Contents

Real Analysis

1 Measure Theory

This section contains:

1.1 Rings and Algebra

Let X be a set of elements, $E_n \subset X$, let $\{E_n\}$ be a sequence of sets.

Definition 1.1.1 (the superior limit of $\{E_n\}$).

$$\begin{aligned} \overline{\lim}\ E_n &= \{x \in X : \ x \textit{ belongs to infinitely } E_n^{'}s\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \end{aligned}$$

Definition 1.1.2 (the inferior limit of $\{E_n\}$).

$$\begin{aligned} \underline{\lim}\ E_n &= \{x \in X : \ x \textit{ belongs to all but finitely many } E_n^{'}s\} \\ &= \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \end{aligned}$$

Note 1.1.1.

$$\underline{\lim} E_n \subseteq \overline{\lim} E_n$$

It will be proved soon.

Definition 1.1.3 (the limit of $\{E_n\}$). *$\{E_n\}$ has a limit if:*

$$\underline{\lim} E_n = \overline{\lim} E_n$$

in this case, we denote it as $\lim\limits_{n\to\infty} E_n$.

Example 1.1.1.

$$E_n = \begin{cases} A & \textit{if } n \textit{ is even} \\ B & \textit{if } n \textit{ is odd} \end{cases}$$

Then $\overline{\lim} E_n = A \bigcup B$, $\underline{\lim} E_n = A \bigcap B$, *and* $\lim E_n$ *exists* $\Leftrightarrow \quad A = B$

Some properties:

Proposition 1.1.1.

1. $\underline{\lim}\ E_n \subseteq \overline{\lim}\ E_n$

2. $(\underline{\lim}E_n)^c = \overline{\lim}E_n^c$

$\left(\overline{\lim}E_n\right)^c = \underline{\lim}E_n^c$

3. $\forall n,\ E_n \subseteq E_{n+1} \quad \Rightarrow \quad \lim E_n = \bigcup_n E_n$

$\forall n,\ E_n \supseteq E_{n+1} \quad \Rightarrow \quad \lim E_n = \bigcap_n E_n$

Proof.

1. $\because \forall x \in \underline{\lim}E_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n \ \therefore \exists k_0$ s.t. $x \in \bigcap_{n=k_0}^{\infty} E_n$, i.e. $x \in E_{k_0}, E_{k_0+1}, \cdots, E_j, \cdots \quad j \geq k_0$

   so, $\forall k, \exists k_1$, s.t. $k_1 \geq max\{k_0, k\}$, $x \in E_{k_1}$, i.e. $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n$, so, $\underline{\lim}\ E_n \subseteq \overline{\lim}\ E_n$.

2. By De Morgan's law, it's obviously.

3. We modify the statement of (3):

   $\forall n,\ E_n \subseteq E_{n+1} \ \Rightarrow \ \lim E_n = \bigcup_n E_n, \qquad F_n \supseteq F_{n+1} \ \Rightarrow \ \lim F_n = \bigcap_n F_n$

   (a) $\because E_n \subseteq E_{n+1} \ \therefore \bigcap_{n=k}^{\infty} E_n = E_k \ \therefore \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n = \bigcup_{k=1}^{\infty} E_k = \bigcup_{n=1}^{\infty} E_n \ \therefore \underline{\lim}E_n = \bigcup_{n=1}^{\infty} E_n$,

   $\because \underline{\lim}E_n \subseteq \overline{\lim}E_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} E_n = \underline{\lim}E_n \therefore \underline{\lim}E_n = \overline{\lim}E_n \therefore \lim E_n = \bigcup_{n=1}^{\infty} E_n$

   (b) $\because F_n \supseteq F_{n+1} \ \therefore \bigcup_{n=k}^{\infty} F_n = F_k \ \therefore \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} F_n = \bigcap_{k=1}^{\infty} F_k = \bigcap_{n=1}^{\infty} F_n \ \therefore \overline{\lim}F_n = \bigcap_{n=1}^{\infty} F_n$,

   $\because \underline{\lim}F_n \subseteq \overline{\lim}F_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n \subseteq \bigcap_{n=k}^{\infty} F_n \subseteq \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} F_n = \underline{\lim}F_n \therefore \underline{\lim}F_n = \overline{\lim}F_n \therefore \lim F_n = \bigcup_{n=1}^{\infty} F_n$

$\square$

Definition 1.1.4 (Characteristic Function of a Set)**.**

$$\chi_E\,(x) = \begin{cases} 1 & if\ x \in E \\ 0 & if\ x \notin E \end{cases}$$

$\chi_E : X \to \mathbb{R}$

Proposition 1.1.2.

1. $\overline{\lim}\ \chi_{E_n} = \chi\ \overline{\lim}E_n$

2. $\underline{\lim}\ \chi_{E_n} = \chi\ \underline{\lim}E_n$

Proof.

1. (a) if $\chi\overline{\lim}E_n = 1$, then $x \in \overline{\lim}E_n = \bigcap\limits_{k=1}^{\infty}\bigcup\limits_{n=k}^{\infty} E_n \Rightarrow \forall k, x \in \bigcup\limits_{n=k}^{\infty} E_n \Rightarrow \forall k, \exists n_k \geqslant k, x \in E_{n_k}, \chi E_{n_k} = 1, \because \chi E_n \leqslant 1, \therefore \overline{\lim}\ \chi E_n = 1$

   (b) if $\chi\overline{\lim}E_n = 0$, then $x \notin \overline{\lim}E_n = \bigcap\limits_{k=1}^{\infty}\bigcup\limits_{n=k}^{\infty} E_n \Rightarrow \exists k, x \notin \bigcup\limits_{n=k}^{\infty} E_n \Rightarrow \exists k, \forall n \geqslant k, x \notin E_n, \chi E_n = 0, \lim\limits_{n\to\infty} \chi E_n = 0, \therefore \lim\limits_{n\to\infty} \chi E_n = \overline{\lim\limits_{n\to\infty}} \chi E_n$, so $\overline{\lim\limits_{n\to\infty}}\ \chi E_n = 0$

2. (a) if $\chi\underline{\lim}E_n = 1$ then $x \in \underline{\lim}E_n = \bigcup\limits_{k=1}^{\infty}\bigcap\limits_{n=k}^{\infty} E_n \Rightarrow \exists k_0, x \in \bigcap\limits_{n=k_0}^{\infty} E_n \Rightarrow \exists k_0, \forall n \geqslant k_0, x \in E_n \Rightarrow \exists k_0, \forall n \geqslant k_0, \chi E_n = 1 \Rightarrow \lim\limits_{n\to\infty} \chi E_n = 1 \because \lim\limits_{n\to\infty} \chi E_n = \underline{\lim}\chi E_n \therefore \underline{\lim}\chi E_n = 1$

   (b) if $\chi\underline{\lim}E_n = 0$ then $x \notin \underline{\lim}E_n = \bigcup\limits_{k=1}^{\infty}\bigcap\limits_{n=k}^{\infty} E_n \Rightarrow \forall k, x \notin \bigcap\limits_{n=k}^{\infty} E_n \Rightarrow \forall k, \exists n_k \geqslant k, x \notin E_{n_k} \Rightarrow \forall k, \exists n_k \geqslant k, \chi E_{n_k} = 0 \Rightarrow \lim\limits_{k\to\infty} \chi E_{n_k} = 0 \because \chi E_n \geqslant 0 \therefore \underline{\lim}\chi E_n = 0$

□

Proposition 1.1.3. $\lim E_n$ *exists* $\Leftrightarrow$ $\lim \chi_{E_n}$ *exists*

Proof. $\lim\limits_{n\to\infty} E_n$ exists $\Leftrightarrow \overline{\lim}E_n = \underline{\lim}E_n \Leftrightarrow \chi\ \overline{\lim}E_n = \chi\ \underline{\lim}E_n \Leftrightarrow \overline{\lim}\ \chi E_n = \underline{\lim}\ \chi E_n \Leftrightarrow \lim\limits_{n\to\infty} \chi E_n$ exist. □

Definition 1.1.5 (Power Set)**.** $\mathcal{P}(x) = \{$*all sunstes of* $X\}$

Definition 1.1.6 (Ring)**.** $A, B \subset X, R$ *is a ring if*

1. $\emptyset \in R$

2. $A, B \in R \Rightarrow A\backslash B \in R$ $(A\backslash B:\ x \in A,$ *but* $x \notin B)$

3. $A, B \in R \Rightarrow A \bigcup B \in R$

Some properties of R:

Proposition 1.1.4.

1. $A_1, ..., A_n \in R \Rightarrow \bigcup\limits_{i=1}^{n} A_i \in R$

2. $A, B \in R \Rightarrow A \cap B \in R$

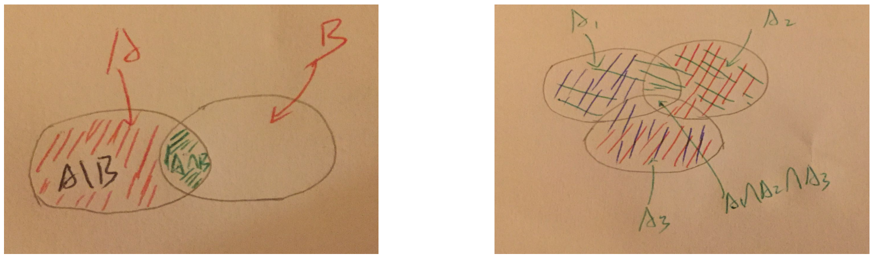
3. $A_1, ..., A_n \in R \Rightarrow \bigcap\limits_{i=1}^{n} A_i \in R$

Proof.

1. it's easily

2. $A \cap B = A \backslash (A \backslash B) \in R, \quad A \cap B = (A \cup B) \backslash [((A \cup B) \backslash B) \cup ((A \cup B) \backslash A)]$ also holds.

3. $\bigcap_{i=1}^{n} A_i = A_n \cap \left(\bigcap_{i=1}^{n-1} A_i \right) = \cdots = A_n \cap A_{n-1} \cap \cdots \cap A_1 \in R$



(a) $A \cap B$ (b) $A_1 \cap A_2 \cap A_3$

□

Proposition 1.1.5 ([1])**.** *If we define $A + B = A \Delta B, \quad A \cdot B = A \cap B$, then $R(\Delta, \cdot)$ is a algebra ring.*

Definition 1.1.7 ($\sigma-$ring)**.** *R is a $\sigma-$ring if*

1. $\emptyset \in R$
2. $A, B \in R \Rightarrow A \backslash B \in R$
3. $A_1, A_2, ... \in R \Rightarrow \bigcup_{n=1}^{\infty} A_n \in RA$

Definition 1.1.8 ($\sigma-$algebra)**.** *R is $\sigma-$algebra if R is a σ-ring & $X \in R$*

Note 1.1.2. *In probability theory, elements in R are "events", $X = \{all\ outcomes\}$*

Example 1.1.2. *Toss a dice: $X = \{1, 2, 3, 4, 5, 6\} \quad R = \mathcal{P}(X)$*

Note 1.1.3. *$\sigma-$ring $\Rightarrow$ ring, $\sigma-$algebra $\Rightarrow$ algebra*

Some properties of $\sigma-$ring R:

Proposition 1.1.6.

1. $A_1, A_2, ... \in R \Rightarrow \bigcap_{n=1}^{\infty} A_n \in R$
2. $A_1, A_2, ... \in R \Rightarrow \overline{\lim} A_n, \underline{lim} A_n \in R$

Proof.

1. Let $A = A_1 \cup A_2 \cup \cdots \in R$, we can get $\bigcap_n A_n = A \backslash \bigcup_n (A \backslash A_n) \in R$

2. it is easily

$\square$

Theorem 1.1.1 (ring)**.** *R is a ring $\Leftrightarrow \emptyset \in R, A, B \in R \Rightarrow A\backslash B \in R$, and $A \bigcap B = \emptyset \Rightarrow A \bigcup B \in R$*

Proof. " $\Rightarrow$ " is easily , " $\Leftarrow$ "$A \cup B = A \cup (A\backslash B) \in R$ $\square$

Theorem 1.1.2 (algebra)**.** *R is a algebra $\Leftrightarrow \emptyset \in R,\ A, B \in R \Rightarrow A \bigcup B \in R$, and $A \in R \Rightarrow A^c \in R$*

Proof. " $\Rightarrow$ " $X\backslash A = A^c \in R$, " $\Leftarrow$ "$A\backslash B = A \cap B^c = (A^c \cup B)^c \in R, \emptyset \in R \Rightarrow (\emptyset)^c = X \in R$ $\square$

Theorem 1.1.3 ($\sigma-$ring)**.** *R is a $\sigma-$ring $\Leftrightarrow \emptyset \in R,\ A, B \in R \Rightarrow A\backslash B \in R, \{A_n\} \subset R$, mutuallly disjoint $\Rightarrow \bigcup\limits_n A_n \in R$*

Proof. " $\Rightarrow$ " is easily

" $\Leftarrow$ "

$B_1 = A_1$

$B_2 = A_2\backslash A_1$

$B_3 = A_3\backslash (A_1 \cup A_2)$

$\ldots$

$B_n = A_n\backslash \left(\bigcup\limits_{i=1}^{n-1} A_i\right)$

$\ldots$ $\square$

Theorem 1.1.4 ($\sigma-$algebra)**.** *R is a $\sigma-$ algebra $\Leftrightarrow \emptyset \in R,\ A, B \in R \Rightarrow A\backslash B \in R, \{A_n\} \subset R$, mutuallly disjoint $\Rightarrow \bigcup\limits_n A_n \in R$ and $X \in R$.*

$D \subset \mathcal{P}(X)$, Let R_0 be the intersection of all rings containing D.

Note 1.1.4.

1. *intersection of rings is a ring.*
2. *$\exists$ ring $\mathcal{P}(x)$ which contains $D \Rightarrow R_0$ is the smallest ring contains D.*

Definition 1.1.9. *R_0 ring generated by D, denoted by $R(D)$: top down; bottom up: perform $\bigcup, \bigcap, \setminus$ repeatelly on elements of D.*

Homework 1.1.

1. *Problem* 1.1.8 *If D is any class of sets, then every set in $\mathcal{P}(D)$ can be covered by countable union of set of D.*
2. *Problem* 1.1.9 *Let D consist of those sets which are either finite or have a finite complement. Then D is an algebra. If X is not finite, then D is not a σ-algebra.*

1.2 Definition of Measure

Let X be a set, and $\mathbf{a}$ be a $\sigma-$algebra on X.

Definition 1.2.1 (Measure)**.** *$\mu : \mathbf{a} \to [0, \infty]$ is a measure if*

1. *$\mu(\emptyset) = 0$*
2. *μ is countably addition:*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu\left(E_n\right) \quad \text{for} \quad E_n \in \mathbf{a}, E_i \bigcap E_j = \emptyset \quad \text{for } i \neq j$$

Definition 1.2.2 (Additive and Subadditive)**.**

1. *(completely) additive: $\mu\left(E \cup F\right) = \mu\left(E\right) + \mu\left(F\right) \quad$ for $E, F \subset \mathbf{a},\ E \bigcap F = \emptyset$*
2. *finitely additive: $\mu\left(E_1 \cup \cdots \cup E_n\right) = \mu\left(E_1\right) + \cdots + \mu\left(E_n\right) \quad$ for $E, \cdots, E_n \subset \mathbf{a},\ E_i \bigcap E_j = \emptyset$ for $i \neq j$*
3. *subadditive: $\mu\left(E \cup F\right) \leq \mu\left(E\right) + \mu\left(F\right) \quad \forall E, F \subset \mathbf{a}$*
4. *finitely subadditive: $\mu\left(\bigcup\limits_{i=1}^{n} E_i\right) \leq \sum\limits_{i=1}^{n} \mu\left(E_i\right) \quad$ for $E_1, \cdots, E_n \subset \mathbf{a}$*
5. *countably subadditive: $\mu\left(\bigcup\limits_{i=1}^{\infty} E_i\right) \leq \sum\limits_{i=1}^{\infty} \mu\left(E_i\right) \quad$ for $E_1, \cdots, E_n, \cdots \subset \mathbf{a}$*
6. *finite measure: if $\mu(X) < \infty$*
7. *$\sigma-$finite measure: if $\exists \{E_n\} \subset \mathbf{a}$ s.t. $\mu\left(E_n\right) < \infty \ \ \forall n$ & $X = \bigcup\limits_{n} E_n$*

Proposition 1.2.1 (Properties for μ measure on $\mathbf{a}$).

1. *finitely additive:* $E_1, \cdots, E_n \in \mathbf{a}, E_i \cap E_j = \emptyset$ *for* $i \neq j \Rightarrow \mu\left(\bigcup\limits_{i=1}^{n} E_i\right) = \sum\limits_{i=1}^{n} \mu(E_i)$
2. *monotone:* $E, F \in \mathbf{a}, E \subset F \Rightarrow \mu(E) \leqslant \mu(F)$
3. $E, F \in \mathbf{a}, \mu(E) < \infty \Rightarrow \mu(F \backslash E) = \mu(F) - \mu(E)$
4. $E, F \in \mathbf{a} \Rightarrow \mu(E \cap F) + \mu(E \cup F) = \mu(E) + \mu(F)$
5. *countably subadditive:* $E_n \in \mathbf{a} \Rightarrow \mu\left(\bigcup\limits_{n} E_n\right) \leqslant \sum\limits_{n} \mu(E_n)$
6. *continuity from below:* $E_n \in \mathbf{a}$ & $E_n \uparrow \Rightarrow \lim \mu(E_n) = \mu(\lim E_n)$
7. *continuity from above:* $E_n \in \mathbf{a},\ E_n \downarrow, \mu(E_{n_0}) < \infty$ *for some* $n_0 \Rightarrow \lim \mu(E_n) = \mu(\lim E_n)$
8. *lower semicontinuous:* $E_n \in \mathbf{a} \Rightarrow \mu(\underline{\lim} E_n) \leqslant \underline{\lim} \mu(E_n)$
9. *upper semicontinuous:* $E_n \in \mathbf{a},\ \mu\left(\bigcup\limits_{n} E_n\right) < \infty \Rightarrow \mu(\overline{\lim} E_n) \geqslant \overline{\lim} \mu(E_n)$
10. *continuous* $E_n \in \mathbf{a},\ limE_n$ *exists* & $\mu\left(\bigcup\limits_{n} E_n\right) < \infty \Rightarrow \mu(\lim E_n) = \lim \mu(E_n)$

Proof.

1. Let $E_{n+1} = E_{n+2} = \cdots = \emptyset$.
2. $F = E \bigcup (F \backslash E)$, $E, (F \backslash E)$ are disjoint. so, $\mu(F) = \mu(E \bigcup (F \backslash E)) = \mu(E) + \mu(F \backslash E) \Rightarrow \mu(E) \leqslant \mu(F)$
3. $\mu(F) = \mu(E) + \mu(F \backslash E)$

   Note 1.2.1. *if* $\mu(E) = \infty$, *then* $\mu(F) = \infty \Rightarrow \mu(F) - \mu(E)$ *is meaningless*

4. $E \cup F = E \cup (F \backslash E), \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F \backslash E) = \mu(E) + \mu[F \backslash (E \bigcap F)]$
   (a) $\mu(E \bigcap F) < \infty$ $\quad \mu(F \backslash E) = \mu[F \backslash (E \bigcap F)] = \mu[F] - \mu(E \bigcap F)$ $\quad \mu(E \cup F) = \mu(E) + \mu(F \backslash E) = \mu(E) + \mu(F) - \mu(E \bigcap F) \Rightarrow \mu(E \bigcap F) + \mu(E \cup F) = \mu(E) + \mu(F)$
   (b) $\mu(E \bigcap F) = \infty$ $\quad E \bigcap F \in E, F,\ \therefore \mu(E) = \infty, \mu(F) = \infty, \mu(E \cup F) = \infty$
5. $F_1 = E_1, F_2 = E_2 - E_1, F_3 = E_3 - (E_1 \cup E_2) \cdots\cdots$ so, $F_n \subseteq E_n, \forall n$ $\ \bigcup\limits_{n} F_n = \bigcup\limits_{n} E_n$, $\{F_n\}$ mutually disjoint

$$\mu\left(\bigcup_{n} E_n\right) = \mu\left(\bigcup_{n} F_n\right) = \sum_{n} \mu(F_n) \leqslant \sum_{n} \mu(E_n)$$

6. $\because \lim\limits_{n\to\infty} E_n = E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash E_2) \cup \cdots$ disjoint union

$$\begin{aligned}\therefore \mu\left(\lim_{n\to\infty} E_n\right) &= \mu(E_1) + \mu(E_2\backslash E_1) + \mu(E_3\backslash E_2) + \cdots \\ &= \lim_{n\to\infty}\left[\mu(E_1) + \mu(E_2\backslash E_1) + \mu(E_3\backslash E_2) + \cdots + \mu(E_n\backslash E_{n-1})\right] \\ &= \lim_{n\to\infty} \mu(E_n)\end{aligned}$$

7. consider E_{n_0} as universal set, so $E_{n_0}\backslash E_n \in$ **a** monotone increasing for $n \geq n_0$ $\because E_n \subseteq E_{n_0} \therefore u(E_n) \leqslant \mu(E_{n_0})$

$$\begin{aligned}\lim \mu(E_{n_0}\backslash E_n) &= \mu\left[\bigcup_n (E_{n_0}\backslash E_n)\right] \\ \lim\left[\mu(E_{n_0}) - \mu(E_n)\right] &= \mu\left[E_{n_0}\backslash\left(\bigcap_n E_n\right)\right] \\ &= \mu(E_{n_0}) - \mu\left(\bigcap_n E_n\right)\end{aligned}$$

$$\therefore \qquad \lim \mu(E_n) = \mu\left(\bigcap_n E_n\right)$$

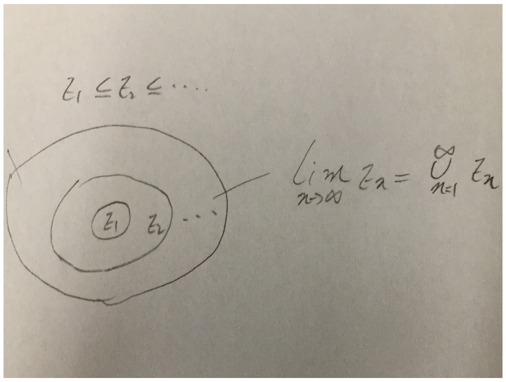
Note 1.2.2. *$\mu(E_{n_0}) < \infty$ is essential, consider the example of $(\mathbb{R}, \mathcal{M}, m)$, and $A_n = [n, \infty), \forall n \in \mathbb{N}$. In this case we have $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}$, but $\lim\limits_{n\to\infty} m(A_n) \not\to m\left(\bigcap\limits_{n=1}^{\infty} A_n\right)$. $\because$ $m(A_n) = \infty,\ \forall n \in \mathbb{N}$, but $m\left(\underbrace{\bigcap\limits_{n=1}^{\infty} A_n}_{\emptyset}\right) = 0$. Claim that $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$, otherwise$\exists n_0$, s.t. $\bigcap\limits_{n=1}^{\infty} A_n = [n_0, \infty) \therefore [n_0, \infty) \subseteq A_{n_0+1} = [n_0+1, \infty)$, it is contradict, so $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$.*

8. $\mu(\underline{\lim} E_n) = \mu\left(\bigcup\limits_{k=1}^{\infty}\bigcap\limits_{n=k}^{\infty} E_n\right) \because \bigcap\limits_{n=k}^{\infty} E_n \subseteq \bigcap\limits_{n=k+1}^{\infty} E_n \quad \therefore \mu\left(\bigcup\limits_{k=1}^{\infty}\bigcap\limits_{n=k}^{\infty} E_n\right) = \lim\limits_{k} \mu\left(\bigcap\limits_{n=k} E_n\right) = \underline{\lim}_{k}\, \mu\left(\bigcap\limits_{n=k} E_n\right) \leqslant \underline{\lim}_{k}\, \mu(E_k)$

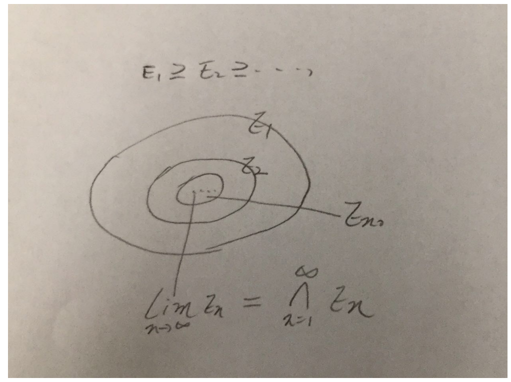
9. $\mu\left(\overline{\lim}\right) = \mu\left(\bigcap\limits_{k=1}^{\infty}\bigcup\limits_{n=k}^{\infty} E_n\right) \quad \because \bigcup\limits_{n=k}^{\infty} E_n \supseteq \bigcup\limits_{n=k+1}^{\infty} E_n \quad \mu\left(\bigcup\limits_{n=1}^{\infty} E_n\right) < \infty \therefore \mu\left(\bigcap\limits_{k=1}^{\infty}\bigcup\limits_{n=k}^{\infty} E_n\right) = \lim \mu\left(\bigcup\limits_{n=k}^{\infty} E_n\right) = \overline{\lim}\mu\left(\bigcup\limits_{n=k}^{\infty} E_n\right) \geqslant \overline{\lim}\mu(E_k)$

10. $\overline{\lim}\mu(E_n) \leqslant \mu\left(\overline{\lim}E_n\right) = \mu(\lim E_n) = \mu(\underline{\lim}E_n) \leqslant \underline{\lim}\mu(E_n) \leqslant \overline{\lim}\mu(E_n) \Rightarrow \mu(\lim E_n) = \lim \mu(E_n)$

$\square$



(a) continuity from below



(b) continuity from above

Homework 1.2.

1. Problem 1.2.5 *Let X consist of a sequence $\{x_m\}$ and let $\{p_m\}$ be a sequence of nonnegative numbers. For any subset $A \subset X$, Let*

$$\mu(A) = \sum_{x_m \in A} p_m$$

Then μ is a $\sigma-$finite measure.

2. Problem 1.2.6 *Given an example of measure μ and a monotone-decreasing sequence $\{E_n\}$ of* **a** *such that $\mu(E_n) = \infty$ for all n, and $\mu\left(\lim_{n\to\infty} E_n\right) = 0$.*

1.3 Outer Measure

Motivation: Constructing Lebesgue measure

$$X \text{ set} \begin{cases} \text{convering sets by union of intervals \& taking infinity} \Rightarrow \text{gives outer measure} \\ \text{Then measure} \end{cases}$$

Definition 1.3.1 (Outer Measure)**.** *$\mu^* : \mathcal{P}(X) \to [0, \infty]$ is out measure if*

1. *$\mu^*(\emptyset) = 0$*
2. *μ^* countably subadditive*
3. *$E, F \in \mathcal{P}(X),\ E \subseteq F \Rightarrow \mu^*(E) \le \mu^*(F)$*

Outer measure $\to$ measure

Definition 1.3.2 (μ^*-measure)**.** *μ^* outer measure on $\mathcal{P}(X), E \in \mathcal{P}(X),\ E$ is μ^*-measure if*

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \backslash B) \quad \forall A \subseteq X$$

Note 1.3.1.

$$\mu^*(A) \le \mu^*(A\cap B) + \mu^*(A\backslash B)$$

always true (due to subadditive)

Theorem 1.3.1. *μ^* outer measure, $\mathbf{a} \equiv \{$ μ^*-measure substets$\}$, then*

1. *$\mathbf{a}$ is σ-algebra*
2. *$\mu^* |\mathbf{a}$ is measure.*

Proof.

1. $\emptyset \in \mathbf{a}$, check $\mu^*(A) \stackrel{?}{=} \mu^*(A\cap\emptyset) + \mu^*(A\backslash\emptyset)$

$$\mu^*(A\cap\emptyset) + \mu^*(A\backslash\emptyset) \ = \mu^*(\emptyset) \ + \mu^*(A) \ = 0 + \mu^*(A) \ = \mu^*(A)$$

2. $E \in \mathbf{a} \Rightarrow E^c \in \mathbf{a}$, check $\mu(A) = \mu^*(A\cap E^c) + \mu^*(A\cap E)$

$$\mu^*(A\cap E^c) = \mu^*(A\backslash E)$$
$$\mu^*(A\backslash E^c) = \mu^*(A\cap E)$$

$\because E \in \mathbf{a}$, so

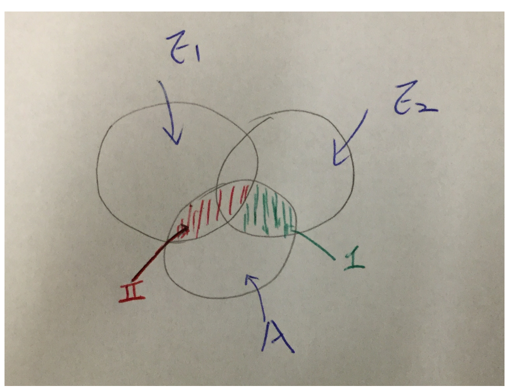
$$\mu(A) = \mu^*(A\cap E) + \mu^*(A\backslash E)$$

so,

$$\mu(A) = \mu^*(A\cap E) + \mu^*(A\backslash E) = \mu^*(A\backslash E^c) + \mu^*(A\cap E^c) = \mu^*(A\cap E^c) + \mu^*(A\backslash E^c)$$

$\therefore E^c \in \mathbf{a}$.

3. $E_1, E_2 \in \mathbf{a} \Rightarrow E_1 \cup E_2 \in \mathbf{a}$, check: $\mu*(A) = \mu^*[A\cap(E_1\cup E_2)] + \mu^*[A\backslash(E_1\cup E_2)] \quad \forall A \subseteq X$



$$A\cap(E_1\cup E_2) = \underbrace{[(A\backslash E_1)\cap E_2]}_{I} \cup \underbrace{(A\cap E_1)}_{II}$$
$$A\backslash(E_1\cup E_2) = [(A\backslash E_1)\backslash E_2]$$

so,

$$\begin{aligned}
\mu^*\left[A\cap\left(E_1\cup E_2\right)\right] &\leqslant \mu^*\left[\left(A\backslash E_1\right)\cap E_2\right]+\mu\left(A\cap E_1\right)\\
\mu^*\left[A\cap\left(E_1\cup E_2\right)\right]+\mu^*\left[A\backslash\left(E_1\cup E_2\right)\right] &\leq \mu^*\left[\left(A\backslash E_1\right)\cap E_2\right]+\mu^*\left(A\cap E_1\right)+\mu^*\left[\left(A\backslash E_1\right)\backslash E_2\right]\\
&= \underbrace{\mu^*\left[\left(A\backslash E_1\right)\cap E_2\right]+\mu^*\left[\left(A\backslash E_1\right)\backslash E_2\right]}_{E_2\in\mathbf{a}}+\mu^*\left(A\cap E_1\right)\\
&= \underbrace{\mu^*\left(A\backslash E_1\right)+\mu^*\left(A\cap E_1\right)}_{E_1\in\mathbf{a}}\\
&= \mu^*\left(A\right)
\end{aligned}$$

From 1 & 2 & 3 $\Rightarrow \mathbf{a}$ is algebra.

Assume that $\{E_n\}\subseteq\mathbf{a}$, disjoint

4. let $S_n=\bigcup\limits_{k=1}^{n}E_k$, by 3 above finite union $S_n\in\mathbf{a}$, then

$$\mu^*\left(A\cap S_n\right)=\sum_{k=1}^{n}\mu^*\left(A\cap E_k\right)\quad\forall A\in X$$

check it by induction,

(a) n=1 it has done already
(b) assume true for $\leq n$
(c) check $n+1$

$$\begin{aligned}
\mu^*\left(A\cap S_{n+1}\right)&=\mu^*\left[\underbrace{\left(A\cap S_{n+1}\right)\cap S_n}_{A\cap S_n}\right]+\mu^*\left[\underbrace{\left(A\cap S_{n+1}\right)\backslash S_n}_{A\cap E_{n+1}}\right]\\
&=\underbrace{\sum_{k=1}^{n}\mu^*\left(A\cap E_k\right)}_{\text{induction hypothesis}}+\mu^*\left(A\cap E_{n+1}\right)\\
&=\sum_{k=1}^{n+1}\mu^*\left(A\cap E_k\right)
\end{aligned}$$

5. Let $S=\bigcup\limits_{n}E_n$ then

$$\mu^*\left(A\cap S\right)=\sum_{n}\mu^*\left(A\cap E_n\right)\quad\forall A\subseteq X$$

check:

(a) $\leq$

$$\begin{aligned}
\mu^*\left[A\cap\left(\bigcup_n E_n\right)\right]&=\mu^*\left[\bigcup_n\left(A\cap E_n\right)\right]\\
&\leq\sum_n\mu^*\left(A\cap E_n\right)
\end{aligned}$$

(b) $\geq$

$$\begin{aligned}\mu^*(A\cap S) &\geqslant \mu^*(A\cap S_n)\\ &= \sum_{k=1}^{n}\mu^*(A\cap E_k)\quad \forall\ n\end{aligned}$$

let $n\to\infty$, it also holds.

6. $S\in\mathbf{a}$

$$\begin{aligned}\mu^*(A) &= \underbrace{\mu^*(A\cap S_n)+\mu^*(A\backslash S_n)}_{S_n\in\mathbf{a}\ \&\ by\ 3}\\ &\geqslant \sum_{k=1}^{n}\mu^*(A\cap E_k)+\underbrace{\mu^*(A\backslash S)}_{A\backslash S\subseteq A\backslash S_n}\end{aligned}$$

let $n\to\infty$

$$\begin{aligned}&\sum_n \mu^*(A\cap E_k)+\mu^*(A\backslash S)\\ &=\underbrace{\mu^*(A\cap S)}_{by\ 4}+\mu^*(A\backslash S)\end{aligned}$$

so,

$$\mu^*(A)=\mu^*(A\cap S)+\mu^*(A\backslash S)$$

$\therefore\ \mathbf{a}\quad\sigma-$algebra

7. Let $A=X$ in 5 then $\mu^*(S)=\sum_n\mu^*(E_n)$, and $\mu^*|\mathbf{a}$ satisfies $\mu^*(\emptyset)=0\ \therefore\ \mu^*|\mathbf{a}$ measure.

□

Homework 1.3.

1. Problem 1.3.1 [2] *Define $\mu^*(E)$ as the number of points in E if E is finite and $\mu^*(E)=\infty$ if E is infinite. Show that μ^* is an outer measure. Determine the measurable sets.*
2. Problem 1.3.3 [2] *Let X have a noncountable number of points. Set $\mu^*(E)=0$ if E is countable, $\mu^*(E)=1$ if E is noncountable. Show that μ^* is an outer measure, and determine the measurable sets.*
3. Problem 1.3.6 [2] *Prove that if an outer measure is finitely additive, then it is a measure.*

1.4 Construction of Outer Measures

Definition 1.4.1 (Sequential Covering Class)**.** *Let X be a set, and $K\subseteq\mathcal{P}(X)$, K is sequential covering class if*

1. $\emptyset\in K$

2. $\forall A \subseteq X, \exists \{E_n\} \subset K \quad s.t. \quad A \subseteq \bigcup\limits_{n=1}^{\infty} E_n$

Example 1.4.1. $X = \mathbb{R}, K = \{(\textit{finite open }) \textit{ interval}\} \bigcup \{\emptyset\}$

Let $\lambda : K \to [0, \infty]$ s.t. $\lambda(\emptyset) = 0$, for $A \subseteq X$, Let

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \lambda(E_n) : A \subseteq \bigcup_n E_n, E_n \in K, \forall n \right\}$$

Theorem 1.4.1. *μ^* is an outer measure.*

Note 1.4.1. *In general, μ^* may not be an extension of λ.*

Proof.

1. $\mu^*(\emptyset) = 0$ let $E_n = \emptyset, \quad \forall n$
2. monotone $A \subseteq B \Rightarrow \mu^*(A) \leqslant \mu^*(B)$

   Note 1.4.2. *Suppose that $A \subseteq B \in \mathbb{R}$. If $supA$ and $supB$ exist, then $supA \leq supB$.*

   Proof. Since $supB$ is an upper bound of B and $A \subseteq B$, it follows that $supB$ is an upper bound of A, so $supA \leq supB$. □

   Similarly, $inf A \geq inf B$.

   We denote that

   $$X_A = \left\{ \sum_{n=1}^{\infty} \lambda(E_n) \middle| A \subseteq \bigcup_n E_n, E_n \in K, \forall n \right\}$$
   $$X_B = \left\{ \sum_{n=1}^{\infty} \lambda(E_n) \middle| B \subseteq \bigcup_n E_n, E_n \in K, \forall n \right\}$$

   To find an arbitrary element of X_B, we need an arbitrary sequential covering class of B, so let $\{E_n\}_{n=1}^{\infty}$ be a sequential covering class of B, i.e. $B \subseteq \bigcup\limits_{n=1}^{\infty} E_n$, then $\sum\limits_{n=1}^{\infty} \lambda(E_n) \in X_B$, notice that $A \subseteq B \subseteq \bigcup\limits_{n=1}^{\infty} E_n$ and hence $\sum\limits_{n=1}^{\infty} \lambda(E_n) \in X_A$, so $X_B \subseteq X_A$. Therefore

   $$\mu^*(A) = \inf(X_A) \leqslant \inf(X_B) = \mu^*(B)$$

3. let $A_n \subseteq X, \forall n$, check $\mu^*\left(\bigcup\limits_n A_n\right) \leqslant \sum\limits_n \mu^*(A_n)$

   Recall that $\inf a_n = a \Leftrightarrow \begin{cases} 1. \ \ a \leq a_n \ \forall n \\ 2. \ \ \forall \varepsilon > 0, \exists a_n < a + \varepsilon \end{cases}$

Let $\varepsilon > 0$

$$\exists \{E_{nk}\} \in K, s.t. A_n \subseteq \bigcup_k E_{nk} \sum_k \lambda(E_{nk}) \leqslant \mu^*(A_n) + \frac{\varepsilon}{2^n}$$

$$A_n \subseteq \bigcup_k E_{nk} \quad \bigcup_n A_n \subseteq \bigcup_{n,k} E_{nk}$$

$$\Rightarrow \quad \mu^*\left(\bigcup_n A_n\right) \leqslant \lambda \sum_{n,k} (E_{nk}) \leqslant \sum_n \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_n \mu^*(A_n) + \varepsilon$$

let $\varepsilon \to 0$, the "=" holds.

$\square$

Example 1.4.2. *X set, $K = \{\emptyset, X, sigletons\}$. Then K is sequence covering class.
And define $\lambda(\emptyset) = 0, \lambda(X) = \#X, \lambda(singlepons) = 1$ then $\mu^* : P(X) \to [0, \infty]$ is $\mu^*(A) = \#A$, μ^* is (counting) measure.*

Homework 1.4.

1. *problem* 1.4.4 *If K is a $\sigma-$algebra and λ is a measure on K, then $\mu^*(A) = \lambda(A)$ for any $A \in K$. (Hint: $\mu^*(A) = inf\{\lambda(E); E \in K, E \supseteq A\}$)*
2. *problem* 1.4.5 *If K is $\sigma-$algebra and λ is a measure on K, then every set in K is μ^*-measure.*

Summary 1.4.1. *X is a set, K is a sequence covering class, $\lambda(\cdot)$*
$\Rightarrow u^*$ outer measure on $\mathcal{P}(x)$
$\Rightarrow u$ measure on $\mathbf{a}$

1.5 Completion of Measures

$(X, \mathbf{a}, \mu)$ measure space, where X be a set, $\mathbf{a}$ be a $\sigma-$algebra, u be a measure.

Definition 1.5.1 (completion)**.** *u is complete if $E \in \mathbf{a}, \mu(E) = 0, N \subseteq E \Rightarrow N \in \mathbf{a}$.*

Example 1.5.1. *$\mathbf{a} = \{\emptyset, X\}$ $\#X \geq 2$. Define $\mu(\emptyset) = \mu(X) = 0, \emptyset \neq \forall E \subsetneqq X, E \notin \mathbf{a}$, i.e. μ is not complete.$\overline{\mathbf{a}} = \mathcal{P}(X), \overline{u} \equiv 0$ complete on $\overline{\mathbf{a}}$*

Example 1.5.2. *μ^* outer measure $\to \mu$ measure on $\mathbf{a}$. Then μ complete on $\mathbf{a}$.*

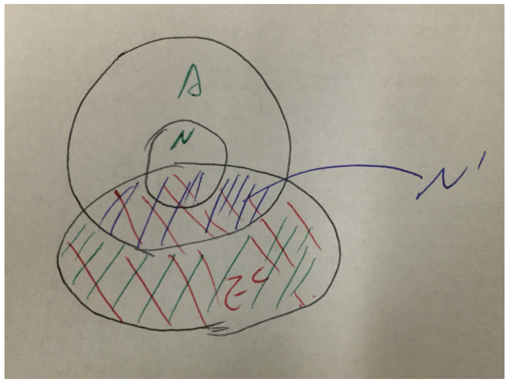
Proof. Let $E \in \mathbf{a}, \mu(E) = 0, \& N \subseteq E$, check $N \in \mathbf{a}$, i.e. check $\mu^*(A) = \mu^*(A \cap N) + \mu^*(A \backslash N)$ $\forall A \subseteq X$ $A \cap N \subseteq A \cap E \subseteq X$, so, $\mu^*(A \cap N) \leqslant \mu^*(A \cap E) \leqslant \mu^*(E)$, $\because \mu^*|a = \mu, \therefore \mu^*(E) = \mu(E) = 0$, so, $\mu^*(A) \geqslant \mu^*(A \backslash N) = 0 + \mu^*(A \backslash N) = \mu^*(A \cap N) + \mu^*(A \backslash N), \therefore N \in \mathbf{a}$. $\square$

Theorem 1.5.1. *μ measure on $\mathbf{a}$*

1. *Let $\overline{\mathbf{a}} = \{E \cup N : E \in \mathbf{a}, N \subseteq A \text{ where } A \in \mathbf{a}, \mu(A) = 0\}$, then $\overline{\mathbf{a}}$ is a $\sigma-$algebra.*
2. *$\overline{\mu}(E \cup N) = \mu(E)$ for $E \cup N \in \overline{\mathbf{a}}$, then $\overline{\mu}$ is complete measure on $\overline{\mathbf{a}}$ & $\overline{\mu}|\mathbf{a} = \mu$*

Proof.

1. (a) $\emptyset = \underbrace{\emptyset}_{\subseteq \mathbf{a}} \cup \underbrace{\emptyset}_{\emptyset \in \mathbf{a}, u(\emptyset)=0} \in \overline{\mathbf{a}}$



(b) $E \cup \underbrace{N}_{\subseteq A \in \mathbf{a}, \mu(A)=0} \in \mathbf{a} \overset{?}{\Rightarrow} (E \cup N)^c \in \overline{\mathbf{a}}$

$(E \cup N)^c = E^c \cap N^c = E^c \backslash N = \underbrace{(E^c \backslash A)}_{\in \mathbf{a}} \cup \underbrace{N'}_{A \in \mathbf{a}, \mu(A)=0} \Rightarrow (E \cup N)^c \in \overline{\mathbf{a}}$

(c) $\underbrace{E_n}_{\in \mathbf{a}} \cup \underbrace{N_n}_{A_n \in \mathbf{a}, \mu(A_n)=0} \in \overline{\mathbf{a}} \ \ \forall n \overset{?}{\Rightarrow} \bigcup_n (E_n \cup N_n) \in \overline{\mathbf{a}}$

$\bigcup_n (E_n \cup N_n) = \underbrace{\left(\bigcup_n E_n\right)}_{\in \mathbf{a}} \cup \underbrace{\left(\bigcup_n N_n\right)}_{\bigcup_n A_n \in \mathbf{a}, \mu\left(\bigcup_n A_n\right) \leqslant \sum_n \mu(A_n) = 0} \Rightarrow \bigcup_n (E_n \cup N_n) \in \overline{\mathbf{a}}$

so, $\overline{\mathbf{a}}$ $\sigma-$algebra.

2. check $\overline{\mu}$ is well-defined, assume $\underbrace{E_1}_{\in \mathbf{a}} \cup \underbrace{N_1}_{\subseteq A_1 \in \mathbf{a}, \mu(A_1)=0} = \underbrace{E_2}_{\in \mathbf{a}} \cup \underbrace{N_2}_{\subseteq A_2 \in \mathbf{a}, \mu(A_2)=0}$

check $\mu(E_1) = \mu(E_2)$
$\because E_1 \subseteq E_1 \cup N_1 = E_2 \cup N_2 \subseteq E_2 \cup A_2 \therefore \mu(E_1) \leqslant \mu(E_2 \cup A_2) \leqslant \mu(E_2) + \mu(A_2) = \mu(E_2)$ i.e. $\mu(E_1) \leqslant \mu(E_2)$, by symmetry, $\mu(E_1) \geqslant \mu(E_2)$, so $\mu(E_1) = \mu(E_2)$.

check $\overline{\mu}$ is measure

(a) $\mu(\emptyset) = \mu\left(\underbrace{\emptyset}_{E} \cup \underbrace{\emptyset}_{N}\right) = \mu(\emptyset) = 0$

(b) let $\left\{\underbrace{E_n}_{\in \mathbf{a}} \cup \underbrace{N_n}_{\subseteq A_n \in \mathbf{a}, \mu(A_n)=0}\right\}$ disjoint, check $\underbrace{\overline{\mu}\left(\bigcup_n (E_n \cup N_n)\right)}_{\underbrace{\overline{\mu}\left(\left(\bigcup_n E_n\right) \bigcup \left(\bigcup_n N_n\right)\right)}_{\mu\left(\bigcup_n E_n\right)}} = \underbrace{\sum_n \overline{\mu}(E_n \cup N_n)}_{\sum_n \mu(E_n)}$

check $\overline{\mu}$ is complete. Assume $\overline{\mu}\left(\underbrace{E}_{\in\mathbf{a}} \quad \cup \quad \underbrace{N}_{\subseteq A\in a,\mu(A)=0}\right) = \mu(E) = 0,\ B \subseteq (E \cup N)$. Check:

$B \in \overline{\mathbf{a}}. B \subseteq E \cup N \subseteq \underbrace{E \bigcup A}_{\in a} \Rightarrow \mu(E \cup A) \leqslant \mu(E) + \underbrace{\mu(A)}_{=0} = \mu(E) \therefore B = \underbrace{\emptyset}_{E} \cup \underbrace{B}_{N} \in \overline{\mathbf{a}}.$

□

Homework 1.5.

1. *problem* 1.5.1 *Let μ be a complete measure. A set of which $\mu(N) = 0$ is called a null set. Show that the class of null sets is a $\sigma-$ring. Is it also a $\sigma-$algebra ?*

2. *Problem* 1.5.2 *Let the conditions of Theorem* 1.5.1 *hold and denote by $\mathbf{a}^*$ the class of all sets of the form $E - N$ where $E \in \mathbf{a}$ and N is any sunset of $\mathbf{a}$ having measure zero. Then $\mathbf{a}^* = \overline{\mathbf{a}}$.*

1.6 The Lebesgue and Lebesgue-Stieltjes Measures

1. $\mathbb{R}^n$ Let $K = \{$ bdd open intervals$\} \bigcup \emptyset$, sequence covering classes. Let λ(bdd open interval) = its volume $\to \mu^*$ lebesgue outer measure on $\mathcal{P}(\mathbb{R}^n) \to \cdots$
$\cdots \to \mu$ lebesgue measure on $\mathbf{a} = \{$ lebesgue measurable subset of $\mathbb{R}^n\}$

2. Lebesgue-Stieltjes measure on $\mathbb{R}$: let $f : \mathbb{R} \to \mathbb{R}$ be $\uparrow$ & right continuous. $K = \{(a, b] : a < b \in \mathbb{R}\} \cup \{\emptyset\}$, sequence covering classes, $\lambda((a, b]) = f(b) - f(a) \to \mu^*$ on $\mathcal{P}(\mathbb{R}) \to \mu_f$ on $\mathbf{a}_f$, $\mu_f((a, b]) = f(b) - f(a)$, Lebesgue-Stieltjes Measure.

Definition 1.6.1 (Lebesgue outer measurable)**.** *Let E be a subset of $\mathbb{R}$. Then Lebesgue outer measurable of E is denoted by*

$$\inf\left\{\sum_k l(I_k) : \{I_k\} \text{ is sequence of open intervals such that } E \subseteq \bigcup_k I_k\right\}$$

Definition 1.6.2 (Lebesgue measurable)**.** *A set $E \in \mathbb{R}$ is lebesgue measurable if for each set $A \in \mathbb{R}$, we have*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$$

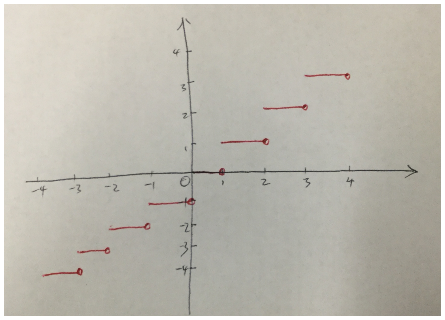
Note 1.6.1. *$f(x) = x$ on $\mathbb{R}$, then μ_f = lebesgue measure.*

Example 1.6.1. *$f(x) = [x]$ $\quad \mathbb{R} \to \mathbb{R} \uparrow$ & right continuous. $\therefore \mu_f$ on $\mathbf{a}_f$ s.t. $\mu_f(E)$ = no. of integers in E.*
Ex. $\left(\frac{1}{2}, 3\right]$ then $\mu_f(E) = \mu(3) - \mu\left(\frac{1}{2}\right) = 3 - 0 = 3$.

Note 1.6.2. *$f \rightleftarrows \mu_f$ for μ on $\mathbb{R}$, let $f(x) = \mu((-\infty, x])$, $\therefore f \uparrow$ & right continuous (distribution function of μ).*

Proposition 1.6.1 (properties of Lebesgue outer measure μ^*)**.**

1. *(translation invariance) E Lebesgue-measurable, $a \in \mathbb{R} \Rightarrow a + E$ Lebesgue-measurable, & $\mu^*(a + E) = \mu^*(E)$*



2. $\mu^*([a,b]) = b - a$

3. $\mu^*(\{x\}) = 0, \forall x \in \mathbb{R}$

4. $\mu^*((a,b]) = \mu^*([a,b)) = \mu^*((a,b)) = b - a$

Proof.

1. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of covering classes that covers E, then $\{a + I_n\}_{n=1}^{\infty}$ be a sequence of covering classes that covers $a + E$, therefore,

$$\mu^*(a+E) \leqslant \sum_{n=1}^{\infty} l(a+I_n) = \sum_{n=1}^{\infty} l(I_n)$$

so,

$$\mu^*(a+E) \leqslant \inf\left\{\sum_{n=1}^{\infty} l(I_n)\right\} = \mu^*(E)$$

now, let $\{I_n\}_{n=1}^{\infty}$ be a sequence of covering classes that covers $a+E$, then $\{I_n - a\}_{n=1}^{\infty}$ be a sequence of covering classes that covers E, therefore,

$$\mu^*(E) \leqslant \sum_{n=1}^{\infty} l(I_n - a) = \sum_{n=1}^{\infty} l(I_n)$$

so,

$$\mu^*(E) \leqslant \inf\left\{\sum_{n=1}^{\infty} l(I_n)\right\} = \mu^*(a+E)$$

so,

$$\mu^*(E) = \mu^*(a+E)$$

2. For any given $\varepsilon > 0$, we have that

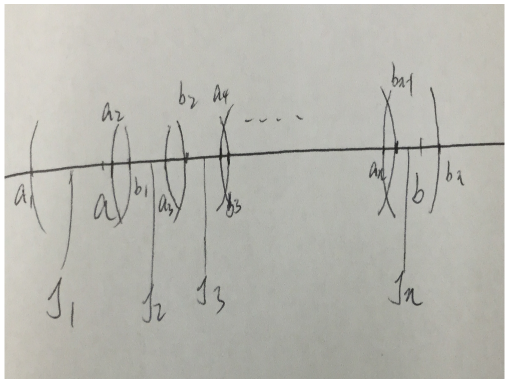
$$[a,b] \subseteq (a-\varepsilon, a+\varepsilon) \cup (a,b) \cup (b-\varepsilon, b+\varepsilon)$$

Thus,

$$\mu^*([a,b]) \leq \mu^*(a-\varepsilon, a+\varepsilon) + \mu^*(a,b) + \mu^*(b-\varepsilon, b+\varepsilon) = b - a + 4\varepsilon$$

as $\varepsilon > 0$, is arbitrary, we can get that

$$\mu^*([a,b]) \leqslant b - a = l(a,b)$$



next, we need to prove that $\mu^*([a,b]) \geq b-a = l(a,b)$.

Let $\{I_k\}_{k=1}^{\infty}$ be any sequence of open intervals that cover $[a,b]$, since $[a,b]$ ia compact, by the Heine-Borel theorem, there is a finite subcollection $\{J_i\}_{i=1}^{n}$ of $\{I_k\}_{k=1}^{\infty}$ still covers $[a,b]$. By reordering and deleting if necessary, we can assume that

$$a \in J_1 = (a_1, b_1)\,,\ b_1 \in J_2 = (a_2, b_2)\,,\ b_2 \in J_3 = (a_3, b_3)\,, \cdots,\ b_{n-1}, b \in J_n = (a_n, b_n)$$

then,

$$b-a < b_n - a_1 = \sum_{i=2}^{n} (b_i - b_{i-1}) + (b_1 - a_1) = \sum_{i=1}^{n} l(J_i) \leqslant \sum_{k=1}^{\infty} l(I_k)$$

then

$$b-a \leqslant \inf\left\{\sum_{k=1}^{\infty} l(I_k)\right\} = \mu^*([a,b])$$

so,

$$\mu^*([a,b]) = b-a$$

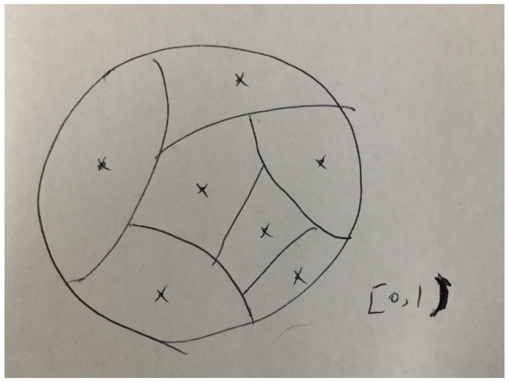
3. it is easily.

4. it is easily.

$\square$

Note 1.6.3 (an example not lebesgue measurable)**.** *$\exists E \subseteq \mathbb{R}$ not Lebesgue measurable,* $\mathbf{a} \subsetneqq \mathcal{P}(E^n)$. *for example $n=1$.*

Proof. let $x, y \in [0,1)$, define $x \equiv y$ if $x-y$ is rational, then " $\equiv$ " is an equivalence relationship. Axiom of choice $\Rightarrow$ let E = subset of $[0,1)$ s.t. contains exactly 1 no. from each equivalence class. Assume E lebesgue measurable, then $\frac{1}{k}+E$ lebesgue measurable, and $\mu\left(\frac{1}{k}+E\right) = \mu(E), \forall k \geqslant 1$. Check $\left\{\frac{1}{k}+E, k \geqslant 1\right\}$ disjoint. Let $x \in \left(\frac{1}{k}+E\right) \cap \left(\frac{1}{l}+E\right)$ for some $k \neq l$, $\because x = \frac{1}{k}+\alpha_1 = \frac{1}{l}+\alpha_2$ where $\alpha_1, \alpha_2 \in E \Rightarrow \underbrace{\frac{1}{k}-\frac{1}{l}}_{\text{rational}} = \alpha_2 - \alpha_1 \Rightarrow \alpha_1 \equiv \alpha_2 \Rightarrow \alpha_1 = \alpha_2 \Rightarrow k = l$ but it is contradictory.



so,

$$\underbrace{\mu\left(\bigcup_{k=1}^{n}\left(\frac{1}{k}\right)+E\right)}_{\leqslant\mu([0,1))=1}=\underbrace{\sum_{k=1}^{n}\mu\left(\frac{1}{k}+E\right)}_{n\cdot\mu(E),\forall n\geqslant 1}$$

Next, $\Rightarrow \mu(E)=0$. Check $[0,1)=\bigcup_{r \text{ rational } \in[0,1)}(r+E)$, " $\supseteq$ " it is easily, " $\subseteq$ " let $x\in[0,1), x\in$ equivalence class $\Rightarrow a\in E, x\equiv a$ for some $a\in E\Rightarrow x=r+a\in r+E$ where r rational, $\Rightarrow x\in$ RHS. Check $(r_1+E)\cap(r_2+E)=\emptyset\ \forall r_1\neq r_2$ rational, let $x\in(r_1+E)\cap(r_2+E), x=r_1+\alpha_1=r_2+\alpha_2$ where $\alpha_1,\alpha_2\in E$. $\Rightarrow \underbrace{r_1-r_2}_{rational}=\alpha_2-\alpha_1\Rightarrow\alpha_1\equiv\alpha_2\Rightarrow\alpha_1=\alpha_2\Rightarrow r_1=r_2$, it is contradictory.

$\therefore \underbrace{\mu([0,1))}_{1}=\sum_{r}\underbrace{\mu(r+E)}_{0}=0$, it is contradictory. □

Homework 1.6 (problem 1.6.3)**.** *The outer Lebesgue measure of a closed bounded interval* $[a,b]$ *on the real line is equal to* $b-a$ *Hint: use the Heine-Borel theorem to replace a countable covering by a finite covering*

1.7 Metric Spaces

Definition 1.7.1 (Metric)**.** *Let* X *be a set,* $\rho: X\times X\to\mathbb{R}$ *is a metric if*

1. $\rho(x,y)\geqslant 0\quad\forall x,y\in X$
2. $\rho(x,y)=0\Leftrightarrow x=y$ *(symmetry)*
3. $\rho(x,y)=\rho(y,x)$
4. $\rho(x,z)\leqslant\rho(x,y)+\rho(y,z)\quad\forall x,y,z\in X$ *(the triangle inequality)*

Definition 1.7.2 (Borel sets)**.** *Denote by* **a** *the* $\sigma-$*ring generated by the class of all the open sets of* X*. The sets of* **a** *are called Borel sets.*

Note 1.7.1. **a** *coincides with the* $\sigma-$*algebra generated by the class of all the open sets of* X*.*

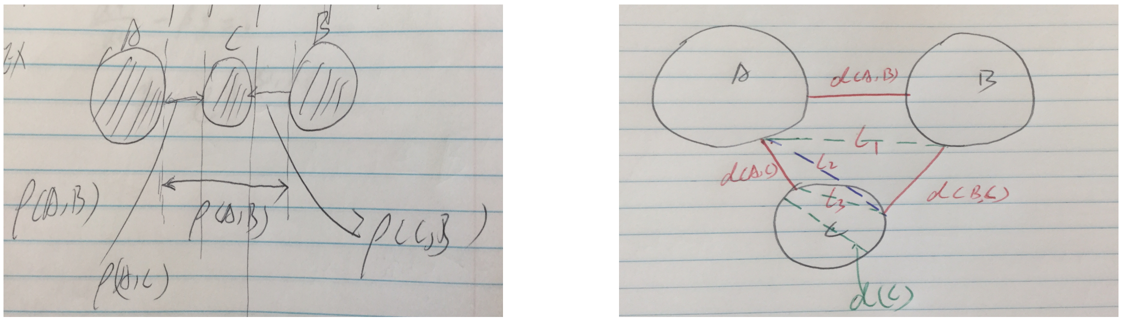
Definition 1.7.3. *Let* (X,ρ) *be metric space,* $x\in X, A\subseteq X, \rho(x,A)=\inf_{y\in A}\rho(x,y)$*.*

Definition 1.7.4. *Let* (X,ρ) *be metric space,* $A,B \subseteq X, \rho(A,B) = \inf\limits_{x\in A, y\in B} \rho(x,y)$.

Definition 1.7.5. *Let* (X,ρ) *be metric space,* $A \subseteq X, d(A) = \sup\limits_{x,y\in A} \rho(x,y)$, *(diamter of* A*)*

Definition 1.7.6. *Let* (X,ρ) *be metric space,* X *bounded if* $d(X) < \infty$.

Note 1.7.2. *triangle inequality may fail for* $\rho(A,B)$.



(a) triangle inequality fail

(b) in general

Example 1.7.1. $\rho(A,B) > \rho(A,C) + \rho(B,C)$*, but* $\rho(A,B) \leqslant \rho(A,C) + \rho(B,C) + d(C)$ *in general.*

$$\begin{aligned}
&\rho(A,B) \leqslant l_1 \leqslant l_2 + \rho(B,C) + \varepsilon, \forall\varepsilon \\
&\Rightarrow \rho(A,B) \leqslant l_2 + \rho(B,C) \\
&l_2 \leqslant \rho(A,C) + \varepsilon + l_3 \ \forall\varepsilon \\
&\Rightarrow l_2 \leqslant \rho(A,C) + l_3 \\
&\therefore \rho(A,B) \leqslant \rho(A,C) + l_3 + \rho(B,C) \\
&\because l_3 \leq d(C) \\
&\therefore \rho(A,B) \leqslant \rho(A,C) + \rho(B,C) + d(C)
\end{aligned}$$

Note 1.7.3. $\rho(A,B) \leqslant \rho(A,x) + \rho(x,B)$

Definition 1.7.7. *Let* (X,ρ) *be metric space,* μ^* *outer measure on* X, μ^* *is metric outer measure if* $\forall A,B \subseteq X$ *s.t.* $\rho(A,B) > 0 \Rightarrow \mu^*(A\cup B) = \mu^*(A) + \mu^*(B)$.

Note 1.7.4.

$$\rho(A,B) > 0 \quad \begin{matrix}\Rightarrow \\ \not\Leftarrow\end{matrix} \quad A \bigcap B = \emptyset$$

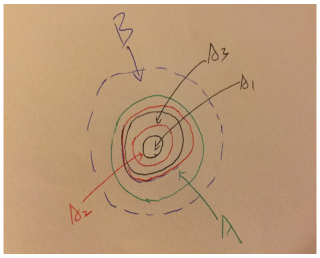
Example 1.7.2 ($\not\Leftarrow$)**.** $A = [0,1], B = (1,2], A\cap B = \emptyset$*, but* $\rho(A,B) = 0$.

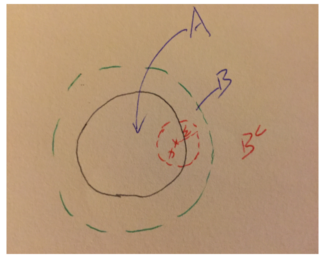
Lemma 1.7.1. μ^* *metric outer measure* $A \subseteq B, B$ *open, let*

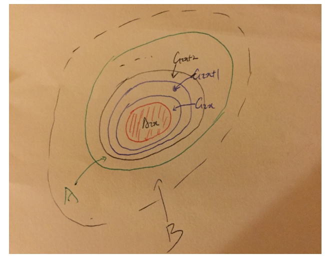
$$A_n = \left\{x \in A : \rho(x, B^c) \geqslant \frac{1}{n}\right\} \quad \forall n$$

then

$$\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A)$$







Proof.

1. $\leq \because \mu^*(A_n) \leq \mu^*(A) \therefore \overline{\lim_{n\to\infty}} \mu^*(A_n) \leqslant \mu^*(A)$
2. $\geq \because A_n \uparrow, \bigcup_n A_n \subseteq A$, check that $\bigcup_n A_n = A$
   Let $x \in A$, so $x \in B, \because B$ open, so $\Rightarrow \underbrace{\exists N_\varepsilon(x) \subseteq B}_{\equiv\{y\in X:\rho(y,x)<\varepsilon\}} \Rightarrow \rho(x, B^c) \geqslant \varepsilon > 0$

   Note 1.7.5. *Suppose that* $\rho(x, B^c) < \varepsilon \Leftrightarrow \inf_{y\in B^c} \rho(x,y) = \varepsilon_1 < \varepsilon \Leftrightarrow \forall \Delta\varepsilon > 0, \exists y_0 \in B^c, \rho(x, y_0) < \varepsilon_1 + \Delta\varepsilon$, *let* $0 < \Delta\varepsilon < \varepsilon - \varepsilon_1 \therefore \rho(x, y_0) < \varepsilon \Rightarrow y_0 \in B$ *but* $y_0 \in B^c$, *it is contradict.*

   Let n be such that $\frac{1}{n} < \varepsilon \therefore \rho(x, B^c) \geqslant \varepsilon > \frac{1}{n} \Rightarrow x \in A_n$.

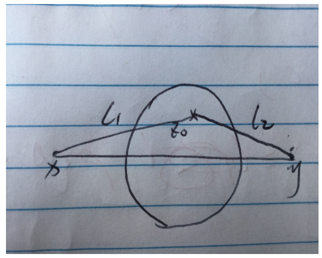
   Let $G_n = A_{n+1} \backslash A_n \forall n \geqslant 1 \therefore G_n = A_{n+1} \backslash A_n \quad \forall n \geqslant 1 \because A_n \uparrow \therefore A = A_{2n} \cup \left(\bigcup_{k=n}^{\infty} G_{2k}\right) \cup \left(\bigcup_{k=n}^{\infty} G_{2k+1}\right) \therefore \mu^*(A) \leqslant \mu^*(A_{2n}) + \sum_{k=n}^{\infty} \mu^*(G_{2k}) + \sum_{k=n}^{\infty} \mu^*(G_{2k+1})$

   We hope that $\sum_{k=1}^{\infty} \mu^*(G_{2k}), \sum_{k=1}^{\infty} \mu^*(G_{2k+1})$ are convergence, then $\sum_{k=n}^{\infty} \mu^*(G_{2k}) + \sum_{k=n}^{\infty} \mu^*(G_{2k+1}) = 0 + 0 = 0 \quad n \to \infty$, then $\mu^*(A) \leqslant \underline{\lim}_{n\to\infty} \mu^*(A_{2n})$ similarity $\mu^*(A) \leqslant \underline{\lim}_{n\to\infty} \mu^*(A_{2n+1})$, so $\mu^*(A) \leqslant \underline{\lim}_{n\to\infty} \mu^*(A_n) \because \underline{\lim}_{n\to\infty} \mu^*(A_n) \leqslant \overline{\lim_{n\to\infty}} \mu^*(A_n)$, so $\mu^*(A) \leqslant \underline{\lim}_{n\to\infty} \mu^*(A_n) \leqslant \overline{\lim_{n\to\infty}} \mu^*(A_n) \leqslant \mu^*(A) \Rightarrow \mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$.

   We check that $\sum_{k=1}^{\infty} \mu^*(G_{2k}), \sum_{k=1}^{\infty} \mu^*(G_{2k+1})$ are convergence.
   $\because A \supseteq A_{2n} \supseteq \bigcup_{k=1}^{n-1} G_{2k} \Rightarrow \mu^*(A) \geqslant \mu^*(A_n) \geqslant \mu^*\left(\bigcup_{k=1}^{n-1} G_{2k}\right)$. Apply $\forall A, B \subseteq X, s.t. \rho(A,B) > 0 \Rightarrow \mu^*(A,B) = \mu^*(A) + \mu^*(B)$, so $\mu^*(A) \geqslant \mu^*(A_n) \geqslant \mu^*\left(\bigcup_{k=1}^{n-1} G_{2k}\right) = \sum_{k=1}^{n-1} \mu^*(G_{2k})$ We need that $\rho(G_{2k}, G_{2k+2}) > 0 \quad \forall k$, let $x \in G_{2k} \Rightarrow x \in A_{2k+1} \Rightarrow \rho(x, B^c) \geqslant \frac{1}{2k+1}, y \in G_{2k+2} \Rightarrow y \notin A_{2k+1} \Rightarrow \rho(y, B^c) < \frac{1}{2k+2}$ otherwise $\rho(y, B^c) \geq \frac{1}{2k+2} \Rightarrow x \in A_{2k+1}$.

   Note 1.7.6. *Let A be any set and x,y any two points. If* $\rho(x, A) \geq \alpha, \rho(y, A) < \beta$ *and* $\alpha > \beta$ *then* $\rho(x,y) > \alpha - \beta$

   Proof. $\because \rho(y, B) = \inf_{z\in A} \rho(y,z) \therefore \forall \varepsilon, \exists z_0 \in A, \; s.t. \; \rho(y, z_0) < \rho(y,B) + \varepsilon < \beta + \varepsilon$, so $\rho(y, z_0) \leqslant \beta$, and $l_1 \geqslant \rho(x, A) \geq \alpha. \therefore \rho(x,y) \geqslant l_1 - l_2 \geqslant \alpha - \beta$ □



$\because \rho(x,y) \geqslant \rho(x,B^c) - \rho(y,B^c) > \frac{1}{2k+1} - \frac{1}{2k+2} > 0$. so $\mu^*(A) \geqslant \mu^*(A_{2n}) \geqslant \mu^*\left(\bigcup_{k=1}^{n-1} G_{2k}\right) = \sum_{k=1}^{n-1} \mu^*(G_{2k}) \overset{n\to\infty}{\Rightarrow} \mu^*(A) \geqslant \overline{\lim_{n\to\infty}} \mu^*(A_n) \geqslant \sum_{k=1}^{\infty} \mu^*(G_k)$.

So $\mu^*(A) \leqslant \underline{\lim_{n\to\infty}}\, \mu^*(A_n) \leqslant \overline{\lim_{n\to\infty}} \mu^*(A_n) \leqslant \mu^*(A) \Rightarrow \mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$.

□

Homework 1.7 (Problem 1.7.8). *Denote by $\mathcal{P}$ the class of all half-open intervals $[a,b)$ on the real line. Show that the $\sigma-ring$ generated by $\mathcal{P}$ coincides with the class $\mathcal{B}$ of the Borel sets on the real line.*

1.8 Metric Outer Measures

Theorem 1.8.1. *(X,ρ) metric space, μ^* metric outer measure on X if and only if closed (open) subsets of X are all measurable.*

Proof.

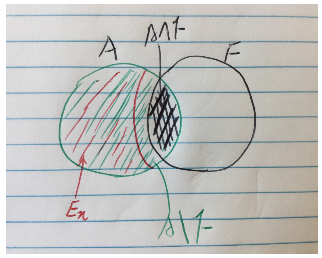
1. $\Leftarrow$ open, $\rho(A,B) > 0$, for $\forall x \in A$, let $u_x = \{y \in X : \rho(y,x) < \frac{1}{2}\rho(A,B)\}$, clearly this set is open, so $U \equiv \bigcup_{x\in A} u_x$ is open, obviously $A \subseteq U, y \in U$, then there exists $x_0 \in A$, such that $\rho(x_0,y) < \frac{1}{2}\rho(A,B) \leqslant \frac{1}{2}\inf_{z\in B}\{\rho(x_0,z)\}$

   Note 1.8.1. *$\forall x \in A$, if $y_0 \in B$, then $\rho(x,y_0) \geqslant \rho(A,B)$*

   Proof. $\rho(x,y_0) \geqslant \inf_{x'\in A, y'\in A} \rho(x',y') = \rho(A,B)$ □

   so, $y \notin B$. so
   $$\mu^*(A\cup B) = \mu^*[(A\cup B)\cap U] + \mu^*[(A\cup B)\backslash U] = \mu^*(A) + \mu^*(B)$$

2. $\Rightarrow$ Let $F \subseteq X$ closed. check that
   $$\mu^*(A) = \mu^*(A\cap F) + \mu^*(A\backslash F), \forall A \subseteq X$$



$\leq$ is already done, we only to show that

$$\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \backslash F), \forall A \subseteq X$$

Let $F \subseteq X$ closed, let $E_n = \{x \in A \backslash F : \rho(x, F) \geq \frac{1}{n}\}$, $A \backslash F \subseteq F^c$, F^c is open, by Lemma 1
$\Rightarrow \lim_{n\to\infty} \mu^*(E_n) = \mu^*(A \backslash F)$
$\because \rho((A \cap F), E_n) \geqslant \rho(F, E_n) \geqslant \frac{1}{n} > 0, \forall n,$
$\therefore \mu^*(A) \geqslant \mu^*((A \bigcap F) \cup E_n) = \mu^*(A \bigcap F) + \mu^*(E_n)\, \mu^*(A \bigcap F) + \mu^*(A \backslash F)$

□

Corollary 1.8.1. *μ^* metric outer measure on $(X, \rho) \Rightarrow$ Borel subsets in X are measure.*

Proof. B = {Borel sets} = $\sigma-$algebra generated by all open subsets. $\mathbf{a}$ = σ-algebra induced by μ^*, i.e. $B \subseteq \mathbf{a}.\{open\} \subseteq \mathbf{a} \Rightarrow B \subseteq \mathbf{a}$ □

Homework 1.8.

1. Problem 1.8.1 *Prove the converse of Corollary 1.8.1, that is, if μ^* is an outer measure and if every open set is measurable, then μ^* is a metric outer measure.*

2. Problem 1.8.3 *Let (X, ρ) be a metric space and let $\{x_n\}$ be a sequence of points in X. Define $\mu^*(E)$ to be the number of points x_n that belong to E. Prove that μ^* is a metric outer measure.*

3. Problem 1.8.4 *Let (X, ρ) be a metric space. Define $\mu^*(E) = 1$ if $E \neq \emptyset$ and $\mu^*(\emptyset) = 0$. Is μ^* a metric outer measure?*

1.9 Construction of Metric Outer Measures

(X, ρ) metric space. K: sequential covering class & $\lambda : K \to [0, \infty) \Rightarrow \mu^*$ outer measure $\to \mu$ measure.

$$K_n = \left\{A \in K : d(A) \leqslant \frac{1}{n}\right\} \cup \{\emptyset\} \quad \forall n \geq 1$$

Note 1.9.1. *K sequence covering class $\nRightarrow K_n$ sequential covering class.*
$K = \{[n, n+1] : n \in \mathbb{Z}\} \cup \{\emptyset\} \in \mathbb{R}$ sequential covering class, but $K_n = \emptyset$ if $n \geq 2$, not sequential covering class.

Example 1.9.1. $\mathbb{R}$ *or* $\mathbb{R}^n$
$K = \{open \ \ intervals\ \} \cup \{\emptyset\}$, *then* K_n *sequential covering class* $\forall n$.

Let $\lambda_n = \lambda|K_n : K_n \to [0, +\infty)$, so $K_n, \lambda_n \sim \mu_n^*$; $\quad K, \lambda \sim \mu^*$. Let μ_n^* outer measure with respectively to K_n, λ_n, ie.

$$\mu_n^*(A) = \inf\left\{\sum_k \lambda(E_k) : A \subseteq \bigcup_k E_k,\ \underbrace{E_k \subseteq K_n}_{d(E_k)\leqslant \frac{1}{n}, \forall k}\right\}, \forall A \subseteq X$$

Note 1.9.2.

1. $K_{n+1} \subseteq K_n$
2. $\mu_n^*(A) \leq \mu_{n+1}^*(A)$

Definition 1.9.1. $\mu_0^*(A) = lim \mu_n^*(A) = sup\ \mu_n^*(A),\ \forall A \subseteq X$.

Theorem 1.9.1. μ_0^* *is metric outer measure.*

Proof.

1. $\mu_0^* : \mathcal{P}(X) \to [0, \infty]$
2. $\mu_0^*(\emptyset) = \lim\limits_{n\to\infty} \mu_n^*(\emptyset) = 0$
3. $A \subseteq B \in X \Rightarrow \mu_n^*(A) \leqslant \mu_n^*(B) \quad \forall n \quad \Rightarrow \lim\limits_{n\to\infty} \mu_n^*(A) \leqslant \lim\limits_{n\to\infty} \mu_n^*(B)$
4. countable subadditivity:
   $\mu_n^*(\bigcup\limits_k E_k) \leqslant \sum\limits_k \mu_n^*(E_k) \Rightarrow \lim\limits_{n\to\infty} \mu_n^*(\bigcup\limits_k E_k) \leqslant \underbrace{\lim\limits_{n\to\infty} \sum\limits_k \mu_n^*(E_k)}_{\sum_k \lim_{n\to\infty} \mu_n^*(E_k)} \Rightarrow \mu_0^*(\bigcup\limits_k E_k) \leqslant \sum\limits_k \mu_0^*(E_k)$

   From 1,2,3,4 μ_0^* is an outer measure.
5. Assume $\rho(A, B) > 0$, check $\mu_0^*(A \cup B) = \mu_0^*(A) + \mu_0^*(B)$

   We only to show that $\mu_0^*(A \cup B) \geqslant \mu_0^*(A) + \mu_0^*(B)$

$$\because \mu_n^*(A \cup B) = \inf\left\{\sum_k \lambda(E_{nk}) : A \cup B \subseteq \bigcup_k E_{nk}, E_{nk} \in K_n\right\}$$

$$\therefore \forall \varepsilon > 0, \forall n, \exists \{E_{nk}\} \subseteq K_n = \left\{A \in K : d(A) \leqslant \frac{1}{n}\right\} \cup \{\emptyset\}$$
$$s.t\ A \cup B \subseteq \bigcup_k E_{nk}\ \&\ \sum_k \lambda(E_{nk}) \leqslant \mu_n^*(A \cup B) + \varepsilon$$

$\because d(E_{nk}) \leq \frac{1}{n}$ & $\rho(A,B) = d > 0 \Rightarrow \rho(A,B) > \frac{1}{n}$ for n large, $\Rightarrow E_{nk}$ cannot intersect both A,B. So decompose $\{E_{nk}\}$ as $\{E'_{nk}\}$ cover A, and $\{E''_{nk}\}$ cover B.

$$\therefore \mu_n^*(A) + \mu_n^*(B) \leqslant \sum_k \lambda\left(E'_{nk}\right) + \sum_k \lambda\left(E''_{nk}\right) = \sum_k \lambda\left(E_{nk}\right) \leqslant \mu_n^*(A \cup B) + \varepsilon$$

$$\therefore \lim_{n\to\infty} (\mu_n^*(A) + \mu_n^*(B)) \leqslant \lim_{n\to\infty} (\mu_n^*(A\cup B) + \varepsilon) \quad \& \quad \varepsilon \to 0$$

$$\therefore \mu_0^*(A) + \mu_0^*(B) \leqslant \mu_0^*(A \cup B)$$

□

X metric space.

$$\begin{array}{ll} K & \lambda \to \mu^* \text{ outer measure} \\ \cup\!\!\mid & \downarrow \\ K_n & \lambda|K_n \to \mu_n^* \uparrow \mu_0^* \text{ metric outer measure} \end{array}$$

Q: $\mu^* = \mu_0$?

Theorem 1.9.2. *K_n sequential covering class $\forall n \geq 1$. $\forall A \in K, \forall \varepsilon > 0, \forall n \geq 1$, $\exists \{E_k\} \subseteq K_n$ s.t. $A \subseteq \bigcup_k E_k$ & $\sum_k \lambda(E_k) \leqslant \lambda(A) + \varepsilon$. Then $\mu^* = \mu_0^*$.*

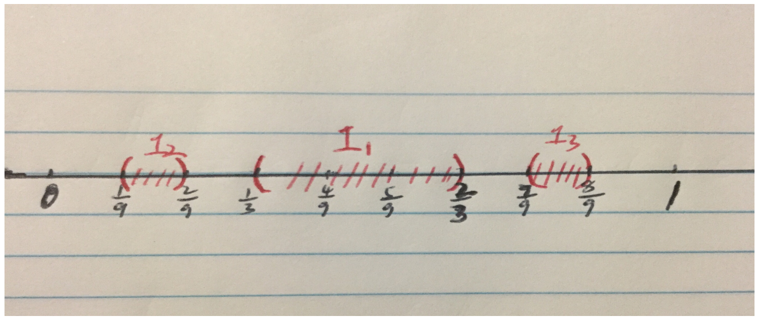
Proof.

1. $\leq$ $\because K_n \subseteq K \Rightarrow \mu^*(A) \leqslant \mu_n^*(A) \quad \forall A, \forall n \geq 1 \Rightarrow \mu^*(A) \leqslant \lim_{n\to\infty} \mu_n^*(A) = \mu_0^*(A)$ i.e. $\mu^*(A) \leqslant \mu_0^*(A)$.
2. $\geq$ $\because A, \forall \varepsilon > 0, \exists \{E_j\} \subseteq K$ s.t. $A \subseteq \bigcup_j E_j$ & $\sum_j \lambda(E_j) \leqslant \mu^*(A) + \frac{\varepsilon}{2}$.
   Hypothesis $\Rightarrow \forall E_j, \exists \{E_{jk}\} \subseteq K_n$ s.t. $E_j \subseteq \bigcup_k E_{jk}$ & $\sum_k \lambda(E_{jk}) \leqslant \lambda(E_j) + \dfrac{\varepsilon}{2^{j+1}}$
   $\because \{E_{jk}\}_{j,k} \subseteq K_n$ & covers A.
   $\therefore \mu_n^*(A) \leqslant \sum_{j,k} \lambda(E_{jk}) \leqslant \sum_j \left[\lambda(E_j) + \dfrac{\varepsilon}{2^{j+1}}\right] = \sum_k \lambda(E_j) + \sum_k \dfrac{\varepsilon}{2^{j+1}} \leqslant \mu^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu^*(A) + \varepsilon$
   Let $\varepsilon \to 0, n \to \infty$, $\lim_{n\to\infty} \mu_n^*(A) \leqslant \mu^*(A) + \varepsilon \Rightarrow \mu_0^* \leqslant \lim_{\varepsilon\to 0} (\mu^*(A) + \varepsilon) \Rightarrow \mu_0^* \leqslant \mu^*(A)$.

so, $\mu^* = \mu_0^*$. □

Note 1.9.3. $\forall A \in K, \forall \varepsilon > 0, \forall n \geq 1$, $\exists \{E_k\} \subseteq K_n$ *s.t.* $A \subseteq \bigcup_k E_k$ & $\sum_k \lambda(E_k) \leqslant \lambda(A) + \varepsilon$. $\to$ *conditions on λ & $K \Rightarrow \mu$ metric outer measure.*

Note 1.9.4. *condition holds for $\mathbb{R}$ or $\mathbb{R}^n$*
$\Rightarrow \mu^*$ Lebesgue metric outer measure
$\Rightarrow$ Borel sets are Lebesgue measurable



Definition 1.9.2. L = {*Lebesgue measurable subsets of* $\mathbb{R}$} *(→ from measure theory)*
B ={*Borel subsets of* $\mathbb{R}$} *(→ from topology)*
m = *Lebesgue measurable on* $\mathbb{R}$.

Relations between L and B, $B \subseteq L \subseteq \mathcal{P}(X)$.

$I_1 = (\frac{1}{3}, \frac{1}{3}), I_2 = (\frac{1}{9}, \frac{2}{9}), I_3 = (\frac{7}{9}, \frac{8}{9}), \cdots$

Definition 1.9.3 (Cantor Set)**.**

$$C = [0,1] \setminus (I_1 \cup I_2 \cup I_3 \cup \cdots)$$

Proposition 1.9.1.

1. *C bounded and closed $\Rightarrow$ compact and Borel $\because$ intersection of closed sets*
2. $m(C) = 0, \quad \because m(C) = 1 - \dfrac{1}{3} - \dfrac{2}{9} - \dfrac{4}{27} - \cdots = 1 - \sum\limits_{n=1}^{\infty} \dfrac{2^{n-1}}{3^n} = 0$

Set Theory: study $\underbrace{\text{number of element}}_{\text{cardinality}}$ of infinite set

A, B sets

Definition 1.9.4. *$\#A = \#B$ if $\exists f : A \to B$ 1-1 and onto, $\#A \leq \#B$ if $\exists f : A \to B$ 1-1.*

Definition 1.9.5. $(\#A) + (\#B) = \#(A \dot\cup B)$ *(disjoint union of A and B)*, $(\#A) \cdot (\#B) = \#(A \times B), \#A^{\#B} = \#\{f : B \to A\}, \aleph_0 = \#\mathbb{N}, \aleph_1 = \#\mathbb{R}$.

Theorem 1.9.3.

1. $\#\mathcal{P}(X) = 2^{\#A}$
2. *$\#A \leq \#B$ and $\#B \leq \#A \Rightarrow \#A = \#B$ (Schröder-Bernstein)*
3. $\#A < 2^{\#A}$
4. *A infinite set $\Leftrightarrow$ A has a subset C s.t. $\#A = \#B$*
5. *A infinite $\Leftrightarrow$ A has a subset B s.t. $\#B = \#\mathbb{N}$*

6. $\aleph_1 = 2^{\aleph_0}$

Note 1.9.5.

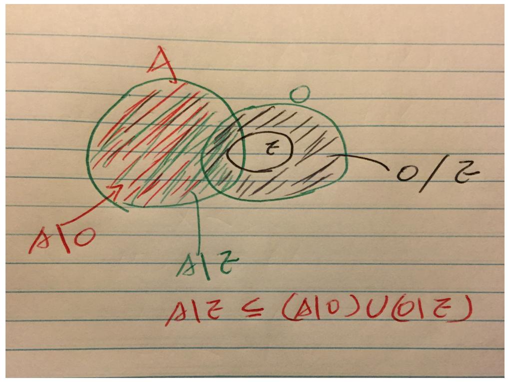
1. *specific sets in* $L \backslash B$ *difficult to give. (typical for modern analysis)*
2. $m|B$ *not complete.* $\because \exists$ *subsets of* C*, not in* $B \because \#2^C = 2^{\alpha_1} > \aleph_1 = \#B$
3. m *is the complete of* $m|B$*, see below.*

Theorem 1.9.4. $E \subseteq \mathbb{R}$ *then*

$$E \subseteq L \Leftrightarrow \forall \varepsilon > 0, \exists \text{ open } O \supseteq E \text{ s.t. } m^*(O\backslash E) < \varepsilon$$

Proof.

1. $\Rightarrow E \in L$, let $\varepsilon > 0$.
   (a) $m^*(E) < \infty$
   By the definition of outer measure, there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ which covers E and for which
   $$\sum_{k=1}^{\infty} \lambda(I_k) < m^*(E) + \varepsilon$$
   Define $O = \bigcup_{k=1}^{\infty} I_k$, then O is an open set containing E. By the definition of the outer measure of O,
   $$m^*(O) \leqslant \sum_{k=1}^{\infty} \lambda(I_k) < m^*(E) + \varepsilon$$
   so that
   $$m^*(O) - m^*(E) < \varepsilon$$
   Because E is measurable and has finite outer measure, therefore
   $$m^*(O\backslash E) = m^*(O) - m^*(E) < \varepsilon.$$
   (b) $m^*(E) = \infty$
   The E may be expressed as the disjoint union of a countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, each of which has finite outer ($E_k = E \cap [k, k+1)$). By the finite measure case, for each index k, there is an open set O_k containing E_k for which $m^*(O_k - E_k) < \frac{\varepsilon}{2^k}$, the set $O = \bigcup_{k=1}^{\infty} O_k$ is open, it contains E and
   $$O\backslash E = \bigcup_{k=1}^{\infty} (O_k\backslash E) \subseteq \bigcup_{k=1}^{\infty} [O_k\backslash E_k]$$
   Therefore
   $$m^*(O\backslash E) \leqslant \sum_{k=1}^{\infty} m^*(O_k\backslash E_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$



2. $\Leftarrow$ Check $m^*(A) \geqslant m^*(A \cap E) + m^*(A \backslash E) \quad \forall E \subseteq X$
   $\because O \in B \subseteq L \therefore m^*(A) = m^*(A \cap O) + m^*(A \backslash O)$
   $\because O \supseteq E \therefore A \cap E \subseteq A \cap O, A \backslash E \subseteq [(A \backslash O) \cup (O \backslash E)]$

$$m^*(A \cap E) \leqslant m^*(A \cap O), m^*(A \backslash E) \leqslant m^*(A \backslash O) + m^*(O \backslash E)$$

   so, whenever $m^*(O \backslash E) =$ finite or infinite, $m^*(A \backslash E) - m^*(O \backslash E) \leqslant m^*(A \backslash O)$ is true.
   $\because m^*(O \backslash E) < \varepsilon \therefore m^*(A \backslash E) - \varepsilon \leqslant m^*(A \backslash E) - m^*(O \backslash E) < m^*(A \backslash O)$

$$\begin{aligned} \therefore m^*(A) &= m^*(A \cap O) + m^*(A \backslash O) \quad \forall O \subseteq X \\ &\geqslant m^*(A \cap E) + m^*(A \backslash E) \quad \forall E \subseteq O \subseteq X \end{aligned}$$

□

Theorem 1.9.5. *$E \subseteq \mathbb{R}$, then*

$$E \in L \Leftrightarrow \forall \varepsilon > 0, \exists \text{ closed } F \subseteq E \text{ s.t. } m^*(E \backslash F) < \varepsilon$$

Proof.

1. $\Rightarrow \because E^c \in L$, Thm 1.9.4 $\Rightarrow \forall \varepsilon, \exists$ open $\underbrace{O \supseteq E^c}_{\underbrace{O^c}_{\equiv F} \subseteq E}$ s.t. $m^*(O \backslash E^c) < \varepsilon$

$$\therefore m^*(E \backslash F) = m^*(E \backslash O^c) = m^*(E \cap O) = m^*\left[O \cap (E^c)^c\right] = m^*(O \backslash E^c) \leq \varepsilon$$

2. $\Leftarrow$ check: $E^c \in L$ by reversing above argument.

□

Theorem 1.9.6. *$A \subseteq \mathbb{R}$, then*

$$A \in L \Leftrightarrow A = C \bigcup N, \text{ where } C \in B, N \in L \ \& \ m(N) = 0.$$

Proof.

1. $\Rightarrow$ by Thm 1.9.5 $\Rightarrow \forall n \geq 1$, $\exists$ closed $F_n \subseteq E$ & $m\left(E \backslash F_n\right) < \dfrac{1}{n}$. Let $F = \bigcup\limits_n F_n \in B$, let $N = E \backslash F \in L$. Check m(N) =0.$m\left(N\right) = m\left(E \backslash F\right) \leqslant m\left(E \backslash F_n\right) < \frac{1}{n}$. $0 \leqslant m\left(N\right) \leqslant \lim\limits_{n \to \infty} \frac{1}{n} = 0 \Rrightarrow m\left(N\right) = 0$

2. $\Leftarrow$ it is trivial.

$\square$

Note 1.9.6.

1. *m is the completion of $m \big| B$ & $\overline{B} = L$*

2. *Thm 1.9.4 – 1.9.6 true for $\mathbb{R}^n$.*

Littlewood principles

Principle I : $E \in L \Leftrightarrow E \sim \bigcup\limits_{i=1}^{n} I_i$, where $I_i s$ are intervals.

Theorem 1.9.7. *$E \subseteq \mathbb{R}, m^*(E) < \infty$, then*

$$E \in L \Leftrightarrow \forall \varepsilon > 0, \exists \text{ finite open intervals } \{I_i\}_{i=1}^n \text{ s.t. } m^*\left(E\Delta\left(\bigcup_{i=1}^{n} I_i\right)\right) < \varepsilon$$

Proof.

1. $\Rightarrow$ by Thm. 1.9.4 $\Rightarrow \forall\varepsilon > 0, \exists$ open $O \supseteq E$ s.t. $m(O \backslash E) < \varepsilon$.

   Note 1.9.7. $O \subseteq \mathbb{R}$,

   $$O \text{ open } \Leftrightarrow O = \bigcup_{n=1}^{\infty} I_n, \text{ where } \{I_n\} \text{ disjoint open intervals.}$$

   Proof.

   (a) $\Leftarrow$ It has already been done.

   (b) $\Rightarrow$ Define $x \sim y$ if $\overline{xy} \subseteq O$, for $x, y \in O$, where $\sim$ is a equivalence relation, each equivalence class is an open interval $\Rrightarrow O = \bigcup\limits_{\alpha} I_\alpha$

   $\because$ correspond each I_α to different rational number in $I_\alpha \Rightarrow \{I_\alpha\}$ countably many $\therefore O = \bigcup\limits_{n=1}^{\infty} I_n$.

   $\square$

$\because m^*(E) < \infty \Rightarrow m(O) < \infty \Rightarrow I_i$ finite intervals
$\because \bigcup\limits_{i=1}^{n} I_i \uparrow O \therefore m\left(\bigcup\limits_{i=1}^{n} I_i\right) \uparrow m(O), \therefore \exists n,\ s.t.\ m\left(O\backslash\left(\bigcup\limits_{i=1}^{n} I_i\right)\right) < \frac{\varepsilon}{2}$

$$\therefore m\left(E\Delta\bigcup_{i=1}^{n} I_i\right) = \underbrace{m\left(E\backslash\left(\bigcup_{i=1}^{n} I_i\right)\right)}_{\leqslant m\left(O\backslash\left(\bigcup\limits_{i=1}^{n} I_i\right)\right)} + \underbrace{m\left(\left(\bigcup_{i=1}^{n} I_i\right)\backslash E\right)}_{\leqslant m(O\backslash E)} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

2. $\Leftarrow$ check $m^*(A) \geqslant m^*(A \cap E) + m^*(A\backslash E) \quad \forall A \subseteq \mathbb{R}$ as in proof of Thm. 1.9.4.

$\square$

Homework 1.9.

1. *Problem* 1.9.7 *If a set $F \in \mathbb{R}^n$ is Lebesgue-measurable and if $\mu(F) < \infty$, then for any $\varepsilon > 0$ there exists an open set E such that $E \supseteq F$ and $\mu(E)$ and $\mu(E) < \mu(F) + \varepsilon$*
2. *Problem* 1.9.14 *The Lebesgue-Stieltjes outer measure is a metric outer measure.*
3. *Problem* 1.9.15 *Let $f(x) = 0$ if $x < 0$, $f(X) = 1$ if $x \geq 0$. Prove that*

$$\mu_f\{(-1,0)\} \leqslant f(0) - f(-1).$$

1.10 Signed Measures

$\mathbf{a}$, $\sigma-$algebra on X

Definition 1.10.1 (signed measure)**.** $\mu : \mathbf{a} \to (-\infty, \infty]$ *or* $[-\infty, \infty)$ *is signed measure if*

1. $\mu(\emptyset) = 0$
2. *μ countably additive*

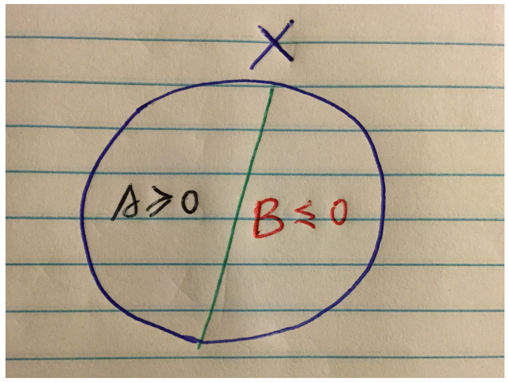
Example 1.10.1. *μ_1, μ_2 measures on $\mathbf{a}$, one of which is a finite measure ($\mu_1(X) < \infty$ or $\mu_2(X) < \infty$), let $\mu(E) = \mu_1(X) - \mu_2(X)$ for $E \in \mathbf{a}$, then μ signed measure.*

μ singed measure, $E \in \mathbf{a}$.

Definition 1.10.2. *$E \geqslant 0$ if $\mu(F) \geqslant 0,\ \forall F \subseteq E, F \in \mathbf{a}$;*
$E \leqslant 0$ if $\mu(F) \leqslant 0,\ \forall F \subseteq E, F \in \mathbf{a}$.

Theorem 1.10.1 (Hahn decomposition)**.** *μ signed measure on $\mathbf{a}$. Then $\exists A, B \in \mathbf{a}$ s.t. $A \geq 0, B \leq 0, X = A \bigcup B, A \bigcap B = \emptyset$.*

We shall need 2 lemmas to prove Thm 1.10.1.



Lemma 1.10.1.

1. $E \geq 0, F \in \mathbf{a}, F \subseteq E \Rightarrow F \geq 0$
2. $E_n \geq 0,\ \forall n\ \Rightarrow\ \bigcup_n E_n \geqslant 0$
3. $E \geq 0, F \in \mathbf{a}, F \subseteq E \Rightarrow \mu(F) \leqslant \mu(E)$

Proof.

1. $\forall A \in \mathbf{a}, A \subseteq F \Rightarrow A \subseteq E \Rightarrow \mu(A) \geq 0$
2. Let $A \subseteq \bigcup_n E_n, A \in \mathbf{a}$, check $\mu(A) \geqslant 0$

$$\begin{aligned}
\mu(A) &= \mu\left[\left(\bigcup_n E_n\right) \cap A\right] \\
&= \mu\left[\bigcup_n (E_n \cap A)\right] \\
&= \underbrace{\mu\left[\bigcup_n \left((E_n \cap A) \setminus \bigcup_{i=1}^{n-1} (E_i \cap A)\right)\right]}_{disjoint\ union} \\
&= \sum_n \mu \underbrace{\left((E_n \cap A) \setminus \bigcup_{i=1}^{n-1} (E_i \cap A)\right)}_{\subseteq E_n} \\
&\geq 0
\end{aligned}$$

$\therefore \bigcup_n E_n \geqslant 0.$

3. $\mu(E) = \underbrace{\mu\underbrace{(E \setminus F)}_{\subseteq E}}_{\geqslant 0} + \mu(F)$

$\square$

Lemma 1.10.2. $E \subseteq F, E, F \in \mathbf{a}$. $|\mu(F)| < \infty \Rightarrow |\mu(E)| < \infty$

Proof. $\because \mu(F) = \mu(E) + \mu(F \backslash E)$

1. if $\mu(E) = \infty$
$$\mu(F) = \underbrace{\mu(E)}_{\infty} + \mu(F \backslash E)$$
$$\because \quad \mu : \mathbf{a} \to (-\infty, \infty]$$
$$\therefore \quad -\infty < \mu(F \backslash E) \leqslant \infty \Rightarrow \quad \mu(F) = \infty$$
it is contradictional.
2. if $\mu(E) = -\infty$, similarly, $\mu(F) = -\infty$ it is contradictional.

$\square$

Proof. (Thm 1.10.1) Assume, $\mu : \mathbf{a} \to (-\infty, \infty]$

Let $b = \inf\{\mu(B_0) : B_0 \leqslant 0, B_0 \in \mathbf{a}\} \quad \therefore -\infty \leq b \leq 0$

Let $B_i \leq 0$ s.t. $\mu(B_i) \to b$

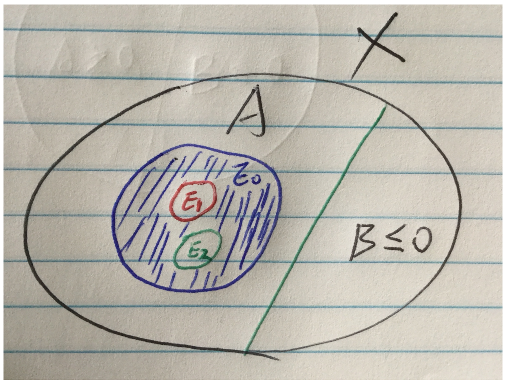
Let $B = \bigcup_j B_j \in \mathbf{a}$ (by lemma 1.10.1.2) $\Rightarrow B \leq 0 \Rightarrow b \leqslant \mu(B) \leqslant \underbrace{\mu(B_j) \to b}_{by \quad Lemma \; 1.10.1.2'} \Rightarrow b = \mu(B) \Rightarrow$
$-\infty < b \leqslant 0$

Let $A = X \backslash B, check A \geq 0$, assume $A \ngeq 0 \therefore \exists E_0 \in \mathbf{a}, E_0 \subseteq A$ s.t. $\mu(E_0) < 0$

It claims that $E_0 \nleq 0$

$\because$ if $E_0 \leq 0$, then $E_0 \bigcup B \leq 0$ $b \leqslant \mu(E_0 \cup B) \underbrace{=}_{E_0 \subseteq A, B = X \backslash A} \underbrace{\mu(E_0)}_{<0} + \underbrace{\mu(B)}_{b} < b$, it is contradictional.

then $\exists E_1 \subseteq E_0, E_1 \in \mathbf{a}$ $s.t.$ $\mu(E_1) > 0$



(1) Let $m_1 \geq 1$ be the smallest s.t. $\mu(E_1) \geqslant \dfrac{1}{m_1}$ & $E_1 \subseteq E_0$

$\because -\infty < \mu(E_0) < 0 \Rightarrow -\infty < \mu(E_1) < \infty$ by Lemma 1.10.2 $\Rightarrow \mu(E_0 \backslash E_1) = \mu(E_0) - \mu(E_1) \leqslant \mu(E_0) - \frac{1}{m_1} < 0$

(2) Let $m_2 \geq 1$ be the smallest s.t. $\exists E_2 \subseteq E_0 \backslash E_1, \mu(E_2) \geq \dfrac{1}{m_2}$ (replace E_1 by $E_0 \backslash E_1$)

$\vdots$

(k) Let $m_k \geqslant 1$ be the smallest s.t. $\exists E_k \subseteq E_0 \backslash \left(\bigcup\limits_{i=1}^{k-1} E_i\right), \mu(E_k) \geqslant \frac{1}{m_k}$

Let $F_0 = E_0 \backslash \left(\bigcup\limits_k E_k\right)$. Check: $F_0 \leq 0$, i.e. Let $F \subseteq F_0$, $F \in \mathbf{a}$, check $\mu(F) \leq 0$.
If $\mu(F) \geq \dfrac{1}{m_k - 1}$, it is contradictional minimality of m_k, so $\mu(F) < \dfrac{1}{m_k - 1}$. $\because \{E_k\}$ disjoint $\Rightarrow \mu\left(\bigcup\limits_k E_k\right) = \sum\limits_k \mu(E_k)$ $\because \bigcup\limits_k E_k \subseteq E_0$ & $-\infty < \mu(E_0) < 0 \underbrace{\Rightarrow}_{by\ Lemma\ 3} \left|\mu\left(\bigcup\limits_k E_k\right)\right| < \infty$ $\therefore$ $\mu\left(\bigcup\limits_k E_k\right) = \sum\limits_k \mu(E_k)$ convergence $\Rightarrow \underbrace{\mu(E_k)}_{> \frac{1}{m_k}} \overset{k\to\infty}{\to} 0 \Rightarrow m_k \to \infty \therefore \mu(F) \leqslant \frac{1}{m_k - 1} \to 0 \Rightarrow \mu(F) \leqslant 0 \Rightarrow F_0 \leqslant 0 \Rightarrow F_0 \cup B \leqslant 0 \Rightarrow b \leqslant \mu(F_0 \cup B) = \mu(F_0) + \mu(B)$, $\because \mu(F_0) = \underbrace{\mu(E_0)}_{<0} - \underbrace{\sum_{k=1}^{\infty} \mu(E_k)}_{>0} < 0, b \leqslant \underbrace{\mu(F_0)}_{<0} + \underbrace{\mu(B)}_{b} < b$, it is contradictional. so $A \geq 0$. $\square$

Proposition 1.10.1. $\mu(B) \leqslant \mu(C) \leqslant \mu(A), \forall\ C \in \mathbf{a}, A, B$ *are defined as Thm 1.10.1*

Proof. $C \in \mathbf{a}, (A \backslash C) \subseteq A \in \mathbf{a}, \mu(A \backslash C) \geqslant 0 \Rightarrow \mu(A) = \mu(A \cap C) + \mu(A \backslash C) \geqslant \mu(A \cap C)$, similarly, $\mu(B) \leqslant \mu(B \cap C)$.
$\therefore \mu(C) = \mu(A \cap C) + \mu(B \cap C) \leqslant \mu(A \cap C) \leqslant \mu(A)$ and $\mu(C) = \mu(A \cap C) + \mu(B \cap C) \geqslant \mu(B \cap C) \geqslant \mu(B)$ $\square$

Theorem 1.10.2 (Jordan decomposition of μ). *μ signed measure on $\mathbf{a}$, then $\exists$ measures μ_1 and μ_2, one is finite measure, s.t. $\mu = \mu_1 - \mu_2$.*

Proof. Let $E \in \mathbf{a}$, define $\mu_1(E) = \mu(E \cap A) \geqslant 0, \mu_2(E) = -\mu(E \cap B) \geqslant 0$, then

1. $\mu_1, \mu_2 : \mathbf{a} \to [0, +\infty]$
2. $\mu_1(\emptyset) = \mu_2(\emptyset) = 0$

3. Let $\{E_n\} \subseteq \mathbf{a}$ disjoint

$$\mu_1\left(\bigcup_n E_n\right) = \mu_1\left(\bigcup_n (E_n \cap A)\right) = \sum_n \mu_1\left(E_n \bigcap A\right) = \sum_n \mu_1(E_n)$$

   similarity for μ_2, so μ_1, μ_2 measures on $\mathbf{a}$.

4. $\because \mu_1(X) = \mu(A), \mu_2(X) = -\underbrace{\mu(B)}_{b}$ finite in proof of Thm 1.10.1 so μ_2 finite measure.

5. $\mu_1(E) - \mu_2(E) = \mu(E \cap A) + \mu(E \cap B) = \mu(E) \therefore \mu_1 - \mu_2 = \mu$

□

Note 1.10.1.

1. *$\mu = \mu_1 - \mu_2$ not unique, $\because \mu = \mu_1 - \mu_2 = (\mu_1 + \mu_0) - (\mu_2 + \mu_0)$ where μ_0 finite measure.*

2. *Hahn decomposition theorem let $\mu^+(E) = \mu(A \cap E), \mu^-(E) = -\mu(B \cap E), \forall E \in \mathbf{a} \therefore \mu = \mu^+ - \mu^-$ Jordan decomposition of μ.*

$a \in \mathbb{R}$

1. $a = a^+ + a^-$ where $a^+ = \frac{1}{2}(|a| + a) \geqslant 0 \quad a^- = \frac{1}{2}(|a| - a) \geqslant 0$
2. $|a| = a^+ + a^-$
3. $a \in [-\infty, \infty] \Rightarrow a = a_1 - a_2$ where $a_1, a_2 \geq 0$ and at least one is finite
4. a_1, a_2 not unique

Note 1.10.2. *Hahn decomposition not unique, but Jordan decomposition is unique.*

Definition 1.10.3. *μ^+ upper variation of μ, μ^- lower variation of μ, $|\mu| = \mu^+ + \mu^-$ total variation of μ, μ is finite if $|\mu|$ is finite, μ is $\sigma - finite$ if $|\mu|$ is $\sigma - finite$.*

Homework 1.10.

1. *Problem* 1.10.3 *Give an example of a signed measure for which the Hahn decomposition is not unique.*

2. *Problem* 1.10.4 *If $X = A_1 \bigcup B_1, X = A_2 \bigcup B_2$ are two Hahn decompositions of a signed measure μ, then for any measurable set E,*

$$\mu(E \cap A_1) = \mu(E \cap A_2), \quad \mu(E \cap B_1) = \mu(E \cap B_2).$$

   Proof. $A_1 \backslash A_2 = (X \backslash B_1) \backslash (X \backslash B_2) = B_2 \backslash B_1$, similarly $A_2 \backslash A_1 = (X \backslash B_2) \backslash (X \backslash B_1) = B_1 \backslash B_2$ $\therefore 0 \leqslant \mu(A_1 \backslash A_2) = \mu(B_2 \backslash B_1) \leqslant 0, 0 \leqslant \mu(A_2 \backslash A_1) = \mu(B_1 \backslash B_2) \leqslant 0$

$$\therefore \mu(A_1 \backslash A_2) = \mu(B_2 \backslash B_1) = \mu(A_2 \backslash A_1) = \mu(B_1 \backslash B_2) = 0$$

$$\begin{aligned}
\mu(E \cap A_1) &= \mu[E \cap [(A_1 \backslash A_2) \cup (A_1 \cap A_2)]] \\
&= \mu[E \cap (A_1 \backslash A_2)] + \mu[E \cap (A_1 \cap A_2)] \\
&= 0 + \mu[E \cap (A_1 \cap A_2)] \\
\mu(E \cap A_2) &= \mu[E \cap [(A_2 \backslash A_1) \cup (A_2 \cap A_1)]] \\
&= \mu[E \cap (A_2 \backslash A_1)] + \mu[E \cap (A_1 \cap A_2)] \\
&= 0 + \mu[E \cap (A_1 \cap A_2)]
\end{aligned}$$

$$\therefore \mu(E \cap A_1) = \mu(E \cap A_2)$$

□

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