - \sigma v^2 \tag{8}$$

Separation of variables:

$$\frac{dv}{P_{\max} - kv - \sigma v^2} = dt \tag{9}$$

You may notice that this is similar to the logistic equation.

Factor the denominator on the left-hand side and re-write the left-hand side (as in "the key step" on the "Analytical solution to the logistic equation" slide in the "Population (Single species, deterministic, Part 1)" module):

$$\begin{aligned} \frac{dv}{P_{\max} - kv - \sigma v^2} &= \frac{-1}{\sigma(\lambda_1 - \lambda_2)}\left[\frac{dv}{v - \lambda_1} - \frac{dv}{v - \lambda_2}\right] \\ &= \frac{-1}{\sigma(\lambda_1 - \lambda_2)} d\ln\left|\frac{v - \lambda_1}{v - \lambda_2}\right| \\ \lambda_{1,2} &\equiv \frac{-k \pm \sqrt{k^2 + 4P_{\max}\sigma}}{2\sigma} \end{aligned} \tag{10}$$

Note $v=\lambda_{1,2}$ are the fixed points, where $dv/dt=0$, and $\lambda_2<0$, $\lambda_1>0$. Since we start with the initial condition, $v(t=0)=0$, from the same graphic argument used for the logistic equation, we have $v(t)- \lambda_1$ is always negative and $v(t)- \lambda_2$ is always positive. Therefore, we have

$$d\ln\left(\frac{\lambda_1 - v}{v - \lambda_2}\right) = -\sigma(\lambda_1 - \lambda_2)dt \tag{11}$$

Integrate, exponentiate, apply the initial condition $v(t=0)=0$, and rearrange, we have

$$v(t) = \frac{-\lambda_1\lambda_2\left(1 - e^{-\sigma(\lambda_1 - \lambda_2)t}\right)}{-\lambda_2 + \lambda_1 e^{-\sigma(\lambda_1 - \lambda_2)t}} \tag{12}$$

Integrate $v(t)$ over time and use the initial condition $x(t=0)=0$, we have:

$$x(t) = \frac{1}{\sigma(\lambda_1 - \lambda_2)}\left[\lambda_1 \ln\left(\frac{\lambda_1 - \lambda_2 e^{\sigma(\lambda_1 - \lambda_2)t}}{\lambda_1 - \lambda_2}\right) - \lambda_2 \ln\left(\frac{\lambda_1 e^{-\sigma(\lambda_1 - \lambda_2)t} - \lambda_2}{\lambda_1 - \lambda_2}\right)\right] \tag{13}$$

$\lambda_{1,2}$ are defined in Eq. (10).