Partial Differential Equations I

Lectures by Chi-Kun Lin

Notes by Yao Zhang

National Chiao Tung University, Fall 2008

Introduction

A brief review of what covered in Partial Differential Equations I are

1. The Single First-Order Equation
2. Second-Order Equations: Hyperbolic Equations for Functions of Two Independent Variables
3. Characteristic Manifolds and Cauchy Problem
4. The Laplace Equation

These notes were live-TeXed, though I edited for typos and added diagrams requiring the Ti*k*Z package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

Acknowledgments

Thank you to all of my friends who will send me suggestions and corrections. My notes will be much improved due to your help.

I would like to especially thank National Chiao Tung University who put their courses in website.

Lecture 1 (pde971_970924)

Text: Partial diff. Eq by Fritz John
Reference:

1. partial diff. Eq Basic theory by M. Taylor
2. partial diff. Eq by C. Evans

Review of vector analysis, Gradient, Divergence, Curl. Denote that $\overrightarrow{F} = (F_x, F_y, F_z)$, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, then

1. inner product: vector $\rightarrow$ scalar

$$div \overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.1}$$

2. outer product: vector $\rightarrow$ vector

$$curl \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_\mathrm{y} & F_z \end{vmatrix} \tag{1.2}$$

3. direct product: scalar $\rightarrow$ vector

$$gradf = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \tag{1.3}$$

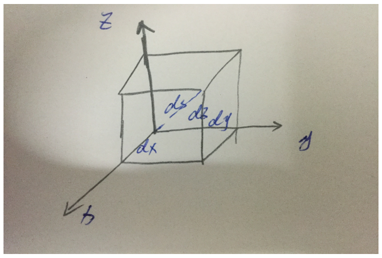
arc-length:

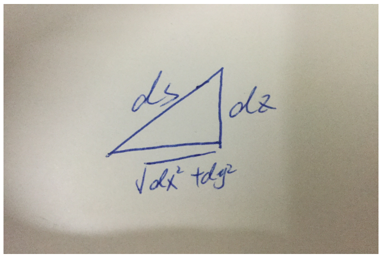
$$ds = d\overrightarrow{r} \cdot d\overrightarrow{r} = dx^2 + dy^2 + dz^2 \tag{1.4}$$

volume:

$$dv = dxdydz \tag{1.5}$$

Right-hand rule: $x \rightarrow y \rightarrow z$. $ds_x = dydz, ds_y = dzdx, ds_z = dxdy$.

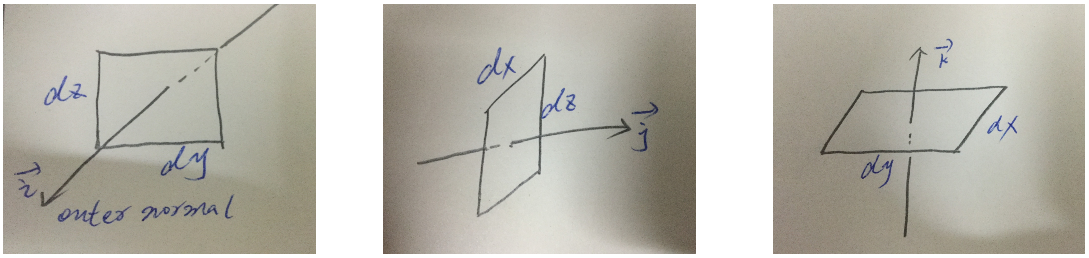




$$\overrightarrow{r} = (x, y, z) = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}, \quad \overrightarrow{i} = \frac{\partial \overrightarrow{r}}{\partial x}, \overrightarrow{j} = \frac{\partial \overrightarrow{r}}{\partial y}, \overrightarrow{k} = \frac{\partial \overrightarrow{r}}{\partial z} \tag{1.6}$$

$$d\overrightarrow{r} = \frac{\partial \overrightarrow{r}}{\partial x}dx + \frac{\partial \overrightarrow{r}}{\partial y}dy + \frac{\partial \overrightarrow{r}}{\partial z}dz \tag{1.7}$$

General Coordinate:



(a) $ds_x = dydz$ (b) $ds_y = dzdx$ (c) $ds_z = dxdy$

$\overrightarrow{r} = (x, y, z) = (x_1, x_2, x_3), (x_1, x_2, x_3) \to (u_1, u_2, u_3), x_i = x_i(u_1, u_2, u_3), i = 1, 2, 3$

$$\frac{\partial (x_1, x_2, x_3)}{\partial (u_1, u_2, u_3)} = \underbrace{\frac{\partial \overrightarrow{r}}{\partial u_1} \cdot \left(\frac{\partial \overrightarrow{r}}{\partial u_2} \times \frac{\partial \overrightarrow{r}}{\partial u_3} \right)}_{triple \quad product} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix} \neq 0 \tag{1.8}$$

Theorem 1.1. *(x_1, x_2, x_3) satisfies the right-hand rule $\Rightarrow$ (u_1, u_2, u_3) satisfies the right-hand rule too.*

$$(dx_1, dx_2, dx_3) = \frac{\partial (x_1, x_2, x_3)}{\partial (u_1, u_2, u_3)} du_1 du_2 du_3 \tag{1.9}$$

basis:

$$\overrightarrow{e_i} = \frac{1}{h_i} \frac{\partial \overrightarrow{r}}{\partial u_i}, \ \underbrace{[\overrightarrow{e_i}]}_{\max well} = 1, \ h_i = \underbrace{\left| \frac{\partial \overrightarrow{r}}{\partial u_i} \right|}_{\text{lame coefficient}}, i = 1, 2, 3 \tag{1.10}$$

Assume:$\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ forms an O.N. system

$$d\overrightarrow{r} = \frac{\partial \overrightarrow{r}}{\partial u_1} du_1 + \frac{\partial \overrightarrow{r}}{\partial u_2} du_2 + \frac{\partial \overrightarrow{r}}{\partial u_3} du_3, ds^2 = d\overrightarrow{r} \cdot d\overrightarrow{r} = \sum_{i,j=1}^{3} g_{ij} du_i du_j \tag{1.11}$$

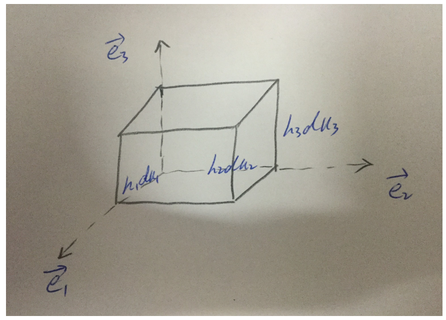
where

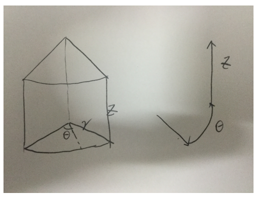
$$\begin{aligned} g_{ij} &= \frac{\partial \overrightarrow{r}}{\partial x_i} \cdot \frac{\partial \overrightarrow{r}}{\partial x_j}, \quad i = 1, 2, 3 \\ g_{ij} &= 0, \ i \neq j, \quad [g_{ij}] = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{bmatrix} \end{aligned} \tag{1.12}$$

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2 \tag{1.13}$$

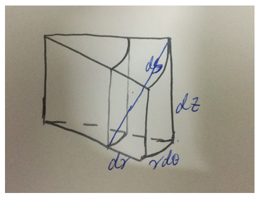
Remark 1.1. $[h_i du_i] = [d\overrightarrow{r}] = L$, recall cgs, MKs, $\{L, M, T\}$

$$\begin{cases} ds_1 = h_2 h_3 du_2 du_3 \\ ds_2 = h_3 h_1 du_3 du_1 \\ ds_3 = h_1 h_2 du_1 du_2 \\ dv = h_1 h_2 h_3 du_1 du_2 du_3 \end{cases} \tag{1.14}$$

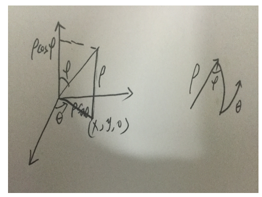




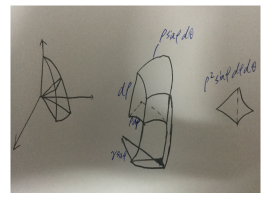
(a) cylindrical coordinate



(b) cylindrical coordinate



(c) spherical coordinate



(d) spherical coordinate

Example 1.1 (cylindrical coordinate system).

$$x = r\cos\theta, y = r\sin\theta, z = z, \overrightarrow{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z) \tag{1.15}$$

$$\begin{aligned}
\overrightarrow{e_r} &= \frac{1}{h_r}\frac{\partial\overrightarrow{r}}{\partial r} = (\cos\theta, \sin\theta, 0) \\
\overrightarrow{e_\theta} &= \frac{1}{h_\theta}\frac{\partial\overrightarrow{r}}{\partial\theta} = (-\sin\theta, cos\theta, 0) \\
\overrightarrow{e_z} &= \frac{1}{h_z}\frac{\partial\overrightarrow{r}}{\partial z} = (0, 0, 1) \\
d\overrightarrow{r} &= d\overrightarrow{r}\cdot\overrightarrow{e_r} + rd\theta\overrightarrow{e_\theta} + dz\overrightarrow{e_z} \\
ds^2 &= dr^2 + r^2 d\theta + dz^2 \\
dv &= rdrd\theta dz \qquad \left(L^3 = LL1L\right)
\end{aligned} \tag{1.16}$$

Example 1.2 (spherical coordinate system).

$$x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi, \overrightarrow{r} = (x, y, z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \tag{1.17}$$

$$\begin{aligned}
\overrightarrow{e_\rho} &= \frac{1}{h_\rho}\frac{\partial\overrightarrow{r}}{\partial\rho} = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \\
\overrightarrow{e_\varphi} &= \frac{1}{h_\varphi}\frac{\partial\overrightarrow{r}}{\partial\varphi} = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \\
\overrightarrow{e_\theta} &= \frac{1}{h_\theta}\frac{\partial\overrightarrow{r}}{\partial\theta} = (-\sin\theta, \cos\theta, 0) \\
d\overrightarrow{r} &= d\rho\overrightarrow{e_\rho} + \rho d\varphi\overrightarrow{e_\varphi} + \rho\sin\varphi d\theta\overrightarrow{e_\theta} \\
ds^2 &= d\rho^2 + \rho d\varphi^2 + \rho^2\sin^2\varphi d\theta^2 \\
dv &= \rho^2\sin\varphi d\rho d\varphi d\theta
\end{aligned} \tag{1.18}$$

The meaning of gradient

Theorem 1.2 (Divergence theorem).

$$\iiint\limits_V div \overrightarrow{A} dv = \iint\limits_S \overrightarrow{A} \cdot \overrightarrow{n} ds \tag{1.19}$$

where $s = \partial v$.

Choose $\overrightarrow{A} = \overrightarrow{c} \cdot f(x, y, z)$, $\quad \overrightarrow{c}$: constant vector, so

$$\overrightarrow{c} \cdot \left(\iiint\limits_V grad\ f dv - \iint\limits_s \overrightarrow{n} ds \right) = \overrightarrow{0}, \quad \forall \overrightarrow{c} \tag{1.20}$$

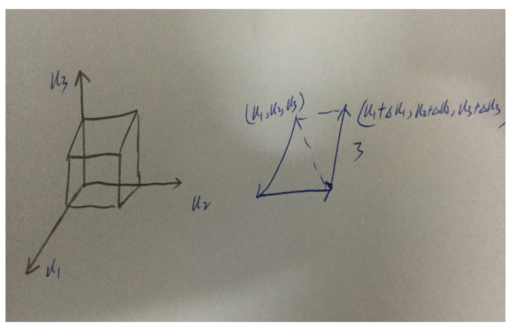
so,

$$\begin{gathered} \iiint\limits_V grad\ f dv = \iint\limits_s \overrightarrow{n} ds, \qquad \frac{[f]}{L} = \frac{1}{L^3} [f] 1 L^2 \\ grad\ f = \lim_{|V| \to 0} \frac{1}{|v|} \iint\limits_s f \overrightarrow{n} ds \end{gathered} \tag{1.21}$$

independent of coordinate

$$gradient \Leftrightarrow directional \quad derivative \tag{1.22}$$

$(u_1, u_2, u_3) \to (u_1 + \Delta u_1, u_2 + \Delta u_2, u_3 + \Delta u_3)$



$$\begin{aligned} \Delta f &= f(u_1 + \Delta u_1, u_2 + \Delta u_2, u_3 + \Delta u_3) - f(u_1, u_2, u_3) \\ &= \frac{\partial f}{\partial u_1} \Delta u_1 + \frac{\partial f}{\partial u_2} \Delta u_2 + \frac{\partial f}{\partial u_3} \Delta u_3 + \cdots \end{aligned} \tag{1.23}$$

$$\Delta \overrightarrow{r} = h_1 \Delta u_1 \overrightarrow{e_1} + h_2 \Delta u_2 \overrightarrow{e_2} + h_3 \Delta u_3 \overrightarrow{e_3} \tag{1.24}$$

$$(\Delta s)^2 = (h_1 \Delta u_1)^2 + (h_2 \Delta u_2)^2 + (h_3 \Delta u_3)^2 \tag{1.25}$$

$$\frac{df}{ds} = \nabla f \cdot \overrightarrow{n} \tag{1.26}$$

$$\frac{\Delta f}{\Delta s} = \left(\frac{1}{h_1} \frac{\partial f}{\partial u_1} \overrightarrow{e_1} + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \overrightarrow{e_2} + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \overrightarrow{e_3} \right) \cdot \frac{\Delta \overrightarrow{r}}{\Delta s} + \cdots \tag{1.27}$$

$$\frac{df}{ds} = \lim_{\Delta s \to 0} \left(\frac{1}{h_1} \frac{\partial f}{\partial u_1} \overrightarrow{e_1} + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \overrightarrow{e_2} + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \overrightarrow{e_3} \right) \cdot \frac{d \overrightarrow{r}}{ds} \tag{1.28}$$

Example 1.3.

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= \frac{\partial f}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{e_\theta} + \frac{\partial f}{\partial z}\overrightarrow{e_z} \\
&= \frac{\partial f}{\partial \rho}\overrightarrow{e_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\overrightarrow{e_\varphi} + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \theta}\overrightarrow{e_\theta}
\end{aligned} \tag{1.29}
$$

Lecture 2 (pde971_970926)

the meaning of divergence

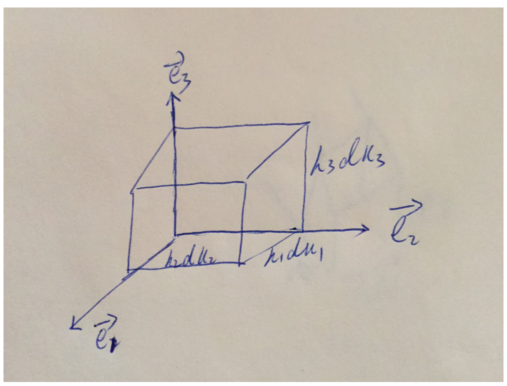
Theorem 2.1 (Gauss Divergence Thm)**.**

$$div\overrightarrow{A} = \lim_{|V|\to 0}\frac{1}{|V|}\iint\limits_{S}\overrightarrow{A}\cdot\overrightarrow{n}\,ds \tag{2.1}$$

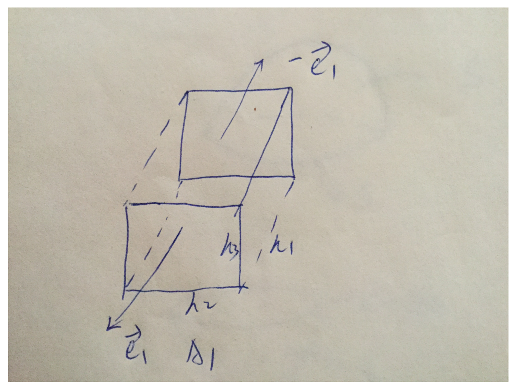
where $S = \partial V, \overrightarrow{A} = (A_x, A_y, A_z)$.

the flux of unit volume.

$$\begin{aligned}\frac{1}{|v|}\iint\limits_{S}\overrightarrow{A}\cdot\overrightarrow{n}\,ds &= \frac{1}{|v|}\iiint\limits_{V}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)dv\\ |v|\to 0 \Rightarrow div\overrightarrow{A} &= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\end{aligned} \tag{2.2}$$



(a)



(b)

$$\begin{aligned}\overrightarrow{A} &= A_1\overrightarrow{e_1}+A_2\overrightarrow{e_2}+A_3\overrightarrow{e_3}\\ div\overrightarrow{A} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\partial}{\partial u_2}(A_2h_3h_1)+\frac{\partial}{\partial u_3}(A_3h_1h_2)\right]\end{aligned} \tag{2.3}$$

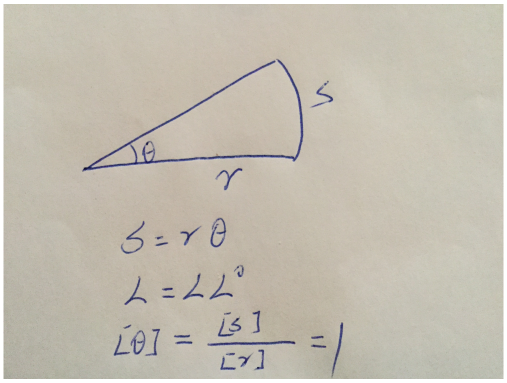
$$\lim_{\Delta h_1\to 0}\left(\frac{A_1h_2h_3-\widetilde{A_1}\widetilde{h_2}\widetilde{h_3}}{h_1h_2h_3}\right) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(Ah_2h_3) \tag{2.4}$$

Examples 2.1.

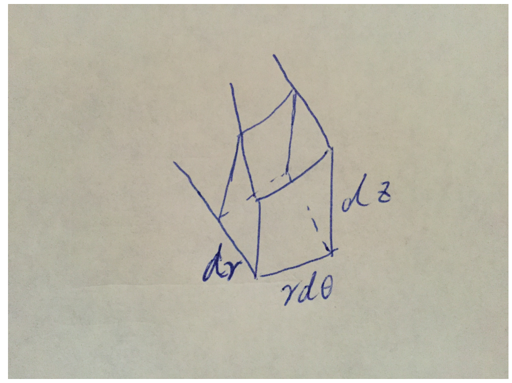
1. cylindrical coordinate system: $\overrightarrow{A} = A_r\overrightarrow{e_r}+A_\theta\overrightarrow{e_\theta}+A_z\overrightarrow{e_z}$

$$\begin{aligned}div\overrightarrow{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial}{\partial\theta}A_\theta+\frac{\partial}{\partial z}A_z\\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r)+\frac{\partial}{\partial\theta}A_\theta+\frac{\partial}{\partial z}(rA_z)\right]\end{aligned} \tag{2.5}$$

$$\frac{[A]}{L} = \frac{1}{L}\frac{1}{L}[L[A]] = \frac{1}{L}\frac{1}{1}[A] \tag{2.6}$$



(a)



(b)

2. spherical coordinate system: $\overrightarrow{A} = A_\rho \overrightarrow{e_\rho} + A_\varphi \overrightarrow{e_\varphi} + A_\theta \overrightarrow{e_\theta}$

$$div \overrightarrow{A} = \frac{1}{\rho^2}\frac{\partial \rho^2 A_\rho}{\partial \rho} + \frac{1}{\rho \sin\varphi}\frac{\partial A_\varphi \sin\varphi}{\partial \varphi} + \frac{1}{\rho\sin\varphi}\frac{\partial A_\theta}{\partial\theta} \tag{2.7}$$

Definition 2.1.

$$\Delta = div\,(grad) = \nabla \cdot \nabla = \nabla^2 \tag{2.8}$$

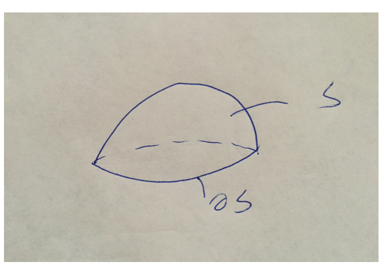
where ∇ : nabla.

Examples 2.2.

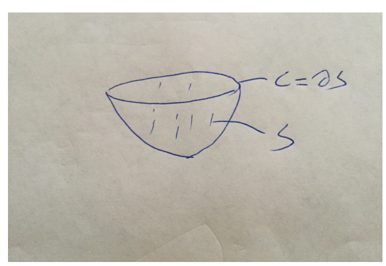
$$\Delta u = \frac{1}{\rho^2 \sin\varphi}\left[\frac{\partial}{\partial\rho}\left(\rho^2 \sin\varphi \frac{\partial u}{\partial \rho}\right) + \frac{\partial}{\partial\varphi}\left(\sin\varphi^2 \frac{\partial u}{\partial\varphi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\theta}\right)\right] \tag{2.9}$$

$$\frac{[u]}{L^2} = \frac{1}{L^2}\frac{1}{L}\left(L^2 1 \frac{[u]}{L}\right) \tag{2.10}$$

the meaning of curl:



(a)



(b)

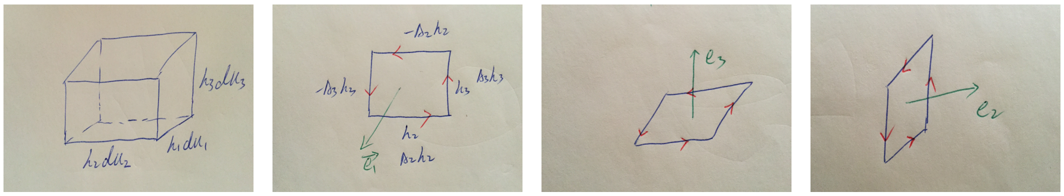
Theorem 2.2 (Stoker Thm)**.**

$$\oint_C \overrightarrow{A} \cdot d\overrightarrow{r} = \iint_S cur\overrightarrow{A} \cdot \overrightarrow{n}\, ds \tag{2.11}$$

where $\partial S = C$*.*

$$cur\overrightarrow{A}\cdot\overrightarrow{n}=\lim_{|S|\to 0}\frac{1}{|S|}\oint_{C}\overrightarrow{A}\cdot dr \tag{2.12}$$

the circulation of unit area.



(a) (b) (c) (d)

$$\overrightarrow{A}=A_1\overrightarrow{e_1}+A_2\overrightarrow{e_2}+A_3\overrightarrow{e_3} \tag{2.13}$$

$$\begin{aligned} curl\overrightarrow{A} &= \frac{1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(A_3h_3)-\frac{\partial}{\partial u_3}(A_2h_2)\right)\overrightarrow{e_1} \\ &+\frac{1}{h_3h_1}\left(\frac{\partial}{\partial u_3}(A_1h_1)-\frac{\partial}{\partial u_1}(A_3h_3)\right)\overrightarrow{e_2} \\ &+\frac{1}{h_1h_2}\left(\frac{\partial}{\partial u_1}(A_2h_2)-\frac{\partial}{\partial u_2}(A_1h_1)\right)\overrightarrow{e_3} \end{aligned} \tag{2.14}$$

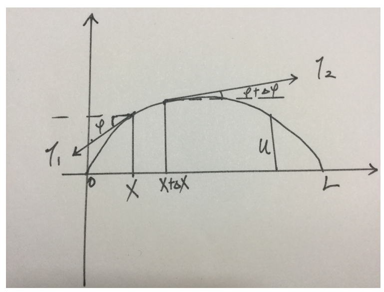
$$curl\overrightarrow{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\overrightarrow{e_1} & h_2\overrightarrow{e_2} & h_3\overrightarrow{e_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix} \tag{2.15}$$

Lecture 3 (pde971_97101)

Some famous PDEs contains:

1. Wave Equation (d'Alembert D. Bernoulli 1-dim , L. Euler 1-dim 2-dim , Pairs Prussia St. Petersburg
2. Heat Equation (Fourier series transform)
3. Euler Equation (Navier-Stokes Equation)
4. Maxwell Equation (Vector Analysis)
5. Boltzmann Equation (Kinetic Theory Probability Statistics)
6. Schrodinger Equation (Functional Analysis)

Wave Equation



u(x,t): displacement at time of the point of the string with abscissa x, $[u] = L$.
$\mathcal{T}_1$: tension force at A $(\to x)$.
$\mathcal{T}_2$: tension force at B $(\to x + \Delta x)$.
$[\mathcal{T}_1] = [\mathcal{T}_2] = [ma] = M\frac{L}{T^2}$.
Assume $\mathcal{T}_1 = \mathcal{T}_2$, Force on $\widehat{AB}$

$$\begin{aligned} \mathcal{T}[sin(\varphi + \Delta\varphi) - sin\varphi] &\approx \mathcal{T}[\tan(\varphi + \Delta\varphi) - \tan\varphi] \\ &= \mathcal{T}[\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}] \\ &\overset{MVT}{=} \mathcal{T}\frac{\partial^2 u}{\partial x^2}(x + \theta x, t)\Delta x \qquad (0 < \theta < 1) \end{aligned} \tag{3.1}$$

u : displacement, $\frac{\partial u}{\partial t}$: velocity(speed), $\frac{\partial^2 u}{\partial t^2}$: acceleration

Newton 2^{nd} law : $F = ma$

ρ : linear density of the string (mass per unit length) $[\ \rho\ ] = \frac{M}{L}$

$$\mathcal{T}\frac{\partial^2 u}{\partial x^2}\Delta x = \rho\Delta x\frac{\partial^2 u}{\partial t^2} \tag{3.2}$$

$$\mathcal{T}\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} \tag{3.3}$$

Homogenous wave eq.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\mathcal{T}}{\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{3.4}$$

where $c^2 = \frac{\mathcal{T}}{\rho}$

$[c]^2 = \frac{[\mathcal{T}]}{[\rho]} = \frac{M^{L}/T^2}{M/_L} = \frac{L^2}{T^2} = [\frac{L}{T}]^2$

$[c] = \frac{L}{T}$: speed

Forcing term

$F_1(x,t)$: force per unit length of the string

$$\begin{aligned} \rho\Delta x\frac{\partial^2 u}{\partial t^2} &= \mathcal{T}\frac{\partial^2 u}{\partial x^2}\Delta x + F_1(x,t)\Delta x \\ \rho\frac{\partial^2 u}{\partial t^2} &= \mathcal{T}\frac{\partial^2 u}{\partial x^2} + F_1 \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\mathcal{T}}{\rho}\frac{\partial^2 u}{\partial x^2} + \frac{F_1}{\rho} \end{aligned} \tag{3.5}$$

Higher dimensional wave eq.

Ex: vibration of membrane

$\Box_c = \frac{\partial^2}{\partial u^2} - c^2\Delta$ $\quad d\,'Alembertian$. Where $\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + ... + \frac{\partial^2}{\partial {x_n}^2}$

$$\begin{cases} \Box_c u = \dfrac{\partial^2 u}{\partial {x_1}^2} - c^2\Delta u = 0 \\ \Box_c u = \dfrac{\partial^2 u}{\partial t^2} - c^2\Delta u = f(x,t) \end{cases} \tag{3.6}$$

where $f(x,t) = f_0(x)e^{i\omega t}$

$\overset{Expect}{\Longrightarrow} u(x,t) = u_0(x)e^{i\omega t}$

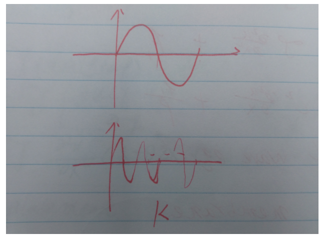
Klein-Gordion

$\Delta u_0 + k^2 u_0 = -\frac{f_0(x)}{c^2}$, $\quad k^2 = \frac{\omega^2}{c^2}$, $\quad$ (Helmholtz eq. scattering: scattering).

$e^{i\omega t}$: transcendental function (dimensionless).

$$\begin{aligned} [\omega t] = 1 \Rightarrow [\omega] &= \frac{1}{[t]} : \text{ frequency} \\ [k^2] &= \frac{[\omega]^2}{[c]^2} = \frac{\frac{1}{T^2}}{\frac{L^2}{T^2}} = \frac{1}{L} \\ [k] &= \frac{1}{L} : \text{wave number} \end{aligned} \tag{3.7}$$

sinkx



Ex: (Geometric optics)

Ex: $\frac{\partial^2 v}{\partial t^2} - c^2(\overrightarrow{x})\Delta v = 0$ c: wave speed

$\frac{[v]}{T^2} = [c]^2\frac{[v]}{L^2} \Rightarrow [c] = \frac{L}{T}$

$v(x, y, z, t) = e^{i\omega t}\psi(x, y, z)$ ω : frequency.

$k = \frac{\omega}{c_0}$ c_0 : average speed. $[k] = \frac{1}{L}$ wave number.

$V(x, y, z, t) = e^{ic_0 kt}\psi(x, y, z) \Rightarrow \Delta\psi + k^2 V(\overrightarrow{x}) = 0$ (Helmholtz eq.)

$m = \frac{c_0}{c}$ reflection index.

Q: How about k large ? geometrical optics $\leftrightarrow$ classical optics

$\psi(x, y, z) = A(x, y, z)e^{iks(x,y,z)}$

Helmholtz eq. $\Rightarrow A(|\nabla s|^2 - n^2) = o(\frac{1}{k})$

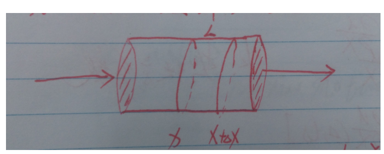
$|\nabla s|^2 - n^2 = 0$ Eikonal Eq.

William Rowam Hamilton 1805 - 1865.

Heat Equation

1-dim heat eq.

Newton's law of cooling



A : sectional area L length

(1) $u(x, t)$: temperature

(2) $\rho(x, t)$: density

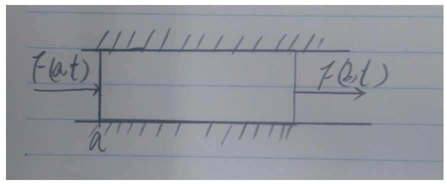
(3) c : specific heat $[c] = \frac{[\text{quantity of heat}]}{[\text{quality}][\text{temperature}]}$

the amount of heat required for a single unit of mass of a substance to be raised by one degree of temperature high temperature $\to$ low temperature

$x \to x + dx$ mass $M = Adx \cdot \rho = \rho Adx$ $[\rho] = \frac{M}{L^3}$

$x \to x + dx$ heat $u(x,t)c\rho \cdot Adx \Rightarrow x = a \to x = b$ heat $Q = \int_a^b u(x,t)c\rho \cdot Adx$

conservation of energy $\frac{dQ}{dt}$ = the term of heat flux + the term of heat source



$F(x,t)$: heat flux

the term of heat flux $= -A(F(b,t) - F(a,t))$

Fourier law : the time rate of heat transfer through a material is proportional to the negative gradient of temperature and to the area at right angles through which the heat flows. $F \propto -\frac{\partial u}{\partial x} \Rightarrow F(x,t) = -k\frac{\partial u}{\partial x}$ k: the heat transfer coefficient

By MVT:$A[k\frac{\partial u}{\partial x}(b,t) - k\frac{\partial u}{\partial x}(a,t)] = \int_a^b \frac{\partial}{\partial x}(k\frac{\partial u}{\partial x}(x,t))Adx$

$\frac{dQ}{dt} = \int_a^b \frac{\partial u}{\partial t}c\rho Adx$

By conservation law $\int_a^b [c\rho\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(k\frac{\partial u}{\partial x})]Adx$ = the term of heat source, assume the term of heat source = 0

$c\rho\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(k\frac{\partial u}{\partial x})Adx = 0 \ \Rightarrow \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \qquad K = \frac{k}{c\rho} > 0$

If q_1 is the heat source $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + q(x,t,u)$ $q = \frac{q_1}{c\rho}$, $\frac{[u]}{T} = [K]\frac{[u]}{L^2} \Rightarrow [K] = \frac{L^2}{T}$

Heat eq. $\frac{\partial u}{\partial t} = k\Delta u + q(x,t,u)$

nonhomogeneous media

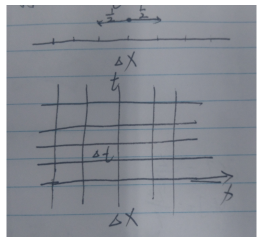
$$\begin{aligned} c(x)\rho(x)\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}[k(x)\frac{\partial u}{\partial x}] - q(x,t,u) = 0 \\ \frac{\partial u}{\partial t} = \frac{1}{\sigma(x)}\frac{\partial}{\partial x}[k(x)\frac{\partial u}{\partial x}] + q(x,t,u) \end{aligned} \tag{3.8}$$

$\sigma(x) = c(x)\rho(x)$: specific heat per unit volume.

Lecture 4 (pde971_971003)

Random walk

Diffusion Eq (Brownian motion)



$t_k = k\Delta t,\ k = 1, 2, 3...\ \ x_k = k\Delta x,\ X(t,x)$: random varible $X(t,x) = \begin{cases} 1, & \text{particle is at } x \\ 0, & \text{particle is not at } x \end{cases}$

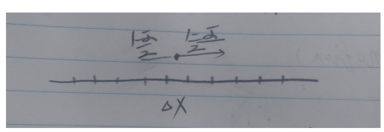
$u(t_k, x_k) = P\{x(t_k, x_k) = 1\}$ probability.

$$\begin{aligned} u(t_{k+1}, x_k) &= \frac{1}{2}u(t_k, x_{k-1}) + \frac{1}{2}u(t_k, x_{k+1}) \\ \underbrace{u(t_{k+1}, x_k) - u(t_k, x_k)}_{LHS} &= \underbrace{\frac{1}{2}[(u(t_k, x_{k-1}) - u(t_k, x_k)) - (u(t_k, x_k) - u(t_k, x_{k-1}))]}_{RHS} \end{aligned} \tag{4.1}$$

difference equation, 1st order difference $\frac{(f(x+\Delta x))-(f(x)-f(x-\Delta x))}{\Delta x^2}$

Assume "$\Delta t = \Delta x^2$", (parabolic scaling)

$\frac{LHS}{\Delta t} = \frac{RHS}{\Delta x^2} \qquad \xrightarrow{\Delta t, \Delta x \to 0} \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$



$c = \bar{\alpha}\frac{\Delta x}{\Delta t},\ v = \frac{1}{2}\frac{(\Delta x)^2}{\Delta t}$

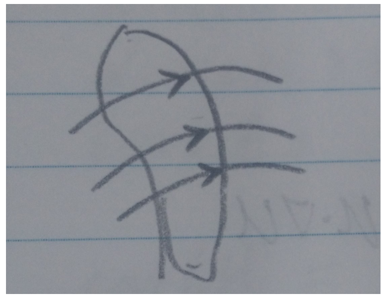
$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$

c: drift velocity , $[c] = \frac{L}{T}$, ν : diffusion coefficient $[\nu] = \frac{L^2}{T}$

Hydrodynamic Equation u : density; $\overrightarrow{n}$: outside normal; f : flux.

conservation law:

A conservation law asserts that the rate of the change of the total amount of subspace contained in a filed domain v is equal to flux of that subspace across the boundary ∂v of v.



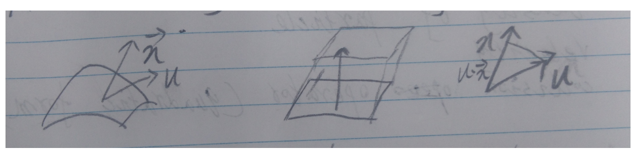
$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{v} u dx = -\int\limits_{\partial v} \overrightarrow{f}\overrightarrow{n} ds \\
&\int_v (\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{f}) dx = 0 \\
&\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{f} = 0
\end{aligned} \tag{4.2}
$$

Euler Equation

fluid mechanic

nonviscous

ρ : mass density; u : velocity.



The volume flow rate across ∂v, $u \cdot \overrightarrow{n} ds$: high.

$\rho u \cdot \overrightarrow{n} ds$ mass

$\frac{d}{dt}\int_v \rho dx = -\int_{\partial v} \rho u \cdot \overrightarrow{n} ds$

$\frac{\partial \rho}{\partial t} + div(\rho \overrightarrow{u}) = 0$

Euler (compressible)

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + div(\rho u) = 0 : \qquad mass \\
\rho \dfrac{Du}{Dt} = -\nabla p + \rho b : \qquad \dfrac{Du}{Dt} = \dfrac{\partial u}{\partial t} + u \cdot \nabla u
\end{cases} \tag{4.3}
$$

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + div(\rho u) = 0 \\
\dfrac{\partial (\rho u)}{\partial t} + div(\rho u \otimes u) + \nabla p = 0
\end{cases} \tag{4.4}
$$

where $p = p(\rho) = A\rho^r$.

Incompressible Euler Eq.

$$\begin{cases} \rho = 1, \quad divu = 0 \ \ \text{constancy of volume} \\ \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0 \end{cases} \tag{4.5}$$

Boltzmann Equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f, f)$$

$f \geq 0$: density of particle

v : velocity

Q : collision operator (quadratic form)

Schrodinger Equation

$$\begin{aligned} &ih\partial_t\psi = H\psi \quad \text{Hamiltoniam} \\ &ih\partial_t\psi + \frac{h^2}{\partial m}\Delta\psi \pm v(|\psi|^2) = 0 \end{aligned} \tag{4.6}$$

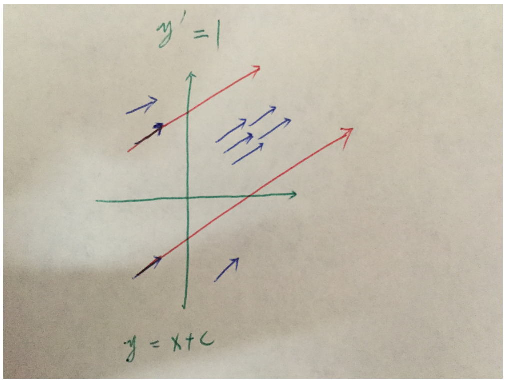
Maxwell Equation

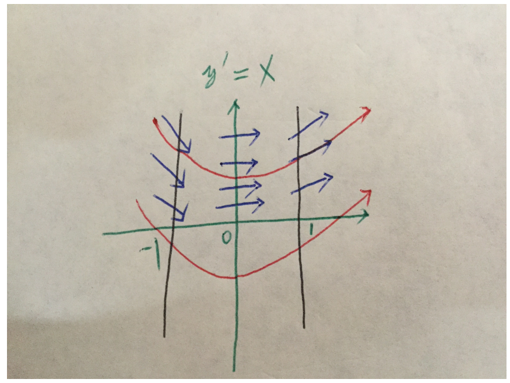
$$\begin{cases} div(\varepsilon E) = 4\pi\rho & \text{E: electronic field} \\ div(\mu H) = 0 & \text{H: magnetic fileld} \\ curlE = \dfrac{1}{c}\dfrac{\partial(\mu H)}{\partial t} & \text{Faraday Law} \\ curlH = \dfrac{1}{c}\dfrac{\partial(\varepsilon E)}{\partial t} + \dfrac{4\pi}{c}I & \text{Ampere Law, } \dfrac{4\pi}{c} \text{ : Maxwell} \end{cases} \tag{4.7}$$

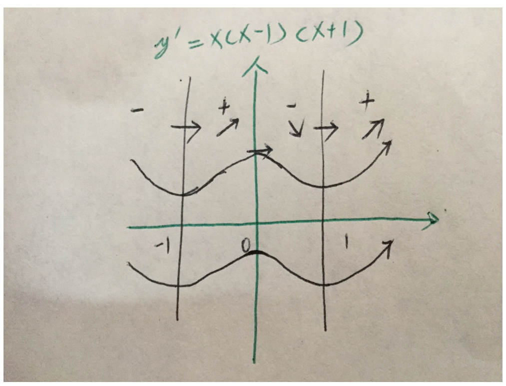
Lecture 5 (pde971_971008)

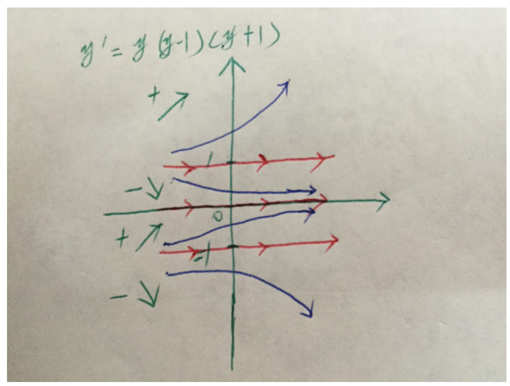
1st order PDE

The integration of 1st order PDE reduces to the integration of a system of ODEs the so called characteristic eq.. The basis of the reduction is a simple geometric analysis of the formulation of surfaces from familiar of curves. method of characteristic 1st PDE(surface) $\Leftrightarrow$ system of ODE. Recall ODE Hamilton (phopagation of light), geometric optics; C. Huygens (1629-1695) Dutch; T.









Young (1773-1829) English; A. Frieshel (1788-1872) French; Fermat, least time principle

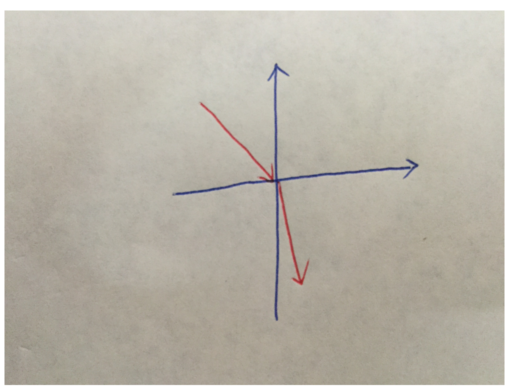
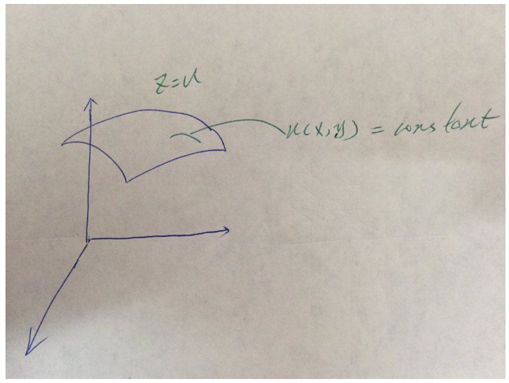


Figure 1: reflection law



$$Q: \quad au_x + bu_y = c \quad (1st\ order\ PDE) \tag{5.1}$$

Assume $u(x,y) = d$ solution surface $du = u_x dx + u_y dy = 0$.

$$\begin{cases} (u_x, u_y, -1) \cdot (dx, dy, du) = 0 \\ (u_x, u_y, -1) \cdot (a, b, c) = 0 \end{cases} \Rightarrow \quad (dx, dy, du) \,//\, (a, b, c) \tag{5.2}$$

(D.E.) Given tangent vector $(a,b,c) \Rightarrow u(x,y) =$? Ans: Hamilton's method of characteristic

$$au_x + bu_y = c \Leftrightarrow \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad (characteristic) \tag{5.3}$$

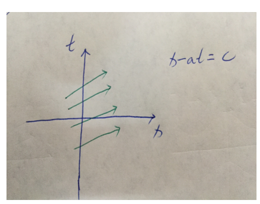
Theorem 5.1. *Every surface $z = u(x,y)$ generated by a one parameter family of characteristic curves ia an integral surface of PDE; conversely, every integral surface $z = u(x,y)$ is generated by one parameters family of characteristic curves.*

single 1st order Eq.: $F(x, u, \partial u) = 0$.

Example 5.1 (Advention)**.**

$$\partial_t u + a \cdot \nabla_x u = 0 \quad (\frac{dx}{dt} = a) \tag{5.4}$$

consider $u(x,y) = f(x - at)$ $(\frac{du}{dt} = 0)$ (traveling wave). $u(x,0) = u_I(x) = f(x)$. consider



$x = x(t, x_0)$, $\frac{dx}{dt} = \dot{x} = a \Rightarrow x = at + x_0$, *i.e.* $x_0 = x - at$. now, for any x

$$\frac{d}{dt} u(x_0 + at, t) = \frac{\partial u}{\partial t} + a \cdot \frac{\partial u}{\partial x} = 0 \tag{5.5}$$

along

$$\begin{cases} \dot{x} = a \\ \dot{u} = 0 \end{cases} \Rightarrow \quad \begin{cases} x = x_0 + at \\ u = u(x_0) \end{cases} \Rightarrow \begin{cases} x_0 = x - at \\ u = u_I(x - at) \end{cases} \tag{5.6}$$

Example 5.2.

$$\frac{du}{dt} = \partial_t u + \frac{dx}{dt} \nabla_x u = 0 \tag{5.7}$$

$$\begin{cases} \partial_t u + a(x) \nabla_x u = 0 \\ u(x,0) = u_I(x) \end{cases} \tag{5.8}$$

$$\begin{cases} \dot{x} = a(x) \\ \dot{u} = 0 \end{cases} \Rightarrow \begin{cases} x = \varphi^t(x_0) \\ u = u_I(x_0) \end{cases} \Rightarrow \begin{cases} x_0 = \left(\varphi^t\right)^{-1}(x) \\ u = u_I(x_0) \end{cases} \tag{5.9}$$

φ^t deformation u

1. solve the characteristic ODE
2. eliminate x_0

$u = u_I\left(\varphi^t(x)\right)$

more general (semi-linear)

$$a(x) \nabla_x u + b(x,u) = 0 \tag{5.10}$$

Remark 5.1.

$$\Delta u + u^p = 0 \tag{5.11}$$

Let $\cdot = \frac{ds}{dt}$, the characteristic eq.

$$\begin{cases} \dot{x} = a(x) \quad x(0) = x_0 \\ \dot{u} = -b(x,u) \quad u_0 = u_I(x_0) \end{cases} \tag{5.12}$$

Example 5.3.

$$\begin{cases} 2xu_x + yu_y = \alpha u \\ u = \varphi(x) \quad on\ y = 1 \end{cases} \tag{5.13}$$

characteristic Eq.

$$\begin{cases} \dot{x} = 2x \quad x(0) = x_0 \\ \dot{y} = y \qquad y(0) = 1 \\ \dot{u} = \alpha u \quad u(0) = \phi(x_0) \end{cases} \Rightarrow \begin{cases} x = e^{2s} x_0 \\ y = e^s \\ u = e^{2s} \varphi(x_0) \end{cases} \tag{5.14}$$

by inverse function thm, eliminatex_0, s

$$\begin{cases} x = y^2 x_0 \\ u = y^\alpha \varphi(x_0) \end{cases} \Rightarrow u = y^\alpha \varphi\left(\frac{x}{y^2}\right) \tag{5.15}$$

Example 5.4 (quasi-linear Eq.)**.**

$$a(x,u) \nabla_x u + b(x,u) = 0 \tag{5.16}$$

Assume $u = \varphi(\xi)\ \ on\ x = x_0(\xi)$, characteristic Eq.

$$\begin{cases} \dot{x} = a(x,u) \quad x(0) = x_0(\xi) \\ \dot{u} = -b(x,u) \quad u(0) = \varphi(\xi) \end{cases} \tag{5.17}$$

solve the characteristic Eq.

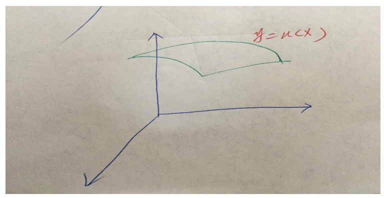
$$\begin{cases} x = x(s,\xi) \\ u = u(s,\xi) \end{cases} \tag{5.18}$$

eliminate s, ξ by I. F. Thm.

Geometric interpretation
The general quasi-linear Eq. may be written as

$$a(x,u)\nabla_x u + b(x,u) = 0 \tag{5.19}$$

A solution $u(x)$ defines an integral curve (surface) $y = u(x)$



The normal is $\begin{pmatrix} -\nabla_x u \\ 1 \end{pmatrix}$, the characteristic vector field is $\begin{pmatrix} a \\ -b \end{pmatrix}$. $normal \cdot vectorfield = 0$.
The equation (PDE) can be interpreted as the condition that integral surface at each point has the property that the vector $(a, -b)$ is tangent to the surface.

Example 5.5 (Semilinear)**.**

$$\begin{cases} u_x + u_y = u^2 \\ u = \dfrac{x-y}{2} \quad on\ x + y = 0 \end{cases} \tag{5.20}$$

Let $x = \xi, y = -\xi$ on $x + y = 0$; $u(0) = \xi,\ on\ x(0) = \xi, y(0) = -\xi$. characteristic Eq.

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \\ \dot{u} = u^2 \end{cases} \Rightarrow \begin{cases} x = s + \xi \\ y = s - \xi \\ y = \dfrac{\xi}{1 - \xi s} \end{cases} \Rightarrow u = \frac{2(x-y)}{4x^2 + y^2} \tag{5.21}$$

Example 5.6 (Eikonal Eq.)**.**

$$u_x^2 + u_y^2 = n^2 \tag{5.22}$$

where n is refraction index.

$$\begin{cases} [n] = 1 \\ [u] = L \end{cases} \Rightarrow [u_x] = [u_y] = [n] \tag{5.23}$$

Characteristic Eq.

$$\begin{cases} \dfrac{dx}{dt} = u_x \\ \dfrac{dy}{dt} = u_y \\ \dfrac{du}{dt} = n^2 \end{cases} \tag{5.24}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}u_x = u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} = u_{xx}u_x + u_{xy}u_y = \frac{1}{2}\left(u_x^2 + u_y^2\right) = \frac{1}{2}\left(n^2\right)_x \tag{5.25}$$

$$\frac{d^2y}{dt^2}=\frac{1}{2}\left(u_x^2+u_y^2\right)_y=\frac{1}{2}\left(n^2\right)_y \tag{5.26}$$

$$\frac{du}{dt}=u_x\frac{dx}{dt}+u_y\frac{dy}{dt}=u_x^2+u_y^2=n^2 \tag{5.27}$$

$$\Rightarrow u\left(x\left(t\right),y\left(t\right)\right)=u\left(x\left(0\right),y\left(0\right)\right)+\int_0^t n\left(\cdot\right)d\tau \tag{5.28}$$

Example 5.7 (Burgers Eq.)**.**

$$\begin{cases}\partial_t u+u\cdot\partial_x u=0\\ u\left(x,0\right)=\varphi\left(x\right)\end{cases} \tag{5.29}$$

characteristic Eq.

$$\begin{cases}\dot{x}=u & x\left(0\right)=\xi\\ \dot{u}=0 & u\left(0\right)=\varphi\left(\xi\right)\end{cases}\Rightarrow\begin{cases}x=\xi+\varphi\left(\xi\right)t\\ u\left(t\right)=\varphi\left(\xi\right)\end{cases} \tag{5.30}$$

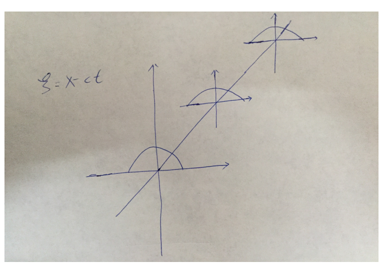
eliminate ξ, say $\xi=z\left(x,t\right)$, then

$$u=\varphi\left(z\left(x,t\right)\right) \tag{5.31}$$

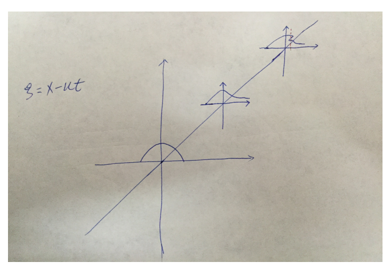
Comparison between linear and nonlinear

$$\begin{gathered}\begin{cases}\partial_t u+c\partial_t u=0\\ u\left(x,t\right)=f\left(x-ct\right)\end{cases}\qquad\qquad\begin{cases}\partial_t u+u\partial_x u=0\\ u\left(x,t\right)=f\left(x-ut\right)\end{cases}\\ I.C.\quad u\left(x,0\right)=\begin{cases}a^2-x^2 & \left|\xi\right|\leqslant a\\ 0,\ \left|\xi\right|>a\end{cases}\end{gathered} \tag{5.32}$$

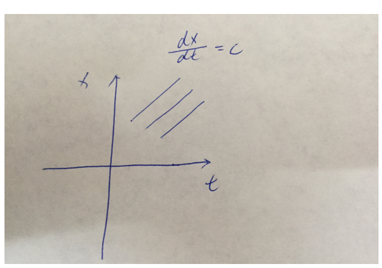
$$\Rightarrow u\left(x,t\right)=\begin{cases}a^2-\xi^2, & \left|\xi\right|\leqslant a\\ 0, & \left|\xi\right|>a\end{cases} \tag{5.33}$$



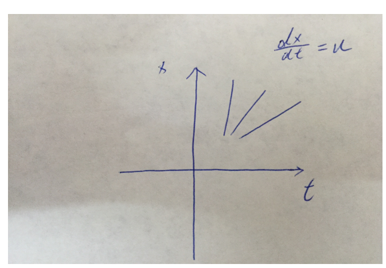
(a)



(b)



(c)



(d)

$$\frac{\partial u}{\partial t} = f' \cdot \frac{\partial}{\partial t}(x - ut) = f' \cdot \left(-u - t\frac{\partial u}{\partial t}\right) \Rightarrow \frac{\partial u}{\partial t} = -\frac{f' \cdot u}{1 + tf'} \tag{5.34}$$

$$\frac{\partial u}{\partial x} = \frac{f'}{}\ 1 + tf', 1 + tf' = 0 \Rightarrow t = -\frac{1}{f'} \quad (blow \quad up) \tag{5.35}$$

Lecture 6 (pde971_971015)

Lecture 7 (pde971_971017)