ANNOUNCEMENTS

- Homework 2 is out on Canvas:
  - Due date: May 1 (midnight) on Canvas
    - Should be submitted as a pdf file
    - Answers for homework 1 are out (check announcements on Canvas)

The worst homework grade will be removed for the final score.

- Quiz 4: April 30, last 10 minutes of class.
  - Material: "Wanderjahre" chapter from Logicomix and lectures 8 and 9 (today's lecture is lecture 9).
  - You will get 2 attempts.
- Exam 1: May 5. Canvas quiz during class time. Covering all month of April.
- Short review next class, but mostly topics from homework, quizzes, and lectures.

LOGICAL REASONING

Formal Proofs

GIVEN A SET OF PREMISES, WHAT NEW INFORMATION CAN WE DERIVE?

- If Superman is able and willing to prevent evil, he would do so
- If Superman is unable to prevent evil, he would be impotent
- If Superman was unwilling to prevent evil, he would be malevolent
- Superman does not prevent evil
- If Superman exists, he is neither impotent nor malevolent
- If all the above premises are true, can we conclude that Superman does not exist?

WHAT WE HAVE SEEN SO FAR

- Definition: A Tautology is a proposition that is always true
- Definition: A Contradiction is a proposition that is always false
- Definition: p and q are equivalent
  - iff $p \leftrightarrow q$ is a tautology
  - iff p and q have the same truth table.
  - Notation:
    - $p \Leftrightarrow q$ or $p \equiv q$

THE TRUTH TABLE METHOD IS NOT SCALABLE

Premises

m	p	q
T	T	T
.	.	.
T	F	T
.	.	.
F	T	T
.	.	.
F	F	T
F	F	F

Conclusion

m	p	q
T	T	T
.	.	.
T	F	T
.	.	.
F	T	T
F	T	F
F	F	T
F	F	F

With n constants, there are 2^n truth assignments.

WHAT WE MUST TACKLE NOW: PROOFS

- How do we get new conclusions from previous premises or previous results?
- "True Logic" developing a deductive system for the language of propositions
- Proofs:
  - Symbolic manipulation of sentences rather than enumeration of truth assignments
  - Benefits:
    - Proofs are usually smaller than truth tables
    - Proofs can often be found with less work
- The foundations of mathematics:
  - Given a set of axioms, or premises, we must use a valid argument (a proof) to obtain a conclusion

PROOFS AS DEFINED IN LOGICOMIX:

- Verification of a mathematical statement
- Starting from a set of agreed-upon first principles
  - Axioms or already proven statements derived from these axioms
- Proceeding by unambiguous and unabridged logical steps or rules of inference (e.g., modus ponens)

THEOREMS

- A theorem is any inference obtained from:
  - a set of premises: Γ
  - axioms or previously proved Theorems and
  - the rules of inference
- **Notation (Hilbert system):**

$$\Gamma \vdash \alpha$$

$$\vdash \alpha \qquad \text{if } \Gamma \text{ is empty}$$

(ZyBook Notation):

$$\frac{\Gamma}{\alpha}$$

$\vdash$ is called the (single) "Turnstile" symbol

EXAMPLE: HILBERT SYSTEM

- Small set of rules of inference and axioms from where all "theorems" can be derived
- Lukasiewicz Axioms and only Modus Ponens
- 3 fixed Axioms (all truth tables are Tautologies) doc cam:

$$\text{A1} \quad \alpha \to (\beta \to \alpha)$$

$$\text{A2} \quad (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$$

$$\text{A3} \quad (\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$$

- Modus Ponens
  - $p, p \to q \vdash q$
- *Goal: Use only axioms and modus ponens (and previous results) to prove all theorems of propositional logic*

IS THIS A VALID PROOF?

Premises:

$p \to q$

$p \to r$

Rule of Inference Modus Ponens:

$p \to q\ , p\ \vdash\ \mathrm{q}$ Since p occurs in the second premise

Conclusion:

$q \to r$

IT IS VERY IMPORTANT TO SPELL OUT THE REASON FOR EACH STEP

$$
\begin{aligned}
Thm1: &\quad \vdash \alpha \to \alpha \\
Thm2: &\quad \alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma \\
R1: &\quad \alpha \vdash \beta \to \alpha \\
R2: &\quad \alpha \to (\beta \to \gamma), \alpha \to \beta \vdash \alpha \to \gamma \\
R3: &\quad \neg\alpha \to \neg\beta \vdash \beta \to \alpha \\
Thm3: &\quad \neg\alpha \vdash \alpha \to \beta \\
Thm4: &\quad \neg\neg\alpha \vdash \alpha \\
Thm5: &\quad \alpha, \neg\alpha \vdash \beta
\end{aligned}
$$

IT IS VERY IMPORTANT TO SPELL OUT THE REASON FOR EACH STEP

Theorem 2 $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$

Proof $1.\beta \to \gamma$ $\quad P$

$2.(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$ $\quad A1$

$3.\alpha \to (\beta \to \gamma)$ $\quad MP1,2$

$4.\ (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ $\quad A2$

$5.(\alpha \to \beta) \to (\alpha \to \gamma)$ $\quad MP3,4$

$6.\alpha \to \beta$ $\quad P$

$7.\alpha \to \gamma$ $\quad MP5,6$

ANOTHER STEP-BY-STEP PROOF

Rule 2 $\alpha \to (\beta \to \gamma), \alpha \to \beta \vdash \alpha \to \gamma$

Proof	$1. \alpha \to (\beta \to \gamma)$	P
	$2. (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	$A2$
	$3. (\alpha \to \beta) \to (\alpha \to \gamma)$	$MP 1, 2$
	$4. \alpha \to \beta$	P
	$5. \alpha \to \gamma$	$MP 3, 4$

PREVIOUS THEOREMS CAN BE USED IN PROOFS

Theorem 4 $\neg\neg\alpha \vdash \alpha$

Proof

$1. \neg\neg\alpha$	P
$2. \neg\alpha \rightarrow \neg\neg\neg\alpha$	$Thm3, 1$
$3. \neg\neg\alpha \rightarrow \alpha$	$R3, 2$
$4. \alpha$	$MP1, 3$

PREVIOUS THEOREMS CAN BE USED IN PROOFS

Theorem 5 $\alpha, \neg\alpha \vdash \beta$

Proof	$1.\neg\alpha$	P
	$2.\alpha \rightarrow \beta$	$Thm3, 1$
	$3.\alpha$	P
	$4.\beta$	$MP2, 3$

PROOF BY CONTRADICTION

Meta – Theorem Let Γ be a set of premises

$$\Gamma, \neg\alpha \vdash \beta, \neg\beta \implies \Gamma \vdash \alpha$$

Meta – Corollary

$$(a)\Gamma, \alpha \vdash \beta, \neg\beta \implies \Gamma \vdash \neg\alpha$$

$$(b)\Gamma, \neg\alpha \vdash \alpha \implies \Gamma \vdash \alpha$$

USING PROOF BY CONTRADICTION

Theorem 6 $\alpha \to \beta, \neg\beta \vdash \neg\alpha$(Modus Tollens)

Proof Assume $\alpha \to \beta, \neg\beta, \neg\neg\alpha$

$1.\alpha \to \beta \qquad P$

$2.\neg\beta \qquad P$

$3.\neg\neg\alpha \qquad P$

$4.\alpha \qquad Thm4, 3$

$5.\beta \qquad MP1, 4$

You get a contradiction with lines 2 and 5

USING PROOF BY CONTRADICTION (2)

Theorem 7 $\alpha \to \beta, \neg\alpha \to \beta \vdash \beta$

Proof Assume $\alpha \to \beta, \neg\alpha \to \beta, \neg\beta$

$1. \alpha \to \beta$	P
$2. \neg\beta$	P
$3. \neg\alpha \to \beta$	P
$4. \neg\alpha$	$Thm6,\ 1, 2$
$5. \beta$	$MP\ 3, 4$

You get a contradiction with lines 2 and 5

USING PROOF BY CONTRADICTION (3)

Theorem 8 $\alpha, \neg\beta \vdash \neg(\alpha \to \beta)$

Proof Assume $\alpha, \neg\beta, \alpha \to \beta$

$$1.\alpha \qquad P$$

$$2.\alpha \to \beta \qquad P$$

$$4.\beta \qquad MP\ 1,2$$

$$5.\neg\beta \qquad P$$

You get a contradiction with lines 4 and 5