# Measure Theory

Lectures by Claudio Landim

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## Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. Taylor published by Cambridge University Press.

These notes were live-TeXed, though I edited for typos and added diagrams requiring the Ti*k*Z package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

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# Lecture 1

## Introduction: a Non-measurable Set

$\lambda$ satisfies the flowing:

0. $\lambda: \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$
1. $\lambda((a,b]) = b - a$
2. $A \subseteq \mathbb{R},\ A + x = \{x + y :\ y \in A\},\ \ \forall A,\ A \subseteq \mathbb{R},\ \forall x \in \mathbb{R}:$
$$\lambda(A + x) = \lambda(A) \tag{1.1}$$
3. $A = \bigcup_{j \geqslant 1} A_j,\ \ A_j \cap A_k = \varnothing:$
$$\lambda(A) = \sum_k \lambda(A_k) \tag{1.2}$$

**Definition 1.1.** $x \sim y,\ x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $[x] = \{y \in \mathbb{R},\ y - x \in \mathbb{Q}\}$.
$\Lambda = \mathbb{R}|_\sim$, only one point represents the equivalence class of $\Omega$ , like $\alpha, \beta$.

$\Omega$ is a class of equivalence class, if $\Omega \subseteq R, \Omega \subseteq (0,1)$

**Claim 1.1.** $\begin{cases} \Omega + q = \Omega + q \\ \Omega + q \cap \Omega + q = \varnothing \end{cases} \quad q, p \in \mathbb{Q}$

*Proof.* Assume that $\Omega + q \cap \Omega + q \neq \varnothing$ then, $x = \alpha + p = \beta + q,\ \alpha, \beta \in \Omega \Rightarrow \alpha - \beta = q - p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p, q \in \mathbb{Q} \Rightarrow (\Omega + q) \cap (\Omega + p) = \varnothing]$. $\square$

**Claim 1.2.** $\Omega + q \subseteq (-1, 2)$, if $-1 < q < 1$.

then we can get

$$\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q) \subseteq (-1, 2) \tag{1.3}$$

**Claim 1.3.** $E \subseteq F \Rightarrow \lambda(E) \leqslant \lambda(F)$

*Proof.* $\because E \subseteq F \therefore F = E \cup (F \backslash E),\ E \cap (F \backslash E) = \varnothing$, then $\lambda(F) = \lambda(E) + \lambda((F \backslash E)) \Rightarrow \lambda(F) \geqslant \lambda(E)$. $\square$

Then,

$$\lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\right) \leqslant \lambda((-1, 2)) = 3 \tag{1.4}$$

and ,

$$\infty \cdot \lambda((\Omega + q)) = \infty \cdot \lambda(\Omega) \leq 3 \Rightarrow \lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\right) = 0 \tag{1.5}$$

**Claim 1.4.** $(0,1) \subseteq \sum\limits_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)$

*Proof.* $\forall$ fixed $x \in (0,1)\,, \exists \alpha \in [x] \cap \Omega,\ \alpha \in (0,1)$, and we know that $\alpha - x = q \in \mathbb{Q},\ - < q < 1 \ \Rightarrow$ $x = \alpha + q,\ x \in \Omega + q$ $\square$

But, we get that:

$$1 = \lambda\left((0,1)\right) \leqslant \lambda\left(\sum_{q\in\mathbb{Q}} \Omega + q\right) = 0 \tag{1.6}$$

it is impossible.

# Lecture 2

**Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions**

**Definition 2.1.** $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is semi-algebra if:

1. $\Omega \subseteq \mathcal{S}$
2. $A, B \in \mathcal{S} \Rightarrow A \bigcap B \in \mathcal{S}$
3. $\forall A \in \mathcal{S} \Rightarrow A^c = \sum_{i=1}^{n} E_j,\ \ \exists E_1, \cdots, E_n \in \mathcal{S},\ E_i, E_j\ (i \neq j)$ disjoint sets, $n$ is finite number

**Example 2.1.** $\Omega = \mathbb{R},\ \ \mathcal{S} = \{\mathbb{R}, \{(a,b), a<b, a,b \in \mathbb{R}\}, \{(-\infty, b], b \in \mathbb{R}\}, \{(a, \infty), a \in \mathbb{R}\}, \varnothing\}$, $(a,b]^c = (-\infty, a] \cup [b, +\infty)$

**Example 2.2.** $\Omega = \mathbb{R}^2$

$\mathcal{S} = \ \ \{\mathbb{R}^2, \{(a_1,b_1) \times (a_2,b_2),\ a_i < b_i, a_i, b_i \in \mathbb{R}, \{(-\infty, b_1] \times (-\infty, b_2], b_i \in \mathbb{R}\}, \{(a_1, \infty) \times (a_2, \infty), a_i \in \mathbb{R}\}, \varnothing\}$

**Definition 2.2.** $a = \mathcal{P}(\Omega)$ is an algebra:

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

**Remark 2.1.** $a$ algebra $\Rightarrow$ $a$ semi-algebra

**Definition 2.3.** $\sigma$-algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$:

1. $\Omega \subseteq \mathcal{S}$
2. $A_j \in \mathcal{S}, j \leq 1 \Rightarrow \bigcap_{j \geqslant 1} A_j \in \mathcal{S}$
3. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

**Remark 2.2.** $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega)$, $a_\alpha$ algebra, $\alpha \in I \Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is an algebra.

*Proof.* check the followings

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

$\square$

**Remark 2.3.** $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega), \alpha \in I, a_\alpha$, $\sigma$-algebra $\Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is a $\sigma$-algebra

*Proof.* check the followings

1. $\Omega \in a$

2. $A_j, j \geq 1 \in a \Rightarrow \bigcap\limits_{j\geqslant 1} A_j \in a$

3. $A \in a \Rightarrow A^c \in a$

$\square$

**Definition 2.4** ( minimal algebra generated by $c$)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is an algebra generated by $c$, and $a = a(c)$:

1. $c \subseteq a$

2. $\forall \mathcal{B}$ is algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.1}$$

**Remark 2.4.** $a(c)$ exits, and $a = a(c) = \bigcap\limits_{\alpha} a_\alpha,\ \forall\alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an algebra.

**Definition 2.5** ( minimal $\sigma$-algebra generated by $c$)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is a $\sigma$-algebra generated by $c$, and $a = a(c)$:

1. $c \subseteq a$

2. $\forall \mathcal{B}$ is $\sigma$-algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.2}$$

**Remark 2.5.** $a(c)$ exits, and $a = a(c) = \bigcap\limits_{\alpha} a_\alpha,\ \forall\alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an $\sigma$-algebra.

**Lemma 2.1.** *$\Omega, f$ semi-algebra $f \subseteq \mathcal{P}(\Omega)$, $a(f)$ algebra generated by $f$ then*
$$A \in a(f) \Leftrightarrow \exists E_j \in f, 1 \leqslant j \leqslant n,\ A = \sum_{j=1}^{n} E_j \tag{2.3}$$

*Proof.*

1. $\Leftarrow$

   $A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f \in a(f)$

   By definition 2.1 and remark 2.6 $\Rightarrow A \in a(f)$

2. $\Rightarrow$

   $A \in a(f) \Rightarrow A = \sum\limits_{j=1}^{n} E_j, E_j \in f$

   Then by remark 2.7, it will be proved easily.

$\square$

**Remark 2.6.** $E, J \in a, E \bigcup F \in a, E \bigcup F = (E^c \bigcap F^c)^c$

**Remark 2.7.** $\mathcal{B} = \left\{ \sum_{j=1}^{n} F_j,\ F_j \in f \right\}$, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ then

1. $\mathcal{B}$ algebra
2. $\mathcal{B} \supseteq f$
3. $\mathcal{B} \supseteq a(f)$

*Proof.* We only prove that $\mathcal{B}$ algebra, then check the following

1. $\Omega \in \mathcal{B}$
2. $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$

   $\because A, B \in \mathcal{B}$, $\therefore A = \sum_{j=1}^{n} E_j,\ E_j \in f,\ B = \sum_{k=1}^{m} F_k,\ F_k \in f$, then

$$\begin{aligned} A \cap B &= \left(\sum_{j=1}^{n} E_j\right) \cap \left(\sum_{k=1}^{m} F_k\right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \\ &\in \mathcal{B} \end{aligned} \tag{2.4}$$

3. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

   $A = \sum_{j=1}^{n} E_j,\ E_j \in f$

   By definition 2.1:

$$\begin{aligned} E_1^c &= \sum_{k_1=1}^{l_1} F_{1,k_1},\ F_{1,j} \in f \\ \cdots &= \cdots \\ E_i^c &= \sum_{k_i=1}^{l_i} F_{i,k_i},\ F_{i,j} \in f \end{aligned} \tag{2.5}$$

   Then, we get that

$$\begin{aligned} A^c &= \left(\sum_{k_1=1}^{l_1} F_{1,k_1}\right) \cap \left(\sum_{k_2=1}^{l_2} F_{2,k_2}\right) \cap \cdots \cap \left(\sum_{k_n=1}^{l_n} F_{n,k_n}\right) \\ &= \sum_{k_1=1}^{l_1} \sum_{k_2=1}^{l_2} \cdots \sum_{k_n=1}^{l_n} \left(F_{1,k_1} \cap F_{2,k_2} \cap F_{n,k_n}\right) \\ &\in \mathcal{B} \end{aligned} \tag{2.6}$$

□

**Definition 2.6.** $c \subseteq \mathcal{P}(\Omega), \varnothing \in c,\ \mu:\ c \to \mathbb{R}_+ \cup \{+\infty\}$. $\mu$ is additive if

1. $\mu(\varnothing) = 0$
2. $E_1, E_2, ..., E_n \in c,\ E = \sum\limits_{j=1}^{n} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j=1}^{n} \mu(E_k)$

**Remark 2.8.**

$$\exists A \in c,\ \mu(A) < \infty,\ A = A \cup \varnothing,\ \mu(A) = \mu(A) + \mu(\varnothing) \Rightarrow \mu(\varnothing) = 0 \tag{2.7}$$

**Remark 2.9.** $c,\ \mu : c \to \mathbb{R}_+ \bigcup +\infty,\ E \subseteq F,\ F \backslash E \in c,\ E, F \in c$

$$F = E \cup (F \backslash E),\ \mu(F) = \mu(E) + (F \backslash E) \tag{2.8}$$

1. $\mu(E) = +\infty,\ \mu(F) = +\infty$
2. $\mu(E) < +\infty,\ \mu(F \backslash E) = \mu(F) - \mu(E)$

so,

$$\mu(E) \leqslant \mu(F) \tag{2.9}$$

**Example 2.3.** Discrete measure: $\Omega,\ c \subseteq \mathcal{P}(\Omega),\ \{x_j,\ j \geqslant 1\},\ x_j \in \Omega,\ \{p_j,\ j \geqslant 1\},\ p_j, \geqslant 0,\ A \in c,$ define that

$$\mu(A) = \sum_{j \geqslant 1} p_j 1\{x_j \in A\} \tag{2.10}$$

then $\mu$ is additive

**Definition 2.7.** $c \in \mathcal{P}(\Omega), \varnothing \in c,\ \mu:\ c \to \mathbb{R}_+ \bigcup +\infty$, $\mu$ is $\sigma$-additive if

1. $\mu(\varnothing) = 0$
2. $E_j \in c,\ j \neq k, E_j \bigcap E_k = \varnothing,\quad E = \sum\limits_{j \geq 1} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j \geq 1} \mu(E_j)$

**Example 2.4.** $\Omega = (0,1),\ c = \{(a,b],\ 0 \leqslant a < b < 1\},\ \mu:\ c \to \mathbb{R}_+ \cup \{+\infty\}$, define that

$$\mu(a,b] = \begin{cases} +\infty & a = 0 \\ b - a & a > 0 \end{cases} \tag{2.11}$$

$(a,b] = \sum\limits_{j=1}^{n} (a_j, b_j)$, we can get that $\mu$ is NOT $\sigma$-additive.

If $x_1 = \frac{1}{2}, x_j > x_{j+1}, x_j \downarrow\to 0$, then

$$\frac{1}{2} = \left(0, \frac{1}{2}\right] = \sum_{j \geqslant 1} (x_{j+1}, x_j] = +\infty \tag{2.12}$$

it is impossible.

**Definition 2.8.** Any non-negative set function $\mu : C \to \mathbb{R}_+ \cup \{+\infty\}$ which is $\sigma - additive$ is called a measure on $C$.

# Lecture 3

## Set Functions

**Definition 3.1.** $c \subseteq \mathcal{P}(\Omega)$, $\mu : c \to \mathbb{R}_+ \bigcup +\infty$:

1. $E \in c$, $\mu$ continuous from below at $E$, if $\forall (E_n)_{n\geqslant 1},\ E_n \in c,\ E_n \uparrow E\ \left(E_n \subseteq E_{n+1},\ \bigcup\limits_{n\geqslant 1} E_n = E\right)$:
$$\mu(E_n) \to \mu(E) \tag{3.1}$$

2. $E \in c$, $\mu$ continuous from above at $E$, if $\forall (E_n)_{n\geqslant 1},\ E_n \in c,\ E_n \downarrow E\ \left(E_{+1} \subseteq E_n,\ \bigcap\limits_{n\geqslant 1} E_n = E\right)$, and $\exists n_0,\ \mu(E_{n_0}) < \infty$:
$$\mu(E_n) \to \mu(E) \tag{3.2}$$

**Remark 3.1.** For a sequence $E_1, E_2, ...$ of sets, we put

$$\limsup E_i = \bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty} E_i\right), \liminf E_i = \bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} E_i\right) \tag{3.3}$$

and if $\{E_i\}$ is such that $\limsup E = \liminf E_i$ we say that the sequence converges to the set

$$E = \limsup E = \liminf E_i \tag{3.4}$$

**Remark 3.2.** 2 need $\exists n_0,\ \mu(E_{n_0}) < \infty$, if not:

$$E_n = [n, +\infty),\ \mu(E_n) = +\infty,\ E_n \downarrow \varnothing,\ \lambda(\varnothing) = 0 \tag{3.5}$$

**Lemma 3.1.** *$a \subseteq \mathcal{P}(\Omega)$, algebra; $\mu :\ a \to \mathbb{R}_+ \cup \{+\infty\}$, additive;*

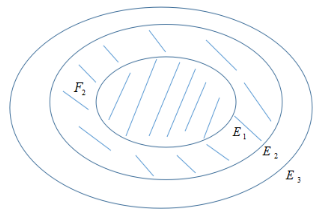
1. *$\mu$ is $\sigma$-additive $\Rightarrow$ $\mu$ continuous at $E$, $\forall E \in a$*
2. *$\mu$ is continuous from below $\Rightarrow$ $\mu$ is $\sigma$-additive*
3. *$\mu$ is continuous from above at $\varnothing$& $\mu$ is finite $\Rightarrow$ $\sigma$-additive*

*Proof.*

1.

 (i) $\mu$ is $\sigma$-additive $\Rightarrow$ $\mu$ conti. from below at $E \in a$. $E \in a, E_n \uparrow E,\ E_n \in a$:

$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \backslash E_1 \\ \vdots &= \vdots \\ F_n &= E_n \backslash E_{n-1} \end{aligned} \tag{3.6}$$



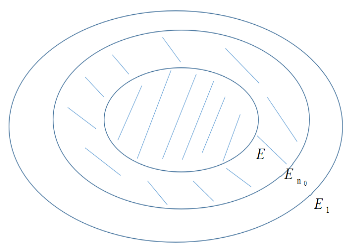

and we can get that

$$F_j \cap F_k = \varnothing, \quad \sum_{k=1}^{n} F_k = E_n, \quad \bigcup_{n\geqslant 1} E_n = \bigcup_{n\geqslant 1} F_n \tag{3.7}$$

so

$$\mu\left(E\right) = \sum_{k\geqslant 1} \mu\left(F_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu\left(F_k\right) = \lim_{n\to\infty} \mu\left(E_n\right) \tag{3.8}$$

(ii) $\mu$ cont. from above $E \in a, E_n \in a, E_n \downarrow E, \mu\left(E_{n_0}\right) < \infty \Rightarrow \mu\left(E_n\right) \downarrow \mu\left(E\right)$




$$\begin{aligned} G_1 &= E_{n_0} \backslash E_{n_0+1} \\ G_2 &= E_{n_0} \backslash E_{n_0+2} \\ \vdots &= \vdots \\ G_k &= E_{n_0} \backslash E_{n_0+k} \end{aligned} \tag{3.9}$$

then $G_k \uparrow E_{n_0} \backslash E, G_k \in a \Rightarrow \mu\left(G_k\right) \uparrow \mu\left(E_{n_0} \backslash E\right)$, so

$$\begin{aligned} \mu\left(E_{n_0} \backslash E\right) &= \lim_{n\to\infty} \mu\left(E_{n_0} \backslash E_{n_0+k}\right) \\ \mu\left(E_{n_0} \backslash E\right) &= \mu\left(E_{n_0}\right) - \mu\left(E\right) \\ \mu\left(E_{n_0}\right) - \mu\left(E\right) &= \lim_{k\to\infty} \left(\mu\left(E_{n_0}\right) - \mu\left(E_{n_0+k}\right)\right) \end{aligned} \tag{3.10}$$

2. $\mu$ cont. below, $E = \sum\limits_{k\geqslant 1} E_k, \ E, E_k \in a$.

   Obs.

$$\sum_{k=1}^{n} E_k \subseteq E \overset{additive}{\Rightarrow} \begin{cases} \mu\left(\sum\limits_{k=1}^{n} E_k\right) \leqslant \mu\left(E\right) \\ \sum\limits_{k=1}^{n} \mu\left(E_k\right) \leqslant \mu\left(E\right) \end{cases} \tag{3.11}$$

then

$$\sum_{k\geqslant 1}\mu\left(E_k\right)\leqslant\mu\left(E\right) \tag{3.12}$$

$F_n=\sum\limits_{k=1}^{n}E_k\in a,\ F_n\uparrow E,$

$$\sum_{k=1}^{n}\mu\left(E_k\right)=\mu\left(F_n\right)\uparrow\mu\left(E\right)\Rightarrow\sum_{k\geqslant 1}\mu\left(E_k\right)=\mu\left(E\right) \tag{3.13}$$

3. $\mu$ cont. at $\varnothing$, $\mu\left(\Omega\right)<\infty$, $E,E_k\in a,E=\sum\limits_{k\geqslant 1}E_k$.

$$F_n=\sum_{k\geqslant m}E_k\in a\ \left(E\backslash\sum_{j=1}^{n-1}E_j\right) \tag{3.14}$$

$F_n\downarrow\varnothing,\mu\left(F_1\right)<\infty,\ \mu\left(F_n\right)\rightarrow 0$

$$\begin{aligned}\mu\left(E\right)&=\mu\left(\sum_{k=1}^{n}E_k\cup\sum_{k>n}E_k\right)\\&=\underbrace{\mu\sum_{k=1}^{n}E_k}_{\rightarrow\sum_{k\geqslant 1}\mu(E_n)}+\underbrace{\mu\sum_{k>n}E_k}_{\rightarrow 0}\\&\rightarrow\sum_{k\geqslant 1}\mu\left(E_n\right)\end{aligned} \tag{3.15}$$

□

**Remark 3.3.** Suppose $E_\alpha,\ \alpha\in I$ is a class of subsets of $X$, and $E_i$ is one set of the class, then

1. $\bigcap\limits_{\alpha\in I}E_\alpha\subseteq E_i\subseteq\bigcup\limits_{\alpha\in I}E_\alpha$
2. $X-\bigcup\limits_{\alpha\in I}E_\alpha=\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$
3. $X-\bigcap\limits_{\alpha\in I}E_\alpha=\bigcup\limits_{\alpha\in I}\left(X-E_\alpha\right)$

*Proof.*

1. This is immediate from the definition.
2. Suppose $x\in X-\bigcup\limits_{\alpha\in I}E_\alpha$ then $x\in X$ and x is not in $\bigcup\limits_{\alpha\in I}E_\alpha$, that is $x$ is not in any $E_\alpha$, $\alpha\in I$ so that $x\in X-E_\alpha$ for every $\alpha\in I$, and $x\in\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$. Conversely if $x\in\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$, then for every $\alpha\in I$, $x$ is in $X$ but not in $E_\alpha$, so $x\in X$ but $x$ is not in $\bigcup\limits_{\alpha\in I}E_\alpha$, that is $x\in\bigcup\limits_{\alpha\in I}\left(X-E_\alpha\right)$.

3. Similar to 2

Remark 3.3 (2) and (3) are also called as de Morgan's Law. □

**Example 3.1.** $(0,1), (a,b], 0 \leqslant a < b < 1$

$$\mu(a,b] = \begin{cases} b-a, & a>0 \\ +\infty, & a=0 \end{cases} \tag{3.16}$$

$\mu$ is additive but NOT $\sigma$-additive

*Proof.* $E_n \downarrow \varnothing,\ \mu(E_{n_0}) < \infty,\ E_n = (a_{n,1}, b_{n,1}] \cup \cdots \cup (a_{n,k_n}, b_{n,k_n}], a_{n,j} < a_{n,j+1}$.

$\begin{cases} a_{n,1} = 0, & \forall n \\ a_{n_0} > 0, & \textit{some } n_0 \end{cases}$ □

**Theorem 3.1** (Extension)**.** *$f \subseteq \mathcal{P}(\Omega)$ semi-algebra, $\mu : f \to \mathbb{R}_+ \cup \{\infty\}$ $\sigma$-additive, then $\exists \nu$ :*

$$\nu : a(f) \to \mathbb{R}_+ \cup \{\infty\} \tag{3.17}$$

*such that:*

1. *$\nu$ $\sigma$-additive*
2. *$\nu(A) = \mu(A), \ \forall A \in f$*
3. *$\mu_1, \mu_2, a(f) \to \mathbb{R}_+ \bigcup \{+\infty\}$, then $\mu_1(A) = \mu_2(A), \forall A \in s \Rightarrow \mu_1(E) = \mu_2(E), \forall E \in a(f)$*

*Proof.* $A \in a(f) \Rightarrow A = \sum_{j=1}^{n} E_j,\ E_j \in f$ by Lemma 2.1.

$$\nu(A) \overset{add}{=} \sum_{j=1}^{n} \nu(E_j) \overset{ext}{=} \sum_{j=1}^{n} \mu(E_j) \tag{3.18}$$

we define that

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j) \tag{3.19}$$

we want to show that $\nu(A) = \sum_{j=1}^{n} \mu(E_j)$ is well-defined:

1. $\nu$ is unique

$$\begin{aligned} A &= \sum_{j=1}^{n} E_j,\ E_j \in f \\ &= \sum_{k=1}^{m} F_k,\ F_k \in f \end{aligned} \tag{3.20}$$

then we will prove that

$$\begin{aligned} \nu(A) &= \sum_{j=1}^{n} \mu(E_j) \\ &= \sum_{k=1}^{m} \mu(F_k) \end{aligned} \tag{3.21}$$

$$\because E_j \subseteq A = \sum_{k=1}^{m} F_k \Rightarrow E_j = E_j \cap \left( \sum_{k=1}^{m} F_k \right) = \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \tag{3.22}$$

$$\therefore \mu\left(E_j\right) = \mu\left( \sum_{k=1}^{m} (E_j \cap F_k) \right) \tag{3.23}$$

then

$$\nu\left(A\right) = \sum_{j=1}^{n} \mu\left(E_j\right) = \sum_{j=1}^{n} \sum_{k=1}^{m} \mu\left(E_j \cap F_k\right) = \sum_{k=1}^{m} \mu\left(F_k\right) \tag{3.24}$$

2. $\nu$ is an additive, $\nu\left(A\right) = \sum\limits_{j=1}^{n} \mu\left(E_j\right)$

   Assume that

$$\begin{cases} A = \sum\limits_{j=1}^{n} E_j, \ E_j \in f \\ B = \sum\limits_{k=1}^{m} F_k, \ F_k \in f \end{cases}, A \cap B = \varnothing \tag{3.25}$$

   We will show that

$$\nu\left(A \cup B\right) = \nu\left(A\right) + \nu\left(B\right) \tag{3.26}$$

$$\because A \cup B = \sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \tag{3.27}$$

   therefore

$$\begin{aligned} \nu\left(A \cup B\right) &= \mu\left( \sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \right) \\ &= \sum_{j=1}^{n} \mu\left(E_j\right) + \sum_{k=1}^{m} \mu\left(F_k\right) \\ &= \nu\left(A\right) + \nu\left(B\right) \end{aligned} \tag{3.28}$$

3. $\nu\left(A\right) = \mu\left(A\right), \ A \in f$ by Eq 3.19

4. $\nu$ is uniqueness, we want to show that:

   Suppose that $\mu_1, \mu_2 : \ a\left(f\right) \to R_+ \cup \{+\infty\}, \forall A \ \in f, \mu_1, \mu_2 \ additive$, then

$$\mu_1\left(A\right) = \mu_2\left(A\right) \Rightarrow \mu_1\left(B\right) = \mu_2\left(B\right), \ \forall B \in a\left(f\right) \tag{3.29}$$

   $\because B \in a\left(f\right), \therefore B = \sum\limits_{j=1}^{n} \mu_1\left(E_j\right), \ E_j \in f$

$$\mu_1\left(B\right) = \sum_{j=1}^{n} \mu_1\left(E_j\right) = \sum_{j=1}^{n} \mu_2\left(E_j\right) = \mu_2\left(B\right) \tag{3.30}$$

Now we proof the extension of $\sigma$-additive, ie: $\mu-\sigma$ additive, $f$ semi-algebra, $\nu-\sigma$ additive, $a(f)$ is a algebra generated by $f$. we want to show that

$$A=\sum_{j\geqslant 1}A_j,\ A,A_j\in a\left(f\right)\Rightarrow \nu\left(A\right)=\sum_{j\geqslant 1}\nu\left(A_j\right) \tag{3.31}$$

by representation of an algebra:

$$A=\sum_{j=1}^{m}E_j,E_j\in f;\quad A_k=\sum_{l=1}^{m_k}E_{k,l},E_{k,l}\in f \tag{3.32}$$

by Eq 3.19:

$$\nu\left(A\right)=\sum_{j=1}^{m}\nu\left(E_j\right),\quad \nu\left(A_k\right)=\sum_{l=1}^{m_k}\nu\left(E_{k,l}\right) \tag{3.33}$$

$$\because E_j=E_j\cap A=E_j\cap\left(\sum_{k\geqslant 1}A_k\right)=E_j\cap\left(\sum_{k\geqslant 1}\sum_{l=1}^{m_k}E_{k,l}\right)=\sum_{k\geqslant 1}\sum_{l=1}^{m_k}\left(E_j\cap E_{k,l}\right) \tag{3.34}$$

therefore

$$\begin{aligned}\nu\left(A\right)&=\sum_{j=1}^{n}\mu\left(E_j\right)\\&=\sum_{j=1}^{n}\sum_{k\geqslant 1}\sum_{l=1}^{m_k}\mu\left(E_j\cap E_{k,l}\right)\\&=\sum_{k\geqslant 1}\underbrace{\sum_{l=1}^{m_k}\mu\left(E_{k,l}\right)}_{\subseteq A_k}\end{aligned} \tag{3.35}$$

Eq 3.35 holds because:

$$E_{k,l}=E_{k,l}\cap A=\sum_{j=1}^{n}\left(E_{k,l}\cap E_j\right) \tag{3.36}$$

and

$$\mu\left(E_{k,l}\right)=\sum_{j=1}^{n}\mu\left(E_{k,l}\cap E_j\right) \tag{3.37}$$

so we can get that

$$\nu\left(A\right)=\sum_{k\geqslant 1}\nu\left(A_k\right) \tag{3.38}$$

□

# Lecture 4

## Caratheodory Theorem

**Theorem 4.1** (Caratheodory Theorem).

$$
\begin{array}{lll}
\sigma - add & \mu : f \to \mathbb{R}_+ \cup \{+\infty\} & f \subseteq \mathcal{P}(\Omega), f \textit{ is semialgebra} \\
\downarrow & \qquad\downarrow & \\
\sigma - add & \nu : a(f) \to \mathbb{R}_+ \cup \{+\infty\} & a(f) \textit{ algebra generated by } f \\
\downarrow & \qquad\downarrow & \\
\sigma - add & \pi : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\} & \mathcal{F}(a) \textit{ is } \sigma - \textit{algebra generated by algebra } a
\end{array} \tag{4.1}
$$

The big picture of the proof:

1. Define the $\pi^*$ outer measure:
$$\pi^* = \inf_{\{E_i\}} \sum_{i \geqslant 1} \nu(E_i) \tag{4.2}$$
2. $\mathcal{M}$ $\sigma$-algebra, $\mathcal{M} \supseteq \mathcal{F}(a)$
3. $$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.3}$$
is $\sigma$-additive, and
$$\pi^*|_a = \nu \tag{4.4}$$
4. (uniqueness) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \bigcup \{+\infty\}$, $\Omega$ is $\sigma$-finite($\mu_1$), if $E_j \uparrow \Omega$, $\mu_1(E_j) < \infty, \forall j,\ E_j \in a$ and $\mu_1|_a = \mu_2|_a$ then implies that
$$\mu_1 = \mu_2 \tag{4.5}$$

Finally, we define $\pi(E) = \pi^*(E),\ \forall E \in \mathcal{F}(a) \subseteq \mathcal{M}$.

Now, let
$$\pi^* :\ \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.6}$$

We will prove $\pi^*$ is an outer measure.

And we will construct a family of subsets $\mathcal{M}$
$$\mathcal{M} \subseteq \mathcal{P}(\Omega) \tag{4.7}$$

we will also prove $\mathcal{M}$ satisfies the following:

1. $\mathcal{M}$ is a $\sigma-$algebra
2. $\mathcal{M} \supseteq a$
3. $\pi^*|_{\mathcal{M}}$ $\sigma-$additive
4. $\pi^*|_a = \nu$

Next, we will define $\pi^*$ and $\mathcal{M}$ .

Step 1

**Definition 4.1** ($\pi^*$)**.** $\pi^*: \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$, $A \in \Omega$, $\{E_i, i \geqslant 1\}, E_i \in a, A \subseteq \bigcup E_i$, $\{E_i\}$ is a covering of A, then we define that

$$\pi^* = \inf_{\{E_i\},A} \sum_{i \geqslant 1} \nu(E_i) \tag{4.8}$$

where $\nu: \ a(f) \to \mathbb{R}_+ \cup \{+\infty\}, \ is\ \sigma$-additive.

**Definition 4.2** (Outer measure)**.** $\mu: \ c \to \mathbb{R}_+ \cup \{+\infty\}, c \subseteq P(\Omega), \varnothing \in c$, $\mu$ is a outer measure if

1. $\mu(\varnothing) = 0$
2. (monotone) $E \subseteq F, \ E, F \in c \ \Rightarrow \mu(E) \leqslant \mu(F)$
3. (subadditive) $E, E_i \in c, \ E \subseteq \bigcup_i E_i \Rightarrow \mu(E) \leqslant \sum_i \mu(E_i)$

**Theorem 4.2.** *$\pi^*$ in 4.1 is a outer measure.*

*Proof.* We will check the conditions in Def 4.2.

1. check $\pi^*(\varnothing) = 0$
   (a) $E_i = \varnothing, \varnothing \subseteq \bigcup_{i \geqslant 1} E_i$ then

   $$\pi^*(\varnothing) = \inf_{\{E_i\},\varnothing} \sum_{i \geqslant 1} \nu(E_i) \leqslant \sum_{i \geqslant 1} \nu(E_i) = 0 \tag{4.9}$$

   (b) $E_i \in a, \{E_i\}, \varnothing \subseteq \bigcup_{i \geqslant 1} E_i$, then

   $$\sum_{i \geqslant 1} \nu(E_i) \geqslant 0 \Rightarrow \pi^*(\varnothing) \geqslant 0 \tag{4.10}$$

2. check $E \subseteq F, \ \pi^*(E) \leqslant \pi^*(F)$

   Let's take any covering of $F$:$\{E_i\}, E_i \in a, F \subseteq \bigcup_{i \geqslant 1} E_i$ is also a covering of $E$, then

   $$\pi^*(E) = \inf_{\{E_i\},E} \sum_{i \geqslant 1} \nu(E_i) \leqslant \pi^*(F) = \inf_{\{E_i\},F} \sum_{i \geqslant 1} \nu(E_i) \tag{4.11}$$

3. check $E \subseteq \bigcup_{i \geqslant 1} E_i, \ \ \pi^*(E) \leqslant \sum_{i \geqslant 1} \pi^*(E_i)$
   (a) $\pi^*(E_i) = \infty$ then

   $$\pi^*(E) \leqslant \sum_{i \geqslant 1} \pi^*(E_i) \tag{4.12}$$

(b) $\pi^*(E_i) < \infty$, then

$$\pi^*(E_i) = \inf_{\{H_{ik}\},\, E_i} \sum_{k\geqslant 1} \nu(H_{ik}) \tag{4.13}$$

then there $\exists \{H_{ik}\} \in a, E_i \subseteq \bigcup\limits_{k\geqslant 1} H_{ik}$ such that

$$\pi^*(E_i) = \inf_{\{H_{ik}\},\, E_i} \sum_{k\geqslant 1} \nu(H_{ik}) \leqslant \sum_{k\geqslant 1} \nu(H_{ik}) \leqslant \pi^*(E_i) + \frac{\varepsilon}{2^i} \tag{4.14}$$

$\{H_{ik}\}$ is a covering of E, then

$$\pi^*(E) \leqslant \sum_{i,k} \nu(H_{ik}) \leqslant \sum_{i\geqslant 1} \left(\pi^*(E_i) + \frac{\varepsilon}{2^i}\right) \leqslant \sum_{i\geqslant 1} \pi^*(E_i) + \varepsilon \tag{4.15}$$

so

$$\pi^*(E) \leqslant \sum_{i\geqslant 1} \pi^*(E_i) \tag{4.16}$$

□

Step 2

**Definition 4.3** (Measurable set $\mathcal{M}$)**.** A set called measurable set $\mathcal{M}$ if $A \in \mathcal{M}$ $\forall E \in \Omega$, we have that

$$\pi^*(E) = \pi^*\left(E \bigcap A\right) + \pi^*\left(E \bigcap A^c\right) \tag{4.17}$$

**Theorem 4.3.** *If $\mathcal{M}$ definited as Def 4.3, then*

1. $\mathcal{M} \supseteq a$
2. $\mathcal{M}$ *is a* $\sigma-$*algebra*

**Remark 4.1.**

$$E \subseteq (E \cap A) \cup (E \cap A^c) \Rightarrow \pi^*(E) \leqslant \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.18}$$

so we only to check $\geq$ in Eq 4.17

*Proof.* $\pi^*$ is an outer measurable by Thm 4.1, then by subadditive of outer measure. □

Now we proof Thm 4.3.

*Proof.*

1. $a \in \mathcal{M}$

   Suppose that $A \in a$, $E \in \Omega$, we will show that

   $$\pi^*(E) \geqslant \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.19}$$

   assume that $\pi^*(E) < \infty$, given $\varepsilon, \exists \{E_i\}, E,$ *such that* $E_i \in a, E \subseteq \bigcup\limits_{i\geqslant 1} E_i$, then

   $$\pi^*(E) \leqslant \sum_{i\geqslant 1} \nu(E_i) \leqslant \pi^*(E) + \varepsilon \tag{4.20}$$

$E_i \cap A \in a, E \cap A \subseteq \bigcup\limits_{i \geqslant 1} (E_i \bigcap A)$, so

$$
\begin{aligned}
\pi^*(E \cap A) &\leqslant \sum_{i \geqslant 1} \nu\left(E_i \bigcap A\right) \\
\pi^*(E \cap A^c) &\leqslant \sum_{i \geqslant 1} \nu\left(E_i \bigcap A^c\right)
\end{aligned}
\tag{4.21}
$$

so

$$
\pi^*(E \cap A) + \pi^*(E \cap A^c) \leqslant \sum_{i \geqslant 1} \nu\left(E_i \bigcap A\right) + \sum_{i \geqslant 1} \nu\left(E_i \bigcap A^c\right) \leq \sum_{i \geqslant 1} \nu(E_i) \leqslant \pi^*(E) + \varepsilon
\tag{4.22}
$$

2. $\mathcal{M}$ is $\sigma$-algebra.

   We need to show that

   (a) $\Omega \in \mathcal{M}$

   It is clearly that:

$$
\pi^*(E) = \pi^*(E \cap \Omega) + \pi^*(E \cap \Omega^c) \tag{4.23}
$$

   (b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$

$$
\because \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.24}
$$

   (c) $A_i \in \mathcal{M} \Rightarrow \bigcup\limits_{i \geqslant 1} A_i \subseteq \mathcal{M}$

   Finite union is closed: $A, B \in \mathcal{F} \Rightarrow A \bigcup B \in M$. Let's take $E \subseteq \Omega$. We will proof that

$$
\pi^*(E) \geqslant \pi^*\left(E \cap \left(A \bigcup B\right)\right) + \pi^*\left(E \cap \left(A \bigcup B\right)^c\right) \tag{4.25}
$$

   $\because A \in \mathcal{M}$,

$$
\therefore \quad \pi^*(E) = \pi^*\left(E \bigcap A\right) + \pi^*\left(E \bigcap A^C\right) \tag{4.26}
$$

   $\because B \in \mathcal{M}$

$$
\begin{aligned}
\therefore \quad \pi^*(E \backslash A) &= \pi^*(E \backslash A \cap B) + \pi^*(E \backslash A \cap B^c) \\
&= \pi^*(E \backslash A \cap B) + \pi^*\left(E \backslash \left(A \bigcup B\right)\right)
\end{aligned}
\tag{4.27}
$$

   then

$$
\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \backslash A \cap B) + \pi^*(E \backslash (A \cup B)) \tag{4.28}
$$

   We want to show

$$
\pi^*(E \cap A) + \pi^*(E \backslash A \cap B) \geqslant \pi^*(E \cap (A \cup B)) \tag{4.29}
$$

   By $\pi^*$ is subadditive, we only to show that

$$
E \cap (A \cup B) \subseteq (E \cap A) \cup (E \backslash A \cap B) \tag{4.30}
$$

   this is because

$$
E \cap (A \cup B) = \underbrace{\{[E \cap (A \cup B)] \cap A\}}_{\subseteq E \cap A} \bigcup \underbrace{\{[E \cap (A \cup B)] \cap A^c\}}_{\subseteq (E \cap A^c) \cap B = (E \backslash A) \bigcap B} \tag{4.31}
$$

Then Eq 4.25 holds. So $\mathcal{M}$ is closed by finite(countable) union.

Now, we will show that countable infinite union is also closed. $A_i \in \mathcal{M}$, we want to show $A = \bigcup_{j \geqslant 1} A_j \in \mathcal{M}$, take $E \subseteq \Omega$,

$$\pi^* (E) \geqslant \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.32}$$

by Eq. 4.25, $\forall\ n$ we know that

$$\begin{aligned} \pi^* (E) &= \pi^* \left( E \cap \left( \bigcup_{j=1}^{n} A_j \right) \right) + \pi^* \left( E \cap \left( \bigcup_{j=1}^{n} A_j^c \right) \right) \\ &\geq \pi^* \left( E \cap \left( \bigcup_{j=1}^{n} A_j \right) \right) + \pi^* (E \backslash A) \end{aligned} \tag{4.33}$$

$\geq$ holds in Eq 4.33 because $(E \backslash A) \subseteq \left( E \backslash \left( \bigcup_{j=1}^{n} A_j \right) \right)$.

Now, we define

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_1 \backslash A_2 \\ F_3 &= A_1 \backslash (A_2 \cup A_3) \\ &\vdots \\ F_n &= A_1 \backslash (A_2 \cup \cdots \cup A_{n-1}) \\ &\vdots \end{aligned} \tag{4.34}$$

It is clear that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{n} F_j \tag{4.35}$$

where $F_j \cap F_k = \varnothing, F_j \in \mathcal{M}$.

Then Eq 4.33 can be written as

$$\pi^* (E) \geqslant \pi^* \left( E \cap \sum_{j=1}^{n} F_j \right) + \pi^* (E \backslash A) \tag{4.36}$$

By Remark 4.2, we have

$$\begin{aligned} \pi^* (E) &\geqslant \pi^* \left( E \cap \left( \sum_{j=1}^{n} F_j \right) \right) + \pi^* (E \backslash A) \\ &= \sum_{j=1}^{n} \pi^* (E \cap F_j) + \pi^* (E \backslash A) \end{aligned} \tag{4.37}$$

$\because$ $n$ is any number in Remark 4.2 , $\therefore$ $\pi^*\left(E \cap \sum_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} \pi^*(E \cap F_j)$, by $\pi^*$ is subadditive

$$\begin{aligned}
\pi^*(E) &\geqslant \pi^*\left(E \cap \sum_j F_j\right) + \pi^*(E \backslash A) \\
&= \sum_{j \geqslant 1} \pi^*(E \cap F_j) + \pi^*(E \backslash A) \\
&\geqslant \pi^*\left(\bigcup_{j \geqslant 1} (E \cap F_j)\right) + \pi^*(E \backslash A) \\
&= \geqslant \pi^*\left(E \cap \left(\bigcup_{j \geqslant 1} F_j\right)\right) + \pi^*(E \backslash A) \\
&= \pi^*(E \cap A) + \pi^*(E \backslash A)
\end{aligned} \tag{4.38}$$

So $\mathcal{M}$ is a $\sigma-$algebra.

□

**Remark 4.2.** $\forall n$, we have that

$$\pi^*\left(E \cap \sum_{j=1}^{n} F_j\right) = \sum_{j=1}^{n} \pi^*(E \cap F_j) \tag{4.39}$$

where $F_j$ defined as Eq 4.34.

*Proof.* By induction

1. $n = 1$, Eq 4.39 holds
2. Support $n$ holds then we will proof $n+1$ holds. $F_k \in \mathcal{M}, F_{n+1} \in \mathcal{M}$, we have that

$$\begin{aligned}
\pi^*\left(E \cap \sum_{j=1}^{n+1} F_j\right) &= \pi^*\left(\left(E \cap \sum_{j=1}^{n+1} F_j\right) \cap F_{n+1}\right) + \pi^*\left(\left(E \cap \sum_{j=1}^{n+1} F_j\right) \cap F_{n+1}^c\right) \\
&= \pi^*(E \cap F_{n+1}) + \underbrace{\pi^*\left(E \cap \sum_{j=1}^{n} F_j\right)}_{by\ \ assumption\ \ = \sum_{j=1}^{n} \pi^*(E \cap F_j)} \\
&= \sum_{j=1}^{n+1} \pi^*(E \cap F_j)
\end{aligned} \tag{4.40}$$

□

By Thm 4.3 we have that $\mathcal{M} \supseteq \mathcal{F}(a)$.

Step 3

**Theorem 4.4.** *$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$ is $\sigma-$ additive, then*

$$\pi^* (A) = \nu (A) , \quad \forall A \in a \tag{4.41}$$

**Remark 4.3.** Eq 4.41 is also

$$\pi^*|_a = v \tag{4.42}$$

Eq 4.2 holds because Thm 4.3, $a \in \mathcal{M}$.

*Proof.* (Thm 4.4)

1. $\pi^* (A) = \nu (A) , \ \forall A \in a$
   (a) check $\pi^* (A) \leqslant \nu (A)$

   Let's $\underbrace{A}_{E_1}, \underbrace{\varnothing}_{E_2}, \underbrace{\varnothing}_{E_3}, \underbrace{\cdots}_{E_j}$

   $$\pi^* (A) = \inf_{\{E_i\},A} \sum_i \nu (E_i) \leqslant \sum_i \nu (E_i) = \nu (A) \tag{4.43}$$

   (b) check $\pi^* (A) \geqslant \nu (A)$

   Let's take

   $$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \backslash E_1 \\ F_3 &= E_3 \backslash (E_1 \cup E_2) \\ &\vdots \\ F_n &= E_n \backslash (E_1 \cup E_2 \cup \cdots \cup E_{n-1}) \\ &\vdots \end{aligned} \tag{4.44}$$

   $F_j \in a, \bigcup\limits_j F_j = \bigcup\limits_j E_j, F_j \cap F_k = \varnothing, A \subseteq \bigcup\limits_{j \geqslant 1} F_j$, so $A = \sum\limits_j F_j \cap A \in a$.

   Because $\nu$ is $\sigma-$additive we have that

   $$\nu (A) = \sum_{j \geqslant 1} \nu (F_j \cap A) \tag{4.45}$$

   $\because F_j \subseteq E_j$

   $$\nu (A) = \sum_{j \geqslant 1} \nu (F_j \cap A) \leqslant \sum_{j \geqslant 1} \nu (E_j) \tag{4.46}$$

   so

   $$\nu (A) \leqslant \inf_{\{E_i\},A} \sum_{j \geqslant 1} \nu (E_j) = \pi^* (A) \tag{4.47}$$

Then, we can get

$$\pi^* (A) = \nu (A), \ \forall A \in a \tag{4.48}$$

2. $\pi^*|_{\mathcal{M}}$ is $\sigma-$additive

   Suppose that $A_j \in \mathcal{M}, A_j \cap A_k = \varnothing$, we want to proof that

   $$\pi^* \left(\sum A_j\right) = \sum_{j \geqslant 1} \pi^* (A_j) \tag{4.49}$$

   (a) check $\pi^* (\sum A_j) \leqslant \sum\limits_{j \geqslant 1} \pi^* (A_j)$ by $\pi^*$ is an outer measure, $\pi^*$ is subadditive

   (b) check $\pi^* (\sum A_j) \geqslant \sum\limits_{j \geqslant 1} \pi^* (A_j)$

   by $\pi^*$ is an outer measure, $\pi^*$ is monotone

   $$\pi^* \left(\sum_{j \geqslant 1} A_j\right) \geqslant \pi^* \left(\sum_{j=1}^{n} A_j\right) \tag{4.50}$$

   by Remark 4.2, we have that

   $$\pi^* \left(\sum_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \pi^* (A_j), \quad \forall n \tag{4.51}$$

   so

   $$\pi^* \left(\sum_{j \geqslant 1} A_j\right) \geqslant \sum_{j \geqslant 1} \pi^* (A_j) \tag{4.52}$$

□

Step 4

**Definition 4.4.** $\Omega$ is $\sigma$-finite($\mu_1$) if $E_j \uparrow \Omega, \mu_1 (E_j) < \infty, \ \forall j, \ E_j \in a$.

**Theorem 4.5** (Uniqueness)**.** *Suppose that* $\mu_1, \mu_2 : \mathcal{F}(a) \to R_+ \cup \{+\infty\}, \Omega$ *is* $\sigma$*-finite*($\mu_1$)*, if* $\mu_1|_a = \mu_2|_a$*, then*

$$\mu_1 = \mu_2, \quad on \ \ \mathcal{F}(a) \tag{4.53}$$

**Definition 4.5.** $\Omega$, $\mathcal{G} \subseteq \mathcal{P}(\Omega), \mathcal{G}$ is a monotone class if

1. $$A_j \in \mathcal{G}, j \geqslant 1, A_j \subseteq A_{j+1} \Rightarrow A = \bigcup_{j \geqslant 1} A_j = \lim_{j \to \infty} A_j \in \mathcal{G} \tag{4.54}$$

2. $$B_j \in \mathcal{G}, j \geqslant 1, B_j \supseteq B_{j+1} \Rightarrow B = \bigcap_{j \geqslant 1} B_j = \lim_{j \to \infty} B_j \in \mathcal{G} \tag{4.55}$$

**Theorem 4.6.** *$\mathcal{G}_\alpha$ is a monotone class, $\alpha \in I$, then the followings hold*

1. *$\bigcap\limits_{\alpha \in I} \mathcal{G}_\alpha$ is a monotone class*
2. *$c \subseteq \mathcal{P}(\Omega) \Rightarrow \mathcal{G}(c) = \bigcap\limits_{\alpha \in I} \mathcal{G}_\alpha$ , i.e. monotone classes generated by class $c$*

**Lemma 4.1.** *$a \subseteq \mathcal{P}(\Omega)$ is an algebra, $\mu(a)$ is monotone class generated by algebra $a$, $\mathcal{F}(a)$ is a $\sigma-$algebra generated by algebra $a$, then*

$$\mu(a) = \mathcal{F}(a) \tag{4.56}$$

*Proof.* It will proof in the next lecture. □

*Proof.* (Thm 4.5) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\}$, $\mu_1(A) = \mu_2(A), \forall A \in a$, $\Omega$ $\sigma$-finite, $\Omega = \bigcup\limits_{j \geqslant 1} E_j, E_j \in a, \mu_j(E_j) < \infty$, then $\mu_1 = \mu_2$ on $\mathcal{F}(a)$.

Fix $E_n$, we denote that

$$\mathcal{B}_n = \{E \in \mathcal{F}(a), \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\} \tag{4.57}$$

We claim that

1. $\mathcal{B}_n \supseteq a$
2. $\mathcal{B}_n$ is a monotone class

We proof $\mathcal{B}_n$ is a monotone class.

1. $\forall A_j \in \mathcal{B}_n, A_j \uparrow A = \bigcup\limits_{j \geqslant 1} A_j$, then

$$\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n) \tag{4.58}$$

   By Remark 3.1

$$\mu_1(A_j \cap E_n) \to \mu_1(A \cap E_n), \mu_2(A_j \cap E_n) \to \mu_2(A \cap E_n) \tag{4.59}$$

2. $\forall B_j \in \mathcal{B}_n, B_j \downarrow B = \bigcap\limits_{j \geqslant 1} B_j$, then

$$\mu_1(B_j \cap E_n) = \mu_2(B_j \cap E_n) \tag{4.60}$$

   By Remark 3.1

$$\mu_1(B_j \cap E_n) \to \mu_1(B \cap E_n), \mu_2(B_j \cap E_n) \to \mu_2(B \cap E_n) \tag{4.61}$$

So we can get that

$$\mathcal{B}_n \supseteq \mathcal{M}(a) \tag{4.62}$$

where $\mathcal{M}(a)$ is a monotone class generated by $a$. Then by Lemma 4.1

$$\mathcal{M}(a) = \mathcal{F}(a) \tag{4.63}$$

And by Eq 4.57,

$$\mathcal{B}_n\left(a\right) \subseteq \mathcal{F}\left(a\right) \tag{4.64}$$

so

$$\mathcal{B}_n\left(a\right) = \mathcal{F}\left(a\right) \tag{4.65}$$

Finally, $\mu_1\left(A\right) = \mu_2\left(A\right), \forall A \in \mathcal{F}\left(a\right)$, by $\mathcal{B}_n = \mathcal{F}(a)$, then $A \in \mathcal{B}_n$. $B_j \uparrow \Omega$, apply Lemma 3.1 again, we have

$$\mu_1\left(A\right) = \mu_2\left(A\right) \tag{4.66}$$

$\square$

# Lecture 5

## Monotone Classes

**Definition 5.1.** Given $\Omega$, define $\mathcal{M}(a) \subseteq \mathcal{P}(\Omega)$ is a monotone class is

1. $A_j \in \mathcal{M}, A_j \uparrow A \left( A_j \subseteq A_j, \bigcup\limits_{j \geqslant 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$
2. $A_j \in \mathcal{M}, A_j \downarrow A \left( A_j \supseteq A_j, \bigcap\limits_{j \geqslant 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$

**Remark 5.1.**

1. $\mathcal{F}$ is $\sigma$-filed($\sigma$-algebra) $\Rightarrow \mathcal{F}$ is a monotone class
2. $\mathcal{M}_\alpha \subseteq P(\Omega)$, $(\alpha \in I)$ is monotone class, then $\mathcal{M} = \bigcap\limits_{\alpha \in I} \mathcal{M}_\alpha$ is a monotone class.

**Notation 5.1.** (Smallest monotone class contain $c$) $\mathcal{M}(c)$ is a monotone class generated by $c$ if

$$c \subseteq \mathcal{M}(\Omega), \mathcal{M}(c) = \bigcap_{\alpha \in I} \mathcal{M}_\alpha \tag{5.1}$$

**Definition 5.2.** $E \subseteq \mathcal{M}(a)$, the set $\mathcal{G}(E)$ is defined as below

$$\mathcal{G}(E) = \{F \in \mathcal{M}(a), E \backslash F, E \cap F, F \backslash E \in \mathcal{M}(a)\} \tag{5.2}$$

**Lemma 5.1.**

1. *If $E \in a \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$*
2. *If $E \in \mathcal{M}(a) \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$*

*Proof.*

1. $E \in a$, we want to show that
   (a) $\mathcal{G}(E) \supseteq a$

   Take $H \in a \subseteq \mathcal{M}(a)$, then

$$\underbrace{E \backslash H}_{\in a}, \underbrace{E \cap H}_{\in a}, \underbrace{H \backslash E}_{\in a} \in \mathcal{G}(a) \tag{5.3}$$

   so $H \in \mathcal{G}(E)$, then $a \subseteq \mathcal{G}(E)$

   (b) $\mathcal{G}(E)$ is a monotone class

   Suppose that $H_k \uparrow H$, $H_k \in \mathcal{G}(E)$,

$$\because E \backslash H_k \in \mathcal{M}(a), \ E \backslash H_k \to E \backslash H, \therefore E \backslash H \in \mathcal{M}(a) \tag{5.4}$$

$$\because E \cap H_k \in \mathcal{M}(a), \; E \cap H_k \to E \cap H, \therefore E \cap H \in \mathcal{M}(a) \tag{5.5}$$

$$\because H_k \backslash E \in \mathcal{M}(a), \; H_k \backslash E \to H \backslash E, \therefore H \backslash E \in \mathcal{M}(a) \tag{5.6}$$

By Eq 5.6, $H \in \mathcal{M}(a)$, and by the definition 5.2, $H \in \mathcal{G}(E)$. So $\mathcal{G}(E)$ is a monotone class. We also get that $\mathcal{G}(E) \supseteq \mathcal{M}(a)$.

2. $E \in \mathcal{M}(a)$, we want to show that

   (a) $\mathcal{G}(E)$ is a monotone class

   $E \in \mathcal{M}(a)$, suppose $H_k \in \mathcal{G}(E), H_k \uparrow H$

   $$\because E \backslash H_k \in \mathcal{M}(a), E \backslash H_k \downarrow E \backslash H \quad \therefore E \backslash H \in \mathcal{M}(a) \tag{5.7}$$

   Similarity:

   $$E \cap H \in \mathcal{M}(a) \tag{5.8}$$

   $$H \backslash E \in \mathcal{M}(a) \tag{5.9}$$

   then we can get $H \in \mathcal{G}(E)$, so $\mathcal{G}(E)$ is a monotone class.

   (b) $\mathcal{G}(E) \supseteq a$

   We need to show $H \in a \Rightarrow H \in \mathcal{G}(E)$.

   By Lemma 5.1.1, we can get that

   $$\mathcal{G}(H) \supseteq \mathcal{M}(a) \tag{5.10}$$

   $\because E \in \mathcal{M}(a), \therefore E \in \mathcal{G}(H)$, by the Def 5.2, $H \backslash E, H \cap E, E \backslash H \in \mathcal{M}(a)$, so we can get $a \in \mathcal{G}(E)$

□

**Theorem 5.1.** *$a$ is a algebra, $a \subseteq \mathcal{P}(\Omega)$. $\mathcal{F}(a)$ is a $\sigma$-algebra generated by $a$, $\mathcal{M}(a)$ is a monotone class generated by $a$, then*

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.11}$$

*Proof.* By remark 5.1, $\mathcal{F}(a)$ is a monotone class, by Notation 5.1 $\mathcal{F}(a) \supseteq a$ and $\mathcal{F}(a) \supseteq \mathcal{M}(a)$.

So we have to show that

$$\mathcal{F}(a) \subseteq \mathcal{M}(a) \tag{5.12}$$

We will show that

1. $\mathcal{M}(a)$ is a algebra

   (a) $\Omega \in \mathcal{M}(a)$ by $\Omega \subseteq a$

   (b) $E \in \mathcal{M}(a) \Rightarrow E^c \in \mathcal{M}(a)$

   By Lemma 5.1.1, let $E = \Omega$, then $\mathcal{M}(a) \subseteq \mathcal{G}(\Omega)$. $\because E \in \mathcal{M}(a)$, so $E \in \mathcal{G}(\Omega)$ . By Definition 5.2, $\mathcal{G}(\Omega) = \{E \in \mathcal{M}(a), E^c, E, \varnothing \in \mathcal{M}(a)\}$

(c) $E, F \in \mathcal{M}(a) \Rightarrow E \cap F \in \mathcal{M}(a)$

By Lemma 5.1.2, $\mathcal{G}(E) \supseteq \mathcal{M}(a)$, so $F \in \mathcal{G}(E)$.

By Def 5.2 $F \in \mathcal{G}(E) = \{F \in \mathcal{M}(a), F \backslash E, F \cap E, E \backslash F \in \mathcal{M}(a)\}$, so $E \bigcap F \in \mathcal{M}(a)$

2. $\mathcal{M}(a)$ is a $\sigma$-algebra i.e. $A_j \in \mathcal{M}(a)\,,\ j \geqslant 1\ \Rightarrow \bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}(a)$

   By $\mathcal{M}(a)$ is a algebra, so $\bigcup\limits_{j=1}^{n} A_j \in \mathcal{M}(a)$.

   $\bigcup\limits_{j=1}^{n} A_j \uparrow \bigcup\limits_{j \geqslant 1} A_j$ and $\mathcal{M}(a)$is a monotone class, so $\bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}(a)$.

So $\mathcal{F}(a) \subseteq \mathcal{M}(a)$.

Above all,

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.13}$$

□

# Lecture 6

## The Lebesgue Measure I

**Definition 6.1.** $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$, we define $\mathcal{S}$ as below:

$$\mathcal{S} = \{\varnothing, \mathbb{R}, (a, b], (a, \infty), (-\infty, b]\} \tag{6.1}$$

**Remark 6.1.** $\mathcal{S}$ as above, then $\mathcal{S}$ is a semialgebra

*Proof.* by Def 2.1. $\square$

**Definition 6.2.** $\mu : \mathcal{S} \to \mathbb{R}_+ \bigcup \{+\infty\}$, additive, and

$$\mu(\varnothing) = 0, \mu((a, b]) = b - a, \mu((-\infty, b]) = +\infty, \mu(\mathbb{R}) = +\infty \tag{6.2}$$

**Theorem 6.1.** *$\mu$ is additive on a semialgebra $\mathcal{S}$ and defined as Def 6.2, then $\mu$ is $\sigma-$additive, i.e.*

$$A = \sum_{j \geqslant 1} A_j \Rightarrow \mu(A) = \sum_{j \geqslant 1} \mu(A_j), \qquad A, A_j \in \mathcal{S} \tag{6.3}$$

**Remark 6.2.** It is difficult to prove Thm 6.1 $(a, b] \cup (c, d]$ is not in the semialgebra $\mathcal{S}$. But, $\mathcal{S} \to a(\mathcal{S})$ with respect to $\mu \to \nu$.

*Proof.*

1. $$\because A = \sum_{j \geqslant 1} A_j \supseteq \sum_{j=1}^{n} A_j \tag{6.4}$$

   By $\nu$ is additive $\Rightarrow$ $\nu$ is monotone & subadditive,

   $$\therefore \nu(A) \geqslant \nu\left(\sum_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \nu(A_j), \quad \forall n \tag{6.5}$$

   so

   $$\therefore \nu(A) \geqslant \sum_{j \geqslant 1} \nu(A_j) \tag{6.6}$$

2. (a) Assume that $A = (a, b], A_j = (a_j, b_j], A = \sum\limits_{j \geqslant 1} A_j$, we want to show that

   $$\nu(A) = b - a \leqslant \sum_{j \geqslant 1} (b_j - a_j) = \sum_{j \geqslant 1} \nu(A_j) \tag{6.7}$$

For any given $\epsilon > 0$, we have that

$$[a+\varepsilon, b] \subseteq (a, b] = \sum_{j\geqslant 1} (a_j, b_j] \subseteq \bigcup_{j\geqslant 1} \left(a_j, b_j + \frac{\varepsilon}{2^j}\right) \tag{6.8}$$

By a set $K$ is compact i.e. $K$ is closed and bounded $\Rightarrow$ Any open cover for $K$ has a finite subcover

$$[a+\varepsilon, b] \subseteq \bigcup_{k\geqslant 1} \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}}\right) \tag{6.9}$$

By $\nu$ is additive $\Rightarrow$ $\nu$ is monotone & subadditive, we have

$$b-a-\varepsilon \leqslant \nu\left([a+\varepsilon, b]\right) = \nu\left(\bigcup_{k=1}^{m} \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}}\right)\right) \leqslant \sum_{k=1}^{m} \nu\left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}}\right) \tag{6.10}$$

so we can get that

$$b-a-\varepsilon \leqslant \sum_{k=1}^{m} \left(b_{jk} - a_{jk} + \frac{\varepsilon}{2^{jk}}\right) \leqslant \sum_{j\geqslant 1} \left(b_j - a_j + \frac{\varepsilon}{2^j}\right) = \sum_{j\geqslant 1} (b-a) + \varepsilon \tag{6.11}$$

so Eq. 6.7 holds.

(b) General case $A \in \mathcal{S}$, $E_n = (-n, n] \uparrow \mathbb{R}$.

$A \cap E_n = \sum\limits_{j\geqslant 1} A_j \cap E_n$.

By $\nu$ is additive on a semi-algebra

$$\nu\left(A \cap E_n\right) = \sum_{j\geqslant 1} \nu\left(A_j \cap E_n\right) \leqslant \sum_{j\geqslant 1} \nu\left(A_j\right) \tag{6.12}$$

By Remark 6.3, let $n \to \infty$, we have

$$\nu\left(A\right) = \lim_{n\to\infty} \nu\left(A \cap E_n\right) \leqslant \sum_{j\geqslant 1} \nu\left(A_j\right) \tag{6.13}$$

□

**Remark 6.3.** $E_n = (-n, n] \uparrow \mathbb{R}$, $\nu$ is additive on a semi-algebra then

$$\nu\left(A\right) = \lim_{n\to\infty} \nu\left(A \cap E_n\right) \tag{6.14}$$

*Proof.*

$$\because E_n \uparrow \mathbb{R}, \therefore A \cap E \uparrow, \therefore \lim_{n\to\infty} \left(A \cap E_n\right) = \bigcup_{n\geqslant 1} \left(A \cap E_n\right) = A \cap \left(\bigcup_{n\geqslant 1} E_n\right) = A \tag{6.15}$$

$\nu$ is additive,

$$\nu\left(A\right) = \nu\left(\bigcup_{n\geqslant 1} A \cap E_n\right) = \nu\left(\lim_{n\to\infty} A \cap E_n\right) \overset{why}{=} \lim_{n\to\infty} \nu\left(A \cap E_n\right) \tag{6.16}$$

why , because we will check via Def 6.1 except $A = (a, b]$

1. $A = \varnothing$
2. $A = \mathbb{R}$
3. $A = (a, \infty)$
   (a) left hand of why in Eq. 6.16

$$\because A \cap E_n = (a, +\infty) \cap (-n, n) = \begin{cases} (a, n) & a \geqslant -n \\ (-n, n) & a < -n \end{cases} \tag{6.17}$$

$$\therefore \lim_{n \to \infty} (A \cap E_n) = (-\infty, +\infty) = \mathbb{R} \tag{6.18}$$

   by Def 6.2

$$\mu \left( \lim_{n \to \infty} (A \cap E_n) \right) = \mu(\mathbb{R}) = +\infty \tag{6.19}$$

   (b) right hand of why in Eq. 6.16

$$\because \nu (A \cap E_n) = \nu \left( \begin{cases} (a, n) & a \geqslant -n \\ (-n, n) & a < -n \end{cases} \right) = \begin{cases} n - a & a \geqslant -n \\ 2n & a < -n \end{cases} \tag{6.20}$$

$$\therefore \lim_{n \to \infty} \nu (A \cap E_n) = \lim_{n \to \infty} \begin{cases} n - a & a \geqslant -n \\ 2n & a < -n \end{cases} = +\infty \tag{6.21}$$

   So Eq 6.16 holds.
4. $A = (-\infty, b]$

$\square$

# Lecture 7

## The Lebesgue Measure II

$\mathcal{S} = \{\varnothing, \mathbb{R}, (a,b], (a,\infty), (-\infty,b]\}$, $\mu : a(\mathcal{S}) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\mu((a,b]) = b - a \tag{7.1}$$

**Theorem 7.1.** *$\mu$ is $\sigma$-additive on $a(\mathcal{S})$*

**Remark 7.1.** $E_k \in (-N,N]$, $\mu$ is finite and $\mu$ is continuous from below at $\varnothing$ (i.e. $E_k \in a, E_k \downarrow \varnothing \Rightarrow \mu(E_k) \to 0$), by Lemma 3.1 can imply Thm 7.1 hold.

*Proof.* Now we want to show that $E_k \downarrow \varnothing, E_k \in a, E_k \in (-N,N]$, then

$$\mu(E_k) \to 0 \tag{7.2}$$

If not, $\exists \delta > 0$, $\exists E_k \downarrow \varnothing, E_k \in a, E_k \in (-N,N]$, such that

$$\mu(E_k) \geqslant 2\delta > 0 \tag{7.3}$$

If $\exists$ a compact set $\{G_k\}$, *s.t.* $G_k \supseteq G_{k+1}, G_k \subseteq E_k$, but

$$\varnothing \neq \bigcap_{k\geqslant 1} G_k \subseteq \bigcap_{k\geqslant 1} E_k = \varnothing \tag{7.4}$$

Then, we will find a sequence of compact sets $\{G_k\}$ by induction.

Our goal is : $E_k \subseteq (-N,N], \mu(E_n) \geqslant 2\delta, (F_k)_{1\leqslant k\leqslant M} G_k = \overline{F_k}$. $F_k$ satisfy the flowing three conditions:

1. $\overline{F_k} \subseteq E_k, \quad 1 \leqslant k \leqslant n-1$
2. $F_{k+1} \subseteq F_k, \quad 1 \leqslant k \leqslant n-1$
3. $\mu(E_n \backslash F_n) \leqslant \frac{\delta}{2} + \frac{\delta}{4} + \cdots + \frac{\delta}{2^n} = \delta$

Now,

1. by $E_1 \in a$, then $E_1$ can be written as

$$E_1 = \sum_{j=1}^{n_1} (a_{1,j}, b_{1,j}] \tag{7.5}$$

define $F_1$ as

$$F_1 = \sum_{j=1}^{n_1} (a_{1,j} + \varepsilon_1, b_{1,j}] \in a \tag{7.6}$$

$\mu(E_1 \backslash F_1) = m_1 \varepsilon_1$.

We will pick a small enough $\epsilon$ to meet $\mu(E_1 \backslash F_1) \leqslant \frac{\delta}{2}$, *i.e.* $m_1\varepsilon_1 \leqslant \frac{\delta}{2}$, and $b_{1,j} - a_{1,j} \geqslant \varepsilon_1$, *i.e.* $\min_j \{b_{1,j} - a_{1,j}\} \geqslant \varepsilon_1$, so we choose $0 < \varepsilon_1 \leqslant \min\left\{\frac{\delta}{2m_1}, \min_{1\leqslant j\leqslant m_1} \{b_{1,j} - a_{1,j}\}\right\}$.

2. We will show $\mu(E_2 \cap F_1)$ have a lower positive bound , i e. $E_2 \cap F_1 \neq \varnothing$

$$2\delta \leqslant \mu(E_2) = \mu(E_2 \cap F_1) + \underbrace{\mu(E_2 \backslash F_1)}_{\leqslant \mu(E_1 \backslash F_1) \leqslant \frac{\delta}{2}} \Rightarrow \mu(E_2 \cap F_1) \geqslant 2\delta - \frac{\delta}{2} > 0 \tag{7.7}$$

by $E_2 \cap F_1 \neq \varnothing, E_2 \cap F_1 \in a$, then $E_2 \cap F_1$ can be written as

$$E_2 \cap F_1 = \sum_{j=1}^{m_2} (a_{2,j}, b_{2,j}] \tag{7.8}$$

Define $F_2$ :

$$F_2 = \sum_{j=1}^{m_2} (a_{2,j} + \varepsilon_2, b_{2,j}] \tag{7.9}$$

choose a small enough $\epsilon_2$ satisfies that

$$F_2 \subseteq \overline{F_2} \subseteq E_2 \cap F_1 \tag{7.10}$$

then $F_2 \subseteq F_1, \overline{F_2} \subseteq E_2$, and $F_2 \subseteq F_1 \Rightarrow \overline{F_2} \subseteq \overline{F_1}$, then we get that

$$\begin{aligned} F_2 &\subseteq \overline{F_2} \subseteq E_2 \\ F_2 &\subseteq F_1 \\ \mu(E_2 \backslash F_2) &\leqslant \frac{\delta}{2} + \frac{\delta}{4} \end{aligned} \tag{7.11}$$

3. assume the $F_n$ satisfies the three conditions as our goal above

$$2\delta \leqslant \mu(E_{n+1}) = \mu(E_{n+1} \cap F_n) + \underbrace{\mu(E_{n+1} \backslash F_n)}_{\mu(E_n \backslash F) \leqslant \delta} \Rightarrow \mu(E_{n+1} \cap F_n) \geqslant \delta > 0 \tag{7.12}$$

by $E_{n+1} \cap F_n \neq \varnothing$ and $E_{n+1} \cap F_n \in a$ then

$$E_{n+1} \cap F_n = \sum_{j=1}^{k_{n+1}} (a_{n+1,j}, b_{n+1,j}] \tag{7.13}$$

then we define $F_{n+1}$ as

$$F_{n+1} = \sum_{j=1}^{k_{n+1}} (a_{n+1,j} + \varepsilon_{n+1}, b_{n+1,j}] \tag{7.14}$$

choose a small enough $\epsilon_{n+1}$ satisfies that

$$F_{n+1} \subseteq \overline{F_{n+1}} \subseteq E_{n+1} \cap F_n \tag{7.15}$$

then $F_{n+1} \subseteq E_{n+1}, F_{n+1} \subseteq F_n$, and $\overline{F_{n+1}} \subseteq \overline{F_n}$, let $\varepsilon_{n+1} = \frac{\delta}{k_{n+1} \cdot 2^{n+1}}$, then $\mu((E_{n+1} \cap F_n) \backslash F_{n+1}) \leqslant \frac{\delta}{2^{n+1}}$.

Then

$$\mu\left(E_{n+1}\backslash F_{n+1}\right)=\mu\left(\left(E_{n+1}\cap F_n\right)\backslash F_{n+1}\right)+\underbrace{\underbrace{\mu\left(\left(E_{n+1}\backslash F_n\right)\backslash F_{n+1}\right)}_{\leqslant\,\mu\left(E_{n+1}\backslash F_n\right)}}_{\leqslant\mu(E_n\backslash F_n)\leqslant\frac{\delta}{2}+\cdots+\frac{\delta}{2^n}}$$
$$\leq\frac{\delta}{2^{n+1}}+\frac{\delta}{2}+\frac{\delta}{4}+\cdots+\frac{\delta}{2^n}=\delta\left(1-\left(\frac{1}{2}\right)^{n+1}\right) \tag{7.16}$$

define $G_k=\overline{F_k}$, then $G_{k+1}=\overline{F_{k+1}}\subseteq\overline{F_k}=G_k$ $G_k$ : satisfies that

(a) $G_{k+1}\subseteq G_k$

(b) $G_k$ compact

(c) $G_k\neq\varnothing$

Why $G_k\neq\varnothing$ because:

$$2\delta\leqslant\mu\left(E_k\right)=\mu\left(E_k\backslash F_k\right)+\mu\left(E_k\cap F_k\right)\leqslant\delta+\mu\left(F_k\right)\Rightarrow\mu\left(F_k\right)\geq\delta \tag{7.17}$$

Then $F_k\neq\varnothing\Rightarrow G_k=\overline{F_k}\neq\varnothing$.

But

$$\varnothing\neq\bigcap_{k\geqslant1}G_k\subseteq\bigcap_{k\geqslant1}E_k=\varnothing \tag{7.18}$$

Above all, $E_k\in(-N,N]$, $\mu$ is finite and $\mu$ is continuous from below at $\varnothing$, then Lebesgue $\mu$ is $\sigma$-additive on $a(\mathcal{S})$. $\square$

# Lecture 8

## Complete Measures

**Definition 8.1.** $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is $\sigma$-algebra, $\mu : \mathcal{F} \to \mathbb{R}_+ \bigcup \infty$ is additive. $(\mu, \mathcal{F})$ is complete if : $A \in \mathcal{F}$ such that $\mu(A) = 0$, $\forall E \subseteq A$ then $E \in \mathcal{F}$.

**Remark 8.1.** In Def 8.1, by monotone $\mu(E) = 0$.

Next, our goal is: $\overline{\mathcal{F}} \supseteq \mathcal{F}$, and $\overline{\mu} : \overline{\mathcal{F}} \to \mathbb{R}_+ \cup \{+\infty\}$: $\begin{cases} \overline{\mu}|_{\mathcal{F}} = \mu, \\ \left(\overline{\mu}, \overline{\mathcal{F}}\right) \ \textit{is complete} \end{cases}$

**Definition 8.2.** $\overline{\mathcal{F}} = \{A \cup N, \ \textit{where } A \in \mathcal{F} \textit{ and } N \subseteq E \in \mathcal{F}, \ \textit{such that } \mu(E) = 0\}$

**Claim 8.1.** $\overline{\mathcal{F}}$ is a $\sigma$-algebra.

*Proof.* We will check :

1. $\Omega \in \overline{\mathcal{F}}$, $\because \Omega = \Omega \cup \varnothing, \varnothing \subseteq \varnothing \in \mathcal{F}$
2. $A \in \overline{\mathcal{F}} \Rightarrow A^c \in \overline{\mathcal{F}}$

   $\because A \subseteq \overline{\mathcal{F}}, A = E \cup N \textit{ where } E \in \mathcal{F}, \ N \subseteq H \in \mathcal{F} \textit{ such that } \mu(H) = 0$

$$\begin{aligned} A^c &= (E \cup N)^c \\ &= \underbrace{[(E \cup N)^c \cap H]}_{\subseteq H} \cup \underbrace{[(E \cup N)^c \cap H^c]}_{\underbrace{E^c \cap N^c \cap H^c}_{\subseteq E^c \cap H^c \in \mathcal{F}}} \end{aligned} \tag{8.1}$$

   by Def 8.2, $A^c \in \overline{\mathcal{F}}$.

3. $A_j = E_j \cup H_j \ \textit{ where } \ E_j \in \mathcal{F}, H_j \subseteq W_j \textit{ where } w_j \in \mathcal{F}, \mu(W_j) = 0$ then $\bigcup\limits_{j \geqslant 1} A_j \in \overline{\mathcal{F}}$

$$\begin{aligned} \because \bigcup_{j \geqslant 1} A_j &= \bigcup_{j \geqslant 1} (E_j \cup H_j) \\ &= \underbrace{\bigcup_{j \geqslant 1} E_j}_{\mathcal{F}} \cup \underbrace{\bigcup_{j \geqslant 1} H_j}_{\subseteq \bigcup\limits_{j \geqslant 1} W_j \triangleq W} \end{aligned} \tag{8.2}$$

   and $\mu(W) = \mu\left(\bigcup\limits_{j \geqslant 1} W_j\right) \leqslant \sum\limits_{j \geqslant 1} \mu(W_j) = 0$

$\square$

We want to define $\overline{\mu}$ *on* $\overline{\mathcal{F}}$ :

$$\because \quad \underbrace{\overline{\mu}(A \cup N)}_{\geqslant \overline{\mu}(A) = \mu(A)} \leqslant \overline{\mu}(A \cup E) \leqslant \underbrace{\overline{\mu}(A) + \overline{\mu}(E)}_{= \mu(A) + \mu(E) = \mu(A)} \tag{8.3}$$

So we give the following definition.

**Definition 8.3.** $\overline{\mu}(A \cup N) = \mu(A)$

*Proof.* By the Def 8.3

1. check $\overline{\mu}$ is well defined

   Assume that $A \cup N = B \cup M,\ where\ A, B \in \mathcal{F}, N \subseteq E \in \mathcal{F}\ where\ \mu(E) = 0,\ M \subseteq F \in \mathcal{F}\ where\ \mu(F) = 0$. We need to show that $\mu(A) = \mu(B)$.

   $$\because A \subseteq A \cup N = B \cup M \subseteq B \cup M \tag{8.4}$$

   by $\mu$ is $\sigma-$additive, then $\mu$ is monotone,

   $$\mu(A) \leqslant \mu(B \cup F) \leqslant \mu(B) + \mu(F) = \mu(B) \tag{8.5}$$

   similarly, $\mu(B) \leqslant \mu(A)$.

2. check $\overline{\mu}|_{\mathcal{F}} = \mu$

   by $A \in \mathcal{F},\ A = A \bigcup \varnothing$ then $\overline{\mu}(A \cup \varnothing) = \mu(A)$

3. check $\overline{\mu}$ is $\sigma-$additive i.e. $A_j \in \overline{\mathcal{F}},\ A = \sum\limits_{j \geqslant 1} A_j \Rightarrow \overline{\mu}(A) = \sum\limits_{j \geqslant 1} \mu(A_j)$

   $$\because A_j \in \overline{\mathcal{F}}, \therefore A_j = E_j \cup N_j\ where\ E_j \in \mathcal{F},\ N_j \subseteq H_j \subseteq \mathcal{F}\ where\ \mu(H_j) = 0 \tag{8.6}$$

   $\therefore A = \sum\limits_{j \geqslant 1} A_j = \sum\limits_{j \geqslant 1} E_j \cup \sum\limits_{j \geqslant 1} N_j$

   $$\therefore \overline{\mu}(A) = \mu\left(\sum_{j \geqslant 1} E_j\right) = \sum_{j \geqslant 1} \mu(E_j) = \sum_{j \geqslant 1} \overline{\mu}(A_j) \tag{8.7}$$

4. check $\left(\overline{\mu}, \overline{\mathcal{F}}\right)$ is complete, i.e. $\overline{\mathcal{F}}\ is\ \overline{\mu}$-complete.

   Assume that $A \subseteq E \in \overline{\mathcal{F}}$ where $\overline{\mu}(E) = 0$. We have to show that $A \in \overline{\mathcal{F}}$.

   $\because E \in \overline{\mathcal{F}} \therefore E = B \cup N\ where\ B \in \mathcal{F},\ N \subseteq H \in \mathcal{F}\ where\ \mu(H) = 0$

   $A = \varnothing \cup A,\ \varnothing \in F, A \subseteq E \subseteq B \cup N \subseteq \underbrace{B}_{\in \mathcal{F}} \cup \underbrace{H}_{\in \mathcal{F}} \in \mathcal{F}$, so $\mu(B \cup N) \leqslant \mu(B) + \mu(N) = 0$ by $\overline{\mu}(E) = \mu(B) = 0, \mu(A) \leqslant \mu(B) \Rightarrow \mu(A) = 0$, so $A \in \overline{\mathcal{F}}$

5. check $\overline{\mu}$ is unique. $\mu : \mathcal{F} \to \mathbb{R}_+ \bigcup \{+\infty\}$,

   And, extension $\overline{\mathcal{F}_\mu} = \{E \cup N,\ where\ E \in \mathcal{F}, N \subseteq H \in \mathcal{F},\ where\ \mu(H) = 0\}$, $\overline{\mu} : \overline{\mathcal{F}_\mu} \to \mathbb{R}_+ \cup \{+\infty\}$.

   Assume that $\nu : \overline{\mathcal{F}_\mu} \to \mathbb{R}_+ \cup \{+\infty\}$, and $\nu(A) = \overline{\mu}(A), \forall A \in \mathcal{F}$. Then we want show that $\nu(B) = \overline{\mu}(B), \forall B \in \overline{\mathcal{F}_\mu}$.

   Let $B \in \overline{\mathcal{F}_\mu}, B = E \cup N\ where\ E \in \mathcal{F},\ N \subseteq H \in \mathcal{F},\ where\ \mu(H) = 0, \nu(H) = \overline{\mu}(H) = \mu(H) = 0$.

   fix B, $\overline{\mu}(B) = \mu(E) \underbrace{=}_{by\ E \in \mathcal{F}} v(E) \leqslant \nu(B)$

   $\nu(B) = \nu(E \cup N) \leqslant \nu(E \cup H) \leqslant \nu(E) + \nu(H) = \nu(E) = \overline{\mu}(B)$, then

   $$\nu(B) = \overline{\mu}(B) \tag{8.8}$$

∎

$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$.

**Claim 8.2.** $\mathcal{M}$ is $\pi^*$-complete.

*Proof.* $\pi^*$-complete, i.e. $A \subseteq B, B \subseteq \mathcal{M}, \pi^*(B) = 0 \Rightarrow A \in \mathcal{M}$

We have to show $\forall E \subseteq \Omega$, $\pi^*(E) \geqslant \pi^*(E \cap A) + \pi^*(E \cap A^c)$

1. $\because E \cap A \subseteq A \subseteq B \ \therefore \pi^*(E \cap A) \leqslant \pi^*(B) = 0$
2. $\pi^*(E \cap A^c) \leqslant \pi^*(E)$

So, $A \in \mathcal{M}$ ∎

# Lecture 9

## Approximation Theorems

Goal: $\pi^*(A) < \infty, A \in \mathcal{M}, F \in \mathcal{F},\ where\ \mathcal{F}\ is\ \sigma - algebra, A \subseteq F, \pi^*(A) = \pi^*(F).$

**Theorem 9.1.** *$a \subseteq \mathcal{P}(\Omega)$, where $a$ is an algebra, $\mathcal{F}$ is a $\sigma-$algebra generated by $a$, $\mathcal{F}(a) = \mathcal{F}$, we have $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+$, where $\mu$ is a measure, and $\mu|_a = v$, $A \subseteq \mathcal{F}, \mu(A) < \infty, \forall \epsilon > 0$, there*

$$\exists\ E \in a,\ s.t.\ \ \mu(E \backslash A) + \mu(A \backslash E) < \varepsilon \tag{9.1}$$

*Proof.* $A \in \mathcal{F}, \mu(A) < \infty$, by Thm 4.1, then

$$\mu(A) = \pi^*(A) = \inf_{\{A_j\} \supseteq A, A_{j \in a}} \sum \nu(A_i) \tag{9.2}$$

but $\mu$ here is $\pi$ in Thm 4.1.

$\forall \epsilon, \exists \{A_i\} \quad A_i \in a,\ A \subseteq \cup A_i, \quad s.t.$

$$\pi^*(A) \leqslant \sum_{j \geqslant 1} \nu(A_i) \leqslant \pi^*(A) + \varepsilon \tag{9.3}$$

so

$$\exists\ m_0, \quad s.t. \sum_{i \geqslant m_0} \nu(A_i) \leqslant \varepsilon \tag{9.4}$$

Let $E = \bigcup\limits_{i=1}^{m_0} A_i \in a$, then we need to proof the following:

$$\pi^*(E \backslash A) \leqslant \varepsilon, \quad \pi^*(A \backslash E) \leqslant \varepsilon \tag{9.5}$$

By Thm 4.2, $\pi^*(A)$ is an out-measure, $\pi^*(A)$ is monotone and by Tmm 4.4, $\pi^*(A)$ is $\sigma$-additive.

$$\begin{aligned}
\therefore \pi^*(E \backslash A) &= \pi^*\left(\bigcup_{i=1}^{n_0} A_i \backslash A\right) \\
&\leqslant \pi^*\left(\bigcup_{i \geqslant 1} A_i \backslash A\right) \\
&= \pi^*\left(\bigcup_{i \geqslant 1} A_i\right) - \pi^*(A) \quad by\ \pi^*(A) = \mu(A) < \infty \\
&\leqslant \sum_{i \geqslant 1} \pi^*(A_i) - \pi^*(A) \\
&= \sum_{i \geqslant 1} \nu(A_i) - \pi^*(A)\ \ by\ \pi^*|_{\mathcal{F}} = \mu,\ \mu|_a = v,\ A_i \in a \therefore \pi^*(A_i) = \nu(A_i) \\
&\leq \varepsilon
\end{aligned} \tag{9.6}$$

On the other hand,

$$\pi^*\left(A\backslash E\right)=\pi^*\left(A\backslash\bigcup_{i=1}^{n_0}A_i\right)\leqslant\pi^*\left(\bigcup_{i\geqslant1}A_i\backslash\bigcup_{j=1}^{n_0}A_j\right)\leqslant\pi^*\left(\bigcup_{j\geqslant n_0+1}^{n_0}A_j\right)\leqslant\sum_{j\geqslant m_0}\left(\bigcup_{j\geqslant n_0+1}^{n_0}A_j\right)\leqslant\varepsilon \tag{9.7}$$

□

**Remark 9.1.** $\Omega$ is $\sigma-$finite$(\mu)$ ( i.e. $\Omega=\bigcup_{i\geqslant1}E_i$ *where* $E_i\in a,\mu\left(E_i\right)<\infty$), $\overline{\mu}:\overline{\mathcal{F}}\to\mathbb{R}_+\cup\{+\infty\}$, $A\in\overline{\mathcal{F}},\forall\varepsilon>0,\exists E\in a$, such that

$$\overline{\mu}\left(E\backslash A\right)+\overline{\mu}\left(A\backslash E\right)<\varepsilon. \tag{9.8}$$

$\Omega$ is topological space (open, closed sets), $\mathcal{B}$ is Borel $\sigma$-algebra set (the smallest $\sigma$ set which contains all open, closed sets in $\Omega$).

**Definition 9.1** (Regular Measure)**.** $\mu:\mathcal{F}\to\mathbb{R}_+\cup\{\infty\}$ where $\mathcal{B}\subseteq\mathcal{F}$, is a measure. Then $\mu$ is a regular measure if: $\forall A\in\mathcal{F},\forall\epsilon>0$, there $\exists F\subseteq A\subseteq G$, where $F\in\mathcal{B}$ closed, $G\in\mathcal{B}$ open, such that:

$$\mu\left(G\backslash F\right)\leqslant\varepsilon \tag{9.9}$$

**Remark 9.2.** $\mu<\infty$ is not necessary.

**Remark 9.3.** $\mu\left(G\backslash A\right)\leqslant\varepsilon$ and $\mu\left(A\backslash F\right)\leqslant\varepsilon$.

**Remark 9.4.** $\mathcal{B}\subseteq\mathcal{F},\ \mu\ is\ regular\ \Rightarrow\ \mathcal{F}\subseteq\overline{\mathcal{B}_\mu}$

*Proof.* $A\in\mathcal{F},n\geq1,$ *by* $\mu$ is regular, then $\exists F_n,G_n\in\mathcal{B},F_n\subseteq\mathcal{B}$, such that $\mu\left(F_n\backslash G_n\right)\leqslant\frac{1}{n}$.

Let's define $F=\bigcup_{n\geqslant1}F_n\in\mathcal{B},\ G=\bigcap_{n\geqslant1}G_n\in\mathcal{B}$, then $F\subseteq F_n\subseteq A\subseteq G_n\subseteq G,\ i.e.\ F\subseteq A\subseteq G$. By

$$G_n\backslash\left(\bigcup_{k\geqslant1}F_k\right)=G_n\cap\left(\bigcup_{k\geqslant1}F_k\right)^c=G_n\cap\left(\bigcap_{k\geqslant1}F_k^c\right)=\bigcap_{k\geqslant1}\left(G_n\cap F_k^c\right)=\bigcap_{k\geqslant1}\left(G_n\backslash F_k\right)\subseteq G_n\backslash F_n \tag{9.10}$$

then

$$\mu\left(G\backslash F\right)\leqslant\mu\left(G_n\backslash\left(\bigcup_{k\geqslant1}F_k\right)\right)\leqslant\mu\left(G_n\backslash F_n\right)\leqslant\frac{1}{n}\to0 \tag{9.11}$$

Finally,

$$A=\underbrace{F}_{\in\mathcal{B}}\cup\underbrace{\left(A\backslash F\right)}_{\subseteq G\backslash F\in\mathcal{B}}\in\mathcal{B}\Rightarrow A\in\overline{\mathcal{B}} \tag{9.12}$$

□

**Theorem 9.2.** *$\mathcal{L}$ is a $\sigma$-algebra generated by $a(\mathcal{S})$, where $\mathcal{S}$ is a set which defined as in Lecture 7, i.e. $\mathcal{S} = \{\varnothing, \mathbb{R}, (a,b], (a,\infty), (-\infty, b]\}$. $\mu : \mathcal{L} \to \mathbb{R}_+ \cup \{\infty\}$, is Lebesgue measure, then $\mu$ is regular measure. (if $A \in \mathcal{L}$, there $\exists F$ closed, $G$ open, $F \subseteq A \subseteq G$ such that $\mu(G \backslash F) \leqslant \varepsilon$).*

*Proof.*

1. goal: $A \in \mathcal{L}, \varepsilon > 0$, there exists G open, such that $A \subseteq G,\ \mu(G \backslash A) \leqslant \varepsilon$.

   Denote $E_n = [-n, n]$, $A_n = A \cap E_n$, then $\mu(A_n) < \infty$. By the construction of Caratheodory Thm 4.1, there $\exists \{B_{n,k}\}_{k \geqslant 1}, B_{n,k} \in a, A_n \subseteq \bigcup\limits_{k \geqslant 1} B_{n,k}$, such that

$$\mu(A_n) \leqslant \sum_{k \geqslant 1} \mu(B_{n,k}) \leqslant \mu(A_n) + \frac{\varepsilon}{2^n} \tag{9.13}$$

   By $B_{n,k} \in a$, $\therefore B_{n,k} = \sum\limits_{j=1}^{l_{n,k}} I_{n,k,j} \subseteq G_{n,k}$, where $I_{n,k,j} = (a_{n,k,j}, b_{n,k,j}]$.

   Then we denote $c_{n,k,j} = b_{n,k,j} + \underbrace{\delta_{n,k,j}}_{>0}, J_{n,k,j} = (a_{n,k,j}, c_{n,k,j})$, then $B_{n,k} \subseteq G_{n,k} = \bigcup\limits_{j=1}^{l_{n,k}} J_{n,k,j}$, then

$$\mu(G_{n,k}) \leqslant \sum_{j=1}^{l_{n,k}} \mu(I_{n,k,j}) + \delta_{n,k,j} = \underbrace{\sum_{j=1}^{l_{n,k}} \mu(I_{n,k,j})}_{\mu(B_{n,k})} + \underbrace{\sum_{j=1}^{l_{n,k}} \delta_{n,k,j}}_{\leqslant \frac{\varepsilon}{2^n 2^k}} \tag{9.14}$$

   $\because B_{n,k} \subseteq G_{n,k}$, *and* $G_{n,k}$ *open set* $\therefore \mu(G_{n,k}) \leqslant \mu(B_{n,k}) + \frac{\varepsilon}{2^n 2^k}$. $\because A_n \subseteq \bigcup\limits_{k \geqslant 1} B_{n,k}, B_{n,k} \subseteq G_{n,k} \therefore A_n \subseteq \bigcup\limits_{k \geqslant 1} G_{n,k} = G_n$.

   On the other hand,

$$\mu(G_n) \leqslant \sum_{k \geqslant 1} \mu(G_{n,k}) \leqslant \sum_{k \geqslant 1} \mu(B_{n,k}) + \frac{\varepsilon}{2^n} \leqslant \mu(A_n) + \frac{2\varepsilon}{2^n} \tag{9.15}$$

   $\because A_n \subseteq G_n$ *open, and* $\mu(G_n) \leqslant \mu(A_n) + \frac{2\varepsilon}{2^n}$.

   Then define $G = \bigcup\limits_{n \geqslant 1} G_n$, *open and* $A = \bigcup\limits_{n \geqslant 1} A_n,\ A \subseteq G$.

$$\begin{aligned} \because \bigcup_{n \geqslant 1} G_n \backslash \bigcup_{k \geqslant 1} A_k &= \bigcup_{n \geqslant 1} G_n \cap \left( \bigcup_{k \geqslant 1} A_k \right)^c = \bigcup_{n \geqslant 1} G_n \cap \left( \bigcap_{k \geqslant 1} A_k^c \right) \\ &= \bigcap_{k \geqslant 1} \left( \bigcup_{n \geqslant 1} G_n \bigcap A_k^c \right) \subseteq \left( \bigcup_{n \geqslant 1} G_n \bigcap A_n^c \right) = \bigcup_{n \geqslant 1} G_n \backslash A_n \end{aligned} \tag{9.16}$$

$$
\begin{aligned}
\therefore \mu(G\backslash A) &= \mu\left(\bigcup_{n\geqslant 1} G_n \backslash \bigcup_{k\geqslant 1} A_k\right) \\
&\leqslant \mu\left(\bigcup_{n\geqslant 1} G_n \backslash A_n\right) \quad by \;\; Eq.\ 9.16 \\
&\leqslant \sum_{n\geqslant 1} \mu(G_n \backslash A_n) \\
&= \sum_{n\geqslant 1} [\mu(G_n) - \mu(A_n)] \quad by\ \mu(A_n) < \infty \\
&\leq 2\varepsilon
\end{aligned}
\tag{9.17}
$$

2. goal: $A \in \mathcal{L}, \varepsilon > 0$, there exists F closed , such that $F \subseteq A,\ \mu(A\backslash F) \leqslant \varepsilon$.

   By above 1, $\exists H,\ A^c \subseteq H,\ H\ open\ set,\ \mu(H\backslash A^c) \leqslant \varepsilon,\ then\ F = H^c \subseteq A,\ F\ closed$ .

   Finally,

$$
\mu(A\backslash F) = \mu(A \cap F^c) = \mu(A \cap H) = \mu(H \cap (A^c)^c) = \mu(H\backslash A^c) \leqslant \varepsilon. \tag{9.18}
$$

□

**Remark 9.5.** $\mathcal{F}_\sigma$: countable union closed sets, $\mathcal{G}_\sigma$: countable injection open sets. $\forall A \in \mathcal{L}$ there $\exists R \in \mathcal{F}_\sigma$ and $S \in \mathcal{G}_\sigma$, such that

$$
R \subseteq A \subseteq S, \quad \mu(S\backslash R) = 0. \tag{9.19}
$$

# Lecture 10

## Integration: Measurable and Simple Functions

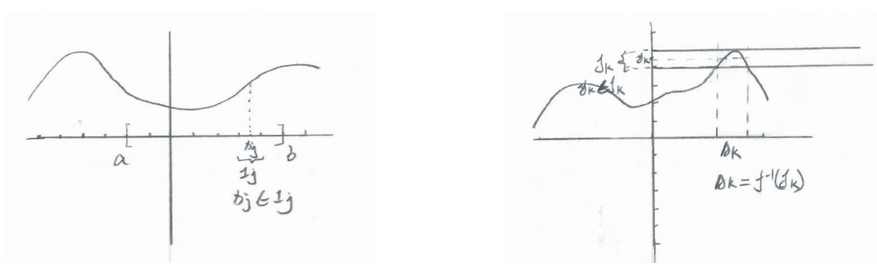
We now assume given $(\Omega, \mathcal{F}, \mu)$ where $\Omega$ is a space, $\mathcal{F}$ a $\sigma-$field of subsets of $\Omega$ and $\mu$ a measure on $\mathcal{F}$.

Before defining such an operator $\mathcal{I}$, we examine the sort of properties $\mathcal{I}$ should have before we would be justified in calling it an integral. Suppose that $\mathcal{A}$ is a class of functions $f : \Omega \to \overline{\mathbb{R}}$, and $\mathcal{I} : \mathcal{A} \to \mathbb{R}$ defines a real number for every $f \in \mathcal{A}$. Then we want $\mathcal{I}$ to satisfy:

1. $f \in \mathcal{A}, f(x) \geqslant 0$, all $x \in \Omega \Rightarrow \mathcal{I}(f) \geqslant 0$, that is $\mathcal{I}$ preserves positivity
2. $f, g \in \mathcal{A},\ \alpha \in \mathbb{R} \Rightarrow \alpha f + g \in \mathcal{A}$ and
$$\mathcal{I}(\alpha f + g) = \alpha \mathcal{I}(f) + \mathcal{I}(g) \tag{10.1}$$
that is $\mathcal{I}$ is linear on $\mathcal{A}$.
3. $\mathcal{I}$ is continuous on $\mathcal{A}$ in some sense, at least we would want to have $\mathcal{I}(f_n) \to 0$ as $n \to \infty$ for any sequence decreasing with $f_n(x) \to 0$ for all $x$ in $\Omega$.

These conditions are satisfied by the elementary integration process, but the Riemann integral does not satisfy the following strengthened form of 3.

- $3'$ If $\{f_n\}$ is an increasing sequence of functions in $\mathcal{A}$, and
$$f_n(x) \to f(x) \quad for \ \ all \ \ x \in \Omega \tag{10.2}$$
then $f \in \mathcal{A}$ and $\mathcal{F}(f_n) \to \mathcal{F}(f)\ \ as\ n \to \infty$




(a) Riemann integral (b) Lebesgue integration

Figure 1: Integration

1. Riemann integral
$$\int f \approx \sum f(x_j) |I_j| \tag{10.3}$$
2. Lebesgue integration
$$I(f) \approx \sum y_k \mu(A_k) = \sum_k y_k \mu\left(f^{-1}(J_k)\right) \tag{10.4}$$
where $A_k = f^{-1}(J_k)$.

In defining measurability we will want to consider functions

$$f : \Omega \to \mathbb{R} \cup \{-\infty, \infty\} = \overline{\mathbb{R}} \tag{10.5}$$

It is possible to define the class of Borel sets $\mathcal{B}$ in $\overline{\mathbb{R}}$ in terms of this topology. However, we adopt the simple procedure of defining the class

$$\overline{\mathcal{B}} = \{A \cup B, A \in \mathcal{B}, B \subseteq \{-\infty, \infty\}\} \tag{10.6}$$

**Proposition 10.1.** *$\overline{\mathcal{B}}$ is a$\sigma$-algebra.*

**Definition 10.1.** A function $f : \Omega \to \overline{\mathbb{R}}$ is said to be $\mathcal{F}-$measurable if and only if

$$f^{-1}(A) \in \mathcal{F} \tag{10.7}$$

for all $A \in \overline{\mathcal{B}}$.

If there is only one $\sigma-$field $\mathcal{F}$ under discussion we may say that $f$ is a measurable function.

**Remark 10.1.**

$$\mathcal{F} \subseteq \mathcal{G} \tag{10.8}$$

**Lemma 10.1.** $(\Omega, \mathcal{F}, \mu)$ *$f : \Omega \to \overline{\mathbb{R}}$, $f$ is measurable each of the following conditions is necessary and sufficient:*

1. $f^{-1}((-\infty, x]) \in \mathcal{F},\ \forall x \in \mathbb{R},$ *i.e.* $\{\omega \in \Omega, f(\omega) \leqslant x\} \in \mathcal{F}$
2. $f^{-1}((-\infty, x)) \in \mathcal{F},\ \forall x \in \mathbb{R},$ *i.e.* $\{\omega \in \Omega, f(\omega) < x\} \in \mathcal{F}$
3. $f^{-1}([x, \infty)) \in \mathcal{F},\ \forall x \in \mathbb{R},$ *i.e.* $\{\omega \in \Omega, f(\omega) \geq x\} \in \mathcal{F}$
4. $f^{-1}((x, \infty)) \in \mathcal{F},\ \forall x \in \mathbb{R},$ *i.e.* $\{\omega \in \Omega, f(\omega) > x\} \in \mathcal{F}$

*Proof.* We only proof (1) in Lemma 10.1

1. $\Rightarrow (-\infty, x] \in \overline{\mathcal{B}}$
2. $\Leftarrow$ If we suppose that the condition is satisfied, and put

$$\mathcal{C} = \left\{A \in \overline{\mathcal{B}}, f^{-1}(A) \in \mathcal{F}\right\} \tag{10.9}$$

then

(a) $\mathcal{C}$ is a $\sigma$-algebra

(b) $\mathcal{C} \supseteq \mathcal{G} = \{(-\infty, x], x \in \mathbb{R}\}$

by *a&b* ,

$$\mathcal{C} \supseteq \mathcal{F}(\mathcal{G}) \supseteq \overline{\mathcal{B}} \tag{10.10}$$

then $\mathcal{C}$ is a $\sigma$-algebra.

- $\overline{\mathbb{R}} \in \mathcal{C}, f^{-1}\left(\overline{\mathbb{R}}\right) = \left\{\omega \in \Omega, f(\omega) \in \overline{\mathbb{R}}\right\} = \Omega \in \mathcal{F}$
- $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}, f^{-1}(A) \in \mathcal{F}$, so $f^{-1}(A^c) \in f^{-1}(A)^c \in \mathcal{F}$

- $A_j \in \mathcal{C} \Rightarrow \bigcup\limits_{j \geqslant 1} A_j \in \mathcal{C}$, then

$$f^{-1}\left(\bigcup_{j\geqslant 1} A_j\right) = \bigcup_j \underbrace{f^{-1}(A_j)}_{\in \mathcal{F}} \in \mathcal{F} \tag{10.11}$$

□

Given $(\Omega, \mathcal{F}, \mu)$ as above. If $\Omega = \bigcup\limits_{i=1}^{n} E_i$ and the sets $E_i$ are disjoint ($E_j \cap E_k = \varnothing,\ j \neq k$), then $E_1, E_2, ..., E_n$ are said to form a (finite) dissection of $\Omega$. They are said to form an $\mathcal{C}$-dissection if, in addition $E_i \in \mathcal{F}(i = 1, 2, ..., n)$.

**Definition 10.2** (Simple Function)**.** A function $f:\ \Omega \to \mathbb{R}$ is called $\mathcal{F}$-simple if it can be expressed as

$$f = \sum_{j=1}^{n} c_j\ 1_{E_j},\ c_j \in \mathbb{R} \tag{10.12}$$

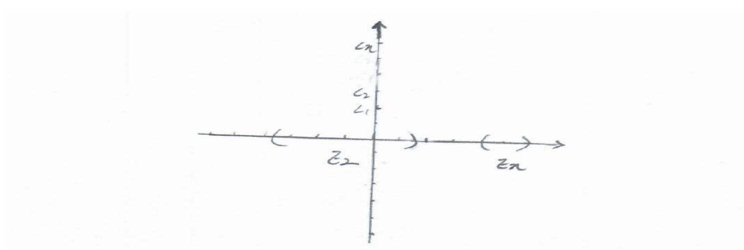
where $1_{E_j}, \Omega \to \overline{\mathbb{R}}$,

$$\omega \mapsto 1_{E_j}(\omega) = \begin{cases} 1,\ \omega \in E_j \\ 0,\ \omega \notin E_j \end{cases} \tag{10.13}$$

and $\sum\limits_{j=1}^{n} E_j = \Omega,\ E_0 = \Omega \backslash \left(\sum\limits_{j=1}^{n} E_j\right) \in \mathcal{F}$.

If there is only one $\sigma-$field $\mathcal{F}$ under discussion we will talk of simple function rather than $\mathcal{F}$-simple functions.

$f^{-1}(A) = \sum\limits_{k, c_k} E_k \in \mathcal{F},\ A \in \overline{\mathcal{B}},\ f: \Omega \to R_+,\ f = \sum\limits_{j=1}^{n} c_j 1_{E_j},\ E_j \in \mathcal{F},\ \{E_1, ..., E_n\}$ *partition of* $\Omega$.



$$I(f) = \sum_{j=1}^{n} c_j \mu(E_j) \tag{10.14}$$

where $c_j \geqslant 0$.

If $f = \sum\limits_{k=1}^{m} d_k 1_{F_k}$.

**Proposition 10.2.** *$E_{j^\circ} \cap F_{k^\circ} \neq \varnothing$, then*

$$\sum_{j=1}^{n} c_j \mu(E_j) = \sum_{k=1}^{n} d_k \mu(F_k) \tag{10.15}$$

*Proof.*

$$\begin{aligned}\mu\left(E_j\right) &= \mu\left(E_j \cap \left(\sum_{k=1}^{m} F_k\right)\right)\\ &= \mu\left(\sum_{k=1}^{m}\left(E_j \cap F_k\right)\right)\\ &= \mu\left(E_j\right) = \sum_{k=1}^{m} \mu\left(E_j \cap F_k\right)\end{aligned} \tag{10.16}$$

then

$$\begin{aligned}\sum_{j=1}^{n} c_j \mu\left(E_j\right) &= \sum_{j=1}^{n}\sum_{k=1}^{m} c_j \mu\left(E_j \cap F_k\right)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{m} d_k \mu\left(E_j \cap F_k\right)\\ &= \sum_{k=1}^{m} d_k \mu\left(F_k\right)\end{aligned} \tag{10.17}$$

$\square$

**Proposition 10.3.**

1. *$f : \Omega \to \overline{\mathbb{R}}_+$ measurable then there exists $(f_n)_{n \geqslant 1}$, $f_n$ simple functions, such that $f_n \geqslant 0$, $f_n \uparrow f$*
2. *$I(f) = \lim_n I(f_n)$*
3. *$f : \Omega \to \overline{\mathbb{R}}$ measurable, $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$, $f^+, f^-$ measurable then $f = f^+ - f^-$, then*

$$I(f) = I\left(f^+\right) - I\left(f^-\right) \tag{10.18}$$

**Example 10.1.** $\Omega = (0, 1]$, $\mathcal{B}, \lambda$, $E = \mathbb{Q} \cap \Omega$, $f = 1_{E^c}$, i.e. $f$ simple, then

$$I(f) = \lambda\left(E^c\right) = 1 \tag{10.19}$$

# Lecture 11

## Measurable Functions

We now assume given $(\Omega, \mathcal{F}, \mu)$, where $\Omega$ is a space, $\mathcal{F}$ a $\sigma$-field of subsets of $\Omega$ and $\mu$ a measure on $\mathcal{F}$.

**Lemma 11.1.** *If $f$ and $g$ are measurable functions: $\Omega \to \overline{\mathbb{R}}$, and $\alpha \in \mathbb{R}$, then*

1. $\alpha f \in \mathcal{F}$*-measurable*
2. $\alpha + f \in \mathcal{F}$*-measurable*
3. $f + g \in \mathcal{F}$*-measurable*
4. $f^2 \in \mathcal{F}$*-measurable*
5. $1/f \in \mathcal{F}$*-measurable*
6. $f^+, f^-, |f| \in \mathcal{F}$*-measurable*
7. $fg \in \mathcal{F}$*-measurable*

*Proof.*

1. $\alpha f \in \mathcal{F}$-measurable, we want to show that $\{\omega : \alpha f(\omega) \leqslant x\} \in \mathcal{F}$
   (a) $\alpha = 0$
   (b) $\alpha > 0,\ \alpha f(\omega) \leqslant x,\ i.e.\ f(\omega) \leqslant {}^{x}\!/_{\alpha}\ by\ \{\omega :\ f(\omega) \leqslant {}^{x}\!/_{\alpha}\} \in\ \mathcal{F},\ \forall\ {}^{x}\!/_{\alpha} \in \mathbb{R}$
   (c) $\alpha < 0,\ \alpha f(\omega) \leqslant x,\ i.e.\ f(\omega) \geqslant {}^{x}\!/_{\alpha}$ then by lemma 10.1.
2. 
3. we want to show $f + g\ \in \mathcal{F}$-measurable, i.e. $\{\omega : f(\omega) + g(\omega) < x\} \in \mathcal{F},\ \forall x \in \mathbb{R}$

$$\{\omega : f(\omega) + g(\omega) < x\} = \bigcup_{r \in \mathbb{Q}} (\{\omega : f(\omega) < r\} \cap \{g(\omega) < x - r\}) \tag{11.1}$$

   by Lemma 10.1, and $\mathcal{F}$ is a $\sigma$-algebra, so $\{\omega : f(\omega) + g(\omega) < x\} \in \mathcal{F}$
4. $f^2 \in \mathcal{F}$-measurable

   Now, we will check $\left\{\omega : f(\omega)^2 < x\right\} \in \mathcal{F}$

$$\left\{\omega : f(\omega)^2 < x\right\} = \begin{cases} \varnothing \in \mathcal{F} & x \leqslant 0 \\ \{\omega \in \Omega, -x < f(\omega) < x\} \in \mathcal{F} & x > 0 \end{cases} \tag{11.2}$$

5. $\frac{1}{f} \in \mathcal{F}$-measurable i.e. $\left\{\omega \in \Omega : \frac{1}{f(\omega)} < x\right\} \in \mathcal{F}$

   by

   (a) $x > 0,\ \{\omega : f(\omega) < 0\} \cup \left\{\omega : f(\omega) > \frac{1}{x}\right\} \in \mathcal{F}$
   (b) $x = 0,\ \{\omega : f(\omega) < 0\} \in \mathcal{F}$

(c) $x < 0$, $\left\{\omega : \frac{1}{x} < f(\omega) < 0\right\} \in \mathcal{F}$

6. $f^+ = \max\{f, 0\}$

$$\left\{\omega \in \Omega : f^+(\omega) < x\right\} = \begin{cases} \varnothing \in \mathcal{F} & x \leqslant 0 \\ \{\omega \in \Omega : f(\omega) < x\} \in \mathcal{F} & x > 0 \end{cases} \tag{11.3}$$

$f^- = \max(-f, 0)$ and $|f| = f^+ + f^-$

7. by $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$

$\square$

**Remark 11.1.**

$$\begin{aligned} \max(f, g) &= \frac{1}{2}\left[f + g + |f - g|\right] \\ \min(f, g) &= f + g - \max(f, g) \end{aligned} \tag{11.4}$$