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10-19-1 Examples of Laplace's equation

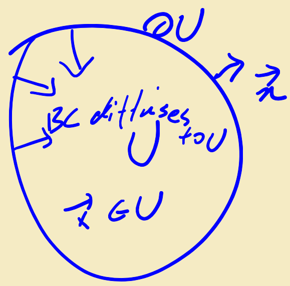
Laplace & Poisson Eqn.

→ wave & heat Eqn: IVPs, evolve in time, hyperbolic & parabolic PDEs

→ now consider time indep. BVPs.

Elliptic problems: Eq Laplace's Eqn

$\Delta u(\vec{x}) = 0$, $\vec{x} \in U \subset \mathbb{R}^n$, U open, subset



BCs: Dirichlet: $u(\vec{y}) = f(\vec{y})$, $\vec{y} \in \partial U$

Neumann: $\frac{\partial u}{\partial n}(\vec{y}) = g(\vec{y})$, $\vec{y} \in \partial U$

↑ derivative normal to boundary

• Functions $u(\vec{x})$ that satisfy $\Delta u(\vec{x})$ are harmonic where does this Eqns come from? many applications.

Ex1. Steady state heat conduction

$u_t = \Delta u$, $\vec{x} \in U$, $\underbrace{u(\vec{x}, 0) = g(\vec{x})}_{IC}$, $\vec{x} \in U$

$u(\vec{y}, t) = f(\vec{y})$ $\vec{y} \in \partial U$
(temp @ boundary)

Diffusing: expect $\lim_{t \to \infty} u(\vec{x}, t) = v(\vec{x})$

satisfies $\Delta v = 0$, $\vec{x} \in U$; $v(\vec{y}) = f(\vec{y})$, $\vec{y} \in \partial U$

Ex2: Electrostatics: $\vec{E}(\vec{x})$: electric field: $\vec{B}(\vec{x})$: magnetic field

Gauss's Law: $\nabla \cdot \vec{E} = 0$, $\vec{x} \in U$

Faraday's Law: $\nabla \times \vec{E} = -\partial_t \vec{B} = \vec{0}$, $\vec{x} \in U$, static case, s.t.

$\vec{E} = \nabla u$, ($u(\vec{x})$ scalar electro static potential) then

Gauss's Law $\Rightarrow$ $\nabla \cdot (\nabla u) = \Delta u = 0$, $\vec{x} \in U$

$\to$ precribe electric potential (voltage) on boundary then

$u(\vec{\eta}) = \rho(\vec{\eta})$, $\vec{\eta} \in \partial U$ (Dirichlet)

$\to$ precribing surface charge (electric filed)

$\vec{n} \cdot \nabla u(\vec{\eta}) = \sigma(\vec{\eta})$

$\frac{\partial u}{\partial n} = \sigma(\vec{\eta})$ (neumann)

unit normal along ∂u

10-19-2 symmetries of Laplace's Eqn

Claim: If $\Delta u(\vec{x}) = 0$ for $\vec{x} \in \mathbb{R}^n$ (u is harmonic on $\mathbb{R}^n$) then

1) $w(\vec{x}) = u(\vec{x} - \vec{\eta})$ is harmonic $\forall \vec{\eta} \in \mathbb{R}^n$ (translation symmetry)

2) $w(\vec{x}) = u(R\vec{x})$ is harmonic for any rotation matrix,

$R \in SO_n$, $R^T R = I$, $\det R = +1$ (rotation symmetry)

Pf: (1) Let $w(\vec{x}) = u(\vec{x} - \vec{\eta})$, $\vec{\eta} \in \mathbb{R}^n$ then

$$\partial_{x_i} w(\vec{x}) = \partial_{x_i} u(\vec{x} - \vec{\eta}) = \partial_{z_i} u(\vec{z}), \quad \vec{z} = \vec{x} - \vec{\eta}$$

$$\Rightarrow \Delta_{\vec{x}} w(\vec{x}) = \Delta_{\vec{z}} u(\vec{z})$$

$$\left[\partial_{z_i} u(\vec{z}) \, \frac{\partial z_i}{\partial x_i} \right]$$

$$\partial_{x_i x_i} w(\vec{x}) = \partial_{z_i z_i} u(\vec{z})$$

$\rightarrow$ so w is harmonic

(2) Let $w(\vec{x}) = u(\vec{y})$, $\vec{y} = R\vec{x}$, $R \in SO_n$, $R = [r_{ij}]$

$$\partial_{x_j} w(\vec{x}) \overset{\text{chain rule}}{=\!=} \sum_{i=1}^{n} \partial_{y_i} u(\vec{y}) \frac{\partial y_i}{\partial x_j}$$

$$= \sum_{i=1}^{n} \partial_{y_i} u(\vec{y}) \, r_{ij}$$

then $\Delta_{\vec{x}} w(\vec{x}) = \sum_{j=1}^{n} \partial_{x_j x_j} w(\vec{x})$

$$= \sum_{j=1}^{n} \partial_{x_j} \left(\sum_{i=1}^{n} \partial_{y_i} u(\vec{y}) \, r_{ij} \right)$$

$$\overset{\text{replace } \partial_{x_j}}{=\!=} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \partial_{y_k} r_{kj} \right) \left(\sum_{i=1}^{n} \partial_{y_i} u(\vec{y}) \, r_{ij} \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \partial_{y_i y_k} u(\vec{y}) \left(\sum_{j=1}^{n} r_{ij} r_{kj} \right)$$

$$R^T R = 1 \quad \Updownarrow \quad \sum_{j=1}^{n} r_{ij} r_{kj} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

$$\Rightarrow \Delta_{\vec{x}} w(\vec{x}) = \sum_{i=1}^{n} \partial_{y_i y_i} u(\vec{y}) = \Delta_{\vec{y}} u(\vec{y}) = 0$$

- Rotation symmetry is a fundamental property of solns to $\Delta u = 0$ on $\mathbb{R}^n$

- seek a fundamental soln: $\Phi(\vec{x}) = \Phi(R\vec{x})$

(analogous to heat eqn case)

with this symmetry

endomorphism

- what form must this soln. take?

$u(\vec{x}) = v(|\vec{x}|) = v(r)$ radial soln

10-21-1 Laplace Eqn fundamental soln

Laplace's / poisson Eqn

Last time: $u(\vec{x})$ is harmonic if $\Delta u(\vec{x}) = 0$

- if $u(\vec{x})$ is harmonic, $u(\vec{x}-\vec{y})$ harmonic ($\vec{y} \in R^n$) translation symmetry

  $u(R\vec{x})$ harmonic (rotation symmetry) $\det(R) = \pm 1$, $R^T R = 1$

- foundament soln: $\Phi(\vec{x}) = \phi(R\vec{x}) \Rightarrow u(\vec{x}) = v(r)$

Foundament soln. to $\Delta u = 0$

$$\Rightarrow \Delta u = \sum_{i=1}^{n} \partial_{x_i x_i} v(r) = \sum_{i=1}^{n} \partial_{x_i}\left(v'(r)\frac{\partial r}{\partial x_i}\right) = \sum_{i=1}^{n} \partial_{x_i}\left(\frac{x_i}{r} v'(r)\right)$$

$$\begin{pmatrix} u(\vec{x}) = v(r) \\ r = \sqrt{\sum_{j=1}^{n} x_j^2} \end{pmatrix} \quad \left(\frac{\partial r}{\partial x_i} = \frac{2x_j}{2\sqrt{\sum_{j=1}^{n} x_j^2}} = \frac{x_j}{r}\right)$$

$$\Delta u = \sum_{i=1}^{n} \frac{1}{r} v'(r) + x_i v'(r)\, \partial_{x_i}\left(\frac{1}{r}\right) + \frac{x_i}{r} v''(r) \frac{\partial r}{\partial x_i}$$

$$\partial_{x_i}\left(x_i \frac{v'(r)}{r}\right) \qquad -\frac{1}{r^2}\frac{\partial r}{\partial x_i} \qquad -\frac{1}{r^2}\frac{2x_i}{2\sqrt{x_1+\cdots}}$$

$$= \frac{v'(r)}{r} + x_i\, \partial_{x_i}\frac{v'(r)}{r}$$

$$= \frac{v'(r)}{r} + x_i \frac{1}{r} \partial_{x_i} v'(r) + x_i v'(r)\, \partial_{x_i}\frac{1}{r}$$

$$= \frac{v'(r)}{r} + x_i \frac{1}{r} v''(r) \frac{\partial r}{\partial x_i} + x_i v'(r)\, \partial_{x_i}\left(\frac{1}{r}\right)$$

$$= \sum_{i=1}^{n} \frac{v'(r)}{r} - \frac{x_i^2}{r^3}v'(r) + \frac{x_i^2}{r^2}v''(r)$$

$$= \frac{1}{r^2}\left(\sum_{i=1}^{n} x_i^2\right)v''(r) + \frac{n}{r}v'(r) - \frac{1}{r^3}\left(\sum_{i=1}^{n} x_i^2\right)v'$$

$$-\frac{1}{r}v'$$

$$\Delta u = v'' + \frac{n-1}{r}v' = 0$$

$$\Rightarrow \frac{v''}{v'} = -\frac{n-1}{r} \Rightarrow (\log v')' = -\frac{n-1}{r}$$

$$\Rightarrow \log v' = -(n-1)\log r$$

$$\Rightarrow v' = \frac{A}{r^{n-1}} \quad (A = e^c)$$

$$\Rightarrow v(r) = \begin{cases} \frac{a}{r^{n-2}} + b, & n \geq 3 \\ a\log r + b, & n = 2 \end{cases}$$

Normalize to get fundament solu $\Phi(\vec{x})$

$$\Phi(\vec{x}) = \begin{cases} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\vec{x}|^{n-2}}, & n \geq 3, \ \vec{x} \in \mathbb{R}^n \setminus \{0\} \\ -\frac{1}{2\pi}\log|\vec{x}|, & n = 2 \end{cases}$$

$\alpha(n)$ = volume of unit sphere S^{n-1} in $\mathbb{R}^n$

$\alpha(2) = \pi$, $\alpha(3) = \frac{4}{3}\pi$, $\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ $\quad \Gamma(z) = \int_0^\infty s^{z-1}e^{-s}\,ds$

Gamma function

$\boxed{\Phi(\vec{x}) \text{ is singular but integrable on bdd sets}}$ 极坐标不证明

$\Delta\Phi$ is singular & not integrable on bdd sets containing origin.

10-21-2 Poisson Eqn Fund. solu

Fund solu of poisson Eqn

Consider $-\Delta u(\vec{x}) = f(\vec{x}), \quad \vec{x} \in \mathbb{R}^n \quad (*)$

where $f \in C_0^2(\mathbb{R}^n) = (C^2(\mathbb{R})$ w/ compact support)

Thm: $u(\vec{x}) = (\Phi * f)(\vec{x}) = \int_{\mathbb{R}^n} \Phi(\vec{x}-\vec{y}) f(\vec{y})\, d\vec{y}$ satisfies $*$

comments: Analogous to heat eqns soln: $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) f(\vec{y})\, dy$

- translating, multiplying, & summing $\Phi(\vec{x}-\vec{y})$ is not harmonic b/c $\Phi(\vec{x}-\vec{y})$ not harmonic at $\vec{x}=\vec{y}$
- $\Delta_{\vec{x}} \Phi(\vec{x}-\vec{y})$ is a δ-distribution

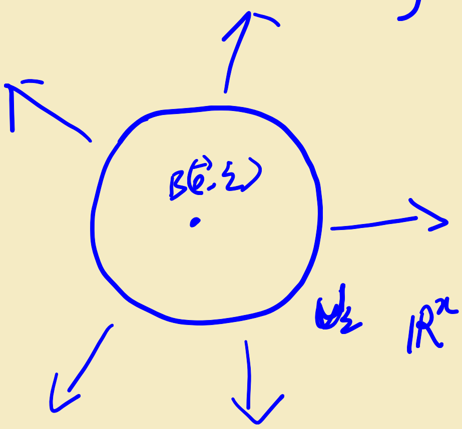
proof: change variables $u(\vec{x}) = \int_{\mathbb{R}^n} \Phi(\vec{y}) f(\vec{x}-\vec{y})\, d\vec{y}$

$$\Rightarrow \Delta u(\vec{x}) = \int_{\mathbb{R}^n} \Phi(\vec{x})\, \Delta_x f(\vec{x}-\vec{y})\, d\vec{y}$$

$$= \int_{\mathbb{R}^n} \Phi(\vec{y})\, \Delta_y f(\vec{x}-\vec{y})\, d\vec{y}$$

$\Rightarrow$ would like to integrate by parts but Φ has singularity at $\vec{y}=\vec{0}$

$\rightarrow$ cut out singularity with open ball $B(\vec{0},\varepsilon)$



$B(\vec{0},\varepsilon)$: open ball @ origin, w/ radius ε

$U_\varepsilon := \mathbb{R}^n \setminus B(\vec{0},\varepsilon)$

$$\Delta u(\vec{x}) = \int_{B(\vec{0},\varepsilon)} \phi(\vec{y})\, \Delta_y f(\vec{x}-\vec{y})\, d\vec{y}$$

$$+ \int_{U_\varepsilon} \Phi(\vec{y})\, \Delta_y f(\vec{x}-\vec{y})\, d\vec{y}$$

- 1st integral $\longrightarrow 0$, $\because f \in C^2$ & Φ integrable @ origin as $\varepsilon \to 0$

· 2nd integral, need Green's 2nd identity.

Divergence Thm: imples $\int_\Omega \Delta u\, d\vec{x} = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds$ and

$$\int_\Omega u\Delta v\, d\vec{x} \overset{IBP}{=\!=} \int_{\partial\Omega} u\frac{\partial v}{\partial n} ds - \int_\Omega \nabla u\cdot\nabla v\, d\vec{x} \quad \text{and}$$

(Green's 1st identity)

$$\int_\Omega v\Delta u\, d\vec{x} \overset{IBP}{=\!=} \int_{\partial\Omega} v\frac{\partial v}{\partial n} ds - \int_\Omega \nabla v\cdot\nabla u\, d\vec{x}$$

$$\overset{\text{subtract}}{\Longrightarrow} \int_\Omega u\Delta v\, d\vec{x} - \int_\Omega v\Delta u\, d\vec{x} = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds \quad \text{(Green's 2nd identity)}$$

Using Green's 2nd:

$$\int_{U_\varepsilon} \Phi(\vec{y})\,\Delta_{\vec{y}} f(\vec{x}-\vec{y})\,d\vec{y} = \int_{\partial B(0,\varepsilon)} \Phi(\vec{y})\frac{\partial f}{\partial n_y}(\vec{x}-\vec{y}) - f(\vec{x},\vec{y})\frac{\partial\Phi(\vec{y})}{\partial n_y}\,ds_y$$

$$+ \int_{U_\varepsilon} \Delta\Phi(\vec{y})\, f(\vec{x}-\vec{y})\,d\vec{y}$$ (contribution from boundary at $|\vec{x}|\to\infty$)

· 2nd integral vanishes b/c Φ harmonic on U_ε

· unit normal n_y is to interior at $B(\vec{0},\varepsilon)$

· $n_y = -\frac{\vec{y}}{\varepsilon} \Rightarrow \frac{\partial\Phi}{\partial n_y}(\vec{y}) = -\frac{\partial\Phi}{\partial r}(\vec{y})\Big|_{|\vec{y}|=\varepsilon} = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$ $(r=|\vec{y}|)$

and $\int_{\partial B(\vec{0},\varepsilon)} \Phi(\vec{y})\frac{\partial f}{\partial n_y}(\vec{x}-\vec{y})\,ds_y \to 0$ as $\varepsilon\to 0$ b/c Φ integrable at origin.

So
$$\Delta u = \lim_{\varepsilon\to0^+}\left(-\int_{\partial B(\vec{0},\varepsilon)} f(\vec{x}-\vec{y})\,ds_y\,\frac{1}{n\alpha(n)\varepsilon^{n-1}}\right) = -\lim_{\varepsilon\to0^+}\underset{\vec{y}\in\partial B(\vec{0},\varepsilon)}{\mathrm{Avg}}\left(f(\vec{x}-\vec{y})\right)$$
$$= -f(\vec{x})$$

b/c f continous $\Rightarrow$ $-\Delta u = f$

10-23-1 Green's ID warm up

poisson Eqn

Thm: $-u(\vec{x}) = f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$ where $f \in C_c^2(\mathbb{R}^n)$

C^2 w/ compact support

was soln $u(\vec{x}) = (\Phi * f)(\vec{x}) = \int_{\mathbb{R}^n} \Phi(\vec{x}-\vec{y}) f(\vec{y})\, d\vec{y}$

Mean value property & max principle

warm up: use Green's 1^{st} identity to prove if $\Delta u = 0$ $(\vec{x} \in \Omega)$

and $u(\vec{y}) = g(\vec{y})$, $\vec{y} \in \partial\Omega$, the solution to this BVP is unique.

Green's 1^{st} ID: $\int_\Omega v \Delta u \, d\vec{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds - \int_\Omega \nabla v \cdot \nabla u \, d\vec{x}$

$\int_\Omega (\nabla u)^2 d\vec{x} = \int_\Omega \nabla u \cdot \nabla u \, d\vec{x} = \int_{\partial\Omega} \cancel{u \frac{\partial u}{\partial n}}_{0} ds - \int_\Omega \cancel{u \Delta u}\, d\vec{x}$ $= 0$

$\Delta u = 0$

$u = u_1 - u_2$: $u = 0$ $\vec{x} \in \partial\Omega$

$\nabla u \equiv 0$

$\Rightarrow u = \text{constant}$

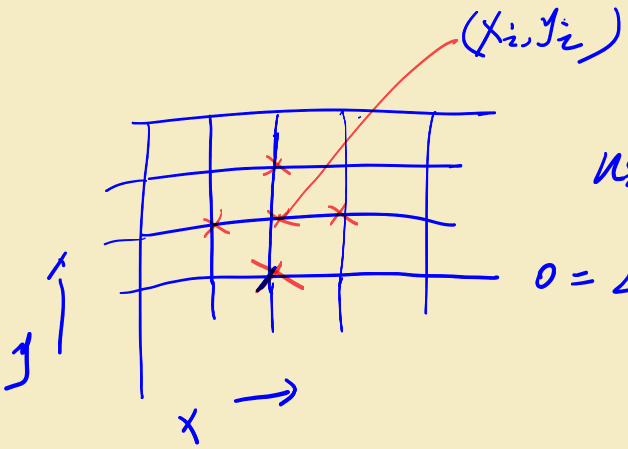
$u_1 - u_2 = 0$ on $\partial\Omega$ $u_1 \equiv u_2$

10-23-2 Mean value property (MVP)

mean value property

$\rightarrow$ $u \in C^2(\Omega)$ harmonic iff for any ball $B(\vec{x}, r) = \{y \in \mathbb{R}^n \mid |\vec{x} - \vec{y}| < r\} \subset \Omega$

$u(\vec{x}) = \int_{\partial B(\vec{x}, r)} u(\vec{y})\, ds_y = \int_{B(\vec{x}, r)} u(\vec{y})\, d\vec{y}$ (avg on surface ball interior equals $u(\vec{x})$ itself) intuition from finite diff. in 2D



$u_{ij} = u(x_i, y_j)$

$0 = \Delta u = u_{xx} + u_{yy}$

$\approx \underbrace{\dfrac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}}_{u_{xx}}$

$+ \underbrace{\dfrac{u_{i,j+1} - 2u_{ij} - u_{i,j-1}}{h^2}}_{u_{yy}}$

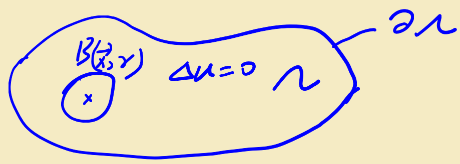
rewrite $\Delta u = 0$

$$\Longrightarrow \quad u_{ij} \approx \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$

$\to$ mean of 4 nearest neighbor's $\Rightarrow$ discrete mean value

Pf of MVP: for $B(\vec{x}, r) \subset \Omega$

if $\Delta u = 0$



$$0 = \int_{B(\vec{x},r)} \Delta u \, d\vec{y} \underset{\text{Green's 1st id}}{=} \int_{\partial B(\vec{x},r)} \frac{\partial u}{\partial n} ds = \oint_{\partial B(\vec{x},r)} \nabla u(\vec{y}) \cdot \underbrace{\left|\frac{\vec{y} - \vec{x}}{r}\right|}_{\text{normal}} ds_y$$

Now set $\phi(r) = \oint_{\partial B(\vec{x}, r)} u(\vec{y}) \, ds_y$

$\to$ note if $u \in C^2(\Omega)$, $\lim_{r \to 0^+} \phi(r) = u(\vec{x})$

just need to show $\phi(r) = \text{constant}$

compute $\phi'(r) = \dfrac{d}{dr} \oint_{\partial B(0,1)} u(\vec{x} + r\vec{z}) \, ds_z = \oint_{\partial B(0,1)} \nabla u(\vec{x} + r\vec{z}) \cdot \vec{z} \, ds_z$

(change of variables $\vec{y} = \vec{x} + r\vec{z}$)

$$= \fint_{\partial B(\vec{x},r)} \nabla u \cdot \frac{\vec{y}-\vec{x}}{r} \, dS_y$$

(change coords back)

$$= 0$$

$\therefore \ \phi'(r) = 0 \quad \therefore \ \phi(r) = \text{constant} = \lim_{s\to 0^+} \phi(s) = u(\vec{x})$

$\longrightarrow \ \Delta u = 0 \ \Longrightarrow \ \fint_{\partial B(\vec{x},r)} u(\vec{y})\, dS_y = u(\vec{x})$

Avg over Ball $B(\vec{x}, r)$ $\quad \displaystyle\int_{B(\vec{x},r)} u(\vec{y})\, d\vec{y} = \int_0^r \underbrace{\int_{\partial B(\vec{x},\rho)} u(\vec{y})\, dS_y}\, d\rho$

$$= u(\vec{x}) \int_0^r n\,\alpha(n)\, \rho^{n-1}\, d\rho$$

$$= \alpha(n)\, r^n\, u(\vec{x})$$

$$\Longrightarrow \ u(\vec{x}) = \fint_{B(\vec{x},\vec{r})} u(\vec{y})\, d\vec{y}$$

10-23-3 MVP ctd and Max principle

Conversely: suppose u has MVP then $\phi'(r) = \frac{r}{n} \fint_{B(\vec{x},r)} \Delta u\, d\vec{y} = 0$

since $\phi(r) = u(\vec{x})$ constant in r

$\fint_{B(\vec{x},r)} \Delta u\, d\vec{y} \equiv 0$ $\quad$ letting $r \to 0$: $\Delta u \equiv 0$ since Δu continuous in Ω.

Consequences: (1) strong / weak max principle

(2) uniqueness thm

(3) $\boxed{\text{size}}$ estimates on harmonic functions

(4) Liouville thm: only bdd harmonic fns are constant

Thm: Max principle: Let $\Omega \subset \mathbb{R}^n$ be open & bdd. suppose

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ harmonic in Ω then

(1) weak $\max_{\vec{x} \in \bar{\Omega}} u(\vec{x}) = \max_{\vec{x} \in \partial\Omega} u(\vec{x})$

(2) strong: If Ω connected then either u = constant in $\bar{\Omega}$ or

$u(\vec{x}) < \max_{\vec{y} \in \partial\Omega} u(\vec{y}) \quad \forall \vec{x} \in \Omega$

10-26-1 Maximum principle Laplace

Last time:

mean value property: $u \in C^2(\Omega)$ harmonic iff for any ball

$B(\vec{x}, r) = \{\vec{y} \in \mathbb{R}^n : |\vec{x} - \vec{y}| < r\} \subset \Omega$

$$u(\vec{x}) = \fint_{\partial B(\vec{x}, r)} u(\vec{y}) \, ds_y = \fint_{B(\vec{x}, r)} u(\vec{y}) \, d\vec{y}$$

Thm: Max principle: Let $\Omega \subset \mathbb{R}^n$ be open & bdd, suppose

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ harmonic in Ω, then

(1) weak: $\max_{\vec{x} \in \bar{\Omega}} u(\vec{x}) = \max_{\vec{x} \in \partial\Omega} u(\vec{x})$

(2) strong: If Ω connected, then either $u \equiv$ constant in $\bar{\Omega}$ or

$u(\vec{x}) < \max_{\vec{y} \in \partial\Omega} u(\vec{y}) \quad \forall \vec{x} \in \Omega$

Pf: prove strong form firstly, weak form follows

suppose Ω connected & $\exists\ \vec{x}_0 \in \Omega$ s.t. $u(\vec{x}_0) = \max\limits_{\vec{x} \in \vec{\Omega}} u(\vec{x}) = M$

choose r s.t. $B(\vec{x}, r) \subset \Omega$, then by MVP

$$u(\vec{x}_0) = M = ⨍_{B(\vec{x}_0, r)} u(\vec{y})\, d\vec{y}$$

But $u(\vec{x}) \leq M$ everywhere so it must be that $u(\vec{x}) \equiv M$ throughout

$B(\vec{x}_0, r)$ so $S = \{\vec{x} \in \Omega : u(\vec{x}) = M\}$ is nonempty and open,

but S is also relatively closed in Ω:

$\rightarrow$ To show this: Let $\vec{x}_n \in S$ converge to $\vec{x} \in \Omega$ as $n \to \infty$

then $u(\vec{x}) = \lim\limits_{n \to \infty} u(\vec{x}_n) = M$ by continuity so $\vec{x} \in S$.

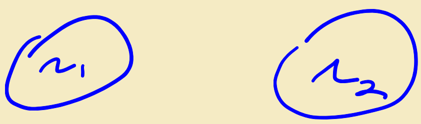
$\rightarrow$ the only non-empty open & closed set in Ω is Ω

itself, so $S \equiv \Omega \Rightarrow u =$ constant on $\bar{\Omega}$.

$\rightarrow$ the weak form follows

$\rightarrow$ min pricinple prove by studying $-u$

$\rightarrow$ If $\Omega = \Omega_1 \cup \Omega_2$



define $u(\vec{x}) = k$ on $\vec{x} \in \Omega_k$, then u is harmonic but

does not satisfy either strong form conclusion

10-26-2 Bounding and uniqueness Laplace's Eqn

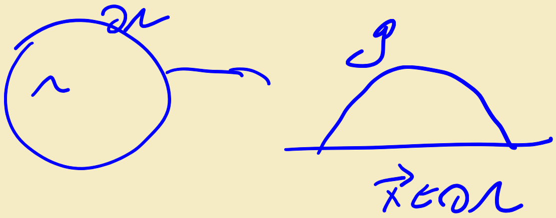
Thm: suppose that $\Delta u = 0$, $\vec{x} \in \Omega$, $u(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$

$g \in C(\partial\Omega)$

$u \in C^2(\Omega) \cap C(\bar{\Omega})$

If Ω connected and $g \geq 0$ $\forall \vec{x} \in \partial\Omega$

and $g > 0$, for some $\vec{x} \in \partial\Omega$, then $u > 0$, $\forall \vec{x} \in \Omega$



Pf: weak min principle implies

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} g$$

strong version gives $u > \min_{\partial\Omega} g$, $\forall \vec{x} \in \Omega$ or $u \equiv$ constant

$\rightarrow$ since $u \neq$ constant, $g \neq$ constant, $u > 0$ on Ω ✓

uniqueness: take $-\Delta u = f$, $\vec{x} \in \Omega$, $f \in C(\Omega)$ $(*)$

$u = g$, $\vec{x} \in \partial\Omega$, $g \in C(\partial\Omega)$

Thm: There is at most one soln $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of $(*)$

Pf: Let u_1, u_2 both be solns. Take $v = u_2 - u_1$, $w = u_1 - u_2$

take $-\Delta v = 0$, Ω; $v = 0$, $\partial\Omega$

$-w = 0$, Ω; $w = 0$, $\partial\Omega$

by weak max principle, $v \leq 0$, $w \leq 0$

$$0 \leq u_1 - u_2 \leq 0 \rightarrow u_1 - u_2 \equiv 0 \rightarrow u_1 \equiv u_2$$

10.26.3 separation of variables Laplace's eqn

separation of variables : Laplace eqn

consider Laplace's eqn on a disk:

$$\Delta u = 0 , \quad \vec{x} \in B(\vec{0}, a) \subset \mathbb{R}^2$$
$$u = f , \quad \vec{x} \in \partial B(\vec{0}, a)$$

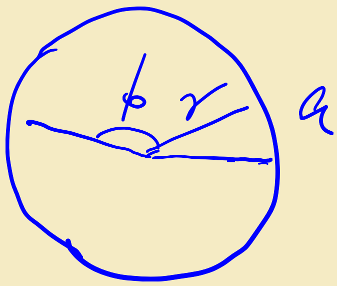
choose polar coordinates : $x = r\cos\phi$, $y = r\sin\phi$

"separate Laplace" $\Longrightarrow$ $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} = 0$

$$0 \le \phi \le 2\pi$$
$$0 < r < a$$

change of variables

$u(a,\phi) = f(\phi)$, $\quad 0 \le \phi < 2\pi$



periodic, $u(r,0) = u(r, 2\pi)$
$u_\phi(r,0) = u_\phi(r, 2\pi)$ (gives continuity & uniqueness)

$\rightarrow$ singularity at $r=0$, require bddness at $r=0$,

$$\lim_{r\to 0^+} u(r,0) = \text{bdd}$$

Let $u(r,\phi) = \mathcal{R}(r) H(\phi)$ $\xrightarrow[\text{PDE}]{\text{plug into}}$ $\dfrac{r^2 \mathcal{R}''(r) + \frac{1}{r}\mathcal{R}'(r)}{\mathcal{R}(r)} + \dfrac{H''(\phi)}{H(\phi)} = 0$

$$\frac{H''(\phi)}{H(\phi)} = -\frac{r^2\mathcal{R}''(r) + \frac{1}{r}\mathcal{R}'(r)}{\mathcal{R}} = -\lambda$$

taking $H''(\phi) = -\lambda H(\phi)$, solve for $\begin{matrix} H(0) = H(2\pi) \\ H'(0) = H'(2\pi) \end{matrix}$:

$$H_n(\phi) = \begin{cases} A_0/2, & \lambda_0 = 0 \\ A_n \cos(n\phi) + B_n \sin(n\phi), & \lambda_n = n^2 \geq 1 \quad n \in \mathbb{Z} \end{cases}$$

$\xrightarrow{\text{eigenvalues}}$ $\lambda_n = n^2$, $r^2\left(R''(r) + \frac{1}{r}R'(r)\right) = n^2 R(r)$, $0 < r < a$

for $n=0$: $r^2 R'' + rR' = 0 \Rightarrow R(r) = C_0 + D_0 \log(r)$

(Recall 2D fund. soln)

Apply BC at $r \to 0^+$: $D_0 = 0$ so $\lim_{r\to 0^+} R(r)$ finite log would blow up

$\Rightarrow R_0(r) = 1$

$n \geq 1$: try $R(r) = r^\alpha \Rightarrow \alpha(\alpha-1) + \alpha = n^2 \Rightarrow \alpha = \pm n$

for bdd soln, take $R_n(r) = r^n$, $n \geq 0$ (r^{-n} blow up)

harmonic solns: $u_n(r,\phi) = \begin{cases} r^n(A_n \cos(n\phi) + B_n \sin(n\phi)) & : n \geq 1 \\ A_0/2 & : n = 0 \end{cases}$

series soln: $u(r,\phi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$

BC: $u(a,\phi) = f(\phi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n\left(A_n \cos(n\phi) + B_n \sin(n\phi)\right)$

using orthogonality of sin & cos:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \begin{pmatrix} \cos(n\phi) \\ \sin n\phi \end{pmatrix} d\phi, \quad n \geq 0$$

10-28-1 Deriving the poisson kernel

seperation of vars: Laplace

$$\begin{cases} \Delta u = 0, & x \in B(0,a) \subseteq \mathbb{R}^2 \\ u = f, & x \in \partial B(0,a) \end{cases}$$

$\Rightarrow$ assume polar coordnates

$\Rightarrow$ assume $u(r,\phi) = R(r)H(\phi)$

$\Rightarrow$ series soln: $u(r,\phi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\phi) + B_n \sin(n\phi)\right)$

$\forall$ given BC $u(a,\phi) = f(\phi)$ $\quad \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \begin{pmatrix} \cos n\phi \\ \sin n\phi \end{pmatrix} d\phi$

$$u(r,\phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi') \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\cos(n\phi)\cos(n\phi') + \sin(n\phi)\sin(n\phi')\right]\right] d\phi'$$

(sub in A_n, B_n)

$$u(r,\phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi') \underbrace{\left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\phi - \phi')]\right]}_{\text{Diriclet kernel}}$$

we can show: $1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\phi - \phi'))$

$$= \frac{a^2 - r^2}{a^2 - 2ar\cos(\phi - \phi') + r^2}$$

$(r/a)^n \qquad \phi - \phi'$

$$1 + 2 \sum_{n=1}^{\infty} R^n \cos(n\theta) = \sum_{n=1}^{\infty} R^n \left(e^{in\theta} + e^{-in\theta}\right) + 1$$

$$= \sum_{n=1}^{\infty} \left[(Re^{i\theta})^n + (Re^{-i\theta})^n\right] + 1$$

$$\underset{\text{series}}{\overset{\text{Geometric}}{=}} \left(\frac{1}{1-\rho e^{i\theta}}\right) + \frac{1}{1-\rho e^{-i\theta}} - 1$$

$$= \frac{2-2\rho\cos\theta}{1-2\rho\cos\theta+\rho^2} - 1$$

$$= \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}$$

$\Rightarrow$ poisson's formula $u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{a^2-r^2}{a^2-2ar\cos(\phi-\phi')+r^2}}_{\text{poisson kernel}} f(\phi')d\phi'$

$$u(r,\phi) = P\left(\frac{r}{a},\cdot\right) * f(\phi)$$

where $P(r/a,\psi) = \frac{1}{2\pi}\frac{1-\left(\frac{r}{a}\right)^2}{1-2\left(\frac{r}{a}\right)\cos\psi+\left(\frac{r}{a}\right)^2}$

10_28_2 poisson kernel remarks Intro to Greens Functions

Remark

$\rightarrow$ setting $r=0$; $u(0,\phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi')d\phi'$ mean value property

$\rightarrow$ setting $f\equiv 1$: $u\equiv 1$ (by max/min principle) so

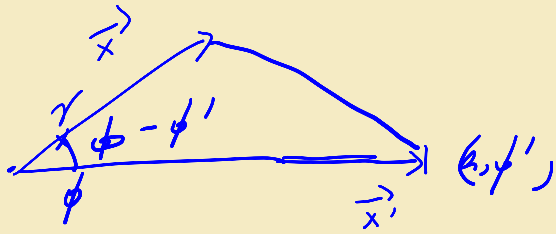
$$\underbrace{\int_0^{2\pi} P\left(\frac{r}{a},\varphi\right)d\varphi = 1}_{\text{normalized}}$$

→ poisson kernel is singular at $r=a$: $\lim_{(r,\phi)\to(a,\theta)} u(r,\phi) = f(\theta)$

→ Geometric interpretation:

$\vec{x} :=$ vector from $\vec{0}$ to (r,ϕ)

$\vec{x}' :=$ vector from $\vec{0}$ to (a,ϕ')



law of cosines

$(\vec{x}-\vec{x}') = r^2+a^2-2ar\cos(\phi-\phi')$

$a^2-r^2 = |\vec{x}'|^2 - |\vec{x}|$ (true for any dimension)

$$u(\vec{x}) = \int_{|\vec{x}|=a} \frac{|\vec{x}'|^2-|\vec{x}|^2}{|\vec{x}'-\vec{x}|^2} u(\vec{x}')\, d(\vec{x}'), \quad |\vec{x}|<a \quad (B(\vec{0},a) \subseteq \mathbb{R}^n)$$

→ poisson kernel is an example of Green's function

→ Green's fun can be convolved with boundary and/or forcing conditions to obtain solns to BVPs and IBVP(s).

→ IBVPs: Heat Eqn, foundation soln

→ BVPs: Laplace + poisson Eqn: posson kernel, etc

Green's functions

→ develop a method to solve PDE by integrating a fundamental soln aginst inhomogeneous.

Ex: $-\Delta u = f$, $\mathbb{R}^n$ has soln $u(\vec{x}) = \int_{\mathbb{R}^n} \Phi(\vec{x}-\vec{y}) f(\vec{y})\, d\vec{y}$

→ can we also incorporate BCs?

10-28-3 1D Poisson Equation

consider BVP: $-u''=f$, $\quad x\in(0,1)$

$u(0)=u(1)=0$

solve by integrating twice. $\quad -u'=\int_0^x f(y)\,dy + C$

$$\to -u(x)=\int_0^x\int_0^z f(y)\,dy\,dz + Cx + D$$

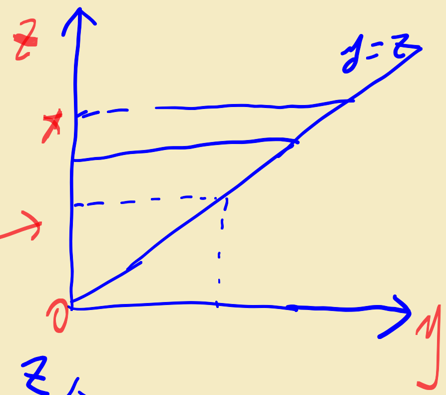
BCs: $u(0)=0=D$

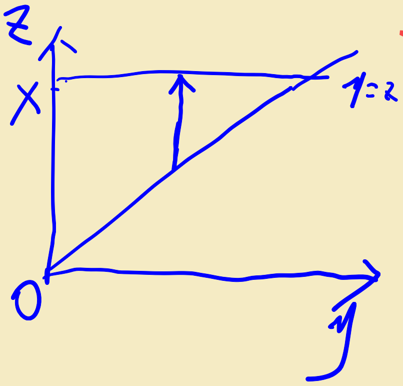
$u(1)=0$ gives eqn for C

first note: $u(x) = -\int_0^x\int_0^z f(y)\,dy\,dz - Cx$

$$= -\int_0^x\int_y^x f(y)\,dz\,dy - Cx$$

$$= -\int_0^x (x-y)f(y)\,dy - Cx$$





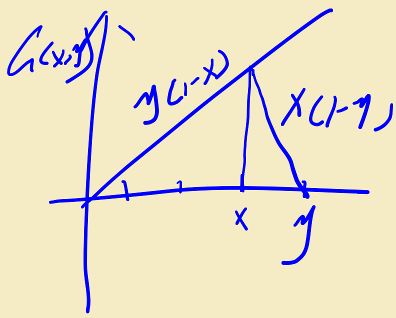
So $u(1)=0$, $\quad C=-\int_0^1(1-y)f(y)\,dy$

$$\to u(x)=\int_0^x (y-x)f(y)\,dy + \int_0^1 x(1-y)f(y)\,dy$$

$$=\int_0^x y(1-x)f(y)\,dy + \int_x^1 x(1-y)f(y)\,dy$$

$$=\int_0^1 G(x,y)f(y)\,dy, \quad 0\le x\le 1$$

$$G(x,y)=\begin{cases} y(1-x), & 0\le y\le x \\ x(1-y), & x\le y\le 1\end{cases}$$



$$\int_0^x (y-xy+xy-x)f(y)\,dy + \int_0^x x(1-y)f(y)\,dy + \int_x^1 x(1-y)f(y)\,dy$$

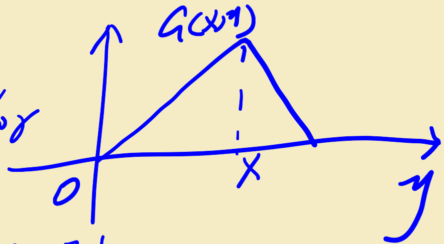
10-30-1-1 Green's function properties

Green's function

BVP: $-u'' = f, \quad x \in (0,1)$

$u(0) = u(1) = 0$

$$\Rightarrow u(x) = \int_0^1 G(x,y) f(y)\,dy \;:\; G(x,y) = \begin{cases} y(1-x), & 0 \le y \le x \\ x(1-y), & x \le y \le 1 \end{cases}$$



· $G(x,y)$ · defines an integral operator

$$\mathcal{G} : C[0,1] \to X = \{ u \in C[0,1] \mid u(0) = u(1) = 0 \}$$

$\mathcal{G}$ is the inverse of $\mathcal{L} = -\frac{d^2}{dx^2} : X \to C[0,1]$

$$(\mathcal{G} f)(x) = \int_0^1 G(x,y) f(y)\,dy$$

Note: Restrict domain of $\mathcal{L}$ to subspace of X with diffa fns

Terminology : $G(x,y)$ is the Green's function for $\begin{matrix} -u'' = f \\ u(0) = u(1) = 0 \end{matrix}$

properties of $G : \underset{x}{[0,1]} \times \underset{y}{[0,1]} \to \mathbb{R}$

(1) $G(x,y) \ge 0$ non-negative

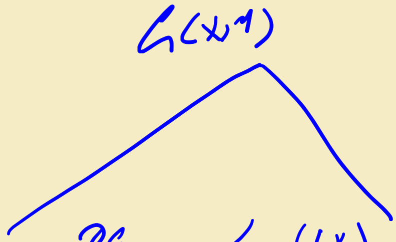
(2) $G(x,y) = G(y,x)$ symmetric

(3) $G(x,y)$ continuous in x & y

(4) G is differentiable except at $x = y$

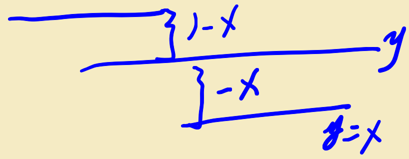
$$\left[\frac{\partial G}{\partial y}\right]_{x=y} = \lim_{x\to y^+}\frac{\partial G}{\partial y} - \lim_{x\to y^-}\frac{\partial G}{\partial y} = -1$$

jump



$$\frac{\partial G}{\partial y} = \begin{cases} (1-x), & y<x \\ -x, & x<y \end{cases}$$

$$\lim_{y\to x}\frac{\partial G}{\partial y} = \begin{cases} 1-x, \\ -x \end{cases}$$



These properties have counterparts for other differential operators L

Note : $\frac{\partial^2 G}{\partial y^2} = 0$ except when $x=y$

$$-\frac{\partial^2 G}{\partial y^2} = \delta(x-y) \quad (\delta(x) \text{ Dirac define fns})$$

$\rightarrow$ $\delta(x)$ assigns zero mass everywhere but $x=0$ where it has unit mass

10-30-1-2 Green's function Boundaries

$\rightarrow$ can relate $G(x,y)$ to fundamental soln

$\Phi(x,y) = -\frac{1}{2}|x-y|$ in 1D

$G(x,y) = -\frac{1}{2}|x-y| + \underbrace{\phi^y(x)}_{\text{Boundary correction}} \Rightarrow -\frac{d^2}{dx^2}\phi^y(x) = 0 \quad \forall y \in (0,1)$

Since $G(0,y) = G(1,y) = 0$ then $\phi^y(x) = -\Phi(x,y)$ at $x=0\ \forall 1$

Need $L\phi^y(x) = 0$, $x \in (0,1)$ and $\phi^y(0) = -\Phi(0,y)$, $\phi^y(1) = -\Phi(1,y)$

$\Rightarrow \phi^y(x) = A(y)x + B(y)$, $\phi^y(0) = B(y) = \frac{y}{2}$

$\Rightarrow \phi^y(1) = A(y) + \frac{y}{2} = \frac{1-y}{2} \rightarrow A(y) = \frac{1}{2} - y$

$\phi^y(x) = \frac{x+y}{2} - xy$

check: $G(x;y) = -\frac{1}{2}|x-y| + \frac{x+y}{2} - xy$

$x<y$: $G(x,y) = \frac{x-y}{2} + \frac{x+y}{2} - xy = x(1-y)$

$y<x$: $G(x,y) = \frac{y-x}{2} + \frac{x+y}{2} - xy = y(1-x)$

$u(x) = \int_0^1 G(x,y) f(y)\, dy$

what about inhom. BCs? $\begin{matrix} u(0) = a \\ u(1) = b \end{matrix}$, $-u'' = f$

Find soln u_i to $-u_i'' = 0$, $\begin{matrix} u(0) = a \\ u(1) = b \end{matrix}$ (BC soln)

$\rightarrow$ interestingly $\frac{\partial G}{\partial y} = \begin{cases} -x, & x<y \\ 1-x, & y<x \end{cases}$ and $\frac{d^2}{dx^2}\left(\frac{\partial G}{\partial y}\right) = 0$ almost everywhere

also $\frac{\partial G}{\partial y}(0,1) = -\frac{\partial G}{\partial y}(1,0) = 0$

whereas $\frac{\partial G}{\partial y}(1^-, 1) = -\frac{\partial G}{\partial y}(0^+, 0) = -1$

$$\Rightarrow u(x) = -b\frac{\partial G}{\partial y}(x,1) + a\frac{\partial G}{\partial y}(x,0) + \int_0^1 G(x,y)f(y)\,dy$$

$$= a + x(b-a) + \int_0^1 G(x,y)f(y)\,dy \qquad \begin{matrix} u(0)=a \\ u(1)=b \end{matrix} \text{ as expected.}$$

$$=$$

$\longrightarrow$ can get BCs satisfied by taking terms involving over Green's fn.

Acting approach: (a) $-u''(y) = 0 \quad y \in (0,1)$
(b) $-G_{yy}(x,y) = \delta(x-y) \quad y \in (0,1)$

multiply (a) by G, (b) by u: subtract & integrate

$$\Rightarrow \int_0^1 [-Gu'' + uG_{yy}]\,dy = -\int_0^1 \delta(x-y)u(y)\,dy = -u(x)$$

$$\Rightarrow u(x) = \int_0^1 [uG_y - u'G]'\,dy = u(1)G_y(x,1) - u'\overset{0}{G(x,1)} - u(0)G_y(x,0)$$
$$+ u'\underset{0}{G(x,0)}$$

$u(x) = u(1-x) + bx$

gives soln satisfying BCs

well-posedness?
existence

uniqueness : u_1, u_2 solves $-u'' = f$
$u(0) = a,\ u(1) = b$

$w = u_1 - u_2$: $\quad -w'' = 0$
$w(0) = w(1) = 0$

$$w = \int_0^1 G(x,y)\, 0\, dy + 0(1-x) + 0x = 0$$

continuous dependence $\quad \|f\|_{L^\infty(0,1)} = \sup_{x\in(0,1)} |f(x)|$,

$$\|f\|_{L^1(0,1)} = \int_0^1 |f(x)|\, dx$$

suppose u_1, u_2 solve BVP $-u_{1,2}'' = f_{1,2}$ then

$$|u_1 - u_2| = \left| \int_0^1 G(x,y)\,[f_1(y) - f_2(y)]\, dy \right| \le \int_0^1 |G(x,y)|\, |f_1 - f_2|\, dy$$
$$\le \frac{1}{4} \|f_1 - f_2\|_\infty$$

10-30-3 Test functions and distributions

Test f-ns & Distributions

- consider smooth test fns on $\mathbb{R}$, $\phi \in D = C_c^\infty(\mathbb{R})$ (compact support)
- a sequence $\{\phi_n\}$ convergences to $\phi \in D$ if
  1. supp $\phi_n \subset K$ and supp $\phi \subset K$ for $K \subset \mathbb{R}$ closed + bdd
  2. $\phi_n^{(j)} \to \phi^{(j)}$ as $n \to \infty$ uniformly on K. $j = 0, 1, 2, 3..$
  call derivitive

that is $\sup_{x \in K} \left| \phi_n^{(j)}(x) - \phi^{(j)}(x) \right| \to 0$ as $n \to \infty$

Distributions: · a distribution f acts on a test fn ϕ to produce a number $f(\phi)$

$f : D \to \mathbb{R}$ has this properties

1) $f(a\phi_1 + b\phi_2) = af(\phi_1) + bf(\phi_2)$, $a, b \in \mathbb{R}$, $\phi_1, \phi_2 \in D$ (linearity)

2) $\phi_n \to \phi$ in D implies $f(\phi_n) \to f(\phi)$ continuity

· we also denote $f(\phi)$ by (f, ϕ)

· note the space D' of distributions is the dual of D

· Some functionals are nonlinear and thus not distributions:

$(f, \phi) = \phi(0)^2$ is not a distribution.

Ex of distributions: for $\phi \in D$

1) $(f_1, \phi) = \int_{\mathbb{R}} \phi(x) dx$

2) $(f_2, \phi) = \int_0^\infty \phi(x) dx$ Heaviside

3) $(f_3, \phi) = (\delta, \phi) = \phi(0)$ delta

4) $(f_4, \phi) = \phi'(0)$

show a own these are linear & continuous

$\to$ don't confuse the generalize fn with the distribution

(1) is associated w genera fn $\tilde{f}_1 = 1$ (not integrable)

(2) is associated with $\tilde{f}_2(x) = H(x)$

3

11-4-1 properties of Distributions

Distributions

$\to$ a distribution f acts on a test function ϕ to produce a number $f(\phi)$, or (f, ϕ)

1) $f(a\phi_1 + b\phi_2) = a f(\phi_1) + b f(\phi_2)$ linearity

2) $\phi_n \to \phi$ in D (space of test fns)

$\Rightarrow f(\phi_n) \to f(\phi)$ (continuity)

Ex: $(\delta, \phi) = \phi(0)$; $(H, \phi) = \int_0^\infty \phi(x)\,dx$

properties:

1) convergence: we say $f_k \to f$ in D' (space of distribution, dual of D) in the sense of distributions if $(f_k, \phi) \to (f, \phi)$ as $k \to \infty$ $\forall \phi \in D(\mathbb{R}^n)$

Ex: heat kernel $\Phi(x,t) \to \delta(x)$ as $t \to 0^+$

2. Derivative: If $\tilde{f} \in C^1(\mathbb{R}^n)$, then $\left(\frac{\partial f}{\partial x_i}, \phi\right) = \int_{\mathbb{R}^n} \frac{\partial \tilde{f}}{\partial x_i}(\vec{x})\phi(\vec{x})\,d\vec{x}$

$\overset{IBP}{=} -\int_{\mathbb{R}^n} \tilde{f}(x) \frac{\partial \phi}{\partial x_i}\,d\vec{x}$

$= -\left(f, \frac{\partial \phi}{\partial x_i}\right)$

in general:

$$\left(\frac{\partial f}{\partial x_i}, \phi\right) = -\left(f, \frac{\partial \phi}{\partial x_i}\right) \quad \forall \phi \in D$$

Ex: $(H'(x), \phi) = -(H, \phi') = -\int_0^\infty \phi'(x)\,dx = \phi(0) = (\delta, \phi)$

$\Rightarrow H' = \delta$

3) translation: translating $f \in D'$ by $\vec{y} \in \mathbb{R}^n$ acts to shift test fn $\phi \in D$ by $-\vec{y} \in \mathbb{R}^n$:

$$(f^{\vec{y}}, \phi) = \int_{\mathbb{R}^n} f(\vec{x} - \vec{y})\phi(\vec{x})\,d\vec{x} = \int_{\mathbb{R}^n} f(\vec{z})\phi(\vec{z} + \vec{y})\,d\vec{y}$$

Ex: $(\delta(\vec{x} - \vec{y}), \phi(\vec{x})) = \phi(\vec{y})$

4) scaling: Take $c \in C^\infty$, $cf \in D'$ is defined:

$$(cf, \phi) = (f, c\phi)$$

Ex: $(x\delta, \phi) = (\delta, x\phi) = 0$

11-4-2 shock waves Rankie-Hugoniot

Ex: shock waves are distributional solns to PDE

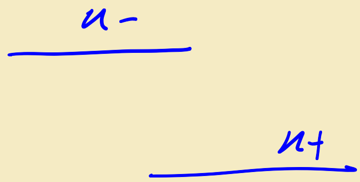
$u_t + f(u)_x = 0$ in the sense of distributions

1): $(u_t, \phi) + (f(u)_x, \phi) = 0 \quad \phi \in D(\mathbb{R}^2)$

$(u, \phi_t) + (f(u), \phi_x) = 0$

consider a shock (jump discontinuity):

$$u(x,t)=\begin{cases} u_-, & x<st \\ u_+, & x>st \end{cases}$$



$s\in\mathbb{R}$

then $f(u)=\begin{cases} f(u_-), & x<st \\ f(u_+), & x>st \end{cases}$

rewrite using Heavisite dst: $u(x,t)=(u_+-u_-)H(x-st)+u_-$

$f(u)=(f(u_+)-f(u_-))H(x-st)+f(u_-)$

thus: $u_t=(u_+-u_-)H'(x-st)(-s)$

$$f(u)_x=(f(u_+)-f(u_-))H'(x-st) \qquad (H'=\delta)$$

$$\Rightarrow (u_t,\phi)+(f(u)_x,\phi)=(-sH'(x-st)(u_+-u_-),\phi)+((f(u_+)-f(u_-))H'(x-st),\phi)$$

$$=(-s(u_+-u_-)+(f(u_+)-f(u_-)))\phi(st)$$

$$=0$$

$\Rightarrow -s(u_+-u_-)+f(u_+)-f(u_-)=0$ (Rankine-Hugoniot condition)

$$s=\frac{f(u_+)-f(u_-)}{u_+-u_-}$$ (relates shock speed to shock jump)

4

11-4-3

General framework of Green's functions

consider the BVP:
$$Lu = f, \quad \vec{x} \in U, \qquad U \subset \mathbb{R}^n$$
$$Bu = g, \quad \vec{x} \in \partial U, \qquad u: U \to \mathbb{R}$$

PDE

BC

L: Linear operator

Bu: Linear combination of u & derivative

e.g. $Bu = u(\vec{x})$

$Bu = \frac{\partial u}{\partial n}(\vec{x})$

$Bu = u(\vec{x}) + \alpha\frac{\partial u}{\partial n}(\vec{x})$

→ fundamental soln of BVP:

$\Phi(\vec{x}, \vec{\eta})$ satisfies $L\Phi(\vec{x}, \vec{\eta}) = \delta(\vec{x} - \vec{\eta})$, $\vec{x}, \vec{\eta} \in \mathbb{R}^n$ in sense of distributions

→ If Φ locally integrable in $\vec{\eta}$ for $\vec{x} \in \mathbb{R}^n$ and $f \in D(\mathbb{R}^n)$ then:

$v(\vec{x}) = \int_{\mathbb{R}^n} \Phi(\vec{x}, \vec{\eta}) f(\vec{\eta}) d\vec{\eta} = (\Phi, f)$ satisfies $Lv = f$ on $\mathbb{R}^n$

$\Rightarrow Lv(\vec{x}) = L(\Phi, f) = (L\Phi, f) = (\delta(\vec{x} - \vec{\eta}), f(\vec{\eta})) = f(\vec{x})$

- to satisfy BC add a soln w to v

$\Rightarrow Lw = 0$ s.t. $u = v + w$ satisfies $Bu = g$

$\Rightarrow$ $Lw = 0$, $\vec{x} \in U$ so. $Lu = Lv + Lw = f$

$Bw = g - Bv$, $\vec{x} \in \partial U$ $\quad Bu = Bv + g - Bv = g$

$\Rightarrow$ can get clear formulation by removing the fundamental soln contribution to the boundary use Green's function.

define $G(\vec{x}, \vec{\eta})$ on $\vec{x}, \vec{\eta} \in \bar{U}$

$$L G(\vec{x}, \vec{\eta}) = \delta(\vec{x} - \vec{\eta}), \quad \vec{x} \in U$$
$$B G(\vec{x}, \vec{\eta}) = 0, \quad \vec{x} \in \partial U$$
for $\vec{\eta} \in \bar{U}$

then $G(\vec{x}, \vec{\eta}) = \Phi(\vec{x}, \vec{\eta}) - \phi^{\vec{\eta}}(\vec{x})$: $L\phi^{\vec{\eta}}(\vec{x}) = 0, \quad \vec{x} \in U$
$B\phi^{\vec{\eta}}(\vec{x}) = B\Phi(\vec{x}, \vec{\eta}), \quad \vec{x} \in \partial U$

Thus, instead define $u = v + w$ s.t. $v(\vec{x}) = \int_U G(\vec{x}, \vec{\eta}) f(\vec{\eta}) d\vec{\eta}$

and $Lw = 0, \quad \vec{x} \in U$
$Bw = g, \quad \vec{x} \in \partial U$

Φ : fundamental soln, $\phi^{(\vec{\eta})}(\vec{x})$ gives homo. boundary

w : if inhomo. boundary.

$\rightarrow$ in most case, estimates of $G(\vec{x}, \vec{\eta})$ are made more easily than computing u

$\rightarrow$ in special cases, G can be found explicitly as well as w

11-6-1 Green's Function setup

Green's Functions: General Framework

General BVP: $Lu = f, \ \vec{x} \in U \subset \mathbb{R}^n$
$Bu = g, \ \vec{x} \in \partial U$

Define $u = v + w$ s.t. $v(\vec{x}) = \int_U G(\vec{x}, \vec{\eta}) f(\vec{\eta}) d\vec{\eta}$

and $Lw = 0, \ \vec{x} \in U$
$Bw = g, \ \vec{x} \in \partial U$, $LG(\vec{x}, \vec{\eta}) = \delta(\vec{x}, \vec{\eta}), \ \vec{x} \in U, \ \vec{\eta} \in \bar{U}$
$BG(\vec{x}, \vec{\eta}) = 0, \ \vec{x} \in \partial U$

$\Rightarrow G(\vec{x}, \vec{\eta}) = \Phi(\vec{x}, \vec{\eta}) - \phi^{\vec{\eta}}(\vec{x})$; $L\phi^{\vec{\eta}}(\vec{x}) = 0, \ \vec{x} \in U$
$B\phi^{\vec{\eta}}(\vec{x}) = B\Phi(\vec{x}, \vec{\eta}), \ \vec{x} \in \partial U$

Green's fn for Laplacian

Consider poisson eqn: $-\Delta u = f, \ \vec{x} \in U$ $\quad U$: open & bdd, ∂U: pw smooth
$u = g, \ \vec{x} \in \partial U \quad f, g \in C$

Recall $\Delta u = 0, \ \vec{x} \in \mathbb{R}^n$ has fund soln

$$\Phi(\vec{x}, \vec{\eta}) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\vec{x} - \vec{\eta}|^{n-2}}, \quad n \geq 3$$

nonzero in general on ∂U for $U \subseteq \mathbb{R}^n$

$\rightarrow$ so define $G(\vec{x}, \vec{\eta}) = \Phi(\vec{x}, \vec{\eta}) - \phi^{\vec{\eta}}(\vec{x})$

$\rightarrow$ need to solve $\Delta \phi^{\vec{\eta}}(\vec{x}) = 0, \ \vec{x} \in U$
$\phi^{\vec{\eta}}(\vec{x}) = \Phi(\vec{x} - \vec{\eta}), \ \vec{x} \in \partial U$ (harmonic on U)

→ and for $u = \int_U G(\vec{x}, \vec{\eta}) f(\vec{\eta}) d\vec{\eta} + w \vec{x}$

$w(\vec{x})$ satisfies $\Delta w = 0, \ \vec{x} \in U$
$w = g, \ \vec{x} \in \partial U$

→ to find $\phi^{\vec{\eta}}(\vec{x})$, we exploit symmetry $\phi^{\vec{x}}(\vec{\eta}) = \phi^{\vec{\eta}}(\vec{x})$

from $\Phi(\vec{x}, \vec{\eta})$ sym

11-6-2 poisson Eqn Green's fn soln.

$-\Delta u = f$ in U
$u = g$ in ∂U

Thm: If $u \in C^2(\bar{U})$ solves poisson eqn then

$$u(\vec{x}) = \int_U G(\vec{x}, \vec{\eta}) f(\vec{\eta}) d\vec{\eta} - \int_{\partial U} \frac{\partial G}{\partial n_y}(\vec{x}, \vec{\eta}) g(\vec{\eta}) dS_{\vec{\eta}}$$

Pf: The 1st integral solves

$-\Delta u = f, \ U$
$u = 0, \ \partial U$

since $-\Delta(G, f) = (\delta, f) = f$
$(G, f) = 0, \ \vec{x} \in \partial U$

by construction

2nd integral is harmonic since $G(\vec{x}, \vec{\eta})$ harmonic in $\vec{x} \in U, \forall \vec{y} \in \partial U$

and G is only modified by differentiating & integrating (linear ops)

→ to show satisfies BC, demonstrate $-\frac{\partial G}{\partial n_y} \to \delta(\vec{x} - \vec{\eta})$
as $\vec{x} \to \partial U$

→ will use Divergence Thm

→ to deal w/ singularity in $G(\vec{x}, \vec{\eta})$ at $\vec{y} = \vec{x}$ define

$V_\varepsilon = U \setminus B(\vec{x}, \varepsilon)$

So for any $u \in C^2(\bar{U})$ then $\displaystyle\int_{V_\varepsilon} u(\vec{y}) \Delta \Phi(\vec{x}, \vec{y}) - \Phi(\vec{x}, \vec{y}) \Delta u(\vec{y}) \, d\vec{y}$

$$= \int_{\partial V_\varepsilon} \left(u(\vec{y}) \frac{\partial \Phi(\vec{x}, \vec{y})}{\partial n_y} \right) - \Phi(\vec{x}, \vec{y}) \frac{\partial u(\vec{y})}{\partial n_y} \, ds_y \quad \text{(by div thm)}$$

→ $\Phi(\vec{x}, \vec{y})$ harmonic when $\vec{y} \neq \vec{x}$ so 1st term on LHS is zero

→ boundary has two parts, ∂U and $\partial B(\vec{x}, \varepsilon)$, will show contributions from $\partial B(\vec{x}, \varepsilon)$ go to zero as $\varepsilon \to 0$

Let $\displaystyle I_\varepsilon = \int_{\partial B(\vec{x}, \varepsilon)} u(\vec{y}) \frac{\partial \Phi}{\partial n_y}(\vec{x}, \vec{y}) \, ds_y \longrightarrow u(\vec{x})$ as $\varepsilon \to 0$ (behaves like an avg as in poisson eqn proof)

Let $\displaystyle J_\varepsilon = \int_{\partial B(\vec{x}, \varepsilon)} \Phi(\vec{x}, \vec{y}) \frac{\partial u}{\partial n_y} \, ds_y \longrightarrow N \varepsilon \log \varepsilon \; : \; n = 2$ $\longrightarrow 0$

$\varepsilon \; : \; n \geq 3$

as $\varepsilon \to 0$

So as $\varepsilon \to 0$: (1) $\displaystyle -\int_U \Phi(\vec{x}, \vec{y}) \Delta u(\vec{y}) \, d\vec{y} = u(\vec{x}) + \int_{\partial U} \left(u(\vec{y}) \frac{\partial \Phi}{\partial n_y}(\vec{x}, \vec{y}) - \Phi(\vec{x}, \vec{y}) \frac{\partial u}{\partial n_y} \right) ds_y$

now using $\phi^{\vec{x}}(\vec{y}) = \Phi(\vec{x}, \vec{y})$ and $\phi^{\vec{x}}(\vec{y})$ has no singularity at $\vec{x} = \vec{y}$

$\to$ Going through calculating again with $\phi^x(\vec{\eta})$, we get $I_\varepsilon \to 0$

(no singularity)

so $-\int_U \phi^x(\vec{\eta})\Delta u(\vec{\eta})\,d\vec{\eta} = \int_{\partial U}\left(u(\vec{\eta})\frac{\partial \phi^x}{\partial \nu_\eta}(\vec{\eta}) - \Phi(\vec{x},\vec{\eta})\frac{\partial u}{\partial \nu_\eta}(\vec{\eta})\right) dS_\eta$ (2)

Substract (2) from (1): $-\int_U G(\vec{x},\vec{\eta})\Delta u(\vec{\eta})\,d\vec{\eta} = u(\vec{x}) + \int_{\partial U} u(\vec{\eta})\frac{\partial G}{\partial \nu_\eta}(\vec{x},\vec{\eta})\,dS_\eta$

when $-\Delta u = f$ on U

$u = g$ on ∂U

$$\Rightarrow u(\vec{x}) = \int_U G(\vec{x},\vec{\eta}) f(\vec{\eta})\,d\vec{\eta} - \int_{\partial U}\frac{\partial G}{\partial \nu_\eta}(\vec{x},\vec{\eta})\, g(\vec{\eta})\,dS_\eta$$

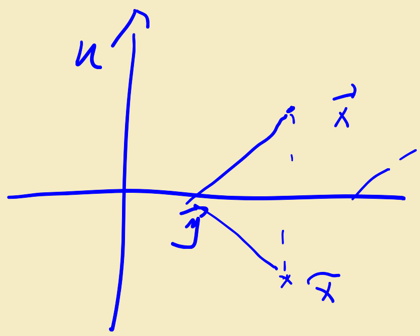
11-6-3 Method of Images

method of images

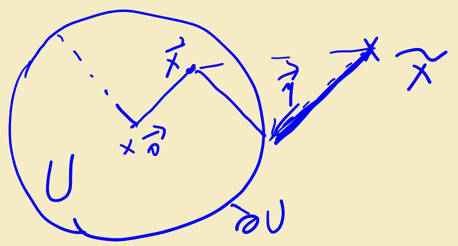
$\to$ Green's function for $-\Delta u$ can be dirrectly calculated

from $\Phi(\vec{x},\vec{\eta})$ for same domain $U \subset \mathbb{R}^n$ approach

(half $-\mathbb{R}^n$)



unit ball



· construct "image points" $\tilde{x}$

outside of U for each $\vec{x} \in U$

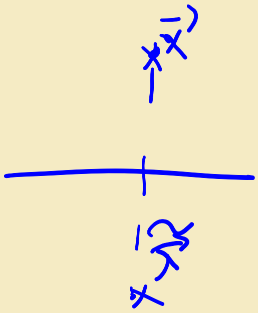
$\cdot\ \Phi(c_{\vec{x}}(\tilde{x}-\vec{y}))$ cancel $\Phi(\vec{x}-\vec{y})$ on ∂U

c_x : scalar factor

$$\therefore G(\vec{x},\vec{y}) = \Phi(\vec{x}-\vec{y}) - \underbrace{\Phi(c_x(\tilde{x}-\vec{y}))}_{\text{harmonic in } \vec{y} \in U}$$

Ex (Half-space)

Let $U = \{\vec{x} \in \mathbb{R}^n \mid x_n > 0\}$, $\vec{x} = (x_1, \dots x_n) \in U$



$\rightarrow$ define image $\tilde{x} = (x_1, \dots x_{n-1}, -x_n) \notin \bar{U}$

(map all $x \in U$ to $\tilde{x} \notin \bar{U}$)

$\rightarrow$ How to chose c_x ?

Require $\Phi(\vec{x}-\vec{y}) - \Phi(c_{\vec{x}}(\tilde{x}-\vec{y})) = 0$, $\vec{x} \in U$, $\vec{y} \in U$

match metrics : $|\vec{x}-\vec{y}| = \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}$

$= |c_x| \sqrt{\sum_{j=1}^{n}(\tilde{x}_j - y_j)^2}$

$= |c_x(\tilde{x}-\vec{y})|$

$y_n = 0$: $\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2 + x_n^2} = |c_x| \sqrt{\sum_{j=1}^{n-1}(\tilde{x}_j - y_j)^2 + \tilde{x}_n}$

$\Rightarrow c_{\vec{x}} = \pm 1$, $\Phi(\vec{x} - \vec{\eta}) = \Phi(\tilde{x} - \vec{\eta})$ on $\vec{x} \in U$, $\vec{\eta} \in \partial U$

So $G(\vec{x}, \vec{\eta}) = \Phi(\vec{x} - \vec{\eta}) - \Phi(\tilde{x} - \vec{\eta})$ (Recall heat eqn w/ Dirichlet BC a half line)

$$-\Delta_{\vec{\eta}} G(\vec{x}, \vec{\eta}) = \delta(\vec{x}, \vec{\eta}) \quad \vec{x}, \vec{\eta} \in U$$

$$G(\vec{x}, \vec{\eta}) = 0, \quad \vec{x} \in U, \vec{\eta} \in \partial U$$

$\rightarrow$ since $\tilde{x} \notin \bar{U}$, $\underbrace{\Phi(\tilde{x} - \vec{\eta})}_{\phi^{\vec{\eta}}(\vec{x})}$ harmonic in $\vec{\eta} \in U$

11-9-1 Green's function Real line

Green's function

Find the Green's fn for $-u'' + q^2 u = f(x)$ $\quad x \in \mathbb{R}$

$$\lim_{x \to \pm\infty} u(x) = 0 \qquad q > 0$$

Reminder: Green's fn is continuous, solves problem with unit δ source at a point

soln: $G(x, \eta)$ solves $-G_{xx} + q^2 G = \delta(x - \eta)$ $\quad x, \eta \in \mathbb{R}$

$$\lim_{x\to\pm\infty} G(x,y) = 0$$

Define: $z = x-y \Rightarrow -G_{zz}+q^2G = \delta(z)$, $\lim_{z\to\pm\infty} G(z+y,y)=0 \quad \forall y\in\mathbb{R}$

$\rightarrow$ solve independent of y & shift.

break domain in two $(z<0,\ z>0)$: $z<0$: $-G_{zz}+q^2G=0$ $\quad z\to-\infty$, $G\to 0$

$$\Rightarrow G(z) = A_-e^{qz}+B_-e^{-qz} \Rightarrow B_-=0$$

$z>0$: $-G_{zz}+q^2G=0 \Rightarrow G(z) = A_+ + B_+e^{-qz}$, $z\to\infty$, $G\to 0$

$A_+=0$

$$G(z) = \begin{cases} A_-e^{qz} & : z<0 \\ B_+e^{-qz} & : z>0 \end{cases}$$

: continuity: $A_- = B_+ = A$

$$\Rightarrow G(z) = Ae^{-q|z|}$$

Require: $\displaystyle\int_{-\varepsilon}^{\varepsilon}(-G_{zz}+q^2G)\,dz = \int_{-\varepsilon}^{\varepsilon}\delta(z)\,dz = 1$

for ε small: $\Rightarrow \overline{[G_z]}^{\text{jump}}_{z=0} = -1 \Rightarrow -qA - qA = -1$

$$\Rightarrow A = \frac{1}{2q} \Rightarrow G(z) = \frac{e^{-q|z|}}{2q}$$

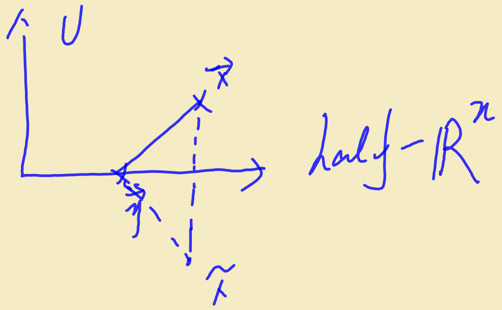
$$\Rightarrow G(x,y) = \frac{e^{-q(x-y)}}{2q}$$

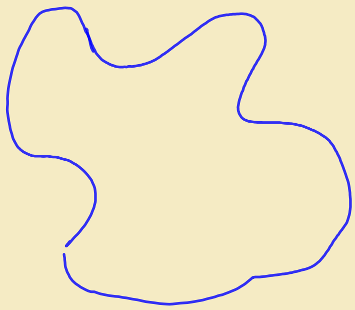
$$u(x) = \int_{-\infty}^{\infty} G(x,y)f(y)\,dy = \int_{-\infty}^{\infty} \frac{e^{-q(x-y)}f(y)}{2q}\,dy$$

$$-\int_{-\varepsilon}^{\varepsilon} G_{zz}\,dz = 1 \Rightarrow -\underbrace{\left[\lim_{z\to 0^+}G_z - \lim_{z\to 0^-}G_z\right]}_{\text{jump } [G_z]_{z=0}} = -1$$

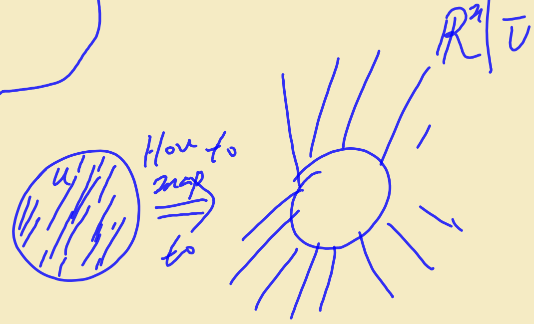
11-9-2 Method of Images Unit Ball

Method of Images





EX (unit Ball): Let $U = B(\vec 0, 1) \subset \mathbb{R}^n$



How to define $\tilde x \notin \bar U$?

give $\vec x \in U$

$$\tilde x = \frac{\vec x}{|\vec x|^2}$$

Green's fn for Laplacian on $B(\vec 0, 1)$ is defined:

$$G(\vec x, \vec \eta) = \Phi(\vec x - \vec \eta) - \Phi\big(|x|(\tilde x - \vec y)\big)$$

Note that $|\tilde x| = \frac{|\vec x|}{|\vec x|^2} = \frac{1}{|\vec x|}$ to check

$G(\vec x, \vec \eta) = 0$ for $\vec x \in U$, $\vec \eta \in \partial U$ need to show

$|\vec x - \vec y| = |\vec x|\,|\tilde x - \vec y|$ for $0 < |\vec x| < 1$, $|\tilde x| = \frac{\vec x}{|\vec x|^2}$ $|\vec y| = 1$

$$\Rightarrow |\vec{x}|^2\,|\tilde{x}-\vec{\eta}|^2 = \left|\frac{\vec{x}}{|\vec{x}|} - |\vec{x}|\,\vec{\eta}\right|^2 = |\vec{\eta}|^2|\vec{x}|^2 - 2\vec{\eta}\cdot\vec{x} + 1$$

b/c $|\vec{\eta}|=1$: $\quad = |\vec{x}-\vec{\eta}|^2$

$\therefore G(\vec{x},\vec{\eta}) = 0 \quad \vec{x}\in U,\ \vec{\eta}\in\partial U$

$\Rightarrow$ also $-\Delta G(\vec{x},\vec{\eta}) = \delta(\vec{x}-\vec{\eta})$ since

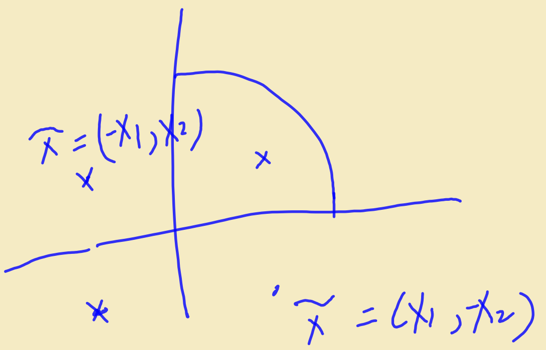
$$-\Delta\Phi(\vec{x}-\vec{\eta}) = \delta(\vec{x}-\vec{\eta}) \quad (\text{on } \vec{x},\vec{\eta}\in U)$$

and $-\Delta\Phi(|\vec{x}|(\tilde{x}-\vec{\eta}))=0$ b/c $\tilde{x}\notin \bar{U},\ \vec{\eta}\in U$

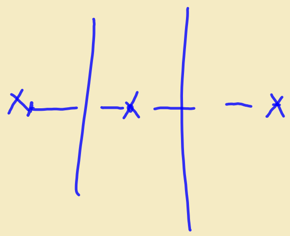
$\therefore G(\vec{x},\vec{\eta})$ is the Green's function for $B(\vec{0},1)$ with $G(\vec{x},\vec{\eta})=0$,

$|\vec{\eta}|=1$

$\vec{x}\in B(\vec{0},1)$



$x_B = \frac{\vec{x}}{|\vec{x}|^2}$



11.5-3 Laplace Eqn unit Ball

Solving BVP using Green's fn on Ball

Assume $\Delta u = 0,\ \vec{x}\in B(\vec{0},1)$

$u = g,\ \vec{x}\in\partial B(\vec{0},1)$

using soln to poisson's eqn: $u(\vec{x}) = -\int_{\partial B(\vec{0},1)} g(\vec{y}) \frac{\partial G}{\partial n}(\vec{x},\vec{y})\, dS_y$

Recall $G(\vec{x},\vec{y}) = \Phi(\vec{y}-\vec{x}) - \Phi\left(|\vec{x}|(\vec{y}-\tilde{x})\right)$, $\tilde{x} = \frac{\vec{x}}{|\vec{x}|^2}$

$$\frac{\partial G}{\partial y_j}(\vec{x},\vec{y}) = \frac{\partial \Phi}{\partial y_j}(\vec{y}-\vec{x}) - \frac{\partial}{\partial y_j}\Phi\left(|\vec{x}|(\vec{y}-\tilde{x})\right)$$

$\frac{\partial \Phi}{\partial y_j}(\vec{y}-\vec{x}) = \frac{1}{n\alpha(n)} \frac{x_j - y_j}{|\vec{x}-\vec{y}|^n}$ and $\frac{\partial \Phi}{\partial y_j} \left| |\vec{x}|(\vec{y}-\tilde{x}) \right|$

$$= -\frac{1}{n\alpha(n)} \frac{y_j|\vec{x}|^2 - x_j}{|\vec{x}|\,|\vec{y}-\tilde{x}|^n}$$

$$\left(\frac{\partial G}{\partial y_1}, \frac{\partial G}{\partial y_2}, \dots, \frac{\partial G}{\partial y_n}\right)$$

$$= -\frac{1}{n\alpha(n)} \frac{y_j|\vec{x}|^2 - x_j}{|\vec{x}-\vec{y}|^n} \quad (\vec{y}\in\partial B)$$

So

$$\frac{\partial G}{\partial n_y}(\vec{x},\vec{y}) = \sum_{j=1}^{n} y_j \frac{\partial G}{\partial y_j}(\vec{x},\vec{y})$$

$$(0, 0, \dots, 1)$$

$$= -\frac{1}{n\alpha(n)} \frac{1}{|\vec{x}-\vec{y}|^n} \sum_{j=1}^{n} y_j \left((y_j - x_j) - y_j|\vec{x}|^2 + x_j\right)$$

$$= -\frac{1}{n\alpha(n)} \frac{1-|\vec{x}|^2}{|\vec{x}-\vec{y}|^n}$$

So $u(\vec{x}) = \frac{1-|\vec{x}|^2}{n\alpha(n)} \int_{\partial B(\vec{0},1)} \frac{g(\vec{y})}{|\vec{x}-\vec{y}|^n}\, dS_y$ $\rightarrow$ change coords to polar reveals this is the poisson kernel integrated with G.

11-11-1 method of Images Neumann Boundary

Neumann BCs in poisson Eqn

$$\Delta u = f, \quad \vec{x} \in \Omega \equiv \mathbb{R}^2 \cap \{x_1 > 0\}$$

$$\frac{\partial u}{\partial n} = 0, \quad \vec{x} \in \partial\Omega \equiv \{\vec{x} \in \mathbb{R}^2 \mid x_1 = 0\}$$

→ If this were Dirichlet problem ⇒ odd reflection

→ Neumann BC ⇒ even reflection (recall heat eq)

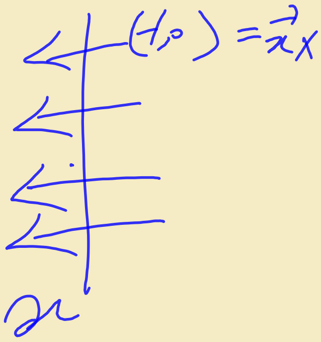
Green's fn: $\Delta G = \delta(\vec{x} - \vec{\eta}), \ \Omega$

$$\frac{\partial G}{\partial n} = 0, \quad \partial\Omega$$

Even reflection of fund. soln:

$$G(\vec{x}, \vec{\eta}) = \Phi(\vec{x} - \vec{\eta}) + \Phi(\hat{x} - \vec{\eta}), \quad \hat{x} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$$

$$\frac{\partial G}{\partial n_x} = \frac{\partial \Phi}{\partial n_x}(\vec{x} - \vec{\eta}) + \frac{\partial \Phi}{\partial n_x}(\hat{x} - \vec{\eta})$$



$$\frac{\partial \Phi}{\partial n_x} = (-1, 0) \cdot \nabla \Phi(\vec{x} - \vec{\eta})$$

$$\frac{\partial \Phi(\hat{x} - \vec{\eta})}{\partial n_x} = (-1, 0) \cdot \nabla \Phi(\hat{x} - \vec{\eta})$$

where $\frac{\partial}{\partial x_1}\left[\Phi\left(\underset{\hat{x}}{\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}} - \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\right)\right] = -\frac{\partial \Phi}{\partial x_1}\left(\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\right)$

chain rule

where $x_1 = 0$, $\vec{x} = \hat{x}$, $\dfrac{\partial G}{\partial n_x} = \dfrac{\partial \Phi}{\partial x_1}\left(\begin{pmatrix}0\\x_2\end{pmatrix} - \begin{pmatrix}y_1\\y_2\end{pmatrix}\right)$

$$+\frac{\partial \Phi}{\partial x_1}\left(\begin{pmatrix}0\\x_2\end{pmatrix} - \begin{pmatrix}y_1\\y_2\end{pmatrix}\right)$$

$$= 0$$

11-11-2 poisson eqn Neumann Boundary

consider $-\Delta u = f$, $\vec{x} \in U$

$-\dfrac{\partial u}{\partial n} = g$, $\vec{x} \in \partial U$

→ In this case of Neumann BCs, not all BVPs are solvable; need a solvability condition (compatibility condition)

$$\int_U f\, d\vec{x} = \int_{\partial U} g\, ds_x \quad \text{(compatibility condition)}$$

to show: note $\displaystyle\int_U f\, d\vec{x} = -\int_U \Delta u\, d\vec{x} = -\int_{\partial U} \frac{\partial u}{\partial n} ds = \int_{\partial U} g\, ds_{\vec{x}}$

Green's function satisfies: $\begin{cases} -\Delta G = \delta(\vec{x}-\vec{y}), & \vec{x} \in U \\ -\dfrac{\partial G}{\partial n} = 0, & \vec{x} \in \partial U \end{cases}$, $\vec{y} \in U$

need to modify to satisfy compatibility:

$$-\Delta G = \delta(\vec{x}-\vec{y}) + c$$

$$\int_U (\delta + c)\, d\vec{x} = \int_{\partial U} 0\, ds_{\vec{x}} \quad c = -\frac{1}{|U|}$$

$|U|$ the volume of U

Apply Green's Thm to G and u:

$$\int_U (G\Delta u - u\Delta G)\,d\vec{x} = \int_{\partial U}\left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)ds_{\vec{x}}$$

$$\Rightarrow u(\vec{x}) = \int_U G(\vec{x}-\vec{\eta})f(\vec{\eta})\,d\vec{\eta} - \int_{\partial U} G(\vec{x}-\vec{\eta})g(\vec{\eta})\,ds_{\vec{\eta}}$$

$$+ \underbrace{\frac{1}{|U|}\int_U u(\vec{\eta})\,d\vec{\eta}}_{\text{avg. of } u \text{ over } U}$$

$\Rightarrow$ soln is unique up to additive constant.

11-16 Green's Fn problem

3. (Green's function) Consider the boundary value problem

$$\begin{aligned} -\Delta u(\vec{x}) &= f(\vec{x}), \quad x \in \Omega \subset \mathbb{R}^3 \\ u(\vec{x}) &= g(\vec{x}), \quad x \in \partial\Omega \end{aligned} \qquad (1)$$

(a) Formulate a boundary value problem for Green's function $G(\vec{x},\vec{\eta}) = \Phi(\vec{x}-\vec{\eta}) - \phi^{\vec{x}}(\vec{\eta})$ for $\vec{x} \in \Omega$ using the fundamental solution $\Phi(\vec{x}) = \frac{1}{4\pi}\frac{1}{|x|}$

(b) Prove the Green's function, if it exists, is unique.

(c) construct Green's function when

$\Omega = B(\vec{0},1) \cap \{X=(x_1,x_2,x_3)\in \mathbb{R}^3 \mid x_3>0\}$

where $B(\vec{0},1)$ is the unit sphere

$$-\Delta G = \delta(\vec{x}-\vec{\eta}), \quad \Omega$$

$$G=0, \quad \vec{x}\in\partial\Omega, \quad \vec{\eta}\in\Omega$$

$$G = \Phi - \Phi^{\vec{x}}(\vec{\eta})$$

$$\Rightarrow -\Delta\Phi + \Delta\Phi^{\vec{x}}(\vec{\eta}) = \delta(\vec{x}-\vec{\eta}) \longrightarrow \Delta\Phi^{\vec{x}}(\vec{\eta})=0$$

$\vec{x},\vec{\eta}\in\Omega$

$$\Phi^{\vec{x}}(\vec{\eta}) = \Phi(\vec{x},\vec{\eta}), \quad \vec{x}\in\partial\Omega, \quad \vec{\eta}\in\Omega$$