CS237 - PROBABILITY IN COMPUTING - FALL 2021 — JOHN BYERS AND ALINA ENE
DUE **tuesday** OCTOBER 12, 2021 AT 10PM VIA GRADESCOPE

H5.

- The assignment is due at Gradescope on **Tuesday** October 12, 2021 at 10pm. Submit early and often.

- Read and sign the collaboration and honesty policy. Submit the signed policy to Gradescope before submitting any work.

- Unless otherwise specified, you can leave your answer in closed form (e.g. $1 - \binom{120}{7}(0.1)^{200}$).

- *Show your work.* Answers without justification will be given little credit. Justify each step in your solutions e.g. by stating that the step follows from an axiom of probability, a definition, algebra, etc.; for example, your answer could include a line like this:

$$\Pr(X \cap Y \cap Z) \cdot \Pr(A \cup B) = \Pr(X \cap Y \cap Z) \cdot (\Pr(A) + \Pr(B)) \qquad (A \text{ and } B \text{ are disjoint})$$

- The syllabus has some pointers on using LaTeX and Python.

PROBLEM 1. You conduct the following experiment: you toss a fair coin until either two heads appear, or you have reached a maximum of 3 tosses. (I.e., if the first two tosses are heads, heads, you would stop tossing the coin. If your first two tosses are tails, tails, you would toss the coin one more time. You never toss more than 3 times.)

(a) List all the possible outcomes of this experiment.

(b) Define the random variable $X$ as the number of heads that appear in an outcome. What are the possible values of $X$?

(c) Find the PDF of $X$.

(d) Find the CDF of $X$.

**Solution:**

PROBLEM 2. You arrive at the bus stop on time. The bus is always late, but it is guaranteed to arrive within 20 minutes of your arrival – it is equally likely to arrive at any time between your arrival (minute 0) and minute 20. In other words, the lateness is a uniformly random number in the continuous interval $[0, 20]$.

(a) Let $X$ be a random variable which denotes the lateness of the bus in minutes. Find the CDF of $X$ and draw a plot of it (don't use software).

(b) Find the PDF of $X$ and plot it by hand.

(c) Whenever the bus is late, you get annoyed. Let $Y$ be a random variable which denotes your annoyance level when the bus arrives. With experience you realize that $Y$ starts at zero but grows with $X$ squared (e.g., if the bus arrives 8.5 minutes late then $Y(8.5) = (8.5)^2$). Find the CDF of $Y$ and plot the CDF of $Y$.

**Solution:**

PROBLEM 3. Alvin wants to donate some money to his favorite charity. To this end, he chooses a card uniformly at random from a standard deck of 52 cards. If the suit of the card is hearts ($\heartsuit$) or if the card is an Ace, he donates \$50.50. Otherwise, if the card is a face card (J,Q,K), but not a heart, he donates \$10.10. Finally, if the card is even (2,4,6,8,10), but not a heart, he donates \$5.05. Otherwise, he donates \$1. Let $X$ be the random variable equal to the amount that Alvin donates.

(a) Is $X$ discrete or continuous?

(b) Find the PDF of $X$.

(c) Find the CDF of $X$.

**Solution:**

PROBLEM 4. Let $X$ be a random variable with the following PDF $f$:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 1 \le x \le 2 \\ kx & \text{if } 2 < x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find all value(s) of $k$ that make $f$ a valid PDF.

(b) Draw a plot of $f(x)$. (Don't use software to plot).

(c) Find the CDF of $X$ and plot it by hand.

(d) Find $\Pr(1.5 \le X \le 2.5)$.

**Solution:**

PROBLEM 5. Let $X$ be a continuous random variable with the following PDF $f$, where $c$ is a constant:

$$f(x) = \begin{cases} cx^2 & \text{if } x \in [-1, 2] \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant $c$ that makes this a valid PDF. Use this value in parts (b) and (c).

(b) Find the CDF of $X$.

(c) Let $Y = 2X - 1$. Find the PDF and CDF of $Y$.

**Solution:**

PROBLEM 6. Goku is preparing for a battle with his rival Vegeta. Based on previous battles, when Goku has a power level of $g$ and Vegeta has a power level of $v$, we can calculate the probability of Goku's victory with the formula:

$$\frac{g}{g + v}$$

Let $W$ be a random variable such that $W = 1$ if Goku wins and $W = 0$ if he loses

(a) Calculate the PDF for $W$ if Goku currently has a power level of $g = 1,100$ while Vegeta has a power level of $v = 1,000$.

(b) Now suppose Goku's friends will revive him (to full power) with the Dragon Balls if he loses the first battle against Vegeta, but only once, as the Dragon Balls disappear for 1 year after use. Goku wants to secure favorable 0.90 odds of winning overall, either by winning the first battle or losing the first but winning the second. If Vegeta starts with a power level of $v = 1,000$ for both battles (if necessary), what power level does Goku need to achieve $\Pr[W = 1] = 0.9$?

**Solution:**