STP 598: Homework 3

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1. Problem 1. Basic Optimization, MLE for IID Poisson Data

Suppose y_i is a count then a very common model is to assume the Poisson disttribuion:

$$P(Y = y \mid \lambda) = \frac{e^{-\lambda}\,\lambda^y}{y!},\ y = 0, 1, 2, \ldots$$

Given $Y_i \sim Poisson(\lambda)$ iid, (that is, $Y_i = y_i$), what is the MLE of λ?

If the random variables are iid, then the joint distribution is the product of the marginal distributions. Let $\mathbf{y}$ be the sample $y_1, \ldots, y_n$

$$L(\lambda|\mathbf{y}) = \prod_{i=1}^{n} \frac{e^{-\lambda}\,\lambda^{y_i}}{y_i!}.$$

We take the log of the likelihood so that we can maximize easily.

$$\log(L(\lambda|\mathbf{y})) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\,\lambda^{y_i}}{y_i!}\right) = -n\lambda - \sum \log(y_i!) + \sum(y_i)\log(\lambda).$$

Differentiate with respect to lambda and set to zero to maximize:

$$\frac{\delta \log(L(\lambda|\mathbf{y}))}{\delta d\lambda} = -n + \frac{\sum y_i}{\lambda} = 0 \implies \hat{\lambda} = \frac{\sum y_i}{n}$$

.

The MLE for the parameter of the *poisson* distribution is the sample mean.

Problem 2. Constrained Optimization, Minimum Variance Portfolio

Suppose we are considering investing in p stocks where the uncertain return on the ith stock is denoted by R_i, $i = 1, 2, \ldots, p$. Let $R = (R_1, R_2, \ldots, R_p)^{'}$. A portfolio is a given by $w = (w_1, w_2, \ldots, w_p)^{'}$ where w_i is the fraction of wealth invested in asset i. The $\{w_i\}$ must satisfy $\sum w_i = 1$. The return on the portfolio is then

$$P = w'R = \sum w_i R_i.$$

We want to find the global **minimum variance portfolio**:

$$\min_w Var(P), \text{ subject to} \sum w_i = 1.$$

If we let $\iota = (1, 1, \ldots, 1)^{'}$, the vector of ones, and $Var(R) = \Sigma$ then our problem is

$$\min_w w^{'}\Sigma w \text{ subject to } w^{'}\iota = 1.$$

Find the global minimum variance portfolio in terms of Σ and ι.

If we know that $\sum w_i = 1$ we can easily say $\sum w_i - 1 = 0$ or $w^{'}\iota - 1 = 0$. We use the Karush-Kuhn-Tucker conditions to minimize. By this we take the derivative with respect to w,

$$\frac{\delta L(\lambda, w)}{\delta w} = \frac{\delta(w^{'}\Sigma w)}{\delta w} + \frac{\delta\lambda(w^{'}\iota - 1)}{\delta w} = 2\Sigma w + \lambda\iota$$

$$\implies \hat{w} = -\lambda\Sigma^{-1}\iota/2.$$

To express this solution in terms of Σ, ι we solve for λ. Premultiply by $\iota^{'}$ and then solve. Since we have that $\iota^{'}w = 1$,

$$\iota^{'}w = -\lambda\iota^{'}\Sigma^{-1}\iota/2 \implies \lambda = -2/(\iota^{'}\Sigma^{-1}\iota).$$

We plug this into our solution of $\hat{w}$ to have:

$$\hat{w} = \frac{-\lambda\Sigma^{-1}\iota/2}{-2/\iota'\Sigma^{-1}\iota} = \frac{\lambda\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota}$$

$$\hat{w}$$

is the w such that $Var(w^{'}R) = Var(P)$ is minimized.