Homework 10

Due Friday, 04/22/2022 11:59 PM on Gradescope

Problem 1. *Online Ads*

You're running a website and are interested in selling advertisements on your website. You have collected enough data on your users to successfully segment them in different demographics $G_1, G_2, ..., G_d$ (These demographics could overlap, meaning there could be users that are members of more than one demographic group). There are n users, for each user u you are given the set D_u of demographic groups they belong to. There are m different advertisers looking to advertise on your site. Decide on a good advertising policy that adhere to the following rules:

- We can show at most one ad to each user u.
- Advertiser a only wants its ads shown to users in a subset $X_a \subseteq \{G_1, G_2, ..., G_n\}$ of demographic groups.
- Advertiser a wants to show at least a total of r_a ads.

You are given as input the number of users n, the demographics D_u for each user u, the number of advertisers m, the preferred demographics groups X_a and required number of ads r_a for each advertiser a.

1. Define an instance of the max flow problem i.e. specify the graph along with edge capacities, so that solving the problem yields a valid ad assignment. (You don't have to write an algorithm or proof in this part.)
2. What is the number of nodes in your graph? What is the maximum number of edges? Express both values in terms of n, m and d. (Write the formula, no explanation needed.)
3. Suppose that somebody finds the maximum flow f in your graph, e.g. by running Ford-Fulkerson, and they provide you with the flow $f(e)$ on each edge. Give a precise description of how to decide which ad to assign to which user given f. (You can either write a very simple algorithm or give a very clear and succinct description.)
4. Suppose that it is indeed possible to meet all advertisers' demands. What is the value of the maximum flow in this case?

Problem 2. *Verifying the flow*

You are given a directed graph $G(V, E)$ with source s, sink t and integer capacities $c(e)$ along the edges. Further, you are also given the value $f(e)$ for each edge, which indicates the amount of flow along edge e. ($f(e)$ was computed by someone else and you get it as an input.)

1. Finde an algorithm that takes G, c and f and verifies whether f is a maximum flow in time $O(n + m)$.

2. Now suppose that f is indeed a maximum flow. The value of the max flow is $val(f)$. Suppose that for one particular edge e^* the capacity $c(e^*)$ is *increased* by 1. That is, $c_{new}(e^*) = c(e^*) + 1$. What are the possible values of the maximum flow in the augmented graph? For each possible new max flow value, please draw an example graph that has that flow. (You may use the notation $e = (u, v)$ if you want to refer to the nodes in e.)

3. One way to find the maximum flow in the capacity-increased graph is to run Ford-Fulkerson on the augmented graph again. The worst-case running time of this is $\Theta(Cm)$ where C is the largest capacity. Find an algorithm, that given G, the original capacities c, the maximum flow f, and the edge e^*, finds the maximum flow in the augmented graph in time $O(m + n)$.