H7.

- The assignment is due at Gradescope on Tuesday November 2, 2021 at 10pm. Submit early and often.
- Read and sign the collaboration and honesty policy. Submit the signed policy to Gradescope before submitting any work.
- Unless otherwise specified, you can leave your answer in closed form (e.g. $1 - \binom{120}{7}(0.1)^{200}$).
- *Show your work.* Answers without justification will be given little credit. Justify each step in your solutions e.g. by stating that the step follows from an axiom of probability, a definition, algebra, etc.; for example, your answer could include a line like this:

$$\Pr(X \cap Y \cap Z) \cdot \Pr(A \cup B) = \Pr(X \cap Y \cap Z) \cdot (\Pr(A) + \Pr(B)) \qquad (A \text{ and } B \text{ are disjoint})$$

- The syllabus has some pointers on using LaTeX and Python.

PROBLEM 1. We roll two standard 6-sided dice. In each part, determine whether the two events are independent.

(a) Let A = "the sum of of the two rolls is odd" and B = "both tosses were greater than 3."

(b) Let C = "the two rolls showed the same number" and D = "the second toss was greater than 4."

PROBLEM 2. A student applies to both BU and Northeastern. Looking at statistics from students who applied to the same schools in the past, she figures out that she has a probability of 0.25 of getting into BU and 0.5 of getting into Northeastern. She also finds that she has a probability of 0.2 of getting into both.

(a) What is the probability that she gets into Northeastern given that she gets admitted to BU?

(b) Are the two events ("getting into BU" and "getting into Northeastern") independent?

(c) How do you think your answer to the previous question might reflect reality?

PROBLEM 3. Let A and B be two independent events. Prove that:

(a) A and $\overline{B}$ are independent.

(b) $\overline{A}$ and $\overline{B}$ are independent.

Let $\Pr(A) = p$ and $\Pr(B) = q$. Find each of the following probabilities, expressed in terms of p and q:

(c) $\Pr(A \mid \overline{B})$.

(d) $\Pr(A \cup \overline{B})$.

PROBLEM 4. Alice wants to send a message to Bob, where the message is a sequence of bits. Alice sends her message through a noisy communication channel that randomly flips the bits: a 0 bit is incorrectly transmitted as a 1 with probability $\frac{1}{4}$, and it is correctly transmitted with probability $\frac{3}{4}$; a 1 bit is incorrectly transmitted as a 0 with probability $\frac{1}{3}$, and it is correctly transmitted with probability $\frac{2}{3}$; each bit is flipped independently from the other bits.

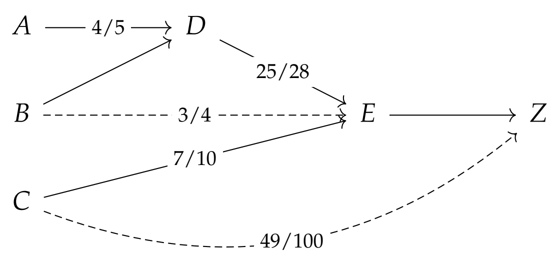
(a) Alice chooses a single bit uniformly at random and sends it to Bob. What is the probability that Bob receives it correctly?

(b) What is the probability that Bob receives the message 1011 correctly?

(c) In an effort to improve the probability that Bob receives the correct message, Alice transmits each bit three times and Bob uses the majority rule to decode. More precisely, Alice transmits a 0 as 000 and a 1 as 111. Bob decodes the three bits received as a 0 if there are at least 2 0s, and as a 1 otherwise. What is the probability that Bob correctly decodes a 0?

(d) Alice chooses a single bit uniformly at random and she uses the scheme in part (c) to send it. What is the probability that the bit was 0 given that Bob received the sequence 101?

PROBLEM 5. Tang Sanzang is a Chinese monk travelling to the west to seek Buddhist scriptures, accompanied by his monster companions Wukong, Bajie, and Wujing. The fellowship encounters a crossroads with three paths (starting at A, B, and C) which cut through a dangerous mountain range filled with hungry monsters, however, they all lead to the same exit Z. They happen to run into a travelling oracle, who draws them a map of the paths, and supplies the probability of survival for various segments. However, he is unable at that moment to measure of the probability of every possible segment. The map produced by the oracle is as shown below:



In the above map, the solid lines are the segments on which the fellowship can travel on (e.g., the path $B \rightsquigarrow Z$ from B to Z is comprised of the segments $B \to D \to E \to Z$). The dashed lines provide the probability of survival on the entire path between the endpoints (e.g., the probability of survival on the path $B \to D \to E$ is $3/4$). For any segment, the probability of surviving on that segment is independent of other segments (recall that the segments are the solid lines).

For each of the three paths $A \rightsquigarrow Z$, $B \rightsquigarrow Z$, $C \rightsquigarrow Z$, find the probability of survival on that path. Which path should the fellowship take in the interest of survival?