Problem 1

$\lambda = \frac{1}{2}$

$P(X>4) = 1 - F(4)$

$\because F(x) = 1 - e^{-\lambda x}$

$\therefore P(X>4) = 1 - (1 - e)^{-\frac{1}{2}(4)} = e^{-2}$

$$P(X>8 \mid X>4) = \frac{1-F(8)}{1-F(4)} = \frac{e^{-\frac{1}{2}(8)}}{e^{-\frac{1}{2}(4)}} = \frac{e^{-4}}{e^{-2}} = e^{-2}$$

Problem 2

$z = \frac{x - \text{mean}}{\text{std}}$

(a). $z = \frac{66-68}{1.45} \approx -1.38$

$P(X<66) = 0.084$

(b). $z = \frac{72-68}{1.45} \approx 2.76$

$P(X<72) = 0.997 \quad \therefore P(X>72) = 1 - 0.997 = 0.003$

(c). $z = \frac{66.5-68}{1.45} = -1.03$

$P(X<66.5) = 0.150$

$z = \frac{71-68}{1.45} = 2.069$

$P(X<71) = 0.981$

$P(X<71) - P(X<66.5$

66.5 $P=0.15$ 71 $P=0.981$

$\therefore P(66.5<X<71) = 0.981 - 0.150 = 0.831$

Problem 3

$T \le 30$

$D=30$ $D=30+30=60$.

0 100

(a). When $T = 30$ mins, distance $d = 60(0.5) = 30$ miles away from the ambulance station, $D = 30+30 = 60$ miles $\therefore P(T>30) = \frac{100-60}{100} = 0.4$

(b). $P(T>t) = 1 - P(T \le t) = 1 - \frac{1}{t-0} = 1 - \frac{1}{t}$

(c).

$$f(T=t) = \begin{cases} 0 & \text{Otherwise} \\ 2\left(\frac{1}{100}\right) = \frac{1}{50} & (0 \le t \le 30) \\ \frac{1}{100} & (30 < t \le 70) \end{cases}$$

$100 - 30 = 70$ miles. $\left(\frac{70}{60}\right)60 = 70$ min

(d). $Ex = \int x f(x)\,dx$

$$Ex(t) = \begin{cases} \int_0^{30} t\left(\frac{1}{50}\right)dt = \frac{t^2}{100}\Big|_0^{30} = 9 \\ \int_{30}^{70} t\left(\frac{1}{100}\right)dt = \frac{t^2}{200}\Big|_{30}^{70} = \frac{4900}{200} - \frac{900}{200} = \frac{4000}{200} = 20. \end{cases}$$

$\therefore Ex(T) = 9 + 20 = 29.$ $\quad Ex(T^2) = \int_0^{30} t^2\left(\frac{1}{50}\right)dt + \int_{30}^{70} t^2\left(\frac{1}{100}\right)dt = \frac{3700}{3}$

$Var(X) = E(X^2) - E(X)^2 \quad \therefore Var(T) = Ex(T^2) - Ex(T)^2 = \frac{3700}{3} - 29^2 \approx 392$

Problem 4 $X = \lceil T \rceil \leq T$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{\lceil T \rceil}(k) = P(k-1 < x \leq k) = \int_{k-1}^{k} \lambda e^{-\lambda y}\,dx = -e^{-\lambda k} + \underbrace{e^{-\lambda(k-1)}}_{-\lambda k+\lambda} \text{ for } k=0,1,\ldots$$

$$= e^{-\lambda(k-1)}(1-e^{-\lambda}) = e^{\lambda(k-1)}(1-e^{\lambda})$$

$p = 1-e^{-\lambda}$

$$\therefore P(X=k) = \underbrace{(1-(1-e^{-\lambda}))^{k-1}(1-e^{-\lambda})}_{(1-p)^{k-1}p} = e^{-\lambda(k-1)}(1-e^{-\lambda}) \text{ for } k=0,1,\ldots$$

0 otherwise

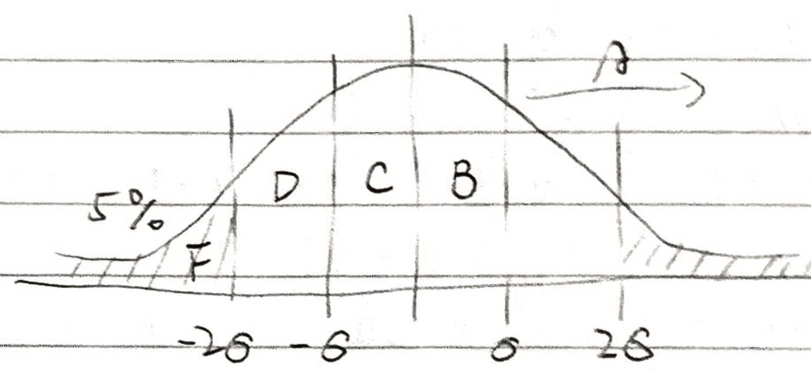
$\therefore X \sim \text{Geometric}(1-e^{-\lambda})$

Problem 5

$(\mu-\sigma, \mu+\sigma)$ 68%

$(\mu-2\sigma, \mu+2\sigma)$ 95%

$(\mu-3\sigma, \mu+3\sigma)$ 99.7%



$F = \frac{1}{2}(5\%) = 2.5\% = \frac{1}{40}$

$B = C = \frac{1}{2}(68\%) = 34\% = \frac{17}{50}$

$D = 50\% - C - F = 50\% - 34\% - 2.5\% = 13.5\% = \frac{27}{200}$

$A = 50\% - B = 50\% - 34\% = 16\% = \frac{4}{25}$

Problem 6 $Ex(T) = 10s$

(a). $P(T \geq 60) \leq \frac{Ex(T)}{60} = \frac{10}{60} = \frac{1}{6}$

(b). $P(|T - Ex(T)| \geq 50) \leq \frac{Var(T)}{50^2} = \frac{25}{50^2} = \boxed{\frac{1}{100}}$

$60-10=50.$

Problem 7

(a). r = estimated conversion rate

R = actual conversion rate $\quad n = \frac{1}{((0.05)(4)(0.01))^2} = 50000$

$P(|r-R| < 0.01) \geq 0.95 \quad \therefore P(|r-R| \geq 0.01) \leq 0.05$

$$\therefore P(|r-R| \geq 0.01) \leq \frac{Var(x)}{0.01^2} = \frac{\frac{1}{n}(p)(1-p)}{0.01^2} = \frac{1}{4(0.01)^2(50000)} = 0.05$$

Q: estimate rate of new cart design.

(b). $P(Q > p+0.01 \mid q < p) \leq P(Q > p+0.01 \mid q = p)$

$$\therefore Pr(|Q-q| \geq 0.01) \leq 0.05 = \frac{Var(Q)}{0.01^2} = \frac{\frac{1}{n}(q)(1-q)}{0.01^2} \rightarrow P(|Q-q| \geq 0.01) \leq \frac{\frac{1}{n}(\frac{1}{4})}{0.01^2} \quad (n = 50000)$$