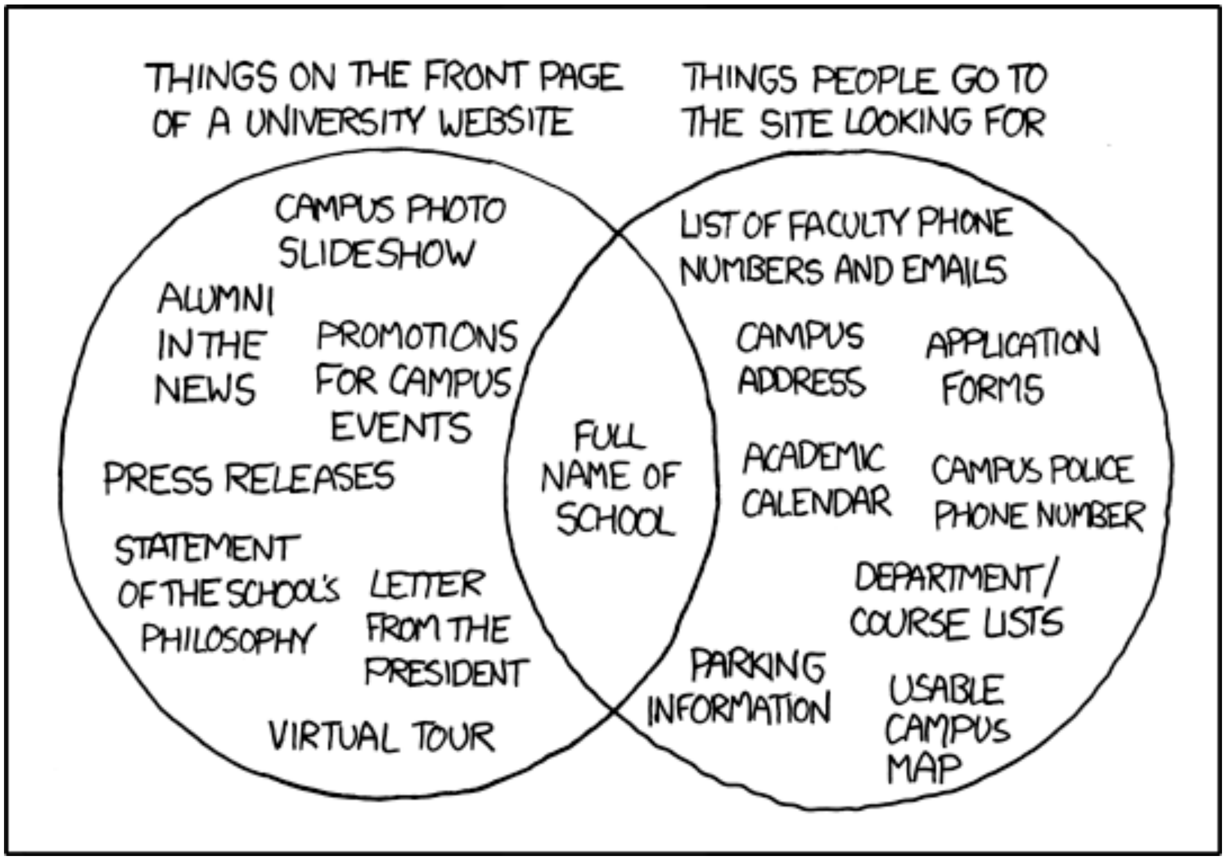
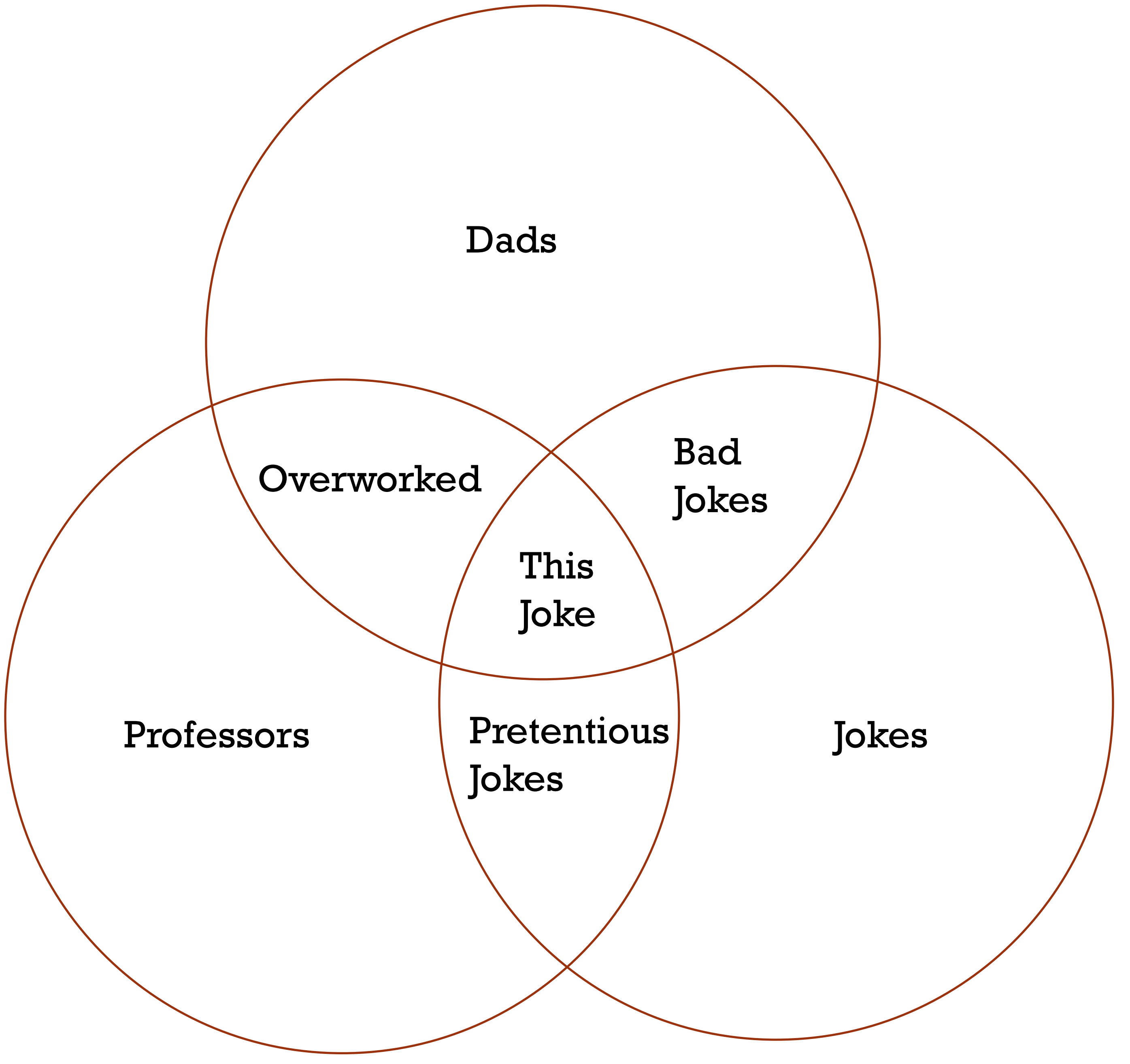
UNIVERSITY WEBSITE



https://xkcd.com/773/

SETS

Some slides courtesy of Tracy Larrabee and
Rosen, Discrete Math and its Applications (Wiley)


Dads
Overworked
Bad
Jokes
This
Joke
Professors
Pretentious
Jokes
Jokes

WHY ARE WE LEARNING ABOUT SETS?

- Isn't this elementary-school material?
  - Everyone knows about Sets, Venn diagrams, the union, intersection, etc.
- Here we are going to formalize the treatment of sets
- Sets play a foundational role in math
  - See Logicomix
  - Russell was dedicated to the proposition that mathematics was just logic + set theory

WHAT IS A SET?

- Is a set is a well-defined collection of objects?
  - The type Boolean in Java
  - The set of movies released in 2020
  - The set of Pokémon
  - The set of all ideas
- Objects in a set are called elements
- Elements of the sets above
  - true
  - Birds of Prey and the Fantabulous Emancipation of one Harley Quinn
  - Pikachu
  - The set of all ideas
- Can a set contain itself?

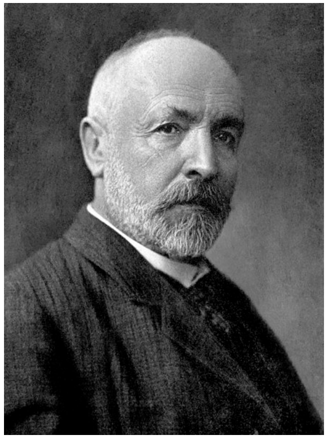
OVERVIEW

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

INTRODUCTION

- Sets are one of the basic building blocks for all of mathematics
  - The theory of sets provides a language that is suited to describing and explaining all types of mathematical structures.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

DEFINITIONS: SETS AND ELEMENTS

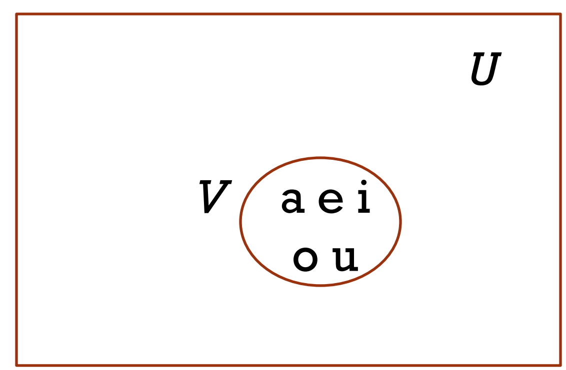


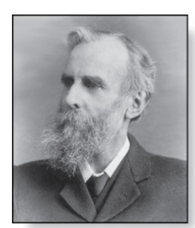
- A *set* is an **unordered** collection of objects.
  - Naïve set theory:
  - Georg Cantor (1845-1918) father of set theory
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$
- Uppercase letters used for sets
  - Lowercase letters for elements
  - ZyBook Participation Activity 1.1.1

UNIVERSAL SET AND EMPTY SET

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but **{} also used.**

Venn Diagram





John Venn (1834-1923)
Cambridge, UK

$= U$

$\{$  $\} = S$

$\in S$

$\notin S$

DESCRIBING A SET: ROSTER METHOD

- $S = \{a,b,c,d\}$
- Ellipses (…) may be used to describe a set without listing all of the members when the pattern is clear.
  - $S = \{a,b,c,d, \ldots, z\}$
- The above ellipses are used for a pattern that repeats and that ends. But you can also use them for infinite patterns:
  - $S = \{1,2,3, \ldots\}$

ROSTER METHOD

- Set of all vowels in the English alphabet:

  $V = \{a, e, i, o, u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots\ldots,99\}$

- Set of all integers less than 0:

  $S = \{\ldots, -3,-2,-1\}$

DESCRIBING A SET: SET-BUILDER NOTATION

- Specify the property or properties that all members must satisfy:

  $S = \{x : x \text{ is a positive integer less than } 100\}$

  $O = \{x : x \text{ is an odd positive integer less than } 10\}$

- Positive rational numbers:

$\mathbb{Q}^{+} = \{x \in \mathbb{R} : x = p/q, \text{ for some positive integers } p,q\}$

SOME IMPORTANT SETS

- $\mathbb{N}$ = natural numbers = { 0, 1, 2, 3, 4, … }
- $\mathbb{Z}$ = Integers= { …,-4,-3,-2,-1,0, 1, 2, 3, 4, … }
- $\mathbb{Z}^+$ = Positive Integers = {1,2,3,4,…}
- $\mathbb{Q}$ = rational numbers = $\{p/q : p \in \mathbb{Z}, q \in \mathbb{Z}^+\}$
- $\mathbb{R}$ = real numbers = any decimal number of arbitrary precision
- Irrational numbers = $\{ x: x \in \mathbb{R}$ and $x \notin \mathbb{Q} \}$
  - (e.g., pi, e,)
- $\emptyset = \{\}$ is the empty set

INTERVAL NOTATION

$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

(caution: ∞ can never have a square bracket)

SETS AS ELEMENTS OF SETS

- Sets can be elements of sets.

  $\{\{1,2,3\}, a, \{b,c\}\}$

  $\{\mathbb{Q}, \mathbb{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

WHICH SETS ARE EQUAL?

- $\{a,b,c,d\}$ and $\{a,b,c,b,c,d\}$
- $\{a,b,c,d\}$ and $\{a,b,\{c,d\}\}$
- $\{a,b,c,d\}$ and $\{c,d,a,b\}$
-  **and**

SET EQUALITY

Definition: Two sets are *equal* **if and only if (iff)** they have the same elements; (Definitions are "iff" even if they only say "if")

If they have the same elements, then they are equal,

If they are equal, then they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if
  - i.e.,

$$\forall x(x \in A \leftrightarrow x \in B)$$

$$\forall x[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$$

  - **We write *A* = *B* if *A* and *B* are equal sets.**

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

PROVING TWO SETS ARE EQUAL

- Let A= {1,5,5,5,3,3,1} and B={1,3,5}
- Prove that A=B
- We want to show that

$$\forall x[(x \in A \to x \in B) \land (x \in B \to x \in A)]$$

- In pseudocode notation (two for loops):
  - for x in A:
    - Show x belongs to B
  - for x in B
    - Show x belongs to A

SET CARDINALITY

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$

WHAT IS THE CARDINALITY OF THE FOLLOWING SETS?

- $|\varnothing| = ?$
- 0
- $|\{\varnothing\}| = ?$
- 1
- $|\ \mathbb{Z}\ | = ?$
- ∞

SUBSETS

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if is true. $\forall x (x \in A \rightarrow x \in B)$
- ZyBook Exercises 1.1

SHOWING A SET IS OR IS NOT A SUBSET OF ANOTHER SET

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then **x** also belongs to B:

$$\forall x(x \in A \to x \in B)$$

- **Showing that A is not a Subset of B**: $A \not\subseteq B$

$$\neg\forall x(x \in A \to x \in B)$$

$$\exists x\neg(x \in A \to x \in B)$$

$$\exists x(x \in A \land x \notin B)$$

- To show that A is not a subset of B, $A \not\subseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

PROVING SETS ARE SUBSETS OF OTHERS

$$\forall x(x \in A \rightarrow x \in B)$$

- {1,5,5,5,3,3,1} ⊆ {1,3,5} ?
- Definition of subset

$$\forall x(x \in A \rightarrow x \in B)$$

  - for x in A:
    - Show x belongs to B

PROVING SETS ARE SUBSETS OF OTHERS

$$\forall x(x \in A \to x \in B)$$

- $\emptyset \subseteq S$?
- Yes, Proof:
- Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  - We will see why this make sense when we talk about logic
- $S \subseteq S$, for every set S?
- Because $a \in S \to a \in S$,

PROVING SETS ARE SUBSETS OF OTHERS

$$\exists x(x \in A \wedge x \notin B)$$

- Is the set of integers with squares less than 100 a subset of $\mathbb{N}$?
- No! We are now going to prove the first set is not a subset of the - 2 belongs to the first set, but not to $\mathbb{N}$
- Proof:
  - -2 belongs to the set of integers with squares less than 100
  - -2 does not belong to $\mathbb{N}$
- The above proof is usually called a counterexample

ANOTHER LOOK AT EQUALITY OF SETS

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$A \subseteq B$ and $B \subseteq A$

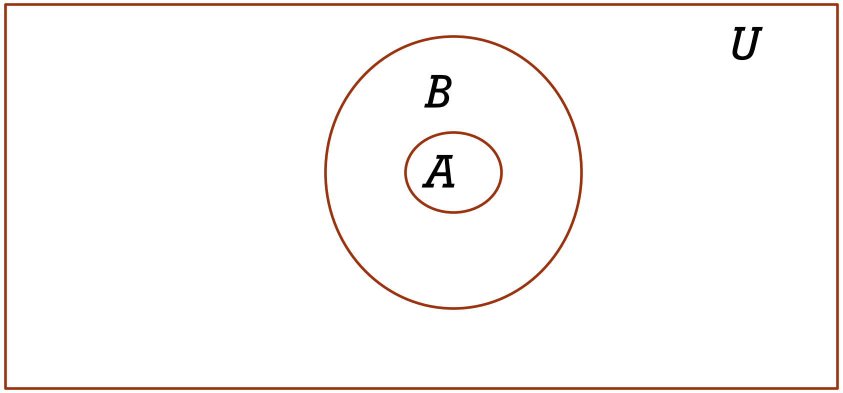
PROPER SUBSETS

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$.

$A \subset B$ ***iff***

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

Venn Diagram



TRUE OR FALSE?

$\mathbb{N} \subset \mathbb{R}$

$\mathbb{Z} \subseteq \mathbb{N}$

$-3 \subseteq \mathbb{R}$

$\{1,2\} \notin \mathbb{Z}^+$

$\varnothing \subseteq \varnothing$

$\varnothing \subset \varnothing$

$\{x\} \subseteq \{x\}$

$\{x\} \in \{x,\{x\}\}$

$\{x\} \subseteq \{x,\{x\}\}$

$\{x\} \in \{x\}$