Lecture 03

Discrete Mathematics

Instructor: Byung-Woo Hong

CHAPTER

2 Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

A *set* is an unordered collection of objects.

Elements

The objects in a set are called the *elements,* or *members,* of the set. A set is said to *contain* its elements.

Membership

We will now introduce notation used to describe membership in sets. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A. Note that lowercase letters are usually used to denote elements of sets.

Examples of Set

The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p \text{ and } q\}.$$

Examples of Set

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**

Equal Sets

Two sets are *equal* if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Example of Equal Sets

The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

Venn Diagram

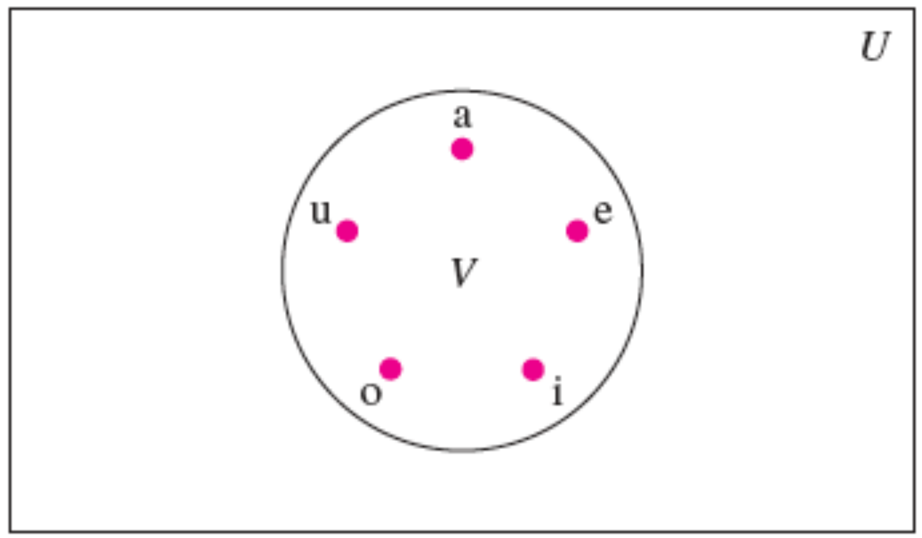


FIGURE 1 **Venn Diagram for the Set of Vowels.**

Solution: We draw a rectangle to indicate the universal set U, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V. Inside this circle we indicate the elements of V with points (see Figure 1).

Empty Set

There is a special set that has no elements. This set is called the **empty set,** or **null set,** and is denoted by Ø. The empty set can also be denoted by { }

Singleton Set

A set with one element is called a **singleton set.**

A common error is to confuse the empty set Ø with the set {Ø}, which is a singleton set. The single element of the set {Ø} is the empty set itself!

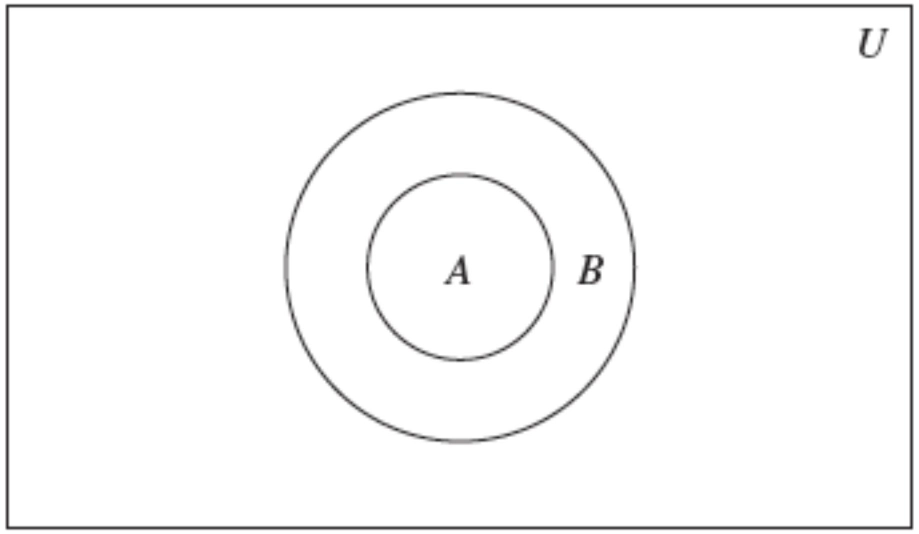
Subset

The set A is said to be a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \rightarrow x \in B)$$

is true.



Subset

For every set S,

$(i)\ \emptyset \subseteq S \quad$ and $\quad (ii)\ S \subseteq S$.

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \to x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \to x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. That is, $\forall x(x \in \emptyset \to x \in S)$ is true. This completes the proof of (i). ◁

Proper Subset

When we wish to emphasize that a set A is a subset of the set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Proof of the Equal Sets

One way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. This turns out to be a useful way to show that two sets are equal. That is, $A = B$, where A and B are sets, if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$, or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

Finite Set and Cardinality

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Infinite Set

A set is said to be *infinite* if it is not finite.

Power Set

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

Examples of Power Set

What is the power set of the set $\{0, 1, 2\}$?

Solution: The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets.

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$P(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

Number of Power Set Elements

If a set has n elements, then its power set has 2^n elements.

Ordered n-tuple

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$. In particular, 2-tuples are called **ordered pairs.**

Cartesian Product

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Examples of Cartesian Product

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$.

Examples of Cartesian Product

The Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Cartesian Product

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

Example of Cartesian Product

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.$$

◀

Set Notation with Quantifiers

$\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

Example

What do the statements $\forall x \in \mathbf{R}\ (x^2 \geq 0)$ and $\exists x \in \mathbf{Z}\ (x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

Truth Set

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the **truth set** of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

Examples of Truth Set

What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Solution: The truth set of P, $\{x \in \mathbf{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q, $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

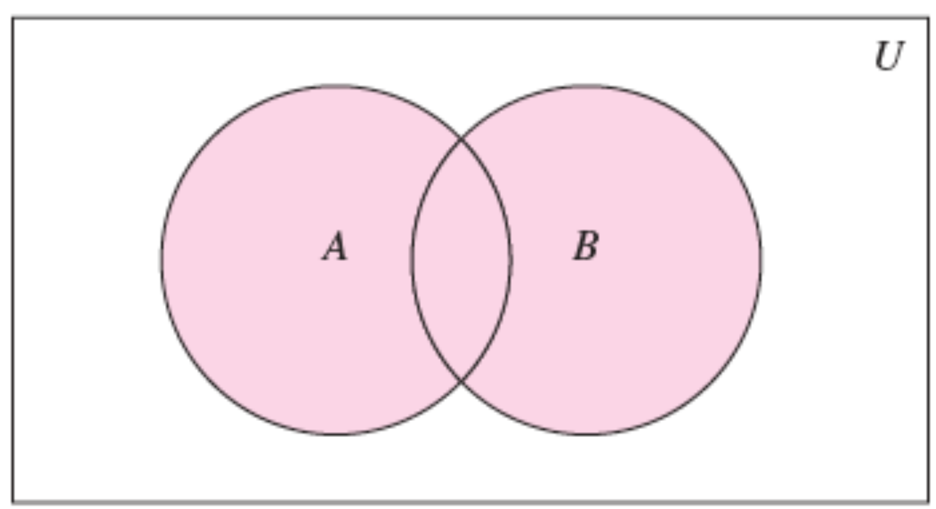
The truth set of R, $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is $\mathbf{N}$, the set of nonnegative integers. ◀

2.2 Set Operations

Union

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

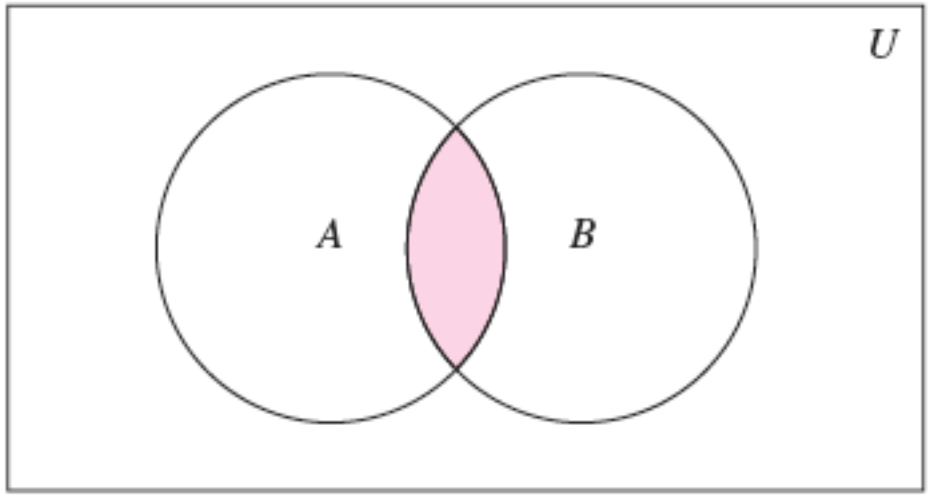


$A \cup B$ is shaded.

Intersection

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



$A \cap B$ is shaded.

Disjoint

Two sets are called *disjoint* if their intersection is the empty set.

Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

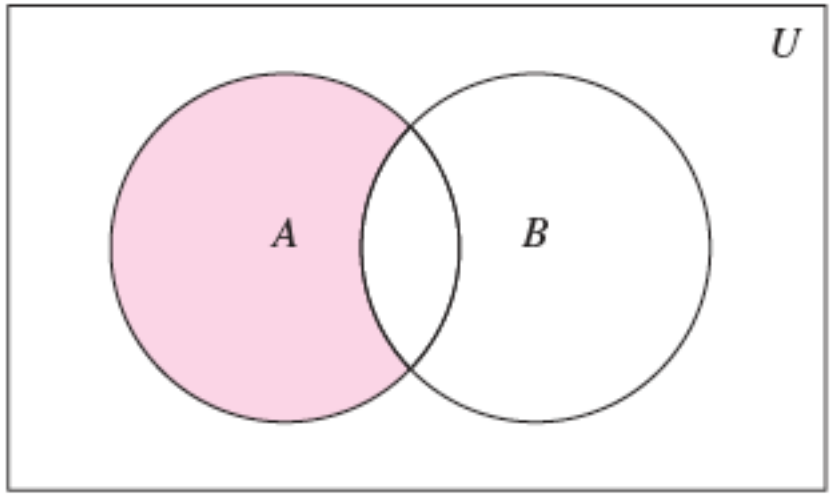
Cardinality

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Difference

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



$A - B$ is shaded.

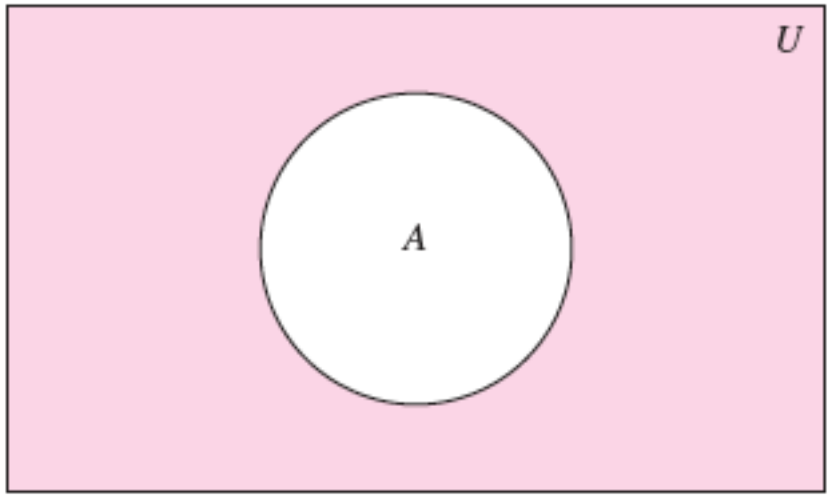
Example

The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ◀

Complement

Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. In other words, the complement of the set A is $U - A$.

$$\overline{A} = \{x \mid x \notin A\}.$$



$\overline{A}$ is shaded.

Example

Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Example of Set Identities

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: To show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. So suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. By the definition of intersection, $\neg((x \in A) \wedge (x \in B))$ is true. Applying De Morgan's law (from logic), we see that $\neg(x \in A)$ or $\neg(x \in B)$. Hence, by the definition of negation, $x \notin A$ or $x \notin B$. By the definition of complement, $x \in \overline{A}$ or $x \in \overline{B}$. It follows by the definition of union that $x \in \overline{A} \cup \overline{B}$. This shows that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. By the definition of union, $x \in \overline{A}$ or $x \in \overline{B}$. Using the definition of complement, we see that $x \notin A$ or $x \notin B$. Consequently, $\neg(x \in A) \vee \neg(x \in B)$ is true. By De Morgan's law (from logic), we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$ holds. We use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved. ◀

Example of Set Identities

Use set builder notation and logical equivalences to establish the second De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Example of Set Identities

Prove the first distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). Consequently, we know that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. ◀

Membership Table

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Example of Set Identities

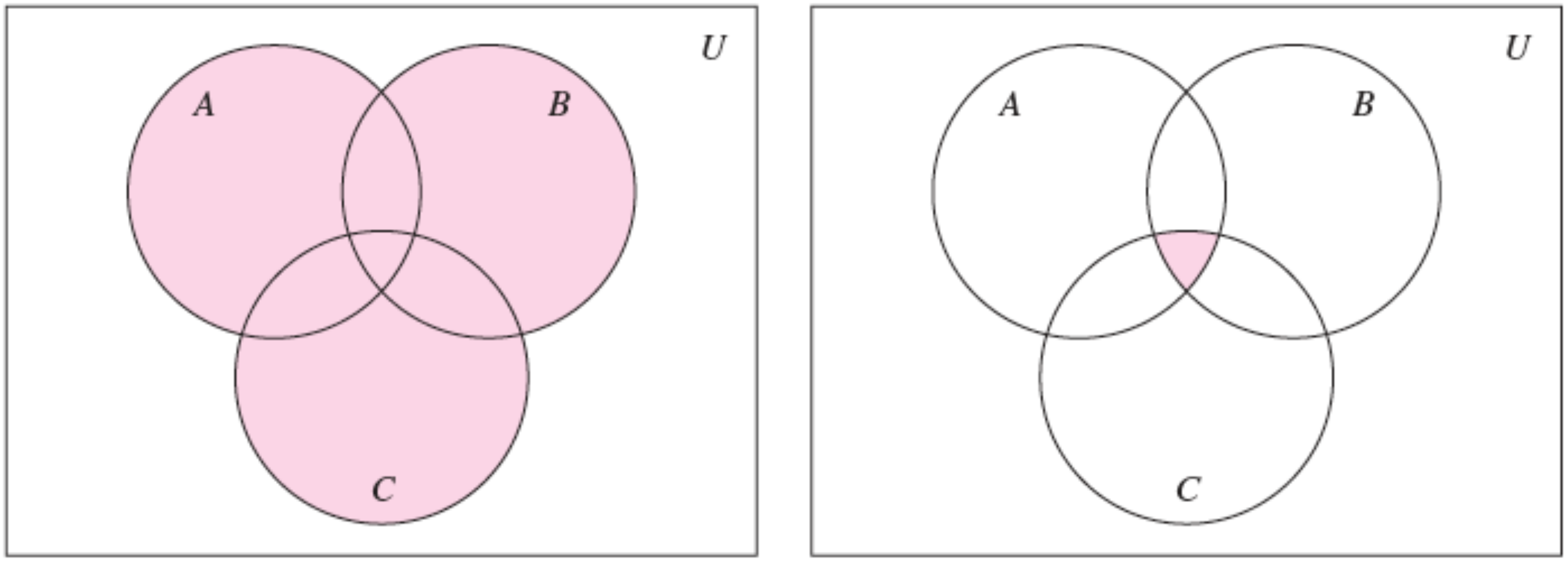
Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Generalized Unions and Intersections



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

Example

Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Solution: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,

$$A \cap B \cap C = \{0\}.$$

Generalized Union

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$$

Generalized Intersection

The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

Example

Suppose that $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$

Computer Representation of Sets

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading because long bit strings are difficult to read.) Similarly, we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000. ◀

Computer Representation of Sets

We have seen that the bit string for the set $\{1, 3, 5, 7, 9\}$ (with universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$) is

10 1010 1010.

What is the bit string for the complement of this set?

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set $\{2, 4, 6, 8, 10\}$. ◀

Computer Representation of Sets

The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set {1, 2, 3, 4, 5, 7, 9}. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

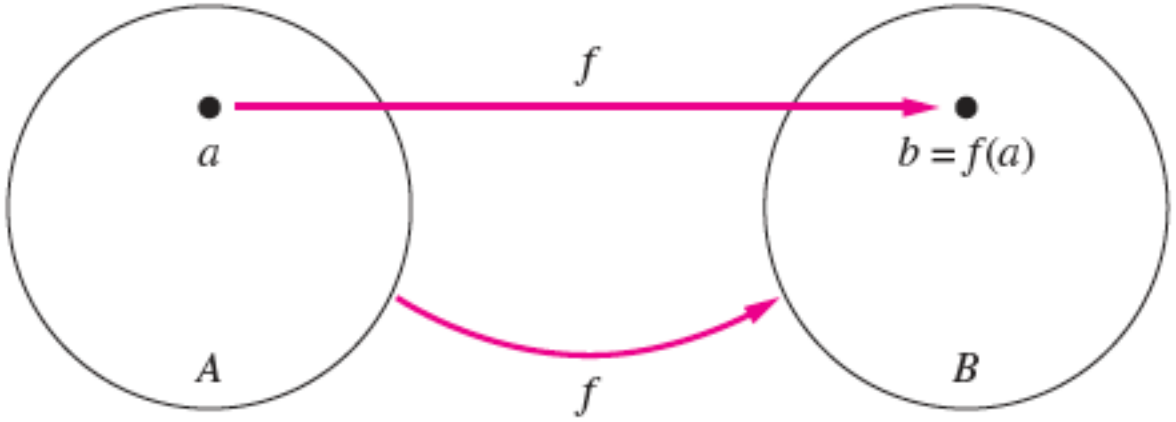
which corresponds to the set {1, 3, 5}. ◀

2.3 Functions

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Domain and Codomain

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range* of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.



Function

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

Image

Let f be a function from the set A to the set B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

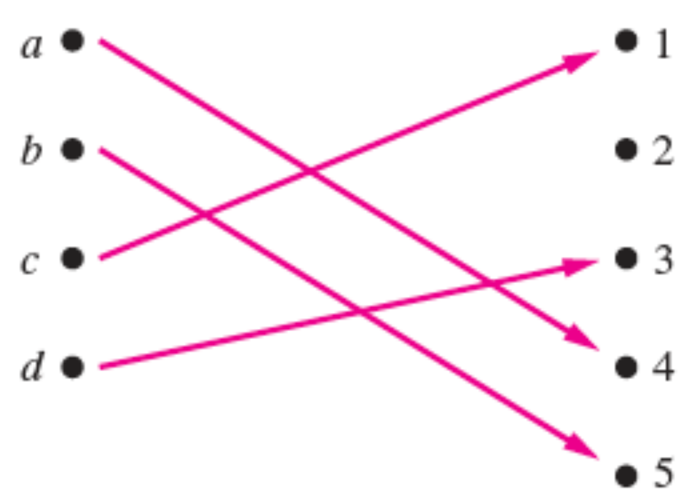
$$f(S) = \{\, t \mid \exists s \in S\ (t = f(s)) \,\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

one-to-one Function

A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

Remark: We can express that f is one-to-one using quantifiers as $\forall a \forall b(f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$, where the universe of discourse is the domain of the function.



Increasing and Decreasing

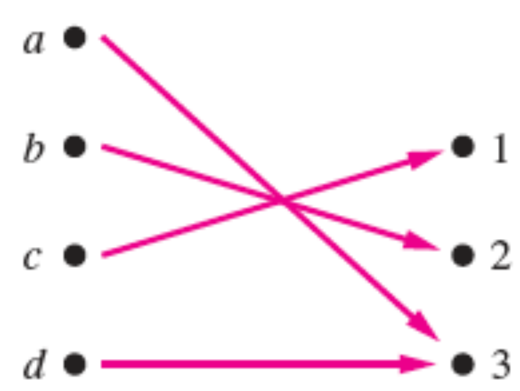
A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word *strictly* in this definition indicates a strict inequality.)

Remark: A function f is increasing if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$, decreasing if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$, and strictly decreasing if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

onto Function

A function f from A to B is called *onto*, or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is onto.

Remark: A function f is onto if $\forall y \exists x (f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.



one-to-one correspondence

The function f is a *one-to-one correspondence,* or a *bijection,* if it is both one-to-one and onto.

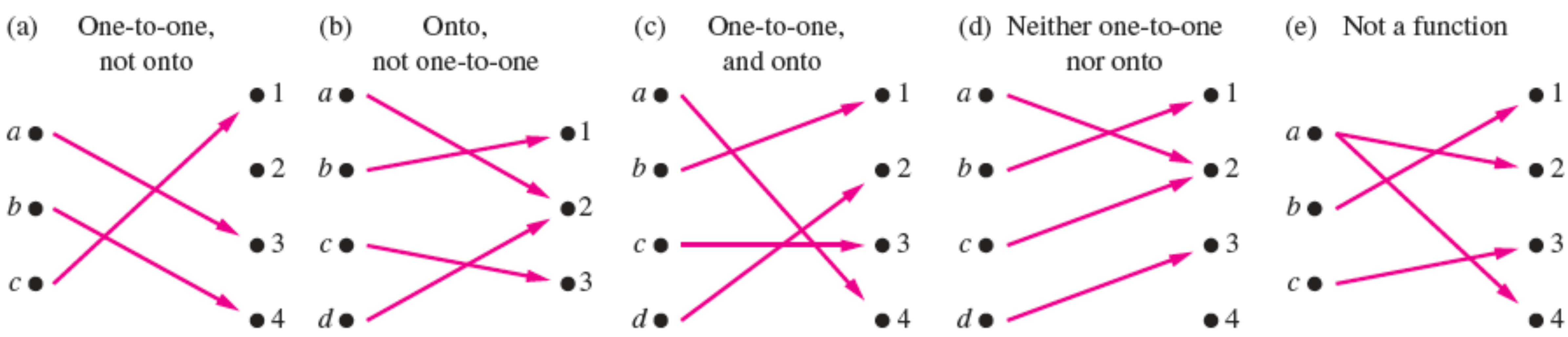
Identity Function

Let A be a set. The *identity function* on A is the function $\iota_A : A \rightarrow A$, where

$$\iota_A(x) = x$$

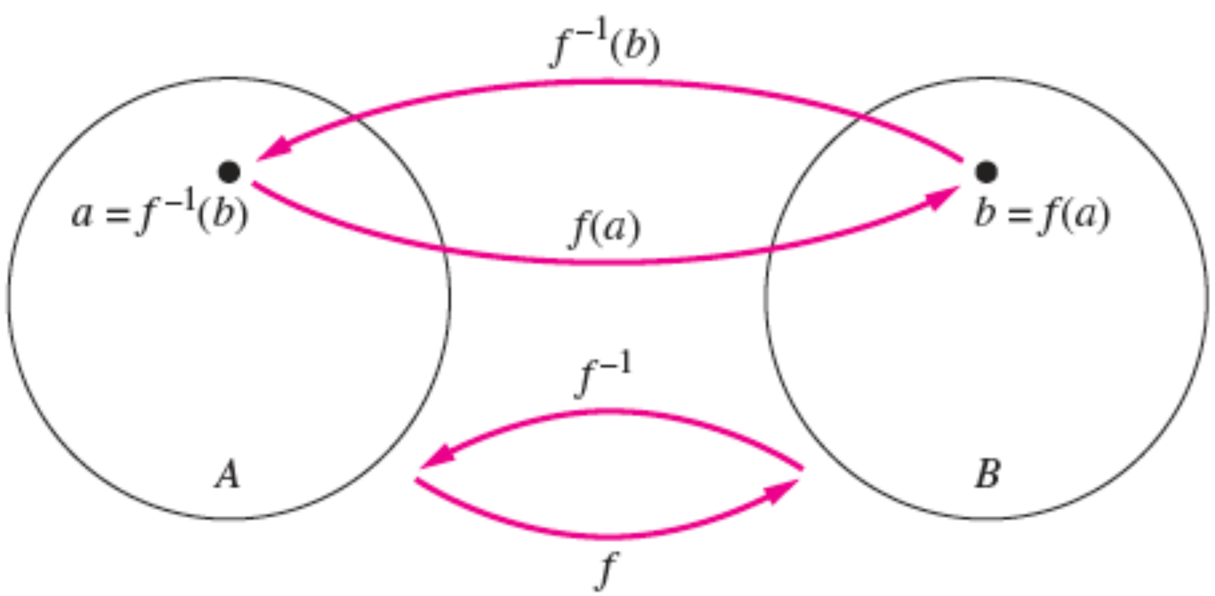
for all $x \in A$. In other words, the identity function ι_A is the function that assigns each element to itself. The function ι_A is one-to-one and onto, so it is a bijection. (Note that ι is the Greek letter iota.)

Correspondences



Inverse Functions

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.



Inverse Function

If a function f is not a one-to-one correspondence, we cannot define an inverse function of f. When f is not a one-to-one correspondence, either it is not one-to-one or it is not onto. If f is not one-to-one, some element b in the codomain is the image of more than one element in the domain. If f is not onto, for some element b in the codomain, no element a in the domain exists for which $f(a) = b$. Consequently, if f is not a one-to-one correspondence, we cannot assign to each element b in the codomain a unique element a in the domain such that $f(a) = b$ (because for some b there is either more than one such a or no such a).

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Example

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as we have shown. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$. ◀

Example

Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. ◀

Example

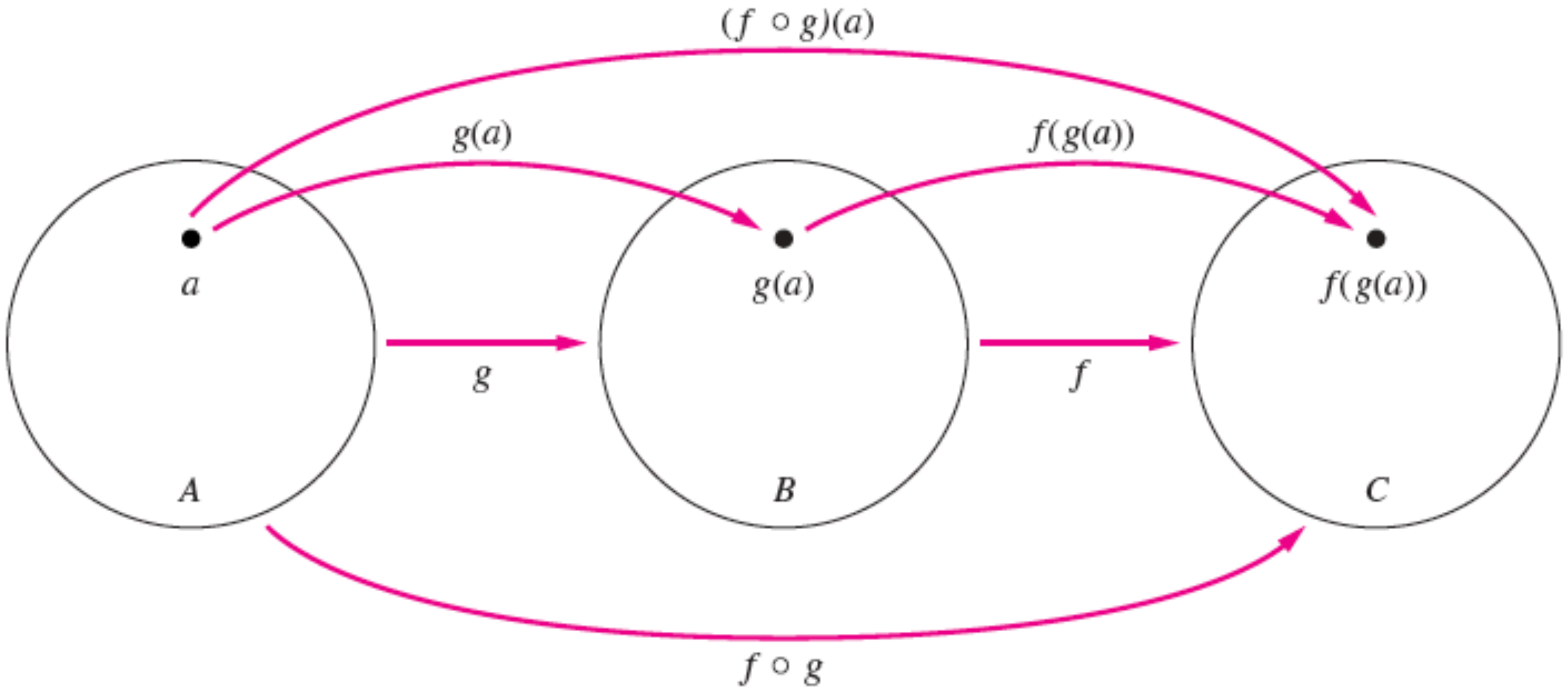
Show that if we restrict the function $f(x) = x^2$ in Example 18 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible.

Solution: The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both x and y are nonnegative, we must have $x = y$. So, this function is one-to-one. Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$. ◀

Composition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$



Example

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g. ◀

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

Identity

suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$. Hence,

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,$$

and

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

Graph

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

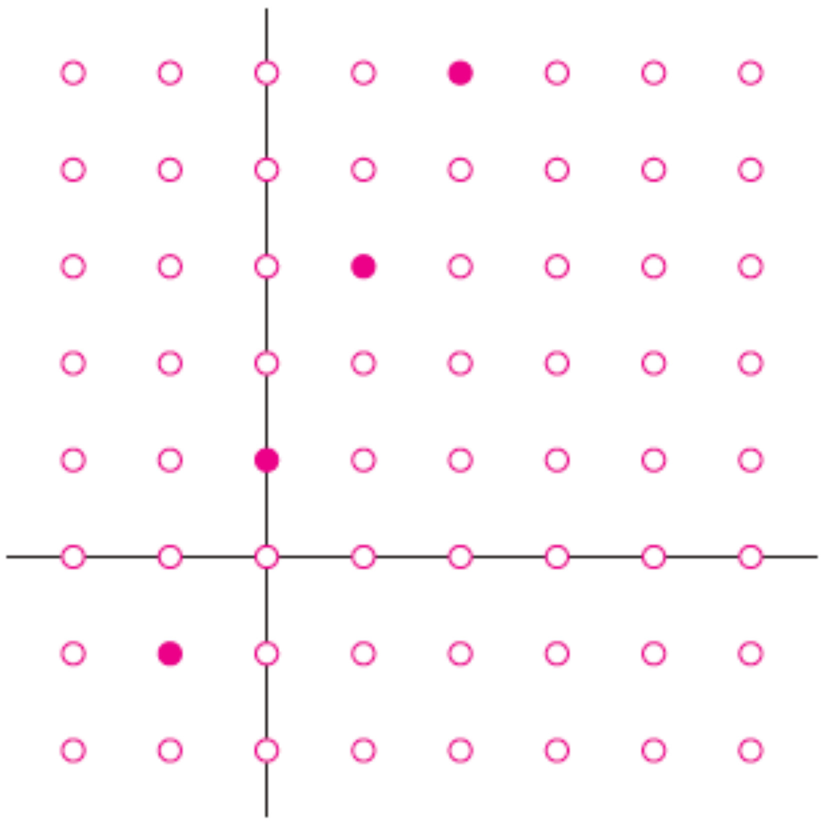


FIGURE 8 **The Graph of $f(n) = 2n + 1$ from Z to Z.**

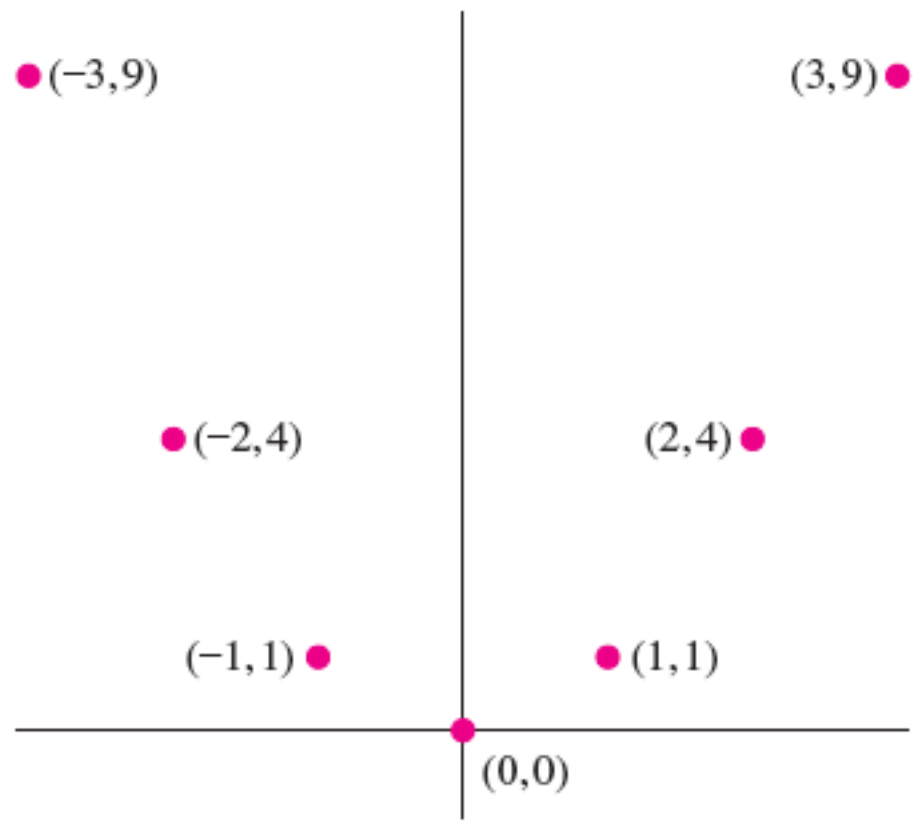
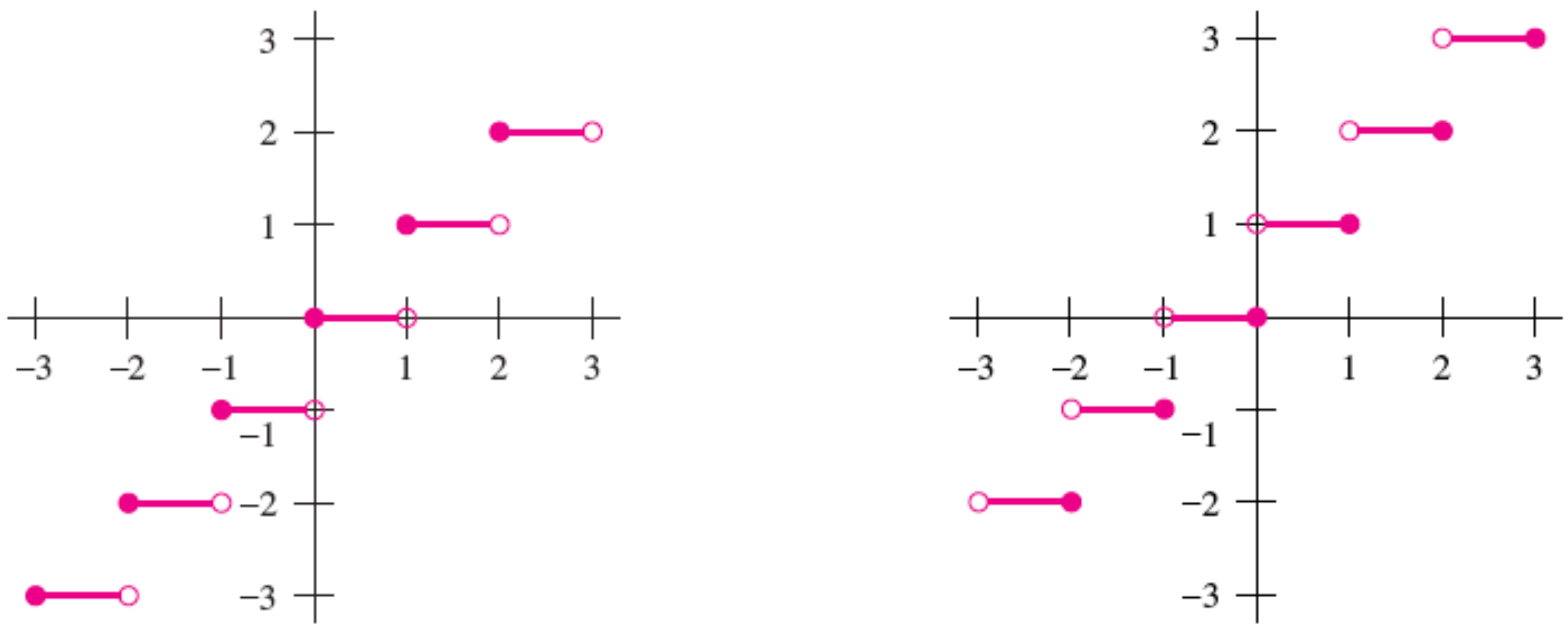


FIGURE 9 **The Graph of $f(x) = x^2$ from Z to Z.**

Floor and Ceiling Function

The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.



(a) $y = [x]$ (b) $y = [x]$

FIGURE 10 **Graphs of the (a) Floor and (b) Ceiling Functions.**

Floor and Ceiling Function

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Properties

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$

Proof: Suppose that $\lfloor x \rfloor = m$, where m is a positive integer. By property (1a), it follows that $m \leq x < m + 1$. Adding n to both sides of this inequality shows that $m + n \leq x + n < m + n + 1$. Using property (1a) again, we see that $\lfloor x + n \rfloor = m + n = \lfloor x \rfloor + n$.

Floor and Ceiling Function

A useful approach for considering statements about the floor function is to let $x = n + \epsilon$, where $n = \lfloor x \rfloor$ is an integer, and ϵ, the fractional part of x, satisfies the inequality $0 \leq \epsilon < 1$. Similarly, when considering statements about the ceiling function, it is useful to write $x = n - \epsilon$, where $n = \lceil x \rceil$ is an integer and $0 \leq \epsilon < 1$.

Example

Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Solution: To prove this statement we let $x = n + \epsilon$, where n is a positive integer and $0 \leq \epsilon < 1$. There are two cases to consider, depending on whether ϵ is less than or greater than or equal to $\frac{1}{2}$. (The reason we choose these two cases will be made clear in the proof.)

We first consider the case when $0 \leq \epsilon < \frac{1}{2}$. In this case, $2x = 2n + 2\epsilon$ and $\lfloor 2x \rfloor = 2n$ because $0 \leq 2\epsilon < 1$. Similarly, $x + \frac{1}{2} = n + (\frac{1}{2} + \epsilon)$, so $\lfloor x + \frac{1}{2} \rfloor = n$, because $0 < \frac{1}{2} + \epsilon < 1$. Consequently, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + n = 2n$.

Next, we consider the case when $\frac{1}{2} \leq \epsilon < 1$. In this case, $2x = 2n + 2\epsilon = (2n + 1) + (2\epsilon - 1)$. Because $0 \leq 2\epsilon - 1 < 1$, it follows that $\lfloor 2x \rfloor = 2n + 1$. Because $\lfloor x + \frac{1}{2} \rfloor = \lfloor n + (\frac{1}{2} + \epsilon) \rfloor = \lfloor n + 1 + (\epsilon - \frac{1}{2}) \rfloor$ and $0 \leq \epsilon - \frac{1}{2} < 1$, it follows that $\lfloor x + \frac{1}{2} \rfloor = n + 1$. Consequently, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + (n + 1) = 2n + 1$. This concludes the proof. ◀

Example

Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Solution: Although this statement may appear reasonable, it is false. A counterexample is supplied by $x = \frac{1}{2}$ and $y = \frac{1}{2}$. With these values we find that $\lceil x + y \rceil = \lceil \frac{1}{2} + \frac{1}{2} \rceil = \lceil 1 \rceil = 1$, but $\lceil x \rceil + \lceil y \rceil = \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil = 1 + 1 = 2$. ◄

Factorial Function

Another function we will use throughout this text is the **factorial function** $f : \mathbf{N} \to \mathbf{Z}^+$, denoted by $f(n) = n!$. The value of $f(n) = n!$ is the product of the first n positive integers, so $f(n) = 1 \cdot 2 \cdots (n-1) \cdot n$ [and $f(0) = 0! = 1$].