Problem 1

$\mu = Ex(X)$ $\sigma = \sqrt{Var(X)} = \sqrt{Ex((X-Ex(X))^2)} = Ex(X^2) - \mu^2$

(a). $Ex(2X^2 - 3X + 4)$

$= 2Ex(X^2) - 3Ex(X) + 4$

$= 2(\mu^2 + \sigma^2) - 3\mu + 4$

$= 2\mu^2 + 2\sigma^2 - 3\mu + 4$

(b). $Var(-3X+4)$

$= (-3^2)Var(X) + Var(4)$

$= 9Var(x) + 0$

$= 9\sigma^2$

Problem 2

(a) $\int_1^2 cx(1-x)^2 dx = 1$

$c\int_1^2 x(1-x)^2 dx = 1$

$c\int_1^2 x(1-2x+x^2) dx = 1$

$c\int_1^2 x - 2x^2 + x^3 dx = 1$

$c\left(\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right)\Big|_1^2 = 1$

$\frac{7}{12}c = 1 \quad c = \frac{12}{7}$

(b). $E(X) = \sum_{x \in R} x \cdot f(x) = \int_1^2 x\left(\frac{12}{7}x(1-x)^2\right)dx$

$= \frac{12}{7}\int_1^2 x^2(1-x)^2$

$= \frac{12}{7}\int_1^2 x^2(1-2x+x^2)dx$

$= \frac{12}{7}\int_1^2 x^2 - 2x^3 + x^4 dx$

$= \frac{12}{7}\left(\frac{x^3}{3} - \frac{2}{4}x^4 + \frac{x^5}{5}\right)\Big|_1^2 = \frac{62}{35}$

(c). $Var(x) = Ex((X - Ex(X))^2) = Ex(X^2) - Ex(X)^2$

$Ex(x^2) = \sum x^2 \cdot f(x) = \int_1^2 x^2\left(\frac{12}{7}x(1-x)^2\right)dx$

$= \frac{12}{7}\int_1^2 x^3(1-x)^2 dx = \frac{12}{7}\int_1^2 x^3(1-2x+x^2)dx$

$= \frac{12}{7}\int_1^2 x^3 - 2x^4 + x^5 dx = \frac{12}{7}\left(\frac{x^4}{4} - \frac{2}{5}x^5 + \frac{x^6}{6}\right)\Big|_1^2 = \frac{111}{35}$

$Var(X) = \frac{111}{35} - \left(\frac{62}{35}\right)^2 = \frac{41}{1225}$

Problem 3

$Pr(X=1) = p \quad Pr(Y=1) = r$

$Ex((X-Y)^2) = Ex(X^2 - 2XY + Y^2) = Ex(X^2) - 2Ex(XY) + Ex(Y^2)$

$= \sum_{i=0}^{1} x^2(P(X=x)) - 2\sum_{i=0}^{1}\sum_{i=0}^{1} xy(P(X=x)P(Y=y)) + \sum_{i=0}^{1} y^2(P(Y=y))$

$= 0^2(P(X=0)) + 1^2(P(X=1)) - 2(0^2 0^2 P(X=0)P(Y=0) +$

$1^2 1^2 P(X=1)P(Y=1)) + 0^2(P(Y=0)) + 1^2(P(Y=1))$

$= 0 + 1(p) - 2(0 + 1(p)(r)) + 0 + 1(r) = p - 2pr + r$

Problem 4

can pair with at most $(n-1)$ people. $x \geq 0$

(a). PDF of $X \sim Binomial(n-1, P=0.1)$ is $\begin{cases} \binom{n-1}{x} p^x (1-p)^{n-x} = \binom{n}{x}(0.1)^x (0.9)^{n-1-x} \\ 0 \quad \text{otherwise} \end{cases}$

(b). $Ex(X=n) = (n-1)p = 0.1(n-1)$

(c). $P(X \geq 2) = 1 - P(X<2) = 1 - P(X=0) - P(X=1) = 1 - \binom{n-1}{0}0.9^{n-1} - \binom{n-1}{1}(0.1)(0.9)^{n-2}$

(d). $Ex = \sum_{k=2}^{n} k \cdot Pr(X=2) = \binom{n}{2} 0.1$

Problem 5 $\quad n \in \{1, 2, \ldots, k\} \quad H = n \quad T = 2n$

(a). $Pr(H) = Pr(T) = \frac{1}{2}$

$Pr(\ell = n) = \frac{1}{k} \quad Pr(\ell = 2n) = \frac{1}{k}$

$Pr(\ell \in \{1, 2, \ldots k\}) = Pr(\ell \in \{2, 4, \ldots 2k\}) = \frac{1}{2}(\frac{1}{k}) = \frac{1}{2k}$

$\therefore \begin{cases} Pr(\ell \in \{1, 3, 5, \ldots\}) = \frac{1}{2k} & \ell \text{ is odd number and } \leq k. \\ Pr(\ell \in \{2, 4, 6, \ldots\}) = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} & \ell \text{ is even number and } \leq k. \\ Pr(\ell \in \{k+1, \ldots, 2k\}) = \frac{1}{2k} & \ell \text{ is even number and } k < \ell \leq 2k \\ Pr(\ell) = 0 & \text{else} \end{cases}$

(b). $E(X) = \sum_{X \in R} X \cdot Pr(X)$

$$= \sum_{\substack{1 \leq X \leq k \\ X \text{ is odd}}} \ell \cdot \overset{\frac{1}{2k}}{Pr(X = \ell)} + \sum_{\substack{1 \leq X \leq k \\ X \text{ is even}}} \ell \cdot \overset{\frac{1}{k}}{Pr(X = \ell)} + \sum_{k < \ell \leq 2k} \ell \cdot \overset{\frac{1}{2k}}{Pr(X = \ell)}$$

$$= \frac{1}{2k}\left(\sum_{x=1}^{k} \ell + \sum_{x=1}^{2k} \ell\right) = \frac{1}{2\cancel{k}}\left(\frac{\cancel{k}(k+1)}{2} + \frac{2\cancel{k}(2k+2)}{\cancel{2}}\right) = \frac{3(k+1)}{4}$$

(c). $E(X) = 30\left(\frac{3(k+1)}{4}\right) = \frac{45(k+1)}{2}$