H8.

- The assignment is due at Gradescope on Monday, November 8, 2021 at 10pm. Submit early and often.
- Read and sign the collaboration and honesty policy. Submit the signed policy to Gradescope before submitting any work.
- Unless otherwise specified, you can leave your answer in closed form (e.g. $1 - \binom{120}{7}(0.1)^{200}$).
- *Show your work.* Answers without justification will be given little credit. Justify each step in your solutions e.g. by stating that the step follows from an axiom of probability, a definition, algebra, etc.; for example, your answer could include a line like this:

$$\Pr(X \cap Y \cap Z) \cdot \Pr(A \cup B) = \Pr(X \cap Y \cap Z) \cdot (\Pr(A) + \Pr(B)) \qquad (A \text{ and } B \text{ are disjoint})$$

- The syllabus has some pointers on using LaTeX and Python.

PROBLEM 1. Suppose we roll a fair die three times, independently.

(a) Find the expectation of the number of sixes appearing in the three rolls.

(b) Find the expectation of the number of odd numbers appearing in the three rolls.

Solution:

PROBLEM 2. In homework 5, we worked with a continuous random variable X with the following PDF:

$$f_X(x) = \begin{cases} \frac{x^2}{3} & \text{if } x \in [-1, 2] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $\mathrm{Ex}[X]$.

There is a powerful theorem nicknamed ”The Law of the Unconscious Statistician” (LOTUS) that states:

$$\mathrm{Ex}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

(b) Use LOTUS to compute $\mathrm{Ex}\left[\frac{1}{X}\right]$.

(c) Is $\mathrm{Ex}\left[\frac{1}{X}\right] = \frac{1}{\mathrm{Ex}[X]}$?

(d) Use LOTUS to compute $\mathrm{Ex}[X^2 + 1]$.

Solution:

PROBLEM 3. A building has 10 floors above the basement and an elevator. Suppose 12 people get into the elevator at the basement, and each chooses a floor uniformly at random to get out, independently of the others. Find the expected number of floors where the elevator makes a stop to let out at least one person.

Solution:

PROBLEM 4. Suppose that the birthday of each of three people is equally likely to be any one of the 365 days of the year, independently of others. Let B_{ij} denote the event that person i has the same birthday as person j, where the labels i and j may be $1, 2$ or 3.

(a) Are the events B_{12} and B_{23} independent?

(b) Are the events B_{12}, B_{23}, and B_{13} pairwise independent?

(c) Are the events B_{12}, B_{23}, and B_{13} mutually independent?

Solution:

PROBLEM 5. The Great Snyderinsky, a local magician, asks the audience to toss two coins, without showing the result to him, and to write their results (either 'H' for heads or 'T' for tails) in two separate envelopes labelled A and B respectively. He also asks the audience to take an extra envelope labelled C and write 'H' in it if the previous two coin tosses gave the same result, and write 'T' in it if the previous two coin tosses were different from each other (e.g. 'H' 'T'). He then tells the audience to show him the content of any two envelopes, and claims that he can use his magical moustache to guess the content of the third envelope. Explain how the trick works in terms of independence. In particular address the following questions. Let

$$X_A = \begin{cases} 1 & \text{if envelope } A \text{ contains HEADS} \\ 0 & \text{if envelope } A \text{ contains TAILS} \end{cases}$$

$$X_B = \begin{cases} 1 & \text{if envelope } B \text{ contains HEADS} \\ 0 & \text{if envelope } B \text{ contains TAILS} \end{cases}$$

$$X_C = \begin{cases} 1 & \text{if envelope } C \text{ contains HEADS} \\ 0 & \text{if envelope } C \text{ contains TAILS} \end{cases}$$

are X_A, X_B and X_C mutually independent? Are they pairwise independent?

PROBLEM 6. Download the HW 8 Jupyter notebook from Piazza. Complete all the exercises in the notebook. Submit the Jupyter notebook with your solutions to the Homework 8 Programming assignment on Gradescope. Your submission should be a single .ipynb file.