

Differential geometry of curves and surfaces

Chiung-Jue Anna Sung

1

Parametrized Curves, Regular Curves and Arc Length:

1. A. Definition: Parametrized Curves B. Examples: Parametrized Differentiable Curves
2. A. Note: Difference Between Curve and Trace B. Definition: Regular Curves C. Example: Helix 1
3. A. Example: Helix 2 B. Definition: The Arc Length of a Regular Curve

The Local Theory of Curves Parametrized by Arc Length:

1. A. Note: Properties of Arc Length B. Recall: Inner Product and Wedge Product C. Definition: Parametrized by Arc Length
2. A. Example: Logarithmic Spiral B. Derivation: Curvature of a Curve Parametrized by Arc Length
3. A. Definition: Curvature B. Examples: Straight Line and Circle C. Definition: Frenet Frame
4. A. Definition: Torsion of a Curve Parametrized by Arc Length B. Derivation: Frenet Formula
5. A. Definition: Normal Plane and Rectifying Plane B. Example: Helix
6. Theorem: A Curve with Positive Curvature Is a Plane Curve if and only if Its Torsion Is Identically Zero
7. Derivation: Curvature and Torsion of a Curve NOT Parametrized by Arc Length 1
8. A. Derivation: Curvature and Torsion of a Curve NOT Parametrized by Arc Length 2 B. Example: Curve in \mathbb{R}^2
9. Definition: Rigid Motion B. Theorem: Fundamental Theorem of the Local Theory of Curves
10. Proof: Fundamental Theorem of the Local Theory of Curves 1
11. Proof: Fundamental Theorem of the Local Theory of Curves 2
12. Proof: Fundamental Theorem of the Local Theory of Curves 3

Isoperimetric Inequality:

1. A. Definition: Closed Curves, Simple Closed Curves and Positive Oriented Simple Closed Curves B. Motivation: Dido's Problem
2. A. Theorem: Isoperimetric Inequality B. Recall: Green's Theorem
3. Proof: Isoperimetric Inequality

Regular Surface:

1. A. Review: Isoperimetric Inequality B. Definition: Regular Surface
2. Derivation: Operation of dx_q
3. A. Note: Regularity Condition of x B. Example: Unit Sphere 1
4. Example: Unit Sphere 2
5. A. Proposition: The Graph of a Differentiable Function Is a Regular Surface B. Definition: Critical Points and Critical Values
6. A. Definition: Regular Values B. Proposition: The Inverse Image of a Regular Value Is a Regular Surface C. Example: Unit Sphere 3 D. Recall: Inverse Function Theorem
7. Proof: The Inverse Image of a Regular Value Is a Regular Surface
8. Derivation: Quadric Surfaces
9. Proposition: A Regular Surface Is Locally the Graph of a Differentiable Function

Change of Parameters:

1. A. Example: Hyperboloid B. Definition: Connected Surfaces
2. Proposition: Change of Parameters
3. A. Proof: Change of Parameters B. Definition: Differentiable Functions on Surfaces

Differentiable Functions on Surfaces:

1. A. Review: Differentiable Functions on Surfaces B. Examples: Height Function and Distance Square Function
2. A. Corollary: Differentiable Functions Between Surfaces B. Definition: Diffeomorphic and Diffeomorphism C. Examples: Symmetric Map and Rotation Map D. Example: A Sphere Is Diffeomorphic to an Ellipsoid
3. Derivation: Surface of Revolution

Parametrized Surfaces and Tangent Vectors:

1. A. Definition: Parametrized Surfaces and Regular Parametrized Surfaces B. Example: The Tangent Surface
2. A. Proposition: A Regular Parametrized Surface Is Locally a Regular Surface B. Definition: Tangent Vectors
3. A. Proposition: Relation Between $T_p(S)$ and $dx_q(R^2)$ B. Proposition: The Tangent Plane to $f^{-1}(a)$ Is the Kernel of df_p

The Tangent Plane and the Differential of a Map:

1. A. Review: The Tangent Plane B. Proof: The Tangent Plane to $f^{-1}(a)$ Is the Kernel of df_p C. Example: Sphere 1
2. A. Example: Sphere 2 B. Proposition: Differential of a Map
3. A. Example: Rotation Map on Unit Sphere B. Definition: Local Diffeomorphism C. Proposition: The Differential Is a Local Isomorphism Implies a Local Diffeomorphism
4. A. Definition: Critical Points B. Example: The Tangent Plane at a Critical Point C. Derivation: Differential of Composite Maps 1
5. A. Derivation: Differential of Composite Maps 2 B. Definition: Orthogonality of Two Surfaces

The First Fundamental Form and the Area of a Surface:

1. A. Definition: The First Fundamental Form B. Example: Unit Sphere 1
2. A. Example: Unit Sphere 2 B. Application: Rhumb Line 1
3. A. Application: Rhumb Line 2 B. Definition: The Area of a Surface 1
4. A. Definition: The Area of a Surface 2 B. Example: Unit Sphere
5. A. Definition: Locally Isometric B. Example: A Plane Is Locally Isometric to a Cylinder

Orientation of a Surface:

1. A. Definition: Orientable Surface B. Example: Surface Covered by One Parametrization
2. A. Example: Surface Covered by Two Parametrizations B. Proposition: A Surface Is Orientable if and only if It Has a Differentiable Unit Normal Vector Field 1
3. A. Proposition: A Surface Is Orientable if and only if It Has a Differentiable Unit Normal Vector Field 2 B. Example: Möbius Band

The Gauss Map and Its Fundamental Properties:

1. A. Proposition: Regular Surface Given by the Inverse Image of a Regular Value Is Orientable
B. Definition: Gauss Map 1
2. A. Definition: Gauss Map 2 B. Examples: Plane and Hyperbolic Paraboloid 1
3. Examples: Hyperbolic Paraboloid 2 and Unit Sphere
4. A. Motivation: The Curvature of a Surface Is Characterized by the Differential of Gauss Map
B. Example: Cylinder
5. A. Definition: Self-Adjoint Linear Map B. Proposition: The Differential of Gauss Map Is a Self-Adjoint Linear Map C. Definition: The Second Fundamental Form
6. A. Definition: Normal Curvature B. Derivation: The Geometric Meaning of the Second Fundamental Form
7. A. Review: The Geometric Meaning of the Second Fundamental Form B. Proposition: Meusnier
C. Example: Unit Sphere
8. A. Definition: Normal Section B. Derivation: Define the Normal Curvature by Normal Section
9. A. Definition: Principal Curvatures and Principal Directions B. Definition: Line of Curvature
C. Proposition: Olinde Rodrigues D. Derivation: Euler Formula
10. A. Definition: Gauss Curvature and Mean Curvature B. Definition: Elliptic, Hyperbolic, Planar and Parabolic Point
11. A. Definition: Umbilical Point B. Example: Study the Gauss Map on $2z = x^2y^2$ at $(0,0,0)$
12. Proposition: A Connected Surface with Every Point Being Umbilical Point Is a Piece of Plane or Sphere 1
13. Proposition: A Connected Surface with Every Point Being Umbilical Point Is a Piece of Plane or Sphere 2
14. A. Proposition: A Connected Surface with Every Point Being Umbilical Point Is a Piece of Plane or Sphere 3 B. Definition: Asymptotic Direction and Asymptotic Curve C. Example: Straight Line 1
15. A. Example: Straight Line 2 and Curve with Positive Curvature B. Observation: There Is NO Asymptotic Direction at an Elliptic Point C. Definition: Dupin Indicatrix

The Gauss Map in Local Coordinates:

1. Derivation: Equations of Weingarten 1
2. A. Derivation: Equations of Weingarten 2 B. Gauss Curvature in terms of the First and Second Fundamental Form
3. A. Mean Curvature in terms of the First and Second Fundamental Form B. Principal Curvatures in terms of the First and Second Fundamental Form C. Proposition: Smoothness of Gauss Curvature, Mean Curvature and Principal Curvatures
4. A. Review: The Formula of Gauss Curvature, Mean Curvature and Principal Curvatures B. Example: Torus 1
5. Example: Torus 2
6. Example: Helicoid
7. Proposition: The Position of a Surface in the Neighborhood of an Elliptic Point or a Hyperbolic Point with respect to the Tangent Plane
8. A. Review: Dupin Indicatrix and Its Graph B. Example: Monkey Saddle
9. A. Examples: $z = y^3$ Rotated About $z = 1$ and Cylinder B. Derivation: Gauss Curvature of a Surface of Revolution 1
10. Derivation: Gauss Curvature of a Surface of Revolution 2
11. A. Review: Gauss Curvature of a Surface of Revolution B. Derivation: Differential Equation of the Asymptotic Curves
12. A. Proposition: The Coordinate Curves Are Asymptotic Curves if and only if $e=g=0$ B. Example: Asymptotic Curves 1
13. A. Example: Asymptotic Curves 2 B. Proposition: The Coordinate Curves Are Asymptotic Curves if and only if $f=F=0$ 1
14. Proposition: The Coordinate Curves Are Asymptotic Curves if and only if $f=F=0$ 2
15. Preview: Local Version of Gauss Bonnet Theorem

2

The Sign of Gauss Curvature:

1. A. Review: Some Setting in the Last Semester B. Example: The Surface Given by $z=h(x,y)$ with $K(p)>0$

2. A. Example: The Surface Given by $z=h(x,y)$ with $K(p)<0$ B. Example: The Surface Given by $z=h(x,y)$ with $K(p)=0$
3. Example: Torus and Monkey Saddle

Geometric Interpretation of Gauss Curvature:

1. A. Orientation Preserving and Orientation Reversing Map B. Proposition: The Gauss Map is Orientation Preserving at Elliptic Point and Orientation Reversing at Hyperbolic Point
2. Proposition: Geometric Interpretation of Gauss Curvature 1
3. A. Proposition: Geometric Interpretation of Gauss Curvature 2 B. Examples: Sphere and Trough-Shaped Surface

Local Convex and Curvature:

1. A. Review: Geometric Interpretation of Gauss Curvature B. Remark: Similar Result for Curvature of Plane Curve C. Tangent Indicatrix D. Locally Convex and Strictly Locally Convex
2. Note: Relations between Curvature and Locally Convex
3. Note: A Critical Point of a Distance Function on a Surface

The Rigidity of the Sphere:

1. Theorem: A Compact Surface Has an Elliptic Point
2. A. Theorem: A Compact Connected Surface with Constant Gauss Curvature Is a Sphere B. Lemma: Three Conditions of a Point to Be an Umbilical Point C. Proof: A Compact Connected Surface with Constant Gauss Curvature Is a Sphere 1
3. A. Proof: A Compact Connected Surface with Constant Gauss Curvature Is a Sphere 2 B. Proof: Three Conditions of a Point to Be an Umbilical Point 1
4. Proof: Three Conditions of a Point to Be an Umbilical Point 2
5. Proof: Three Conditions of a Point to Be an Umbilical Point 3
6. Proof: Three Conditions of a Point to Be an Umbilical Point 4

Vector Field:

1. A. Vector Field B. Trajectory of a Vector Field C. Examples: $w=(x,y)$ and $w=(y,-x)$
2. A. Theorem: Existence and Uniqueness of the Trajectory of a Vector Field B. Theorem: Existence of the Local Flow a Vector Field
3. A. Ruled Surface, Ruling and Directrix B. Examples: Plane, Cylinder, Cone and Hyperboloid of Revolution

Ruled Surface:

1. Line of Striction
2. A. Condition of a Ruled Surface Given by the Line of Striction as Directrix being a Regular Surface B. Gauss Curvature of a Ruled Surface Given by the Line of Striction as Directrix
3. A. Developable Surface B. Developable Surface Has Gauss Curvature Zero at Regular Points

Developable Surface:

1. Two Subclasses of Developable surface 1
2. A. Two Subclasses of Developable surface 2 B. Example: The Envelope of the Family of Tangent Planes Along a Curve of a Surface
3. The Envelope of the Family of Tangent Planes Along a Curve of a Surface Is Developable

Minimal Surface:

1. A. Review: Developable Surface B. Minimal Surface C. Normal Variation D. Interpretation of Minimality 1
2. Interpretation of Minimality 2
3. A. Interpretation of Minimality 3 B. Proposition: A Parametrized Surface Is Minimal if and only if $A'(0)=0$
4. A. Proof: A Parametrized Surface Is Minimal if and only if $A'(0)=0$ B. Isothermal Parametrized Surface C. Theorem: If x Is an Isothermal Parametrized Surface Then $x_u u x_v v = 2(a^2)HN$ 1
5. A. Theorem: If x Is an Isothermal Parametrized Surface Then $x_u u x_v v = 2(a^2)HN$ 2 B. Harmonic Function C. Corollary: An Isothermal Parametrized Surface Is Minimal if and only if Its Coordinate Functions Are Harmonic D. Introduction: Development of Minimal Surface

6. A. Examples: Catenoid and Helicoid B. Proposition: Any Minimal Surface of Revolution Is an Open Subset of a Plane or a Catenoid C. Proposition: Any Ruled Minimal Surface Is an Open Subset of a Plane or a Helicoid D. Theorem: There Is No Compact Minimal Surface

The Intrinsic Geometry of Surfaces: Isometries and Conformal:

1. A. Parametrizations for Catenoid and Helicoid B. Isometry and Local Isometry C. Example: Local Isometric; The Cylinder and Plane in R^2 1
2. A. Example: Local Isometric; The Cylinder and Plane in R^2 2 B. Example: Every Helicoid Is Locally Isometric to Catenoid C. Proposition: Two Regular Surfaces Have the Same First Fundamental Form in Domain U if and only if They Are Locally Isometric 1
3. A. Proposition: Two Regular Surfaces Have the Same First Fundamental Form in Domain U if and only if They Are Locally Isometric 2 B. Conformal
4. A. Review: Isometry and Conformal B. Note: A Conformal Map Preserves the Angle between Two Tangent Vectors C. Proposition: A Parametrization Is Conformal if and only if It is Isothermal D. Theorem: Any Two Regular Surfaces Are Locally Conformal
5. A. Proposition: A Criterion for Local Conformal B. Stereographic Projection 1
6. Stereographic Projection 2

The Intrinsic Geometry of Surfaces: Gauss Remarkable Theorem:

1. Christoffel Symbols 1
2. A. Christoffel Symbols 2 B. Christoffel Symbols in Terms of the First Fundamental Form
3. A. All Geometric Concepts and Properties Expressed in Terms of the Christoffel Symbols Are Invariant under Isometry B. Codazzi-Mainardi Equations
4. Codazzi-Mainardi Equations and Gauss Formula 1
5. Codazzi-Mainardi Equations and Gauss Formula 2
6. A. Codazzi-Mainardi Equations and Gauss Formula 3 B. Gauss's Theorema Egregium
7. Codazzi-Mainardi Equations and Gauss Formula 4
8. A. Gauss Curvature in Terms of the First Fundamental Form B. Example: Surface of Revolution C. Proof: Gauss's Theorema Egregium D. Example: Catenoid and Helicoid
9. A. Review: Gauss's Theorema Egregium B. Counterexample: The Converse of Gauss's Theorema Egregium Is Not True 1

10. A. Counterexample: The Converse of Gauss's Theorema Egregium Is Not True 2 B. Theorem: Fundamental Theorem of Surface (Bonnet)
11. Example: Is There a Regular Surface with the Given Differentiable Functions E, F, G, e, f, g
1

Parallel Transport and Geodesics:

1. A. Example: Is There a Regular Surface with the Given Differentiable Functions E, F, G, e, f, g 2 B. Covariant Derivative
2. A. General Formula of the Covariant Derivative B. Example: Covariant Derivative of a Vector Field on a Plane C. Parallel Vector Field D. Proposition: There Exists a Unique Parallel Vector Field along a Curve with Given Initial Value
3. A. Proposition: The Inner Product of Two Parallel Vector Fields Is Constant B. Example: The Tangent Vector Field of a Meridian Is a Parallel Vector Field on a Sphere
4. Parallel Transport
5. A. Parameterized Geodesic and Geodesic B. Algebraic Value and Geodesic Curvature
6. A. Geometric Interpretation of Geodesic Curvature B. Example: Geodesic Curvature of a Circle on a Unit Sphere

Algebra Value of the Covariant Derivative:

1. A. Example: The Normal Curvature and the Geodesic Curvature of the Circle on the Elliptic Parabolic B. Lemma: The Differentiable Extension of a Determination
2. A. Lemma: Relation between the Covariant Derivative of Two Unit Vector Fields and the Variation of the Angle That They Form B. Note: The Geodesic Curvature Is the Rate of Change of the Angle That the Tangent to the Curve Makes with a Parallel Vector Field C. Proposition: An Expression for the Algebraic Value in Terms of the First Fundamental Form and the Variation of the Angle 1
3. Proposition: An Expression for the Algebraic Value in Terms of the First Fundamental Form and the Variation of the Angle 2
4. Proposition: Liouville's Formula 1
5. A. Proposition: Liouville's Formula 2 B. Geodesic Equations
6. Geometric Interpretation of Geodesic

Geodesic Equations:

1. A. Example: Geodesics of a Cylinder B. Example: Geodesics of a Surface of Revolution 1
2. Example: Geodesics of a Surface of Revolution 2
3. A. Example: Geodesics of a Sphere B. Geodesic Parametrization and Geodesic Coordinates

Surfaces of constant Gaussian curvature:

1. A. Review: Geodesic Parametrization and Geodesic Coordinates B. Theorem: Any Point of a Surface of Constant Gauss Curvature Is Contained in a Coordinates Neighborhood That Is Isometric to an Open Set of a Plane, a Sphere or a Pseudo-Sphere 1
2. A. Theorem: Any Point of a Surface of Constant Gauss Curvature Is Contained in a Coordinates Neighborhood That Is Isometric to an Open Set of a Plane, a Sphere or a Pseudo-sphere 2
B. Simple Closed Piecewise Regular Parametrized Curve
3. A. Closed Vertices and Regular Arcs B. Differentiable Functions That Measure the Positive Angle from x_u to the Tangent of a Simple Closed Curve

Gauss-Bonnet Theorem for Simple Closed Curves and Curvilinear Polygons:

1. A. Proposition: Theorem of Turning Tangents B. The Integral of a Differentiable Function over a Bounded Region on an Oriented Surface C. Theorem: Local Version of Gauss-Bonnet Theorem 1
2. Theorem: Local Version of Gauss-Bonnet Theorem 2
3. A. Theorem: Local Version of Gauss-Bonnet Theorem 3 B. Theorem: Global Gauss-Bonnet Theorem

Gauss-Bonnet Theorem:

1. A. Triangulation B. Euler Characteristic Number C. Proposition: Every Regular Region of a Regular Surface Admits a Triangulation
2. Proof: Global Gauss-Bonnet Theorem 1
3. Proof: Global Gauss-Bonnet Theorem 2
4. A. Theorem: Gauss-Bonnet Theorem for Orientable Compact Surface B. Example: Sphere with Radius r C. Example: Convex Surface in R^3 D. Example: Polar Cap 1
5. A. Example: Polar Cap 2 B. Euler Characteristic Number and Genus C. Theorem: Diffeomorphic Surfaces Have the Same Euler Characteristic Number and Two Compact Oriented Surfaces with the Same Euler Characteristic Number Are Diffeomorphic

6. A. Theorem: A compact Oriented Surface with Positive Gauss Curvature Is Diffeomorphic to a Standard Sphere B. Four Color Map Theorem

Clairaut's Theorem:

1. A. Proposition: A Regular Compact Connected Oriented Surface Which Is Not Homeomorphic to a Sphere Has Some Points Such That the Gauss Curvature Is Positive, Negative and Zero B. Proposition: Clairaut's Theorem 1
2. Proposition: Clairaut's Theorem 2
3. Surface of Revolution and Hyperbolic Models

Hyperbolic Models:

1. Hyperbolic Models: Pseudo-Sphere, Upper Half-Plane and Poincare Disc
2. Geodesics of Upper Half-plane: By Clairaut's Theorem
3. Geodesics of Upper Half-plane: By Geodesic Equations

Mobius Transformation and Non-Euclidean Geometry:

1. A. Mobius Transformation B. Mobius Transformation from Upper Half-plane to Upper Half-Plane Is an Isometry 1
2. A. Mobius Transformation from Upper Half-plane to Upper Half-Plane Is an Isometry 2 B. Five Postulates for Euclidean Geometry
3. The Parallel Postulate and Non-Euclidean Geometry

October 31, 2025