Multipoint distributions of the KPZ fixed point with compactly supported initial conditions

Yuchen Liao* Zhipeng Liu†

September 3, 2025

Abstract

The KPZ fixed point is a universal limiting space-time random field for the Kardar-Parisi-Zhang universality class. While the joint law of the KPZ fixed point at a fixed time has been studied extensively, the multipoint distributions of the KPZ fixed point in the general space-time plane are much less well understood. More explicitly, formulas were only available for the narrow wedge initial condition [JR21, Liu22] and the flat initial condition [Liu22] for the multipoint distributions, and the half-Brownian and Brownian initial conditions [JR22, Rah25] for the two-point distributions. In this paper, we obtain the first formula for the space-time joint distributions of the KPZ fixed point with general initial conditions of compact support. We also verify that the equal time degenerated version of our formula matches the path integral formula in [MQR21] for the KPZ fixed point.

The formula is obtained through taking $1:2:3$ KPZ scaling limit of the multipoint distribution formulas for the totally asymmetric simple exclusion process (TASEP). A key novelty is a probabilistic representation of the kernel encoding the initial condition for TASEP, which was first defined through an implicit characterization in [Liu22].

1 Introduction

1.1 Background

The Kardar–Parisi–Zhang (KPZ) universality class contains a broad family of random growth models in $(1+1)$-dimensions, including models from directed polymers, interacting particle systems, stochastic partial differential equations and so on. Since the seminal work [KPZ86], the KPZ universality class has become a central object of study in probability theory, statistical mechanics, and mathematical physics. For a more thorough introduction, we refer to the surveys [Cor12, Qua12, Zyg22] and the references therein.

A hallmark of this class is the universal $1:2:3$ scaling exponent for height fluctuations, spatial correlations and temporal correlations and a conjectural universal scaling limit for all the models in the universality class. More precisely, it is conjectured that the random height functions $H(x,t)$ describing the evolutions of different models will all converge to a universal limiting space-time field $\mathcal{H}(\alpha,\tau)$, under the following scaling:

$$\lim_{\varepsilon\to 0} c_3\varepsilon^{\frac{1}{2}} H(c_2\alpha\varepsilon^{-1}, c_3\tau\varepsilon^{-\frac{3}{2}}; \mathfrak{h}^{\varepsilon}) = \mathcal{H}(\alpha,\tau;\mathfrak{h}), \tag{1.1}$$

where $\mathfrak{h}^{\varepsilon}$ and $\mathfrak{h}$ are the initial conditions for the height functions before and after the limit with $\mathfrak{h}^{\varepsilon}\to\mathfrak{h}$ in a proper sense. A central question in this area is to understand $\mathcal{H}(\alpha,\tau;\mathfrak{h})$.

*School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R.China. Email: ycliao@ustc.edu.cn

†Department of Mathematics, University of Kansas, Lawrence, KS 66045. Email: zhipeng@ku.edu

The field $\mathcal{H}(\alpha,\tau;\mathfrak{h})$ is known as the KPZ fixed point. It was first constructed in [MQR21], and can be described as a $1:2:3$ scaling invariant Markov process on the space of upper semicontinuous functions on $\mathbb{R}$ with explicit formulas for its transition probability. Convergence to the KPZ fixed point has only been shown for a few special models, see [MQR21, NQR20, MQR25, ACH24a]. An alternative description is through a Hopf-Lax type variational formula, with the driving force given by the directed landscape $\mathcal{L}(y,s;x,t)$. This is another universal limiting object in the KPZ universality class first constructed in [DOV22]. Convergence to the directed landscape are shown for a few special models in [DOV22, DV21, Wu23, ACH24b, DZ24].

It is well known (see, e.g., [BDJ99, Joh00, ACQ11]) that for special initial conditions, the one point marginals of $\mathcal{H}(\alpha,\tau)$ are described by the Tracy-Widom distribution and its relatives. Extensions to joint laws of multiple spatial points at equal time was obtained in [PS02, BFPS07, BFS08, BFP10], leading to explicit descriptions of the spatial process $\mathcal{H}(\cdot,\tau)$ for special initial conditions. In the breakthrough work [MQR21], the authors were able to find explicit Fredholm determinant formulas for the joint laws of $\mathcal{H}(\alpha_1,\tau;\mathfrak{h}),\cdots,\mathcal{H}(\alpha_m,\tau;\mathfrak{h})$, starting from general upper semicontinuous initial conditions. This leads a complete description of the Markovian dynamics of the fixed point.

Joint laws along the time direction, or more generally in space-time, are much less known until recently. For the narrow wedge initial condition, a formula for the multi-time distribution was obtained by [JR21], which builds on the earlier work of two-time formulas in [Joh17, Joh19]. A different multipoint formula which works for both the narrow wedge and the flat initial conditions and possibly equal time parameters, was obtained in [Liu22]. We remark that a direct proof of the equivalence between the two formulas for the narrow wedge initial conditions is still missing due to the complicatedness of both formulas. Two-time formulas for half-Brownian or Brownian initial conditions were also obtained recently in [JR22, Rah25]. Besides these distribution formulas, there are also results on the correlation or tail properties of KPZ models at two times, see [dNLD17, dNLD18, LD17, Joh20, FS16, FO19, CGH21]. We point out that all these mentioned results on the multi-time problems are studying the KPZ fixed point on $\mathbb{R}$ and with special initial conditions. It is also worth mentioning the related work [BL19, BL21, Lia22] for the multipoint distributions of TASEP models in periodic domain.

1.2 Main results

The main goal of this paper is to describe the space-time joint distributions of the two-dimensional random field $\mathcal{H}(\alpha,\tau;\mathfrak{h})$, with sufficiently general initial conditions $\mathfrak{h}$, in the same spirit as in [MQR21]. We start with introducing the spaces of initial conditions we will consider. The largest possible space of initial conditions from which the KPZ fixed point will be almost surely finite at all positive time is the following:

$$\mathrm{UC} := \left\{\mathfrak{h}:\mathbb{R}\to[-\infty,\infty) \text{ upper semicontinuous}, \mathfrak{h}\not\equiv-\infty \text{ and } \limsup_{x\to\pm\infty}\frac{\mathfrak{h}(\mathsf{x})}{\mathsf{x}^2}\le 0\right\}. \tag{1.2}$$

For technical reasons, We will mostly work with the dense subspace of UC consisting of functions that are $-\infty$ outside of a compact set.

Definition 1.1 (The function spaces of initial conditions and topology)**.** *Define*

$$\mathrm{UC}_c := \{\mathfrak{h}\in\mathrm{UC} : \textit{there exists } L>0 \textit{ such that } \mathfrak{h}(\mathsf{x})=-\infty \textit{ for all } |\mathsf{x}|>L\}. \tag{1.3}$$

The space is equipped with the topology of local Hausdorff convergence of hypographs. We will call functions $\mathfrak{h}\in\mathrm{UC}_c$ compactly supported, where the support of $\mathfrak{h}\in\mathrm{UC}$ is defined as

$$\mathrm{supp}(\mathfrak{h}) := \overline{\{\mathsf{x}\in\mathbb{R}:\mathfrak{h}(\mathsf{x})\neq-\infty\}}, \tag{1.4}$$

and $\overline{A}$ means the closure of the set A.

Our main results are formulas for the joint distributions of the KPZ fixed point starting with initial condition $\mathfrak{h} \in \mathrm{UC}_c$, at arbitrary many distinct space-time points $(\alpha_1, \tau_1), \cdots, (\alpha_m, \tau_m)$. To state the result, we introduce the following total ordering $\prec$ on the space-time plane $\mathbb{R} \times \mathbb{R}$:

$$(\alpha_1, \tau_1) \prec (\alpha_2, \tau_2) \Longleftrightarrow \tau_1 < \tau_2, \text{ or } \tau_1 = \tau_2 \text{ and } \alpha_1 < \alpha_2. \tag{1.5}$$

Theorem 1.2. *Let $\mathfrak{h} \in \mathrm{UC}_c$. Then for any $m \geq 1$ and any m space-time points $(\alpha_1, \tau_1) \prec \cdots \prec (\alpha_m, \tau_m) \in \mathbb{R} \times \mathbb{R}_+$, we have the following formula for the multi-point distribution of the KPZ fixed point $\mathcal{H}(\alpha, \tau; \mathfrak{h})$:*

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m} \{\mathcal{H}(\alpha_\ell, \tau_\ell; \mathfrak{h}) \leq \beta_\ell\}\right) = \oint_0 \frac{\mathrm{d}z_1}{2\pi\mathrm{i} z_1(1-z_1)} \cdots \oint_0 \frac{\mathrm{d}z_{m-1}}{2\pi\mathrm{i} z_{m-1}(1-z_{m-1})} \mathbf{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1}). \tag{1.6}$$

The function $\mathbf{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1})$ is defined as a Fredholm determinant in Definition 2.1. An equivalent definition through a series expansion will be discussed in Section 2.2.

Similar as in the narrow wedge case [JR21, Liu22], our multipoint formula for the KPZ fixed point with a general initial condition has the form of contour integrals of a Fredholm determinant. The Fredholm determinant $\mathbf{D}_{\mathfrak{h}}$ has a block diagonal kernel acting on nested Airy-type contours. The dependency on the initial condition is only through the top-left corner of the kernel, characterized by a function $\chi_{\mathfrak{h}}(\eta, \xi)$ defined on certain Airy contours, see Section 2.1.1 for its definition. For the narrow wedge case, our formula matches with the one in [Liu22].

The function $\chi_{\mathfrak{h}}(\eta, \xi)$ is defined in terms of Brownian motion hitting expectations, an idea highly inspired by [MQR21]. Indeed, $\chi_{\mathfrak{h}}(\eta, \xi)$ should be understood as the Brownian hitting operators in [MQR21] written in Fourier-like spaces. Nevertheless we stress that our results do not follow directly from [MQR21]. In the multi-time situation, direct connections to determinantal point processes and the Eynard-Mehta theorem are lost and the bi-orthoganalization procedure here arises in a different way and takes a different form. On the contrary, our results are, in some sense, more general. Indeed if we set the time parameters to be the same (which is allowed in the assumption of the theorem), the right-hand side of (1.6) can be shown to recover the formulas in [MQR21], after some quite non-trivial manipulations. We refer to Section 6.2 for the details, see also [LO25] which treats the special narrow wedge case.

1.3 Outline of the proof and some discussions

Theorem 1.2 is proved by taking a $1:2:3$ scaling limit of the corresponding multipoint distribution formulas of the totally asymmetric simple exclusion process (TASEP). The starting point is an algebraic formula obtained in [Liu22, Theorem 2.1] for the multipoint (space-time) distribution of TASEP starting from any right-finite initial condition. The dependency of the TASEP formula on the initial condition is encoded in a function $\mathrm{ch}_Y(v, u)$, which is characterized by an implicit reproducing-type property, see Definition 3.1. For the step and (pseudo) flat initial condition, an explicit form of $\mathrm{ch}_Y(v, u)$ in terms of symmetric functions was obtained in [Liu22] and is suitable for asymptotics, thus leading to the corresponding multipoint formula for the KPZ fixed point starting from the narrow wedge and flat initial conditions after taking limits.

A key novelty of this paper is that we find an explicit probabilistic representation of the function $\mathrm{ch}_Y(v, u)$, through a hitting expectation with respect to geometric random walks, see Theorem 3.4. The probabilistic representation is suitable for asymptotic analysis and leads to the Brownian hitting representation in the limit. For technical reasons, we first take the limit of the TASEP formula under the assumption that the KPZ fixed point starts with initial conditions consisting of finitely many narrow wedges, and then extend the formula to compactly supported initial condition at the level of the KPZ fixed point, using a density argument and the continuity of the law of the KPZ fixed point with respect to initial conditions.

Finally we comment on our assumptions on the initial conditions. It would be desirable if one can get a formula that works for all initial conditions $\mathfrak{h} \in \mathrm{UC}$, in particular, the flat initial condition $\mathfrak{h} \equiv 0$. The reason we choose to restrict to the subspace UC_c is not merely a technical issue. There are genuine structural difficulties in this generality: the characteristic function $\chi_{\mathfrak{h}}(\eta, \xi)$ of the initial condition (see Definition 2.2) may not be well-defined pointwisely in general. Indeed for the flat initial condition $\mathfrak{h} \equiv 0$, one can show from our formula that $\chi_{\mathfrak{h}}(\eta, \xi)$ is the dirac delta kernel $\delta_{\eta=-\xi}$, when properly interpreted. This suggests that in general, one should conjugate our formula to real spaces so that the limit when the support goes to infinity exists, at the level of operators acting on real spaces. We choose to stick to contour integral/Fourier type kernels as it makes the algebraic structure of the multi-time formula much nicer and the analogy between narrow wedge and general initial conditions much more transparent. We leave it as a future project to extend our formula to any $\mathfrak{h} \in \mathrm{UC}$, with a kernel acting on real spaces.

Notation and conventions

Throughout the paper, we will mostly use english letters $x, t, h, u, v, w, \cdots$ for the pre-limit (TASEP) formulas and greek letters $\alpha, \tau, \beta, \xi, \eta, \zeta, \cdots$ for the limiting (KPZ fixed point) formulas. A detailed summary of the notation we use is in the following table.

Notation	**Pre-limit (TASEP) formulas**	**Limiting (KPZ fixed point) formulas**
time, space, height	t, x, h	τ, α, β
initial height function	$\mathsf{h}(\cdot)$	$\mathfrak{h}(\cdot)$
the height function	$H(x, t; \mathsf{h})$	$\mathcal{H}(\alpha, \tau; \mathfrak{h})$
integration contours	$\Sigma_{\mathrm{L}}, \Sigma_{\mathrm{R}}$	$\Gamma_{\mathrm{L}}, \Gamma_{\mathrm{R}}$
integration variables	u, v, w	ξ, η, ζ

Organization of the paper

The rest of the paper is organized as follows. In Section 2 we present the formulas for the main part $\mathbf{D}_{\mathfrak{h}}$ appearing in Theorem 1.2, both as a Fredholm determinant in Section 2.1, and as a Fredholm series expansion in Section 2.2. Then in Section 3 we present and prove the corresponding pre-limit formulas for TASEP, in particular in Section 3.1 we prove that the characteristic function of the initial condition is given by a random walk hitting expectation. In Section 4 we prove convergence of the TASEP formulas to the KPZ fixed point formulas, under the assumption that the initial condition of the KPZ fixed point consists of finitely many narrow wedges. We then extend the KPZ fixed point formula to any compactly supported initial condition in Section 5. Finally in Section 6 we show that at equal-time, our formula reduces to a genuine Fredholm determinant, which is then shown to be equivalent to the path integral formula of [MQR21].

Acknowledgements

YL acknowledges the support of a Junior Fellowship at Institut Mittag-Leffler in Djursholm, Sweden, during the Fall semester of 2024, funded by the Swedish Research Council under grant no. 2021-06594. The work of ZL is partially supported by NSF DMS-2246683.

2 Multipoint distribution formula for the KPZ fixed point

In this section we explain in details the function $\mathbf{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1})$ appearing on the right-hand side of (1.6). Proofs of the formula will be deferred to Section 4 and Section 5.

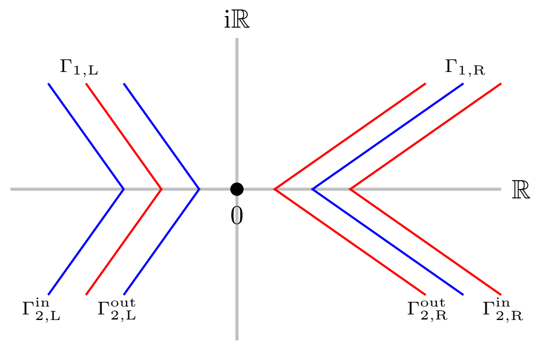

Figure 1: Illustration of the contours for $m = 2$: S_1 is the union of the red contours and S_2 is the union of the blue contours.

2.1 Fredholm determinant representation of $\mathbf{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1})$

The function $\mathbf{D}_{\mathfrak{h}}$ is defined in the same way as in [Liu22] for the narrow wedge initial condition, except for the part that involves the initial condition $\mathfrak{h}$ which only appears in the top-left corner of the integral kernel. Below we introduce the Fredholm determinant representation of the function $\mathbf{D}_{\mathfrak{h}}$.

Denote two regions of the complex plane

$$\mathbb{C}_{\mathrm{L}} := \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) < 0\}, \quad \text{and} \quad \mathbb{C}_{\mathrm{R}} := \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) > 0\}. \tag{2.1}$$

Let $\Gamma^{\mathrm{out}}_{m,\mathrm{L}}, \cdots, \Gamma^{\mathrm{out}}_{2,\mathrm{L}}, \Gamma_{1,\mathrm{L}}, \Gamma^{\mathrm{in}}_{2,\mathrm{L}}, \cdots, \Gamma^{\mathrm{in}}_{m,\mathrm{L}}$ be $2m-1$ "nested" contours in the region $\mathbb{C}_{\mathrm{L}}$. They are all unbounded contours from $\infty e^{-2\pi i/3}$ to $\infty e^{2\pi i/3}$. Moreover, they are located from the right (corresponding to the superscript "out") to the left ("in"). The superscripts "out" and "in" should be understood with respect to the point $-\infty$. Similarly, let $\Gamma^{\mathrm{out}}_{m,\mathrm{R}}, \cdots, \Gamma^{\mathrm{out}}_{2,\mathrm{R}}, \Gamma_{1,\mathrm{R}}, \Gamma^{\mathrm{in}}_{2,\mathrm{R}}, \cdots, \Gamma^{\mathrm{in}}_{m,\mathrm{R}}$ be $2m-1$ "nested" contours from left to right on the half plane $\mathbb{C}_{\mathrm{R}}$. They are from $\infty e^{-\pi i/5}$ to $\infty e^{\pi i/5}$. Their superscripts "out" and "in" could be understood with respect to the point $+\infty$. Note that the angles for the left contours and right contours are chosen differently. The choice of the angles guarantees super-exponential decay of the kernel along the contours even if $\tau_i = \tau_{i+1}$ for some i. See [Liu22, LZ25] for more discussions on the choices of the angles. See Figure 1 for an illustration of the contours.

We define

$$\Gamma_{\ell,\mathrm{L}} := \Gamma^{\mathrm{out}}_{\ell,\mathrm{L}} \cup \Gamma^{\mathrm{in}}_{\ell,\mathrm{L}}, \qquad \Gamma_{\ell,\mathrm{R}} := \Gamma^{\mathrm{out}}_{\ell,\mathrm{R}} \cup \Gamma^{\mathrm{in}}_{\ell,\mathrm{R}}, \qquad \ell = 2, \cdots, m,$$

and

$$\mathrm{S}_1 := \Gamma_{1,\mathrm{L}} \cup \Gamma_{2,\mathrm{R}} \cup \cdots \cup \begin{cases} \Gamma_{m,\mathrm{L}}, & \text{if } m \text{ is odd}, \\ \Gamma_{m,\mathrm{R}}, & \text{if } m \text{ is even}, \end{cases}$$

and

$$\mathrm{S}_2 := \Gamma_{1,\mathrm{R}} \cup \Gamma_{2,\mathrm{L}} \cup \cdots \cup \begin{cases} \Gamma_{m,\mathrm{R}}, & \text{if } m \text{ is odd}, \\ \Gamma_{m,\mathrm{L}}, & \text{if } m \text{ is even}. \end{cases}$$

We introduce a measure on these contours. Let

$$\mathrm{d}\mu(\zeta) = \mathrm{d}\mu_{\boldsymbol{z}}(\zeta) := \begin{cases} \dfrac{-z_{\ell-1}}{1 - z_{\ell-1}} \dfrac{\mathrm{d}\zeta}{2\pi i}, & \zeta \in \Gamma^{\mathrm{out}}_{\ell,\mathrm{L}} \cup \Gamma^{\mathrm{out}}_{\ell,\mathrm{R}}, \quad \ell = 2, \cdots, m, \\ \dfrac{1}{1 - z_{\ell-1}} \dfrac{\mathrm{d}\zeta}{2\pi i}, & \zeta \in \Gamma^{\mathrm{in}}_{\ell,\mathrm{L}} \cup \Gamma^{\mathrm{in}}_{\ell,\mathrm{R}}, \quad \ell = 2, \cdots, m, \\ \dfrac{\mathrm{d}\zeta}{2\pi i}, & \zeta \in \Gamma_{1,\mathrm{L}} \cup \Gamma_{1,\mathrm{R}}. \end{cases} \tag{2.2}$$

Let Q_1 and Q_2 be as follows:,

$$Q_1(j) := \begin{cases} 1 - z_j, & \text{if } j \text{ is odd and } j < m, \\ 1 - \dfrac{1}{z_{j-1}}, & \text{if } j \text{ is even}, \\ 1, & \text{if } j = m \text{ is odd}, \end{cases} \qquad Q_2(j) := \begin{cases} 1 - z_j, & \text{if } j \text{ is even and } j < m, \\ 1 - \dfrac{1}{z_{j-1}}, & \text{if } j \text{ is odd and } j > 1, \\ 1, & \text{if } j = m \text{ is even, or } j = 1. \end{cases} \tag{2.3}$$

Definition 2.1. *We define*

$$\mathrm{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1}) = \det\left(\mathrm{I} - \mathrm{K}_1 \mathrm{K}_{\mathfrak{h}}\right),$$

where the operators

$$\mathrm{K}_1 : L^2(\mathrm{S}_2, \mathrm{d}\mu) \to L^2(\mathrm{S}_1, \mathrm{d}\mu), \qquad \mathrm{K}_{\mathfrak{h}} : L^2(\mathrm{S}_1, \mathrm{d}\mu) \to L^2(\mathrm{S}_2, \mathrm{d}\mu)$$

are defined by their kernels

$$\mathrm{K}_1(\zeta, \zeta') := \left(\delta_i(j) + \delta_i(j + (-1)^i)\right) \frac{\widehat{\mathrm{f}}_i(\zeta)}{\zeta - \zeta'} Q_1(j) \tag{2.4}$$

and

$$\mathrm{K}_{\mathfrak{h}}(\zeta', \zeta) := \begin{cases} \left(\delta_j(i) + \delta_j(i - (-1)^j)\right) \frac{\widehat{\mathrm{f}}_j(\zeta')}{\zeta - \zeta'} Q_2(i), & i \geq 2, \\ \delta_j(1) \widehat{\mathrm{f}}_1(\zeta') \chi_{\mathfrak{h}}(\zeta', \zeta), & i = 1, \end{cases} \tag{2.5}$$

for any $\zeta \in (\Gamma_{i,\mathrm{L}} \cup \Gamma_{i,\mathrm{R}}) \cap \mathrm{S}_1$ and $\zeta' \in (\Gamma_{j,\mathrm{L}} \cup \Gamma_{j,\mathrm{R}}) \cap \mathrm{S}_2$ with $1 \leq i, j \leq m$. Here the function

$$\widehat{\mathrm{f}}_i(\zeta) := \begin{cases} \mathrm{f}_i(\zeta), & \mathrm{Re}(\zeta) < 0, \\ \dfrac{1}{\mathrm{f}_i(\zeta)}, & \mathrm{Re}(\zeta) > 0, \end{cases} \tag{2.6}$$

with

$$\mathrm{f}_i(\zeta) := \begin{cases} \mathrm{e}^{-\frac{1}{3}(\tau_i - \tau_{i-1})\zeta^3 + (\alpha_i - \alpha_{i-1})\zeta^2 + (\beta_i - \beta_{i-1})\zeta}, & i = 2, \cdots, m, \\ \mathrm{e}^{-\frac{1}{3}\tau_1 \zeta^3 + \alpha_1 \zeta^2 + \beta_1 \zeta}, & i = 1. \end{cases} \tag{2.7}$$

The kernel $\chi_{\mathfrak{h}}(\zeta', \zeta)$ is defined in Section 2.1.1, see Definition 2.2.

2.1.1 The characteristic function $\chi_{\mathfrak{h}}$

The dependency on the initial condition of the entire formula is through the function $\chi_{\mathfrak{h}}$ defined on $\mathbb{C}_{\mathrm{R}} \times \mathbb{C}_L$. Recall that we always use ξ and η to denote a variable on the Γ-contours on $\mathbb{C}_{\mathrm{L}}$ and $\mathbb{C}_{\mathrm{R}}$ respectively throughout the paper. Note that $\chi_{\mathfrak{h}}$ is a function on $((\Gamma_{1,\mathrm{L}} \cup \Gamma_{1,\mathrm{R}}) \cap \mathrm{S}_2) \times ((\Gamma_{1,\mathrm{L}} \cup \Gamma_{1,\mathrm{R}}) \cap \mathrm{S}_1) = \Gamma_{1,\mathrm{R}} \times \Gamma_{1,\mathrm{L}}$. So we will use the notation $\chi_{\mathfrak{h}}(\eta, \xi)$ in the paper. The functions $\chi_{\mathfrak{h}}$ is defined using a Brownian motion hitting expectation as follows:

Definition 2.2. *Let $\mathbf{B}(t)$ be a two-sided Brownian motion with diffusivity constant 2. Let $\mathfrak{h} \in \mathrm{UC}_c$ and $\boldsymbol{\tau}_{\pm}$ be the hitting time of $\mathbf{B}$ to the hypograph of the positive (respectively, negative) part of $\mathfrak{h}$, i.e.,*

$$\boldsymbol{\tau}_+ := \inf\{\alpha \geq 0 : \mathbf{B}(\alpha) \leq \mathfrak{h}(\alpha)\}, \quad \boldsymbol{\tau}_- := \sup\{\alpha \leq 0 : \mathbf{B}(\alpha) \leq \mathfrak{h}(\alpha)\}. \tag{2.8}$$

Then for any $\xi \in \mathbb{C}_{\mathrm{L}}$ and $\eta \in \mathbb{C}_{\mathrm{R}}$ we define

$$\begin{aligned} \chi_{\mathfrak{h}}(\eta, \xi) := &\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{+s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(-\boldsymbol{\tau}_+ \xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+ < \infty}\right] \\ &+ \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-s\xi} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(\boldsymbol{\tau}_- \eta^2 + \mathbf{B}(\boldsymbol{\tau}_-)\eta\right) \mathbf{1}_{\boldsymbol{\tau}_- > -\infty}\right] \\ &- \int_{\mathbb{R}} \mathrm{d}s\, \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(-\boldsymbol{\tau}_+ \xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+ < \infty}\right] \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(\boldsymbol{\tau}_- \eta^2 + \mathbf{B}(\boldsymbol{\tau}_-)\eta\right) \mathbf{1}_{\boldsymbol{\tau}_- > -\infty}\right]. \end{aligned} \tag{2.9}$$

Proposition 2.3. *The function $\chi_{\mathfrak{h}}(\eta,\xi)$ is well-defined and analytic for $\zeta \in \mathbb{C}_{\mathrm{L}}$ and $\eta \in \mathbb{C}_{\mathrm{R}}$. Moreover if $\mathfrak{h}(\alpha) = -\infty$ for $|\alpha| > L$, we have*

$$|\chi_{\mathfrak{h}}(\eta,\xi)| \leq CL^{\frac{3}{2}} \mathrm{e}^{\mathrm{Re}(\eta-\xi)\cdot \max_{\alpha\in\mathbb{R}} \mathfrak{h}(\alpha)} \cdot \left(\mathrm{e}^{cL(|\xi|^2+|\eta|^2)} + \frac{1}{\mathrm{Re}(\eta-\xi)} \right), \tag{2.10}$$

for some constants $c, C > 0$ depending only on $\mathfrak{h}$.

Proof. We verify the statement for the first term appearing on the right-hand side of (2.9), the other two terms can be treated similarly. For the well-definedness it suffices to show that the integral over the s-variable converges absolutely. First we split the integral into two parts depending on whether $s \leq \mathfrak{h}(0)$ or not. For $s \leq \mathfrak{h}(0)$ clearly we have $\boldsymbol{\tau}_+ = 0$ and $\mathbf{B}(\boldsymbol{\tau}_+) = s$, hence

$$\int_{-\infty}^{\mathfrak{h}(0)} \mathrm{d}s\, \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s} \left[\exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+<\infty} \right] = \int_{-\infty}^{\mathfrak{h}(0)} \mathrm{d}s\, \mathrm{e}^{-s\xi+s\eta} = \frac{\mathrm{e}^{\mathfrak{h}(0)(\eta-\xi)}}{\eta-\xi}, \tag{2.11}$$

with the integral converging absolutely since $\mathrm{Re}(\xi) < \mathrm{Re}(\eta)$ by our assumption. For the other part note that $\mathfrak{h} \in \mathrm{UC}_c$ is bounded above, namely, $\mathfrak{h}(\alpha) \leq \beta$ for some $\beta \in \mathbb{R}$. Define a new stopping time

$$\boldsymbol{\sigma}_+ := \inf\{\alpha \geq 0 : \mathbf{B}(\alpha) \leq \beta\},$$

then clearly we have $\boldsymbol{\sigma}_+ \leq \boldsymbol{\tau}_+$. Hence $\mathbb{P}_{\mathbf{B}(0)=s}(\boldsymbol{\tau}_+ < +\infty) \leq \mathbb{P}_{\mathbf{B}(0)=s}(\boldsymbol{\sigma}_+ < +\infty)$. Thus

$$\begin{aligned} &\mathbb{E}_{\mathbf{B}(0)=s} \left[\left| \exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+<\infty} \right| \right] \\ &\leq \mathbb{E}_{\mathbf{B}(0)=s} \left[\exp\left(\boldsymbol{\tau}_+ \cdot \mathrm{Re}(-\xi^2) - \beta \cdot \mathrm{Re}(\xi)\right) \mathbf{1}_{\boldsymbol{\tau}_+<\infty} \right] \\ &\leq \exp(L|\xi|^2 - \beta\mathrm{Re}(\xi)) \cdot \mathbb{E}_{\mathbf{B}(0)=s} \left[\mathbf{1}_{\boldsymbol{\tau}_+<\infty} \right] \leq \exp(L|\xi|^2 - \beta\mathrm{Re}(\xi)) \cdot \mathbb{E}_{\mathbf{B}(0)=s} \left[\mathbf{1}_{\boldsymbol{\sigma}_+<\infty} \right]. \end{aligned} \tag{2.12}$$

The density of $\boldsymbol{\sigma}_+$ can be computed using the reflection principle, see, e.g., [Dur19, (7.4.6)]. We have

$$\mathbb{P}_{\mathbf{B}(0)=s}(\boldsymbol{\sigma}_+ \in \mathrm{d}T) = \frac{|s-\beta|}{\sqrt{4\pi T^3}} \mathrm{e}^{-\frac{(s-\beta)^2}{4T}}\ \mathrm{d}T, \tag{2.13}$$

whenever $s > \beta$. Thus for $\mathfrak{h}$ supported on $[-L, L]$, we have

$$\begin{aligned} &\int_{\mathfrak{h}(0)}^{\infty} \mathrm{d}s\ |\mathrm{e}^{s\eta}| \cdot \mathbb{E}_{\mathbf{B}(0)=s} \left[\left| \exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \right| \mathbf{1}_{\boldsymbol{\tau}_+<\infty} \right] \\ &\leq \mathrm{e}^{L|\xi|^2} \int_{\mathfrak{h}(0)}^{\beta+1} \mathrm{d}s\, \mathrm{e}^{s\mathrm{Re}(\eta) - \beta\mathrm{Re}(\xi)} + \mathrm{e}^{L|\xi|^2} \int_{\beta+1}^{\infty} \mathrm{d}s \int_0^L \mathrm{d}T\, \mathrm{e}^{s\mathrm{Re}(\eta)} \cdot \frac{|s-\beta|}{\sqrt{4\pi T^3}} \mathrm{e}^{-\frac{(s-\beta)^2}{4T} - \beta\cdot\mathrm{Re}(\xi)} \\ &\leq C_1 \mathrm{e}^{L|\xi|^2 + \mathrm{Re}(\eta)} \mathrm{e}^{\beta\cdot\mathrm{Re}(\eta-\xi)} + C_2 \mathrm{e}^{L|\xi|^2} \int_{\beta+1}^{\infty} \mathrm{d}s \int_0^L \mathrm{d}T\ \mathrm{e}^{s\mathrm{Re}(\eta) - \frac{(s-\beta)^2}{8T} - \beta\cdot\mathrm{Re}(\xi)}, \end{aligned} \tag{2.14}$$

where C_1 and C_2 are two constants that depend on β, and in the last step we used the elementary inequality

$$\frac{|s-\beta|}{\sqrt{4\pi T^3}} \mathrm{e}^{-\frac{(s-\beta)^2}{4T}} = \frac{|s-\beta|}{\sqrt{4\pi T^3}} \mathrm{e}^{-\frac{(s-\beta)^2}{8T}} \cdot \mathrm{e}^{-\frac{(s-\beta)^2}{8T}} \leq \max_{T\in[0,\infty)} \left\{ \frac{|s-\beta|}{\sqrt{4\pi T^3}} \mathrm{e}^{-\frac{(s-\beta)^2}{8T}} \right\} \cdot \mathrm{e}^{-\frac{(s-\beta)^2}{8T}} \leq \frac{C}{(s-\beta)^2} \cdot \mathrm{e}^{-\frac{(s-\beta)^2}{8T}}.$$

Now bounding $\mathrm{e}^{-\frac{(s-\beta)^2}{8T}}$ by $\mathrm{e}^{-\frac{(s-\beta)^2}{8L}}$ for $T \in [0, L]$ we have

$$\int_{\beta+1}^{\infty} \mathrm{d}s \int_0^L \mathrm{d}T\ \mathrm{e}^{s\mathrm{Re}(\eta) - \frac{(s-\beta)^2}{8T} - \beta\cdot\mathrm{Re}(\xi)} \leq L\mathrm{e}^{-\beta\cdot\mathrm{Re}(\xi)} \int_{\beta+1}^{\infty} \mathrm{d}s\ \mathrm{e}^{s\mathrm{Re}(\eta) - \frac{(s-\beta)^2}{8L}} \leq CL^{\frac{3}{2}} \mathrm{e}^{\beta\cdot\mathrm{Re}(\eta-\xi) + 2L|\eta|^2}.$$

Thus we conclude that

$$\begin{aligned}&\left|\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+<\infty}\right]\right| \\ &\le CL^{\frac{3}{2}} \mathrm{e}^{\mathrm{Re}(\eta-\xi)\cdot \max_{\alpha\in\mathbb{R}} \mathfrak{h}(\alpha)} \cdot \left(\mathrm{e}^{cL(|\xi|^2+|\eta|^2)} + \frac{1}{\mathrm{Re}(\eta-\xi)}\right).\end{aligned}$$

The estimates for the other two parts are similar. The analyticity in ξ and η is an easy consequence of the above bound and the dominated convergence theorem.

□

For convenience we were using the origin 0 as the starting point of the Brownian motions in the hitting formula. The next proposition shows that this is not necessary and one can start with any point on the real line $\mathbb{R}$ and get the same characteristic function $\chi_{\mathfrak{h}}$.

Proposition 2.4. *One can change the starting point of the Brownian motions in the definition of the function $\chi_{\mathfrak{h}}$ defined in* (2.9)*. More precisely, for any $\omega \in \mathbb{R}$ one has*

$$\begin{aligned}\chi_{\mathfrak{h}}(\eta,\xi) &= \mathrm{e}^{\omega\eta^2}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{+s\eta} \cdot \mathbb{E}_{\mathbf{B}(\omega)=s}\left[\exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+<\infty}\right] \\ &\quad + \mathrm{e}^{-\omega\xi^2}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-s\xi} \cdot \mathbb{E}_{\mathbf{B}(\omega)=s}\left[\exp\left(\boldsymbol{\tau}_-\eta^2 + \mathbf{B}(\boldsymbol{\tau}_-)\eta\right) \mathbf{1}_{\boldsymbol{\tau}_->-\infty}\right] \\ &\quad - \int_{\mathbb{R}} \mathrm{d}s\, \mathbb{E}_{\mathbf{B}(\omega)=s}\left[\exp\left(\boldsymbol{\tau}_-\eta^2 + \mathbf{B}(\boldsymbol{\tau}_-)\eta - \boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{|\boldsymbol{\tau}_\pm|<\infty}\right],\end{aligned} \tag{2.15}$$

where the hitting times $\boldsymbol{\tau}_\pm$ are now defined as

$$\boldsymbol{\tau}_+ := \inf\{\alpha \ge \omega : \mathbf{B}(\alpha) \le \mathfrak{h}(\alpha)\}, \quad \boldsymbol{\tau}_- := \sup\{\alpha \le \omega : \mathbf{B}(\alpha) \le \mathfrak{h}(\alpha)\}. \tag{2.16}$$

In particular if $\mathrm{supp}(\mathfrak{h}) \subset [-L, L]$ *for some $L > 0$, then*

$$\chi_{\mathfrak{h}}(\eta,\xi) = \mathrm{e}^{-L\eta^2}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(-L)=s}\left[\exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right) \mathbf{1}_{\boldsymbol{\tau}_+\le L}\right]. \tag{2.17}$$

The proof of Proposition 2.4 will be given in Section 5.2. We point out that it is purely a result about the Brownian motion but we are not able to find it in the literature.

2.2 An equivalent series expansion formula

Due to the block diagonal structure of the kernel $\mathbf{K}_1$ and $\mathbf{K}_{\mathfrak{h}}$, the Fredholm determinant $\mathrm{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1})$ admits a series expansion, which we will be working with more frequently in the subsequent sections. To introduce the formula, we first introduce a few notation. Given $W = (w_1, \cdots, w_n) \in \mathbb{C}^n$ and $W' = (w'_1, \cdots, w'_m) \in \mathbb{C}^m$, we denote

$$W \sqcup W' := (w_1, \cdots, w_n, w'_1, \cdots, w'_m) \in \mathbb{C}^{m+n}. \tag{2.18}$$

Assume in addition that $n = m$ and $w_i \ne w'_j$ for all $1 \le i, j \le n$, we denote

$$\mathrm{C}(W; W') := \det\left[\frac{1}{w_i - w'_j}\right]_{1\le i,j\le n} = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{1\le i<j\le n}(w_j - w_i)(w'_j - w'_i)}{\prod_{1\le i,j\le n}(w_i - w'_j)}, \tag{2.19}$$

which is the usual Cauchy determinant. The Cauchy determinant $\mathrm{C}(W \sqcup W'; \widehat{W} \sqcup \widehat{W}')$ is defined in the same way with the combined variables $W \sqcup W'$ and another set of variables $\widehat{W} \sqcup \widehat{W}'$ with the same dimension.

Proposition 2.5 (Series expansion for $\mathbf{D}_{\mathfrak{h}}(z_1,\cdots,z_{m-1})$)**.** *Alternatively, we have*

$$\mathbf{D}_{\mathfrak{h}}(z_1,\cdots,z_{m-1}) = \sum_{\substack{n_\ell\geq 0,\\ 1\leq\ell\leq m}} \frac{1}{(n_1!\cdots n_m!)^2}\mathrm{D}_{\mathfrak{h}}^{(\mathbf{n})}(z_1,\cdots,z_{m-1}), \tag{2.20}$$

where $\mathbf{n}=(n_1,\cdots,n_m)\in(\mathbb{Z}_{\geq 0})^m$*, and*

$$\begin{aligned} &\mathrm{D}_{\mathfrak{h}}^{(\mathbf{n})}(z_1,\cdots,z_{m-1}) = \mathrm{D}_{\mathfrak{h}}^{(\mathbf{n})}(z_1,\cdots,z_{m-1};(\alpha_1,\tau_1,\beta_1),\cdots,(\alpha_m,\tau_m,\beta_m)) \\ &= \prod_{\ell=1}^{m-1}(1-z_\ell)^{n_\ell}\left(1-z_\ell^{-1}\right)^{n_{\ell+1}}\left(\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\int_{\Sigma_{\ell,\mathrm{L}}}\mathrm{d}\mu_{\boldsymbol{z}}(\xi_{i_\ell}^{(\ell)})\int_{\Sigma_{\ell,\mathrm{R}}}\mathrm{d}\mu_{\boldsymbol{z}}(\eta_{i_\ell}^{(\ell)})\right)\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\frac{\mathrm{f}_\ell(\xi_{i_\ell}^{(\ell)})}{\mathrm{f}_\ell(\eta_{i_\ell}^{(\ell)})} \\ &\quad\cdot\det\left[\chi_{\mathfrak{h}}(\eta_i^{(1)},\xi_j^{(1)})\right]_{1\leq i,j\leq n_1}\cdot\prod_{\ell=1}^{m-1}\mathrm{C}\left(\boldsymbol{\xi}^{(\ell)}\sqcup\boldsymbol{\eta}^{(\ell+1)};\boldsymbol{\eta}^{(\ell)}\sqcup\boldsymbol{\xi}^{(\ell+1)}\right)\cdot\mathrm{C}(\boldsymbol{\xi}^{(m)};\boldsymbol{\eta}^{(m)}), \end{aligned} \tag{2.21}$$

with $\boldsymbol{\xi}^{(\ell)}=(\xi_1^{(\ell)},\cdots,\xi_{n_\ell}^{(\ell)})$ *and* $\boldsymbol{\eta}^{(\ell)}=(\eta_1^{(\ell)},\cdots,\eta_{n_\ell}^{(\ell)})$*, for* $1\leq\ell\leq m$*. Here*

$$\mathrm{f}_\ell(w) := \mathrm{e}^{-\frac{1}{3}(\tau_\ell-\tau_{\ell-1})w^3+(\alpha_\ell-\alpha_{\ell-1})w^2+(\beta_\ell-\beta_{\ell-1})w},$$

for $1\leq\ell\leq m$*, with the convention that* $\tau_0=\alpha_0=\beta_0:=0$*.*

We remark that (2.21) looks slightly different from the version in [Liu22, (2.27)] because we use the function C instead of the Vandermonde type products Δ. This formula also appears in [LZ25, Proposition 3.1] when $\chi_{\mathfrak{h}}(\eta,\xi)=1/(\eta-\xi)$ for the narrow wedge initial condition. The equivalence of Proposition 2.5 and Theorem 1.2 follows from [Liu22, Proposition 2.9], see also [BL19, Lemma 4.8] and [BL21, Lemma 5.6].

3 Multipoint distribution formulas for TASEP

Our formulas for the KPZ fixed point are obtained by taking $1:2:3$ scaling limit of the analogous formulas for the totally asymmetric simple exclusion process, which we discuss in this section. The totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$ is a continuous-time Markov chain $X_t=(\mathrm{x}_i(t))_{i\geq 1}$, consisting of particles on $\mathbb{Z}$ performing independent Poisson random walks subject to the exclusion rule. Each particle tries to jump to its right neighbor after an independent exponential waiting time with rate 1 but the jump is forbidden if the target site is occupied. The exponential clock is reset after each jump attempt. We will assume there is a right-most particle with index 1 and label the particles from right to left, so the i-th particle at time t has location $\mathrm{x}_i(t)$ and

$$\cdots<\mathrm{x}_3(t)<\mathrm{x}_2(t)<\mathrm{x}_1(t).$$

The initial configuration is denoted by $Y=(y_i)_{i\geq 1}:=(\mathrm{x}_i(0))_{i\geq 1}$. Our key observation, following the work [Liu22], is that the initial condition Y can be encoded in a two-variable function $\mathrm{ch}_Y(v,u)$, defined as an expectation involving random walk hitting problems. We begin by introducing this key object.

3.1 Characteristic function of the initial condition

In this subsection, we discuss how to characterize the initial condition in the TASEP formulas, and provide a probabilistic representation of the characterization.

A key property of the TASEP is that the distribution of any finite set of the rightmost particles, up to a fixed label N, is independent of the state of particles to their left. Conversely, a TASEP model with

N particles can be embedded into a TASEP model with infinitely many particles, where the N rightmost particles correspond to the N-particle system, and the states of all other particles are arbitrary. This feature will be used when we characterize the initial condition of TASEP with finitely many particles.

Consider the following two simply connected regions of $\mathbb{C}$:

$$\Omega_{\mathrm{L}} := \left\{ w \in \mathbb{C} : |w+1| < \frac{1}{2} \right\}, \quad \Omega_{\mathrm{R}} := \left\{ w \in \mathbb{C} : |w| < \frac{1}{2} \right\}. \tag{3.1}$$

The following characterization of the initial condition is from [Liu22], see Proposition 2.13 and the subsequent discussion in that paper for further details.

Definition 3.1. *Let $Y = (y_1 > y_2 > \cdots > y_N)$ where N is a fixed integer. We say ch_Y is a characteristic function of Y, if it satisfies the following two conditions:*

1. $\mathrm{ch}_Y : \Omega_{\mathrm{R}} \times (\Omega_{\mathrm{L}} \backslash \{-1\}) \to \mathbb{C}$ *is analytic.*
2. *For any $1 \le i \le N$, one has*

$$\oint_0 v^{-i}(v+1)^{y_i+i} \cdot \mathrm{ch}_Y(v,u) \frac{\mathrm{d}v}{2\pi \mathrm{i}} = -u^{-i}(u+1)^{y_i+i}. \tag{3.2}$$

Remark 3.2. *As pointed out in Proposition 2.13 of [Liu22] and the comments thereafter, for any given Y, there are infinitely many characteristic functions. However, the law of the TASEP model such as the multipoint distributions we are interested in here does not depend on the choice of ch_Y. This non-uniqueness comes from the nature of the TASEP model, as we discussed at the beginning of this subsection.*

Remark 3.3. *One could formally extend the concept of the characteristic "function" $\mathrm{ch}_Y(v,u)$ to an infinite system with particles labeled on $\mathbb{Z}_+$ and $Y = (\ldots, y_3, y_2, y_1)$ by defining $\mathrm{ch}_Y(v,u)$ to satisfy (3.2) for all $i \in \mathbb{Z}_+$. We could extend it even further to a TASEP with particle labeled on $\mathbb{Z}$, while the first condition is absorbed into the second condition by allowing $i \in \mathbb{Z}$ in (3.2). The issue for these extensions is that ch_Y is not necessarily well defined as a function because of the convergence issue.*

In [Liu22], the author derived a characteristic function expressed in terms of symmetric functions for any Y, which is well-suited for asymptotic analysis under the step or flat initial condition. As a result, the author obtained the multipoint distribution of the KPZ fixed point for both the narrow-wedge and flat initial conditions. However, the characteristic function presented in [Liu22] is not suitable for asymptotic analysis with general initial conditions. One main contribution of this paper is the following characteristic function ch_Y defined by an expectation involving random walk hitting problems, which turns out to be suitable for asymptotic analysis. The idea is heavily inspired by the seminal work [MQR21] but the formula does not follow directly from their results.

Theorem 3.4. *Let $(G_k)_{k\ge 0}$ be a geometric random walk with transition probability given by*

$$\mathbb{P}(G_{k+1} = x | G_k = y) := \frac{1}{2^{y-x}} \mathbf{1}_{x<y}, \tag{3.3}$$

and τ is the hitting time of G to the strict epigraph of Y, namely

$$\tau := \min\{m \ge 0 : G_m > y_{m+1}\}. \tag{3.4}$$

Then the following function ch_Y is a characteristic function of Y

$$\mathrm{ch}_Y(v,u) := \sum_{z \in \mathbb{Z}} (2u+2)^z \cdot \mathbb{E}_{G_0 = z} \left[\frac{2}{(2v+2)^{G_\tau + 1}} \cdot \left(\frac{-v}{v+1} \right)^\tau \mathbf{1}_{\tau < N} \right]. \tag{3.5}$$

Proof. First we check the analyticity of $\mathrm{ch}_Y(v,u)$ in $\Omega_{\mathrm{R}} \times (\Omega_{\mathrm{L}} \backslash \{0\})$. Recall that $v \in \Omega_{\mathrm{R}} =: \{|w| < 1/2\}$ and $u \in \Omega_{\mathrm{L}} =: \{|w+1| < 1/2\}$. We claim that $\mathbb{P}_{G_0=z}(\mathbf{1}_{\tau<N}) = 0$ whenever $z \le y_N + N - 1$. To see this note that since the random walk G is moving strictly to the negative side, one has $G_k \le G_0 - k$ for all $k \ge 0$. Thus if $G_0 \le y_N + N - 1$, then $G_k \le y_N + N - k - 1 \le y_{k+1}$ for all $0 \le k \le N-1$, meaning that $\tau \ge N$. Therefore

$$\mathrm{ch}_Y(v,u) = \sum_{z \ge y_N + N} (2u+2)^z \cdot \mathbb{E}_{G_0=z}\left[\frac{2}{(2v+2)^{G_\tau+1}} \cdot \left(\frac{-v}{v+1}\right)^{\tau} \mathbf{1}_{\tau<N}\right]. \tag{3.6}$$

Now note that

$$\mathbb{E}_{G_0=z}\left[\frac{2}{(2v+2)^{G_\tau+1}} \cdot \left(\frac{-v}{v+1}\right)^{\tau} \mathbf{1}_{\tau<N}\right] = \sum_{k=0}^{N-1} \sum_{w=y_{k+1}+1}^{y_k-1} \frac{2}{(2v+2)^{w+1}} \cdot \left(\frac{-v}{v+1}\right)^{k} \mathbb{P}_{G_0=z}[\tau=k, G_\tau = w],$$

which is clearly analytic in v for $v \in \Omega_{\mathrm{R}}$ as a finite sum of functions analytic in v. Moreover since

$$y_N + 1 \le G_\tau \le z, \qquad \left|\frac{-v}{v+1}\right| < 1, \qquad |2v+2| > 1,$$

for any $0 \le \tau \le N-1$ and $v \in \Omega_{\mathrm{R}}$, we have

$$\left|\mathbb{E}_{G_0=z}\left[\frac{2}{(2v+2)^{G_\tau+1}} \left(\frac{-v}{v+1}\right)^{\tau} \mathbf{1}_{\tau<N}\right]\right| \le \frac{2}{|2v+2|^{z+1}} \sum_{k=0}^{N-1} \sum_{w=y_{k+1}+1}^{y_k-1} \mathbb{P}_{G_0=z}[\tau=k, G_\tau=w] \le \frac{1}{|2v+2|^z}. \tag{3.7}$$

Thus by the estimates $|2(u+1)| < 1 < |2(v+1)|$ for $u \in \Omega_{\mathrm{L}}$ and $v \in \Omega_{\mathrm{R}}$ we have

$$\left|\sum_{z \ge y_N+N} (2u+2)^z \cdot \mathbb{E}_{G_0=z}\left[\frac{2}{(2v+2)^{G_\tau+1}} \cdot \left(\frac{-v}{v+1}\right)^{\tau} \mathbf{1}_{\tau<N}\right]\right| \le \sum_{z \ge y_N+N} \frac{|2u+2|^z}{|2v+2|^z}, \tag{3.8}$$

which converges uniformly on compact sets in $\Omega_{\mathrm{R}} \times (\Omega_{\mathrm{L}} \backslash \{-1\})$. This implies $\mathrm{ch}_Y(v,u)$ is analytic on $\Omega_{\mathrm{R}} \times (\Omega_{\mathrm{L}} \backslash \{-1\})$.

Next we verify that the right-hand side of (3.5) satisfies (3.2) for all $1 \le i \le N$. A taylor expansion of u^{-i} at -1 gives

$$-u^{-i}(u+1)^{y_i+i} = (-1)^{i+1} \sum_{j=0}^{\infty} \binom{i+j-1}{j} (u+1)^{y_i+i+j},$$

where the series converges absolutely for $u \in \Omega_{\mathrm{L}}$. From (3.8) we have seen that the right-hand side of (3.5) converges absolutely as a Laurent series in u for $u \in \Omega_{\mathrm{L}} \backslash \{-1\}$. Hence it is sufficient to show that

$$\oint_0 \frac{\mathrm{d}v}{2\pi \mathrm{i}} v^{-i}(v+1)^{y_i+i} \cdot \frac{2^z}{v+1} \cdot \mathbb{E}_{G_0=z}\left[\frac{1}{(2v+2)^{G_\tau}} \left(\frac{-v}{v+1}\right)^{\tau} \mathbf{1}_{\tau<N}\right] = (-1)^{i+1} \mathbf{1}_{z \ge y_i+i} \binom{z-y_i-1}{i-1}.$$

Interchanging the contour integration and the expectation (which is justified by (3.7)), the above is equivalent to

$$2^z \cdot \mathbb{E}_{G_0=z}\left[\oint_0 \frac{\mathrm{d}v}{2\pi \mathrm{i}} \frac{(v+1)^{y_i+i-1-G_\tau-\tau}}{(-v)^{i-\tau}} \frac{1}{2^{G_\tau}} \mathbf{1}_{\tau<N}\right] = -\mathbf{1}_{z \ge y_i+i} \binom{z-y_i-1}{i-1},$$

which is, using the assumption that $i \le N$,

$$2^z \cdot \mathbb{E}_{G_0=z}\left[-\binom{G_\tau - y_i - 1}{i-\tau-1} \frac{1}{2^{G_\tau}} \mathbf{1}_{\tau<i}\right] = -\mathbf{1}_{z \ge y_i+i} \binom{z-y_i-1}{i-1}. \tag{3.9}$$

Below we use induction to prove (3.9) for any $1 \le i \le N$.

When $i = 1$, the expectation on the left-hand side of (3.9) is nonzero if and only if $\tau = 0$, which is equivalent to $z \geq y_1 + 1$. Moreover, when $\tau = 0$, $G_\tau = G_0 = z$. Thus (3.9) holds.

Assuming the identity is true for $i-1$, we want to show it holds for i.

Note that when $z \geq y_1 + 1$, we have $\tau = 0$ and both sides are equal.

When $z \leq y_1$, we define $\hat{G}_k = G_{k+1}$ and $\hat{y}_k = y_{k+1}$ for $k = 0, 1, \cdots$. Then we have, by induction,

$$2^{\hat{z}} \cdot \mathbb{E}_{\hat{G}_0 = z} \left[- \binom{\hat{G}_{\hat{\tau}} - \hat{y}_{i-1} - 1}{i - \hat{\tau} - 2} \frac{1}{2^{\hat{G}_{\hat{\tau}}}} \mathbf{1}_{\hat{\tau} < i-1} \right] = -\mathbf{1}_{\hat{z} \geq \hat{y}_{i-1} + i - 1} \binom{\hat{z} - \hat{y}_{i-1} - 1}{i-2},$$

here $\hat{\tau} := \tau - 1$. Thus we have

$$\sum_{\hat{z}=-\infty}^{z-1} \frac{1}{2^{z-\hat{z}}} \cdot \mathbb{E}_{\hat{G}_0 = \hat{z}} \left[- \binom{\hat{G}_{\hat{\tau}} - \hat{y}_{i-1} - 1}{i - \hat{\tau} - 2} \frac{1}{2^{\hat{G}_{\hat{\tau}}}} \mathbf{1}_{\hat{\tau} < i-1} \right] = - \sum_{\hat{z}=-\infty}^{z-1} \frac{1}{2^z} \mathbf{1}_{\hat{z} \geq \hat{y}_{i-1} + i - 1} \binom{\hat{z} - \hat{y}_{i-1} - 1}{i-2}. \tag{3.10}$$

Using the Markov property for the left-hand side of (3.10), we have

$$\mathbb{E}_{G_0 = z} \left[- \binom{G_\tau - y_i - 1}{i - \tau - 1} \frac{1}{2^{G_\tau}} \mathbf{1}_{\tau < i} \right] = - \frac{\mathbf{1}_{z \geq y_i + i}}{2^z} \sum_{\hat{z} = y_i + i - 1}^{z-1} \binom{\hat{z} - y_i - 1}{i-2} = - \frac{\mathbf{1}_{z \geq y_i + i}}{2^z} \binom{z - y_i - 1}{i-1},$$

where we used the identity $\sum_{m=k}^{n} \binom{m-1}{k-1} = \binom{n}{k}$ in the last step. This finishes the induction and the proof. $\square$

3.2 Multipoint distribution of TASEP with general initial configurations

The following theorem is essentially [Liu22, Theorem 2.1], where the integrand $\mathcal{D}_Y(z_1, \cdots, z_{m-1})$ is defined by a Fredholm determinant formula or a series expansion formula and the initial condition information is encoded in a characteristic function of Y. It was proved that the choice of characteristic functions does not affect the value of $\mathcal{D}_Y$ function. We will present the formula with a new characteristic function ch_Y as in (3.5) in Theorem 3.4 that is suitable for asymptotic analysis.

Theorem 3.5. *Given $Y = (\cdots < y_2 < y_1) \in \mathbb{Z}^{\mathbb{Z}_+}$. Consider TASEP with initial particle locations $X_0 = Y$. Let $(k_1, t_1), \cdots, (k_m, t_m)$ be m distinct points in $\mathbb{Z}_+ \times \mathbb{R}_+$. Then for any integers $a_1, \cdots, a_m$,*

$$\mathbb{P}_Y \left(\bigcap_{\ell=1}^{m} \{ \mathrm{x}_{k_\ell}(t_\ell) \geq a_\ell \} \right) = \oint_0 \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1 (1 - z_1)} \cdots \oint_0 \frac{\mathrm{d}z_{m-1}}{2\pi \mathrm{i} z_{m-1}(1 - z_{m-1})} \mathcal{D}_Y(z_1, \cdots, z_{m-1}), \tag{3.11}$$

where $\mathbb{P}_Y$ denotes the probability given $X(0) = Y$. The function $\mathcal{D}_Y(z_1, \cdots, z_{m-1})$ is defined as a Fredholm determinant in Definition 3.6 , or equivalently as a series in Definition 3.7.

3.2.1 Fredholm determinant representation of $\mathcal{D}_Y(z_1, \cdots, z_{m-1})$

The definition of $\mathcal{D}_Y(z_1, \cdots, z_{m-1})$ is very similar to its limiting counterpart $\mathbf{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1})$ defined in Section 2.1 and 2.2, either as a Fredholm determinant $\det(\mathrm{I} - \mathcal{K}_1 \mathcal{K}_Y)$ or as a Fredholm series expansion. We will only use the series expansion formula in this paper but we present both formulas here for completeness and possible later uses.

3.2.1.1 Spaces of the operators

We will define the operators on two specific spaces of nested contours with complex measures depending on $\boldsymbol{z} = (z_1, \cdots, z_{m-1})$, where $z_\ell \neq 1$ for each $1 \leq \ell \leq m-1$. Recall the definition of the two regions Ω_{L} and Ω_{R} from (3.1).

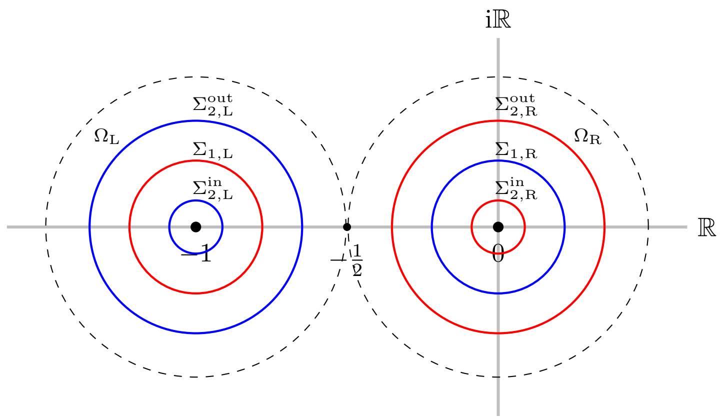

Figure 2: Illustration of the contours for $m = 2$: The regions Ω_{L} and Ω_{R} are the interior of the two dashed circles, from left to right; the three contours around -1 from outside to inside are $\Sigma_{2,\mathrm{L}}^{\mathrm{out}}, \Sigma_{1,\mathrm{L}}, \Sigma_{2,\mathrm{L}}^{\mathrm{in}}$ respectively; the three contours around 0 from outside to inside are $\Sigma_{2,\mathrm{R}}^{\mathrm{out}}, \Sigma_{1,\mathrm{R}}, \Sigma_{2,\mathrm{R}}^{\mathrm{in}}$ respectively. $\mathcal{S}_1$ is the union of the red contours, and $\mathcal{S}_2$ is the union of the blue contours.

Suppose $\Sigma_{m,\mathrm{L}}^{\mathrm{out}}, \cdots, \Sigma_{2,\mathrm{L}}^{\mathrm{out}}$, $\Sigma_{1,\mathrm{L}}$, $\Sigma_{2,\mathrm{L}}^{\mathrm{in}}, \cdots, \Sigma_{m,\mathrm{L}}^{\mathrm{in}}$ are $2m-1$ nested simple closed contours, from outside to inside, in Ω_{L} enclosing the point -1. Similarly, $\Sigma_{m,\mathrm{R}}^{\mathrm{out}}, \cdots, \Sigma_{2,\mathrm{R}}^{\mathrm{out}}$, $\Sigma_{1,\mathrm{R}}$, $\Sigma_{2,\mathrm{R}}^{\mathrm{in}}, \cdots, \Sigma_{m,\mathrm{R}}^{\mathrm{in}}$ are $2m-1$ nested simple closed contours, from outside to inside, in Ω_{R} enclosing the point 0. See Figure 2 for an illustration of the contours. These contours are all counterclockwise oriented.

We define

$$\Sigma_{\ell,\mathrm{L}} := \Sigma_{\ell,\mathrm{L}}^{\mathrm{out}} \cup \Sigma_{\ell,\mathrm{L}}^{\mathrm{in}}, \qquad \Sigma_{\ell,\mathrm{R}} := \Sigma_{\ell,\mathrm{R}}^{\mathrm{out}} \cup \Sigma_{\ell,\mathrm{R}}^{\mathrm{in}}, \qquad \ell = 2, \cdots, m, \tag{3.12}$$

and

$$\mathcal{S}_1 := \Sigma_{1,\mathrm{L}} \cup \Sigma_{2,\mathrm{R}} \cup \cdots \cup \begin{cases} \Sigma_{m,\mathrm{L}}, & \text{if } m \text{ is odd,} \\ \Sigma_{m,\mathrm{R}}, & \text{if } m \text{ is even,} \end{cases}$$

and

$$\mathcal{S}_2 := \Sigma_{1,\mathrm{R}} \cup \Sigma_{2,\mathrm{L}} \cup \cdots \cup \begin{cases} \Sigma_{m,\mathrm{R}}, & \text{if } m \text{ is odd,} \\ \Sigma_{m,\mathrm{L}}, & \text{if } m \text{ is even.} \end{cases}$$

We introduce a measure on these contours in the same way as in (2.2). Let

$$\mathrm{d}\mu(w) = \mathrm{d}\mu_{\boldsymbol{z}}(w) := \begin{cases} \dfrac{-z_{\ell-1}}{1-z_{\ell-1}} \dfrac{\mathrm{d}w}{2\pi\mathrm{i}}, & w \in \Sigma_{\ell,\mathrm{L}}^{\mathrm{out}} \cup \Sigma_{\ell,\mathrm{R}}^{\mathrm{out}}, \quad \ell = 2, \cdots, m, \\ \dfrac{1}{1-z_{\ell-1}} \dfrac{\mathrm{d}w}{2\pi\mathrm{i}}, & w \in \Sigma_{\ell,\mathrm{L}}^{\mathrm{in}} \cup \Sigma_{\ell,\mathrm{R}}^{\mathrm{in}}, \quad \ell = 2, \cdots, m, \\ \dfrac{\mathrm{d}w}{2\pi\mathrm{i}}, & w \in \Sigma_{1,\mathrm{L}} \cup \Sigma_{1,\mathrm{R}}. \end{cases}$$

3.2.1.2 Operators $\mathcal{K}_1$ and $\mathcal{K}_Y$

Now we introduce the operators $\mathcal{K}_1$ and $\mathcal{K}_Y$ to define $\mathcal{D}_Y(z_1, \cdots, z_{m-1})$ in Theorem 3.5. We assume that $\boldsymbol{z} = (z_1, \cdots, z_{m-1})$ is the same as in Section 3.2.1.1. Let

$$Q_1(j) := \begin{cases} 1 - z_j, & \text{if } j \text{ is odd and } j < m, \\ 1 - \dfrac{1}{z_{j-1}}, & \text{if } j \text{ is even,} \\ 1, & \text{if } j = m \text{ is odd,} \end{cases} \qquad Q_2(j) := \begin{cases} 1 - z_j, & \text{if } j \text{ is even and } j < m, \\ 1 - \dfrac{1}{z_{j-1}}, & \text{if } j \text{ is odd and } j > 1, \\ 1, & \text{if } j = m \text{ is even, or } j = 1. \end{cases}$$

Definition 3.6. *We define*

$$\mathcal{D}_Y(z_1,\cdots,z_{m-1}) = \det\left(\mathrm{I} - \mathcal{K}_1\mathcal{K}_Y\right),$$

where the two operators

$$\mathcal{K}_1 : L^2(\mathcal{S}_2, \mathrm{d}\mu) \to L^2(\mathcal{S}_1, \mathrm{d}\mu), \qquad \mathcal{K}_Y : L^2(\mathcal{S}_1, \mathrm{d}\mu) \to L^2(\mathcal{S}_2, \mathrm{d}\mu)$$

are defined by their kernels

$$\mathcal{K}_1(w,w') := \left(\delta_i(j) + \delta_i(j+(-1)^i)\right) \frac{\widehat{f}_i(w)}{w-w'} Q_1(j), \tag{3.13}$$

and

$$\mathcal{K}_Y(w',w) := \begin{cases} \left(\delta_j(i) + \delta_j(i-(-1)^j)\right) \dfrac{\widehat{f}_j(w')}{w'-w} Q_2(i), & i \geq 2, \\ \delta_j(1)\widehat{f}_j(w')\mathrm{ch}_Y(w';w), & i = 1, \end{cases} \tag{3.14}$$

for any $w \in (\Sigma_{i,\mathrm{L}} \cup \Sigma_{i,\mathrm{R}}) \cap \mathcal{S}_1$ and $w' \in (\Sigma_{j,\mathrm{L}} \cup \Sigma_{j,\mathrm{R}}) \cap \mathcal{S}_2$ with $1 \leq i,j \leq m$. Here ch_Y is the characteristic function given by (3.5)*. The function*

$$\widehat{f}_i(w) := \begin{cases} f_i(w), & w \in \Omega_{\mathrm{L}} \setminus \{-1\}, \\ \dfrac{1}{f_i(w)}, & w \in \Omega_{\mathrm{R}} \setminus \{0\}, \end{cases}$$

with

$$f_i(w) := \begin{cases} w^{k_i-k_{i-1}}(w+1)^{-(a_i-a_{i-1})-(k_i-k_{i-1})}\mathrm{e}^{(t_i-t_{i-1})w}, & i = 2,\cdots,m, \\ w^{k_1}(w+1)^{-a_1-k_1}\mathrm{e}^{t_1 w}, & i = 1, \end{cases} \tag{3.15}$$

for all $w \in (\Omega_{\mathrm{L}} \setminus \{-1\}) \cup (\Omega_{\mathrm{R}} \setminus \{0\})$.

3.2.2 Series expansion formula for $\mathcal{D}_Y(z_1,\cdots,z_{m-1})$

We will be working with the following series expansion formulas, which is equivalent to the Fredholm determinant formula in the previous section, by [Liu22, Proposition 2.9]. We use the same notation and conventions as in Section 2.2 for the Cauchy determinants.

Definition 3.7 (Alternative definition of $\mathcal{D}_Y$)**.** *We have an alternative definition of $\mathcal{D}_Y$ below*

$$\mathcal{D}_Y(z_1,\cdots,z_{m-1}) := \sum_{\boldsymbol{n}\in(\mathbb{Z}_{\geq 0})^m} \frac{1}{(\boldsymbol{n}!)^2} \mathcal{D}_Y^{(\mathbf{n})}(z_1,\cdots,z_{m-1}), \tag{3.16}$$

with $\boldsymbol{n}! = n_1!\cdots n_m!$ for $\boldsymbol{n} = (n_1,\cdots,n_m)$. Here

$$\begin{aligned} &\mathcal{D}_Y^{(\mathbf{n})}(z_1,\cdots,z_{m-1}) = \mathcal{D}_Y^{(\mathbf{n})}(z_1,\cdots,z_{m-1};(x_1,t_1,a_1),\cdots,(x_m,t_m,a_m)) \\ &= \prod_{\ell=1}^{m-1} (1-z_\ell)^{n_\ell} \left(1-z_\ell^{-1}\right)^{n_{\ell+1}} \left(\prod_{\ell=1}^{m} \prod_{i_\ell=1}^{n_\ell} \int_{\Sigma_{\ell,\mathrm{L}}} \mathrm{d}\mu_{\boldsymbol{z}}(u_{i_\ell}^{(\ell)}) \int_{\Sigma_{\ell,\mathrm{R}}} \mathrm{d}\mu_{\boldsymbol{z}}(v_{i_\ell}^{(\ell)}) \right) \prod_{\ell=1}^{m} \prod_{i_\ell=1}^{n_\ell} \frac{f_\ell(u_{i_\ell}^{(\ell)})}{f_\ell(v_{i_\ell}^{(\ell)})} \\ &\quad \cdot \det\left[\mathrm{ch}_Y(v_i^{(1)}, u_j^{(1)})\right]_{1\leq i,j\leq n_1} \cdot \prod_{\ell=1}^{m-1} \mathrm{C}\left(U^{(\ell)} \sqcup V^{(\ell+1)}; V^{(\ell)} \sqcup U^{(\ell+1)}\right) \cdot \mathrm{C}(U^{(m)}; V^{(m)}), \end{aligned} \tag{3.17}$$

with $U^{(\ell)} = (u_1^{(\ell)},\cdots,u_{n_\ell}^{(\ell)})$ and $V^{(\ell)} = (v_1^{(\ell)},\cdots,v_{n_\ell}^{(\ell)})$, and the functions f_ℓ defined in (3.15) *for $1 \leq \ell \leq m$.*

4 Convergence of the TASEP formula

In this section we will take proper scaling limit of the TASEP formulas (see Theorem 3.5), to get the corresponding KPZ fixed point formulas. We start with the setup for the proper rescaling.

4.1 TASEP height function and $1:2:3$ rescaling

The TASEP particle configurations can be encoded into the corresponding height functions $H(x,t)$ defined as the unique function $\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ satisfying the following conditions:

1. $H(0,0)=0$,
2. $H(x+1,t)=H(x,t)+\widehat{\eta}(x,t)$ for all $x\in\mathbb{Z}$, where
$$\widehat{\eta}(x,t)=\begin{cases}1 & \text{if there is a particle at site } x \text{ at time } t,\\ -1 & \text{if there is no particle at site } x \text{ at time } t,\end{cases} \tag{4.1}$$
3. $H(\cdot,t)$ is piecewise linear with constant slopes between consecutive integers.

The dynamics of the height function is as follows. Each local maximum of the height function turns into a local minimum after an independent exponential time with rate 1, and after each flip of max to min the height at the flip descreases by 2 while the height at the other integer points remain unchanged. The values at general $x\in\mathbb{R}$ are then determined by linear interpolations. Note that here we follow the convention in [MQR21] where the height function decreases in time, instead of increasing as in some other literature. More explicitly, for the TASEP Markov chain $X_t=(\mathrm{x}_i(t))_{i\geq 1}$ we define its height function as

$$H(x,t):=-2\left(X_t^{-1}(x-1)-X_0^{-1}(-1)\right)-x,\quad \text{for } x\in\mathbb{Z}, \tag{4.2}$$

where

$$X_t^{-1}(u):=\inf\{k\in\mathbb{Z}:\mathsf{x}_k(t)\leq u\}.$$

In particular the initial height function h corresponding to the initial particle configuration Y is

$$\mathsf{h}(x):=H(x,0)=-2\left(Y^{-1}(x-1)-Y^{-1}(-1)\right)-x,\quad \text{for } x\in\mathbb{Z}.$$

We will use $Y(\mathsf{h})$ or $\mathsf{h}(Y)$ to represent the initial particle configuration Y corresponding to the initial height function h and vice versa. Under this identification we can express the joint distribution of particle configurations using the height functions and vice versa, for example

$$\mathbb{P}_Y\left(\bigcap_{\ell=1}^m\{\mathrm{x}_{k_\ell}(t_\ell)\geq a_\ell\}\right)=\mathbb{P}_{\mathsf{h}(Y)}\left(\bigcap_{\ell=1}^m\{H(a_\ell,t_\ell)\leq -a_\ell-2k_\ell\}\right). \tag{4.3}$$

Now we introduce the proper rescaling for the TASEP height function so that it will converge to the KPZ fixed point. For $\varepsilon>0$ we define the rescaled TASEP height function $\mathcal{H}^\varepsilon(\alpha,\tau)$ for $(\alpha,\tau)\in\mathbb{R}\times\mathbb{R}_+$ as follows:

$$\mathcal{H}^\varepsilon(\alpha,\tau):=\varepsilon^{\frac12}\left(H(2\varepsilon^{-1}\alpha,2\varepsilon^{-\frac32}\tau)+\varepsilon^{-\frac32}\tau\right). \tag{4.4}$$

In particular $\mathcal{H}^\varepsilon(\alpha,0)=\varepsilon^{\frac12}\cdot H(2\varepsilon^{-1}\alpha,0)=:\mathfrak{h}^\varepsilon(\alpha)$. It was shown in [MQR21, Theorem 3.13] that if $\mathfrak{h}^\varepsilon\to\mathfrak{h}$ in UC as $\varepsilon\to 0$, then for any positive integer m one has $(\mathcal{H}^\varepsilon(\cdot,\tau_1;\mathfrak{h}^\varepsilon),\cdots,\mathcal{H}^\varepsilon(\cdot,\tau_m;\mathfrak{h}^\varepsilon))$ converges in distribution to $(\mathcal{H}(\cdot,\tau_1;\mathfrak{h}),\cdots,\mathcal{H}(\cdot,\tau_m;\mathfrak{h}))$ in the topology of UC^m, where $\mathcal{H}(\cdot,\cdot;\mathfrak{h})$ is the KPZ fixed point starting from the initial condition $\mathfrak{h}$. This in particular implies

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{H}(\alpha_\ell,\tau_\ell;\mathfrak{h})\le\beta_\ell\}\right)=\lim_{\varepsilon\to0}\mathbb{P}_{Y(\mathfrak{h}^\varepsilon)}\left(\bigcap_{\ell=1}^{m}\left\{\mathrm{x}_{\frac12\varepsilon^{-\frac32}\tau_\ell-\varepsilon^{-1}\alpha_\ell-\frac12\varepsilon^{-\frac12}\beta_\ell}(2\varepsilon^{-\frac32}\tau_\ell)\ge2\varepsilon^{-1}\alpha_\ell\right\}\right). \tag{4.5}$$

We will use (4.5) and Theorem 3.5 to prove Theorem 1.2. Our strategy is to first assume that the initial condition $\mathfrak{h}$ is a linear combination of finitely many narrow wedges and prove convergence of the TASEP approximations for such initial conditions. Then we use the density of such initial conditions to extend (1.6) to all $\mathfrak{h}\in\mathrm{UC}_c$.

4.2 Finitely many narrow wedges and approximations

Definition 4.1 (Multiple narrow wedges)**.** *Define the space of initial height functions consisting of finitely many narrow wedges:*

$$m\mathrm{NW}:=\{\mathfrak{h}\in\mathrm{UC}:\mathfrak{h}(\omega):=\sum_{k=0}^{M-1}\theta_k\mathbf{1}_{\omega=\omega_k}-\infty\mathbf{1}_{\omega\notin\{\omega_k:0\le k\le M-1\}},M\in\mathbb{Z}_+,\theta_k\in\mathbb{R},\omega_0>\cdots>\omega_{M-1}\}. \tag{4.6}$$

We will also be working with the following subspace of $m\mathrm{NW}$ *consisting of normalized height functions:*

$$m\mathrm{NW}_0:=\{\mathfrak{h}\in m\mathrm{NW}:\omega_0=\theta_0=0\}. \tag{4.7}$$

We start with proving Theorem 1.2 under the additional assumption that the initial condition $\mathfrak{h}$ for the KPZ fixed point is in $m\mathrm{NW}_0$, namely it takes the form

$$\mathfrak{h}(\omega):=\sum_{k=0}^{M-1}\theta_k\mathbf{1}_{\omega=\omega_k}-\infty\mathbf{1}_{\omega\notin\{\omega_k:0\le k\le M-1\}}, \tag{4.8}$$

where $M\in\mathbb{Z}_+$, $0=\omega_0>\omega_1>\cdots>\omega_{M-1}$ and $\theta_0=0$. Note that if $\mathfrak{h}(\omega)=\sum_k\theta_k\mathbf{1}_{\omega\neq\omega_k}-\infty\mathbf{1}_{\omega\notin\{\omega_k:k\}}\in m\mathrm{NW}$, then $\widehat{\mathfrak{h}}(\cdot):=\mathfrak{h}(\cdot+\omega_0)-\theta_0\in m\mathrm{NW}_0$. By the invariance property of the KPZ fixed point and also the structure of $\chi_{\mathfrak{h}}$ one can extend the formula to $\mathfrak{h}\in m\mathrm{NW}$ from $m\mathrm{NW}_0$, see Section 5.3 for explanations.

We approximate $\mathfrak{h}\in m\mathrm{NW}_0$ by the following sequence of height functions $\{\mathfrak{h}^\varepsilon\}_{\varepsilon>0}$:

$$\mathfrak{h}^\varepsilon(\omega):=\varepsilon^{1/2}\mathsf{h}^\varepsilon(2\varepsilon^{-1}\omega), \tag{4.9}$$

where h^ε is piecewise linear with slope ±1 such that $\mathsf{h}^\varepsilon(2\varepsilon^{-1}\omega_k)=\varepsilon^{-1/2}\theta_k+O(1)$ for each $0\le k\le M-1$. In terms of TASEP particle configurations, h^ε corresponds to setting the occupation functions $\widehat{\eta}(x,0)$ defined in (4.1) as:

$$\widehat{\eta}(x,0):=\begin{cases}+1, & \text{if } \varepsilon^{-1}(\omega_k+\omega_{k+1})+\varepsilon^{-\frac12}\frac{\theta_{k+1}-\theta_k}{2}\le x<2\varepsilon^{-1}\omega_k \text{ for some } 0\le k\le M-1,\\ -1, & \text{if } 2\varepsilon^{-1}\omega_k\le x<\varepsilon^{-1}(\omega_{k-1}+\omega_k)+\varepsilon^{-\frac12}\frac{\theta_k-\theta_{k-1}}{2} \text{ for some } 0\le k\le M-1.\end{cases} \tag{4.10}$$

Roughly, we are putting densely packed particles between $2\varepsilon^{-1}\omega_k$ and $\varepsilon^{-1}(\omega_k+\omega_{k+1})+\varepsilon^{-\frac12}\frac{\theta_k-\theta_{k+1}}{2}$ and no particles between $2\varepsilon^{-1}\omega_k$ and $\varepsilon^{-1}(\omega_k+\omega_{k-1})+\varepsilon^{-\frac12}\frac{\theta_{k-1}-\theta_k}{2}$, for $0\le k\le M-1$. Here ω_M is understood as $-\infty$ and ω_{-1} is understood as $+\infty$.

The following proposition implies Theorem 1.2 under the additional assumption that $\mathfrak{h}\in m\mathrm{NW}_0$.

Proposition 4.2. *Given $\mathfrak{h} \in m\mathrm{NW}_0$. Let $(\mathfrak{h}^\varepsilon)_{\varepsilon>0}$ be the approximating sequence of initial height functions for TASEP defined as in* (4.9) *and* (4.10)*. Given $z_1, \cdots, z_m \in \mathbb{C}$ with $|z_i| = r < 1$ for $1 \le i \le m-1$. To lighten the notation we will suppress the dependency on ε at most places and write*

$$\mathcal{D}_{Y^\varepsilon}(z_1, \cdots, z_{m-1}) := \mathcal{D}_{Y(\mathsf{h}^\varepsilon)}(z_1, \cdots, z_{m-1}; \mathbf{k}^\varepsilon, \mathbf{a}^\varepsilon, \mathbf{t}^\varepsilon),$$

where $\mathcal{D}_Y(z_1, \cdots, z_{m-1}) = \mathcal{D}_Y(z_1, \cdots, z_{m-1}; \mathbf{k}, \mathbf{a}, \mathbf{t})$ is defined in Section 3.2.1. Here we use boldface letters to denote vectors, for example, $\mathbf{k} := (k_1, \cdots, k_m)$. Assume the parameters satisfy

$$k_\ell^\varepsilon := \frac{1}{2}\varepsilon^{-\frac{3}{2}}\tau_\ell - \varepsilon^{-1}\alpha_\ell - \frac{1}{2}\varepsilon^{-\frac{1}{2}}\beta_\ell + O(1), \quad a_\ell^\varepsilon := 2\varepsilon^{-1}\alpha_\ell + O(1), \quad t_\ell^\varepsilon := 2\varepsilon^{-\frac{3}{2}}\tau_\ell, \quad \text{for } 1 \le \ell \le m. \tag{4.11}$$

Then we have

$$\lim_{\varepsilon \to 0} \prod_{\ell=1}^{m-1} \oint \frac{\mathrm{d}z_\ell}{2\pi \mathrm{i} z_\ell (1-z_\ell)} \mathcal{D}_{Y^\varepsilon}(z_1, \cdots, z_{m-1}) = \prod_{\ell=1}^{m-1} \oint \frac{\mathrm{d}z_\ell}{2\pi \mathrm{i} z_\ell (1-z_\ell)} \mathrm{D}_{\mathfrak{h}}(z_1, \cdots, z_{m-1}).$$

Proposition 4.2 is a consequence of the following two lemmas and the dominated convergence theorem.

Lemma 4.3. *Let $\mathcal{D}_{Y^\varepsilon}^{(\mathbf{n})}$ and $\mathrm{D}_{\mathfrak{h}}^{(\mathbf{n})}$ be as in* (3.17) *and* (2.21)*, where $\mathfrak{h}$ is given by* (4.8) *and $Y^\varepsilon = Y(\mathsf{h}^\varepsilon)$ is described in* (4.9)*. Then for each $\mathbf{n} \in (\mathbb{Z}_{\ge 0})^m$ and $(z_1, \cdots, z_{m-1}) \in (\mathbb{D}(0,1))^{m-1}$, we have*

$$\lim_{\varepsilon \to 0} \mathcal{D}_{Y^\varepsilon}^{(\mathbf{n})}(z_1, \cdots, z_{m-1}; \mathbf{k}^\varepsilon, \mathbf{a}^\varepsilon, \mathbf{t}^\varepsilon) = \mathrm{D}_{\mathfrak{h}}^{(\mathbf{n})}(z_1, \cdots, z_{m-1}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}). \tag{4.12}$$

Lemma 4.4. *There exists constant $C > 0$ such that*

$$\left| \mathcal{D}_{Y^\varepsilon}^{(\mathbf{n})}(z_1, \cdots, z_{m-1}; \mathbf{k}^\varepsilon, \mathbf{a}^\varepsilon, \mathbf{t}^\varepsilon) \right| \le \prod_{\ell=1}^{m-1} \frac{(1+|z_{\ell+1}|)^{2n_{\ell+1}}}{|z_\ell|^{n_{\ell+1}} |1-z_\ell|^{n_{\ell+1}-n_\ell}} \cdot \prod_{\ell=1}^{m} n_\ell^{n_\ell} \cdot C^{n_1 + \cdots + n_m}, \tag{4.13}$$

for any $\mathbf{n} = (n_1, \cdots, n_m) \in (\mathbb{Z}_{\ge 0})^m$ and $(z_1, \cdots, z_{m-1}) \in (\mathbb{D}(0,1))^{m-1}$.

The remaining of this section is organized as follows: we will first prove a uniform bound for $\mathrm{ch}_{Y^\varepsilon}$ and the pointwise convergence of $\mathrm{ch}_{Y^\varepsilon}$ to $\chi_{\mathfrak{h}}$ in Section 4.2.1. Then we will use these results to prove Lemma 4.3 in Section 4.3.1 and Lemma 4.4 in Section 4.3.2, these complete the proof of Proposition 4.2.

4.2.1 Pointwise convergence of the characteristic function

For the approximating sequence of height functions h^ε described in (4.10), we denote Y^ε the corresponding particle configurations for TASEP. It consists of exactly M clusters of densely packed particles. To lighten the notation we denote temporarily the indices of the rightmost particle of each cluster by $\mathtt{t}_0, \cdots, \mathtt{t}_{M-1}$, from right to left. We have

$$\mathtt{t}_i := -\lfloor \varepsilon^{-1}\omega_i \rfloor - \lfloor \frac{1}{2}\varepsilon^{-\frac{1}{2}}\theta_i \rfloor + 1, \quad y_{\mathtt{t}_i} := 2\lfloor \varepsilon^{-1}\omega_i \rfloor, \quad 0 \le i \le M-1. \tag{4.14}$$

Recall that we assume $\omega_0 = \theta_0 = 0$. Thus

$$\mathtt{t}_0 = 1, \quad \text{ and } y_{\mathtt{t}_0} = 0. \tag{4.15}$$

The goal of this section is to analyze the asymptotic behaviors of the characteristic function $\mathrm{ch}_{Y^\varepsilon}(v,u)$. We write

$$u = -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\xi, \quad v = -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\eta. \tag{4.16}$$

The main result of this section is summarized in the following proposition:

Proposition 4.5. *Under the same assumption as in Proposition 4.2, we have*

(a) For any $\xi \in \mathbb{C}_{\mathrm{L}}, \eta \in \mathbb{C}_{\mathrm{R}}$ fixed,

$$\lim_{\varepsilon \to 0} \frac{1}{2}\varepsilon^{\frac{1}{2}} \cdot \mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\eta, -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\xi\right) = \chi_{\mathfrak{h}}(\eta, \xi). \tag{4.17}$$

(b) Assume that $\varepsilon > 0$, and $\xi \in \mathbb{C}_{\mathrm{L}}, \eta \in \mathbb{C}_{\mathrm{R}}$ satisfy $0 < |1 + \varepsilon^{\frac{1}{2}}\xi| < 1$. Then the following estimate holds

$$\begin{aligned}&\left|\frac{1}{2}\varepsilon^{\frac{1}{2}} \cdot \mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\eta, -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\xi\right)\right| \\ &\leq \frac{1}{\mathrm{Re}(\eta)}\left(1 + (M-1)\frac{|1 - \varepsilon\eta^2|^{\mathtt{t}_{M-1}}}{|1 + \varepsilon^{\frac{1}{2}}\eta|^{2\mathtt{t}_{M-1} + y_{\mathtt{t}_{M-1}+1}}} \cdot \frac{|1 + \varepsilon^{\frac{1}{2}}\xi|^{y_{\mathtt{t}_{M-1}} + t_{M-1}}}{(2 - |1 + \varepsilon^{\frac{1}{2}}\xi|)^{\mathtt{t}_{M-1}-1}}\right).\end{aligned} \tag{4.18}$$

As a corollary, if we further assume that $|\varepsilon^{\frac{1}{2}}\xi| < 100^{-1}$ and $|\varepsilon^{\frac{1}{2}}\eta| < 100^{-1}$, then we have

$$\left|\frac{\varepsilon^{\frac{1}{2}}}{2}\mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2} + \frac{\varepsilon^{\frac{1}{2}}}{2}\eta, -\frac{1}{2} + \frac{\varepsilon^{\frac{1}{2}}}{2}\xi\right)\right| \leq \frac{\mathrm{e}^{C(|\xi|^2 + |\eta|^2 + |\xi| + |\eta| + 1)}}{\mathrm{Re}(\eta)}, \tag{4.19}$$

where C is a constant that only depends on the parameters M and ω_i, θ_i, $0 \leq i \leq M-1$.

Proof. We will prove part (b) first. Note that the geometric random walk moves strictly downwards, so it can only go above the boundary at the beginning of each cluster, namely

$$\mathbb{P}(\tau \notin \{\mathtt{t}_0, \cdots, \mathtt{t}_{M-1}\}) = 0.$$

Here τ is defined as in (3.4) and the indices $\mathtt{t}_0, \cdots, \mathtt{t}_{M-1}$ are as in (4.14). Hence

$$\begin{aligned}&\frac{1}{2}\varepsilon^{\frac{1}{2}} \cdot \mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\eta, -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\xi\right) \\ &= \varepsilon^{\frac{1}{2}} \sum_{z \in \mathbb{Z}} (1 + \varepsilon^{\frac{1}{2}}\xi)^z \cdot \mathbb{E}_{G_0 = z}\left[(1 + \varepsilon^{\frac{1}{2}}\eta)^{-G_\tau - \tau - 1} \cdot (1 - \varepsilon^{\frac{1}{2}}\eta)^\tau \mathbf{1}_{\tau \leq \max_\ell \{\mathtt{t}_\ell\}}\right] \\ &= \varepsilon^{\frac{1}{2}} \sum_{k=0}^{M-1} \sum_{\substack{z_i \leq y_{\mathtt{t}_i} \\ 0 \leq i \leq k-1}} \sum_{z_k > y_{\mathtt{t}_k}} \frac{(1 + \varepsilon^{\frac{1}{2}}\xi)^{z_0}}{(1 + \varepsilon^{\frac{1}{2}}\eta)^{z_k + 1}} \cdot \frac{(1 - \varepsilon^{\frac{1}{2}}\eta)^{\mathtt{t}_k}}{(1 + \varepsilon^{\frac{1}{2}}\eta)^{\mathtt{t}_k}} \cdot \prod_{i=0}^{k-1} p_{\mathtt{t}_{i+1} - \mathtt{t}_i}(z_{i+1} - z_i),\end{aligned} \tag{4.20}$$

where $p_{t-s}(z-y)$ is the transition probability $\mathbb{P}(G_t = z | G_s = y)$ for the geometric random walk $(G_k)_{k \geq 0}$ defined as in (3.3). It admits the following expression:

$$p_{t-s}(z-y) = 2^{z-y}\binom{y - z - 1}{t - s - 1},$$

for $t - s \in \mathbb{Z}_+$ and $z - y \in \mathbb{Z}_-$. Then (4.20) implies

$$\begin{aligned}&\left|\frac{1}{2}\varepsilon^{\frac{1}{2}} \cdot \mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\eta, -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\xi\right)\right| \\ &\leq \varepsilon^{\frac{1}{2}} \sum_{k=0}^{M-1} \sum_{\substack{z_i \leq y_{\mathtt{t}_i} \\ 0 \leq i \leq k-1}} \sum_{z_k > y_{\mathtt{t}_k}} \frac{|1 + \varepsilon^{\frac{1}{2}}\xi|^{z_0}}{|1 + \varepsilon^{\frac{1}{2}}\eta|^{z_k + 1}} \cdot \frac{|1 - \varepsilon^{\frac{1}{2}}\eta|^{\mathtt{t}_k}}{|1 + \varepsilon^{\frac{1}{2}}\eta|^{\mathtt{t}_k}} \cdot \prod_{i=0}^{k-1} p_{\mathtt{t}_{i+1} - \mathtt{t}_i}(z_{i+1} - z_i).\end{aligned} \tag{4.21}$$

(b) We bound each term on the right-hand side of (4.21) corresponding to index k. For $k=0$ we have

$$\varepsilon^{\frac{1}{2}} \sum_{z_0=y_{\mathsf{t}_0}+1}^{\infty} \frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{z_0}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{z_0+1}} = \frac{\varepsilon^{\frac{1}{2}}}{|1+\varepsilon^{\frac{1}{2}}\eta| - |1+\varepsilon^{\frac{1}{2}}\xi|}. \tag{4.22}$$

Assume $k \geq 1$, we have

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}} \sum_{\substack{z_i \leq y_{\mathsf{t}_i} \\ 0\leq i \leq k-1}} \sum_{z_k > y_{\mathsf{t}_k}} \frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{z_0}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{z_k+1}} \cdot \frac{|1-\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}} \cdot \prod_{i=0}^{k-1} p_{\mathsf{t}_{i+1}-\mathsf{t}_i}(z_{i+1}-z_i) \\
&\leq \varepsilon^{\frac{1}{2}} \sum_{z_0 \leq y_{\mathsf{t}_0}, z_k > y_{\mathsf{t}_k}} \frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{z_0}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{z_k+1}} \cdot \frac{|1-\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}} \cdot p_{\mathsf{t}_k-\mathsf{t}_0}(z_k - z_0) \\
&= \varepsilon^{\frac{1}{2}} \frac{|1-\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}} \cdot \sum_{\delta = y_{\mathsf{t}_k}-y_{\mathsf{t}_0}+1}^{\mathsf{t}_0-\mathsf{t}_k} |1+\varepsilon^{\frac{1}{2}}\eta|^{-\delta} p_{\mathsf{t}_k-\mathsf{t}_0}(\delta) \left(\sum_{z_0 = y_{\mathsf{t}_k}-\delta+1}^{y_{\mathsf{t}_0}} \frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{z_0}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{z_0+1}} \right) \\
&= \varepsilon^{\frac{1}{2}} \frac{|1-\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}} \cdot \sum_{\delta = y_{\mathsf{t}_k}-y_{\mathsf{t}_0}+1}^{\mathsf{t}_0-\mathsf{t}_k} |1+\varepsilon^{\frac{1}{2}}\eta|^{-\delta} p_{\mathsf{t}_k-\mathsf{t}_0}(\delta) \left(\frac{\frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{y_{\mathsf{t}_k}-\delta+1}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{y_{\mathsf{t}_k}-\delta+1}} - \frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{y_{\mathsf{t}_0}+1}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{y_{\mathsf{t}_0}+1}}}{|1+\varepsilon^{\frac{1}{2}}\eta| - |1+\varepsilon^{\frac{1}{2}}\xi|} \right).
\end{aligned} \tag{4.23}$$

By the assumptions of ξ and η, we have

$$|1-\varepsilon^{\frac{1}{2}}\eta| < |1+\varepsilon^{\frac{1}{2}}\eta|, \quad |1+\varepsilon^{\frac{1}{2}}\xi| < 1 < |1+\varepsilon^{\frac{1}{2}}\eta|. \tag{4.24}$$

Hence

$$\left| \frac{1-\varepsilon^{\frac{1}{2}}\eta}{1+\varepsilon^{\frac{1}{2}}\eta} \right|^{\mathsf{t}_k} \leq 1, \quad \left| \frac{1+\varepsilon^{\frac{1}{2}}\xi}{1+\varepsilon^{\frac{1}{2}}\eta} \right|^{y_{\mathsf{t}_0}+1} \leq \left| \frac{1+\varepsilon^{\frac{1}{2}}\xi}{1+\varepsilon^{\frac{1}{2}}\eta} \right|^{y_{\mathsf{t}_k}-\delta+1}, \quad \text{for all } y_{\mathsf{t}_k} - y_{\mathsf{t}_0} + 1 \leq \delta, \tag{4.25}$$

where we are using the fact that $\mathsf{t}_k \geq 0$ and $y_{\mathsf{t}_0} \geq y_{\mathsf{t}_k} - \delta + 1$ for all $y_{\mathsf{t}_k} - y_{\mathsf{t}_0} + 1 \leq \delta$. Thus we conclude that the right-hand side of (4.23) is bounded above by

$$\frac{\varepsilon^{\frac{1}{2}}}{|1+\varepsilon^{\frac{1}{2}}\eta| - |1+\varepsilon^{\frac{1}{2}}\xi|} \cdot \frac{|1-\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{\mathsf{t}_k}} \cdot \frac{|1+\varepsilon^{\frac{1}{2}}\xi|^{y_{\mathsf{t}_k}+1}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{y_{\mathsf{t}_k}+1}} \cdot \sum_{\delta = y_{\mathsf{t}_k}-y_{\mathsf{t}_0}+1}^{\mathsf{t}_0-\mathsf{t}_k} |1+\varepsilon^{\frac{1}{2}}\xi|^{-\delta} p_{\mathsf{t}_k-\mathsf{t}_0}(\delta). \tag{4.26}$$

The summation over δ is bounded above by

$$\sum_{\delta \leq \mathsf{t}_0-\mathsf{t}_k} |1+\varepsilon^{\frac{1}{2}}\xi|^{-\delta} p_{\mathsf{t}_k-\mathsf{t}_0}(\delta) = \left(\frac{|1+\varepsilon^{\frac{1}{2}}\xi|}{2-|1+\varepsilon^{\frac{1}{2}}\xi|} \right)^{\mathsf{t}_k-\mathsf{t}_0}, \tag{4.27}$$

which follows from a standard moment generating function computation for the geometric random walk. Finally, note that $|1+\varepsilon^{\frac{1}{2}}\eta| \geq 1+\varepsilon^{\frac{1}{2}}\mathrm{Re}(\eta) > 1$ for $\eta \in \mathbb{C}_{\mathrm{R}}$ and $|1+\varepsilon^{\frac{1}{2}}\xi| < 1$ by our assumption, we have

$$\frac{\varepsilon^{\frac{1}{2}}}{|1+\varepsilon^{\frac{1}{2}}\eta| - |1+\varepsilon^{\frac{1}{2}}\xi|} \leq \frac{\varepsilon^{\frac{1}{2}}}{1+\varepsilon^{\frac{1}{2}}\mathrm{Re}(\eta) - 1} = \frac{1}{\mathrm{Re}(\eta)}. \tag{4.28}$$

Using the bound from (4.28) in (4.26) and summing over k, we arrive at the desired estimate (4.18).

For (4.19), we note the following simple inequality

$$C_1|z| < \log(|1+z|) < C_2|z|, \quad \text{for all } z \text{ satisfying } |z| < 100^{-1}, \tag{4.29}$$

for some constants C_1 and C_2 that are independent of z. Therefore by our assumption,

$$\frac{|1-\varepsilon\eta^2|^{\mathtt{t}_{M-1}}}{|1+\varepsilon^{\frac{1}{2}}\eta|^{2\mathtt{t}_{M-1}+y_{\mathtt{t}_{M-1}+1}}} \le \mathrm{e}^{C_2|\eta^2|\varepsilon\mathtt{t}_{M-1}-(|C_1|+|C_2|)|\eta||\varepsilon^{1/2}(2\mathtt{t}_{M-1}+y_{\mathtt{t}_{M-1}+1})|} \le \mathrm{e}^{C(|\eta^2|+|\eta|+1)}, \tag{4.30}$$

for some large constant C by using (4.15). For the other factor, we note that

$$x(2-x)\ge 1-c^2 \quad \text{ when } 1-c<x<1+c \text{ and } 0<c<1. \tag{4.31}$$

Therefore

$$|1+\varepsilon^{\frac{1}{2}}\xi|(2-|1+\varepsilon^{\frac{1}{2}}\xi|)\ge 1-\varepsilon|\xi^2|, \tag{4.32}$$

and

$$(|1+\varepsilon^{\frac{1}{2}}\xi|(2-|1+\varepsilon^{\frac{1}{2}}\xi|))^{-\mathtt{t}_{M-1}+1} \le (1-\varepsilon|\xi^2|)^{-\mathtt{t}_{M-1}+1} \le^{C_1|\xi^2|\varepsilon(-\mathtt{t}_{M-1}+1)} \le \mathrm{e}^{C(|\xi^2|+1)}, \tag{4.33}$$

for some constant C by using (4.15). Finally,

$$|1+\varepsilon^{1/2}\xi|^{2\mathtt{t}_{M-1}+y_{\mathtt{t}_{M-1}-1}} \le \mathrm{e}^{(|C1|+|C_2|)|\xi|\cdot\varepsilon^{1/2}|2\mathtt{t}_{M-1}+y_{\mathtt{t}_{M-1}-1}|} \le \mathrm{e}^{C(|\xi|+1)}, \tag{4.34}$$

for some constant C by using (4.15). Combining the above estimates, we obtain (4.19).

(a) Now we prove part (a). We start with rewriting $\chi_{\mathfrak{h}}(\eta,\xi)$ under the assumption that $\mathfrak{h}\in m\mathrm{NW}_0$, recall the definition of $\chi_{\mathfrak{h}}(\eta,\xi)$ from (2.9). For $\mathrm{supp}(\mathfrak{h})=\{\omega_0,\cdots,\omega_{M-1}\}$ with $0=\omega_0>\cdots>\omega_{M-1}$, one has $\mathbb{P}(\boldsymbol{\tau}_+\neq 0)=0$ where $\boldsymbol{\tau}_+$ is defined in (2.16). Thus it is easy to check that the first and third term on the right-hand side of (2.9) are both equal to $\frac{\mathrm{e}^{\mathfrak{h}(0)(\eta-\xi)}}{\eta-\xi}$ and they cancel each other. On the other hand since $\mathbb{P}(\boldsymbol{\tau}_-\notin\{\omega_0,\cdots,\omega_{M-1}\})=0$, the second term on the right-hand side of (2.9) is given by

$$\sum_{k=0}^{M-1}\int_{\substack{\mathrm{s}_i\ge\theta_i,0\le i\le k-1;\\ \mathrm{s}_k<\theta_k}} \mathrm{e}^{\mathrm{s}_k\eta+\omega_k\eta^2-\mathrm{s}_0\xi}\cdot\prod_{i=0}^{k-1}\mathrm{p}_{\omega_i-\omega_{i+1}}(\mathrm{s}_{i+1}-\mathrm{s}_i)\,\mathrm{ds}_0\cdots\mathrm{ds}_k, \tag{4.35}$$

where $\mathrm{p}_{\omega-\omega'}(\mathrm{s}-\mathrm{s}')=\frac{1}{\sqrt{4\pi(\omega-\omega')}}\mathrm{e}^{-\frac{(\mathrm{s}-\mathrm{s}')^2}{4(\omega-\omega')}}$ is the transition density of a Brownian motion with diffusivity constant 2. Thus for $\mathfrak{h}\in m\mathrm{NW}_0$, we have

$$\chi_{\mathfrak{h}}(\eta,\xi)=\sum_{k=0}^{M-1}\int_{\substack{\mathrm{s}_i\ge\theta_i,0\le i\le k-1;\\ \mathrm{s}_k<\theta_k}} \mathrm{e}^{\mathrm{s}_k\eta+\omega_k\eta^2-\mathrm{s}_0\xi}\cdot\prod_{i=0}^{k-1}\mathrm{p}_{\omega_i-\omega_{i+1}}(\mathrm{s}_{i+1}-\mathrm{s}_i)\,\mathrm{ds}_0\cdots\mathrm{ds}_k. \tag{4.36}$$

Now we fix $\xi\in\mathbb{C}_{\mathrm{L}},\eta\in\mathbb{C}_{\mathrm{R}}$, and consider (4.20). Use the following scaling and recall that $\omega_0=\theta_0=0$,

$$\mathtt{t}_i:=-\lfloor\varepsilon^{-1}\omega_i\rfloor-\lfloor\frac{1}{2}\varepsilon^{-\frac{1}{2}}\theta_i\rfloor+1,\quad z_i:=2\lfloor\varepsilon^{-1}\omega_i\rfloor-\lfloor\varepsilon^{-\frac{1}{2}}(\mathrm{s}_i-\theta_i)\rfloor,\quad y_{\mathtt{t}_i}:=2\lfloor\varepsilon^{-1}\omega_i\rfloor. \tag{4.37}$$

We write (4.20) as a multiple Riemann sum

$$\begin{aligned}&\frac{1}{2}\varepsilon^{\frac{1}{2}}\cdot\mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\eta,-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\xi\right)\\ &=\sum_{k=0}^{M-1}\int_{\substack{\mathrm{s}_i\ge\theta_i,0\le i\le k-1;\\ \mathrm{s}_k<\theta_k}}\frac{(1+\varepsilon^{\frac{1}{2}}\xi)^{z_0}}{(1+\varepsilon^{\frac{1}{2}}\eta)^{z_k+2\mathtt{t}_k+1}}\cdot(1-\varepsilon\eta^2)^{\mathtt{t}_k}\cdot\prod_{i=0}^{k-1}\varepsilon^{-\frac{1}{2}}p_{\mathtt{t}_{i+1}-\mathtt{t}_i}(z_{i+1}-z_i)\,\mathrm{ds}_0\cdots\mathrm{ds}_k.\end{aligned} \tag{4.38}$$

Note that when $\mathrm{s}_0,\ldots,\mathrm{s}_k$ are all fixed, the factors in the integrand all converge as $\varepsilon\to 0$:

$$\begin{aligned}(1+\varepsilon^{\frac{1}{2}}\xi)^{z_0}&\to\mathrm{e}^{-\mathrm{s}_0\xi},\\ (1+\varepsilon^{\frac{1}{2}}\eta)^{z_k+2\mathtt{t}_k+1}&\to\mathrm{e}^{-\mathrm{s}_k\eta},\\ (1-\varepsilon\eta^2)^{\mathtt{t}_k}&\to\mathrm{e}^{\omega_k\eta^2},\\ \varepsilon^{-\frac{1}{2}}p_{\mathtt{t}_{i+1}-\mathtt{t}_i}(z_{i+1}-z_i)&\to\mathrm{p}_{\omega_i-\omega_{i+1}}(\mathrm{s}_{i+1}-\mathrm{s}_i),\end{aligned} \tag{4.39}$$

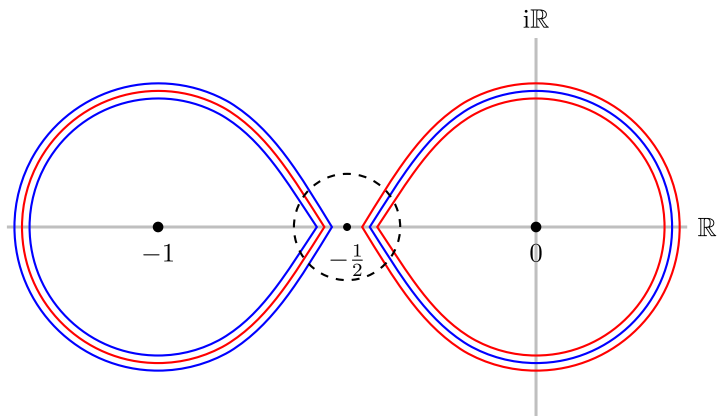

Figure 3: The deformed u, v-contours. Inside the dashed circle is the region $\{w \in \mathbb{C} : |w + \frac{1}{2}| < \frac{1}{200}\}$, where we deform the contours to be the same as their limiting counterparts shown in Figure 1.

where the last convergence follows from the local central limit theorem or a direct computation using the formulas. Thus, we formally obtain that the limit of (4.38) is equal to (4.35), therefore (4.17) follows.

In order to rigorously show the above convergence, we need to show that (4.38) is uniformly bounded and the dominated convergence theorem applies. Note that the right hand side of (4.38) is the same as that of (4.21). Therefore (4.19) gives a uniform bound for (4.38). This completes the proof. □

4.3 Convergence of the series expansion

In this section we prove Lemma 4.3 and 4.4. We will make the additional assumption that $0 < \tau_1 < \cdots < \tau_m$ to make the presentation lighter. The convergence results and arguments in this section still work if $\tau_i = \tau_{i+1}$ for some i but the contours need to be chosen carefully to make sure that the integrand has the desired super-exponential decay. Alternatively one can directly work with the limiting KPZ fixed point formula which is continuous with respect to the limit $\tau_{i+1} \to \tau_i$ and our choices of the angles in Figure 1 guarantees the convergence of the formula (2.21) even when some time parameters are equal.

We will deform the u, v contours so that locally near the critical point $-\frac{1}{2}$, they look like the limiting contours for ξ, η. More concretely, let Γ_{L} be a contour in the left half-plane going from $\infty e^{-2\pi i/3}$ to $e^{2\pi i/3}$ and Γ_{R} be a contour in the right half-plane going from $\infty e^{-\pi i/5}$ to $e^{\pi i/5}$ (see Figure 1). For each $\varepsilon > 0$, we deform the u-contour Σ_L and v-contour Σ_{R} (see Figure 2) so that the corresponding contours $\Gamma_{\mathrm{L}}^{\varepsilon}$ and $\Gamma_{\mathrm{R}}^{\varepsilon}$ for the rescaled variables $\xi := 2\varepsilon^{-\frac{1}{2}}\left(u + \frac{1}{2}\right)$ and $\eta := 2\varepsilon^{-\frac{1}{2}}\left(v + \frac{1}{2}\right)$ satisfy

$$\begin{aligned} \widehat{\Sigma}_{\mathrm{L}} \subset \widehat{\Omega}_{\mathrm{L}}^{\varepsilon}, \quad \Gamma_{\mathrm{L}}^{\varepsilon} \cap \{\zeta \in \mathbb{C} : |\zeta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}\} = \Gamma_{\mathrm{L}} \cap \{\zeta \in \mathbb{C} : |\zeta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}\}, \\ \widehat{\Sigma}_{\mathrm{R}} \subset \widehat{\Omega}_{\mathrm{R}}^{\varepsilon}, \quad \Gamma_{\mathrm{R}}^{\varepsilon} \cap \{\zeta \in \mathbb{C} : |\zeta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}\} = \Gamma_{\mathrm{R}} \cap \{\zeta \in \mathbb{C} : |\zeta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}\}, \end{aligned} \tag{4.40}$$

where

$$\widehat{\Omega}_{\mathrm{L}}^{\varepsilon} := \{-\varepsilon^{-\frac{1}{2}} + \varepsilon^{-\frac{1}{2}} z : |z| < 1\}, \quad \widehat{\Omega}_{\mathrm{R}}^{\varepsilon} := \{\varepsilon^{-\frac{1}{2}} + \varepsilon^{-\frac{1}{2}} z : |z| < 1\}. \tag{4.41}$$

See Figure 3 for an illustration of the deformed contours. Note that

$$u \in \Omega_{\mathrm{L}} \iff \xi \in \widehat{\Omega}_{\mathrm{L}}^{\varepsilon}, \quad v \in \Omega_{\mathrm{R}} \iff \eta \in \widehat{\Omega}_{\mathrm{R}}^{\varepsilon}.$$

Recall the functions f_i defined in (3.15). We now introduce some rescaled versions of them. For each $1 \le i \le m$, $\zeta \in \mathbb{C}_{\mathrm{L}} \cup \mathbb{C}_{\mathrm{R}}$ and $\varepsilon > 0$ sufficiently small, define

$$\mathrm{f}_i^{\varepsilon}(\zeta) := (1 - \varepsilon^{\frac{1}{2}}\zeta)^{k_i^{\varepsilon} - k_{i-1}^{\varepsilon}} (1 + \varepsilon^{\frac{1}{2}}\zeta)^{(k_{i-1}^{\varepsilon} - k_i^{\varepsilon}) + (a_{i-1}^{\varepsilon} - a_i^{\varepsilon})} e^{\frac{1}{2}(t_i^{\varepsilon} - t_{i-1}^{\varepsilon})\varepsilon^{\frac{1}{2}}\zeta}. \tag{4.42}$$

Here

$$k_\ell^\varepsilon := \frac{1}{2}\varepsilon^{-\frac{3}{2}}\tau_\ell - \varepsilon^{-1}\alpha_\ell - \frac{1}{2}\varepsilon^{-\frac{1}{2}}\beta_\ell + O(1), \quad a_\ell^\varepsilon := 2\varepsilon^{-1}\alpha_\ell + O(1), \quad t_\ell^\varepsilon := 2\varepsilon^{-\frac{3}{2}}\tau_\ell, \quad \text{for } 1 \le \ell \le m, \tag{4.43}$$

with the convention that $k_0^\varepsilon = a_0^\varepsilon = t_0^\varepsilon := 0$. It is straightforward to check that

$$\frac{\mathrm{f}_i^\varepsilon(\xi)}{\mathrm{f}_i^\varepsilon(\eta)} = \frac{f_i\left(-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\xi\right)}{f_i\left(-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\eta\right)}, \quad \forall \xi \in \mathbb{C}_{\mathrm{L}}, \eta \in \mathbb{C}_{\mathrm{R}},$$

where f_i's are defined in (3.15) with the parameters chosen as in (4.43). We begin by stating the needed estimates and asymptotics for the functions f_i^ε.

Lemma 4.6. *Assume $0 < \tau_1 < \cdots < \tau_m$. Let f_i^ε be defined as in (4.42) for $1 \le i \le m$. The following holds.*

(a) For any $\zeta \in \mathbb{C}_{\mathrm{L}} \cup \mathbb{C}_{\mathrm{R}}$ fixed, we have

$$\lim_{\varepsilon\to 0} \mathrm{f}_i^\varepsilon(\zeta) = \mathrm{f}_i(\zeta) =: \exp\left(-\frac{1}{3}(\tau_i - \tau_{i-1})\zeta^3 + (\alpha_i - \alpha_{i-1})\zeta^2 + (\beta_i - \beta_{i-1})\zeta\right), \quad 1 \le i \le m. \tag{4.44}$$

(b) There exists constants $c, C > 0$ such that

$$|\mathrm{f}_i^\varepsilon(\xi)| \le C\mathrm{e}^{-c(\tau_i-\tau_{i-1})|\xi|^3}, \forall \xi \in \Gamma_{\mathrm{L}}^\varepsilon, \quad \textit{and} \quad |\mathrm{f}_i^\varepsilon(\eta)| \ge C^{-1}\mathrm{e}^{c(\tau_i-\tau_{i-1})|\eta|^3}, \forall \eta \in \Gamma_{\mathrm{R}}^\varepsilon. \tag{4.45}$$

Proof. To lighten the notation we temporarily denote $\tau = \tau_i - \tau_{i-1}$, $\alpha = \alpha_i - \alpha_{i-1}$ and $\beta = \beta_i - \beta_{i-1}$. Write

$$\mathrm{f}_i^\varepsilon(\zeta) = \exp\left(\varepsilon^{-\frac{3}{2}}\tau \cdot g_3(\zeta) + \varepsilon^{-1}\alpha \cdot g_2(\zeta) + \varepsilon^{-\frac{1}{2}}\beta \cdot g_1(\zeta)\right),$$

where

$$\begin{aligned} g_1(\zeta) &:= -\frac{1}{2}\log(1-\varepsilon^{\frac{1}{2}}\zeta) + \frac{1}{2}\log(1+\varepsilon^{\frac{1}{2}}\zeta), \\ g_2(\zeta) &:= -\log(1-\varepsilon^{\frac{1}{2}}\zeta) - \log(1+\varepsilon^{\frac{1}{2}}\zeta), \\ g_3(\zeta) &:= \frac{1}{2}\log(1-\varepsilon^{\frac{1}{2}}\zeta) - \frac{1}{2}\log(1+\varepsilon^{\frac{1}{2}}\zeta) + \varepsilon^{\frac{1}{2}}\zeta. \end{aligned} \tag{4.46}$$

Part (a) follows from a straightforward Taylor expansion of (4.46). For part (b) we will assume $\zeta = \eta \in \Gamma_{\mathrm{R}}^\varepsilon$, the other case is similar. We split into two cases depending on whether $|\eta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}$ or not. Using the elementary bound

$$|\log(1-z) + \sum_{k=1}^{n-1}\frac{z^k}{k}| \le \frac{|z|^n}{n(1-|z|)}, \quad \forall |z| < 1,$$

we have for $|\eta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}$:

$$\left|g_3(\eta) + \frac{1}{3}\varepsilon^{\frac{3}{2}}\eta^3\right| \le \varepsilon^2|\eta|^4 \le \frac{\varepsilon^{\frac{3}{2}}}{100}|\eta|^3.$$

Thus

$$\mathrm{Re}\left(g_3(\eta)\right) \ge \mathrm{Re}\left(-\frac{1}{3}\varepsilon^{\frac{3}{2}}\eta^3\right) - \left|g_3(\eta) + \frac{1}{3}\varepsilon^{\frac{3}{2}}\eta^3\right| \ge -\frac{1}{3}\varepsilon^{\frac{3}{2}}\mathrm{Re}(\eta^3) - \frac{1}{100}\varepsilon^{\frac{3}{2}}|\eta|^3 \ge c'\varepsilon^{\frac{3}{2}}|\eta|^3,$$

for some $c' > 0$, due to our choice of the contour Γ_{R}. Similar argument shows that

$$|g_2(\eta)| \le C'\varepsilon|\eta|^2, \quad |g_1(\eta)| \le C'\varepsilon^{\frac{1}{2}}|\eta|,$$

for some $C' > 0$. Thus for $|\eta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}$, we have

$$|\mathrm{f}_i^\varepsilon(\eta)| = \exp\left(\varepsilon^{-\frac{3}{2}}\tau \cdot \mathrm{Re}(g_3(\eta)) + \varepsilon^{-1}\alpha \cdot \mathrm{Re}(g_2(\eta)) + \varepsilon^{-\frac{1}{2}}\beta \cdot \mathrm{Re}(g_3(\eta))\right)$$
$$\ge \exp(c\tau|\eta|^3 - C'|\alpha||\eta|^2 - C'|\beta||\eta|) \ge C\exp(c'|\eta|^3),$$

for some constants $c, c', C, C' > 0$. On the other hand it is elementary to check that for $\eta \in \widehat{\Omega}_{\mathrm{R}}^\varepsilon \backslash \{|\eta| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}\}$, we have

$$\mathrm{Re}(g_3(\eta)) \ge c_3 > 0, \quad |g_2(\eta))| \le C_2, \quad |g_1(\eta)| \le C_1.$$

Thus for such η we have

$$|\mathrm{f}_i^\varepsilon(\eta)| \ge \exp(c_3\varepsilon^{-\frac{3}{2}} - C_2\varepsilon^{-1} - C_1\varepsilon^{-\frac{1}{2}}) \ge \exp(c_3'\varepsilon^{-\frac{3}{2}}) \ge \exp(c_3'|\eta|^3/8),$$

since in this region $\frac{1}{100}\varepsilon^{-\frac{1}{2}} \le |\eta| \le 2\varepsilon^{-\frac{1}{2}}$. This completes the proof of part (b) and the lemma. $\square$

4.3.1 Proof of Lemma 4.3

Introduce the change of variables

$$u_{i_\ell}^{(\ell)} = -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\xi_{i_\ell}^{(\ell)}, \quad v_{i_\ell}^{(\ell)} = -\frac{1}{2} + \frac{1}{2}\varepsilon^{\frac{1}{2}}\eta_{i_\ell}^{(\ell)}, \tag{4.47}$$

for $1 \le \ell \le m$ and $1 \le i_\ell \le n_\ell$. It is easy to check that under this change of variables we have

$$\mathrm{C}\left(U^{(\ell)} \sqcup V^{(\ell+1)}; V^{(\ell)} \sqcup U^{(\ell+1)}\right) = \left(\frac{2}{\varepsilon}\right)^{\frac{n_\ell+n_{\ell+1}}{2}} \mathrm{C}\left(\boldsymbol{\xi}^{(\ell)} \sqcup \boldsymbol{\eta}^{(\ell+1)}; \boldsymbol{\eta}^{(\ell)} \sqcup \boldsymbol{\xi}^{(\ell+1)}\right), \tag{4.48}$$

$$\mathrm{C}\left(U^{(m)}; V^{(m)}\right) = \left(\frac{2}{\varepsilon}\right)^{\frac{n_m}{2}} \mathrm{C}\left(\boldsymbol{\xi}^{(m)}; \boldsymbol{\eta}^{(m)}\right), \tag{4.49}$$

for $1 \le \ell \le m-1$. As in (4.40), we split the ξ, η contours depending on whether they lie in the region $\Omega_0^\varepsilon := \{w \in \mathbb{C} : |w| \le \frac{1}{100}\varepsilon^{-\frac{1}{2}}\}$ or not, and deform the contours so that $\Gamma_{\mathrm{L/R}}^\varepsilon$ agree with $\Gamma_{\mathrm{L/R}}$ inside Ω_0^ε. Recall the definition of $\mathcal{D}_{Y^\varepsilon}^{(\mathbf{n})}$ in (3.17) and $\mathrm{D}_{\mathfrak{h}}^{(\mathbf{n})}$ in (2.21). Let A be the event that all the variables $\xi_{i_\ell}^{(\ell)}, \eta_{i_\ell}^{(\ell)}$ lie inside Ω_0^ε for $1 \le \ell \le m$ and $1 \le i_\ell \le n_\ell$, and A^c be the complement. Define

$$\begin{aligned}\mathcal{D}_{Y^\varepsilon}^{(\mathbf{n},\mathrm{main})} := &\prod_{\ell=1}^{m-1}(1-z_\ell)^{n_\ell}\left(1-z_\ell^{-1}\right)^{n_{\ell+1}}\left(\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\int_{\Gamma_{\ell,\mathrm{L}}}\mathrm{d}\mu_{\boldsymbol{z}}(\xi_{i_\ell}^{(\ell)})\int_{\Gamma_{\ell,\mathrm{R}}}\mathrm{d}\mu_{\boldsymbol{z}}(\eta_{i_\ell}^{(\ell)})\right)\mathbf{1}_A \\ &\left(\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\frac{\mathrm{f}_\ell^\varepsilon(\xi_{i_\ell}^{(\ell)})}{\mathrm{f}_\ell^\varepsilon(\eta_{i_\ell}^{(\ell)})}\right)\cdot\det\left[\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\mathrm{ch}_{Y^\varepsilon}\left(-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\eta_j^{(1)}, -\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\xi_i^{(1)}\right)\right]_{1\le i,j\le n_1} \\ &\cdot\prod_{\ell=1}^{m-1}\mathrm{C}\left(\boldsymbol{\xi}^{(\ell)} \sqcup \boldsymbol{\eta}^{(\ell+1)}; \boldsymbol{\eta}^{(\ell)} \sqcup \boldsymbol{\xi}^{(\ell+1)}\right)\cdot\mathrm{C}(\boldsymbol{\xi}^{(m)}; \boldsymbol{\eta}^{(m)}),\end{aligned} \tag{4.50}$$

and $\mathcal{D}_{Y^\varepsilon}^{(\mathbf{n},\mathrm{error})} := \mathcal{D}_{Y^\varepsilon}^{(\mathbf{n})} - \mathcal{D}_{Y^\varepsilon}^{(\mathbf{n},\mathrm{main})}$. Note that

$$\begin{aligned}\mathcal{D}_{Y^\varepsilon}^{(\mathbf{n},\mathrm{error})} = &\prod_{\ell=1}^{m-1}(1-z_\ell)^{n_\ell}\left(1-z_\ell^{-1}\right)^{n_{\ell+1}}\left(\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\int_{\Gamma_{\ell,\mathrm{L}}^\varepsilon}\mathrm{d}\mu_{\boldsymbol{z}}(\xi_{i_\ell}^{(\ell)})\int_{\Gamma_{\ell,\mathrm{R}}^\varepsilon}\mathrm{d}\mu_{\boldsymbol{z}}(\eta_{i_\ell}^{(\ell)})\right)\mathbf{1}_{A^c} \\ &\left(\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\frac{\mathrm{f}_\ell^\varepsilon(\xi_{i_\ell}^{(\ell)})}{\mathrm{f}_\ell^\varepsilon(\eta_{i_\ell}^{(\ell)})}\right)\cdot\det\left[\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\mathrm{ch}_{Y^\varepsilon}\left(-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\eta_j^{(1)}, -\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\xi_i^{(1)}\right)\right]_{1\le i,j\le n_1} \\ &\cdot\prod_{\ell=1}^{m-1}\mathrm{C}\left(\boldsymbol{\xi}^{(\ell)} \sqcup \boldsymbol{\eta}^{(\ell+1)}; \boldsymbol{\eta}^{(\ell)} \sqcup \boldsymbol{\xi}^{(\ell+1)}\right)\cdot\mathrm{C}(\boldsymbol{\xi}^{(m)}; \boldsymbol{\eta}^{(m)}).\end{aligned} \tag{4.51}$$

We claim that:

$$\lim_{\varepsilon\to 0}\mathcal{D}^{(\mathbf{n},\mathrm{main})}_{Y^\varepsilon}=\mathrm{D}^{(\mathbf{n})}_{\mathfrak{h}},\quad \lim_{\varepsilon\to 0}\mathcal{D}^{(\mathbf{n},\mathrm{error})}_{Y^\varepsilon}=0. \tag{4.52}$$

For the first part of the claim, note that the integrand on the right-hand side of (4.50) converges pointwise to the integrand on the right-hand side of (2.21) by Lemma 4.6(a) and Proposition 4.2(a). On the other hand the cubic exponential decay bound for $|\mathrm{f}^\varepsilon_i|$ from Lemma 4.6(b) and the quadratic exponential growth estimate (4.19) implies that the integrand on the right-hand side of (4.50) has a cubic exponential decay in every variable $\eta^{(\ell)}_{i_\ell},\xi^{(\ell)}_{i_\ell}$ as they go to ∞, uniform in ε. Thus the dominated convergence theorem applies and the first claim is proved.

For the second part of the claim, note that on A^c, at least one of the variables, say $\xi^{(1)}_1$, lies outside of $\widehat{\Omega}^\varepsilon_0$. Then Lemma 4.6(b) implies that $|\mathrm{f}^\varepsilon_1(\xi^{(1)}_1)|\le C\exp(-c\varepsilon^{-\frac{3}{2}})$ for all $\xi^{(1)}_1\in\Gamma^\varepsilon_{1,\mathrm{L}}\backslash\widehat{\Omega}^\varepsilon_0$. On the other hand Proposition 4.2(b) implies that

$$\left|\frac{1}{2}\varepsilon^{\frac{1}{2}}\mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\eta^{(1)}_j,-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\xi^{(1)}_1\right)\right|\le C\exp(c\varepsilon^{-1}), \tag{4.53}$$

for some $c,C>0$ and all $1\le j\le n_1$. The other parts of the integrand remain bounded. Thus

$$|\mathcal{D}^{(\mathbf{n},\mathrm{error})}_{Y^\varepsilon}|\le C\varepsilon^{-(n_1+\cdots+n_m)}\cdot\exp(-c\varepsilon^{-\frac{3}{2}})\to 0,\quad \text{as } \varepsilon\to 0. \tag{4.54}$$

This proves the second part of the claim and Lemma 4.3 follows.

4.3.2 Proof of Lemma 4.4

Recall the expression (3.17) for $\mathcal{D}^{(\mathbf{n})}_{Y^\varepsilon}$. We first rewrite it slightly by writing $\mathrm{f}^\varepsilon_i(\zeta)=\mathrm{f}^\varepsilon_i(\zeta)^{\frac{1}{2}}\cdot\mathrm{f}^\varepsilon_i(\zeta)^{\frac{1}{2}}$ and putting one of the square root inside the first determinant, to get

$$\begin{aligned}\mathcal{D}^{(\mathbf{n})}_{Y^\varepsilon}=&\prod_{\ell=1}^{m-1}(1-z_\ell)^{n_\ell}\left(1-z_\ell^{-1}\right)^{n_{\ell+1}}\left(\prod_{\ell=1}^{m}\prod_{i_\ell=1}^{n_\ell}\int_{\Gamma^\varepsilon_{\ell,\mathrm{L}}}\mathrm{d}\mu_{\boldsymbol{z}}(\xi^{(\ell)}_{i_\ell})\int_{\Gamma^\varepsilon_{\ell,\mathrm{R}}}\mathrm{d}\mu_{\boldsymbol{z}}(\eta^{(\ell)}_{i_\ell})\right)\prod_{\ell=2}^{m}\prod_{i_\ell=1}^{n_\ell}\frac{\mathrm{f}^\varepsilon_\ell(\xi^{(\ell)}_{i_\ell})}{\mathrm{f}^\varepsilon_\ell(\eta^{(\ell)}_{i_\ell})}\\&\left(\prod_{i_1=1}^{n_1}\frac{\mathrm{f}^\varepsilon_1(\xi^{(1)}_{i_1})^{1/2}}{\mathrm{f}^\varepsilon_1(\eta^{(1)}_{i_1})^{1/2}}\right)\cdot\det\left[\frac{\mathrm{f}^\varepsilon_1(\xi^{(1)}_i)^{1/2}}{\mathrm{f}^\varepsilon_1(\eta^{(1)}_j)^{1/2}}\cdot\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\mathrm{ch}_{Y^\varepsilon}\left(-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\eta^{(1)}_j,-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\xi^{(1)}_i\right)\right]_{1\le i,j\le n_1}\\&\cdot\prod_{\ell=1}^{m-1}\mathrm{C}\left(\boldsymbol{\xi}^{(\ell)}\sqcup\boldsymbol{\eta}^{(\ell+1)};\boldsymbol{\eta}^{(\ell)}\sqcup\boldsymbol{\xi}^{(\ell+1)}\right)\cdot\mathrm{C}(\boldsymbol{\xi}^{(m)};\boldsymbol{\eta}^{(m)}),\end{aligned} \tag{4.55}$$

Here the choice of square root does not matter as long as we make the same choice for the two. The advantage of this rewriting is that

$$\left|\frac{\mathrm{f}^\varepsilon_1(\xi^{(1)}_i)^{1/2}}{\mathrm{f}^\varepsilon_1(\eta^{(1)}_j)^{1/2}}\cdot\frac{1}{2}\varepsilon^{\frac{1}{2}}\mathrm{ch}_{Y^\varepsilon}\left(-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\eta^{(1)}_j,-\frac{1}{2}+\frac{1}{2}\varepsilon^{\frac{1}{2}}\xi^{(1)}_i\right)\right|\le c,\quad \forall 1\le i,j\le n_1, \tag{4.56}$$

for some constant $c>0$ independent of ε and $\boldsymbol{\xi}^{(1)},\boldsymbol{\eta}^{(1)}$. Thus by Hadamard's inequality

$$\left|\det\left[\frac{\mathrm{f}^\varepsilon_1(\xi^{(1)}_i)^{1/2}}{\mathrm{f}^\varepsilon_1(\eta^{(1)}_j)^{1/2}}\cdot\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\mathrm{ch}_{Y^\varepsilon}\left(-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\eta^{(1)}_j,-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon^{\frac{1}{2}}\xi^{(1)}_i\right)\right]_{1\le i,j\le n_1}\right|\le n_1^{\frac{n_1}{2}}\cdot c^{n_1}. \tag{4.57}$$

The same arguments imply that

$$|\mathrm{C}(\boldsymbol{\xi}^{(m)};\boldsymbol{\eta}^{(m)})|\le n_m^{\frac{n_m}{2}}\cdot D^{n_m}, \tag{4.58}$$

and

$$|\mathrm{C}\left(\boldsymbol{\xi}^{(\ell)}\sqcup\boldsymbol{\eta}^{(\ell+1)};\boldsymbol{\eta}^{(\ell)}\sqcup\boldsymbol{\xi}^{(\ell+1)}\right)|\le(n_\ell+n_{\ell+1})^{\frac{n_\ell+n_{\ell+1}}{2}}\cdot D^{\frac{n_\ell+n_{\ell+1}}{2}}\le n_\ell^{\frac{n_\ell}{2}}n_{\ell+1}^{\frac{n_{\ell+1}}{2}}\cdot(2D)^{\frac{n_\ell+n_{\ell+1}}{2}}, \tag{4.59}$$

for $1 \le \ell \le m-1$. Here D is chosen to be the reciprocal of the minimal distance between different Γ^ε contours, which can be made postive. Thus

$$\begin{aligned}
|\mathcal{D}^{(\mathbf{n})}_{Y^\varepsilon}| \le & \prod_{\ell=1}^{m-1} \frac{|1-z_\ell|^{n_\ell+n_{\ell+1}}}{|z_\ell|^{n_{\ell+1}}} \cdot (C')^{n_1+\cdots+n_m} \cdot \prod_{\ell=1}^{m} n_\ell^{n_\ell} \cdot \prod_{i_1=1}^{n_1} \int_{\Gamma^\varepsilon_{1,\mathrm{L}}} \frac{\mathrm{d}|\xi^{(1)}_{i_1}|}{2\pi} \int_{\Gamma^\varepsilon_{1,\mathrm{R}}} \frac{\mathrm{d}|\eta^{(1)}_{i_1}|}{2\pi} \frac{|\mathrm{f}^\varepsilon_1(\xi^{(1)}_{i_1})|^{\frac12}}{|\mathrm{f}^\varepsilon_1(\eta^{(1)}_{i_1})|^{\frac12}} \\
& \cdot \prod_{\ell=2}^{m} \prod_{i_\ell=1}^{n_\ell} \left(\frac{1+|z_\ell|}{|1-z_\ell|}\right)^2 \int_{\Gamma^\varepsilon_{\ell,\mathrm{L}}} \frac{\mathrm{d}|\xi^{(\ell)}_{i_\ell}|}{2\pi} \int_{\Gamma^\varepsilon_{\ell,\mathrm{R}}} \frac{\mathrm{d}|\eta^{(\ell)}_{i_\ell}|}{2\pi} \frac{|\mathrm{f}^\varepsilon_\ell(\xi^{(\ell)}_{i_\ell})|}{|\mathrm{f}^\varepsilon_\ell(\eta^{(\ell)}_{\ell_1})|} \\
\le & \prod_{\ell=1}^{m-1} \frac{(1+|z_{\ell+1}|)^{2n_{\ell+1}}}{|z_\ell|^{n_{\ell+1}}|1-z_\ell|^{n_{\ell+1}-n_\ell}} \cdot \prod_{\ell=1}^{m} n_\ell^{n_\ell} \cdot C^{n_1+\cdots+n_m},
\end{aligned} \tag{4.60}$$

for some constant $C>0$, since each of the integrals is bounded by some finite constant. This completes the proof of Lemma 4.4.

5 From multiple narrow wedges to compactly supported initial conditions

In this section we extend the multipoint formula (1.6) to any $\mathfrak{h} \in \mathrm{UC}_c$ using a density argument. We choose to work at the level of the KPZ fixed point formula which enjoys more symmetry and nicer decay properties.

5.1 Density and Approximation

Our starting point is the following proposition asserting that our formula is continuous with respect to the initial condition on the space UC_c.

Proposition 5.1. *Given $\{\mathfrak{h}^n\}_{n\ge1} \subset \mathrm{UC}$ with $\mathrm{supp}(\mathfrak{h}^n) \subset [-L,L]$ and $\sup_{\alpha\in\mathbb{R}} \mathfrak{h}^n(\alpha) \le \beta$ for all $n \ge 1$. Assume $\mathfrak{h}^n \to \mathfrak{h}$ in UC and $\mathrm{supp}(\mathfrak{h}) \subset [-L,L]$, $\sup_{\alpha\in\mathbb{R}} \mathfrak{h}(\alpha) \le \beta$. Then*

$$\lim_{n\to\infty} \chi_{\mathfrak{h}^n}(\eta,\xi) = \chi_{\mathfrak{h}}(\eta,\xi), \tag{5.1}$$

for all $\xi \in \mathbb{C}_{\mathrm{L}}$ and $\eta \in \mathbb{C}_{\mathrm{R}}$. Recall the kernel $\chi_{\mathfrak{h}}$ was defined in Definition 2.2, equation (2.9). Consequently

$$\lim_{n\to\infty} \prod_{\ell=1}^{m-1} \oint \frac{\mathrm{d}z_\ell}{2\pi\mathrm{i} z_\ell(1-z_\ell)} \mathrm{D}_{\mathfrak{h}^n}(z_1,\cdots,z_{m-1}) = \prod_{\ell=1}^{m-1} \oint \frac{\mathrm{d}z_\ell}{2\pi\mathrm{i} z_\ell(1-z_\ell)} \mathrm{D}_{\mathfrak{h}}(z_1,\cdots,z_{m-1}). \tag{5.2}$$

Proof. It suffices to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}_n\xi^2 - \mathbf{B}(\boldsymbol{\tau}_n)\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}\right] = \int_{\mathbb{R}} \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}\xi^2 - \mathbf{B}(\boldsymbol{\tau})\xi)\mathbf{1}_{\boldsymbol{\tau}<\infty}\right],$$

where

$$\boldsymbol{\tau} := \inf\{\alpha \ge 0 : \mathbf{B}(\alpha) \le \mathfrak{h}(\alpha)\}, \quad \boldsymbol{\tau}_n := \inf\{\alpha \ge 0 : \mathbf{B}(\alpha) \le \mathfrak{h}^n(\alpha)\},$$

for a Brownian motion $\mathbf{B}(\alpha)$ with diffusivity constant 2. Here we have suppressed the + sign in the subscript of the hitting time to lighten the notation. The convergence of the other two parts in the definition of $\chi_{\mathfrak{h}}$ can be proved in the same way. From [MQR21][(B.20)] we know

$$\mathbb{P}_{\mathbf{B}(0)=s}(\mathbf{B}(\boldsymbol{\tau}_n) \in \mathrm{d}b, \boldsymbol{\tau}_n \in \mathrm{d}T) \to \mathbb{P}_{\mathbf{B}(0)=s}(\mathbf{B}(\boldsymbol{\tau}) \in \mathrm{d}b, \boldsymbol{\tau} \in \mathrm{d}T) \quad \text{weakly as } n \to \infty.$$

Take a smooth function $0 \le g_\beta \le 1$ such that $g_\beta(x) \equiv 1$ for $x \le \beta$ and $g_\beta(x) \equiv 0$ for $x \ge \beta + 1$. Since $\mathfrak{h}(\alpha) \le \beta$ and $\mathfrak{h}^n(\alpha) \le \beta$ for all $\alpha \in \mathbb{R}$ and $n \ge 1$, we have

$$\mathbf{1}_{\boldsymbol{\tau}_n<\infty}(1 - g_\beta(\mathbf{B}(\boldsymbol{\tau}_n))) \equiv 0, \quad \mathbf{1}_{\boldsymbol{\tau}<\infty}(1 - g_\beta(\mathbf{B}(\boldsymbol{\tau}))) \equiv 0.$$

Hence

$$\begin{aligned}\mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}_n\xi^2 - \mathbf{B}(\boldsymbol{\tau}_n)\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}\right] &= \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}_n\xi^2 - \mathbf{B}(\boldsymbol{\tau}_n)\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}g_\beta(\mathbf{B}_{\boldsymbol{\tau}_n})\right],\\ \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}\xi^2 - \mathbf{B}(\boldsymbol{\tau})\xi)\mathbf{1}_{\boldsymbol{\tau}<\infty}\right] &= \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}\xi^2 - \mathbf{B}(\boldsymbol{\tau})\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}g_\beta(\mathbf{B}_{\boldsymbol{\tau}})\right].\end{aligned}$$

By the weak convergence of $(\boldsymbol{\tau}_n, \mathbf{B}(\boldsymbol{\tau}_n))$ to $(\boldsymbol{\tau}, \mathbf{B}(\boldsymbol{\tau}))$ we know

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}_n\xi^2 - \mathbf{B}(\boldsymbol{\tau}_n)\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}g_\beta(\mathbf{B}_{\boldsymbol{\tau}_n})\right] = \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}\xi^2 - \mathbf{B}(\boldsymbol{\tau})\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}g_\beta(\mathbf{B}_{\boldsymbol{\tau}})\right],$$

since the function $f(T, b) := \exp(-T\xi^2 - b\xi)\mathbf{1}_{T<\infty}g_\beta(b) : [0, \infty] \times \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Therefore

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}_n\xi^2 - \mathbf{B}(\boldsymbol{\tau}_n)\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty})\right] = \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}\xi^2 - \mathbf{B}(\boldsymbol{\tau})\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}\right].$$

Finally, using the bound obtained in Proposition 2.3 and the dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}_n\xi^2 - \mathbf{B}(\boldsymbol{\tau}_n)\xi)\mathbf{1}_{\boldsymbol{\tau}_n<\infty}\right] = \int_{\mathbb{R}} \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp(-\boldsymbol{\tau}\xi^2 - \mathbf{B}(\boldsymbol{\tau})\xi)\mathbf{1}_{\boldsymbol{\tau}<\infty}\right].$$

This completes the proof of (5.1). (5.2) follows from the dominated convergence theorem and a similar bound as in Lemma 4.4 whose proof is almost identical and we omit it here. □

To extend (1.6) to all $\mathfrak{h} \in \mathrm{UC}_c$ using Propsition 5.1, we also need the following proposition asserting that the space mNW is dense in UC_c.

Proposition 5.2. *Let* $\mathfrak{h} \in \mathrm{UC}_c$. *Assume that* $\mathrm{supp}(\mathfrak{h}) \subset [-L, L]$ *and* $\max_{\alpha\in\mathbb{R}} \mathfrak{h}(\alpha) = \beta < \infty$. *Then there exists a sequence* $\{\mathfrak{h}^{(n)}\}_{n\ge1} \subset \mathrm{NW}$, *such that* $\mathrm{supp}(\mathfrak{h}) \subset [-L, L]$, $\max_{\alpha\in\mathbb{R}} \mathfrak{h}^n \le \beta$ *for all* $n \ge 1$ *and*

$$\mathfrak{h}^n \to \mathfrak{h} \text{ in UC}, \quad \text{as } n \to \infty,$$

Proof. We will use the following characterization (see, e.g., [MQR21, Section 3.1]): a sequence $\{\mathfrak{h}^n\}_{n\ge1} \subset \mathrm{UC}$ converges to $\mathfrak{h} \in \mathrm{UC}$ locally if and only if for any $\mathsf{x} \in \mathbb{R}$, one has

1. $\limsup_{n\to\infty} \mathfrak{h}(\mathsf{x}_n) \le \mathfrak{h}(\mathsf{x})$, for all $\mathsf{x}_n \to \mathsf{x}$;
2. There exists $\mathsf{x}_n \to \mathsf{x}$ such that $\liminf_{n\to\infty} \mathfrak{h}(\mathsf{x}_n) \ge \mathfrak{h}(\mathsf{x})$.

It suffices to consider the restrictions of the functions on $[-L, L]$. For each n consider the dyadic intervals $I_{n,k} := [\frac{k}{2^n} \cdot L, \frac{k+1}{2^n} \cdot L]$, for $k = -2^n, -2^n + 1, \cdots, 2^n - 1$. On each interval $I_{n,k}$ the maximum of $\mathfrak{h}$ exists, let $m_{n,k} \in I_{n,k}$ be one of the argmax, and set

$$\mathfrak{h}^n(\alpha) := \sum_{k=1}^{2^n-1} \mathfrak{h}(m_{n,k})\mathbf{1}_{\alpha=m_{n,k}} - \infty \cdot \mathbf{1}_{\alpha\ne m_{n,k},\forall k},$$

here if some $m_{n,k}$ is the argmax of two consecutive intervals, then it should appear only once in the sum. Now it is straightforward to check that for each α

$$\limsup_{\alpha^n\to\alpha} \mathfrak{h}^n(\alpha^n) \le \mathfrak{h}(\alpha),$$

and there exists $\alpha^n \to \alpha$ such that

$$\liminf_{\alpha^n\to\alpha} \mathfrak{h}^n(\alpha^n) \ge \mathfrak{h}(\alpha).$$

Therefore $\mathfrak{h}^n \to \mathfrak{h}$ locally in UC, and hence globally since $\mathfrak{h}^n, \mathfrak{h}$ are supported in $[-L, L]$. □

5.2 Proof of Proposition 2.4

Recall that what we have shown in Proposition 5.2 is that the formula (1.6) holds for initial condition $\mathfrak{h} \in m\mathrm{NW}_0$. We would like to prove it for $\mathfrak{h} \in m\mathrm{NW}$ by a shift argument. To this end it is more convenient to have a more general version of the characteristic function $\chi_{\mathfrak{h}}$, defined through (2.15), instead of the original (2.9). The goal of this section is to prove the equivalence of the two, i.e., Proposition 2.4. To this end, we denote

$$\begin{aligned}
\chi_{\mathfrak{h}}^{\omega}(\eta,\xi) &:= \mathrm{e}^{-\omega\xi^2}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-s\xi}\cdot \mathbb{E}_{\mathbf{B}(\omega)=s}\Big[\exp\big(\boldsymbol{\tau}_-^{\omega}\eta^2+\mathbf{B}(\boldsymbol{\tau}_-^{\omega})\eta\big)\,\mathbf{1}_{\boldsymbol{\tau}_-^{\omega}>-\infty}\Big]\\
&\quad + \mathrm{e}^{\omega\eta^2}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{+s\eta}\cdot \mathbb{E}_{\mathbf{B}(\omega)=s}\Big[\exp\big(-\boldsymbol{\tau}_+^{\omega}\xi^2-\mathbf{B}(\boldsymbol{\tau}_+^{\omega})\xi\big)\,\mathbf{1}_{\boldsymbol{\tau}_+^{\omega}<\infty}\Big]\\
&\quad - \int_{\mathbb{R}} \mathrm{d}s\, \mathbb{E}_{\mathbf{B}(\omega)=s}\Big[\exp\big(\boldsymbol{\tau}_-^{\omega}\eta^2+\mathbf{B}(\boldsymbol{\tau}_-^{\omega})\eta-\boldsymbol{\tau}_+^{\omega}\xi^2-\mathbf{B}(\boldsymbol{\tau}_+^{\omega})\xi\big)\,\mathbf{1}_{|\boldsymbol{\tau}_\pm^{\omega}|<\infty}\Big]\\
&:= \chi_{\mathfrak{h}}^{\omega,1}(\eta,\xi)+\chi_{\mathfrak{h}}^{\omega,2}(\eta,\xi)-\chi_{\mathfrak{h}}^{\omega,3}(\eta,\xi),
\end{aligned}\tag{5.3}$$

where

$$\boldsymbol{\tau}_+^{\omega} := \inf\{\alpha\ge\omega : \mathbf{B}(\alpha)\le\mathfrak{h}(\alpha)\},\quad \boldsymbol{\tau}_-^{\omega} := \sup\{\alpha\le\omega : \mathbf{B}(\alpha)\le\mathfrak{h}(\alpha)\}.$$

we claim that $\chi_{\mathfrak{h}}^{\omega} = \chi_{\mathfrak{h}}^{\omega_1}$, for any $\omega\in\mathbb{R}$. First assume $\mathfrak{h}\in m\mathrm{NW}$, say

$$\mathfrak{h}(\alpha) = \sum_{i=1}^{k}\theta_i\mathbf{1}_{\alpha=\omega_i} - \infty\mathbf{1}_{\alpha\ne\omega_i,\,\forall i},$$

where $\omega_1>\cdots>\omega_k$. To lighten the notation we will fix η,ξ and denote

$$\begin{aligned}
E^{+}(\omega,s) &:= \mathbb{E}_{\mathbf{B}(\omega)=s}\Big[\exp\big(-\boldsymbol{\tau}_+^{\omega}\xi^2-\mathbf{B}(\boldsymbol{\tau}_+^{\omega})\xi\big)\,\mathbf{1}_{\boldsymbol{\tau}_+^{\omega}<\infty}\Big],\\
E^{-}(\omega,s) &:= \mathbb{E}_{\mathbf{B}(\omega)=s}\Big[\exp\big(+\boldsymbol{\tau}_-^{\omega}\eta^2+\mathbf{B}(\boldsymbol{\tau}_-^{\omega})\eta\big)\,\mathbf{1}_{\boldsymbol{\tau}_-^{\omega}>-\infty}\Big].
\end{aligned}$$

For $\omega\in[\omega_1,\infty)$, we have

$$\chi_{\mathfrak{h}}^{\omega,1}(\eta,\xi) = \mathrm{e}^{-\omega\xi^2}\int_{\mathbb{R}}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}b_1\,\mathrm{e}^{-s\xi}\mathrm{p}_{\omega-\omega_1}(b_1-s)\,E^{-}(\omega_1,b_1) = \mathrm{e}^{-\omega_1\xi^2}\int_{\mathbb{R}}\mathrm{d}b_1\,\mathrm{e}^{-b_1\xi}\,E^{-}(\omega_1,b_1) = \chi_{\mathfrak{h}}^{\omega_1,1}(\eta,\xi),$$

where we are using the Markov property and the simple identity

$$\int_{\mathbb{R}}\mathrm{d}s\,\mathrm{e}^{sW}\mathrm{p}_t(r-s) = \mathrm{e}^{tW^2+rW}.\tag{5.4}$$

Recall that $\mathrm{p}_{t-s}(x-y)$ is the transition density of a Brownian motion with diffusivity constant 2. On the other hand

$$\chi_{\mathfrak{h}}^{\omega,2}(\eta,\xi) = \chi_{\mathfrak{h}}^{\omega,3}(\eta,\xi) = \mathbf{1}_{\omega=\omega_1}\frac{\mathrm{e}^{\omega_1(\eta^2-\xi^2)+\theta_1(\eta-\xi)}}{\eta-\xi},\quad \forall\omega\ge\omega_1.$$

Thus $\chi_{\mathfrak{h}}^{\omega}=\chi_{\mathfrak{h}}^{\omega_1}$ for all $\omega\le\omega_1$. We proceed by induction. Assume $\chi_{\mathfrak{h}}^{\omega}=\chi_{\mathfrak{h}}^{\omega_1}$ holds for all $\omega\ge\omega_i$. Now let $\omega\in[\omega_{i+1},\omega_i)$. We will show $\chi_{\mathfrak{h}}^{\omega}=\chi_{\mathfrak{h}}^{\omega_i}$, which is equal to $\chi_{\mathfrak{h}}^{\omega_1}$ by the induction hypothesis. By the same argument as above we have

$$\chi_{\mathfrak{h}}^{\omega,2}(\eta,\xi) = \chi_{\mathfrak{h}}^{\omega_i,2}(\eta,\xi) + \mathbf{1}_{\omega=\omega_{i+1}}\frac{\mathrm{e}^{\omega_{i+1}(\eta^2-\xi^2)+\theta_{i+1}(\eta-\xi)}}{\eta-\xi}.\tag{5.5}$$

On the other hand, by the Markov property and the identity (5.4) we have

$$\begin{aligned}\chi_{\mathfrak{h}}^{\omega_i,1}(\eta,\xi) &= \frac{\mathrm{e}^{\omega_i(\eta^2-\xi^2)+\theta_i(\eta-\xi)}}{\eta-\xi} + \mathrm{e}^{-\omega_i\xi^2}\int_{\theta_i}^{\infty}\mathrm{d}b_i\int_{\mathbb{R}}\mathrm{d}s\,\mathrm{e}^{-b_i\xi}\mathrm{p}_{\omega_i-\omega}(s-b_i)\,E^-(\omega,s)\\ &= \frac{\mathrm{e}^{\omega_i(\eta^2-\xi^2)+\theta_i(\eta-\xi)}}{\eta-\xi} + \chi_{\mathfrak{h}}^{\omega,1} - \mathrm{e}^{-\omega_i\xi^2}\int_{-\infty}^{\theta_i}\mathrm{d}b_i\int_{\mathbb{R}}\mathrm{d}s\,\mathrm{e}^{-b_i\xi}\mathrm{p}_{\omega_i-\omega}(s-b_i)\,E^-(\omega,s),\end{aligned}\tag{5.6}$$

where in the second equality we used Fubini's theorem and the identity (5.4) to show that

$$\mathrm{e}^{-\omega_i\xi^2}\int_{\mathbb{R}}\mathrm{d}b_i\int_{\mathbb{R}}\mathrm{d}s\,\mathrm{e}^{-b_i\xi}\mathrm{p}_{\omega_i-\omega}(s-b_i)\,E^-(\omega,s) = \mathrm{e}^{-\omega\xi^2}\int_{\mathbb{R}}\mathrm{d}s\,\mathrm{e}^{-s\xi}\cdot E^-(\omega,s) = \chi_{\mathfrak{h}}^{\omega,1}.$$

Finally,

$$\begin{aligned}\chi_{\mathfrak{h}}^{\omega,3}(\eta,\xi) &= \int_{\mathbb{R}}\mathrm{d}s\,E^-(\omega,s)\cdot E^+(\omega,s)\\ &= \int_{\mathbb{R}}\mathrm{d}s\,E^-(\omega,s)\cdot\left(\int_{\mathbb{R}}\mathrm{d}b_i\,\mathrm{p}_{\omega_i-\omega}(s-b_i)E^+(\omega_i,b_i) + \mathbf{1}_{\omega=\omega_{i+1}}\cdot\mathrm{e}^{-\omega_{i+1}\xi^2-s\xi}\cdot\mathbf{1}_{s\le\theta_{i+1}}\right).\end{aligned}$$

We split the b_i-integral into two parts depending on whether $b_i\le\theta_i$ or $b_i>\theta_i$. For $b_i>\theta_i$ we have

$$\int_{\mathbb{R}}\mathrm{d}s\,E^-(\omega,s)\mathrm{p}_{\omega_i-\omega}(s-b_i) = E^-(\omega_i,b_i),$$

by the Markov property. For $b_i\le\theta_i$ we have

$$E^+(\omega_i,b_i) = \mathrm{e}^{-\omega_i\xi^2-b_i\xi}\mathbf{1}_{b_i\le\theta_i}.$$

Thus

$$\begin{aligned}\chi_{\mathfrak{h}}^{\omega,3}(\eta,\xi) &= \int_{\theta_i}^{\infty}\mathrm{d}s\,E^+(\omega_i,b_i)E^-(\omega_i,b_i) + \mathrm{e}^{-\omega_i\xi^2}\int_{-\infty}^{\theta_i}\mathrm{d}b_i\int_{\mathbb{R}}\mathrm{d}s\,\mathrm{e}^{-b_i\xi}\mathrm{p}_{\omega-\omega_i}(s-b_i)\,E^-(\omega,s)\\ &\quad+\mathbf{1}_{\omega=\omega_{i+1}}\frac{\mathrm{e}^{\omega_{i+1}(\eta^2-\xi^2)+\theta_{i+1}(\eta-\xi)}}{\eta-\xi}.\end{aligned}\tag{5.7}$$

Combine (5.5), (5.6) and (5.7) we conclude that

$$\chi_{\mathfrak{h}}^{\omega} = \chi_{\mathfrak{h}}^{\omega,1}+\chi_{\mathfrak{h}}^{\omega,2}-\chi_{\mathfrak{h}}^{\omega,3} = \chi_{\mathfrak{h}}^{\omega_i,1}+\chi_{\mathfrak{h}}^{\omega_i,2}-\chi_{\mathfrak{h}}^{\omega_i,3} = \chi_{\mathfrak{h}}^{\omega_i},$$

for all $\omega\in[\alpha_{i+1},\alpha_i)$. Thus by induction $\chi_{\mathfrak{h}}^{\omega}=\chi_{\mathfrak{h}}$ for all $\omega\in\mathbb{R}$ and $\mathfrak{h}\in m\mathrm{NW}$.

For general $\mathfrak{h}\in\mathrm{UC}_c$ supported on $[-L,L]$, bounded above by β, we use Proposition (5.2) to find a sequence $\{\mathfrak{h}^n\}_{\ge1}\subset m\mathrm{NW}$ bounded above by β whose support are contained in $[-L,L]$, such that $\mathfrak{h}^n\to\mathfrak{h}$ in UC. Then by Proposition 5.1 we know $\chi_{\mathfrak{h}^n}\to\chi_{\mathfrak{h}}$ as $n\to\infty$. A minor variant of Proposition 5.1 with the same proof shows that $\chi_{\mathfrak{h}^n}^{\omega}\to\chi_{\mathfrak{h}}^{\omega}$ as $n\to\infty$ for any $\omega\in\mathbb{R}$ as well. Thus

$$\chi_{\mathfrak{h}}^{\omega} = \lim_{n\to\infty}\chi_{\mathfrak{h}^n}^{\omega} = \lim_{n\to\infty}\chi_{\mathfrak{h}^n} = \chi_{\mathfrak{h}},$$

for all $\omega\in\mathbb{R}$ and $\mathfrak{h}\in\mathrm{UC}_c$. This completes the proof of Proposition 2.4.

5.3 Proof of Theorem 1.2

Recall that in Proposition 4.2, we have shown Theorem 1.2 for $\mathfrak{h}\in m\mathrm{NW}_0$. We first extend it to $\mathfrak{h}\in m\mathrm{NW}$ using Propsition 2.4. There exists $\alpha,\beta\in\mathbb{R}$ such that $\mathfrak{h}^{\alpha,\beta}:=\mathfrak{h}(\cdot+\alpha)+\beta\in m\mathrm{NW}_0$. Thus by the invariance property of the KPZ fixed point (see, e.g., [MQR21][Theorem 4.5]) we have

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{H}(\alpha_\ell,\tau_\ell;\mathfrak{h})\le\beta_\ell\}\right) = \mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{H}(\alpha_\ell-\alpha,\tau_\ell;\mathfrak{h}^{\alpha,\beta})\le\beta_\ell+\beta\}\right).\tag{5.8}$$

Apply Proposition 4.2 to the right-hand side of (5.8) and compare the resulting formula with the right-hand side of (1.6) , we see that in order for Theorem 1.2 to hold for $\mathfrak{h} \in m\mathrm{NW}$, it suffices to show that

$$\chi_{\mathfrak{h}}(\eta,\xi) = \mathrm{e}^{\alpha(\eta^2-\xi^2)+\beta(\xi-\eta)}\chi_{\mathfrak{h}^{\alpha,\beta}}(\eta,\xi). \tag{5.9}$$

Now we prove (5.9). By the definition of the characteristic function in (2.9), we have

$$\begin{aligned}\chi_{\mathfrak{h}^{\alpha,\beta}}(\eta,\xi) = &\int_{\mathbb{R}} \mathrm{d}s\,\mathrm{e}^{-s\xi}\cdot\mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(\boldsymbol{\tau}_-^{\alpha,\beta}\eta^2+\mathbf{B}(\boldsymbol{\tau}_-^{\alpha,\beta})\eta\right)\mathbf{1}_{\boldsymbol{\tau}_-^{\alpha,\beta}>-\infty}\right]\\ &+\int_{\mathbb{R}} \mathrm{d}s\,\mathrm{e}^{+s\eta}\cdot\mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(-\boldsymbol{\tau}_+^{\alpha,\beta}\xi^2-\mathbf{B}(\boldsymbol{\tau}_+^{\alpha,\beta})\xi\right)\mathbf{1}_{\boldsymbol{\tau}_+^{\alpha,\beta}<\infty}\right]\\ &-\int_{\mathbb{R}} \mathrm{d}s\,\mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(\boldsymbol{\tau}_-^{\alpha,\beta}\eta^2+\mathbf{B}(\boldsymbol{\tau}_-^{\alpha,\beta})\eta\right)\mathbf{1}_{\boldsymbol{\tau}_-^{\alpha,\beta}>-\infty}\right]\cdot\mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(-\boldsymbol{\tau}_+^{\alpha,\beta}\xi^2-\mathbf{B}(\boldsymbol{\tau}_+^{\alpha,\beta})\xi\right)\mathbf{1}_{\boldsymbol{\tau}_+^{\alpha,\beta}<\infty}\right],\end{aligned}$$

where

$$\boldsymbol{\tau}_+^{\alpha,\beta} := \inf\{\mathsf{x}\ge 0 : \mathbf{B}(\mathsf{x}) \le \mathfrak{h}(\mathsf{x}+\alpha)+\beta\}, \quad \boldsymbol{\tau}_-^{\alpha,\beta} := \sup\{\mathsf{x}\le 0 : \mathbf{B}(\mathsf{x}) \le \mathfrak{h}(\mathsf{x}+\alpha)+\beta\}.$$

Note that $\mathbf{B}(\mathsf{x}) \le \mathfrak{h}(\mathsf{x}+\alpha)+\beta \iff \widehat{\mathbf{B}}(\mathsf{x}+\alpha) \le \mathfrak{h}(\mathsf{x}+\alpha)$ where $\widehat{\mathbf{B}}(\mathsf{x}) := \mathbf{B}(\mathsf{x}-\alpha)-\beta$. The invariance of the Brownian motion implies that

$$\mathrm{Law}_{\mathbf{B}(0)=s}(\boldsymbol{\tau}_+^{\alpha,\beta},\mathbf{B}(\boldsymbol{\tau}_+^{\alpha,\beta})) = \mathrm{Law}_{\widehat{\mathbf{B}}(\alpha)=s-\beta}(\widehat{\boldsymbol{\tau}}_+-\alpha,\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_+)+\beta),$$

where

$$\widehat{\boldsymbol{\tau}}_- := \sup\{\mathsf{x}\le\alpha : \widehat{\mathbf{B}}(\mathsf{x}) \le \mathfrak{h}(\mathsf{x})\}.$$

Thus

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{d}s\,\mathrm{e}^{-s\xi}\cdot\mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(\boldsymbol{\tau}_-^{\alpha,\beta}\eta^2+\mathbf{B}(\boldsymbol{\tau}_-^{\alpha,\beta})\eta\right)\mathbf{1}_{\boldsymbol{\tau}_-^{\alpha,\beta}>-\infty}\right]\\ &=\int_{\mathbb{R}} \mathrm{d}s\,\mathrm{e}^{-s\xi}\cdot\mathbb{E}_{\widehat{\mathbf{B}}(\alpha)=s-\beta}\left[\exp\left((\widehat{\boldsymbol{\tau}}_--\alpha)\eta^2+(\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_-)+\beta)\eta\right)\mathbf{1}_{\widehat{\boldsymbol{\tau}}_->-\infty}\right]\\ &=\mathrm{e}^{\beta(\eta-\xi)-\alpha\eta^2}\int_{\mathbb{R}} \mathrm{d}s\,\mathrm{e}^{-s\xi}\cdot\mathbb{E}_{\widehat{\mathbf{B}}(\alpha)=s}\left[\exp\left(\widehat{\boldsymbol{\tau}}_-\eta^2+\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_-)\eta\right)\mathbf{1}_{\widehat{\boldsymbol{\tau}}_->-\infty}\right].\end{aligned}$$

Applying similar arguments to the second and third term we see that

$$\begin{aligned}\mathrm{e}^{\alpha(\eta^2-\xi^2)+\beta(\xi-\eta)}\chi_{\mathfrak{h}^{\alpha,\beta}}(\eta,\xi) = &\,\mathrm{e}^{-\alpha\xi^2}\int_{\mathbb{R}} \mathrm{d}s\,\mathbb{E}_{\widehat{\mathbf{B}}(\alpha)=s}\left[\exp\left(\widehat{\boldsymbol{\tau}}_-\eta^2+\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_-)\eta\right)\mathbf{1}_{\widehat{\boldsymbol{\tau}}_->-\infty}\right]\\ &+\mathrm{e}^{\alpha\eta^2}\int_{\mathbb{R}} \mathrm{d}s\,\mathbb{E}_{\widehat{\mathbf{B}}(\alpha)=s}\left[\exp\left(-\widehat{\boldsymbol{\tau}}_+\xi^2-\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_+)\xi\right)\mathbf{1}_{\widehat{\boldsymbol{\tau}}_+<\infty}\right]\\ &-\int_{\mathbb{R}} \mathrm{d}s\,\mathbb{E}_{\widehat{\mathbf{B}}(\alpha)=s}\left[\exp\left(\widehat{\boldsymbol{\tau}}_-\eta^2+\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_-)\eta-\widehat{\boldsymbol{\tau}}_+\xi^2-\widehat{\mathbf{B}}(\widehat{\boldsymbol{\tau}}_+)\xi\right)\mathbf{1}_{|\widehat{\boldsymbol{\tau}}_\pm|<\infty}\right],\end{aligned}$$

which is equal to $\chi_{\mathfrak{h}}^{\alpha}$, and hence $\chi_{\mathfrak{h}}$, by Proposition 2.4. Finally, for a general $\mathfrak{h} \in \mathrm{UC}_c$, we can choose a sequence $\{\mathfrak{h}^n\}_{n\ge1} \subset m\mathrm{NW}$ such that $\mathfrak{h}^n \to \mathfrak{h}$ and they satisfy the conditions in Proposition 5.2. By the continuity of the law of the KPZ fixed point with respect to the initial condition, we know

$$\lim_{n\to\infty}\mathbb{P}\left(\bigcap_{\ell=1}^m\{\mathcal{H}(\alpha_\ell,\tau_\ell;\mathfrak{h}^n)\le\beta_\ell\}\right) = \mathbb{P}\left(\bigcap_{\ell=1}^m\{\mathcal{H}(\alpha_\ell,\tau_\ell;\mathfrak{h})\le\beta_\ell\}\right).$$

Hence by Proposition 5.2 we have

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{H}(\alpha_\ell,\tau_\ell;\mathfrak{h})\le\beta_\ell\}\right) &= \lim_{n\to\infty}\mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{H}(\alpha_\ell,\tau_\ell;\mathfrak{h}^n)\le\beta_\ell\}\right)\\
&= \lim_{n\to\infty}\prod_{\ell=1}^{m-1}\oint\frac{\mathrm{d}z_\ell}{2\pi\mathrm{i}z_\ell(1-z_\ell)}\mathrm{D}_{\mathfrak{h}^n}(z_1,\cdots,z_{m-1})\\
&= \prod_{\ell=1}^{m-1}\oint\frac{\mathrm{d}z_\ell}{2\pi\mathrm{i}z_\ell(1-z_\ell)}\mathrm{D}_{\mathfrak{h}}(z_1,\cdots,z_{m-1}).
\end{aligned}$$

This completes the proof of Theorem 1.2.

6 Reduction in the equal-time case

The goal of this section is to simplify the multipoint formula (1.6) under the additional assumption that all the time parameters $\tau_1,\cdots,\tau_m$ are the same, say, equal to 1. We will show in Section 6.1 that one can get rid of the additional contour integrals with respect to the parameters z_i's in (1.6), and get a genuine Fredholm determinant formula for the equal-time multipoint distribution. To the best of our knowledge this Fredholm determinant formula is not written in the literature. Then in Section 6.2 we will show that our Fredholm determinant formula is equivalent to the path integral formula in [MQR21].

6.1 A new Fredholm determinant formula for the equal-time multipoint distribution of the KPZ fixed point

The following minor modification of [LO25, Proposition 2.1] holds for the KPZ fixed point with a general initial condition $\mathfrak{h}$ that is compactly supported.

Proposition 6.1. *Let $\alpha_1<\cdots<\alpha_m$. Consider the KPZ fixed point starting from initial condition $\mathfrak{h}\in\mathrm{UC}_c$. Then the following formula for the multipoint distribution at space-time points $(\alpha_1,1),\cdots,(\alpha_m,1)$ holds:*

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m}\mathcal{H}(\alpha_\ell,1;\mathfrak{h})\le\beta_\ell\right)=\det(\mathrm{I}+\mathbf{T}_{\mathfrak{h}})_{L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})},\tag{6.1}$$

where the operator $\mathbf{T}_{\mathfrak{h}}:L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})\to L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})$ has the following kernel:

$$\mathbf{T}_{\mathfrak{h}}(i,\zeta;j,\eta):=\left(\prod_{\ell=1}^{i}\int_{\Gamma_{\ell,\mathrm{L}}^{\mathrm{in}}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\right)\frac{\prod_{\ell=1}^{i}\mathrm{F}_\ell(\xi_\ell)\cdot\chi_{\mathfrak{h}}(\eta,\xi_1)}{\prod_{\ell=1}^{i-1}(\xi_\ell-\xi_{\ell+1})\cdot(\xi_i-\zeta)}\cdot\frac{1}{\mathrm{f}_j(\eta)}.\tag{6.2}$$

Here $\Gamma_{1,\mathrm{L}}^{\mathrm{in}}:=\Gamma_{1,\mathrm{L}}$ and

$$\mathrm{F}_i(\zeta)=\frac{\mathrm{f}_i(\zeta)}{\mathrm{f}_{i-1}(\zeta)}:=\begin{cases}\mathrm{e}^{-\frac{1}{3}\zeta^3+\alpha_1\zeta^2+\beta_1\zeta}, & i=1,\\ \mathrm{e}^{(\alpha_i-\alpha_{i-1})\zeta^2+(\beta_i-\beta_{i-1})\zeta}, & 2\le i\le m.\end{cases}\tag{6.3}$$

The proof of Proposition 6.1 is based on the following lemma, whose proof is almost identical to [LO25, Lemma 2.4] and is omitted here. Note that the only difference (modulo obvious change of notation) between (6.5) and [LO25, (2.31)] is that the Cauchy determinant $\mathrm{C}(\boldsymbol{\eta},\boldsymbol{\xi})$ is replaced by $\det\left(\chi_{\mathfrak{h}}(\eta_{\ell_i}^{(i)},\xi_{\ell_j}^{(j)})\right)$.

Lemma 6.2. *Under the same assumption as in Proposition 6.1, we have*

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m}\mathcal{H}(\alpha_\ell,1;\mathfrak{h})\le\beta_\ell\right)=\sum_{n_1\ge\cdots\ge n_m\ge0}\frac{1}{(n_1!\cdots n_m!)^2}\widehat{\mathrm{D}}_{\mathfrak{h}}^{(\mathbf{n})},\tag{6.4}$$

where $\mathbf{n}=(n_1,\cdots,n_m)\in\mathbb{Z}_+^m$ *and*

$$\begin{aligned}\widehat{\mathrm{D}}_{\mathfrak{h}}^{(\mathbf{n})}=&\left(\prod_{i=1}^{m}\frac{n_i!}{k_i!}\right)^2\left(\prod_{i=1}^{m}\prod_{\ell_i=1}^{k_i}\int_{\Gamma_{1,\mathrm{L}}}\frac{\mathrm{d}\xi_{\ell_i}^{(i)}}{2\pi\mathrm{i}}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta_{\ell_i}^{(i)}}{2\pi\mathrm{i}}\right)\det\left[\chi_{\mathfrak{h}}(\eta_{\ell_i}^{(i)},\xi_{\ell_j}^{(j)})\right]_{\substack{(i,\ell_i),(j,\ell_j)\\1\le i,j\le m,1\le\ell_i\le k_i}}\\&\cdot\prod_{i=1}^{m}\det\left[h_i(\xi_a^{(i)},\eta_b^{(i)})\right]_{a,b=1}^{k_i}\cdot\prod_{i=1}^{m}\prod_{\ell_i=1}^{k_i}\frac{1}{\mathrm{f}_i(\eta_{\ell_i}^{(i)})}.\end{aligned}\tag{6.5}$$

Here $k_i:=n_i-n_{i+1}\ge0$ *with the convention that* $n_{m+1}:=0$. *The functions* $h_i(\xi,\eta)$ *are defined for all* $(\xi,\eta)\in\Gamma_{1,\mathrm{L}}\times\Gamma_{1,\mathrm{R}}$ *as follows:*

$$h_i(\xi,\eta):=\begin{cases}\dfrac{\mathrm{F}_1(\xi)}{\xi-\eta}, & i=1,\\ \displaystyle\prod_{\ell=2}^{i}\int_{\Gamma_{\ell,\mathrm{L}}^{\mathrm{in}}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\frac{\mathrm{F}_1(\xi)\cdot\prod_{\ell=2}^{i}\mathrm{F}_\ell(\xi_\ell)}{(\xi-\xi_2)\cdot\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1})\cdot(\xi_i-\eta)}, & 2\le i\le m.\end{cases}\tag{6.6}$$

Proof of Proposition 6.1. Apply a generalized Andreief's identity (see, e.g., [LO25, Lemma 1.2]) to the ξ-integrals, we get

$$\begin{aligned}\widehat{\mathrm{D}}_{\mathfrak{h}}^{(\mathbf{n})}&=\prod_{i=1}^{m}\frac{(n_i!)^2}{k_i!}\left(\prod_{i=1}^{m}\prod_{\ell_i=1}^{k_i}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta_{\ell_i}^{(i)}}{2\pi\mathrm{i}}\right)\det\left[\int_{1,\mathrm{L}}\frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}}h_i(\xi_1,\eta_{\ell_i}^{(i)})\cdot\frac{\chi_{\mathfrak{h}}(\eta_{\ell_j}^{(j)},\xi_1)}{\mathrm{f}_j(\eta_{\ell_j}^{(j)})}\right]_{(i,\ell_i),(j,\ell_j)}\\&=\prod_{i=1}^{m}\frac{(n_i!)^2}{k_i!}\left(\prod_{i=1}^{m}\prod_{\ell_i=1}^{k_i}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta_{\ell_i}^{(i)}}{2\pi\mathrm{i}}\right)\det\left[\mathbf{T}_{\mathfrak{h}}(i,\eta_{\ell_i}^{(i)};j,\eta_{\ell_j}^{(j)})\right]_{(i,\ell_i),(j,\ell_j)},\end{aligned}\tag{6.7}$$

where $\mathbf{T}_{\mathfrak{h}}(i,\zeta;j,\eta)$ is defined as in (6.2). Thus we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{\ell=1}^{m}\mathcal{H}(\alpha_\ell,1;\mathfrak{h})\le\beta_\ell\right)&=\sum_{k_1,\cdots,k_m\ge0}\frac{1}{k_1!\cdots k_m!}\left(\prod_{i=1}^{m}\prod_{\ell_i=1}^{k_i}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta_{\ell_i}^{(i)}}{2\pi\mathrm{i}}\right)\det\left[\mathbf{T}_{\mathfrak{h}}(i,\eta_{\ell_i}^{(i)};j,\eta_{\ell_j}^{(j)})\right]_{(i,\ell_i),(j,\ell_j)}\\&=\det(\mathrm{I}+\mathbf{T}_{\mathfrak{h}})_{L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})}.\end{aligned}$$

□

6.2 Equivalence with the path integral formula of [MQR21]

The goal of this section is to prove that our new equal-time multipoint formula (6.1) is equivalent to the known ones. We will show that it is equivalent to the following path integral formula obtained in [MQR21]:

Proposition 6.3 (Proposition 4.3 of [MQR21])**.**

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m}\mathcal{H}(\alpha_\ell,1;\mathfrak{h})\le\beta_\ell\right)=\det(\mathrm{I}-\mathbf{K}_{1,\alpha_1}^{\mathrm{hypo}(\mathfrak{h})}+\mathbf{1}_{\le\beta_1}\mathrm{e}^{(\alpha_1-\alpha_2)\partial^2}\mathbf{1}_{\le\beta_2}\cdots\mathbf{1}_{\le\beta_m}\mathrm{e}^{(\alpha_m-\alpha_1)\partial^2}\mathbf{K}_{1,\alpha_1}^{\mathrm{hypo}(\mathfrak{h})})_{L^2(\mathbb{R})}.\tag{6.8}$$

We will first express the path integral kernel in terms of contour integrals, the result is summarized in the following proposition.

Proposition 6.4. *Assume* $\mathfrak{h} \in \mathrm{UC}_c$. *Let* $\mathbf{S}_{\mathfrak{h}} := -\mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} + \mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \mathbf{1}_{\le\beta_2} \cdots \mathbf{1}_{\le\beta_m} \mathrm{e}^{(\alpha_m-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}$. *Then we have the following contour integral representation for the kernel of* $\mathbf{S}_{\mathfrak{h}}$*:*

$$\begin{aligned} \mathbf{S}_{\mathfrak{h}}(\lambda,\mu) = &-\mathbf{1}_{\lambda>\beta_1} \int_{\Gamma_{1,\mathrm{L}}} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \frac{\mathrm{f}_1(\xi)}{\mathrm{f}_1(\eta)} \cdot \chi_{\mathfrak{h}}(\eta,\xi) \cdot \mathrm{e}^{(\mu-\beta_1)\xi - (\lambda-\beta_1)\eta} \\ &+ \sum_{i=2}^{m} \mathbf{1}_{\lambda\le\beta_1} \left(\prod_{\ell=1}^{i} \int_{\Gamma^{\mathrm{in}}_{\ell,\mathrm{L}}} \frac{\mathrm{d}\xi_i}{2\pi\mathrm{i}} \right) \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \frac{\prod_{\ell=1}^{i} \mathrm{F}_\ell(\xi_\ell) \cdot \chi_{\mathfrak{h}}(\eta,\xi_1)}{\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1}) \cdot (\xi_i-\eta)} \frac{\mathrm{e}^{(\mu-\beta_1)\xi_1-(\lambda-\beta_1)\xi_2}}{\mathrm{f}_i(\eta)}. \end{aligned} \tag{6.9}$$

Proof. First note that by writing $\mathbf{1}_{\le\beta_m} = 1 - \mathbf{1}_{>\beta_m}$, we have

$$\begin{aligned} \mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \cdots \mathbf{1}_{\le\beta_m} \mathrm{e}^{(\alpha_m-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} &= \mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \cdots \mathbf{1}_{\le\beta_{m-1}} \mathrm{e}^{(\alpha_{m-1}-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} \\ &\quad - \mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \cdots \mathbf{1}_{\le\beta_{m-1}} \mathrm{e}^{(\alpha_{m-1}-\alpha_m)} \mathbf{1}_{>\beta_m} \mathrm{e}^{(\alpha_m-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}, \end{aligned}$$

where we are using the semigroup property

$$\mathrm{e}^{(\alpha_{m-1}-\alpha_m)\partial^2} \mathrm{e}^{(\alpha_m-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} = \mathrm{e}^{(\alpha_{m-1}-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}.$$

Repeating this argument for $\mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \cdots \mathbf{1}_{\le\beta_i} \mathrm{e}^{(\alpha_i-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}$ with $i = m-1, \cdots, 2$, we see that

$$\begin{aligned} &- \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} + \mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \mathbf{1}_{\le\beta_2} \cdots \mathbf{1}_{\le\beta_m} \mathrm{e}^{(\alpha_m-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} \\ &= -\mathbf{1}_{>\beta_1} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} - \sum_{i=2}^{m} \mathbf{1}_{\le\beta_1} \mathrm{e}^{(\alpha_1-\alpha_2)\partial^2} \cdots \mathbf{1}_{\le\beta_{i-1}} \mathrm{e}^{(\alpha_{i-1}-\alpha_i)\partial^2} \mathbf{1}_{>\beta_i} \mathrm{e}^{(\alpha_i-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}. \end{aligned} \tag{6.10}$$

Now recall the definition of $\mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}$ from [MQR21, (4.5)]:

$$\mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1} = \left(S^{\mathrm{hypo}(\mathfrak{h}^-)}_{1,-\alpha_1}\right)^* S_{1,\alpha_1} + S^*_{1,-\alpha_1} S^{\mathrm{hypo}(\mathfrak{h}^+)}_{1,\alpha_1} - \left(S^{\mathrm{hypo}(\mathfrak{h}^-)}_{1,-\alpha_1}\right)^* S^{\mathrm{hypo}(\mathfrak{h}^+)}_{1,\alpha_1}, \tag{6.11}$$

where

$$S_{t,x}(p,q) = S^*_{t,x}(q,p) = \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \mathrm{e}^{\frac{t}{3}\eta^3 + x\eta^2 + (p-q)\eta} = \int_{\Gamma_{1,\mathrm{L}}} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \mathrm{e}^{-\frac{t}{3}\xi^3 + x\xi^2 - (p-q)\xi}, \tag{6.12}$$

and

$$S^{\mathrm{hypo}(\mathfrak{h}^+)}_{t,x}(p,q) := \mathbb{E}_{\mathbf{B}(0)=p}\left[S_{t,x-\boldsymbol{\tau}_+}(\mathbf{B}(\boldsymbol{\tau}_+),q)\mathbf{1}_{\boldsymbol{\tau}_+<\infty}\right]. \tag{6.13}$$

We have

$$\begin{aligned} &S^*_{1,-\alpha_1} S^{\mathrm{hypo}(\mathfrak{h}^+)}_{1,\alpha_1}(p,q) \\ &= \int_{\mathbb{R}} \mathrm{d}s \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \mathrm{e}^{\frac{1}{3}\eta^3 - \alpha_1\eta^2 + (s-p)\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\int_{\Gamma_{1,\mathrm{L}}} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \mathrm{e}^{-\frac{1}{3}\xi^3 + (\alpha_1-\boldsymbol{\tau}_+)\xi^2 + (q-\mathbf{B}(\boldsymbol{\tau}_+))\xi}\right] \\ &= \int_{\Gamma_{1,\mathrm{L}}} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-\frac{1}{3}\xi^3 + \alpha_1\xi^2 + q\xi}}{\mathrm{e}^{-\frac{1}{3}\eta^3 + \alpha_1\eta^2 + p\eta}} \cdot \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{s\eta} \cdot \mathbb{E}_{\mathbf{B}(0)=s}\left[\exp\left(-\boldsymbol{\tau}_+\xi^2 - \mathbf{B}(\boldsymbol{\tau}_+)\xi\right)\right], \end{aligned}$$

where the change of order of integration is justified by Proposition 2.3. A similar computation for the other two terms imply that

$$\mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}(p,q) = \int_{\Gamma_{1,\mathrm{L}}} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-\frac{1}{3}\xi^3 + \alpha_1\xi^2 + q\xi}}{\mathrm{e}^{-\frac{1}{3}\eta^3 + \alpha_1\eta^2 + p\eta}} \cdot \chi_{\mathfrak{h}}(\eta,\xi), \tag{6.14}$$

where $\chi_{\mathfrak{h}}(\eta,\xi)$ is defined in (2.9). Thus

$$\mathrm{e}^{(\alpha_i-\alpha_1)\partial^2} \mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}(p,q) = \int_{\Gamma_{1,\mathrm{L}}} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \int_{\Gamma_{1,\mathrm{R}}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-\frac{1}{3}\xi^3 + \alpha_1\xi^2 + q\xi}}{\mathrm{e}^{-\frac{1}{3}\eta^3 + \alpha_i\eta^2 + p\eta}} \cdot \chi_{\mathfrak{h}}(\eta,\xi), \tag{6.15}$$

for $2 \le i \le m$. On the other hand, for $2 \le i \le m$, the heat kernel $\mathrm{e}^{(\alpha_{i-1}-\alpha_i)\partial^2}$ can be expressed as

$$\mathrm{e}^{(\alpha_{i-1}-\alpha_i)\partial^2}(p,q) = \frac{1}{\sqrt{4\pi(\alpha_i-\alpha_{i-1})}}\mathrm{e}^{-\frac{(p-q)^2}{4(\alpha_i-\alpha_{i-1})}} = \int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{d}\xi_i}{2\pi\mathrm{i}} \mathrm{e}^{(q-p)\xi_i} \cdot \mathrm{e}^{(\alpha_i-\alpha_{i-1})\xi_i^2}, \tag{6.16}$$

for any $c \in \mathbb{R}$. Thus the convolution $\mathrm{e}^{(\alpha_1-\alpha_2)\partial^2}\mathbf{1}_{\le\beta_2}\mathrm{e}^{(\alpha_2-\alpha_3)\partial^2}$ has the following kernel:

$$\begin{aligned}\mathrm{e}^{(\alpha_1-\alpha_2)\partial^2}\mathbf{1}_{\le\beta_2}\mathrm{e}^{(\alpha_2-\alpha_3)\partial^2}(p,q) &= \int_{c_2+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_2}{2\pi\mathrm{i}}\int_{-\infty}^{\beta_2}\mathrm{d}r_2\int_{c_3+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_3}{2\pi\mathrm{i}}\mathrm{e}^{-p\xi_2+r_2\xi_2-r_2\xi_3+q\xi_3}\cdot\mathrm{e}^{(\alpha_2-\alpha_1)\xi_2^2+(\alpha_3-\alpha_2)\xi_3^2}\\ &= \int_{c_2+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_2}{2\pi\mathrm{i}}\int_{c_3+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_3}{2\pi\mathrm{i}}\frac{\mathrm{e}^{(\beta_1-p)\xi_2+(q-\beta_3)\xi_3}}{\xi_2-\xi_3}\cdot\mathrm{F}_2(\xi_2)\cdot\mathrm{F}_3(\xi_3),\end{aligned}$$

where $\mathrm{F}_i(\zeta) =: \mathrm{e}^{(\alpha_i-\alpha_{i-1})\zeta^2+(\beta_i-\beta_{i-1})\zeta}$ and $c_2 > c_3$. Similarly for any $2 \le i \le m$ we have

$$\mathrm{e}^{(\alpha_1-\alpha_2)\partial^2}\mathbf{1}_{\le\beta_2}\cdots\mathbf{1}_{\le\beta_{i-1}}\mathrm{e}^{(\alpha_{i-1}-\alpha_i)\partial^2}(p,q) = \left(\prod_{\ell=2}^{i}\int_{c_\ell+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\right)\frac{\mathrm{e}^{(\beta_1-p)\xi_2+(q-\beta_i)\xi_i}}{\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1})}\cdot\prod_{\ell=2}^{i}\mathrm{F}_\ell(\xi_\ell), \tag{6.17}$$

where $c_2 > \cdots > c_i$. Thus

$$\begin{aligned}&\mathrm{e}^{(\alpha_1-\alpha_2)\partial^2}\mathbf{1}_{\le\beta_2}\cdots\mathbf{1}_{\le\beta_{i-1}}\mathrm{e}^{(\alpha_{i-1}-\alpha_i)\partial^2}\mathbf{1}_{>\beta_i}\mathrm{e}^{(\alpha_i-\alpha_1)\partial^2}\mathbf{K}^{\mathrm{hypo}(\mathfrak{h})}_{1,\alpha_1}(p,q)\\ &= \left(\prod_{\ell=2}^{i}\int_{c_\ell+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\right)\int_{\beta_i}^{\infty}\mathrm{d}r\int_{\Gamma_{1,\mathrm{L}}}\frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta}{2\pi\mathrm{i}}\frac{\mathrm{e}^{(\beta_1-p)\xi_2+(r-\beta_i)\xi_i}}{\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1})}\prod_{\ell=2}^{i}\mathrm{F}_\ell(\xi_\ell)\frac{\mathrm{e}^{-\frac{1}{3}\xi_1^3+\alpha_1\xi_1^2+q\xi_1}}{\mathrm{e}^{-\frac{1}{3}\eta^3+\alpha_i\eta^2+r\eta}}\chi_{\mathfrak{h}}(\eta,\xi_1)\\ &= \left(\prod_{\ell=2}^{i}\int_{c_\ell+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\right)\int_{\Gamma_{1,\mathrm{L}}}\frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta}{2\pi\mathrm{i}}\frac{\mathrm{e}^{(\beta_1-p)\xi_2+(q-\beta_1)\xi_1}}{\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1})\cdot(\eta-\xi_i)}\prod_{\ell=2}^{i}\mathrm{F}_\ell(\xi_\ell)\cdot\frac{\mathrm{f}_1(\xi_1)}{\mathrm{f}_i(\eta)}\cdot\chi_{\mathfrak{h}}(\eta,\xi_1).\end{aligned} \tag{6.18}$$

Combining (6.18) with (6.10), we arrive at the desired expression (6.9) for $\mathbf{S}_{\mathfrak{h}}(\lambda,\mu)$. □

Finally we rewrite $\mathbf{T}_{\mathfrak{h}}$ properly to match with $\mathbf{S}_{\mathfrak{h}}$. Deform $\Gamma_{1,\mathrm{L}}$ and $\Gamma^{\mathrm{in}}_{2,\mathrm{L}}$ into two vertical lines $c_i+\mathrm{i}\mathbb{R}$, $i=1,2$, with $0 > c_1 > c_2$. Then we have

$$\frac{1}{\xi_1-\xi_2} = \int_{-\infty}^{0}\mathrm{d}\lambda\,\mathrm{e}^{\lambda(\xi_1-\xi_2)},$$

for any $\xi_1 \in \Gamma_{1,\mathrm{L}}$ and $\xi_2 \in \Gamma^{\mathrm{in}}_{2,\mathrm{L}}$. Now we write $\mathbf{T}_{\mathfrak{h}} := L_1L_2$, where $L_1 : L^2(\mathbb{R}) \to L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})$ has the following kernel:

$$L_1(i,\zeta;\lambda) = \begin{cases}-\mathrm{e}^{-\lambda\zeta}\mathbf{1}_{\lambda>0}, & i=1,\\ \displaystyle\prod_{\ell=2}^{i}\int_{\Gamma^{\mathrm{in}}_{\ell,\mathrm{L}}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\frac{\prod_{\ell=2}^{i}\mathrm{F}_\ell(\xi_\ell)\mathrm{e}^{-\lambda\xi_2}}{\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1})\cdot(\xi_i-\zeta)}\mathbf{1}_{\lambda\le0}, & 2\le i\le m,\end{cases} \tag{6.19}$$

and $L_2 : L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}}) \to L^2(\mathbb{R})$ has the following kernel:

$$L_2(\lambda;j,\eta) := \int_{\Gamma_{1,\mathrm{L}}}\frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}}\frac{\mathrm{f}_1(\xi_1)}{\mathrm{f}_j(\eta)}\mathrm{e}^{\lambda\xi_1}\cdot\chi_{\mathfrak{h}}(\eta,\xi_1). \tag{6.20}$$

Then we have

$$\det(\mathrm{I}+\mathbf{T}_{\mathfrak{h}})_{L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})} = \det(\mathrm{I}+L_1L_2) = \det(\mathrm{I}+L_2L_1) := \det(\mathrm{I}+\widehat{\mathbf{S}}_{\mathfrak{h}})_{L^2(\mathbb{R})},$$

where

$$
\begin{aligned}
\widehat{\mathbf{S}}_{\mathfrak{h}}(\lambda,\mu) &= \sum_{i=1}^{m}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta}{2\pi\mathrm{i}}L_2(\lambda;i,\eta)L_1(i,\eta;\mu)\\
&= -\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta}{2\pi\mathrm{i}}\int_{\Gamma_{1,\mathrm{L}}}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\frac{\mathrm{f}_1(\xi)}{\mathrm{f}_1(\eta)}\mathrm{e}^{\lambda\xi-\mu\eta}\cdot\chi_{\mathfrak{h}}(\eta,\xi)\cdot\mathbf{1}_{\mu>0}\\
&+\sum_{i=2}^{m}\int_{\Gamma_{1,\mathrm{R}}}\frac{\mathrm{d}\eta}{2\pi\mathrm{i}}\left(\prod_{\ell=1}^{i}\int_{\Gamma_{\ell,\mathrm{L}}^{\mathrm{in}}}\frac{\mathrm{d}\xi_\ell}{2\pi\mathrm{i}}\right)\frac{\prod_{\ell=1}^{i}\mathrm{F}_\ell(\xi_\ell)}{\prod_{\ell=2}^{i-1}(\xi_\ell-\xi_{\ell+1})\cdot(\xi_i-\eta)}\frac{\mathrm{e}^{\lambda\xi_1-\mu\xi_2}}{\mathrm{f}_i(\eta)}\cdot\chi_{\mathfrak{h}}(\eta,\xi_1)\cdot\mathbf{1}_{\mu\le 0}.
\end{aligned}
\tag{6.21}
$$

Comparing (6.21) with (6.9) we see that $\widehat{\mathbf{S}}_{\mathfrak{h}}(\lambda,\mu)=\mathbf{S}_{\mathfrak{h}}(\mu+\beta_1,\lambda+\beta_1)$. Thus

$$
\det(\mathrm{I}+\mathbf{S}_{\mathfrak{h}})_{L^2(\mathbb{R})}=\det(\mathrm{I}+\widehat{\mathbf{S}}_{\mathfrak{h}})_{L^2(\mathbb{R})}=\det(\mathrm{I}+\mathbf{T}_{\mathfrak{h}})_{L^2(\{1,\cdots,m\}\times\Gamma_{1,\mathrm{R}})}.
$$

This completes the proof of the equivalence between Proposition 6.1 and 6.3.

References

[ACH24a] A. Aggarwal, I. Corwin, and M. Hegde. KPZ fixed point convergence of the ASEP and stochastic six-vertex models. *arXiv:2412.18117*, 2024.

[ACH24b] A. Aggarwal, I. Corwin, and M. Hegde. Scaling limit of the colored ASEP and stochastic six-vertex models. *arXiv:2403.01341*, 2024.

[ACQ11] G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions. *Comm. Pure Appl. Math.*, 64(4):466–537, 2011.

[BDJ99] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.

[BFP10] J. Baik, P. L. Ferrari, and S. Péché. Limit process of stationary TASEP near the characteristic line. *Comm. Pure Appl. Math.*, 63(8):1017–1070, 2010.

[BFPS07] A. Borodin, P. Ferrari, M. Prähofer, and T. Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.*, 129(5-6):1055–1080, 2007.

[BFS08] A. Borodin, Patrik L. Ferrari, and T. Sasamoto. Transition between Airy_1 and Airy_2 processes and TASEP fluctuations. *Comm. Pure Appl. Math.*, 61(11):1603–1629, 2008.

[BL19] J. Baik and Z. Liu. Multipoint distribution of periodic TASEP. *J. Amer. Math. Soc.*, 32(3):609–674, 2019.

[BL21] J. Baik and Z. Liu. Periodic TASEP with general initial conditions. *Probab. Theory Related Fields*, 179(3-4):1047–1144, 2021.

[CGH21] I. Corwin, P. Ghosal, and A. Hammond. KPZ equation correlations in time. *Ann. Probab.*, 49(2):832–876, 2021.

[Cor12] I. Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(1):1130001, 76, 2012.

[dNLD17] J. de Nardis and P. Le Doussal. Tail of the two-time height distribution for KPZ growth in one dimension. *J. Stat. Mech. Theory Exp.*, (5):053212, 72, 2017.

[dNLD18] J. de Nardis and P. Le Doussal. Two-time height distribution for 1d KPZ growth: the recent exact result and its tail via replica. *J. Stat. Mech. Theory Exp.*, 2018(9):18, 2018.

[DOV22] D. Dauvergne, J. Ortmann, and B. Virág. The directed landscape. *Acta Math.*, 229(2):201–285, 2022.

[Dur19] R. Durrett. *Probability. Theory and examples*, volume 49 of *Camb. Ser. Stat. Probab. Math.* Cambridge: Cambridge University Press, 5th edition edition, 2019.

[DV21] D. Dauvergne and B. Virág. The scaling limit of the longest increasing subsequence. *arXiv:2104.08210*, 2021.

[DZ24] D. Dauvergne and L. Zhang. Characterization of the directed landscape from the KPZ fixed point. *arXiv:2412.13032*, 2024.

[FO19] P. L. Ferrari and A. Occelli. Time-time covariance for last passage percolation with generic initial profile. *Math. Phys. Anal. Geom.*, 22(1):Paper No. 1, 33, 2019.

[FS16] P. L. Ferrari and H. Spohn. On time correlations for KPZ growth in one dimension. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 12:Paper No. 074, 23, 2016.

[Joh00] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.

[Joh17] K. Johansson. Two Time Distribution in Brownian Directed Percolation *Comm. Math. Phys.*, 351:441–492, 2017.

[Joh19] K. Johansson. The two-time distribution in geometric last-passage percolation. *Probab. Theory Related Fields*, 175(3-4):849–895, 2019.

[Joh20] K. Johansson. Long and short time asymptotics of the two-time distribution in local random growth. *Math. Phys. Anal. Geom.*, 23(4):Paper No. 43, 34, 2020.

[JR21] K. Johansson and M. Rahman. Multitime distribution in discrete polynuclear growth. *Commun. Pure Appl. Math.*, 74(12):2561–2627, 2021.

[JR22] K. Johansson and M. Rahman. On inhomogeneous polynuclear growth. *Ann. Probab.*, 50(2):559 – 590, 2022.

[KPZ86] M. Kardar, G. Parisi, and Y. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56(9):889–892, 1986.

[LD17] P. Le Doussal. Maximum of an Airy process plus Brownian motion and memory in Kardar-Parisi-Zhang growth. *Phys. Rev. E*, 96(6):060101, 2017.

[Lia22] Y. Liao. Multi-point distribution of discrete time periodic TASEP. *Probab. Theory Related Fields*, 182(3-4):1053–1131, 2022.

[Liu22] Z. Liu. Multipoint distribution of TASEP. *Ann. Probab.*, 50(4):1255–1321, 2022.

[LO25] Z. Liu and A. Ortiz. On the multipoint distribution formulas of the parabolic Airy process. *arXiv:2507.20487*, 2025.

[LZ25] Z. Liu and R. Zhang. An upper tail field of the KPZ fixed point. *Commun. Math. Phys.*, 406(8):53, 2025.

[MQR21] K. Matetski, J. Quastel, and D. Remenik. The KPZ fixed point. *Acta Math.*, 227(1):115–203, 2021.

[MQR25] K. Matetski, J. Quastel, and D. Remenik. Polynuclear growth and the Toda lattice. *J. Eur. Math. Soc.* , 2025.

[NQR20] M. Nica, J. Quastel, and D. Remenik. One-sided reflected Brownian motions and the KPZ fixed point. *Forum Math. Sigma*, 8:Paper No. e63, 16, 2020.

[PS02] M. Prähofer and H. Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.*, 108:1071–1106, 2002.

[Qua12] J. Quastel. Introduction to KPZ. In *Current developments in mathematics, 2011*, pages 125–194. Int. Press, Somerville, MA, 2012.

[Rah25] M. Rahman. Revisiting the temporal law in KPZ random growth. *arXiv:2504.19975*, 2025.

[Wu23] X. Wu. The KPZ equation and the directed landscape. *arXiv:2301.00547*, 2023.

[Zyg22] N. Zygouras. Some algebraic structures in KPZ universality. *Probab. Surv.*, 19:590–700, 2022.