

General relativity and black holes



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General relativity primer

Lagrangian for standard model of particle physics

1	$-\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \frac{1}{2}ig_s^2 (\bar{q}_i^\sigma \gamma^\mu q_j^\sigma) g_\mu^a + \bar{G}^a \partial^2 G^a + g_s f^{abc} \partial_\mu \bar{G}^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- -$
2	$M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_w^2} M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h [\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-)] + \frac{2M^4}{g^2} \alpha_h - ig c_w [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - Z_\nu^0 (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + Z_\mu^0 (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+)] - igs_w [\partial_\nu A_\mu (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - A_\nu (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+)] - \frac{1}{2}g^2 W_\mu^+ W_\mu^- W_\nu^+ W_\nu^- + \frac{1}{2}g^2 W_\mu^+ W_\nu^- W_\mu^+ W_\nu^- + g^2 c_w^2 (Z_\mu^0 W_\mu^+ Z_\nu^0 W_\nu^- - Z_\mu^0 Z_\mu^0 W_\nu^+ W_\nu^-) + g^2 s_w^2 (A_\mu W_\mu^+ A_\nu W_\nu^- - A_\mu A_\mu W_\nu^+ W_\nu^-) + g^2 s_w c_w [A_\mu Z_\nu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - 2A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^-] - \frac{1}{8}g^2 \alpha_h [H^4 + (\phi^0)^4 + 4(\phi^+ \phi^-)^2 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - g M W_\mu^+ W_\mu^- H - \frac{1}{2}g \frac{M}{c_w^2} Z_\mu^0 Z_\mu^0 H - \frac{1}{2}ig [W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W_\mu^- (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \frac{1}{2}g [W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - W_\mu^- (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \frac{1}{2}g \frac{1}{c_w} (Z_\mu^0 (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - ig \frac{s_w^2}{c_w} M Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + igs_w M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - ig \frac{1-2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \frac{1}{4}g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \frac{1}{4}g^2 \frac{1}{c_w^2} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) - \frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W_\mu^+ \phi^- - W_\mu^- \phi^+) - g^2 \frac{s_w}{c_w} (2c_w^2 - 1) Z_\mu^0 A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma \partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma \partial \nu^\lambda - \bar{u}_j^\lambda (\gamma \partial + m_u^\lambda) u_j^\lambda - \bar{d}_j^\lambda (\gamma \partial + m_d^\lambda) d_j^\lambda + igs_w A_\mu [-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3}(\bar{d}_j^\lambda \gamma^\mu d_j^\lambda)] + \frac{ig}{4c_w} Z_\mu^0 [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (\frac{4}{3}s_w^2 - 1 - \gamma^5) u_j^\lambda) + (\bar{d}_j^\lambda \gamma^\mu (1 - \frac{8}{3}s_w^2 - \gamma^5) d_j^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1 + \gamma^5) C_{\lambda\kappa} d_j^\kappa)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}_j^\kappa C_{\lambda\kappa}^\dagger \gamma^\mu (1 + \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m_e^\lambda}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] -$
3	$\frac{g}{2} \frac{m_e^\lambda}{M} [H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 - \gamma^5) d_j^\kappa) + m_u^\lambda (\bar{u}_j^\lambda C_{\lambda\kappa} (1 + \gamma^5) d_j^\kappa)] + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) u_j^\kappa) - m_u^\kappa (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \gamma^5) u_j^\kappa)] - \frac{g}{2} \frac{m_u^\lambda}{M} H(\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_d^\lambda}{M} H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_u^\lambda}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_d^\lambda}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda) + \bar{X}^+(\partial^2 - M^2) X^+ + \bar{X}^-(\partial^2 - M^2) X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2}) X^0 + \bar{Y} \partial^2 Y + ig c_w W_\mu^+ (\partial_\mu \bar{X}^0 X^- - \partial_\mu \bar{X}^+ X^0) + igs_w W_\mu^+ (\partial_\mu \bar{Y} X^- -$
4	$\frac{g}{2} \frac{m_e^\lambda}{M} [H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 - \gamma^5) d_j^\kappa) + m_u^\lambda (\bar{u}_j^\lambda C_{\lambda\kappa} (1 + \gamma^5) d_j^\kappa)] + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) u_j^\kappa) - m_u^\kappa (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \gamma^5) u_j^\kappa)] - \frac{g}{2} \frac{m_u^\lambda}{M} H(\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_d^\lambda}{M} H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_u^\lambda}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_d^\lambda}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda) + \bar{X}^+(\partial^2 - M^2) X^+ + \bar{X}^-(\partial^2 - M^2) X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2}) X^0 + \bar{Y} \partial^2 Y + ig c_w W_\mu^+ (\partial_\mu \bar{X}^0 X^- - \partial_\mu \bar{X}^+ X^0) + igs_w W_\mu^+ (\partial_\mu \bar{Y} X^- -$
5	$\frac{g}{2} \frac{m_e^\lambda}{M} [H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 - \gamma^5) d_j^\kappa) + m_u^\lambda (\bar{u}_j^\lambda C_{\lambda\kappa} (1 + \gamma^5) d_j^\kappa)] + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) u_j^\kappa) - m_u^\kappa (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \gamma^5) u_j^\kappa)] - \frac{g}{2} \frac{m_u^\lambda}{M} H(\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_d^\lambda}{M} H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_u^\lambda}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_d^\lambda}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda) + \bar{X}^+(\partial^2 - M^2) X^+ + \bar{X}^-(\partial^2 - M^2) X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2}) X^0 + \bar{Y} \partial^2 Y + ig c_w W_\mu^+ (\partial_\mu \bar{X}^0 X^- - \partial_\mu \bar{X}^+ X^0) + igs_w W_\mu^+ (\partial_\mu \bar{Y} X^- -$

gluon (strong force)

W and Z bosons
(weak force)

action

$$S = \int \mathcal{L} \sqrt{-g} d^4x$$

Lagrange equations

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) + \dots = 0$$

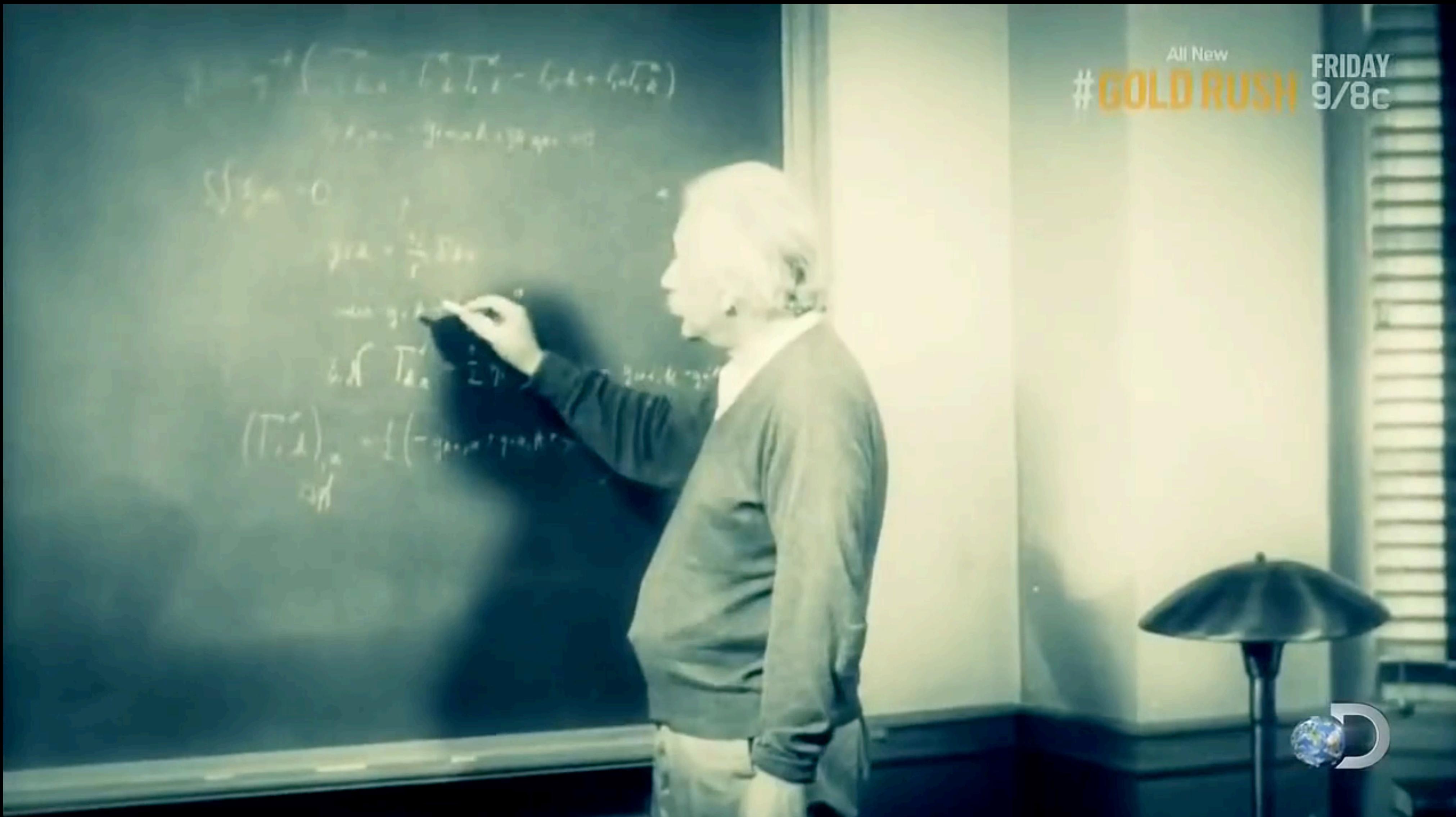
weak interactions +
Higgs

Higgs ghosts

Faddeev-Popov ghosts

slides from black hole primer for
undergrads

Basic idea of general relativity: GRAVITY = SPACETIME CURVATURE



A general relativity primer

Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

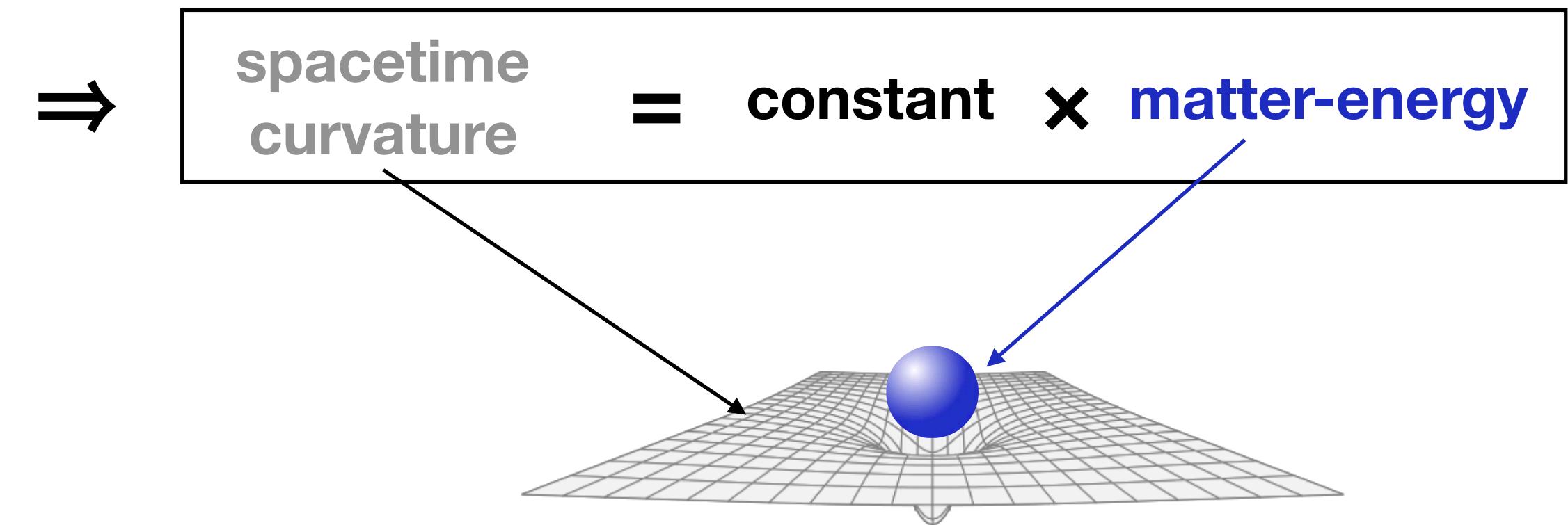
Ricci curvature Metric Ricci scalar Stress-energy

A general relativity primer

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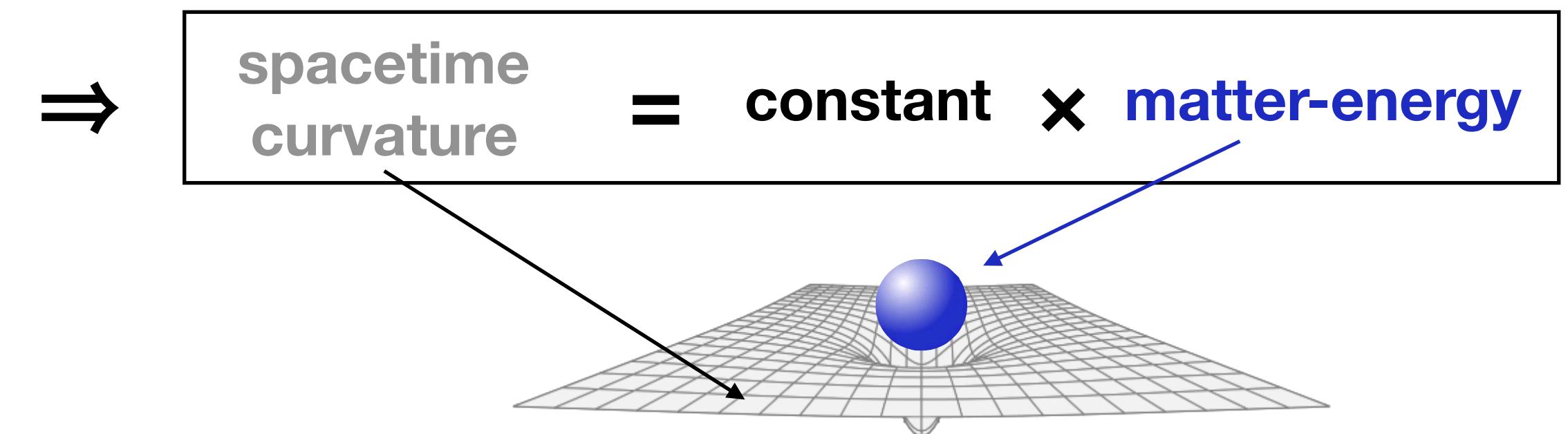


A general relativity primer

Einstein's field equation

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Ricci curvature Metric Ricci scalar Stress-energy



Solution to field equation gives

$$g_{\mu\nu} \quad ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Metric Line element



Newtonian analogue

$$\nabla^2 \phi = 4\pi G\rho \quad \text{Poisson equation}$$

For a free particle:

$$\delta S = 0 \rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

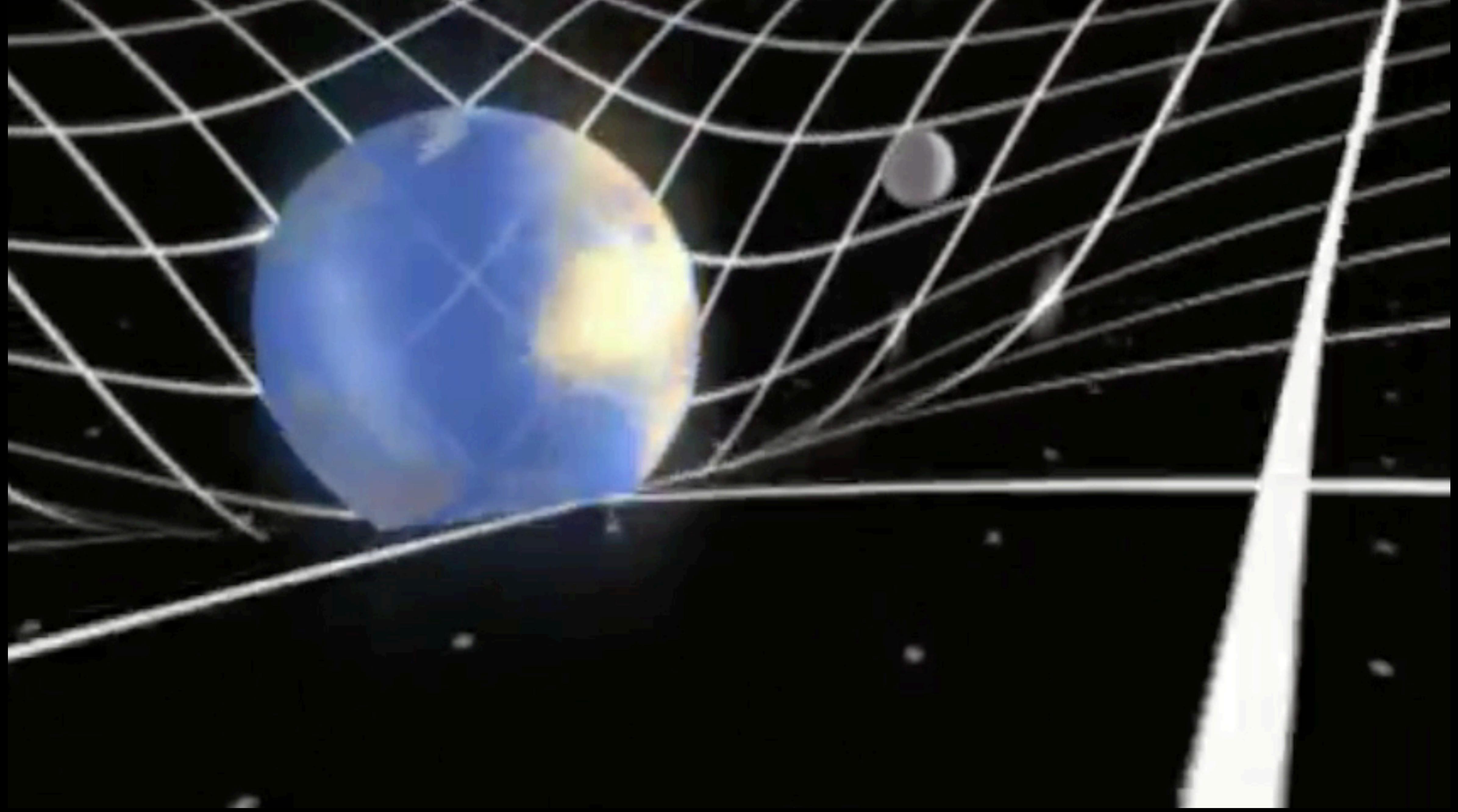
Geodesic equation



Gravity visualized: <https://www.youtube.com/watch?v=MTY1Kje0yLg&list>



Gravity visualized: <https://www.youtube.com/watch?v=MTY1Kje0yLg&list>



The Elegant Universe. Nova / PBS

A concise tutorial on general relativity

DOI: 10.1119/1.12853

General relativity primer

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In this tutorial article the physical ideas underlying general relativity theory are discussed and the basic mathematical techniques (tensor calculus, Riemann curvature) needed to describe them are developed. The general relativity field equations are presented and are used in several applications including a discussion of black holes.

I. INTRODUCTION

A. Purpose and outline

Special relativity theory (SRT) is a part of the intellectual toolbox of all physicists and a feature of the physicist's education even at the undergraduate level. The novel concepts of SRT, so shocking in 1905, hold no special terror now. The same, regrettably, cannot be said for the general relativity theory (GRT), Einstein's relativistic theory of gravity. The imagery of space-time curvature, and such exotica as black holes, give GRT such a recondite aura that it is too often regarded as hopelessly mystical, even by students and teachers who accept quantum mechanics as a perfectly reasonable description of the world. It is my goal in this article to show that this viewpoint on GRT is unjustified, that relativistic gravity is intuitively accessible and that space-time curvature is a natural conceptual basis for it. More specifically this article presents the mathematical and

Clearly in a small article covering a large subject, sacrifices must be made. The most regrettable sacrifice will be the omission of all but a cursory discussion of the stress-energy tensor, the "source" of the gravitational field. Also omitted will be many mathematical details, some of them formal and elegant, some of them tricky and technical, some of them useful for reducing very difficult calculations to merely difficult ones. Missing too will be most of the applications of GRT to problems of current interest. A useful discussion is given, however, of that aspect of GRT that stimulates the most interest and confusion: black holes.

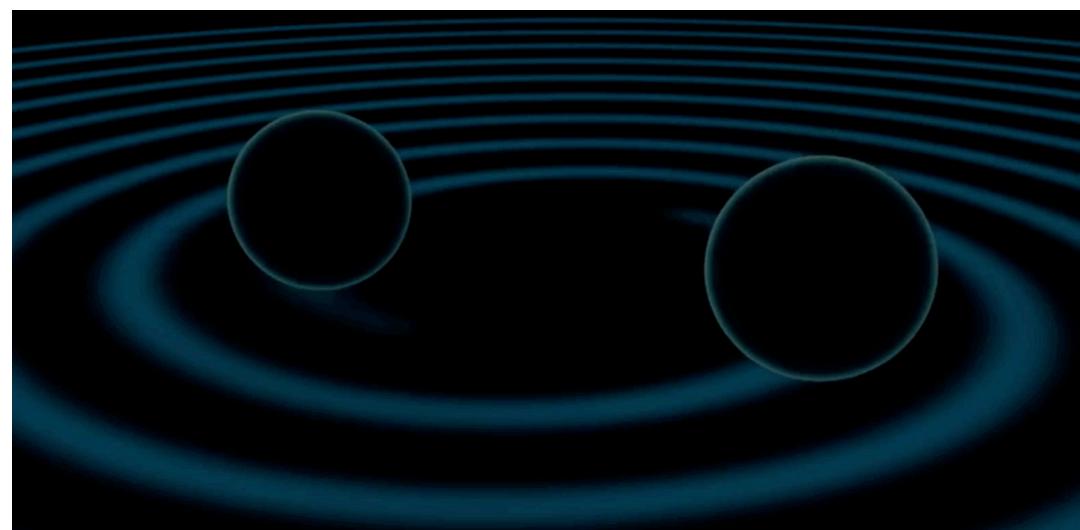
I assume that the reader comes to this article with two prerequisites: First, a familiarity is required with partial differential equations and their application in physics, as would certainly result from, say, a junior- or senior-level course in electrodynamics. Experience with partial differential equations will be necessary for an appreciation of the meaning of the GRT field equations; specific techniques

General relativity is a classical theory

- ✳ All equations of motion are **deterministic**. No probabilities involved
- ✳ Once we specify the initial positions and velocities of particles, everything is determined!
- ✳ Evolving the gravitational field and matter dynamics for astrophysical situations can be challenging → 

~~$\Psi(x, t)$~~

x^μ, v^μ



from GR undergraduate course, introduction

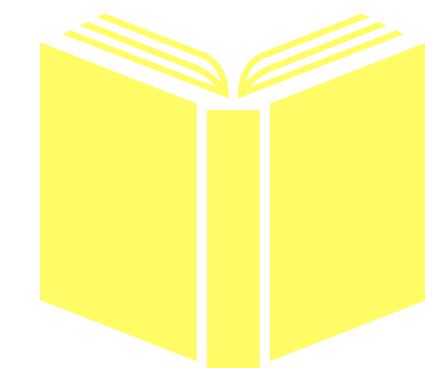
Once initial conditions are given, the physical truth is perfectly determined

Some *important properties* of gravitational interaction

Gravity is **unscreened:** there are no negative gravitational charges. It is not possible to shield the gravitational field

Gravity is a **long-range interaction.** There is no characteristic length scale for gravitational interactions

Gravity is the **weakest of fundamental interactions between elementary particles.**



These explain why gravity plays such a pivotal role in the universe

Gravity governs large scale structure formation in the universe

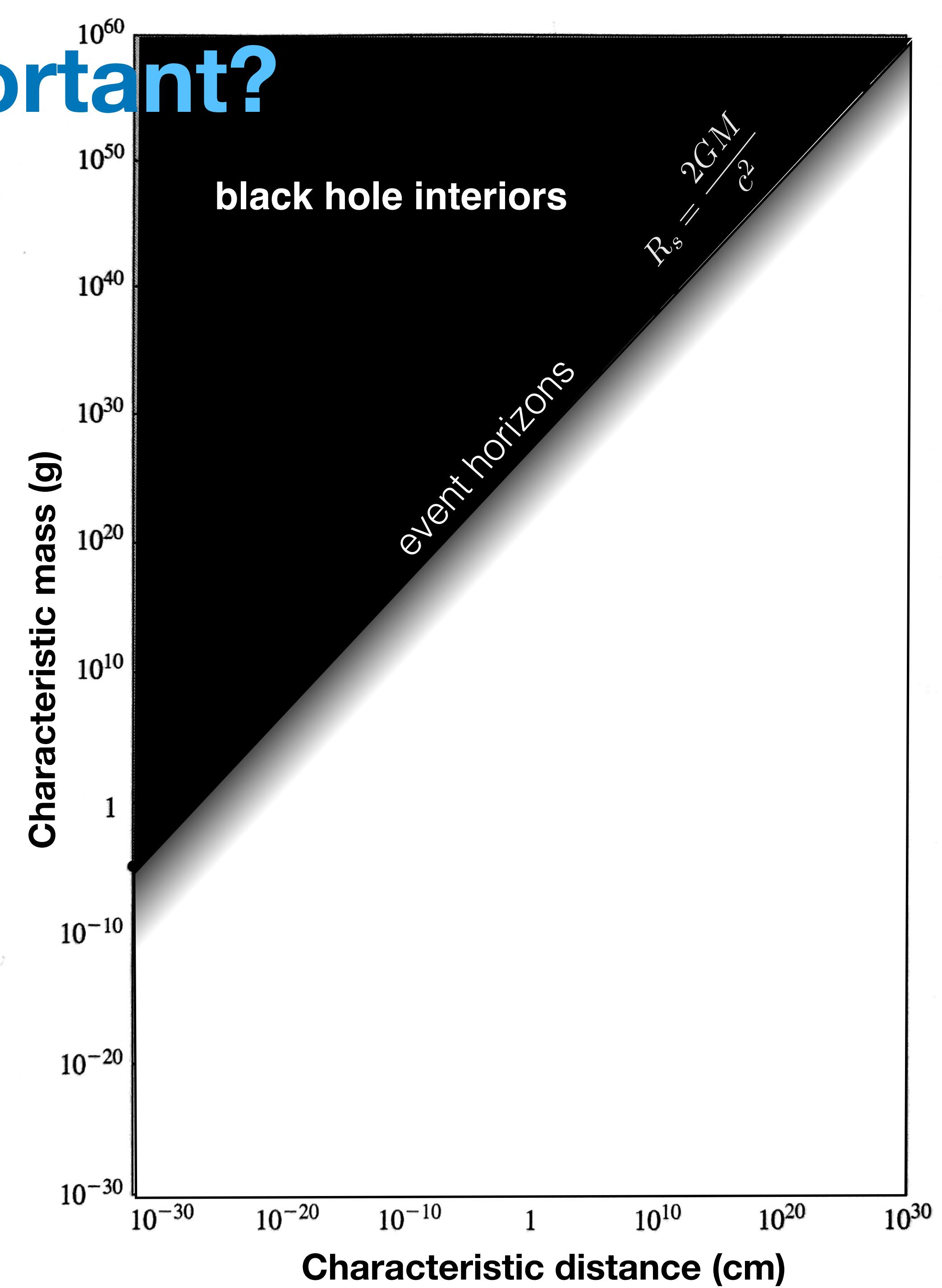
**Phenomena for which general
relativity is important**

When is general relativity important? vs. Newtonian gravity

Given object of mass M and size R

GR (general relativity) is important when

$$\frac{GM}{Rc^2} \rightarrow 1$$

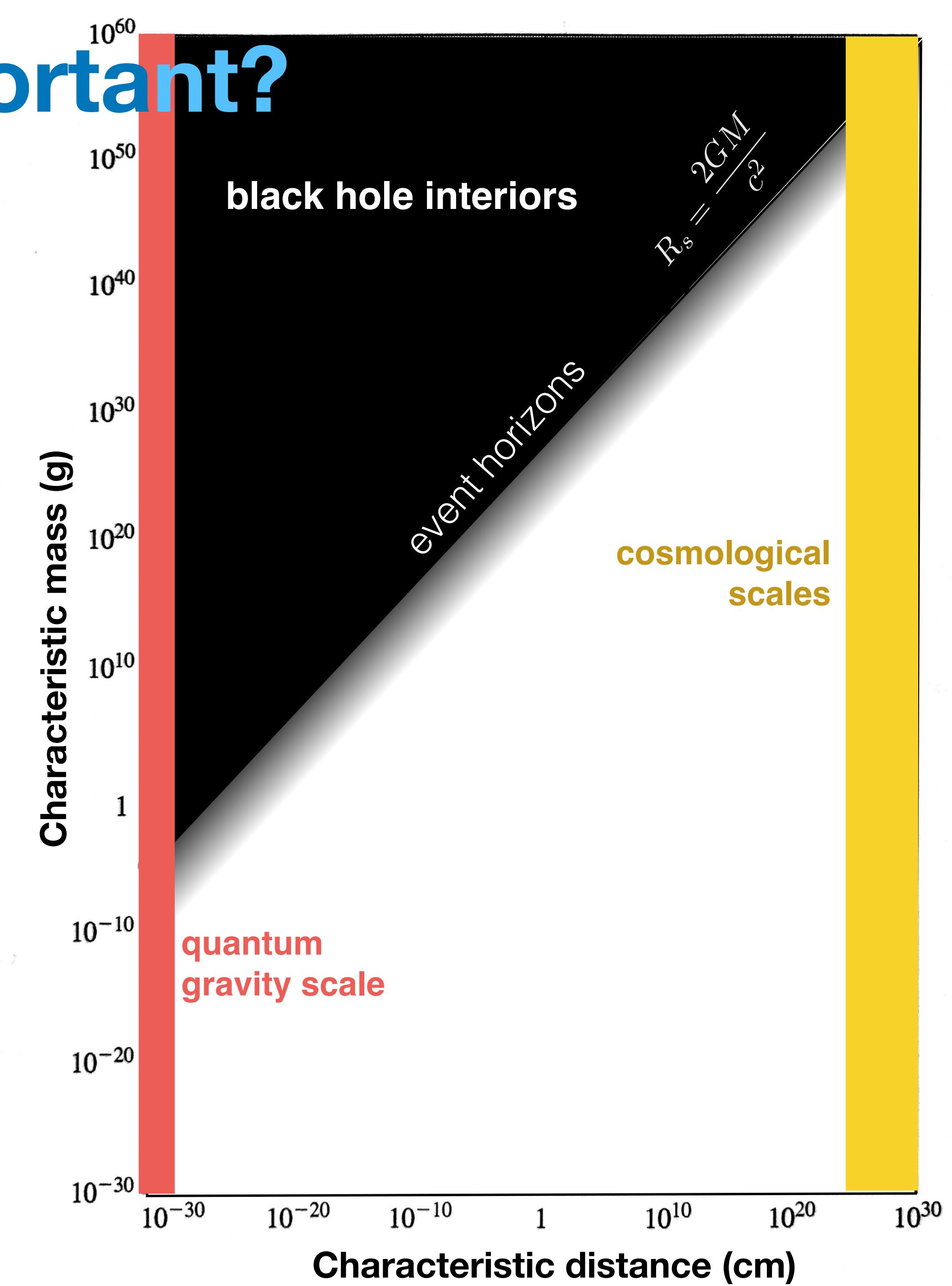


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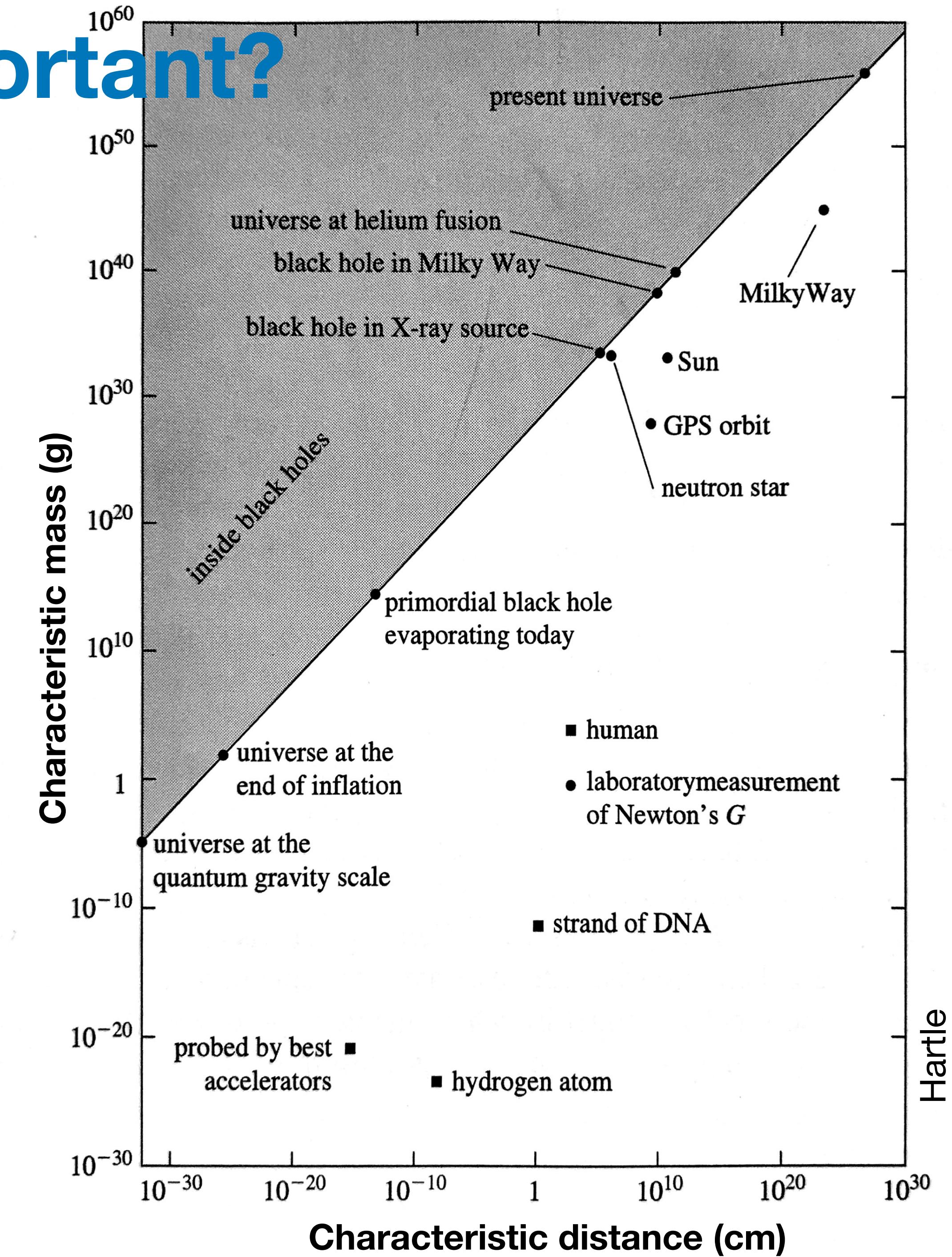


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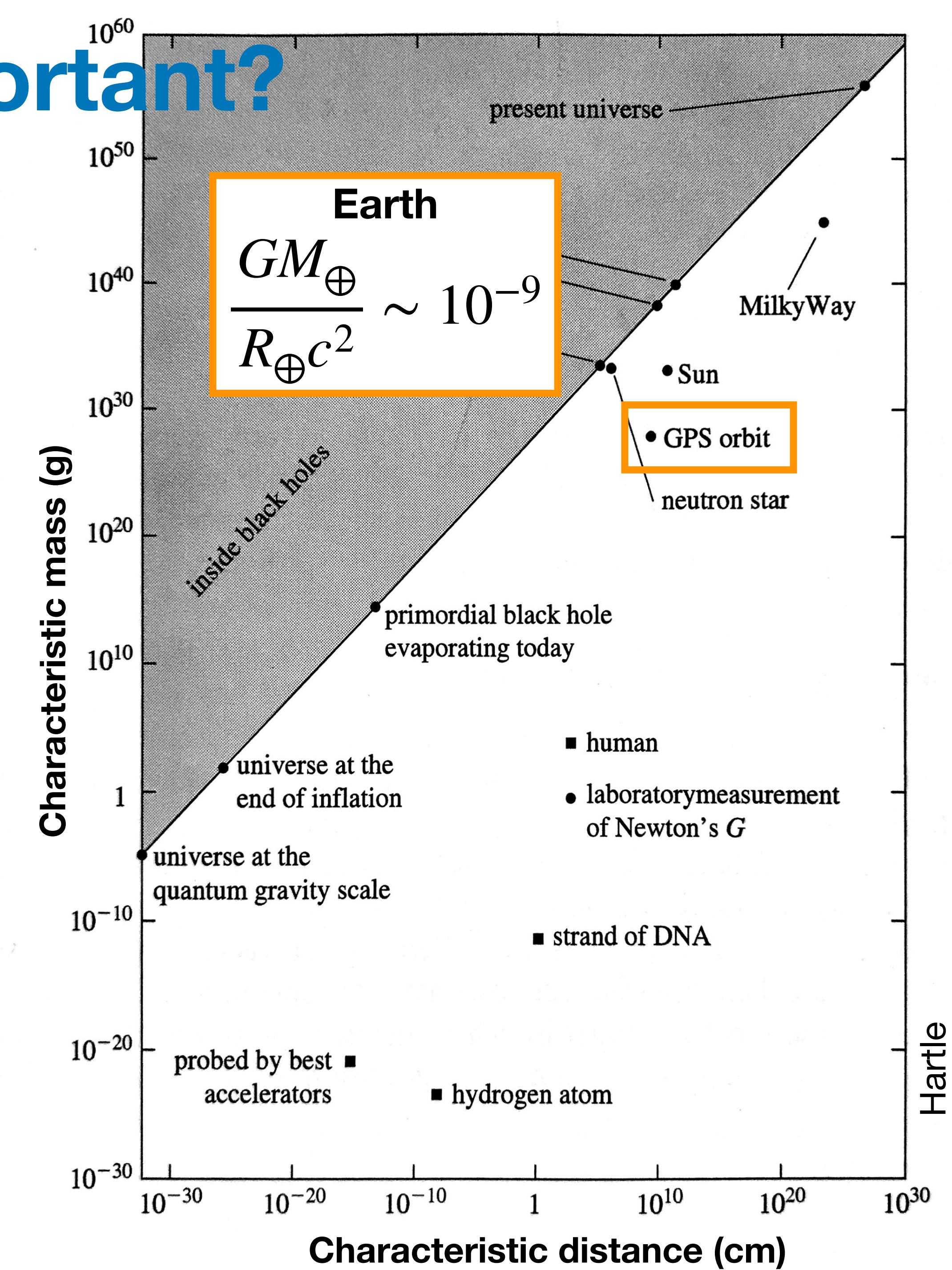
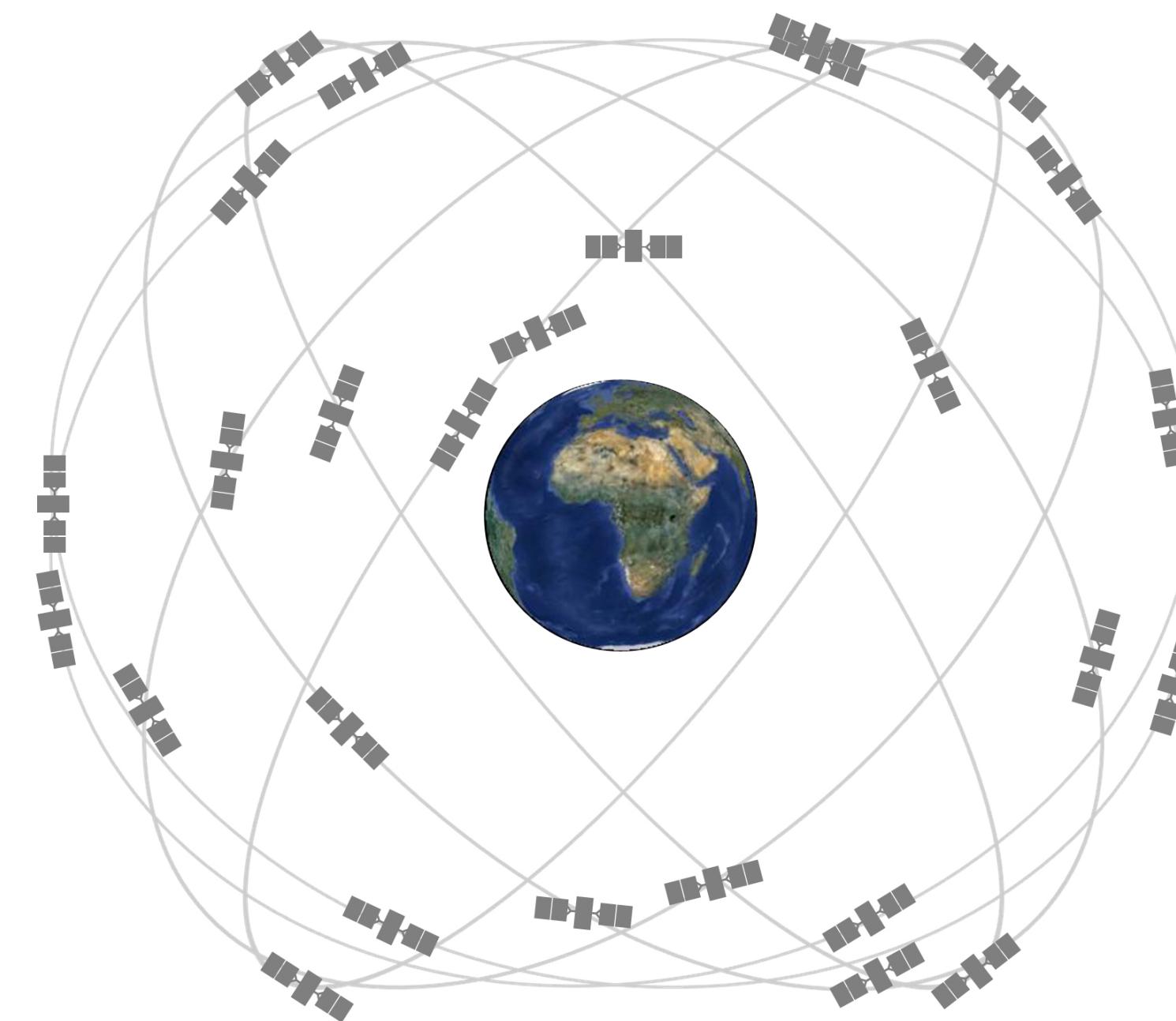


When is general relativity important? vs. Newtonian gravity

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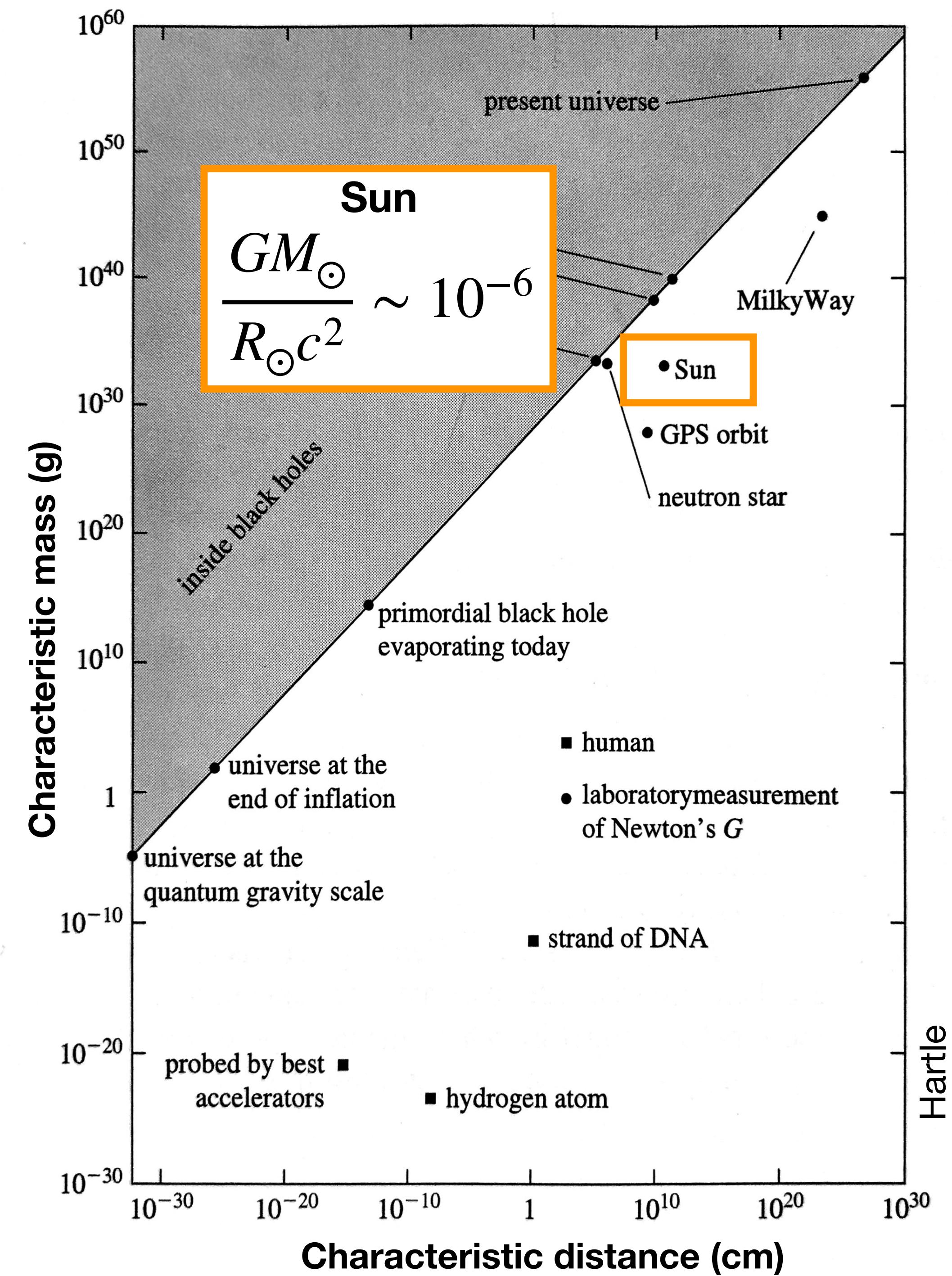
$$\frac{GM}{Rc^2} \rightarrow 1$$



Solar System

Precession of perihelion of Mercury

Bending of path of light rays passing
near the Sun



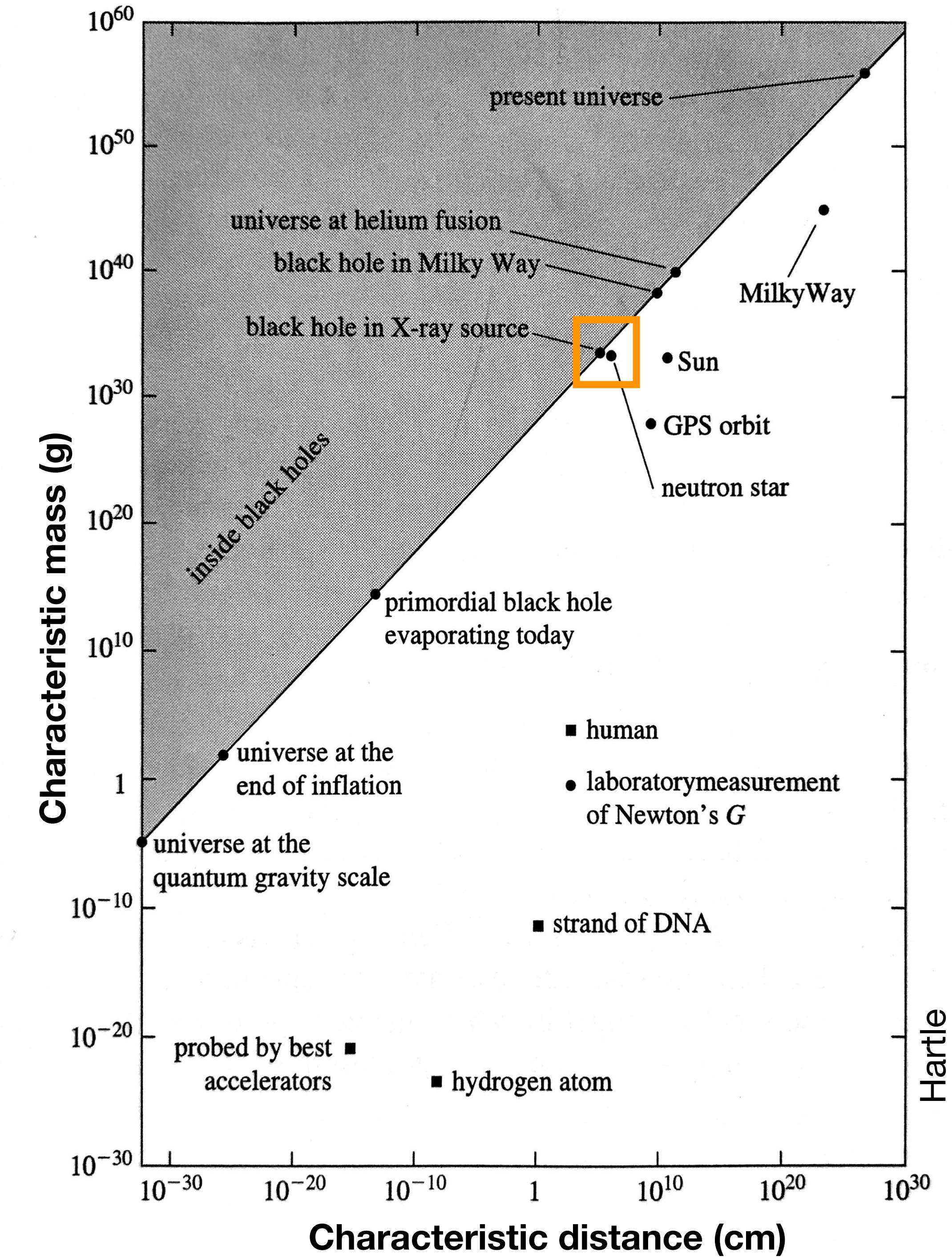
Neutron stars

$M \lesssim 3M_{\odot}$ (maximum mass)

radius ~ 10 km



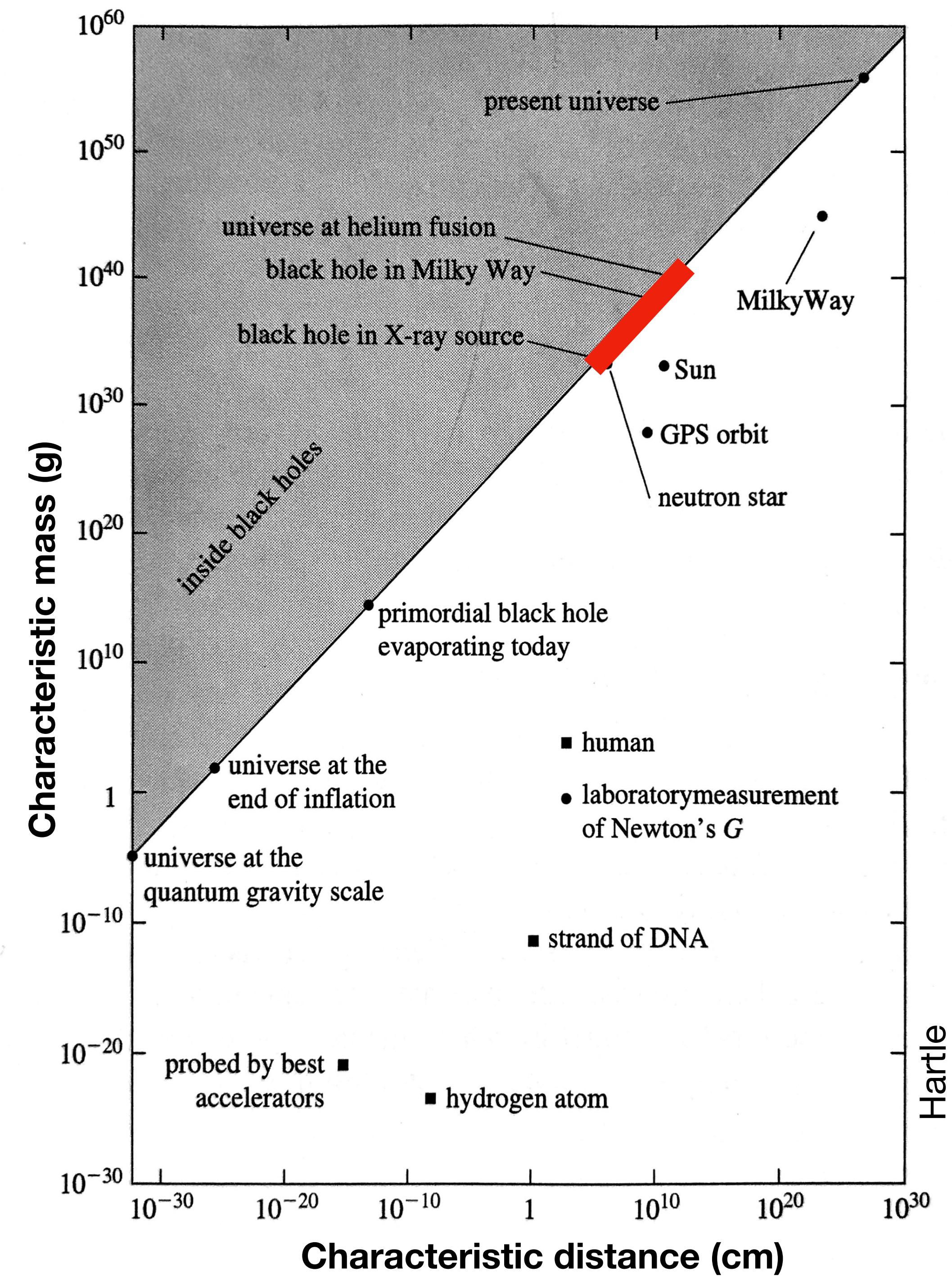
$$\frac{GM}{Rc^2} \sim 0.1$$



Black holes

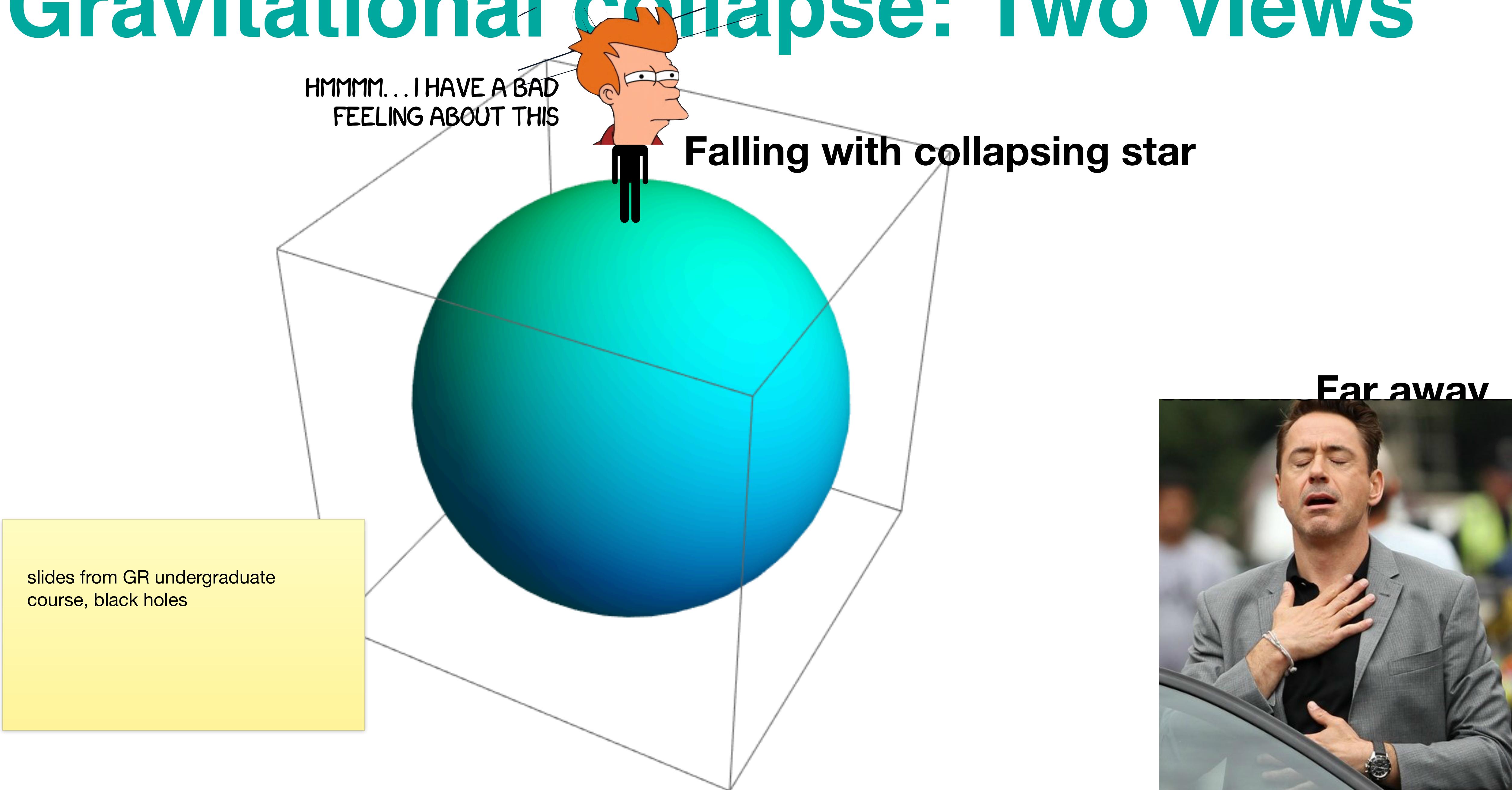
$$M \gtrsim 3M_{\odot}$$

$$\frac{GM}{Rc^2} = 0.5$$

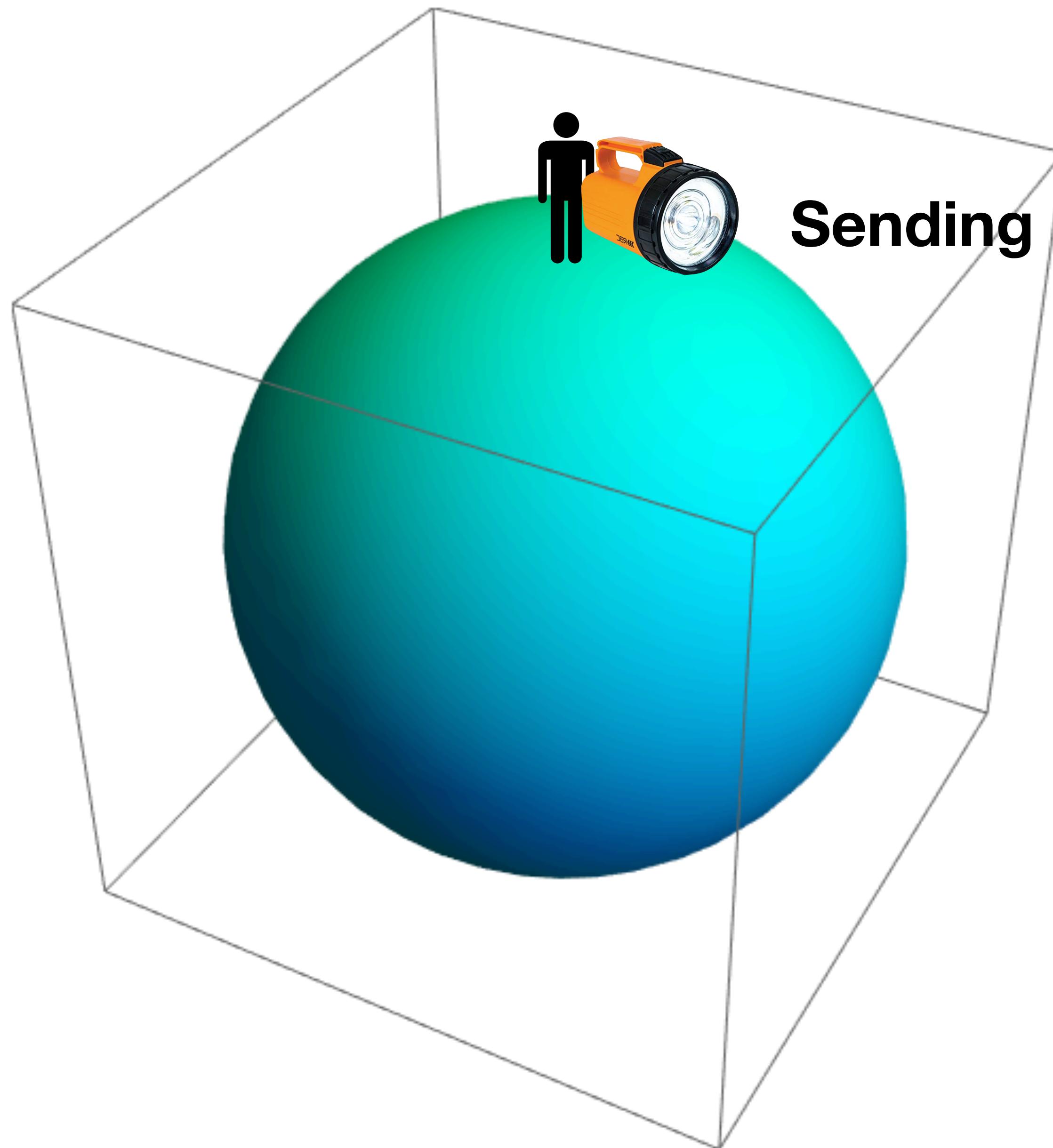


**Analysis of gravitational
collapse leading to a BH**

Gravitational collapse: Two views



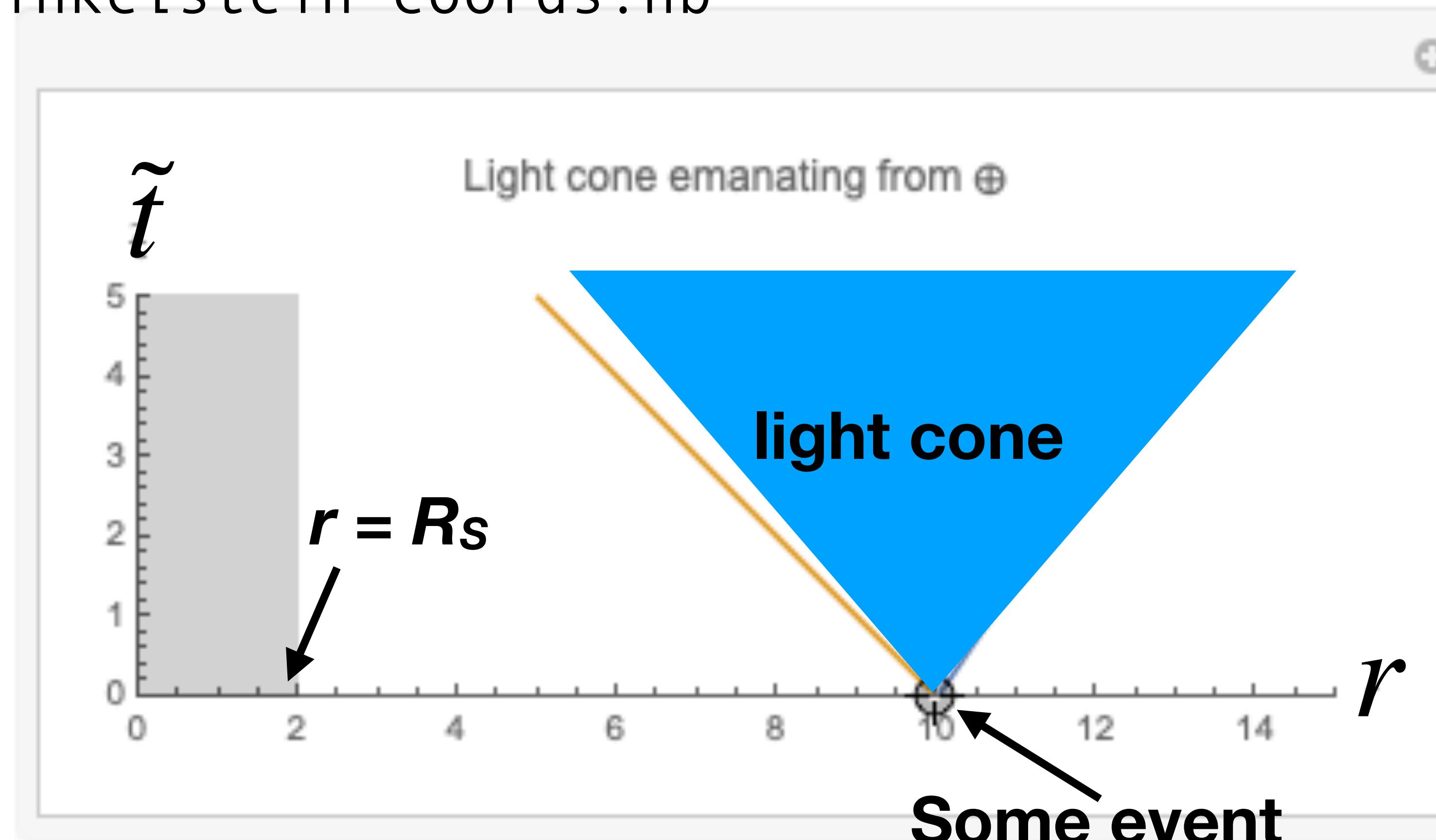
Gravitational collapse: View from inside



Sending light rays at regular intervals

Light rays in Schwarzschild spacetime

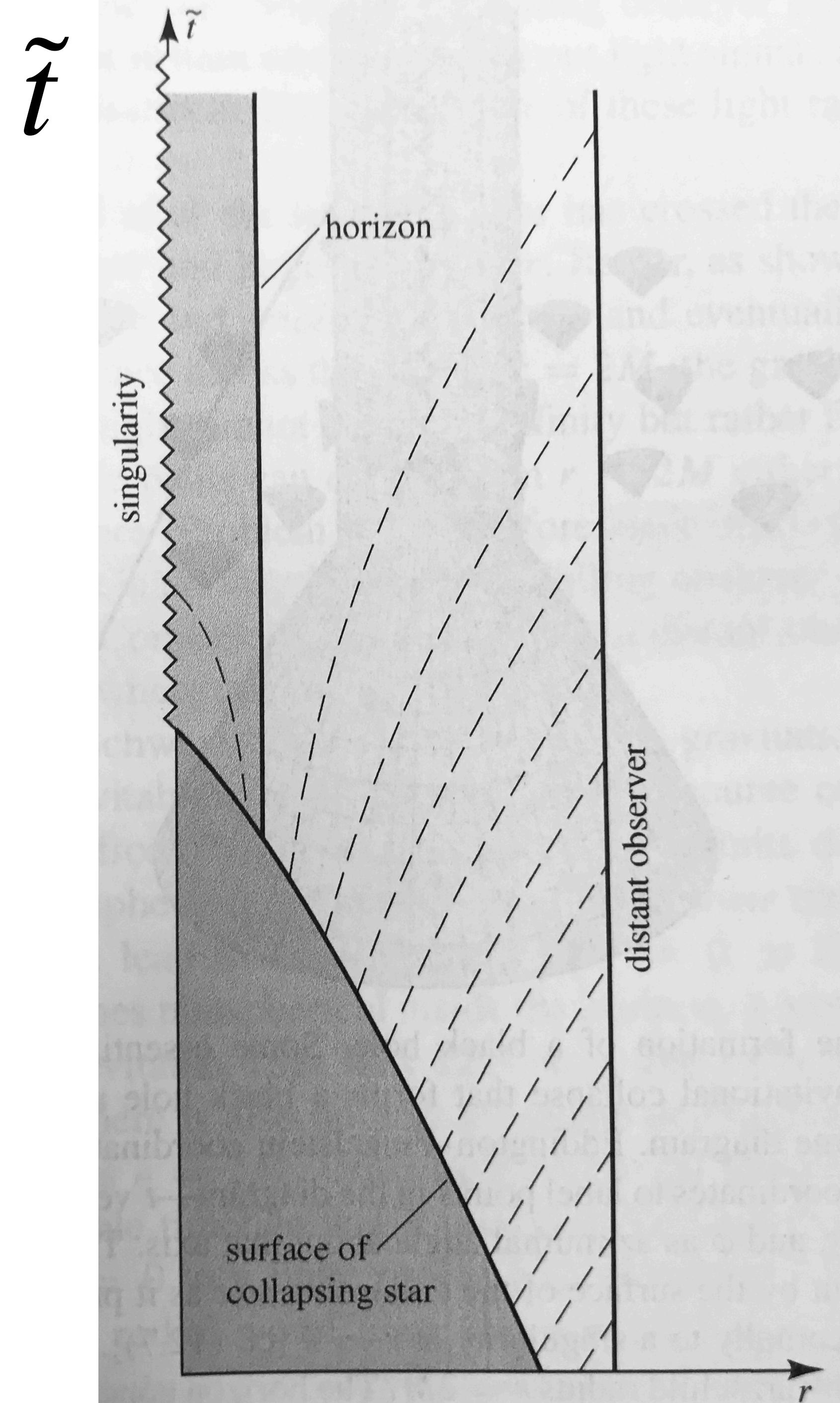
GOTO Mathematica: light-rays near black holes, Eddington-Finkelstein coords.nb



Schwarzschild radius

$$R_S = \frac{2GM}{c^2}$$

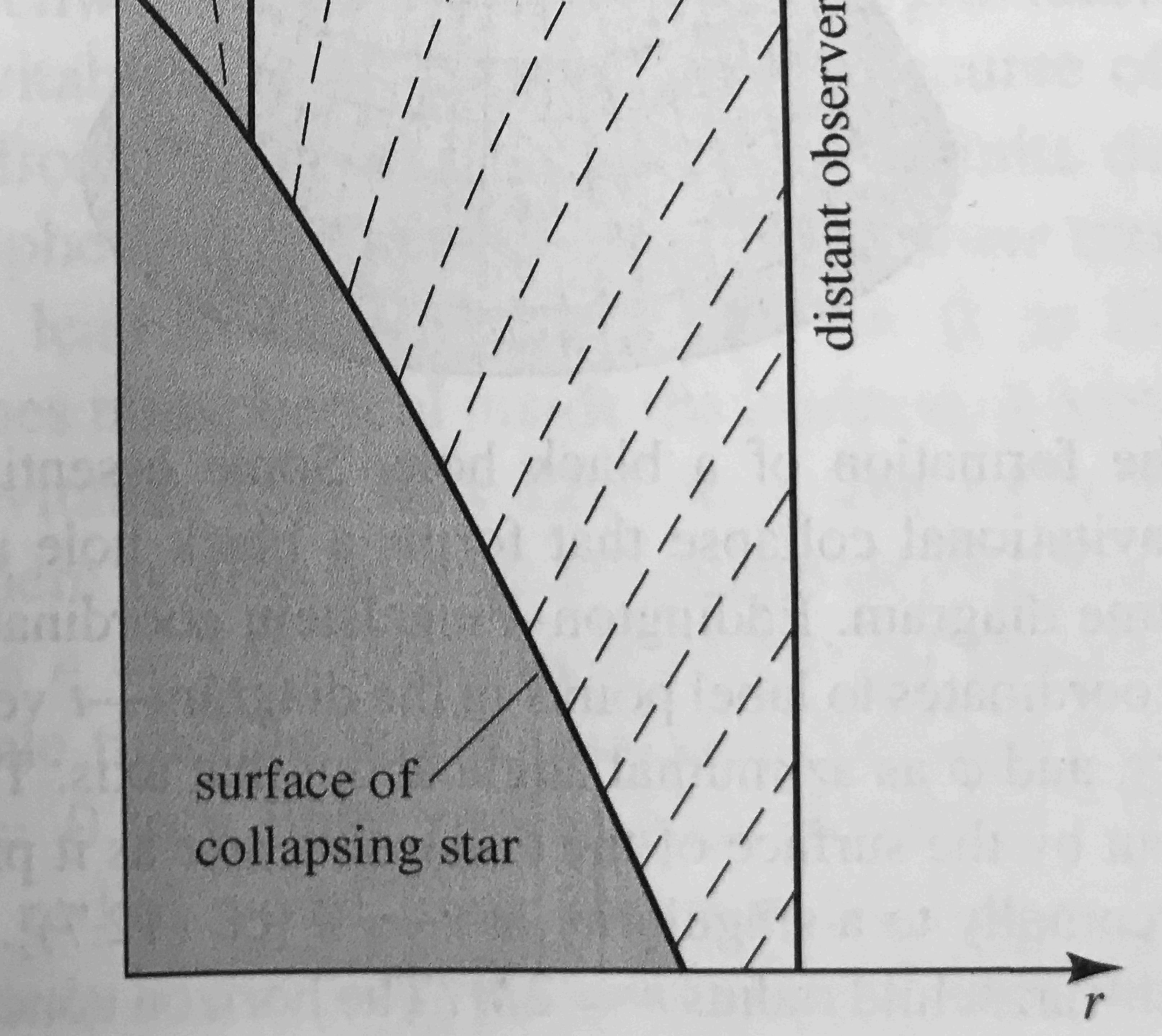
Spacetime diagram of gravitational collapse



r

Hartle

View from inside

\tilde{t} 

surface of
collapsing star

 r

Hartle

\tilde{t}

once inside
 $r_s=2M$, falling
observer
cannot
communicate
with distant
one anymore

**Collapse to
singularity at
 $r=0$ is
inevitable**

singularity

horizon

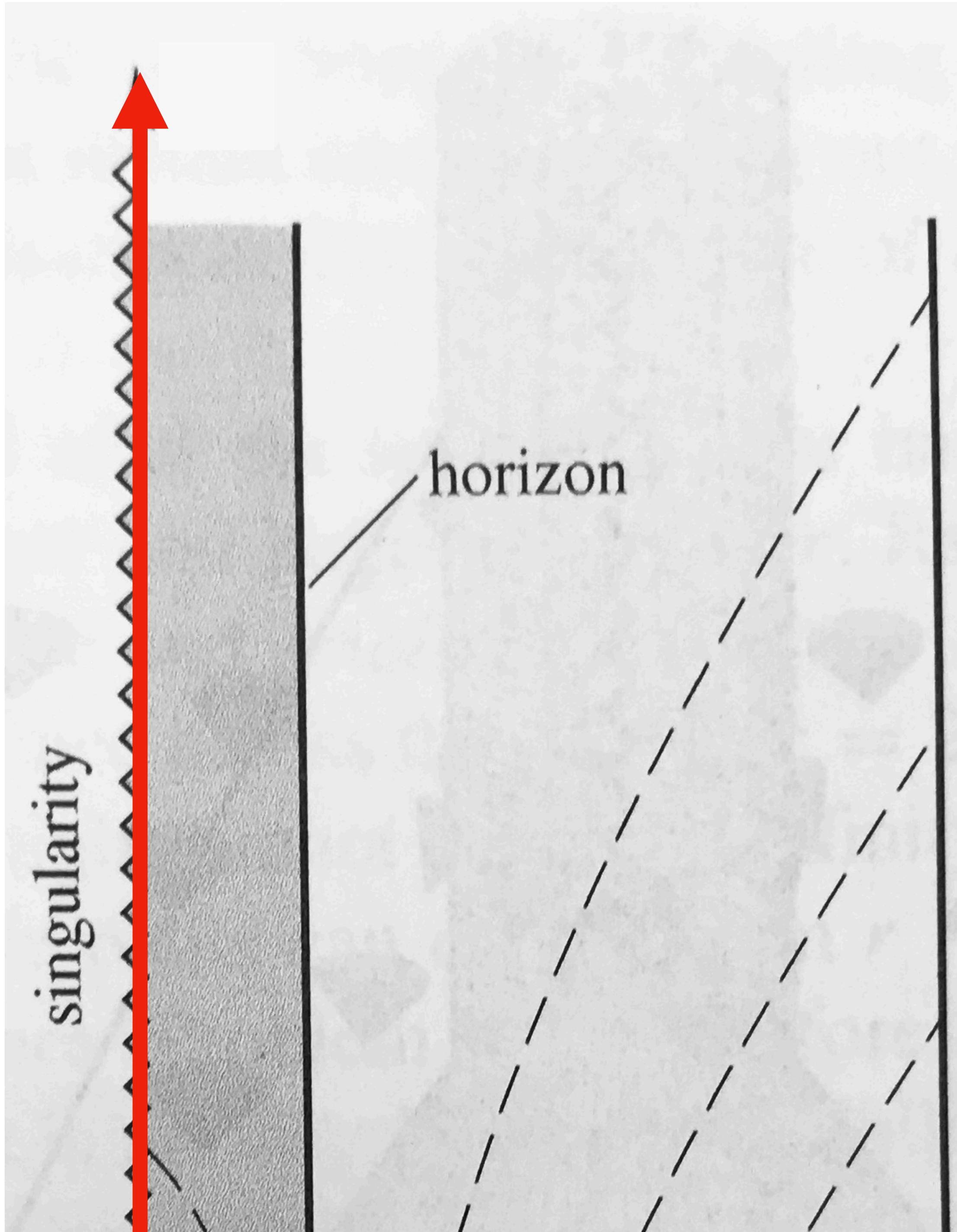
distant observer

r

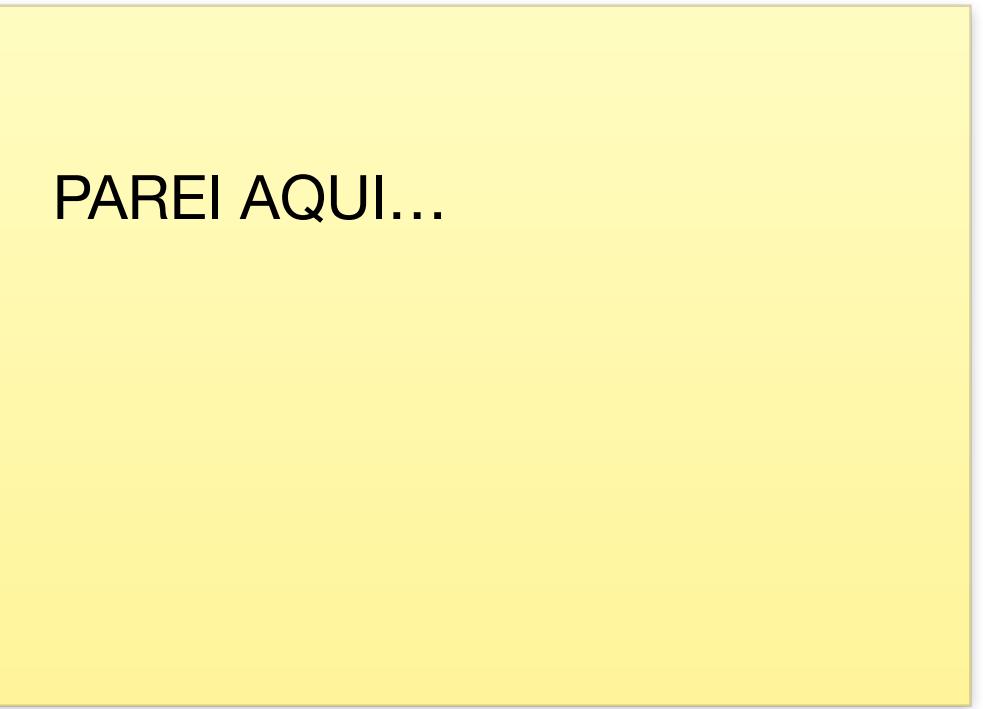
Hartle

**Singularity
hidden
from
observers
outside
black hole**

\tilde{t}



distant observer



Hartle

View from *inside*: Conclusion

Death crushed at singularity ($r=0$)

In a finite proper time

View from *outside*: Conclusion

Distant observer sees gravitational collapse:

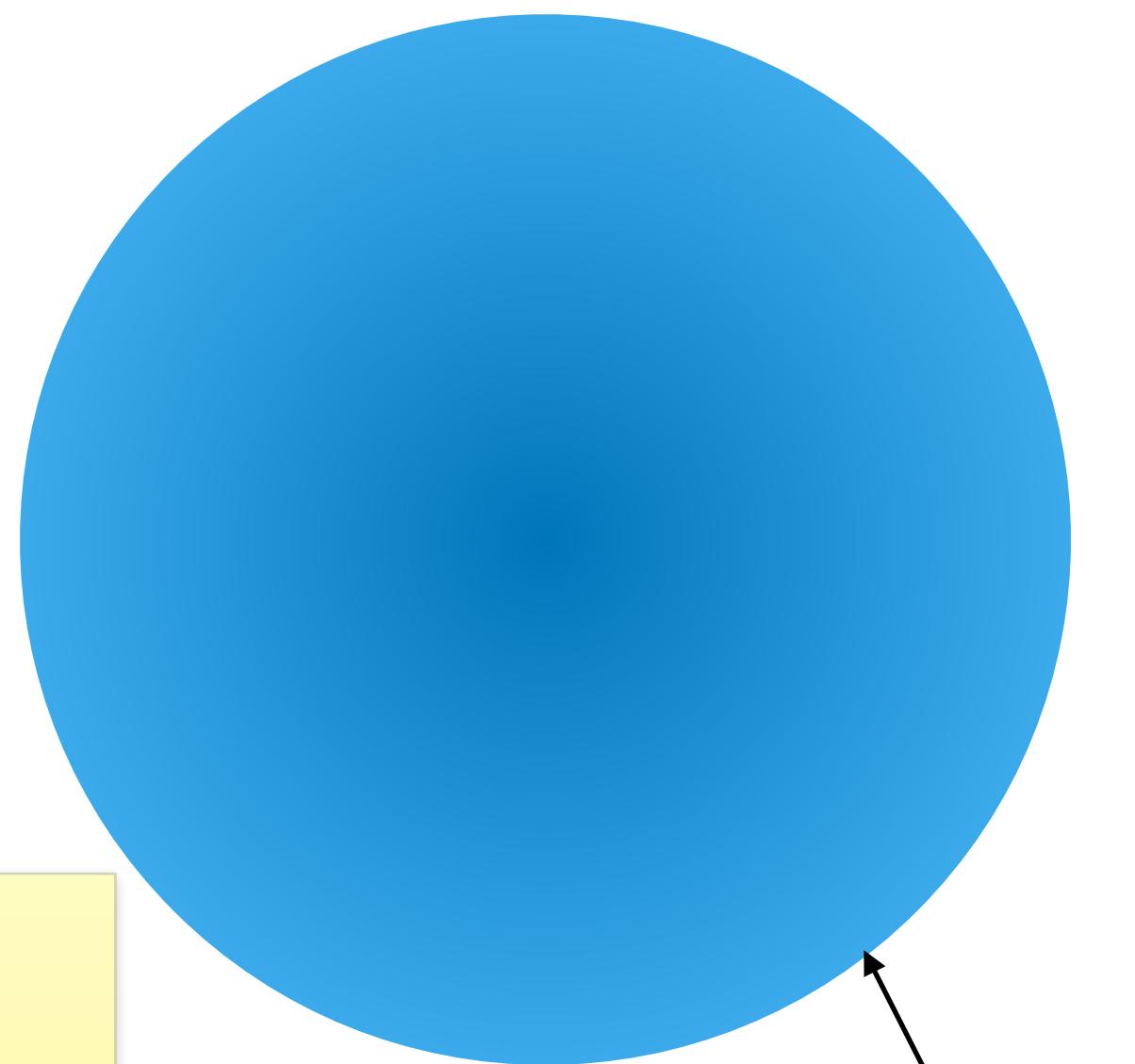
- **slow down**
- **get redshifted**
- **darken**

**All records of star's history and its properties
will be erased from the exterior geometry**

**Reduced to one number: mass M
(Schwarzschild spacetime)**

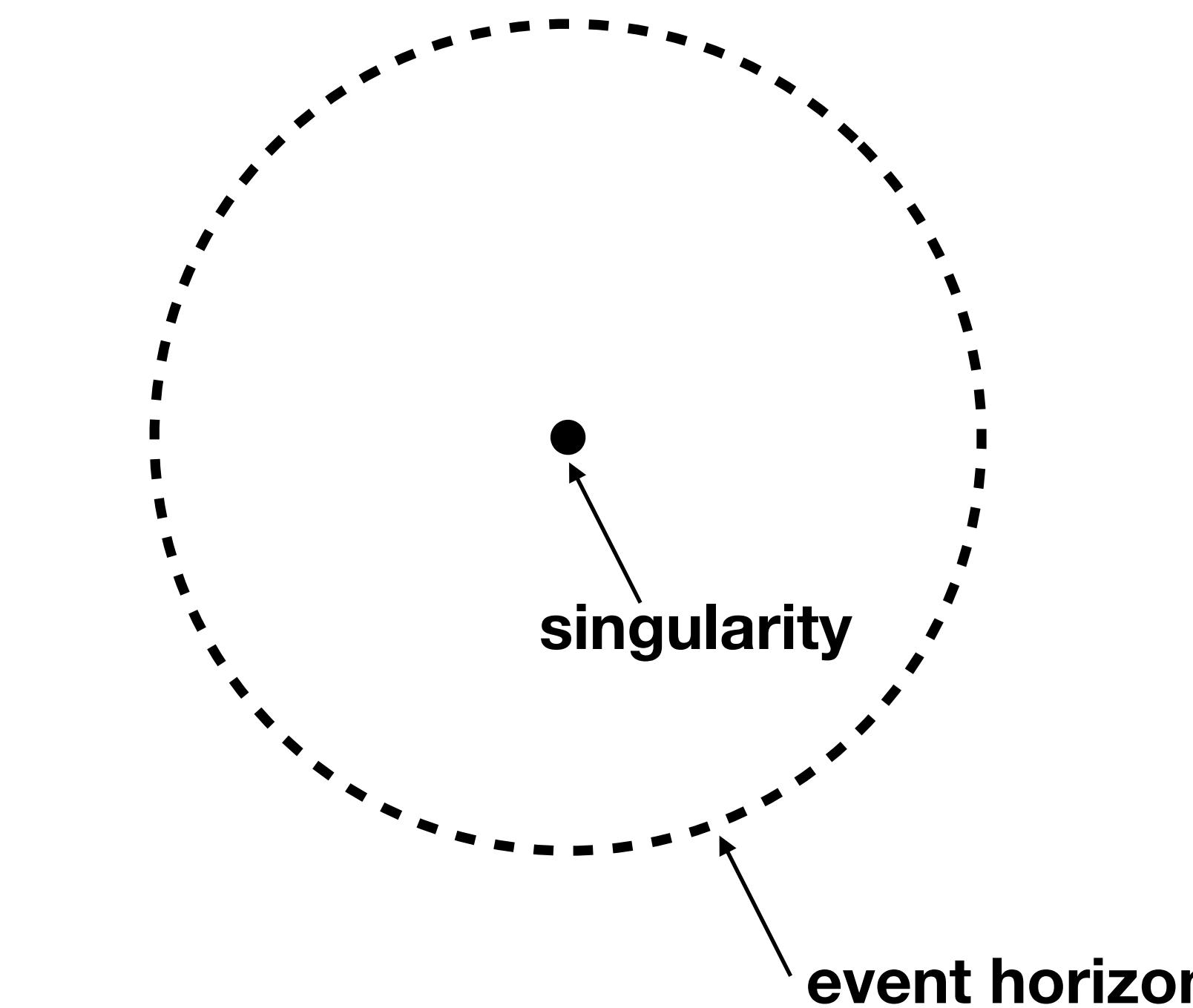
What is a black hole? Remarkable prediction of general relativity

Normal object



from black hole primer for
undergrads

Black hole



Event horizon: one-way membrane, matter/energy can fall in, but nothing gets out

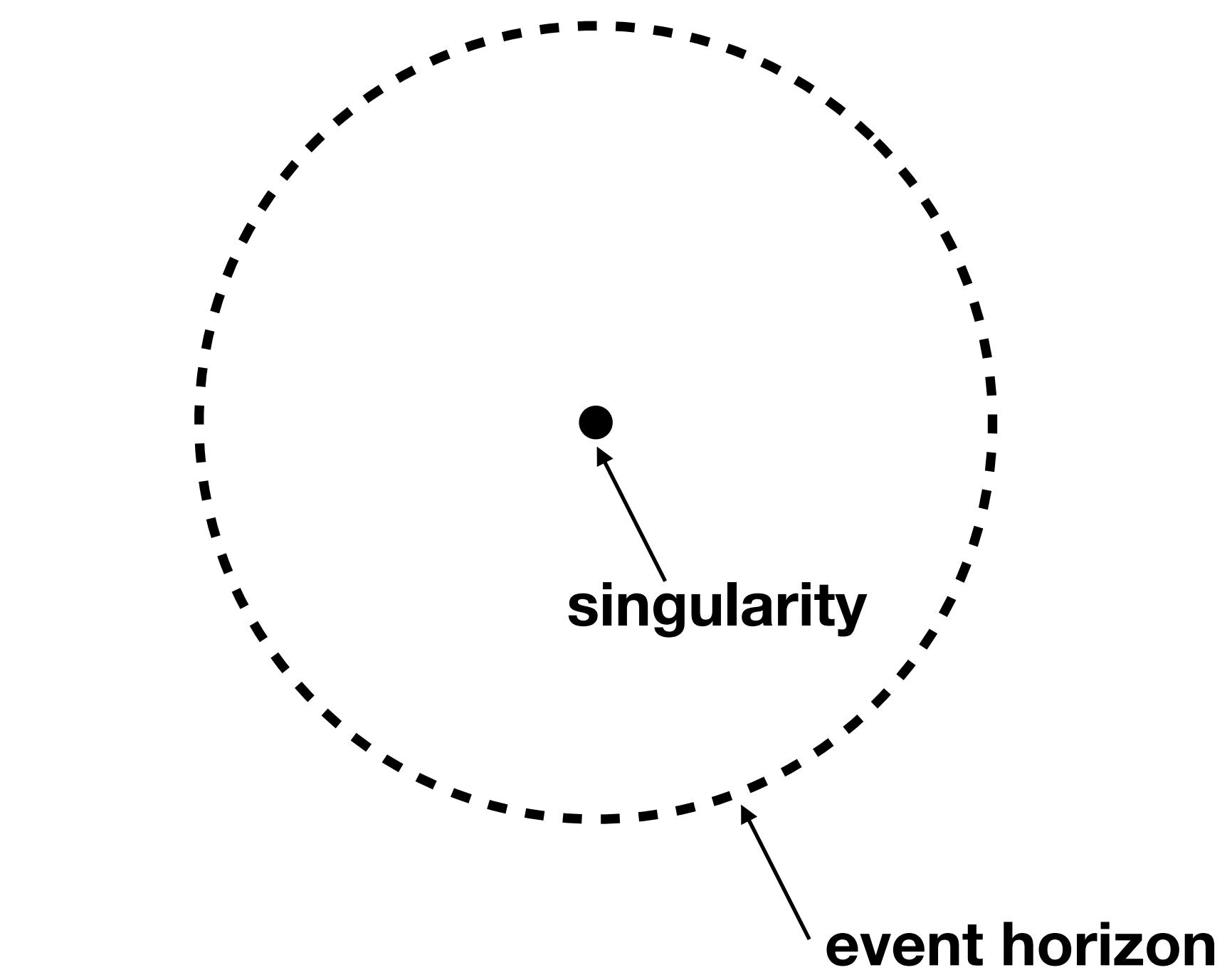
**Region inside event horizon
causally cut-off from outside**

Radius of event horizon:

$$R_S = \frac{2GM}{c^2} = 2.95 \left(\frac{M}{M_\odot} \right) \text{ km}$$

Schwarzschild radius

Black hole



What is a black hole?



**Massive, compact
astronomical object:
gravity so strong that it
traps everything that falls
inside the event horizon**

*Once inside, nothing
escapes*

What is a black hole?

Massive, compact astronomical objects with gravity so strong that it traps everything that crosses its boundary inside the event horizon.

Once inside, nothing escapes



What is a black hole?

Massive, compact
astronomical object:
gravity so strong that it
traps everything that falls
inside the event horizon

Once inside, you can't escape



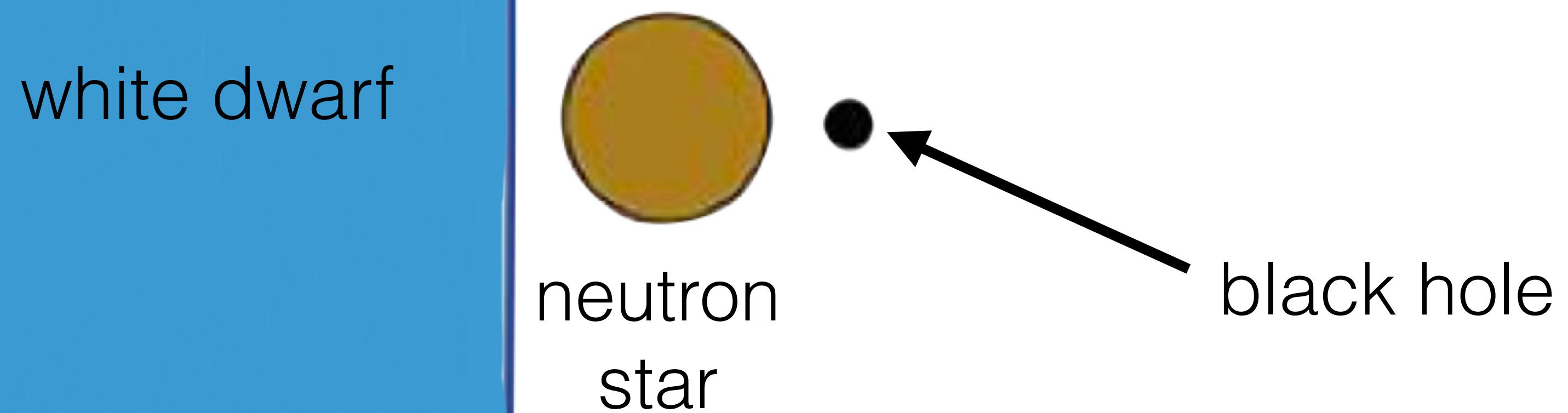
What is a

Massive, compact
astronomic
gravity so
traps every
inside the

*Once it
escapes*



Black holes are the most compact objects in nature



Radii of different objects with $M = 1.2 M_{\text{Sun}}$

Thorne

Generic features of gravitational collapse

Formation of spacetime singularity:

- ◆ unavoidable once star crosses $r=2M$ (singularity theorems)
- ◆ point where theory breaks down

Formation of event horizon:

- ◆ singularities inside event horizon, hidden from external observers
- ◆ *Cosmic censorship conjecture*: singularities always hidden inside event horizon even for nonspherical collapse

Area increase:

- ◆ if mass falls in a black hole, its area will increase
- ◆ analogous to entropy in thermodynamic \rightarrow laws of black hole thermodynamics

Growth of black holes

If particles fall into the black hole

- M increases
- Schwarzschild radius $r_s = 2M$ increases
- surface area increases

Growth of black holes

If particles fall into the black hole

- M increases
- Schwarzschild radius $r_s = 2M$ increases
- surface area increases

There is no limit to how big a BH can grow. From astrophysics:

- $M_{\min} = 3.6 \text{ M}_{\odot}$
- $M_{\max} \sim 10^{10} \text{ M}_{\odot}$

A black hole has no hair

All black hole solutions of Einstein's equation completely characterized by only three externally observable classical parameters:

Mass M

Spin: angular momentum J

Charge Q

$$J \equiv a \frac{GM^2}{c}$$
$$-1 \leq a \leq 1 \text{ spin parameter}$$

No-hair theorem All other information (“hair”=metaphor) disappears behind the event horizon, therefore permanently inaccessible to external observers

Types of black holes

Mass M

Spin a

Charge Q

Schwarzschild spacetime

Kerr spacetime

Reissner–Nordström spacetime

Schwarzschild black hole

- Simplest black hole
- Spherically symmetric spacetime
- Relatively “easy” to handle analytically

Schwarzschild geometry in Schwarzschild coordinates

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Kerr black hole

Conservation of angular momentum leads to spinning black holes

Rotational energy deforms spacetime → Kerr spacetime

Kerr metric considerably more complex than Schwarzschild

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2$$

Boyer-Lindquist coords.

$$a \equiv J/M, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2Mr + a^2$$

A Kerr black hole is spinning

Spin

$$j \equiv \frac{J}{J_{\max}} = \frac{a}{M}$$

$$J_{\max} = \frac{GM^2}{c}$$

**angular
velocity**

$$\Omega_H = \frac{jc}{2r_H}$$

Period

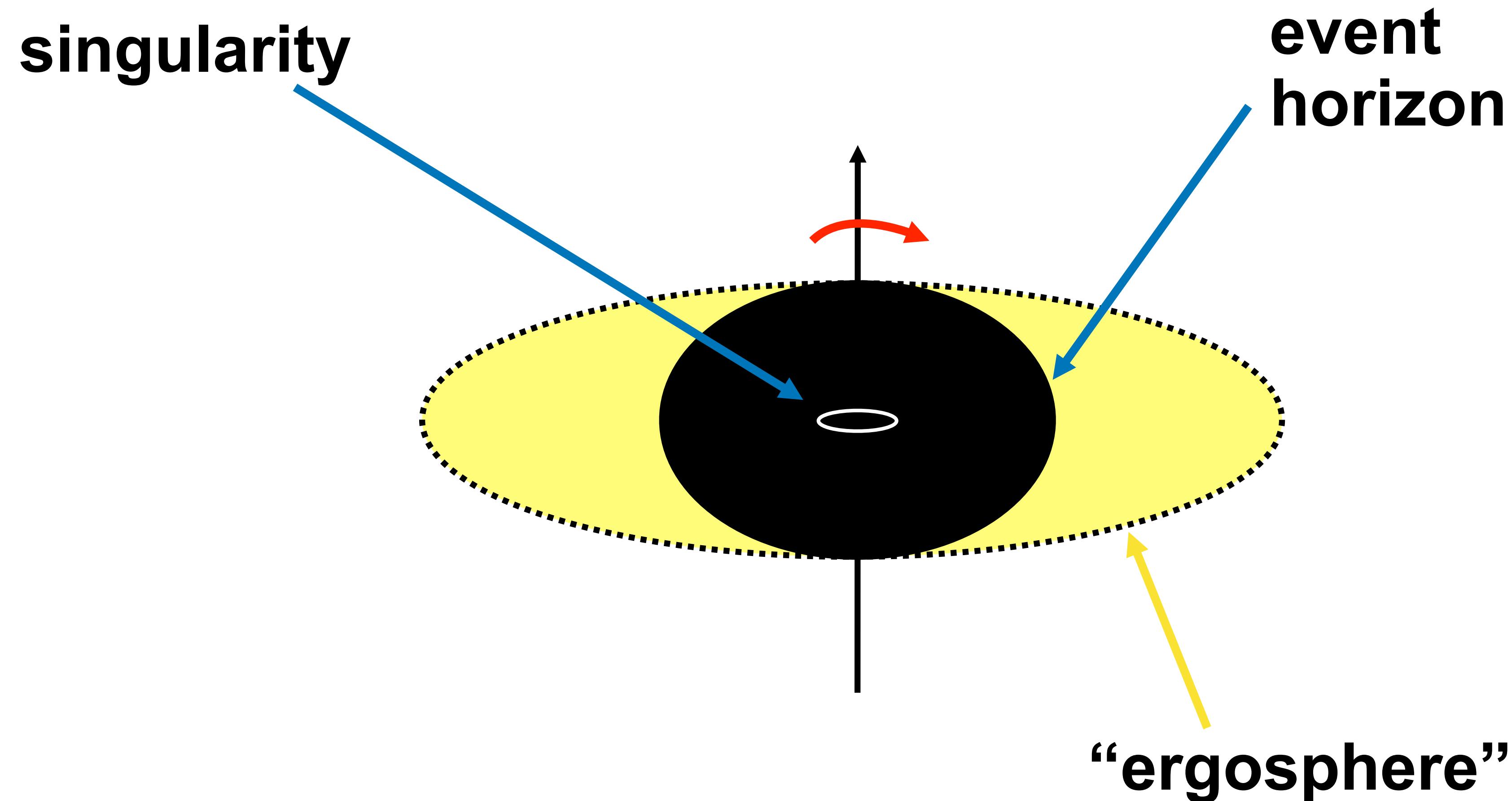
$$P_H = \frac{2\pi}{\Omega_H} = \frac{4\pi r_H}{jc}$$

Velocity

$$v_H = \frac{4\pi r_H}{P_H} = jc$$

$$j = 1 \quad \Rightarrow \quad v_H = c$$

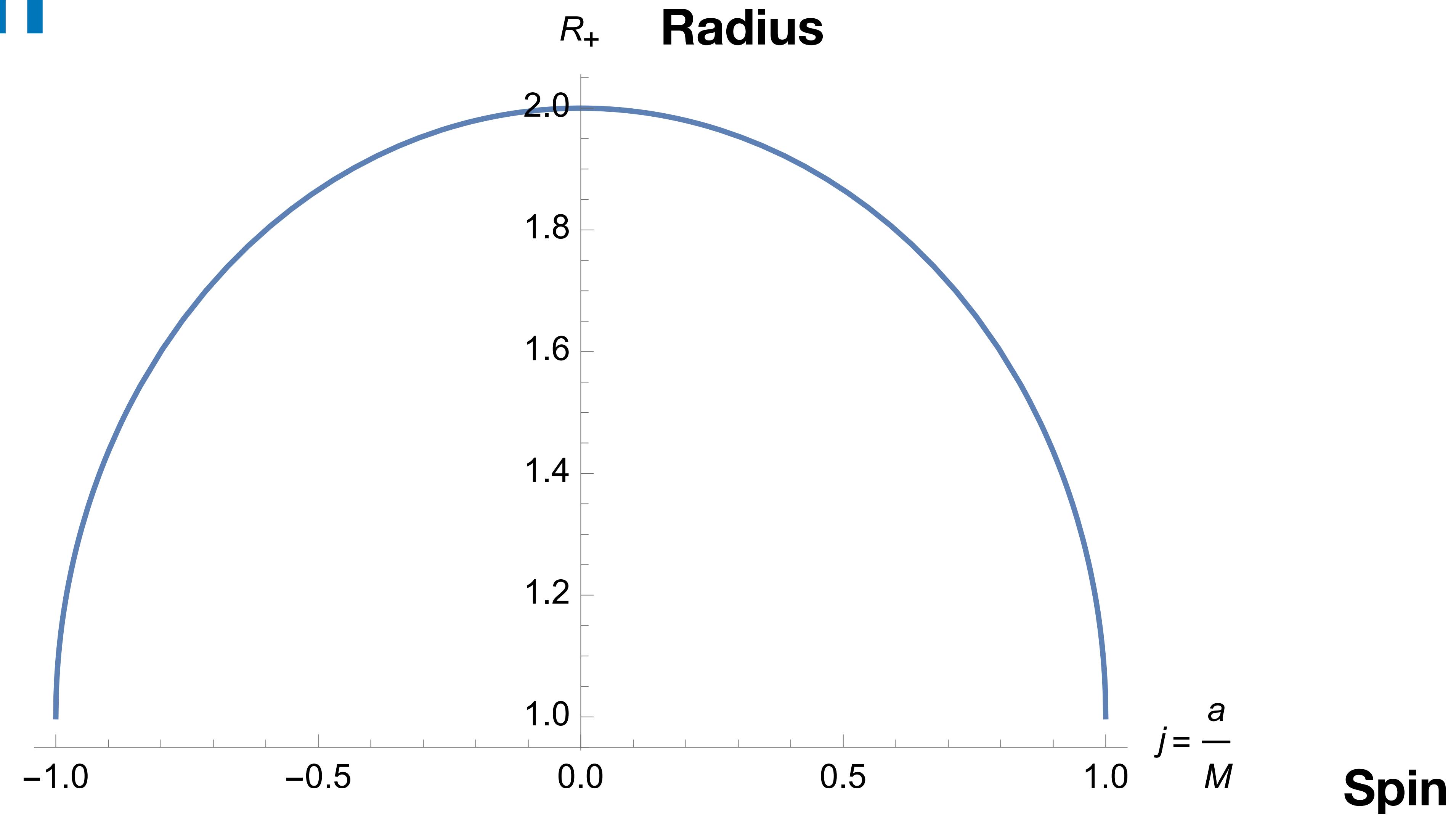
Structure of a Kerr black hole



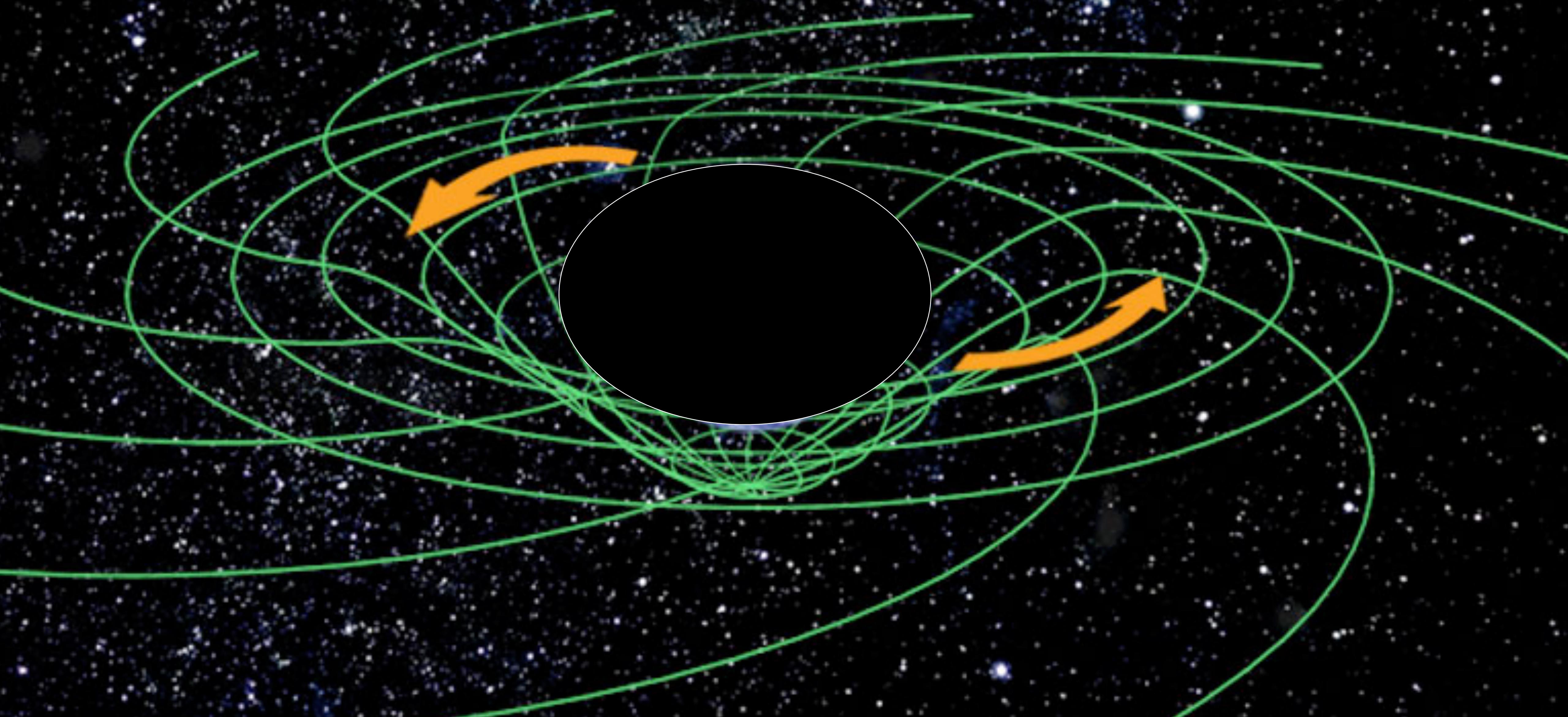
Event horizon radius
decreases with spin:

$$R_H = \frac{GM}{c^2} \quad \text{for } a/M = 1 \text{ (maximal spin)}$$

Event horizon radius as a function of spin



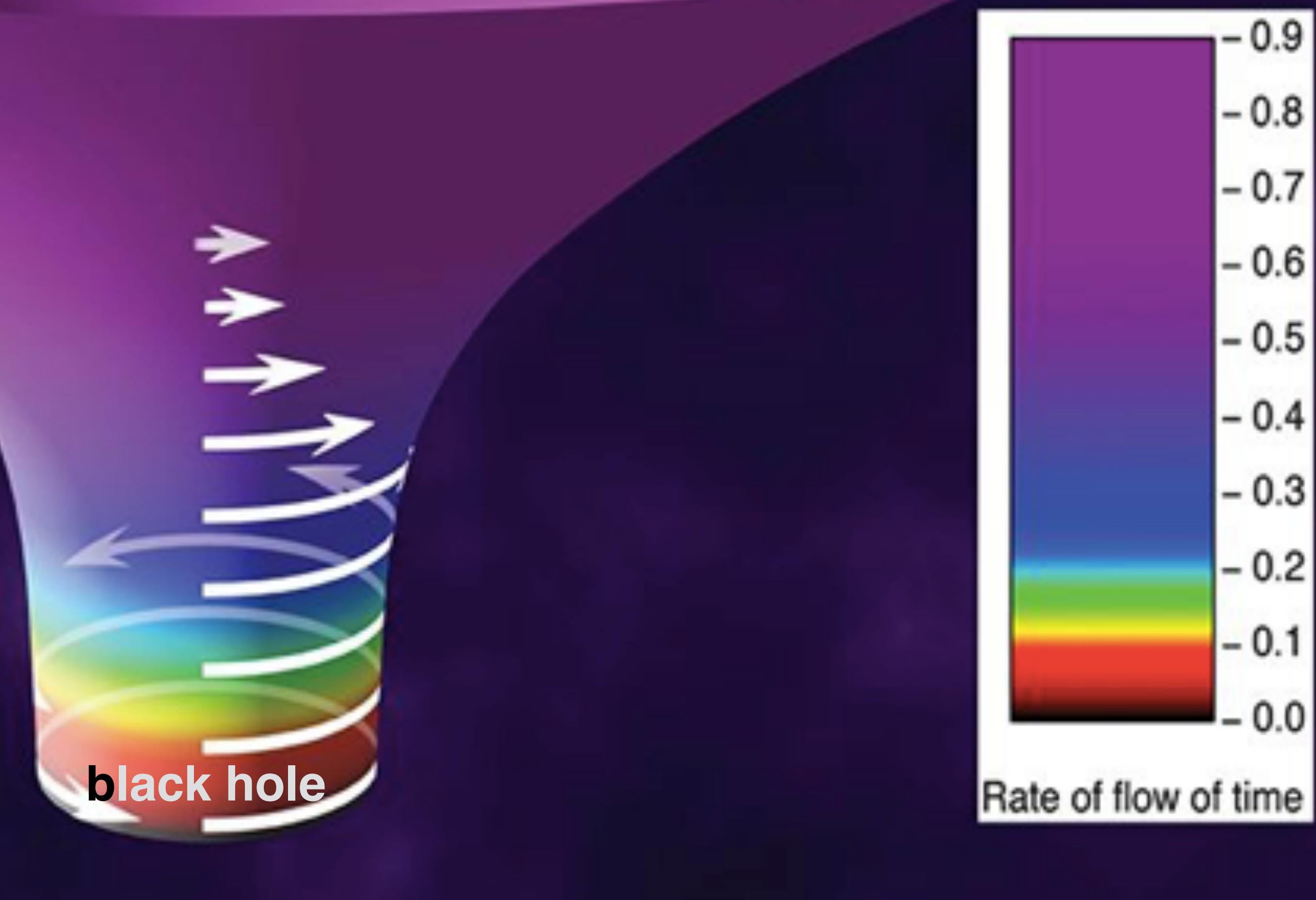
Spacetime around a Kerr black hole



Black hole spin generates spacetime whirlwind (non-Newtonian effect)

Huge energy stored in rotating spacetime

Credit: Thorne



Frame-dragging effect

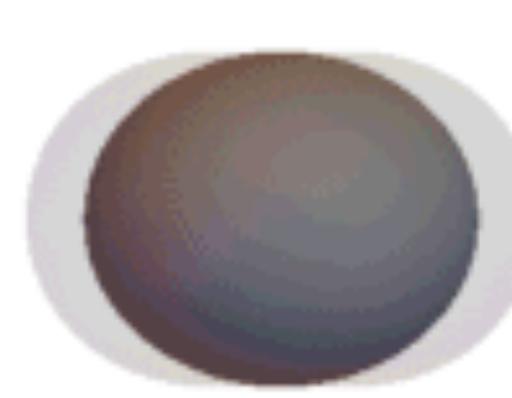


Penrose effect:
rotational energy can
be extracted
(more about this in
next lecture)

● : FREFO, ● : ZAMO

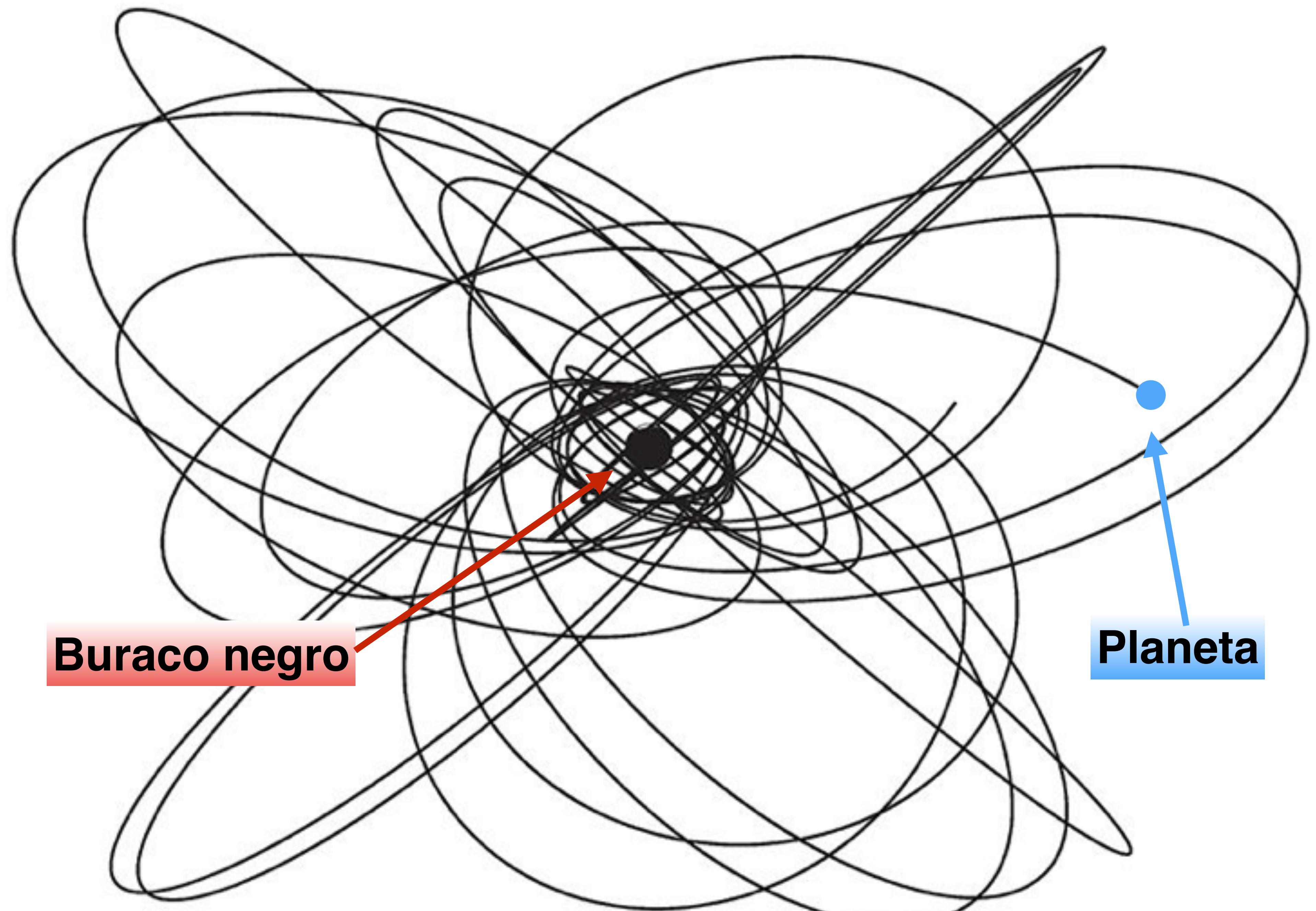
a=0.9, r0=7, v0=0.4, i0=arctan(5/6)

yukterez



x ↴
z
y ↴
x

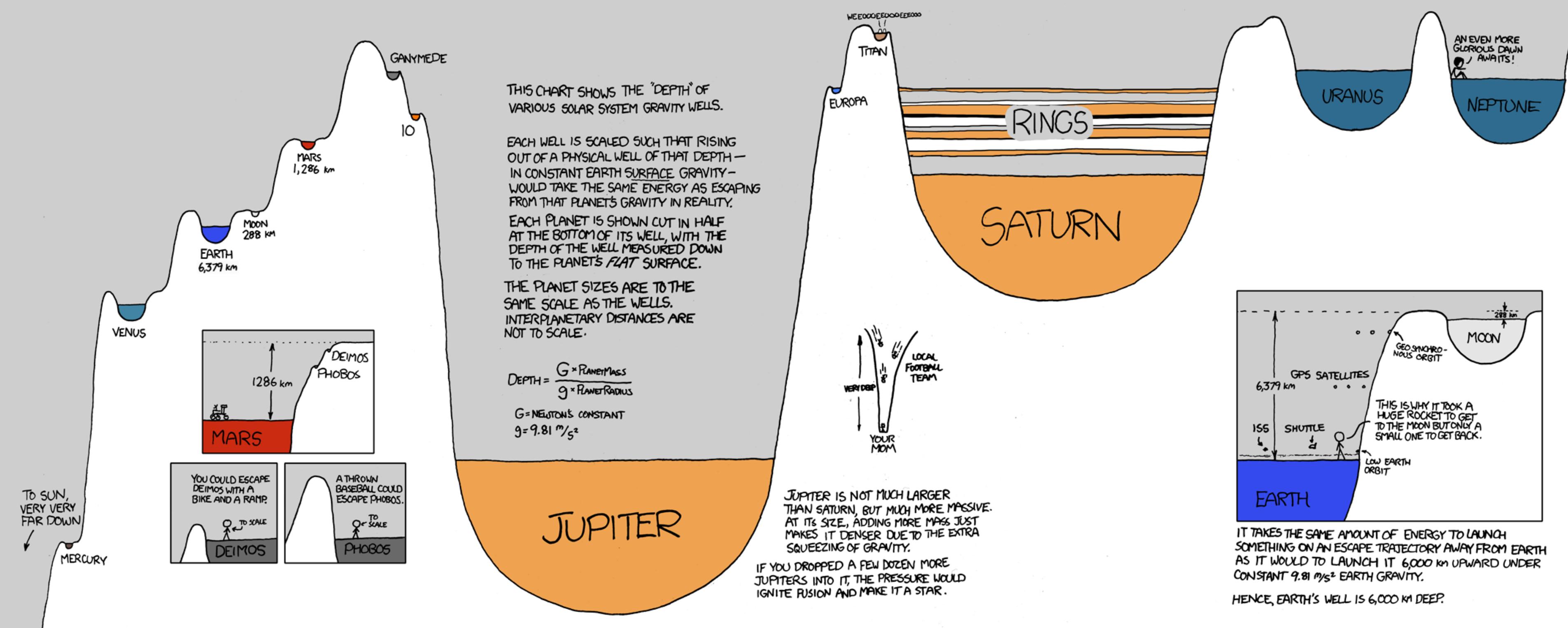
t coord = 0	GM/c ³	Ř crc.φ = 7.074	GM/c ²	L axial = 2.37176	GMm/c	ω fdrag = 0.0051386	c ³ /G/M
z propr = 0	GM/c ³	Σ crc.θ = 7.	GM/c ²	L polar = 1.9558	GMm/c	v fdrag = 0.0429707	c
γ total = 1.2898	dt/dz	Δ rad.f = 5.98415	GM/c ²	p r.mom = 0	mc	Ω fdrag = 0.0362663	c
ξ gravt = 1.18212	dt/dz	E kinet = 0.0910895	mc ²	R carts = 7.05762	GM/c ²	v obsvd = 0.366484	c
r coord = 7.	GM/c ²	E poten = -0.15591	mc ²	x carts = 7.05762	GM/c ²	v escpe = 0.533285	c
φ longd = 0	rad	E total = 0.935179	mc ²	y carts = 0	GM/c ²	v delay = 0.364686	c
θ lattd = 1.5708	rad	CarterQ = 3.82514	GMm/c	z carts = 0	GM/c ²	v local = 0.4	c

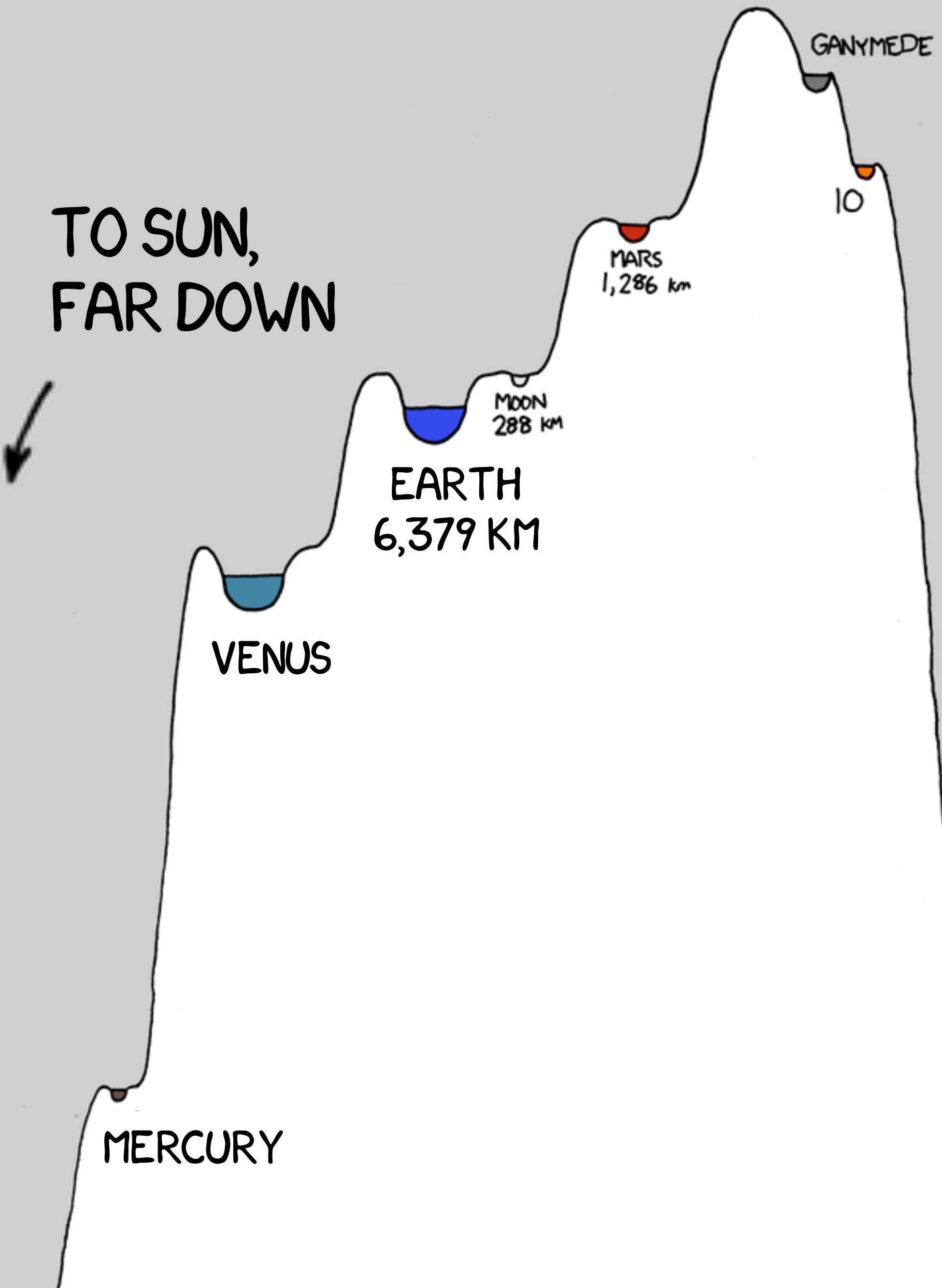


GRAVITY WELLS

SCALED TO EARTH SURFACE GRAVITY

<https://xkcd.com/681/>





GRAVITY WELLS

SCALED TO EARTH SURFACE GRAVITY

THIS CHART SHOWS THE "DEPTH" OF VARIOUS SOLAR SYSTEM GRAVITY WELLS.

EACH WELL IS SCALED SUCH THAT RISING OUT OF A PHYSICAL WELL OF THAT DEPTH — IN CONSTANT EARTH SURFACE GRAVITY — WOULD TAKE THE SAME ENERGY AS ESCAPING FROM THAT PLANET'S GRAVITY IN REALITY.

EACH PLANET IS SHOWN CUT IN HALF AT THE BOTTOM OF ITS WELL, WITH THE DEPTH OF THE WELL MEASURED DOWN TO THE PLANET'S FLAT SURFACE.

THE PLANET SIZES ARE TO THE SAME SCALE AS THE WELLS.
INTERPLANETARY DISTANCES ARE NOT TO SCALE.

$$\text{DEPTH} = \frac{G \times \text{PlanetMass}}{g \times \text{PlanetRadius}}$$

G = NEWTON'S CONSTANT
 $g = 9.81 \text{ m/s}^2$

JUPITER



DEPTH
GRAVITY
WELL

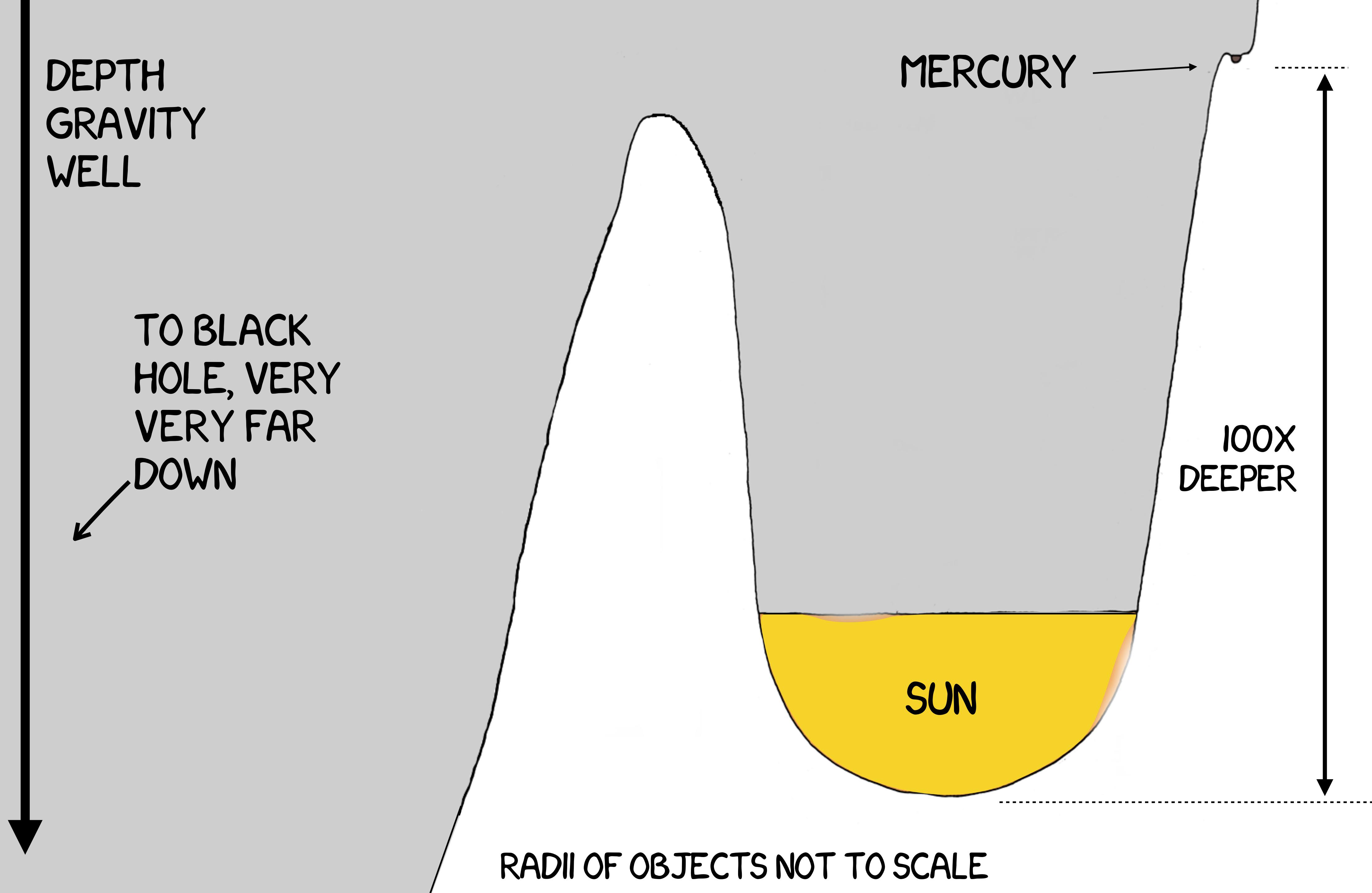
TO BLACK
HOLE, VERY
VERY FAR
DOWN

MERCURY

100X
DEEPER

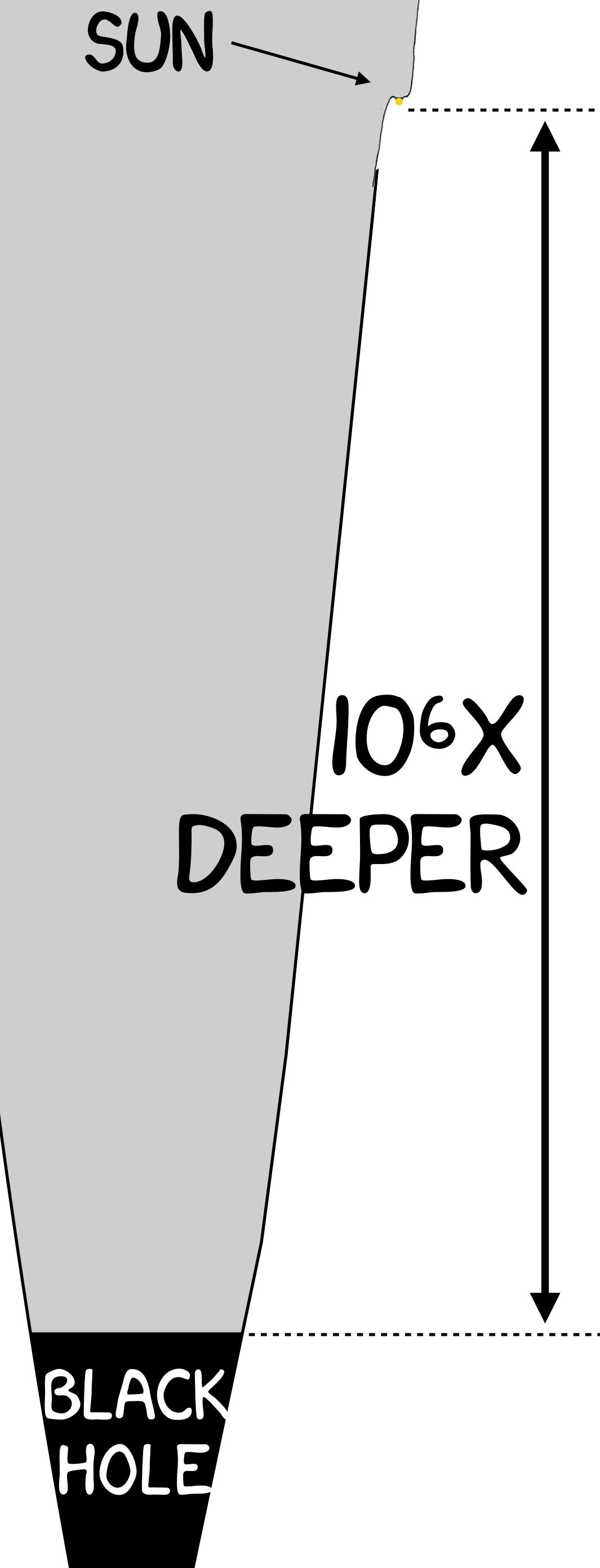
SUN

RADIi OF OBJECTS NOT TO SCALE



**BLACK HOLES HAVE DEEP,
RELATIVISTIC GRAVITY
WELLS**

DEPTH
GRAVITY
WELL



Classical vs quantum black holes

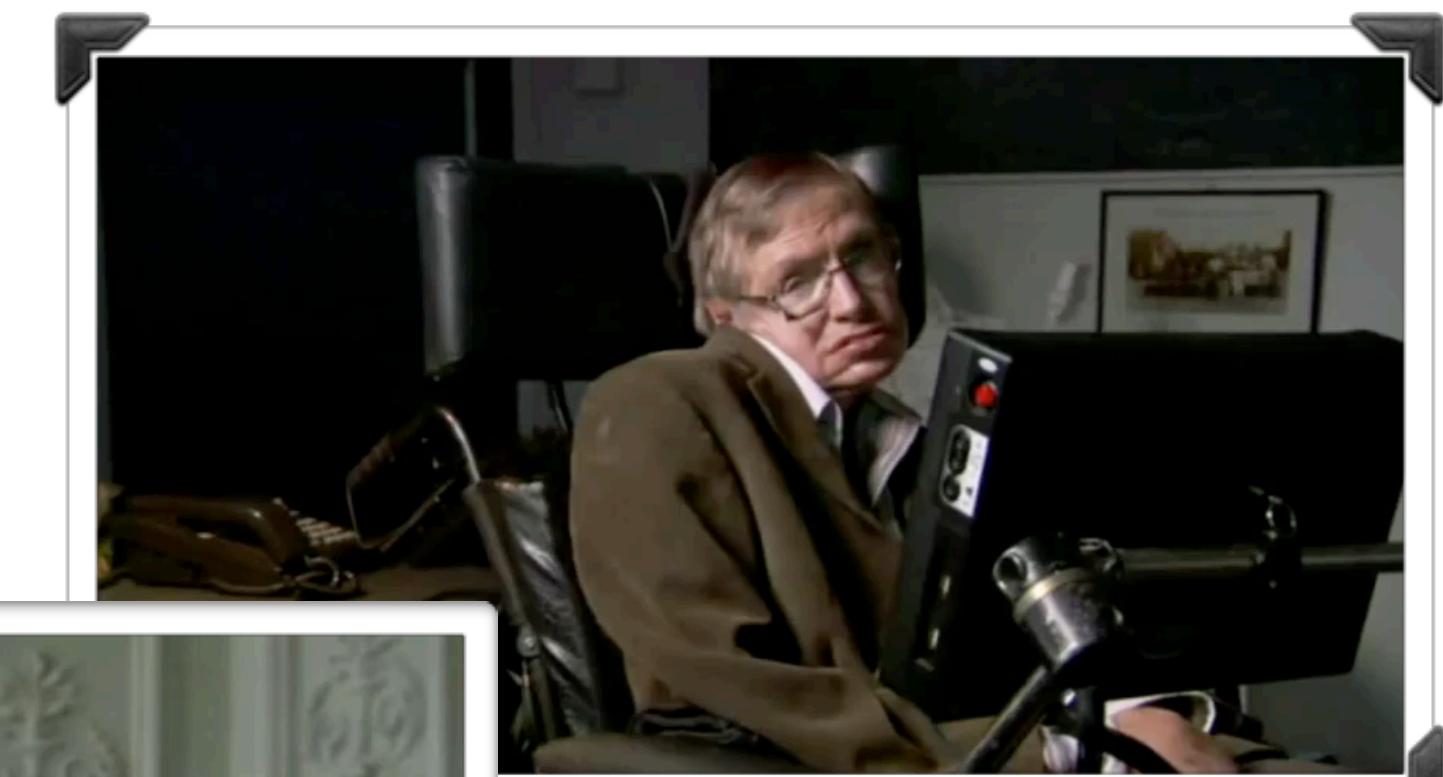
Black holes from general relativity are classical objects

Quantum BHs: need quantum gravity theory

Quantum BHs have weird properties:

- Hawking radiation
- Information paradox

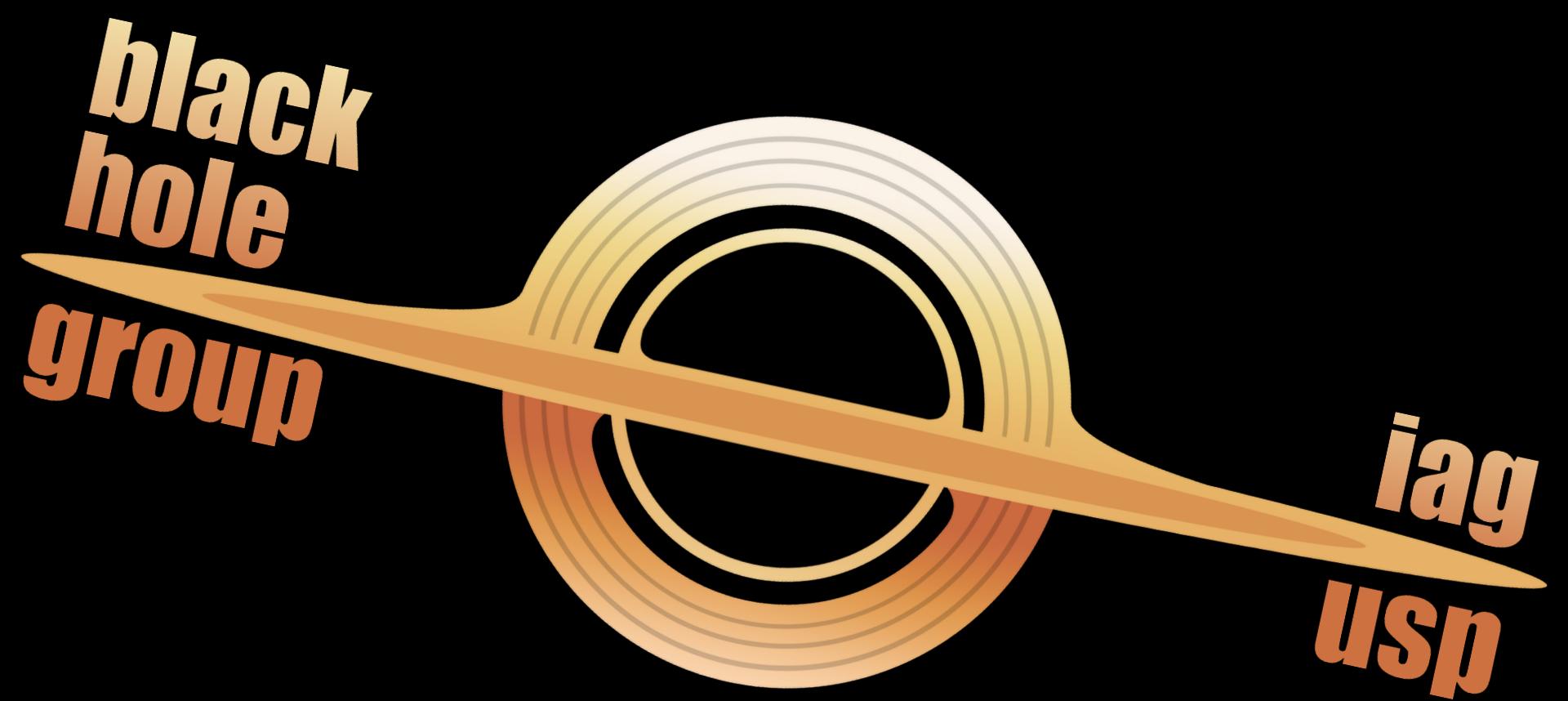
Will not talk about them



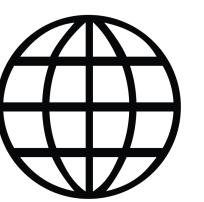
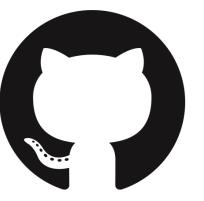
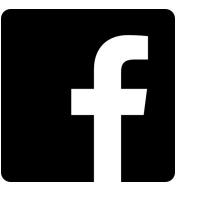
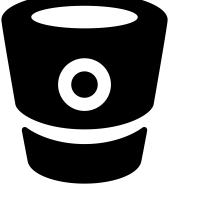
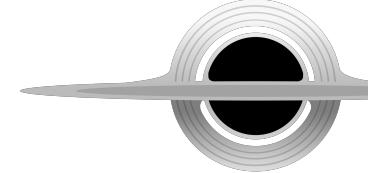
Credit: BBC



<https://kahoot.it/>



Next: Accretion flows

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