Suppose we have a metric Jap(E) that depends on a superameter E.

(In our case, E=m/M.) We want to approximate Jap for E<<1.

e.g., we can write it as a Taylor series

$$\widetilde{g}_{\alpha\beta} = \widetilde{g}_{\alpha\beta}(0) + \varepsilon \frac{1}{d\varepsilon} \widetilde{g}_{\alpha\beta}(0) + \underbrace{\varepsilon^{2} d^{2}}_{d\varepsilon^{2}} \widetilde{g}_{\alpha\beta}(0) + O(\varepsilon^{3})$$

$$= g_{\alpha\beta} = h_{\alpha\beta}^{(1)} = h_{\alpha\beta}^{(2)}$$

For short, 1ct's define hap = $\frac{g}{g} a p - g_{\alpha p} = \sum_{n \geq 1} e^n h_{\alpha p}^{(n)}$ So $\frac{g}{g} a p = g_{\alpha p} + h_{\alpha p}$.

The exact unities satisfies Gap[g] = BATAp. What equations do him satisfy?

Note that Gap[g] = 15

To start, let's rewrite the EFE as an exact equation for hap.

constructed from Rapps[g], which is defined from \$\tilde{v}_p - \tilde{v}_p \tilde{v}_r \rightarrow we'll rewrite \$\tilde{v}_r\$ in terms of \$\tilde{v}_r\$.

So, define \$C^*p_T\$ such that

 $(\tilde{\nabla}_{p} - \nabla_{p}) \omega^{q} = C^{q}_{p} x \omega^{q}$ for any ω^{q} compatible with g_{qp}

We can also write $C^{\alpha}_{\beta\gamma} = \tilde{\Gamma}^{\alpha}_{\beta\gamma} - \Gamma^{\gamma}_{\beta\gamma}$ Choistoffel associated Choistoffel associated with $\tilde{g}_{\alpha\beta}$ with $g_{\alpha\beta}$

Proof: $C^{\alpha}_{\rho Y} \omega^{\gamma} = \nabla_{\rho} \omega^{\gamma} - \nabla_{\rho} \omega^{\gamma}$ $= \partial_{\rho} \omega^{\gamma} + \tilde{\Gamma}^{\gamma}_{\rho Y} \omega^{\gamma} - \partial_{\rho} \omega^{\gamma} - \Gamma^{\gamma}_{\rho Y} \omega^{\gamma}$ $= (\tilde{\Gamma}^{\gamma}_{\rho Y} - \Gamma^{\gamma}_{\rho Y}) \omega^{\gamma}$ $\Rightarrow C^{\gamma}_{\rho Y} = \tilde{\Gamma}^{\gamma}_{\rho Y} - \Gamma^{\gamma}_{\rho Y}$

We'll use this to show Cap = 1 gam (Vphon + Vrhpm - Vmhpr)

Proof: in a local inertial frame of gap, $\Gamma_{pr}^{qr} = 0$ $\Rightarrow C_{pr}^{qr} = \widetilde{\Gamma}_{pr}^{qr} \quad \text{in this frame}$ $= \frac{1}{2} \widetilde{g}_{pr}^{qr} \left(\partial_{p} \widetilde{g}_{pr} + \partial_{r} \widetilde{g}_{pr} - \partial_{rr} \widetilde{g}_{pr} \right)$ $= \frac{1}{2} \widetilde{g}_{pr}^{qr} \left(\nabla_{p} \widetilde{g}_{rr} + \nabla_{r} \widetilde{g}_{pr} - \nabla_{rr} \widetilde{g}_{pr} \right) \quad \text{since } \Gamma_{pr}^{qr} = 0$ $= \frac{1}{2} \widetilde{g}_{pr}^{qr} \left(\nabla_{p} h_{rr} + \nabla_{r} h_{pr} - \nabla_{rr} h_{pr} \right) \quad \text{since } \nabla_{rr} g_{pr} = 0$

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Rars[3] Rars[3]
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Next, we show Rapes = Rapes + 2 Ver Casip + 2 Caper Chesp where

Proof: By definition, $\tilde{R}^{\alpha}_{\beta\beta\delta}\omega^{\beta} = (\tilde{\nabla}_{\beta}\tilde{\nabla}_{\delta} - \tilde{\nabla}_{\delta}\tilde{\nabla}_{\delta})\omega^{\alpha}$ And $\tilde{\nabla}_{\beta}\tilde{\nabla}_{\delta}\omega^{\alpha} = \nabla_{\beta}(\nabla_{\delta}\omega^{\alpha} + C^{\alpha}_{\delta\mu}\omega^{\mu}) - C^{\mu}_{\beta\delta}(\nabla_{\mu}\omega^{\alpha} + C^{\alpha}_{\mu\nu}\omega^{\nu})$ $+ C^{\alpha}_{\beta\mu}(\nabla_{\delta}\omega^{\mu} + C^{\mu}_{\delta\nu}\omega^{\nu})$

 $\Rightarrow (\tilde{\nabla}_{7}\tilde{\nabla}_{6} - \tilde{\nabla}_{6}\tilde{\nabla}_{7})\omega^{\alpha} = 2\tilde{\nabla}_{67}\tilde{\nabla}_{67}\omega^{\alpha}$ $= 2\nabla_{67}\nabla_{67}\omega^{\alpha} + 2\nabla_{67}(C^{\alpha}_{52\mu}\omega^{\mu}) - 2C^{\mu}_{676}(\nabla_{\mu}\omega^{\alpha} + C^{\alpha}_{\mu\nu}\omega^{\nu})$ $+ 2C^{\alpha}_{\mu[7}(\nabla_{67}\omega^{\mu} + C^{\alpha}_{67}\omega^{\nu})$

= R & pr & w & + 2 (V[x C = 5] m) w + 2 C = [5 V x] w + + 2 C = [x V s] w + 2 C = [5] v w

RAS[g]

RASEGI

Next, Rps, = Ras = Rps + 2 V[x Casip + 2 Care Cusip

We can write the EFE in terms of this as Rop - : gap gue Rux = 8 TTap.

Note: This isn't quite in terms of hop because Cops involves gop. We can write gop as an expansion in powers of hop:

gap = gap - hap + O(h2)

Convention: we use gop and gop to raise

and lower indices, so hop = gong prhur

Let's expand the entire egn. this way. For a tensor nonstructed from gap + hop, define $S^nF[h] = \frac{1}{n!} \frac{d^n}{d\lambda^n} F[g+\lambda h]|_{\lambda=0}$, st $F[g+h] = F[g] + SF[h] + S^2F[h] + \cdots$

linear m h quadratic in 1

e.g.,

g m = g m + Sg m + ...

where 8g 7 = - h 9B

Start with

~h

Rap[g] = Rap[g] + 2 V[n C" x]p + 2 C M x [n C x]p

SRap = 2 Van SC alp.

= 1 gm (Vm Vahra + Vm Vahra - Vm Vrhaa - Vax hra - Va Vahra + Vy Vrhaa

 $\Rightarrow SRap = -\frac{1}{2} \Box h ap - \frac{1}{2} \nabla a \nabla_{\beta} h + \nabla^{\mu} \nabla_{(\alpha} h \beta) \mu \qquad \text{where } \Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ and $h = g^{\mu\nu} h_{\mu\nu}$

By using the Ricci identity on this, we can rewrite Shap as

 $SR_{\gamma p} = -\frac{1}{2} \prod_{h \neq p} - R_{\alpha} \stackrel{\mu}{}_{\beta} \stackrel{\nu}{}_{h \mu \nu} + R_{(\alpha} \stackrel{\mu}{}_{h p) \mu} + \nabla_{(\alpha} \nabla^{\mu} \hat{h}_{\beta) \mu}$ where $\hat{h}_{qp} = h_{qp} - \frac{1}{2} g_{qp} h$ is the trace-reverse of hyp

because $g^{qp} \hat{h}_{qp} = -g^{qp} h_{qp}$

What we want is 86 ap = S(Rap - 1 gap gun Rus) = SRap - 1 hap R - 1 gap Sgun Rus - 1 gap gun SRus Let's specialize to a vacuum bankground; Rus [g] = 0

Then SGop = SRap - 2 gap gur SRus.

A quart calculation yields SGgp = - 1 Dhap - Ramp how + Var Php) u - 2 gap Vu Vr hur

Getting 526 pp is more luborious, but the idea is the same: just find the quadratic at 526 p = 52 (Rap - ± gap g, m Ruy) terms in Rap to get 52 Rap , then look.

The result is $S^2G_{qp} = S^2R_{qp} - \frac{1}{2}g_{qp}g^{\mu\nu}S^2R_{\mu\nu} - \frac{1}{2}h_{qp}g^{\mu\nu}SR_{\mu\nu} + \frac{1}{2}g_{qp}h^{\mu\nu}SR_{\mu\nu}$ where $S^2R_{qp} = -\frac{1}{2}\nabla_r \hat{h}^{\mu\nu}(2\nabla_{lq}h_{p)\mu} - \nabla_{\mu}h_{qp}) + \frac{1}{4}\nabla_q h^{\mu\nu}\nabla_p h_{\mu\nu}$ $-\frac{1}{2}h^{\mu\nu}(2\nabla_{\mu}\nabla_{lq}h_{p)\nu} - \nabla_{\mu}\nabla_r h_{qp} - \nabla_q\nabla_p h_{\mu\nu})$ $+ \nabla^r h^{\mu}_p \nabla_{lp}h_{\mu lq}\alpha$

The EFE is now Graffy) + 8 Gap [h] + 82 Gap [h] + $O(h^3) = 8\pi$ Tap

Now substitute hap = $Eh^{(1)}_{ap} + E^2h^{(2)}_{ap} + O(E^3)$ $\Rightarrow E S Gap [h^{(2)}] + E^2 (S Gap [h^{(2)}] + S^2 Gap [h^{(2)}]) + O(E^3) = 8\pi$ Tap

If we suppose that $T_{qp} = E T_{qp}^{(r)} + E^2 T_{qp}^{(r)} + O(E^3)$, then we can equate coefficients of powers of E:

 $SG_{ap}[h^{(n)}] = 8\pi T_{ap}^{(n)}$ $SG_{ap}[h^{(n)}] = 8\pi T_{ap}^{(n)} - S^2G_{ap}[h^{(n)}]$

For most of these lectures we'll focus on the first-order equation and just write hop as hop

Gauge freedom

We can after the form of the field equations using the theory's gauge freedom. The nonperturbative equations are musicant under wordmate transformations; the perturbative equations are invariant under infinitesimal coordinate transformations.

Say $\chi^{\alpha} \rightarrow \chi^{\alpha'} = \chi^{\alpha'} = \xi^{\alpha'} + \mathcal{O}(\epsilon^2)$. How does gap transform?

 $\tilde{g}_{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x^{\alpha}}, \frac{\partial x^{\beta}}{\partial x^{\beta}}, \tilde{g}_{\alpha\beta}(x^{\beta})$

 $= \frac{\partial}{\partial x^{\alpha'}} \left(\frac{1}{2} \chi^{\mu} + \varepsilon \, \xi^{\mu} \right) \frac{\partial}{\partial x^{\beta'}} \left(\chi^{\nu} + \varepsilon \, \xi^{\nu} \right) \, \tilde{g}_{\mu\nu} \left(\chi^{\kappa'} + \varepsilon \, \xi^{\delta'} \right) \, + O(\varepsilon^2)$ = $(S_x^{\mu} + \epsilon \partial_x \tilde{z}^{\mu})(S_p^{\nu} + \epsilon \partial_p \tilde{z}^{\nu})[\tilde{g}_{\mu\nu}(x^{\nu}) + \epsilon \tilde{z}^{\nu} \partial_{\nu} \tilde{g}_{\mu\nu}(x^{\nu})] + o(\epsilon^2)$ = gap + [3 3 8 gap + 2 3 3 gup + 2 3 3 gup + 2 3 3 gup] + 0 (62)

But gap = gap + Ehap + O(e2)

 $\Rightarrow \tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta} + \epsilon \left[\tilde{3}^{\delta} \partial_{\gamma} g_{\alpha\beta} + \partial_{\alpha} \tilde{3}^{\mu} g_{\mu\rho} + \partial_{\beta} \tilde{3}^{\mu} g_{\alpha\mu} \right] + \mathcal{O}(\epsilon^{2})$

This is figging. We can also write it as 30 Trgap + Va3" gus + VB3" gan = Va3p+VB3a

So gap = gap + & hap + 0 (62)

where hap = hap + £ 3 9 xp | DT, = £ 3 To p... = hap + 7 + 3 p + Vp 3 x

In general, for a tensor . Tx = To p. + ET, g. + Ole

So if To g. = 0, from To

We can check that $8G_{orp}[\nabla \mu_{3}^{2} + \nabla \mu_{3}^{2} \mu] = 0$ if $G_{orp}[g] = 0$ [σ [σ [σ [σ [σ] σ] σ [σ [σ] σ] σ [σ] σ [σ] σ [σ] σ] σ [σ] σ [σ] σ] σ] σ [σ] σ] σ [σ] σ] σ] σ [σ] σ] σ] σ] σ [σ] σ] σ] σ] σ [σ] σ]

We can use this freedom to: ", simplify the equation.

For example, we can impose the Lorenz gange condition To hid =0. With this condition we have SG & Blie] == = = ([hap + 2 Ra " p" hur)

OF EXP[h] = - 16 17 Typ

How do we know we can impose To harp. = 0?

Say hap is in an unspecified gauge, and we want hap to be in the Lorenz gauge. Then Ps(Former) =0

€ Tp (hap - 2 gap gur how) =0

(Jahnen - 2 Va (que ham) = 0

⇒ ∇p hapla + ∇p (∇α3β + ∇β3α) - ∇α hold-2 ∇α (2√μ3μ) = 0

TaTB3B+RXBQ3X

⇒ □3° = -Vañorp

This is a hyperbolic equation that we can always solve => we can always impose the Lorenz gauge condition.

Point particle approximation

What we've said about perturbation theory so far is very generic. Now let's assume that at leading order, the small object in an EMRI can be approximated as a point mass. That is, Top is a delta function supported & on the worldine 8:

 $T_{\alpha\beta}^{ei} = m \int_{\gamma} u_{\alpha} u_{\beta} \frac{S^{4}(\chi^{\mu} - \chi^{\mu}(T))}{\sqrt{-g_{\alpha}}} dT$

e.g., in local coordinates, comoving with x', $T_{x\beta}^{(1)} = m \int_{X} S_{x}^{T} S_{y}^{T} S(T-T') S^{3}(x^{a}) dx$ = m 8 8 8 8 8 8 (xa)

We'll return to whether this approximation makes sonse, For mon we'll assume it does .

```
x of y ox = r sino coso (or or + ox
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                 = 2 Strait do
                                                            =\frac{L_2/r^2}{\sqrt{E^2-V(r(\psi))}}
X=P-6-20054 1P
 = P dy = 2 dx =
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  3PX
                                                             = -ersin4
1+ecosp
                                         dt = dt dr
                  = JE2-V(r(4))
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⇒ て(X) = ∫ 益 d*

= State dir day

= J = - ersin4 dip