Contents

Real Analysis

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2 Integration

2.1 Definition of Measurable Functions

$(X, \mathbf{a}, \mu)$ measure space, $X_0 \subseteq X, X_0 \in \mathbf{a}$.

Definition 2.1.1. *$f : X_0 \to \mathbb{R}$ is measurable if $\forall$ open $E \subseteq \mathbb{R}, f^{-1}(E) \in \mathbf{a}$.*

Definition 2.1.2. $\underbrace{f^{-1}(A)}_{inverse\ image\ of\ A\ under\ f} = \{x \in X : f(x) \in A\}$

Example 2.1.1. *$f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = 0, \forall x \in \mathbb{R}$, let $A \in\subseteq \mathbb{R}$,*

$$f^{-1}(A) = \begin{cases} R, \ if\ 0 \in A \\ \emptyset, \ if\ 0 \notin A \end{cases} \tag{2.1.1}$$

Proposition 2.1.1.

1. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
2. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
3. $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$

Definition 2.1.3. *$f : X_0 \to [-\infty, \infty]$ is measurable if $\forall$ open $E \subseteq \mathbb{R}, f^{-1}(E) \in \mathbf{a}$ & $f^{-1}(\{-\infty\}) \in \mathbf{a}$ & $f^{-1}(\{\infty\}) \in \mathbf{a}$.*

Theorem 2.1.1. *$f : X_0 \to \mathbb{R}$, then the following are equivalence:*

1. *f is measurable*
2. $f^{-1}((-\infty, c)) \in \mathbf{a}, \forall c \in \mathbb{R}$
3. $f^{-1}((-\infty, c]) \in \mathbf{a}, \forall c \in \mathbb{R}$
4. $f^{-1}([c, \infty)) \in \mathbf{a}, \forall c \in \mathbb{R}$
5. $f^{-1}((c, \infty)) \in \mathbf{a}, \forall c \in \mathbb{R}$
6. $f^{-1}(B) \in \mathbf{a}, \ \forall \ Borel, \ B \subseteq \mathbb{R}$

Proof.

1. (1) $\Rightarrow$ (2) by definition 2.1.1.
2. (2) $\Rightarrow$ (3)

$$f^{-1}\left((-\infty,c]\right)=f^{-1}\left(\bigcap_{n=1}^{\infty}\left(-\infty,c+\frac{1}{n}\right)\right)=\bigcap_{n=1}^{\infty}\underbrace{f^{-1}\left(-\infty,c+\frac{1}{n}\right)}_{\in\mathbf{a}}\in\mathbf{a} \tag{2.1.2}$$

3. (3) $\Rightarrow$ (4)

$$f^{-1}\left([c,\infty)\right)=f^{-1}\left(R\backslash\left(-\infty,c\right)\right)=\underbrace{f^{-1}\left(R\right)}_{\in\mathbf{a}}\backslash\underbrace{f^{-1}\left((-\infty,c)\right)}_{\in\mathbf{a}}\in\mathbf{a} \tag{2.1.3}$$

4. (4) $\Rightarrow$ (5)

$$f^{-1}\left([c,\infty)\right)=f^{-1}\left(\bigcap_{n=1}^{\infty}\left(c-\frac{1}{n},\infty\right)\right)=\bigcap_{n=1}^{\infty}\underbrace{f^{-1}\left(c-\frac{1}{n},\infty\right)}_{\in\mathbf{a}}\in\mathbf{a} \tag{2.1.4}$$

5. (5) $\Rightarrow$ (6) Let $\beta=\left\{A\subseteq R,f^{-1}\left(A\right)\in\mathbf{a}\right\}$
   (a) β is $\sigma-$algebra
   (b) $\beta\supseteq\{[c,\infty)\}\quad by\ \bigcap\limits_{n=1}^{\infty}\left(c-\frac{1}{n},\infty\right)=[c,\infty)$
   $\beta\supseteq\{(-\infty,a)\}\quad by\ R\backslash\left[a,\infty\right)=(-\infty,a)$
   $\beta\supseteq\{(b,\infty)\}\quad by\ \ (5)$
   $\beta\supseteq\{(b,a)\}\Rightarrow\beta\supseteq\{open\}$
   (c) $\beta\supseteq\{Borel\}$
6. (6) $\Rightarrow$ (1) it is trivial.

$\square$

Definition 2.1.4. $(X,\mathbf{a})\,,X_0\in\mathbf{a}$, $f:X_0\to[-\infty,\infty]$ *is measurable if* $f^{-1}\left(\{-\infty\}\right),f^{-1}\left(\{\infty\}\right)\&\ f^{-1}\left(O\right)\in\mathbf{a}\ \ \forall O\in\mathbb{R}$, *open.*

Proposition 2.1.2. (X,ρ) *metric space,* μ^* *metric outer measure,* μ *induced measure on* $\mathbf{a}$, $X_0\subseteq X$ *in* $\mathbf{a}$, $f:X_0\to\mathbb{R}$ *continuous* $\Rightarrow f$ *measurable.*

Proof. Let $O\subseteq\mathbb{R}\Rightarrow f^{-1}\left(O\right)$ open in X_0. $\therefore\ f^{-1}\left(O\right)=\underbrace{E}_{\in\mathbf{a}}\cap\underbrace{X_0}_{\in\mathbf{a}}$ for some open $E\quad\Rightarrow f$ measurable. $\square$

Homework 2.1.

1. *Problem* 2.1.8
2. *Problem* 2.1.9
3. *Problem* 2.1.10

2.2 Operations on measurable function

Lemma 2.2.1. *$f, g: X \to [-\infty, \infty]$ measurable, then* $\left\{x \in X : f(x) \underbrace{<}_{\Rightarrow >, \neq, =, \leqslant, \geqslant} g(x)\right\} \in \mathbf{a}$

Proof. Let $\{r_n\}$ be all rational numbers in $\mathbb{R}$.

$$\bigcup_n \left(\underbrace{\{x \in X : f(x) < r_n\}}_{f^{-1}((-\infty, r_n))} \cap \underbrace{\{x \in X : r_n < g(x)\}}_{g^{-1}((r_n, \infty))} \right) \in \mathbf{a} \tag{2.2.1}$$

□

Theorem 2.2.1. *f, g measurable, then*

1. *$f + g$ measurable*
2. *$f - g$ measurable*
3. *$f \cdot g$ measurable*
4. *$\dfrac{f}{g}$ measurable, if $g \neq 0, \forall x \in X$*

Proof.

1.
$$\begin{gathered} (f+g)^{-1}(\{-\infty\}) = f^{-1}(\{-\infty\}) \cup g^{-1}(\{-\infty\}) \in \mathbf{a} \\ (f+g)^{-1}(\{\infty\}) = f^{-1}(\{\infty\}) \cup g^{-1}(\{\infty\}) \in \mathbf{a} \end{gathered} \tag{2.2.2}$$

   let $c \in \mathbb{R}$

$$\begin{aligned} (f+g)^{-1}((-\infty, c)) &= \{x : f(x) + g(x) < c\} \\ &= \{x : f(x) < c - g(x)\} \end{aligned} \tag{2.2.3}$$

   check: $c - g$ is measurable

$$\begin{aligned} \because \quad & (c - g(x))^{-1}((-\infty, c_1)) \\ = & \{x : c - g(x) < c_1\} \\ = & \{x : g(x) > c - c_1\} \\ = & g^{-1}((c - c_1, \infty)) \in \mathbf{a} \end{aligned} \tag{2.2.4}$$

   by Lemma 2.2.1,

$$(f+g)^{-1}((-\infty, c)) \in \mathbf{a} \tag{2.2.5}$$

2. similar as 1

3. $\because f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$
check: h measure $\Rightarrow h^2$ measure

$$\left(h^2\right)^{-1}((\infty, c)) = \left\{x : h^2(x) < c\right\} = \begin{cases} \emptyset & if\ c < 0 \\ \left\{ x : \underbrace{h(x) > \sqrt{c}}_{h^{-1}((-\sqrt{c},\infty))} \cap \underbrace{h(x) < -\sqrt{c}}_{h^{-1}((-\infty,\sqrt{c}))} \right\} & if\ c \geqslant 0 \end{cases} \tag{2.2.6}$$

so $\left(h^2\right)^{-1}((\infty, c)) \in \mathbf{a}$

4. check $\dfrac{1}{g}$ measure

$$\begin{aligned} \left(\frac{1}{g(x)}\right)^{-1}((-\infty, c)) &= \left\{x : \frac{1}{g(x)} < c\right\} \\ &= \begin{cases} \left\{x : \frac{1}{c} < g(x)\right\} = g^{-1}\left(\left(\frac{1}{c}, \infty\right)\right) & if\ c < 0 \\ \{x : g(x) < 0\} = g^{-1}((-\infty, 0)) & if\ c = 0 \\ \underbrace{\{x : g(x) < 0\}}_{g^{-1}((-\infty,0))} \cup \underbrace{\left\{x : \dfrac{1}{c} < g(x)\right\}}_{g^{-1}\left(\left(\frac{1}{c},\infty\right)\right)} & if\ c > 0 \end{cases} \quad \in \mathbf{a} \end{aligned} \tag{2.2.7}$$

by 3 $\Rightarrow f \cdot \frac{1}{g}$ measurable.

□

Theorem 2.2.2. $\{f_n\}$ *measurable* $\Rightarrow \sup\limits_n f_n, \inf\limits_n f_n, \overline{\lim\limits_n} f_n, \underline{\lim\limits_n} f_n$ *measurable.*

Note 2.2.1.

$$x_n \leqslant a \quad \forall n \Leftrightarrow \sup_n x_n \leqslant a \tag{2.2.8}$$

1. $\Leftarrow \quad \sup\limits_n x_n \leqslant a \Rightarrow x_n \leqslant a$

2. $\Rightarrow \quad x_n \leqslant a$ *suppose that* $\sup\limits_n x_n = a + b > a,\ b > 0$ *then* $\exists n_{n_0}$ *s.t.* $x_{n_0} > a + \frac{b}{2} > a$, *it is contradictional, so* $\sup\limits_n x_n \leq a$.

Proof.

$$\begin{aligned} \left(\sup_n \mathrm{f}_n\right)^{-1}((-\infty, c]) &= \bigcap_n \{x : f_n(x) \leqslant c\} \\ &= \bigcap_n f_n^{-1}((-\infty, c]) \in \mathbf{a} \quad \forall c \in \mathbb{R} \end{aligned} \tag{2.2.9}$$

$\therefore \sup\limits_n f_n$ measurable.

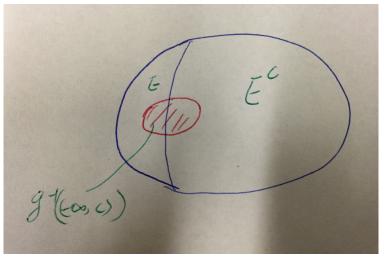
$$\begin{aligned}\inf_n f_n &= -\sup_n (-f_n)\\ \overline{\lim_n} f_n &= \inf_k \sup_{n\geqslant k} f_n \\ \underline{\lim_n} f_n &= \sup_k \inf_{n\geqslant k} f_n\end{aligned} \tag{2.2.10}$$

so, they all are measurable functions. $\square$

$(X, \mathbf{a}, \mu)$, $P(x)$ a.e. (*almost everywhere*) if $\{x \in X : P(x)\ \ false\}$ & $\mu(E) = 0$

Definition 2.2.1. *$f, g : X \to [-\infty, \infty]$ measurable, $f = g$ a.e. if* $\mu\left(\underbrace{\{x \in X : f(x) \neq g(x)\}}_{\triangleq E}\right) = 0$

Lemma 2.2.2. *$f, g : X \to [-\infty, \infty]$, f measurable, $f = g$ a.e., μ complete $\Rightarrow$ g measurable.*



Proof.

$$\begin{aligned} g^{-1}((-\infty, c)) &= \{x : g(x) < c\} \\ &= \left(\underbrace{\{x : g(x) < c\} \cap E}_{\in \mathbf{a}}\right) \cup \left(\underbrace{\underbrace{\{x : g(x) < c\} \cap E^c}_{=\{x : f(x) < c\} \cap E^c}}_{f^{-1}((-\infty,c))\cap E^c \in \mathbf{a}}\right) \\ &\in \mathbf{a}\end{aligned} \tag{2.2.11}$$

$\square$

Theorem 2.2.3. *$\{f_n\}$ measurable,*

1. *$f_n \to g$ (pointwise on X) $\Rightarrow$ g measurable.*
2. *$f_n \to g$ a.e. & μ complete $\Rightarrow$ g measurable.*

Proof.

1. $\because g = \overline{\lim_n} f_n$ measurable by Theorem 2.2.2.

2. let

$$h(x) = \begin{cases} \lim_n f_n(x) & if\ f\ con. \\ 0 & if\ f\ div. \end{cases} \tag{2.2.12}$$

let

$$\begin{aligned} E &= \left\{ x : \lim_n f_n(x)\ not\ exsits \right\} \\ &= \left\{ x : \underline{lim}_n f_n(x) < \overline{\lim_n} f_n(x) \right\} \in \mathbf{a} \end{aligned} \tag{2.2.13}$$

so,

$$h = \left(\lim_{n\to\infty} f_n \right) \cdot \chi_E \tag{2.2.14}$$

need: $E \in \mathbf{a} \Leftrightarrow \chi_E$ measurable

(a) $\Rightarrow$

$$\begin{aligned} \chi_E^{-1}((-\infty, c)) &= \{x : \chi_E(x) < c\} \\ &= \begin{cases} \emptyset & if\ c \leqslant 0 \\ E^c & if\ 0 < c < 1 \\ X & if\ c > 1 \end{cases} \end{aligned} \tag{2.2.15}$$

so χ_E measurable.

(b) $\Leftarrow$

$$\begin{aligned} \chi_E^{-1}\left(\left(-\infty, \frac{1}{2}\right)\right) &= \left\{ x : \chi_E(x) < \frac{1}{2} \right\} = E^c \in \mathbf{a} \\ \chi_E^{-1}\left(\left(\frac{1}{2}, \infty\right)\right) &= \left\{ x : \chi_E(x) > \frac{1}{2} \right\} = E \in \mathbf{a} \end{aligned} \tag{2.2.16}$$

$$\therefore h = \left(\left(\lim_n f_n\right)\right) \cdot \chi_E\ measurable \tag{2.2.17}$$

$h = g\ a.e.\ \mu$ complete $\Rightarrow$ g measurable by Lemma 2.2.2.

□

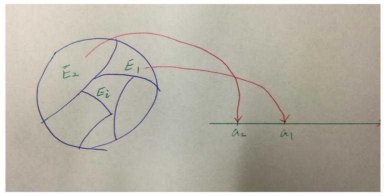
Definition 2.2.2. *$f : X \to \mathbb{R}$ is a simple function if*

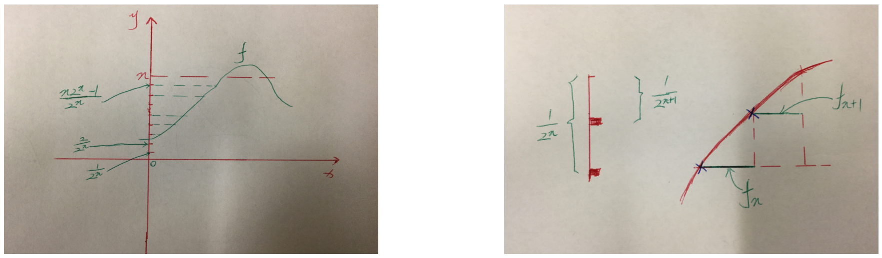
$$f = \sum_{i=1}^{n} a_i \chi_{E_i} \tag{2.2.18}$$

where $a_i \in \mathbb{R},\ E_i \in \mathbf{a},\ X = \bigcup_{i=1}^{n} E_i,\ E_i \cap E_j = \emptyset,\ \forall i \neq j$

Lemma 2.2.3. *f simple $\Leftrightarrow f$ measurable & $\#f(X) < \infty$.*

Theorem 2.2.4. *$f \geq 0$ measurable then $\exists\ f_n \uparrow f$ (pointwise).*





(a) (b)

Proof. let

$$
\begin{aligned}
f_n(x) &= \begin{cases} \frac{i-1}{2^n} & if\, \frac{i-1}{2^n} \leqslant f(x) < \frac{i}{2^n} \\ n & if\;\, f(x) > n \end{cases} \\
&= \sum_{i=1}^{n \cdot 2^n} \frac{i-1}{2^n} \chi_{E_{n_i}}(x) + n \chi_{F_n}
\end{aligned}
\tag{2.2.19}
$$

where $E_{n_i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \in \mathbf{a} \quad i = 1, 2, ..., n \cdot 2^n$ and $F_n = f^{-1}([n, \infty)) \in \mathbf{a}, \therefore f_n(x)$ simple $\therefore f_n(x) \geq 0$ simple, $f_n \uparrow, f_n \leq f$. check: $f_n(x) \to f(x)\ \ \forall x \in X$, fix $x \in X$

1. $f(x) = \infty$

   $$f_n(x) = n\ \forall n \ \ \therefore \underbrace{f_n(x)}_{n} \to \underbrace{f(x)}_{\infty}$$

2. $0 \leq f(x) < \infty$

   $$\exists n_0\ s.t.\ f(x) < n_0 \ \ \therefore 0 \leqslant f(x) - f_n(x) \leqslant \tfrac{1}{2^n}\ as\ n \to \infty\ \forall n \geqslant n_0$$

□

Corollary 2.2.1. *f measurable $\Rightarrow \exists f_n$ simple, s.t. $f_n \to f$ (pointwise on X).*

Proof. $\because f = f^+ - f^-$, where $f^+ = \frac{1}{2}(f + |f|) \geqslant 0, f^- = \frac{1}{2}(|f| - f) \geqslant 0$ measurable, Theorem 2.2.4 $\Rightarrow \exists f_n \geq 0$ simple $f_n \uparrow f^+, \exists g_n \geq 0$ simple $g_n \uparrow f^-$. $\therefore f_n - g_n$ simple, $f_n - g_n \to f^+ - f^- = f$. □

Homework 2.2.

1. *Problem* 2.2.2

2. *Problem* 2.2.8
   hint: f bdd. & measurable $\Rightarrow\ \exists$ *simple* f_n *s.t.* $f_n \to f$ *uniform on* $X \ \Leftrightarrow\ \forall \varepsilon > 0, \exists N,\ s.t.\ n > N,\ \sup\limits_n |f_n(x) - f(x)| < \varepsilon$

2.3 Egoroff's Theorem

$(X, \mathbf{a}, \mu)$ $f_n \to f$ on X, f, f_n measurable & real-valued.

Convergence

1. pointwise: $\forall x \in X\ f_n \to f$ i.e. $\forall x \in X, \forall \varepsilon > 0,\ \exists N,\ s.t.\ n > N,\ |f_n(x) - f(x)| < \varepsilon$
2. uniform: $\forall \varepsilon > 0, \exists N,\ s.t.\ n > N \Rightarrow \sup\limits_{x \in X} |f_n(x) - f(x)| < \varepsilon$
3. almost where: $\mu(\{x \in X, f_n(x) \nrightarrow f(x)\}) = 0$ i.e. $\exists E \in \mathbf{a}, \mu(E) = 0$ s.t. $\forall x \in X \backslash E, \forall \varepsilon > 0, \exists N,\ s.t.\ n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$
4. almost uniformly: $\forall \varepsilon > 0, \exists E \in \mathbf{a},\ s.t.\ \mu(E) < \varepsilon$ & $f_n \to f$ uniformly on $X \backslash E$
   i.e. $\forall \varepsilon > 0, \exists E \in \mathbf{a},\ s.t.\ \mu(E) < \varepsilon$ & $\forall \delta > 0, \exists N,\ s.t.\ n > N \Rightarrow \underset{X \backslash E}{sup} |f_n(x) - f(x)| < \delta$

Theorem 2.3.1. *$f_n \to f$ almost uniformly $\Rightarrow f_n \to f$ a.e.*

Proof. $\forall m \geqslant 1, \exists E_m \in \mathbf{a}\ s.t.\ \mu(E_m) < \frac{1}{m}$& $f_n \to f$ uniformly on $X \backslash E_m$. Let $F = \bigcup\limits_m E_m^c \therefore f_n \to f$ pointwise on F.
check $\mu(F^c) = 0$

$$\mu(F^c) = \mu\left(\bigcap_m E_m\right) \leqslant \mu(E_m) < \frac{1}{m} \to 0 \quad as\ m \to \infty \tag{2.3.1}$$

$\therefore f_n \to f \quad a.e.$ ☐

Theorem 2.3.2 (Egoroff)**.** *$\mu(X) < \infty$ then $f_n \to f$ a.e. $\Rightarrow f_n \to f$ almost uniformly.*

Note 2.3.1.

1. *$f_n \to f$ pointwise on $[a, b] \nRightarrow f_n \to f$ uniformly on $[a, b]$, for example let $f_n(x) = x^n$ on $[0, 1]$*

   $$f_n(x) = \begin{cases} 0 & if\ 0 \leqslant x < 1 \\ 1 & if\ x = 1 \end{cases} \tag{2.3.2}$$

   $\because \sup\limits_{x \in [0,1]} |f_n(x) - f(x)| = 1 \nrightarrow 0 \quad \therefore f_n \nrightarrow f$ *uniformly on* $[0, 1]$.
   but let $\epsilon > 0$, *let* $E = \left[1 - \frac{\varepsilon}{2}, 1\right] \therefore m(E) = \frac{\varepsilon}{2} < \varepsilon$ *on* $[0, 1] \backslash E, f_n \to f$ *uniformly,* $\therefore f_n \to f$ *almost uniformly on* $[0, 1]$.
2. *$\mu(E) = \infty$, the Egoroff's theorem may fail.*

Proof. Fix $k \geq 1$, let $E_n^k = \bigcap\limits_{m=n}^{\infty} \left\{x \in X : |f_n(x) - f(x)| < \frac{1}{k}\right\} \in \mathbf{a}\ \forall n \geqslant 1$, $\therefore E_n^k \uparrow \subseteq \underbrace{\bigcup\limits_n E_n^k}_{F} \in \mathbf{a} \therefore$

$\left(E_n^k\right)^c \downarrow \subseteq F^c$, $\because \mu$ finite measurable $\Rightarrow \mu\left(\left(E_n^k\right)^c\right) \downarrow \mu(F^c)$.

$$\begin{aligned} \because\quad & \{x : f_m(x) \to f(x)\} \subseteq F \\ \therefore\quad & F^c \subseteq \{x : f_m(x) \not\to f(x)\} \\ \therefore\quad & \mu(F^c) \leqslant \mu(\{x : f_m(x) \not\to f(x)\}) = 0 \\ \therefore\quad & \qquad \mu(F^c) = 0 \end{aligned} \tag{2.3.3}$$

$\forall \varepsilon > 0, \forall k \geq 1, \exists n_k$ *s.t.* $n \geq n_k \Rightarrow \mu\left(\left(E_n^k\right)^c\right) < \frac{\varepsilon}{2^k} \therefore \mu\left(\left(E_{n_k}^k\right)^c\right) < \frac{\varepsilon}{2^k}$.

Let $E = \bigcup\limits_k \left(E_{n_k}^k\right)^c \in \mathbf{a}$, check

1. $\mu(E) \leqslant \varepsilon$
2. $f_n \to f$ uniformly on E^c

1. $\mu(E) = \mu\left(\bigcup\limits_k \left(E_{n_k}^k\right)^c\right) \leqslant \sum\limits_k \mu\left(\left(E_{n_k}^k\right)^c\right) \leqslant \sum\limits_k \frac{\varepsilon}{2^k} = \varepsilon$
2. $\forall x \in E^c = \bigcap\limits_k E_{n_k}^k$, for $\varepsilon > 0$, let k_0 be s.t. $\frac{1}{k_0} < \varepsilon \Rightarrow x \in E_{n_{k_0}}^{k_0} \therefore |f_m(x) - f(x)| < \frac{1}{k_0} < \varepsilon\ \ \forall m \geqslant n_{k_0}$ *i.e.* $\sup\limits_{x \in E^c} |f_m(x) - f(x)| \leqslant \varepsilon\ \forall m \geqslant n_{k_0} i.e. f_m \to f\ uniformly\ on\ E^c$

□

Homework 2.3. *Problem* 2.3.1

2.4 Convergence in Measure

$(X, \mathbf{a}, \mu)$, $f_n : X \to [-\infty, \infty]$, measurable & $\mu(\{x : f(x) = \pm\infty\}) = 0\ \forall n$ i.e. f_n real-valued a.e., $f : X \to [-\infty, \infty]$, measurable

Definition 2.4.1. *$f_n \to f$ in measure if* $\forall \varepsilon > 0, \mu(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}) \to 0$ *as* $n \to \infty$ *i.e.* $\forall \varepsilon > 0, \forall \delta > 0,\ \exists N,$ *s.t.* $n > N \Rightarrow \mu(\{x : |f_n - f| \geqslant \varepsilon\}) < \delta$.

Proposition 2.4.1. *1.* $\left.\begin{matrix} f_n \to f & in\ measure \\ f_n \to g & in\ measure \end{matrix}\right\} \Rightarrow f = g\ a.e.$

2. $f_n \to f\ \ in\ measure \Rightarrow f\ real-valued\ a.e.$

3. $f_n \to f\ \ in\ measure \Rightarrow |f_n| = |f|\ \ in\ measure$

4. $f_n \to f\ in\ measure\ \&\ g_n \to g\ in\ measure, a, b \in \mathbb{R} \Rightarrow af_n + bg_n \to af + bg\ in\ measure$

5. $\mu(X) < \infty$, $\left.\begin{matrix} f_n \to f \\ g_n \to g \end{matrix}\right\}$ *in measure* $\Rightarrow f_n g_n \to fg$ *in measure*

6. $\begin{cases} f_n \to f \\ g_n \to g \end{cases}$ *in measure* & $g_n, g \neq 0$ *a.e.* $\Rightarrow \frac{f_n}{g_n} \to \frac{f}{g}$

Proof.

1. check $\mu\left(\underbrace{\{x : f(x) \neq g(x)\}}_{=\underbrace{\left(\bigcup_n \left\{x : |f-g| > \frac{1}{m}\right\} \forall m, n\right)}_{\subseteq \left\{x : |f-f_n| > \frac{1}{2m}\right\} \cup \left\{x : |f_n - g| > \frac{1}{2m}\right\}}}\right) = 0,$

   (otherwise $\frac{1}{m} < |f-g| \leqslant |f-f_n| + |f_n - g| \leqslant \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m} \to\leftarrow$)

$$\begin{aligned}
\therefore \quad & \mu\left(\bigcup_m \left\{x : |f-g| > \frac{1}{m}\right\}\right) \\
& \leq \sum_m \mu\left(\left\{x : |f-g| > \frac{1}{m}\right\}\right) \\
& \leq \sum_m \left[\underbrace{\mu\left(\left\{x : |f-f_n| > \frac{1}{2m}\right\}\right)}_{\to 0} + \underbrace{\mu\left(\left\{x : |f_n - g| > \frac{1}{2m}\right\}\right)}_{\to 0}\right] \\
\therefore \quad & \mu(\{x : f(x) \neq g(x)\}) = 0
\end{aligned} \tag{2.4.1}$$

2. check $\mu(\{x : f(x) = \pm\infty\}) = 0$. Let $E_n = \{x : f_n(x) = \pm\infty\}$, let $E = \bigcup_n E_n$, $\because \mu(E_n) = 0 \ \forall n \ \therefore \mu(E) \leqslant \sum_n \mu(E_n) = 0$, let $\epsilon = 1$

$$\begin{aligned}
\because \quad & \{x : f(x) = \pm\infty\} \subseteq \underbrace{(\{x : f(x) = \pm\infty\} \cap E^c)}_{\subseteq \left\{x : \left|\underbrace{f_n}_{\in\mathbb{R}} - \underbrace{f}_{\equiv\pm\infty}\right| \geqslant 1\right\} \cap E^c} \cup E \\
\therefore \quad & \mu(\{x : f(x) = \pm\infty\}) \leqslant \mu(\{|f_n - f| \geqslant 1\} \cap E^c) + \underbrace{\mu(E)}_{=0} \\
& \leqslant \mu(\{|f_n - f| \geqslant 1\}) < \delta \text{ if } n \text{ large} \\
& \to 0
\end{aligned} \tag{2.4.2}$$

3. it is simple

4. it is simple

5. we take a backward method to check it

   (a) we need to show

$$f_n g_n = \frac{1}{4}\left[\underbrace{(f_n+g_n)^2}_{\to(f+g)^2 \ in\ measure\ by(b)} - \underbrace{(f_n-g_n)^2}_{\to(f-g)^2 \ in\ measure\ by(b)}\right] \tag{2.4.3}$$

   (b) we need to show $f_n^2 \to f^2$

$$\begin{aligned} \because f_n - f \to 0 \quad in\ measure \Rightarrow & \underbrace{(f_n-f)^2 \to 0 \quad in\ measure}_{by\ c} \\ & = f_n^2 - 2f_n f + f^2 \to 0\ in\ measure \end{aligned} \tag{2.4.4}$$

$$\because f_n f \to 2f \quad in\ measure\ by\ d \tag{2.4.5}$$

$$\begin{aligned} \therefore (f_n-f)^2 + 2f_n f = f_n^2 - 2f_n f + f^2 + 2f_n f \to 2f^2\ in\ measure \\ \Rightarrow f_n^2 \to f^2\ in\ measure \end{aligned} \tag{2.4.6}$$

   (c) we need to show that $f_n \to 0$ in measure $\Rightarrow f_n^2 \to$ in measure
   Let $\epsilon > 0, \mu\left(\left\{x : f_n^2 > \varepsilon\right\}\right) = \mu\left(\left\{x : f_n > \sqrt{\varepsilon}\right\}\right) \to 0$

   (d) we need to show $f_n \to f$ in measure $\Rightarrow f_n g \to fg$ in measure

$$\mu\left(\underbrace{\underbrace{\{x : |f_n g - fg| > \varepsilon\}}_{\{x : |f_n - f|\,|g| > \varepsilon\}}}_{\subseteq \underbrace{(\{x:|f_n-f|\,|g|>\varepsilon\}\cap E)}_{\subseteq\{x:|f_n-f|>\frac{\varepsilon}{2}\}}\cup E^c}\right) \to 0 \tag{2.4.7}$$

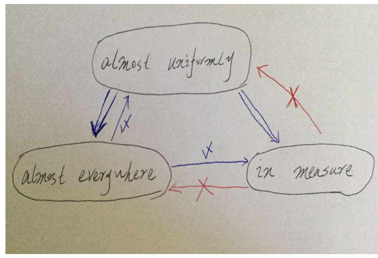
   (e) If $\forall \delta > 0, \exists c > 0$ s.t. $\mu\left(\{x : |g(x)| > c\}\right) < \delta$ i.e g measurable real-valued a.e. true and $\mu(X) < \infty$ we can get g almost bounded, let $E = \{x : |g(x)| \leqslant c\}$

   (f) $\forall n \geq 1$, let $E_n = \{x : |g(x)| \leqslant n\}$ $\because$ $E_n \uparrow \bigcup\limits_n E_n = \{x : g(x) \in \mathbb{R}\}$ $\therefore$ $\mu(E_n) \uparrow$
   $\mu\left(\bigcup\limits_n E_n\right) \underbrace{=}_{X=\left(\bigcup\limits_n E_n\right)\cup\{x:g(x)=\pm\infty\}\&g\ real-valued\ a.e.\ by\ 2} \mu(X)$
   so $\exists n\ s.t.\ \mu(X) - \mu(E_n) < \delta, \because \mu(X) < \infty \therefore \mu(X \backslash E_n) = \mu(\{x : |g(x)| > n\}) < \delta.$

6. by myself

□

Relationship among convergence a.e., almost uniformly, in measure.



Theorem 2.4.1. *$f_n \to f$ almost uniformly $\Rightarrow f_n \to f$ in measure.*

Proof. $\forall \delta > 0,\ \exists E \in \mathbf{a}$ *s.t.* $\mu(E) < \delta$ $f_n \to f$ *uniformly on* $E^c \Leftrightarrow\ \ \forall \varepsilon > 0, \exists N,$ *s.t.* $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon \ \ \forall x \in E^c \quad \therefore \exists N,$ *s.t.* $n > N \ \{x : |f_n(x) - f(x)| \geqslant \varepsilon\} \subseteq E \ \ \therefore \forall n > N, \mu(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}) \leqslant \mu(E) < \delta$ i.e. $\mu(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}) \to 0$. □

Corollary 2.4.1. *$\mu(X) < \infty, f_n \to f$ a.e. $\Rightarrow f_n \to f$ in measure.*

Proof. By Egoroff's thm & Theorem. 2.4.1 □

Example 2.4.1. *$X = \mathbb{R}, f_n(x) = \chi_{(n,\infty)} \therefore f_n \to 0$ a.e. but $f_n(x) \not\to 0$ in measure $\because \mu\left(\left\{x \in \mathbb{R} : |f_n(x)| > \frac{1}{2}\right\}\right) = \mu((n,\infty)) = \infty \not\to 0$*

Theorem 2.4.2. *$f_n \to f$ in measure $\begin{array}{cc} \Rightarrow & \exists f_{n_k} \to f \text{ almost uniformly} \\ \Rightarrow & f_{n_k} \to f \text{ in measure} \end{array}$ $\{f_n\}$ real-valued a.e. measure.*

Proof. It will be proved. □

Definition 2.4.2. *$\{f_n\}$ is Cauchy in measure if $\forall \varepsilon > 0, \forall \delta > 0, \exists N$ s.t. $n, m > N \Rightarrow \mu(\{x : |f_n - f_m| > \varepsilon\}) < \delta$*

Note 2.4.1. *$f_n \to f$ in measure $\Rightarrow \{f_n\}$ cauchy in measure*

Proof. $\{x : |f_n - f_m| > \varepsilon\} \subseteq \left\{x : |f_n - f| > \frac{\varepsilon}{2}\right\} \cup \left\{x : |f - f_m| > \frac{\varepsilon}{2}\right\}$ otherwise $\varepsilon < |f_n - f_m| \leqslant |f_n - f| + |f - f_m| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
$\therefore \mu(\{x : |f_n - f_m| > \varepsilon\}) \leqslant \mu\left(\left\{x : |f_n - f| > \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |f - f_m| > \frac{\varepsilon}{2}\right\}\right) < \delta$ if n, m large □

Theorem 2.4.3. *$\{f_n\}$ Cauchy in measure, then $\exists$ measurable f, $\exists f_{n_k}$ s.t. $f_{n_k} \to f$ almost uniformly.*

Proof. It will be proved. □

Corollary 2.4.2. *$\{f_n\}$ Cauchy in measure $\Rightarrow \exists f$ measurable s.t. $f_n \to f$ in measure.*

Proof. $\because \{x : |f_n - f| > \varepsilon\} \subseteq \left\{x : |f_n - f_{n_k}| > \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_{n_k} - f| > \frac{\varepsilon}{2}\right\}$

$$\therefore \mu(\{x : |f_n - f| > \varepsilon\}) \leqslant \underbrace{\mu\left(\left\{x : |f_n - f_{n_k}| > \frac{\varepsilon}{2}\right\}\right)}_{\to 0 \ if \ n, n_k \ large} + \underbrace{\mu\left(\left\{x : |f_{n_k} - f| > \frac{\varepsilon}{2}\right\}\right)}_{\to 0 \ if \ n, n_k \ large} \to 0.$$ □

Theorem 2.4.4 (Theorem 2.4.3)**.** *If a sequence $\{f_n\}$ of a.e. real-valued, measurable functions is Cauchy in measure, then there is a measurable function f and a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges to f almost uniformly.*

Proof. For $k \geq 1$, let $\epsilon = \delta = \frac{1}{2^k}$, $\therefore \exists n_k$ s.t. $m, n \geqslant n_k \Rightarrow \mu\left(\left\{x : |f_n(x) - f_m(x)| \geqslant \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$

$$\begin{array}{ccccccccccc} & & & n_1 & & n_2 & & & n_3 & & \\ & & & \downarrow & & \downarrow & & & \downarrow & & \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \cdots \end{array}$$

May assume $n_k \uparrow$.

check: $\{f_{n_k}\}$ convergence almost uniformly.

check: $\{f_{n_k}\}$ Cauchy uniformly on E_m.

Let $E_k = \left\{x : \left|f_{n_k}(x) - f_{n_{k+1}}(x)\right| < \frac{1}{2^k}\right\} \in \mathbf{a}$, $F_n = \bigcap\limits_{k=n}^{\infty} E_k \in \mathbf{a}$

Note 2.4.2.

1. *$\{g_n\}$ on X. Define $\{g_n\}$ Cauchy uniformly on X*
   if $\forall \varepsilon > 0, \exists N$ s.t.$n, m \geqslant N \Rightarrow \sup\{|g_n(x) - g_m(x)| : x \in X\} < \varepsilon$.
2. *$\{g_n\}$ convergence uniformly on $X \Leftrightarrow \{g_n\}$ Cauchy uniformly on X.*

Let $x \in F_m$, if $h > j > m$ then $x \in E_{h-1}, \ldots, E_j$

$$\therefore \quad \begin{aligned} &\left|f_{n_{h-1}}(x) - f_{n_h}(x)\right| < \frac{1}{2^{h-1}} \\ &\left|f_{n_{h-2}}(x) - f_{n_{h-1}}(x)\right| < \frac{1}{2^{h-2}} \\ &\cdots \\ &\left|f_{n_j}(x) - f_{n_{j+1}}(x)\right| < \frac{1}{2^j} \end{aligned}$$

$$\begin{aligned} \left|f_{n_h}(x) - f_{n_j}(x)\right| &= \left|f_{n_h}(x) - f_{n_{h-1}}(x) + f_{n_{h-1}}(x) - f_{n_{h-2}}(x) + \cdots + f_{n_{j+1}}(x) - f_{n_j}(x)\right| \\ &\leq \left|f_{n_h}(x) - f_{n_{h-1}}(x)\right| + \left|f_{n_{h-1}}(x) - f_{n_{h-2}}(x)\right| + \cdots + \left|f_{n_{j+1}}(x) - f_{n_j}(x)\right| \end{aligned} \tag{2.4.8}$$

so

$$\begin{aligned} &\underset{x \in F_m}{sup} \left|f_{n_h}(x) - f_{n_j}(x)\right| \leqslant \\ &\quad \sup_{x \in F_m} \left[\left|f_{n_h}(x) - f_{n_{h-1}}(x)\right| + \left|f_{n_{h-1}}(x) - f_{n_{h-2}}(x)\right| + \cdots + \left|f_{n_{j+1}}(x) - f_{n_j}(x)\right|\right] \\ &\quad \leq \frac{1}{2^{h-1}} + \frac{1}{2^{h-2}} + \cdots + \frac{1}{2^j} = \frac{1}{2^j} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{j-1}} \end{aligned} \tag{2.4.9}$$

if n, j large enough , then $\sup\limits_{x \in F_m} \left|f_{n_h}(x) - f_{n_j}(x)\right|$ arbitrarily small $\Rightarrow f_{n_k}$ convergence uniformly on F_m.

$F_m \uparrow \bigcup_m F_m = A$, define

$$f(x) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(x) & if\ x \in A \Rightarrow x \in F_m\ for\ some\ m \\ 0 & if\ x \in A^c \end{cases} \tag{2.4.10}$$

Then

1. f measurable $\because f = \underbrace{\chi_A}_{measurable} \cdot \underbrace{\left(\lim\limits_{k\to\infty} f_{n_k}\right)}_{measurable}$
2. $f_{n_k} \to f$ uniformly on F_m
3. $f_{n_k} \to f$ almost uniformly $\quad\because\quad \mu(F_m^c) = \mu\left(\bigcup\limits_{k=n}^{\infty} E_k^c\right) \leqslant \sum\limits_{k=m}^{\infty} \mu(E_k^c) \leqslant \sum\limits_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}$

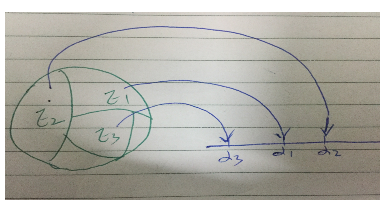
□

Homework 2.4.

1. 2.4.2$(a)(b)(e)$
2. 2.4.4
3. 2.4.5

2.5 Integrals of Simple Function

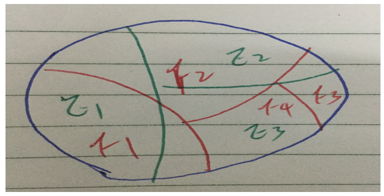
Measure function $(X, \mathbf{a}, \mu)$, simple function: $f = \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i}$, where $\alpha_i \in \mathbb{R}$ (*not necessarily distinct*), $E_i \in \mathbf{a}, \{E_1 \cdots E_n\}$ partition of X.



Definition 2.5.1.

$$\int_X f d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i) \tag{2.5.1}$$

f integrable if $\mu(E_i) < \infty$ for $\alpha_i \neq 0$.



Check: if $f = \sum_i \alpha_i \chi_{E_i} = \sum_j \beta_i \chi_{F_j}$, $\{E_i\}\{F_j\}$ partition, then

$$\sum_i \alpha_i \mu(E_i) = \sum_j \beta_j \mu(F_j) \tag{2.5.2}$$

If $E_i \cap F_j \neq \emptyset$, then $\alpha_i = \beta_j = \gamma_{ij}$

$$\sum_i \alpha_i \mu(E_i) = \sum_i \alpha_i \sum_j \mu(E_i \cap F_j) = \sum_j \sum_i \gamma_{ij} \mu(E_i \cap F_j) \underbrace{=}_{similary} \sum_i \beta_j \mu(F_j) \tag{2.5.3}$$

Note 2.5.1. *f simple, $E \in \mathbf{a} \Rightarrow \chi_E f$ simple*

Proof. let $f = \sum_i \alpha_i \chi_{E_i} \Rightarrow \chi_E f = \sum_i \alpha_i \underbrace{\chi_E \chi_{E_i}}_{\chi_{E \cap E_i}} + 0 \cdot \chi_{E^c}$ ☐

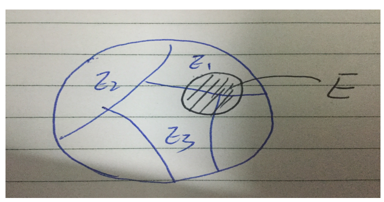


Figure 1: Note 2.5.2

Note 2.5.2. *f simple, integrable, $E \in \mathbf{a} \Rightarrow \chi_E f$ integrable*

Proof. $\int \chi_E f d\mu = \sum_i \alpha_i \mu(E \cap E_i) \leqslant \sum_i \alpha_i \mu(E_i) = \int f d\mu < \infty$ ☐

f integrable on $X, E \in \mathbf{a}$

Definition 2.5.2. $\int_E f d\mu = \int \chi_E f d\mu$

Note 2.5.3. *f simple, integrable, $E \in \mathbf{a}, \mu(E) = 0 \Rightarrow \int_E f d\mu = 0$*

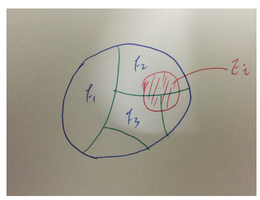
$(X, \mathbf{a}, \mu)$ measure space, $f : X \to \mathbb{R}$, f simple: $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \alpha_i \in \mathbb{R}, E_i \in \mathbf{a}, \bigcup_i E_i = X \{E_i\}$ disjoint. f integrable if $\mu(E_i) = \infty \Rightarrow \alpha_i = 0$ then $\int_X f d\mu = \sum_i \alpha_i \mu(E_i)$.

Proposition 2.5.1. *f, g simple integrable, $a, b \in \mathbb{R}$*

1. *$af+bg$ simple, integrable & $\int af+bg = a\int f + b\int g$*
2. *$f \geqslant 0$ a.e. $\Rightarrow \int f \geqslant 0$*
3. *$f \geqslant g$ a.e. $\Rightarrow \int f \geqslant \int g$*
4. *$|f|$ simple, integrable & $\int |f| \geqslant \int |f|$*
5. *$m \leqslant f \leqslant M$ a.e. on $E \in \mathbf{a}$ with $\mu(E) < \infty$ then $m\mu(E) \leqslant \int_E f \leqslant M\mu(E)$*
6. *$f \geqslant 0$ a.e. $E \subseteq F,\ E, F \in \mathbf{a} \Rightarrow \int_E f \leqslant \int_F f$*
7. *$E = \bigcup\limits_m E_m, \{E_m\} \subseteq \mathbf{a}$ disjoint $\Rightarrow \int_E f = \sum\limits_m \int_{E_m} f$*

Proof.

1. $f = \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i},\ g = \sum\limits_{j=1}^{m} \beta_j \chi_{F_j}$



$\because \{E_i \cap F_j : 1 \leqslant j \leqslant m\}$ *partition of* $E_i \Rightarrow \chi_{E_i} = \sum\limits_j \chi_{E_i \cap F_j}$

$$\begin{aligned} \therefore af+bg &= \sum_i a\alpha_i \chi_{E_i} + \sum_i b\beta_j \chi_{F_j} \\ &= \sum_{ij} a\alpha_i \cdot \chi_{E_i \cap F_j} + \sum_{ij} b\beta_j \cdot \chi_{F_j \cap E_i} \\ &= \sum_{ij} (a\alpha_i + b\beta_j)\chi_{E_i \cap F_j} \end{aligned} \tag{2.5.4}$$

If $\mu(E_i \cap F_j) = \infty$ for some i & j then $\mu(E_i) = \mu(F_j) = \infty \Rightarrow \alpha_i = \beta_j = 0 \Rightarrow a\alpha_i + b\beta_j = 0$, $\therefore$ simple, integrable &

$$\begin{aligned} \int af+bg &= \sum_{ij} (a\alpha_i + b\beta_j) \times \mu(E_i \cap F_j) \\ &= a\sum_i \alpha_i \sum_j \mu\left(E_i \bigcap F_j\right) + b\sum_j \beta_j \sum_i \mu\left(F_j \bigcap E_i\right) \\ &= a\sum_i \alpha_i \mu(E_i) + b\sum_j \beta_j \mu(F_j) \\ &= a\int f + b\int g \end{aligned} \tag{2.5.5}$$

2. $\because f = \sum\limits_i \alpha_i \chi_{E_i} \geq 0$ a.e., If $\mu(E_i) > 0$, then $\alpha_i \geq 0 \therefore \int f = \sum\limits_i \alpha_i \mu(E_i) \geq 0$

3. by (1) & (2)

4. $\because f = \sum\limits_i \alpha_i \chi_{E_i} \Rightarrow |f| = \sum\limits_i |\alpha_i| \chi_{E_i}$, simple, integrable,

$$\int |f| = \sum_i |\alpha_i| \mu(E_i) \geqslant \left| \sum_i \alpha_i \mu(E_i) \right| = \left| \int f \right| \tag{2.5.6}$$

5. $\underbrace{m\chi_E}_{simple, intergrable} \leqslant \chi_E f \leqslant \underbrace{M\chi_E}_{simple, intergrable} \quad a.e \overset{(3)}{\Rightarrow} m\mu(E) \leqslant \int_E f \leqslant M\mu(E)$

6. $\chi_E \leqslant \chi_F \Rightarrow \chi_E f \leqslant \chi_F f \;\; a.e. \;\; \overset{(3)}{\Rightarrow} \int_E f \leqslant \int_F f$

7. $\because f = \sum\limits_i \alpha_i \chi_{F_i}$

$$\begin{aligned} \therefore \int_E f &= \int \chi_E f \\ &= \int \sum_i \alpha_i \chi_{E \cap F_i} \\ &= \sum_i \alpha_i \mu(E \cap F_i) \\ &= \sum_i \alpha_i \sum_m \mu(E_m \cap F_i) \\ &= \sum_m \underbrace{\sum_i \alpha_i \mu(E_m \cap F_i)}_{\int_{E_m} f} \\ &= \sum_m \int_{E_m} f \end{aligned} \tag{2.5.7}$$

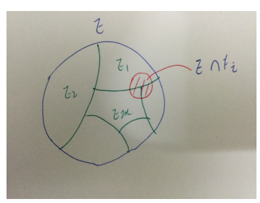


Figure 2: $\{E_m \bigcap F_i\}$ partition of $E \bigcap F_i$

□

$\{f_n\}$ simple, integrable

Definition 2.5.3. $\{f_n\}$ *Cauchy in mean if*

$$\int |f_n - f_m| \to 0, \quad as \;\; n, m \to \infty \tag{2.5.8}$$

$i.e. \forall \epsilon > 0, \exists\ N,\ s.t.\ n, m > N, \int |f_n - f_m| < \varepsilon.$

Lemma 2.5.1. *$\{f_n\}$ simple, integrable, Cauchy in mean $\Rightarrow \exists f$ a.e. real-valued, measurable s.t. $f_n \to f$ in measure*

Note 2.5.4. *$f_n \to f$ in mean $\Rightarrow f_n \to f$ in measure, proof to be continued, similarly area $\to 0 \Rightarrow$ width $\to 0$*

Proof. check $\{f_n\}$ Cauchy in measure

check $\forall \varepsilon > 0, \mu \left\{ \underbrace{|f_n(x) - f_m(x)|}_{E_{mn}} \geqslant \varepsilon \right\} \to 0$ as $n, m \to \infty$

$\chi_{E_{mn}} \cdot |f_n - f_m| \geqslant \varepsilon \cdot \chi_{E_{mn}} \Rightarrow \underbrace{\int |f_n - f_m|}_{\to 0} \geqslant \int_{E_{mn}} |f_n - f_m| \underbrace{\geqslant}_{need\ \mu(E_{mn})<\infty} \int \varepsilon \chi_{E_{mn}} = \underbrace{\varepsilon \mu (E_{mn})}_{\to 0}$

$\because |f_n - f_m| = \sum_i \alpha_i \chi_{E_i}$ simple, integrable, and $E_{mn} \underbrace{\subseteq}_{|f_n - f_m| \geqslant \varepsilon > 0} \bigcup_{\alpha_i \neq 0} E_i$, $\therefore \mu(E_{mn}) \leqslant \sum_{\alpha_i \neq 0} \mu(E_i) < \infty$ ☐

Homework 2.5.

1.
2.

2.6 Integrable functions

$f : X \to [-\infty, \infty]$

Definition 2.6.1. *f integrable if $\exists \{f_n\}$ simple, integrable, s.t.*

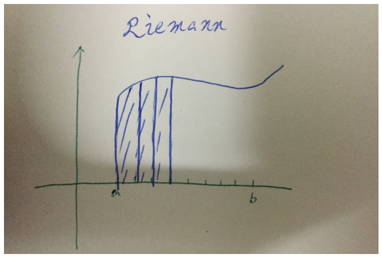
1. *$\{f_n\}$ Cauchy in mean*
2. *$\{f_n\} \to f$ a.e.*

Definition 2.6.2.

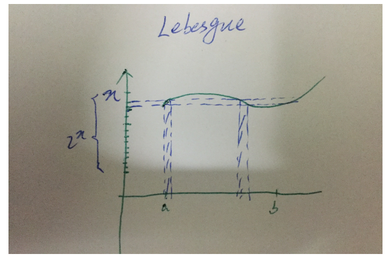
$$\int_X f = \lim_{n\to\infty} \int f_n \tag{2.6.1}$$

Note 2.6.1.

1. *under $(1) \& (2)$, $\lim\limits_{n\to\infty} \int f_n$ exists*



(a)



(b)

Proof. $\because \left|\int f_n - \int f_m\right| \leqslant \int |f_n - f_m| \to 0$ as $n, m \to \infty$ $\therefore \left\{\int f_n\right\}$ is Cauchy, so $\lim\limits_{n\to\infty} \int f_n$ exists. $\square$

2. *$\mu(X)$ may be ∞, f may be unbounded, i.e proper & improper integrals all done at once*

Theorem 2.6.1. *f integrable $\Leftrightarrow \exists \{f_n\}$ simple, integrable s.t.*

(1) *$\{f_n\}$ Cauchy in mean*
(2)$'$ *$f_n \to f$ in measure*

Proof.

1. $\Rightarrow$ (1) & (2) $\Rightarrow$ (1) & (2)$'$
   $\because$ (1) + *Lemma*2.5.1 $\Rightarrow$

   (a) $f_n \to g$ in measure ($\Rightarrow f_{n_k} \to g$ in measure)

   (b) $\{f_n\}$ Cauchy in measure

   by previous Thm $\Rightarrow \exists f_{n_k}$ *s.t.* $f_{n_k} \to h$ almost uniformly $\Rightarrow f_{n_k} \to h$ *a.e.* $\& f_{n_k} \to h$ in measure, (2) $\Rightarrow f_{n_k} \to f$ *a.e.* $\therefore h = f$ *a.e.*, $\therefore g = h$ *a.e.* $\therefore f = g$ *a.e.*
   $\therefore f_n \to f$ in measure, i.e. (2)$'$ holds.

2. $\Leftarrow$ (1) & (2)$'$ $\Rightarrow$ (1) & (2)
   check $\exists f_{n_k}$ satisfy (1) & (2)
   $\because$ (2)$'$ $\exists f_{n_k}$ s.t. $f_{n_k} \to h$ almost uniformly, $\therefore \{f_{n_k}\}$ satisfies (1).
   check $\{f_{n_k}\}$ satisfies (2)
   previous thm $\exists f_{n_k}$ s.t. $f_{n_k} \to h$ almost uniformly $\Rightarrow f_{n_k} \to h$ a.e & $f_{n_k} \to h$ in measure, and (2)$'$ $\Rightarrow f_{n_k} \to f$ in measure, $\therefore f = h$ a.e..
   $\therefore f_{n_k} \to f$ a.e., i.e. (2) holds.

$\square$

Lemma 2.6.1. *f, f_n as Definition 2.6.1, 2.6.2, Let $\lambda(E) = \lim\limits_{n} \int_E f_n$ for $E \in \mathbf{a}$, then $\lambda : \mathbf{a} \to \mathbb{R}$ is a singed measure.*

Note 2.6.2. *f integrable $\Rightarrow E \to \int_E f$ signed measure.*

Proof. $\lambda(\emptyset) = \lim_n \underbrace{\int_\emptyset f_n}_{0} = 0$

Let $E = \bigcup_i E_i, E_i \in \mathbf{a}$ disjoint, check $\lambda(E) = \sum_i \lambda(E_i)$, $\left|\lambda(E) - \sum_i \lambda(E_i)\right|$.
$\lambda(E) = \lim_n \int_E f_n = \lim_n \sum_i \int_{E_i} f_n, \sum_i \lambda(E_i) = \sum_i \lim_n \int_{E_i} f_n$, note $\lim_n \int_E f_n$ exists uniformly in E,

$$\sup_E \left|\int_E f_n - \int_E f_n\right| \leqslant \sup_E \int_E |f_n - f_m| \leqslant \int |f_n - f_m| \to 0 \quad as\ n, m \to \infty \tag{2.6.2}$$

$\therefore \{\int_E f_n\}$ Cauchy uniformly in E. Advanced Calculus $\Rightarrow \lim_n \int_E f_n$ uniformly in E,

$$\begin{aligned}
&\left|\lambda(E) - \sum_{i=1}^m \lambda(E_i)\right| \\
&\leqslant \underbrace{\left|\lambda(E) - \int_E f_n\right|}_{<\varepsilon/3,\ n\ fixed\ large} + \underbrace{\left|\int_E f_n - \sum_{i=1}^m \int_{E_i} f_n\right|}_{<\varepsilon/3\ if\ mlarge} + \underbrace{\left|\sum_{i=1}^m \int_{E_i} f_n - \sum_{i=1}^m \lambda(E_i)\right|}_{=\underbrace{\left|\underbrace{\int_{\bigcup_{i=1}^m E_i} f_n}_{\lambda\left(\bigcup_{i=1}^m E_i\right)} - \underbrace{\sum_{i=1}^m \lim_n \int_{E_i} f_n}_{\underbrace{\lim_n \sum_{i=1}^m \int_{E_i} f_n}_{\lim_n \int_{\bigcup_{i=1}^m E_i} f_n}}\right|}_{<\varepsilon/3}} < \varepsilon
\end{aligned} \tag{2.6.3}$$

$\square$

Theorem 2.6.2. *$f : X \to [-\infty, \infty]$ measurable, $\{f_n\}, \{g_n\}$ simple, integrable, Cauchy in mean, $f_n \to f$ & $g_n \to f$ a.e., then*

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int g_n \left(= \int f\right) \tag{2.6.4}$$

Proof.

1. Let $E \in \mathbf{a}, s.t.\ \mu(E) < \infty$, check $\lim_n \int_E f_n = \lim_n \int_E g_n$.

$$\because \left|\lim_n \int_E f_n - \lim_n \int_E g_n\right| = \left|\lim_n \int_E f_n - g_n\right| \leqslant \overline{\lim_n} \int_E |f_n - g_n| \tag{2.6.5}$$

Let $E \supseteq E_n = \{x \in X : |(f_n\chi_E)(x) - (f\chi_E)(x)| \geqslant \varepsilon\} \in \mathbf{a}$ for $\varepsilon > 0$

$$\begin{aligned}
\int_E |f_n - g_n| &= \underbrace{\int_{E\bigcap E_n} |f_n - g_n|} + \int_{E\setminus E_n} |f_n - g_n| \\
&\leqslant \underbrace{\int_{E_n} |f_n|} + \textstyle\int_{E_n} |g_n| \\
&\leqslant \underbrace{\int_{E_n} |f_n - f_N|}_{\leqslant\varepsilon} + \underbrace{\int_{E_n} |f_N|} \\
&\leqslant \underbrace{\int c\chi_{E_n}} \\
&\underbrace{= c\mu(E_n)}_{\leqslant\varepsilon}
\end{aligned} \tag{2.6.6}$$

($\because f_n \to f$ *a.e.* $\Rightarrow f_n\chi_E \to f\chi_E$ *a.e* $\because \mu(E) < \infty \Rightarrow f_n\chi_E \to f\chi_E$ *in measure* $\Rightarrow \mu(E_n) \to 0$)
Let $\underbrace{E_n}_{\subseteq E} = \{x \in X : |(f_n\chi_E)(x) - (g_n\chi_E)(x)| \geqslant \varepsilon\} \in \mathbf{a}$ for $\varepsilon > 0$, $\because f_n \to f a.e. \& g_n \to f$ *a.e.* $\Rightarrow (f_n - g_n)\chi_E \to 0$ *a.e.* $\because \mu(E) < \infty \Rightarrow (f_n - g_n)\chi_E \to 0$ *in measure* $\Rightarrow \mu(E_n) \to 0$.

$\int_{E\setminus E_n} |f_n - g_n| \leqslant \int_E \varepsilon = \varepsilon\mu(E)$. $\int_E |f_n - g_n| \leqslant 4\varepsilon + \varepsilon\mu(E) \to 0$ as $\varepsilon \to 0$.

2. $E \in \mathbf{a}, E = \bigcup_j E_j, E_j \in a, \& \mu(E_j) < \infty$
   Let $F_1 = E_1,\ F_2 = E_2\ E_1,\ E_3 = E_1(E_1 \bigcup E_2), \cdots, \therefore F_i \in \mathbf{a}, \mu(F_j) < \infty\ \forall j$ disjoint $\&\ E = \bigcup_j F_j \therefore \lim_n \int_E f_n = \sum_j \lim_n \int_{F_j} f_n = \sum_j \lim_n \int_{F_j} g_n = \lim_n \int_E g_n$.

3. Let $N(f_n) = \{x \in X : f_n(x) \neq 0\}, N(g_n) = \{x \in X : g_n(x) \neq 0\}, N = \bigcup_m (N(f_n) \cup N(g_n))$.
   $\because f_n, g_n$ simple $\therefore \mu(N(f_n)), \mu(N(g_n)) < \infty \forall n$. $N(f_n) = \bigcup_{\alpha_j \neq 0} E_j \Rightarrow \mu(N(f_n)) \leqslant \sum_{\alpha_j \neq 0} \mu(E_j) <$
   $\infty \overset{2}{\Rightarrow} \lim_n \underbrace{\int_N f_n}_{\underbrace{\int \chi_N f_n}_{\lim_n \int f_n}} = \lim_n \underbrace{\int_N g_n}_{\underbrace{\int \chi_N g_n}_{\lim_n \int g_n}}$

□

Note 2.6.3. *f integral, $E \in \mathbf{a} \Rightarrow \chi_E f$ integral, $E \in \mathbf{a}$*

Proof. Let $\{f_n\}$ s.t. satisfy def. of integrability of f, $\chi_E f_n$ s.t. satisfy def. of integrability of $\chi_E f \Rightarrow \chi_E f$ integrable. □

Definition 2.6.3. $\int_E f = \int \chi_E f$ *if f integrable, $E \in \mathbf{a}$*

Special case:

1. $X = \mathbb{R}^n, \mu =$ Lebesgue Measure, $\int_E f dx$
2. $X = \mathbb{R}, \mu = \mu_g$ Lebesgue-Stieltjes measure, where $g \uparrow$, right continue on $\mathbb{R}$ $\int_E f dg$

Covers:

1. proper
2. improper of two types
3. multiple
4. Stieltjes integral

Homework 2.6.

1. 2.6.2 *If f is integrable, then the set $N(f) = \{x; f(x) \neq 0\}$ is σ-finite.*

   Proof. $\because \{f(x) \neq 0\} = \{|f(x)| > 0\} = \bigcup\limits_{n=1}^{\infty} \left\{|f(x)| \geqslant \frac{1}{n}\right\}$.
   $\therefore \infty > \int |f| d\mu \geqslant \int_{\left\{|f(x)| \geqslant \frac{1}{n}\right\}} f d\mu \geqslant \frac{1}{n} \mu \left(\left\{|f(x)| \geqslant \frac{1}{n}\right\} \right), \ \forall n \in \mathbb{N}$.
   $\therefore \mu \left(\left\{|f(x)| \geqslant \frac{1}{n}\right\} \right) < \infty, \forall n \in \mathbb{N}$. □

2. 2.6.5 *A function φ on a real interval (a, b) is called a step function*
3. 2.5.6

2.7 Elementary Properties of Integrals

$(X, \mathbf{a}, \mu)$

Definition 2.7.1. *f integrable if $\exists \{f_n\}$ simple, integrable,*
 (a) $\{f_n\}$ Cauchy in mean.
 (b) $f_n \to f$ a.e..
Then $\int f = \lim\limits_n \int f_n$.

Theorem 2.7.1. *f integrable, g measurable*

1. *$f = g$ a.e. $\Rightarrow g$ integrable & $\int f = \int g$.*
2. *f, g integrable, $\alpha\beta \in \mathbb{R}$, then $\alpha f + \beta g$ integrable & $\int \alpha f + \beta g = \alpha \int f + \beta \int g$.*
3. *$f \geq 0$ a.e. $\Rightarrow \int f \geq 0$.*
4. *$f \geq g$ a.e. $\Rightarrow \int f \geq \int g$.*
5. *f integrable $\Leftrightarrow |f|$ integrable & $\left|\int f\right| \leq \int |f|$.*

6. $E \in \mathbf{a}, \mu(E) < \infty, m \le f \le M$ *a.e. on* $E \Rightarrow m\mu(E) \le \int_E f \le M\mu(E)$
7. $f \ge 0$ *a.e.*, $E, F \in \mathbf{a}, E \subseteq F \Rightarrow \int_E f \le \int_F f$.
8. f *integrable,* $m > 0 \Rightarrow \mu\left(\underbrace{\{x : |f(x)| \geqslant m\}}_{E}\right) < \infty$.

Proof.

1. prove by Definition 2.7.1.
2. $\{f_n\}$ satisfies $(a)\,\&\,(b)$ for f, $\{g_n\}$ satisfies $(a)\,\&\,(b)$ for $g \Rightarrow \{\alpha f_n + \beta g_n\}$ satisfies $(a)\,\&\,(b)$ for $\alpha f + \beta g$. $\therefore \int \alpha f + \beta g = \lim_n \int \alpha f_n + \beta g_n = \lim_n \alpha f_n + \lim_n \beta g_n = \alpha \lim_n f_n + \beta \lim_n g_n = \alpha \int f + \beta \int g$.
3. Let $\{f_n\}$ satisfy $(a), (b)$ for $f \Rightarrow \{|f_n|\}$ satisfies $(a), (b)$ for $|f|$. $\therefore \underbrace{|f|}_{=f\ a.e.}$ integrable, $\overset{(1)}{\Rightarrow} \int f = \int |f| = \lim_n \int |f_n| \geqslant 0$.
4. by $(2)\,\&\,(3)$.
5. as in (3) $|f|$ is integrable, $\because \underbrace{\left|\int f_n\right|}_{|\int f|} \leqslant \underbrace{\int |f_n|}_{\int |f|} \quad \forall n$.

   Note 2.7.1. $\Leftarrow$ *also true.*

   Note 2.7.2. $f^+ = \frac{|f|+f}{2}, f^- = \frac{|f|-f}{2}, f = f^+ - f^-, |f| = f^+ + f^-, 0 \leqslant f^+, f^- \leqslant |f|$.
   f *integrable* $\Leftrightarrow f^+, f^-$ *integrable* $\Leftrightarrow |f|$ *integrable.*
   $\Leftarrow$ *prove later.*

   Note 2.7.3. *For proper Riemann integrable,* f *integrable* $\underset{\not\Leftarrow}{\Rightarrow}$ $|f|$ *integrable.*

   Example 2.7.1. $f(x) = \begin{cases} 1, & if\ x\ rational \\ -1, & if\ x\ irrational \end{cases}$ *on* $[0,1]$. *but* $\int_0^1 f$ *not Riemann integrable, note that* $\underbrace{\int_0^1 f}_{=-1(\because f=-1\ a.e.)}$ *is Lebesgue integrable.*

   Note 2.7.4. *For improper Riemann integral,* f *integrable* $\underset{\not\Rightarrow}{\Leftarrow}$ $|f|$ *integrable, similarity to series.*

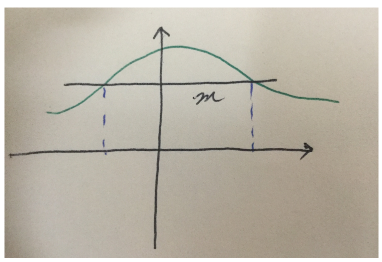
   Example 2.7.2. $f(x) = \frac{\sin x}{x}$ *on* $[1, \infty)$ *then* $\int_1^\infty f$ *exists, but* $\int_1^\infty |f| = \infty$.

   Example 2.7.3. $f(x) = \frac{\sin \frac{1}{x}}{x}$ *on* $(0, 1]$ *then* $\int_0^1 f$ *exists, but* $\int_0^1 |f| = \infty$.

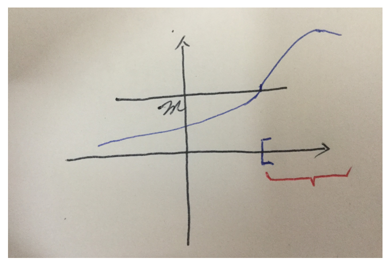
6. $\because \underbrace{m \cdot \chi_E}_{simple,\text{int}\,egrable} \leqslant f \cdot \chi_E \leqslant \underbrace{M \cdot \chi_E}_{simple,\text{int}\,egrable} \Rightarrow m\mu(E) \leqslant \int_E f \leqslant M \cdot \mu(E)$.

7. $\because \chi_E \leqslant \chi_F \Rightarrow \chi_E \cdot f \leqslant \chi_F \cdot f \; a.e. \Rightarrow \int_E f \leqslant \int_F f$.

8. $\because |f| \geqslant m on E \Rightarrow \chi_E \cdot |f| \geqslant m \cdot \chi_E \overset{?}{\Rightarrow} \int_E |f| \geqslant m\mu(E)$, (it is wrong).
proof. $\because E \subseteq \underbrace{\{x : f(x) \neq 0\}}_{\sigma-finite\ by\ homework\ 2.6.2} \Rightarrow E\sigma$-finite $\Rightarrow \exists E_n \in \mathbf{a}\; s.t. E_n \uparrow, \mu(E_n) < \infty \;\&\; E = \bigcup_n E_n \Rightarrow \mu(E_n) \uparrow \mu(E)$.
$|f| \geqslant m$ on $E_n \Rrightarrow \chi_{E_n} |f| \geqslant m \cdot \chi_{E_n} \Rightarrow \underbrace{\int_{E_n} |f|}_{\leqslant \int |f|} \geqslant \underbrace{m\mu(E_n)}_{\to m\mu(E)} \therefore \int |f| \geqslant m\mu(E) \Rightarrow \mu(E) \leqslant \frac{1}{m} \int |f| < \infty$.



(a)



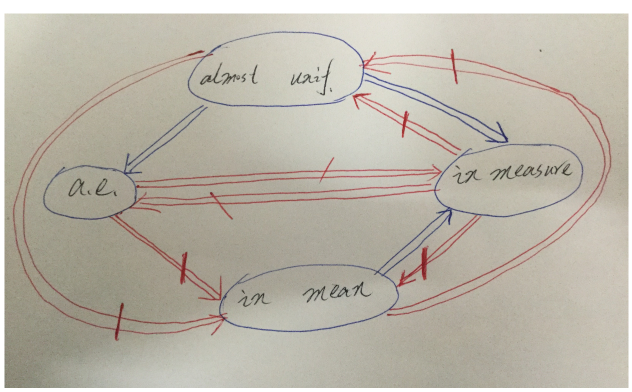
(b)

$\square$

$\{f_n\}, f$ integrable.

Definition 2.7.2. *$f_n \to f$ in mean if $\int |f_n - f| \to 0$ as $n \to \infty$. $\{f_n\}$ Cauchy in mean if $\int |f_n - f_m| \to 0$ as $n, m \to \infty$.*

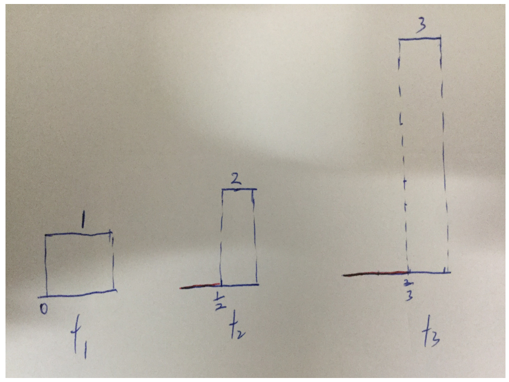
Note 2.7.5. *$f_n \to f$ in mean $\Leftrightarrow \{f_n\}$ Cauchy in mean. $\Leftarrow$ proved in Sec.*??



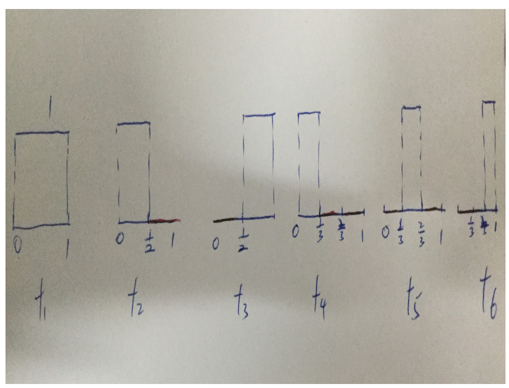
Theorem 2.7.2. *$f_n \to f$ in mean $\Rightarrow f_n \to f$ in measure.*

Proof. For $\varepsilon > 0$, let $E_n = \{x : |f_n(x) - f(x)| \geqslant \varepsilon\}$.
$$\therefore \underbrace{\int \chi_{E_n} \cdot |f_n - f|}_{\leqslant \int |f_n - f|} \underbrace{\geqslant}_{\because \mu(E_n) < \infty \text{ by } (8) \text{ of Thm.2.7.1}} \underbrace{\int \varepsilon \cdot \chi_{E_n}}_{= \varepsilon \cdot \chi_{E_n}} \ \forall n \ \Rightarrow \mu(E_n) \to 0 \ \therefore \ f_n - f \textit{ in measure.}$$
□

Example 2.7.4. *$f \equiv 0$ on $[0,1]$, $\therefore f_n \to f$ a.e. almost uniformly in measure, but $\int_0^1 |f_n| = 1 \ \forall n \therefore f_n \not\to f$ in measure.*



Example 2.7.5. *$f \equiv 0$ on $[0,1]$, $f_n \to f$ in mean. but $f_n(x) \not\to 0, \forall x \in [0,1]$, $\therefore f_n \not\to f$ a.e., $f_n \not\to f$ almost uniformly.*



Theorem 2.7.3. *f integrable, $f \geq 0$ a.e., then $f = 0$ a.e. $\Leftrightarrow \int f = 0$.*

Proof.

1. $\Rightarrow$
2. $\Leftarrow \because \{f_n\}$ simple, integrable, Cauchy in mean & $f_n \to f$ in measure. $\therefore \{|f_n|\}$ simple, integrable, Cauchy in mean & $|f_n| \to \underbrace{|f|}_{f}$ in measure.

   $\therefore \int |f_n| \to \underbrace{\int f}_{=0}$, *i.e.* $f_n \to 0$ *in mean* $\Rightarrow f_n \to 0$ *in measure* $\Rightarrow |f_n| \to 0$ *in measure* $\Rightarrow f = 0$ *a.e.*.

$\square$

Theorem 2.7.4. *f integrable, $f > 0$ a.e. on $E \in \mathbf{a}$, if $\int_E f = 0$, then $\mu(E) = 0$.*

Proof. Let $E_n = \left\{x \in E : |f(x)| \geqslant \frac{1}{n}\right\}$ for $n \geq 1$.
$\therefore \underbrace{\int \chi_{E_n} \cdot f}_{\leqslant \int_E f = 0} \underbrace{\geqslant}_{\mu(E_n) < \infty by (8)} \underbrace{\int \frac{1}{n} \cdot \chi_{E_n}}_{=\frac{1}{n}\mu(E_n)} \Rightarrow \mu(E_n) = 0 \forall n \therefore E_n \uparrow \bigcup_n E_n \underbrace{=}_{a.e.} E \Rightarrow \mu(E) = 0.$ $\square$

Theorem 2.7.5. *f integrable & $\int_E f = 0$ $\forall E \in \mathbf{a} \Rightarrow f = 0$ a.e..*

Proof.

1. Let $E = \{x : f(x) > 0\}$.

$$\int_E f = 0 \underbrace{\Rightarrow}_{by\ Thm. 2.7.4} \mu(E) = 0 \tag{2.7.1}$$

2. Let $F = \{x : f(x) < 0\}$. consider $-f \underbrace{\Rightarrow}_{as\ above} \mu(F) = 0$.

$\therefore f = 0$ a.e.. $\square$

Homework 2.7.

1. *2.7.2 If f is an integrable function, g is a simple function, and $|f(x)| \geqslant |g(x)|$, the g is integrable. (hints : by Theorem2.7.1(5)(8))*
2. *2.7.3*
3. *2.7.6*

2.8 Sequences of Integrable Functions

Theorem 2.8.1 (Main theorem). *$\{f_n\}$ integrable & Cauchy in mean $\Rightarrow \exists f$ integrable $f_n \to f$ in mean.*

Lemma 2.8.1. *f integrable, $\{f_n\}$ simple, integrable, Cauchy in mean & $f_n \to f$ a.e., then $f_n \to f$ in mean.*

Proof. Fix $n \geq 1$, consider $\{|f_n - f_m|\}_{m=1}^{\infty}$, simple, integrable, Cauchy in mean & $|f_n - f_m| \to |f_n - f| \Rightarrow \lim_m \int |f_n - f_m| = \int |f_n - f| \Rightarrow \underbrace{\lim_{n,m} \int |f_n - f_m|}_{=0} = \lim_n \int |f_n - f|$ *i.e.* $f_n \to f$ in mean.

(Cauchy in mean: $\int ||f_n - f_m| - |f_n - f_l|| \leqslant \int |f_m - f_l| \to 0$ *as* $m, l \to \infty$.) $\square$

Note 2.8.1. *f integrable $\Rightarrow \exists\{f_n\}$ simple, integrable, Cauchy in mean & $f_n \to f$ a.e. (or in measure or in mean).*

Theorem 2.8.2. *$\{f_n\}$ integrable, Cauchy in mean & $f_n \to f$ a.e. $\Rightarrow f$ integrable & $f_n \to f$ in mean.*

Proof. by Theorem 2.6.1 f is integrable.

1. assume $f_n \to f$ in measure.
   Idea use Lemma 2.8.1 replace f_n by simple $\widetilde{f_n}$ & then use Lemma again.
   Lemma 2.8.1 $\Rightarrow \exists\ \widetilde{f_n}$, simple, integrable, $\left\{\widetilde{f_n}\right\}$ Cauchy in mean, $\widetilde{f_n} \to f_n$ a.e. *s.t.*

   $$\int \left|\widetilde{f_n} - f_n\right| < \frac{1}{n^2} \tag{2.8.1}$$

   check $\widetilde{f_n} \to f$ in measure.
   let $E_n = \left\{x : \underbrace{\left|\widetilde{f_n}(x) - f_n(x)\right|}_{\text{int}\, egrable} \geqslant \frac{1}{n}\right\} \because \chi_{E_n} \cdot \frac{1}{n} \leqslant \chi_{E_n} \cdot \left|\widetilde{f_n} - f_n\right| \Rightarrow \underbrace{\int_{E_n} \frac{1}{n}}_{\frac{1}{n}\mu(E_n)} \leqslant \underbrace{\int_{E_n} \left|\widetilde{f_n} - f_n\right|}_{\leqslant \int \left|\widetilde{f_n} - f_n\right| \leqslant \frac{1}{n^2}} \Rightarrow$
   $\mu(E_n) \leqslant \frac{1}{n} \to 0$.
   $\forall \varepsilon > 0$, let $n > \dfrac{1}{\varepsilon}$,
   $\because \left\{x : \left|\widetilde{f_n}(x) - f_n(x)\right| \geqslant \varepsilon\right\} \subseteq \left\{x : \left|\widetilde{f_n}(x) - f_n(x)\right| \geqslant \frac{1}{n}\right\} \therefore \mu\left(\left\{x : \left|\widetilde{f_n}(x) - f_n(x)\right| \geqslant \varepsilon\right\}\right) \leqslant$
   $\underbrace{\mu\left(\left\{x : \left|\widetilde{f_n}(x) - f_n(x)\right| \geqslant \dfrac{1}{n}\right\}\right)}_{=\mu(E_n) \to 0}$.

   $$\left.\begin{array}{c} \widetilde{f_n} - f_n \to 0 \ in\ measure \\ f_n - f \to 0 \ in\ measure \end{array}\right\} \Rightarrow \widetilde{f_n} \to f \ in\ measure$$

   by Theorem 2.4.2 and 2.3.1 $\Rightarrow \exists \widetilde{f_{n_k}} \to f$ *a.e.* by Lemma 2.8.1 $\Rightarrow \widetilde{f_{n_k}} \to f$ in mean by triangle inequality $\Rightarrow \left\{\widetilde{f_n}\right\}$ Cauchy in mean $\Rightarrow f_n \to f$ in mean.

2. **Idea:** pass to subsequence & use (1).
   $\because \{f_n\}$ Cauchy in mean $\overset{Theorem\ 2.7.2}{\Rightarrow} \{f_n\}$ Cauchy in measure $\overset{Theorem\ 2.4.2\ Corollary 2.4.2}{\Rightarrow} \exists g, f_{n_k}$
   s.t. $f_{n_k} \to g$ almost uniformly $\Rightarrow \begin{cases} f_{n_k} \to g & a.e. \\ f_{n_k} \to g \ in\ measure \end{cases}$.

$\because f_n \to f$ *a.e.* $\therefore f_{n_k} \to f$ *a.e.* $\therefore f = g$ *a.e.* & $f_{n_k} \to f$ in measure.
$\because \{f_{n_k}\}$ integrable, Cauchy in mean, and $f_{n_k} \to f$ in measure $\overset{(1)}{\Rightarrow} f$ integrable $\& f_{n_k} \to f$ in mean.
$\because \{f_n\}$ Cauchy in mean. $\because \int |f_n - f| \leqslant \underbrace{\int |f_n - f_{n_k}|}_{\varepsilon} + \underbrace{\int |f_{n_k} - f|}_{\varepsilon} \therefore f_n \to f$ in mean. ☐

Proof. (Thm. 2.8.1)
$\because \{f_n\}$ Cauchy in mean $\Rightarrow \{f_n\}$ Cauchy in measure $\Rightarrow \exists f$, s.t. $f_n \to f$ in measure. by theorem 2.8.2 (1) $\Rightarrow f$ integrable & $f_n \to f$ in mean. $\square$

f integrable on $[a, b]$

Definition 2.8.1. $\underbrace{F(x)}_{indefinite\ integrable\ of\ f} = \int_a^x f(t)\,dt$ *for* $x \in [a, b]$*. Then*

1. *F is continuous on $[a, b] : \forall x \in [a, b], \forall \varepsilon > 0,\ \exists \delta\,(dependent\ on\ x, \varepsilon) > 0, s.t. |x - y| < \delta :$*
$$|F(x) - F(y)| < \varepsilon \tag{2.8.2}$$
2. *F is uniformly continuous on $[a, b] : \forall \varepsilon > 0,\ \exists \delta\,(dependent\ on\ \varepsilon) > 0, s.t. |x - y| < \delta, x, y \in [a, b] :$*
$$|F(x) - F(y)| < \varepsilon \tag{2.8.3}$$
3. *F is absolutely continuous on $[a, b] : \forall \varepsilon > 0, \exists \delta > 0, s.t. \forall (a_i, b_i) \subseteq [a, b], disjoint, \sum\limits_i |b_i - a_i| < \delta :$*
$$\sum_i |F(b_i) - F(a_i)| < \varepsilon \tag{2.8.4}$$
$\left(E = \bigcup\limits_i (a_i, b_i)\right)$

Example 2.8.1. *f integrable on $[a, b]$, $F(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$ then F is absolutely continuous.*

$(X, \mathbf{a}, \mu)$

Definition 2.8.2. *$\lambda : \mathbf{a} \to \mathbb{R}$ is absolute continuous with the respect to μ if $\forall \varepsilon > 0, \exists \delta > 0,$ s.t. $\forall E \in \mathbf{a}$ with $\mu(E) < \infty \Rightarrow |\lambda(E)| < \varepsilon$. (notation: $\lambda \ll \mu$.)*

Theorem 2.8.3. *f integrable on X. $\lambda(E) = \int_E f d\mu$ for $E \in \mathbf{a}$. Then*

1. *λ is a finite signed measure*
2. *$\lambda \ll \mu$*

Proof.

1. Hahn decomposition of X:
Find $A, B \in \mathbf{a}$ s.t. $X = A \bigcup B, A \bigcap B = \emptyset. \forall E \subseteq A, E\mathbf{a} \Rightarrow \lambda(E) \geq 0, \forall F \subseteq B, F \in \mathbf{a} \Rightarrow \lambda(F) \leq 0$.
Let $A = \{x : f(x) \geqslant 0\} \in \mathbf{a}, B = \{x : f(x) < 0\} \in \mathbf{a} \therefore A \cap B = \emptyset, A \cup B = X$.

Let $E \in \mathbf{a}, E \subseteq A, \lambda(E) = \int_E f = \int \underbrace{\chi_E f}_{\geqslant 0}$.

Jordan decomposition of λ :

$\lambda^+(E) = \lambda(E \cap A) = \int_{E\cap A} f \geqslant 0, \forall E \in \mathbf{a}, \lambda^-(E) = -\lambda(E \cap B) = -\int_{E\cap B} f \geqslant 0, \forall E \in \mathbf{a}$.

$\because \lambda = \lambda^+ - \lambda^-, |\lambda| |E| = \lambda^+(E) + \lambda^-(E) = \underbrace{\int_{E\cap A}}_{finite} f - \underbrace{\int_{E\cap B}}_{finite} f \in \mathbb{R}. \therefore |\lambda|$ is finite.

2. Let $\{f_n\}$ simple, integrable, Cauchy in mean, $f_n \to f$ a.e. $\Rightarrow f_n \to f$ in mean.

$$|\lambda(E)| = \left|\int_E f\right| \leqslant \underbrace{\left|\int_E f - f_n\right|}_{\underbrace{\leqslant \int_E |f - f_n|}_{\leqslant \underbrace{\int |f-f_n|}_{<\frac{\varepsilon}{2}\ if\ n\ large}}} + \underbrace{\left|\int_E f_n\right|}_{\underbrace{\leqslant \int_E |f_n| \, (|f_n| \leqslant c)}_{\leqslant \underbrace{c\mu(E)}_{< \underbrace{c\delta}_{<\frac{\varepsilon}{2}\ let\ \delta<\frac{\varepsilon}{2c}}}}} < \varepsilon \tag{2.8.5}$$

□

Corollary 2.8.1. *f integrable, $E_n, E \in \mathbf{a}$ s.t. $E_n \to E$, then $\int_{E_n} f \to \int_E f$.*

Proof.

$$\because E_n \to E \Rightarrow \begin{array}{c} \lambda^+(E_n) \to \lambda^+(E) \\ -\lambda^-(E_n) \to \lambda^-(E) \\ \hline \lambda(E_n) \to \lambda(E) \end{array} \tag{2.8.6}$$

□

Corollary 2.8.2. *f integrable, $E_n \in \mathbf{a}$, s.t. $\mu(E_n) \to 0 \Rightarrow \int_{E_n} f \to 0$.*

Proof. fix $\varepsilon > 0$, then $\exists \delta > 0$, s.t. $\forall E \in \mathbf{a}$ with $\mu(E) < \delta \Rightarrow \lambda(E) < \varepsilon$.

$\because \mu(E_n) \to 0 \therefore \exists N, s.t. n > N \Rightarrow \underbrace{\mu(E_n)}_{\Rightarrow \underbrace{|\lambda(E_n)| < \varepsilon}_{=\left|\int_{E_n} f\right|}} < \delta$ □

function of bounded variation: $\gamma : [a,b] \to \mathbb{R}^2$. length of γ : $\sup \left\{ \sum_j \|\gamma(t_j) - \gamma(t_{j-1})\| \right\}, a = t_0 < t_1 < \cdots < t_n = b$.

$f : [a,b] \to \mathbb{R}$

Definition 2.8.3.

1. *total variation of f over* $[a,b]$:

$$T_a^b(f) = \sup\left\{\sum_j |\gamma(x_j) - \gamma(x_{j-1})| : a = x_0 < x_1 < \cdots < x_n = b\right\} \tag{2.8.7}$$

2. *positive variation of f over* $[a,b]$:

$$P_a^b(f) = \sup\left\{\sum_j (\gamma(x_j) - \gamma(x_{j-1}))^+ : a = x_0 < x_1 < \cdots < x_n = b\right\} \tag{2.8.8}$$

3. *negative variation of f over* $[a,b]$:

$$N_a^b(f) = \sup\left\{\sum_j (\gamma(x_j) - \gamma(x_{j-1}))^- : a = x_0 < x_1 < \cdots < x_n = b\right\} \tag{2.8.9}$$

4. *bounded variation of f over* $[a,b]$:

$$BV[a,b] = \left\{f : [a,b] \to R : T_a^b < \infty\right\} \tag{2.8.10}$$

Proposition 2.8.1.

1. $f \uparrow$ *on* $[a,b]$, *then*

$$\begin{aligned} T_a^b(f) &= f(b) - f(a) \\ P_a^b(f) &= f(b) - f(a) \\ N_a^b(f) &= 0 \end{aligned} \tag{2.8.11}$$

2. $f \downarrow$ *on* $[a,b]$, *then*

$$\begin{aligned} T_a^b(f) &= f(a) - f(b) \\ P_a^b(f) &= 0 \\ N_a^b(f) &= f(a) - f(b) \end{aligned} \tag{2.8.12}$$

   1 & 2 *combined, f monotone on* $[a,b]$, $\Rightarrow T_a^b(f) = P_a^b(f) + N_a^b(f) < \infty \therefore f \in BV[a,b]$.

3. $\max\left\{P_a^b(f), N_a^b(f)\right\} \leqslant T_a^b(f) \leqslant P_a^b(f) + N_a^b(f)$

4. $f \in BV[a,b] \Rightarrow \begin{cases} P_a^b(f) + N_a^b(f) = T_a^b(f) \\ P_a^b(f) - N_a^b(f) = f(b) - f(a) \end{cases}$

5. $a \leqslant c \leqslant b \Rightarrow T_a^b(f) = T_a^c(f) + T_c^b(f) \Rightarrow T_a^x(f) \uparrow$ *over* x. *Similarity for* $P_a^b(f), N_a^b(f)$.

6. $T_a^b(f+g) \leqslant T_a^b(f) + T_a^b(g)$

7. $T_a^b(cf) = |c| \cdot T_a^b(f)$

Note 2.8.2. 6 & 7 $\Rightarrow BV\,[a,b]$ *is a vector space.*

8. $T_a^b\,(f)=0 \Leftrightarrow f=$ *constant function on* $[a,b]\,.$

   Note 2.8.3. $T_a^b\,(\cdot)$ *is almost a norm on* $BV\,[a,b]\,.$

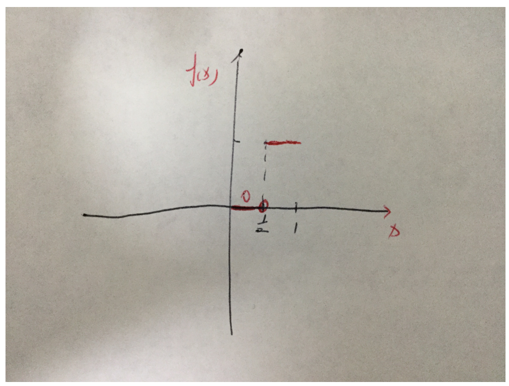
9. $f\in BV\,[a,b] \Leftrightarrow f=g-h$ *for some* $g,h\uparrow$ *on* $[a,b]\,.$

10. $f_n\to f$ *pointwise on* $[a,b] \Rightarrow T_a^b\,(f)\leqslant \underline{\lim} T_a^b\,(f_n)\,,$ *i.e.* $f\to T_a^b\,(f)$ *is lower semicontinuous.*

11. f *absolute continuous on* $[a,b]$ $\underset{\not\Leftarrow}{\Rightarrow}$ f *of bound variation on* $[a,b]$.

    Note 2.8.4. f *uniformly continuous on* $[a,b]$ $\underset{\not\Leftarrow}{\not\Rightarrow}$ *of bound variation on* $[a,b]\,.$

    Example 2.8.2. $f\,(x)=\chi_{[\frac{1}{2},1]}$ *on* $[a,b]\,.$ $f\uparrow \Rightarrow f\in BV\,[a,b]$ *but* f *not* $(uniformly\ absolutely)$ *continuous.*



12. f *satisfies Lipschitz condition on* $[a,b]\,,$ $($*i.e.* $\exists M>0$ *s.t.* $|f\,(x)-f\,(y)|\leqslant M\,|x-y|\quad \forall x,y\in [a,b])\Rightarrow \begin{cases} f \text{ absolytele continious.} \\ f\in BV\,[a,b]. \end{cases}$

Proof.

1. by definition
2. by definition
3. using $\sup\{a_n+b_n\}\leqslant \sup\{a_n\}+\sup\{b_n\}\,.$ $\because a_n\leqslant \sup\{a_n\}\,, b_n\leqslant \sup\{b_n\}\Rightarrow a_n+b_n\leqslant \sup\{a_n\}+\sup\{b_n\}\Rightarrow \sup\{a_n+b_n\}\leqslant \sup\{a_n\}+\sup\{b_n\}$
4.

$$\begin{aligned} p &= \sum_{i=1}^{n}\left(f\,(x_i)-f\,(x_{i-1})\right)^{+} \\ n &= \sum_{i=1}^{n}\left(f\,(x_i)-f\,(x_{i-1})\right)^{-} \\ t = p+n &= \sum_{i=1}^{n}\left|f\,(x_i)-f\,(x_{i-1})\right| \end{aligned} \tag{2.8.13}$$

we can get that

$$p-n=f(b)-f(a) \Rightarrow p=n+f(b)-f(a) \ \text{ and } n=p-[f(b)-f(a)] \tag{2.8.14}$$

denote that

$$\begin{aligned} P &= \sup p \\ N &= \sup n \\ T &= \sup t \end{aligned} \tag{2.8.15}$$

from Eq.2.8.14, we can get

$$\sup\{p\}=\sup\{n+f(b)-f(a)\}=\sup\{n\}+f(b)-f(a) \Rightarrow P=N+f(b)-f(a) \tag{2.8.16}$$

i.e. $P-N=f(b)-f(a)$.

Note 2.8.5. $\sup\{c+c_n\} \leqslant c+\sup\{c_n\}$
if $<$ holds $\Rightarrow \sup\{c+c_n\}=c+r_1, c+\sup\{c_n\}=c+r_2 \Rightarrow r_2>r_1,$ *fix* $\varepsilon=\frac{r_2-r_1}{2}$ $\exists n_0$ *s.t.* $c+c_{n_0} \geqslant c+r_2-\frac{r_2-r_1}{2}=c+\frac{r_2+r_1}{2}>c+r_1 \geqslant c+c_{n_0}$ *i.e.* $c+c_{n_0}>c+c_{n_0}$ *it is impossible, so* $\sup\{c+c_n\}=c+\sup\{c_n\}$.

$t=p+n=p+p-[f(b)-f(a)] \Rightarrow T=2P-[f(b)-f(a)]=2P-(P-N)=P+N.$

5. Let $f:[a,b]\to\mathbb{R}$ be a function and $\Pi=\{x_0,x_1,\cdots,x_n\}$ a partition of $[a,b]$. We denote that $t_\pi(f)=\sum\limits_{k=0}^{k-1}|f(x_{k+1})-f(x_k)|$ and set $T_a^b(f)=\sup\limits_{\pi} t_\pi(f)$.

Note 2.8.6. *Let $f:[a,b]\to\mathbb{R}$ be a function and π any partition of $[a,b]$. If π' is any refine of Π then $t_\pi(f)\leqslant t_{\pi'}(f)$.*

Proof. Since any refinement of π be obtained by adding points to π one at a time. It is enough to prove the Note in the case when we add just one point. Take $\pi=\{x_0,x_1,\cdots,x_n\}$ and take point c to π and denote the result π'. Assume that $x_j\le c<x_{j+1}$ for some j $0\leqslant j\leqslant n-1$, then the triangle inequality gives that

$$|f(x_{j+1})-f(x_j)|=|f(x_{j+1})-f(c)+f(c)-f(x_j)|\leqslant|f(x_{j+1})-f(c)|+|f(c)-f(x_j)|$$

and hence

$$\begin{aligned} t_\pi(f) &= \sum_{k=0}^{n}|f(x_{k+1})-f(x_k)| \\ &= |f(x_{j+1})-f(x_j)|+\sum_{k=0,k\neq j}^{n}|f(x_{k+1})-f(x_k)| \\ &\le |f(x_{j+1})-f(c)|+|f(c)-f(x_j)|+\sum_{k=0,k\neq j}^{n}|f(x_{k+1})-f(x_k)| \\ &= t_{\pi'}(f) \end{aligned}$$

□

Remark 2.8.1. *Note.2.8.6 above assures us that adding points to a partition π will only make the sum $t_\pi(f)$ large or perhaps leave it unchanged.*

Note 2.8.7. *Let $f:[a,b]\to\mathbb{R}$ and c an arbitrary point in (a,b). Then $f\in BV[a,b]$ if $f\in BV[a,c]$ and $f\in BV[c,b]$, furthermore, if $f\in BV[a,b]$, then $T_a^b(f)=T_a^c(f)+T_c^b(f)$.*

Proof. Assume that $f\in BV[a,b]$. We will show that $f\in BV[a,c]$, the proof is similar to prove that $f\in BV[c,b]$. Take an arbitrary partition π of $[a,c]$ and add the point b to π and denote the result π', which is a partition of $[a,b]$ we then have

$$t_{\pi'}(f)=t_\pi(f)+|f(b)-f(c)|\leqslant T_a^b(f)\Leftrightarrow t_\pi(f)\leqslant T_a^b(f)-|f(b)-f(c)| \tag{2.8.17}$$

since $T_a^b(f)$ is finite, the sums of $t_\pi(f)$ are bounded above and thus $\sup_\pi t_\pi(f)$ is finite, thus $f\in BV[a,c]$.

Now, assume that $f\in BV[a,c]$ and $f\in BV[c,b]$. Let π be any partition of $[a,b]$. And the point c to π and denote the result π_1. Then $\pi_1=\pi'\cup\pi''$ where π' is a partition of $[a,c]$ and π'' is a partition of $[c,b]$. Then by Note.2.8.6 we have

$$t_\pi(f)\leqslant t_{\pi_1}(f)=t_{\pi'}(f)+t_{\pi''}(f)\leqslant T_a^c(f)+T_c^b(f)$$

and since both $T_a^c(f)$ and $T_c^b(f)$ are finite, the sum $t_\pi(f)$ are bounded above and thus $f\in BV[a,b]$ and

$$T_a^b(f)\leqslant T_a^c(f)+T_c^b(f). \tag{2.8.18}$$

Now we take any two partitions π' and π'' of $[a,c]$ and $[c,b]$ respectively and let π be the union of π' and π'' then π is a partition of $[a,b]$, we have

$$t_{\pi'}(f)+t_{\pi''}(f)=t_\pi(f)\leqslant T_a^b(f)$$

and thus $t_{\pi'}(f)\leqslant T_a^b(f)-t_{\pi''}(f)$. For any fixed partition π'' of $[c,b]$ the number $T_a^b(f)-t_{\pi''}$ is an upper bound for the sums $t_{\pi'}(f)$ and therefore

$$T_a^c(f)\leqslant T_a^b(f)-t_{\pi''}(f)$$

that is equivalent to

$$t_{\pi''}(f)\leqslant T_a^b(f)-T_a^c(f)$$

thus $T_a^b(f)-T_a^c(f)$ is an upper bound for the sums $t_{\pi''}$ and therefore

$$T_c^b\leqslant T_a^b(f)-T_a^c(f)$$

whence

$$T_a^c(f)+T_c^b\leqslant T_a^b(f) \tag{2.8.19}$$

But by Eq.2.8.18 then we must have

$$T_a^c(f)+T_c^b=T_a^b(f) \tag{2.8.20}$$

□

6. by triangle inequality.

7. by linearity of the absolute value.

8. by algebra.

9.

   (a) $\Rightarrow$ Let g $(x) = P_a^x(f) \uparrow, h(x) = N_a^x(f) - f(a) \uparrow$ by (5). $g(x) - h(x) = P_a^x(f) - N_a^x(f) + f(a) = f(x)$ by (4).

   (b) by $(1), (7), (6)$.

   Note 2.8.8. *f continuous $\Rightarrow g, h$ can be chosen to be continuous on $[a, b]$.*

10. to be continued

11. to be continued

12. to be continued

□

Homework 2.8.

1. 2.8.2

2. 2.8.4

3. 2.8.4

2.9 Lebesgue's Bounded Convergence Theorem

DCT (dominated convergence thm)

Note 2.9.1.

1. $f_n \to f,\ f_n$ *integrable* $\forall n \not\Rightarrow f$ *integrable*

2. $f_n \to f,\ f_n, f$ *integrable* $\forall n \not\Rightarrow \int f_n \to \int f$

In order to be true, need extra conditions:

1. $\{f_n\}$ Cauchy in mean

2. (DCT) $|f_n| \leqslant g$ a.e. for some integrable g

3. (MCT) $0 \leqslant f_n \uparrow f$ a.e.

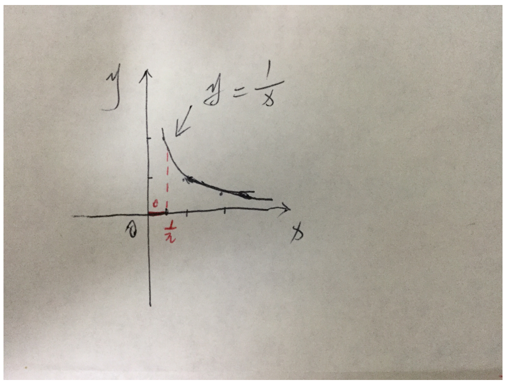
Example 2.9.1.

$$f_n(x)=\begin{cases}\frac{1}{x} & if\ \frac{1}{n}\leqslant x\leqslant 1\\ 0 & otherwise\end{cases}\tag{2.9.1}$$

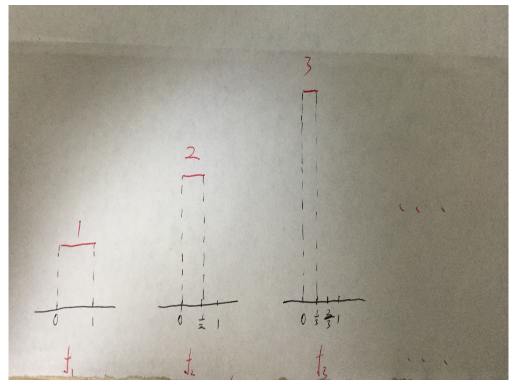
$f_n(x)$ integrable,

$$f(x)=\begin{cases}\frac{1}{x} & if\ 0<x\leqslant 1\\ 0 & if\ x=0\end{cases}\tag{2.9.2}$$

$f_n\to f \quad \underbrace{pointwise}_{a.e.\ or\ in\ measure} \quad on\ [0,1],\ but\ f\ not\ integrable.$



(a) Example 2.9.1



(b) Example 2.9.2

Example 2.9.2. *$f\equiv 0$ on $[0,1]$, then f_n, f integrable $\forall n$, $f_n\to f$ a.e. in measure, almost uniformly, but $\int_0^1 f_n=1\forall n\not\to\int_0^1 f=0$*

Theorem 2.9.1 (DCT)**.** *$\{f_n\}, g$ integrable on X, $f_n\to f$ in measure or a.e. $|f_n|\leqslant g$ a.e. $\forall n\ \Rightarrow f$ integrable & $f_n\to f$ in mean. $\left(\text{ i.e. } \int|f_n-f|\to 0\Rightarrow\int f_n\to\int f\right)$*

Proof.

1. assume $f_n\to f$ in measure. Check: $\{f_n\}$ Cauchy in mean.

   Let $E=\bigcup_n\left\{\underbrace{x: f_n(x)\neq 0}_{\bigcup_m |f_n(x)|\geqslant\frac{1}{m}}\right\}$. $\because \int_X|f_n-f_m|=\int_{X\setminus E}\left|\underbrace{f_n}_{=0}-\underbrace{f_m}_{=0}\right|+\int_E|f_n-f_m|$. $\because E$ is σ-finite by problem 2.6.2. Let $E=\bigcup_k E_k$, where $E_k\in\mathbf{a}, E_k\uparrow, \mu(E_k)<\infty, \forall k$. Let $F_k=E\setminus E_k, \downarrow(\emptyset), \int_{F_k}g\to\int_\emptyset g=0$. $(F_k\ finite)$

$$\therefore\int_X|f_n-f_m|=\int_E|f_n-f_m|=\underbrace{\int_{F_k}|f_n-f_m|}_{\leqslant\underbrace{\int_{F_k}|f_n|+\int_{F_k}|f_m|}_{\leqslant\underbrace{2\int_{F_k}|g|}_{<\varepsilon, if\ k\ large}}}+\int_{E_k}|f_n-f_m|$$

Let $G_{mn} = \{x : |f_n(x) - f_m(x)| \geqslant \varepsilon\}$. $\because f_n \to f$ *in measure* $\Rightarrow \{f_n\}$ *Cauchy in measure* $\therefore$ $\mu(G_{mm}) \to 0$ *as* $m, n \to \infty \Rightarrow \int_{G_{mn}} |\delta| \to 0$

$$\int_{E_k} |f_n - f_m| = \underbrace{\int_{E_k \setminus G_{mn}} |f_n - f_m|}_{\leqslant \int_{E_k} \varepsilon} + \underbrace{\int_{E_k \cap G_{mm}} |f_n - f_m|}_{\leqslant \underbrace{\int_{G_{mn}} |f_n| + \int_{G_{mn}} |f_m|}_{\leqslant\ 2 \underbrace{\int_{G_{mn}} |g|}_{<\varepsilon\ if\ m,n\ large}}}$$

$$\therefore \int_X |f_n - f_m| \to 0$$

2. $f_n \to f$ a.e. check $\mu(E_n) \to 0$ as $n \to \infty$.
   $\because E_n \downarrow \bigcap_n E_n \subseteq \{x : f_n \not\to f(x)\} \because f_n \to f a.e. \Rightarrow \mu(\{x : f_n(x) \not\to f(x)\}) = 0 \therefore \mu\left(\bigcap_n E_\mathrm{n}\right) = 0$, $E_n \downarrow \bigcap_n E_n \underbrace{\Rightarrow}_{need\ \mu(E_n)<\infty} \mu(E_n) \downarrow \mu\left(\bigcap_n E_n\right) = 0$.
   Let $x \in E_n \Rightarrow \exists j \geqslant n s.t.\ \underbrace{|f_j(x) - f(x)|}_{\leqslant \underbrace{|f_j(x)| + |f(x)|}_{\underbrace{\leqslant}_{a.e.} 2\int |g(x)|}} \geqslant \varepsilon \Rightarrow |g(x)|' \geqslant' \frac{\varepsilon}{2} \Rightarrow E_n' \subseteq' \left\{x : |g(x)| \geqslant \frac{\varepsilon}{2}\right\}$

$$\therefore \mu(E_n) \leqslant \mu\left(\left\{x : g(x) \geqslant \frac{\varepsilon}{2}\right\}\right) \underbrace{<}_{by\ g\ integrable} \infty$$

□

Homework 2.9.

1. *2.9.2*
2. *2.9.3*
3. *2.9.4*

2.10 Applications of Lebesgue Bounded Convergence Theorem

Theorem 2.10.1. *f measurable, g integrable, $|f| < g$ a.e. $\Rightarrow f$ integrable*

Note 2.10.1. *$f \to \int f$ not continuous, i.e. $f_n \to f$ a.e. or in measure $\not\Rightarrow \int f_n \to \int f$*

need extra one of the following conditions :

1. $\{f_n\}$ Cauchy i mean.

2. $|f_n| \leqslant g$ a.e. $\forall n,\ g$ integrable

3. (MCT) $0 \leqslant f_n \nearrow f$

Note 2.10.2. *False for Riemann integral*

Example 2.10.1. $f(x) = \begin{cases} 0 \; if \;\; x \; rational \\ 1 \quad otherwise \end{cases}$ *on* $[0,1], g \equiv 1.\ |f| \leqslant g.$ *g integrable, but f not Riemann integral.*

Proof. (Thm 2.10.1) check $|f|$ integrable. $\because |f| \geq 0$, measurable $\therefore \exists$ simple f_n s.t. $0 \leq f_n \nearrow |f|$ a.e.

$$\therefore \underbrace{f_n}_{simple} \leqslant |f| \leqslant \underbrace{g}_{\text{int}\, egrable} \quad a.e. \tag{2.10.1}$$

by problem 2.7(2), f_n simple $f_n \to |f|, |f_n| \leqslant g$ *a.e.*, g integrable $\therefore f_n$ integrable, by Theorem 2.9.1 $\underbrace{DCT}_{f_n \to |f|, |f_n| \leqslant g a.e. f_n \;\; integrable} \Rightarrow |f|$ *integrable*, $\therefore f$ integrable. ☐

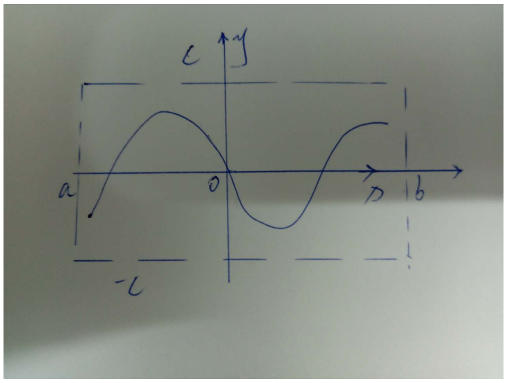
Definition 2.10.1. *f measurable. f is essentially bounded if $\exists c > 0$, s.t. $|f| \leqslant c$ a.e.*

$$ess. \sup f = \inf \{c : |f| \leqslant c \; a.e.\}$$

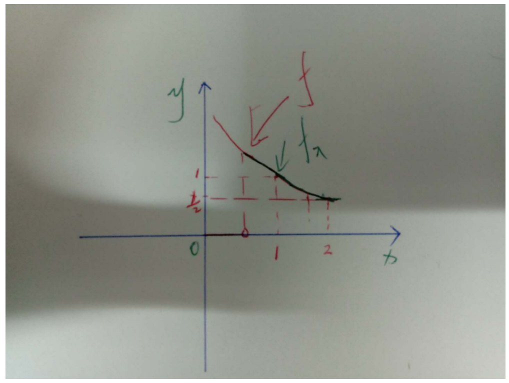
Corollary 2.10.1. *f integrable, g measurable, essential bounded (g may not be integrable) $\Rightarrow f \cdot g$ integrable.*

Proof. Let $|g| \leqslant c$ a.e. $\therefore |f \cdot g| \leqslant \underbrace{c \cdot |f|}_{integrable}$ *a.e.*, Theorem 2.10.1 $\Rightarrow f \cdot g$ integrable. ☐

Corollary 2.10.2. *$E \in a, \mu(E) < \infty, f$ measurable, essential bounded on $E \Rightarrow \int_E f$ exists.*



Proof. $|f| \leqslant c$ a.e. on $E \Rightarrow \underbrace{\chi_E |f|}_{=|\chi_E f|} \leqslant \underbrace{c \cdot \chi_E}_{\int c\chi_E = c\mu(E) < \infty}$ *a.e.*. Theorem 2.10.1 $\Rightarrow \chi_E f$ integrable, i.e. $\int_E f$ exists. ☐



Theorem 2.10.2 (MCT)**.** *$0 \leqslant f_n \nearrow f$ a.e. $\{f_n\}$ integrable $\Rightarrow \int f_n \nearrow \int f$ i.e. $\lim\limits_{n\to\infty} \int f_n d\mu = \int f d\mu$.*

Example 2.10.2. $f_n(x) = \begin{cases} \frac{1}{x} & if\ \frac{1}{n} \leqslant x \leqslant 1 \\ 0 & if\ 0 \leqslant x < \frac{1}{n} \end{cases}, f(x) = \begin{cases} \frac{1}{x} & if\ 0 < x \leqslant 1 \\ 0 & if\ x = 0 \end{cases}$ $0 \leqslant f_n \nearrow f$ $\int_0^1 f_n < \infty \forall n$ & $\int_0^1 f_n \nearrow \int_0^1 f = \infty$

Proof. (Theorem 2.10.2 MCT)

1. f integrable
   $\therefore 0 \leqslant f_n \leqslant \underbrace{f}_{integrable}$ a.e. , $f_n \to f$ a.e., DCT Theorem 2.9.1 $\Rightarrow \int f_n \to \int f$

2. f not integrable, i.e. $\int$ f $= \infty$
   check $\lim\limits_{n\to\infty} \int f_n = \infty$ assume that $\lim\limits_{n\to\infty} \int f_n < \infty$ $\therefore$ $\{f_n\}$ integrable, $f_n \to f$ a.e. & $\underbrace{\{f_n\}\ Cauchy\ in\ mean}_{\int |f_n - f_m| \underbrace{=}_{assume\ n \geqslant m} \int (f_n - f_m) = \underbrace{\underbrace{\int f_n - \int f_m}_{\to 0\ as\ n.m\ large}}_{by\ assume\ \lim_{n\to\infty} \int f_n < \infty}} \underbrace{\Rightarrow}_{by Theorem 2.8.2} f$ integrable, $\to\leftarrow$, $\therefore \lim\limits_{n\to\infty} \int f_n = \infty$.

$\square$

Lemma 2.10.1 (Fatou's Lemma)**.** *$f_n \geqslant 0$ a.e., integrable, $\forall n \Rightarrow \int \underline{\lim}_n f_n \leqslant \underline{\lim}_n \int f_n$*

Note 2.10.3. *$f \to \int f$ is lower semicontinuous.*

Proof. (Fatou's Lemma)

1. $\underline{\lim} \int f_n = \infty$, it is obviously.

2. $\underline{\lim} \int f_n < \infty$

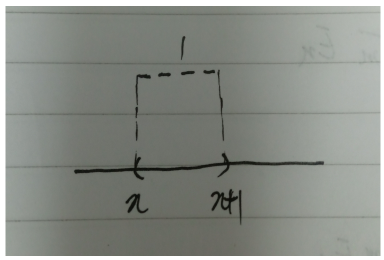
   $f = \underline{\lim} f_n = \lim_n \underbrace{\inf_{j \geqslant n} f_j}_{g_n} \because 0 \leqslant g_n \nearrow f$ *a.e.* MCT Theorem 2.10.2 $\Rightarrow \underbrace{\int \underbrace{g_n}_{\leqslant f_n}}_{\leqslant \int f_n} \nearrow \underbrace{\int f}_{= \int \underline{lim} f_n} \Rightarrow$

   $\lim \int g_n = \int \underline{\lim} f_n \leqslant \underline{\lim} \int f_n \quad \forall n.$

$\square$

Note 2.10.4. $\int \underline{\lim}_n f_n \leqslant \underline{\lim}_n \int f_n$, $<$ *may happen*

Example 2.10.3. $f_n = \chi_{(n,n+1)} \ \forall n \in R \ \Rightarrow \ \int f_n = 1$



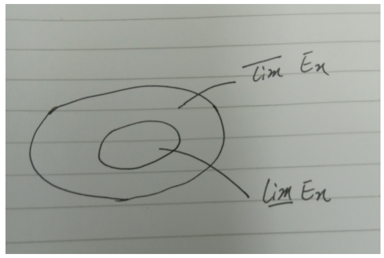
$\underline{\lim}_n = \lim_k \underbrace{\inf_{n \geqslant k} f_n}_{0} = 0 \quad \therefore \int \underline{\lim} f_n = 0 < \underline{\lim} \int f_n = 1$

Note 2.10.5. *MCT* $\rightleftarrows$ *Fatou Lemma*

Test:

1. Borel-Cantelli Lemma: $E_n \in \mathbf{a}, \sum_{\mathbf{n}} \mu(\mathbf{E_n}) < \infty \Rightarrow \mu\left(\overline{\lim}\ \mathbf{E_n}\right) = \mathbf{0}$
2. $E_n \in \mathbf{a}, f_n = \chi_{E_n}$ then f_n convergence a.e. $\Leftrightarrow \ \mu\left(\overline{\lim}\ E_n \backslash \underline{\lim}\ E_n\right) = 0$
3. $f : \mathbb{R} \to \mathbb{R}$ differential $\Rightarrow f'$ is Lebesgue measurable.
4. $(X, \mathbf{a}, \mu)$ $\sigma - finite$, $f, f_n : X \to \mathbb{R}$ measurable, $f_n \to f$ *a.e.* then $\exists N,\ E_1, E_2, \cdots \in a$ *s.t.* $\mu(N) = 0,\ 0 < \mu(E_i) < \infty,\ \forall i,\ X = N \cup E_1 \cup E_2 \cup \cdots$ disjoint, $f_n \to f$ uniformly on each E_i.
5. $|f|$ integrable $\Rightarrow f^+$ & f^- integrable
6. $f, f_n : X \to \mathbb{R}$ integrable, $f_n \to f$ uniformly on $X \not\Rightarrow \int |f_n - f| \to 0$
7. f integral, $a \leqslant \int_E f \leqslant b$, $\forall E \in \mathbf{a}$ with $\mu(E) < \infty$
8. Find f measurable, nonintegrable s.t. $\chi_E f$ integrable $\forall E \in \mathbf{a}, \mu(E) < \infty$

Proof.



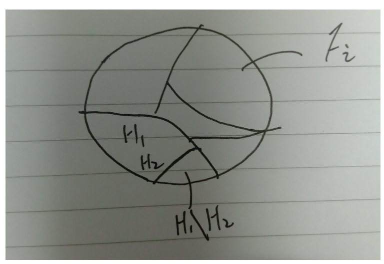
1. $\because \overline{\lim}\ E_n = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=k} E_n \therefore \mu\left(\overline{\lim}\ E_n\right) \leqslant \mu\left(\bigcup\limits_{n=k} E_n\right) \leqslant \sum\limits_{n=k}^{\infty} \mu\left(E_n\right) \to 0$

2. f_n convergence a.e. $\Leftrightarrow \mu\left(\overline{\lim}\ E_n \backslash \underline{\lim}\ E_n\right) = 0$. $\underbrace{\overline{\lim}\ \underbrace{f_n}_{\chi_{E_n}}}_{\chi_{\overline{\lim} E_n}} = \underbrace{\underline{\lim}\ \underbrace{f_n}_{\chi_{E_n}}}_{\chi_{\underline{\lim} E_n}}$

3. $\because$ *f continious on R $\Rightarrow$ f measurable $\Rightarrow$ $f\left(x+\frac{1}{n}\right)$ measurable*

   Note 2.10.6. $f'(x) = \lim\limits_{y \to x} \frac{f(y)-f(x)}{y-x}$

   $$f'(x) = \lim_{n\to\infty} \underbrace{\frac{f\left(x+\frac{1}{n}\right) - f(x)}{\frac{1}{n}}}_{measurable} \therefore f'(x) \text{ measurable.}$$



4. $X = \bigcup\limits_i F_i$ $F_i \in \mathbf{a}, \mu(F_i) < \infty\ \forall i$ may assume $\{F_i\}$ $\because f_n \to f$ *a.e. on* F_i.

   Egorof Theorem 2.3.2 $\Rightarrow$ $f_n \to f$ almost uniformly on F_i. For $\varepsilon = 1, \exists H_1 \subseteq F_i, \mu(H_1) < 1$ & $f_n \to f$ uniformly on $F_i \backslash H_1$

   Egorof Theorem 2.3.2 $\Rightarrow$ For $\varepsilon = \dfrac{1}{2}, \exists H_2 \subseteq H_1, H_2 \in \mathbf{a}, \mu(H_2) < \dfrac{1}{2}$ & $f_n \to f$ uniformly on $H_1 \backslash H_2$

   $$\vdots$$

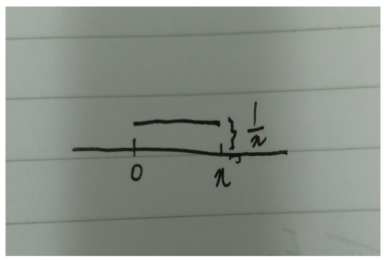
   For $\varepsilon = \dfrac{1}{n}, \exists H_k \subseteq H_{k-1}, H_k \in \mathbf{a}, \mu(H_k) < \dfrac{1}{n}$ & $f_n \to f$ uniformly on $H_{k-1} \backslash H_k$

   $$\vdots$$

$$F_i = \left(\underbrace{F_i \backslash H_1}_{uniformly}\right) \cup \left(\underbrace{H_1 \backslash H_2}_{uniformly}\right) \cup \left(\underbrace{H_2 \backslash H_3}_{uniformly}\right) \cup \cdots \cup \cdots \cup \left(\underbrace{\bigcap_k H_k}_{\mu\left(\bigcap_k H_k\right) \leqslant \mu(H_k) < \frac{1}{k} \to 0}\right) \quad \text{disjoint.}$$

Let $N = \bigcup_i \left(\bigcap_k H_k\right), \mu(N) = 0$

5. Let $E = \{x \in X, f(x) \geqslant 0\}, f^+ = |f| \chi_E, f^- = |f| \chi_{X \backslash E},\ |f|$ integrable $\Rightarrow \underbrace{|f| \chi_E}_{f^+}$



6

6. $f_n = \frac{1}{n}\chi_{[0,n]}$ *on* $\mathbb{R}$, $f \equiv 0$. $f_n \to 0$ *uniformly on* $\mathbb{R}$, *But* $\int |f_n - f| = \frac{1}{n}\int \chi_{\lfloor 0,n \rfloor} = 1\ \forall n$
7. Let $N = \{x \in X : f(x) \neq 0\}$, f integrable $\Rightarrow$ N *is* $\sigma - finite$ *i.e.* $N = \bigcup_n E_n$ $, E_n \in \mathbf{a}, \mu(E_n) < \infty\ \forall n$, may assume $E_n \uparrow N \Rightarrow \int_{E_n} f \to \int_N f$

$$a \leqslant \underbrace{\int_{E_n} f}_{\to \underbrace{\int_N f}_{=\int \chi_N f = \int f}} \leqslant b\ \forall n$$

8. (a) Let $X = \{0,1\}, \mathbf{a} = \rho(x) = \{\emptyset, \{0\}, \{1\}, X\}\ \mu : \mathbf{a} \to \mathbb{R}, \mu(\emptyset) = 0, \mu(\{0\}) = 0, \mu(\{1\}) = \infty, \mu(X) = \infty. f : X \to \mathbb{R},\ f(0) = 0, f(1) = 1, f = \chi_{\{1\}}, \therefore \int f = 1\mu(\{1\}) = \infty$ $E = \emptyset, \int \chi_\emptyset f = 0 < \infty, E = \{0\}, \int \chi_{\{0\}} f = \int 0 = 0 < \infty$
   (b) $X = \mathbb{R}, \mathbf{a} = \{Lebesgue\ measurable\ sets\}, \mu = Lebesgue$ measure, $f \equiv 1$ on $\mathbb{R}$ $\int f = \int 1 = \infty, \mu(E) < \infty \Rightarrow \int_E f < \infty$
   (c) $X = \mathbb{N}, \mathbf{a} = \rho(\mathbb{N}), \mu =$ counting measure, $f \equiv 1$ on $\mathbb{N}, \int f = \infty \Rightarrow \int_E f = \mu(E) < \infty$

□

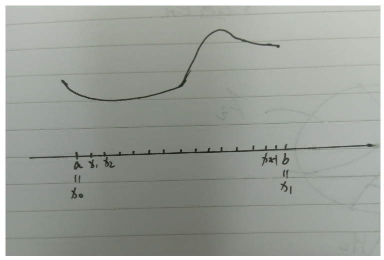
Example 2.10.4. $X = \mathbb{R},\ \mathbf{a} = \{Lebesgue\ \ measurable\ \ sets\}, \mu =$ *Lebesgue measure*, $f \equiv 1$ *on* $\mathbb{R}$, $\int f = \int 1 = \infty$, $\mu(E) < \infty \Rightarrow \int_E f < \infty$

Example 2.10.5. $X = \mathbb{R}, \mathbf{a} = \rho(N),\ \mu =$ *counting measure*, $f \equiv 1$ *on* $\mathbb{N}, \int f = \infty \Rightarrow \int_E f = \mu(E) < \infty$

Homework 2.10.

1. *2.10.2*
2. *2.10.3*
3. *2.10.4*

2.11 The Riemann Integration



f bounded function on $[a, b]$.
for any partition $\pi : a = x_0 < x_1 < \cdots < x_n = b, |\pi| = \max\{x_i - x_{i-1} : 1 \leqslant i \leqslant n\}, S_\pi$: upper Darboux sum, s_π : lower Darboux sum, T_π : Riemann sum.

1. Darboux integrable: $\int_a^b f(x)\,dx \equiv \lim\limits_{|\pi| \to 0} S_\pi = \lim\limits_{|\pi| \to 0} s_\pi$.
2. Riemann integrable: $\int_a^b f(x)\,dx \equiv \lim\limits_{|\pi| \to 0} T_\pi$.
3. Lebesgue integrable: $\int_{[a,b]} f(x)\,dx$.

Note 2.11.1. *From advanced calculus,* $(1)\,\&\,(2)$ *the same.*

Theorem 2.11.1.

Example 2.11.1. *content...*

Example 2.11.2. *content...*

Theorem 2.11.2.

Note 2.11.2. *content...*

Example 2.11.3. *content...*

Note 2.11.3. *content...*

Example 2.11.4. *content...*

Note 2.11.4. *content...*

Proof. (Thm 2.11.1) content... □

Proof. (Thm 2.11.2) content... □

2.12 The Radon-Nikodym Theorem

2.13 The Lebesgue Decomposition

2.14 The Lebesgue Integral on the Real Line

2.15 Product of Measures

2.16 Fubini's Theorem

[1] A. Friedman , "Foundations of Modern Analysis," *Holt, Rinehart, and Winston* , 1970.

[2] P. WU , " Real Analysis," https://ir.nctu.edu.tw/handle/11536/108280, 2010.

Dec 30, 2018