Problem 1

(a). $Pr(X=6)=\frac{1}{6}$ $\therefore Ex(\#$ appearing in the 3 rolls$)=3(\frac{1}{6})=\frac{1}{2}$

(b). $Pr(X=1)=Pr(X=3)=Pr(X=5)=\frac{1}{6}$ $\therefore Pr(X\in\{1,3,5\})=3(\frac{1}{6})=\frac{1}{2}=0.5$

$\therefore Ex(\#$ odd numbers appearing in the 3 rolls$)=3(\frac{1}{2})=\frac{3}{2}=1.5$

Problem 2

(a). $Ex[X]=\int_{-1}^{2}x(\frac{x^2}{3})\,dx=\int_{-1}^{2}\frac{x^3}{3}dx=\frac{x^4}{12}\Big|_{-1}^{2}=\frac{16}{12}-\frac{1}{12}=\frac{15}{12}=1.25$

(b). $Ex[\frac{1}{X}]=\int_{-1}^{2}\frac{1}{x}(\frac{x^2}{3})\,dx=\int_{-1}^{2}\frac{x}{3}dx=\frac{x^2}{6}\Big|_{-1}^{2}=\frac{4}{6}-\frac{1}{6}=\frac{3}{6}=0.5$

(c). $Ex[\frac{1}{X}]=0.5$

$\frac{1}{Ex[X]}=\frac{1}{1.25}=0.8\neq0.5$ $\therefore Ex[\frac{1}{X}]\neq\frac{1}{Ex[X]}$

(d). $Ex[X^2+1]=\int_{-1}^{2}(x^2+1)(\frac{x^2}{3})\,dx=\int_{-1}^{2}(\frac{x^4}{3}+\frac{x^2}{3})\,dx=\int_{-1}^{2}\frac{x^4}{3}dx+\int_{-1}^{2}\frac{x^2}{3}dx$

$=\frac{x^5}{15}\Big|_{-1}^{2}+\frac{x^3}{9}\Big|_{-1}^{2}=\frac{32}{15}-(-\frac{1}{15})+\frac{8}{9}-(-\frac{1}{9})=\frac{33}{15}+1=\frac{48}{15}=3.2$

Problem 3

Ex(# floors where the elevator makes a stop to let out at least one person) = 10 Pr(at least one person get off)

$\because$ Pr(at least one person get off) = 1 − Pr(no one get off)

$\because$ Pr(get off for each person) $=\frac{1}{10}$

$\therefore$ Pr(not get off for each person) $=1-\frac{1}{10}=\frac{9}{10}$

$\therefore$ Pr(at least one person get off) $=1-(\frac{9}{10})^{12}$

$\therefore$ Ex(# floors where the elevator makes a stop to let out at least one person) $=10(1-(\frac{9}{10})^{12})\approx7.176$

Problem 4

(a). Pr(have birthday one day i for each of three people) $=\frac{1}{365}$

$Pr(B_{12})=Pr(B_{23})=\sum_{i=1}^{365}(\frac{1}{365})^2=365(\frac{1}{365})^2=\frac{1}{365}$

$Pr(B_{12}\cap B_{23})=$ Pr(all three people have same birthday)

$=\sum_{i=1}^{365}(\frac{1}{365})^3=365(\frac{1}{365})^3=(\frac{1}{365})^2$

$Pr(B_{12})Pr(B_{23})=(\frac{1}{365})(\frac{1}{365})=(\frac{1}{365})^2=Pr(B_{12}\cap B_{23})$

so, independent

(b). $Pr(B_{12}) = Pr(B_{23}) = Pr(B_{13}) = \sum_{i=1}^{365} (\frac{1}{365})^2 = 365 (\frac{1}{365})^2 = \frac{1}{365}$

$Pr(B_{12} \cap B_{23}) = Pr(B_{23} \cap B_{13}) = Pr(B_{12} \cap B_{13}) = Pr($all three people have same birthday$) = \sum_{i=1}^{365} (\frac{1}{365})^3 = 365 (\frac{1}{365})^3 = (\frac{1}{365})^2$

$\therefore Pr(B_{12} \cap B_{23}) = Pr(B_{12}) Pr(B_{23}) = (\frac{1}{365})^2$

$Pr(B_{23} \cap B_{13}) = Pr(B_{23}) Pr(B_{13}) = (\frac{1}{365})^2$

$Pr(B_{12} \cap B_{13}) = Pr(B_{12}) Pr(B_{13}) = (\frac{1}{365})^2$

$\therefore$ So pairwise independent

(C). $Pr(B_{12}) = Pr(B_{23}) = Pr(B_{13}) = \sum_{i=1}^{365} (\frac{1}{365})^2 = 365 (\frac{1}{365})^2 = \frac{1}{365}$

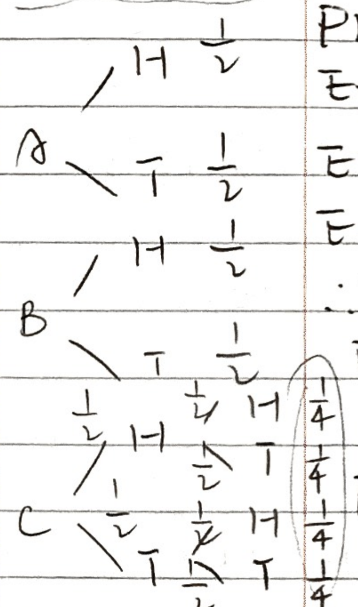
$Pr(B_{12} \cap B_{23} \cap B_{13}) = \sum_{i=1}^{365} (\frac{1}{365})^3 = 365 (\frac{1}{365})^3 = (\frac{1}{365})^2$

$\downarrow \neq$

$Pr(B_{12}) Pr(B_{23}) Pr(B_{13}) = (\frac{1}{365})(\frac{1}{365})(\frac{1}{365}) = (\frac{1}{365})^3 \neq (\frac{1}{365})^2$

So, not mutually independent

Pr



Problem 5

$Ex(X_A) = Ex(X_B) = \cancel{Ex(X_C)} = 1(\frac{1}{2}) + 0(\frac{1}{2}) = \frac{1}{2}$

$Ex(X_A^2) = Ex(X_B^2) \cancel{= Ex(X_C^2)} = 1^2(\frac{1}{2}) + 0^2(\frac{1}{2}) = \frac{1}{2}$

$Ex(X_A)^2 = Ex(X_B)^2 \cancel{= Ex(X_C)^2} = (\frac{1}{2})^2 = \frac{1}{4}$

$\therefore Var(X_A) = Var(X_B) \cancel{= Var(X_C)} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$Ex(X_A + X_B + X_C) = 3(\frac{1}{2})(\frac{1}{2})(\frac{1}{4}) + 1(\frac{1}{2})(\frac{1}{2})(\frac{1}{4}) + 1(\frac{1}{2})(\frac{1}{2})(\frac{1}{4}) + 1(\frac{1}{2})(\frac{1}{2})(\frac{1}{4})$

$= \frac{3}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{9}{16}$

$Ex((X_A + X_B + X_C)^2) = 3^2(\frac{1}{2})(\frac{1}{2})(\frac{1}{4}) + 1^2(\frac{1}{2})(\frac{1}{2})(\frac{1}{4}) + 1^2(\frac{1}{2})(\frac{1}{2})(\frac{1}{4}) + 1^2(\frac{1}{2})(\frac{1}{2})(\frac{1}{4})$

$= \frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{12}{16} = \frac{3}{4}$

C.

H (1): (H, H) $Pr = \frac{1}{4}$; (T, T) $Pr = \frac{1}{4}$

T (0): (H, T) $Pr = \frac{1}{4}$; (T, H) $Pr = \frac{1}{4}$

$Ex(X_A + X_B + X_C)^2 = (\frac{9}{16})^2 = \frac{81}{256}$

$Var(X_A + X_B + X_C) = \frac{3}{4} - \frac{81}{256} = \frac{111}{256} \neq Var(X_A) + Var(X_B) + Var(X_C) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

So, not mutually independent

$\Omega = \{(H,H,H)^3, (T,T,H)^1, (H,T,T)^1, (T,H,T)^1\}$

$Ex(X_A + X_B) = 2(\frac{1}{2})(\frac{1}{2}) + 1(\frac{1}{2})(\frac{1}{2}) + 1(\frac{1}{2})(\frac{1}{2}) + 0(\frac{1}{2})(\frac{1}{2}) = \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = 1$

$Ex((X_A + X_B)^2) = 2^2(\frac{1}{2})(\frac{1}{2}) + 1^2(\frac{1}{2})(\frac{1}{2}) + 1^2(\frac{1}{2})(\frac{1}{2}) + 0^2(\frac{1}{2})(\frac{1}{2}) = 1 + \frac{2}{4} = \frac{6}{4}$

$Ex(X_A + X_B)^2 = 1^2 = 1$

$Var(X_A + X_B) = \frac{6}{4} - 1 = \frac{2}{4} = \frac{1}{2} = Var(X_A) + Var(X_B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\begin{array}{cccc} 2 & 1 & 1 & 0 \\ (H,H), & (T,H), & (H,T), & (T,T) \end{array}$$

A	B	C
H	H	H
T	T	H
H	T	T
T	H	T

$Ex(X_A+X_C) = Ex(X_B+X_C) = 2(\frac{1}{2})(\frac{1}{4}) + 1(\frac{1}{2})(\frac{1}{4}) + 1(\frac{1}{2})(\frac{1}{4}) + 0(\frac{1}{2})(\frac{1}{4})$

$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

$Ex((X_A+X_C)^2) = Ex((X_B+X_C)^2) = 2^2(\frac{1}{2})(\frac{1}{4}) + 1^2(\frac{1}{2})(\frac{1}{4}) + 1^2(\frac{1}{2})(\frac{1}{4}) + 0^2(\frac{1}{2})(\frac{1}{4})$

$= \frac{1}{2} + \frac{1}{8} + \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$

$Ex(X_A+X_C)^2 = Ex(X_B+X_C)^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$Var(X_A+X_C) = Var(X_B+X_C) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

$Ex(X_C) = 1(\frac{1}{4}) + 1(\frac{1}{4}) + 0(\frac{1}{4}) + 0(\frac{1}{4}) = \frac{2}{4} = \frac{1}{2}$

$Ex(X_C^2) = 1^2(\frac{1}{4}) + 1^2(\frac{1}{4}) + 0^2(\frac{1}{4}) + 0^2(\frac{1}{4}) = \frac{2}{4} = \frac{1}{2}$

$Ex(X_C) = (\frac{1}{2})^2 = \frac{1}{4}$ $\quad \therefore Var(X_C) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore Var(X_A+X_C) = \frac{1}{2} \quad = \quad Var(X_A) + Var(X_C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\quad Var(X_B+X_C) = \frac{1}{2} \quad = \quad Var(X_B) + Var(X_C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

so, pairwise independent

Since X_A, X_B, and X_C are not mutually independent, but pairwise independence, so the magician tells the audience to show the any two envelopes, but not any one, so that the mutually dependence can help on the guessing the content of the third one.