ANNOUNCEMENTS

- Homework 2 is out on Canvas:
  - Due date: May 1 (midnight) on Canvas
    - Should be submitted as a pdf file
    - Answers for homework 1: discussion sections and office hours of TAs (you can ask for specific problems)
- Quiz 3: April 23, last 10 minutes of class.
  - Material: "The Sorcerer's Apprentice" chapter from Logicomix and lectures 6 and 7 (today's lecture is lecture 7).
  - You will get 2 attempts.
- Exam 1: May 5. Canvas quiz during class time. Covering all month of April

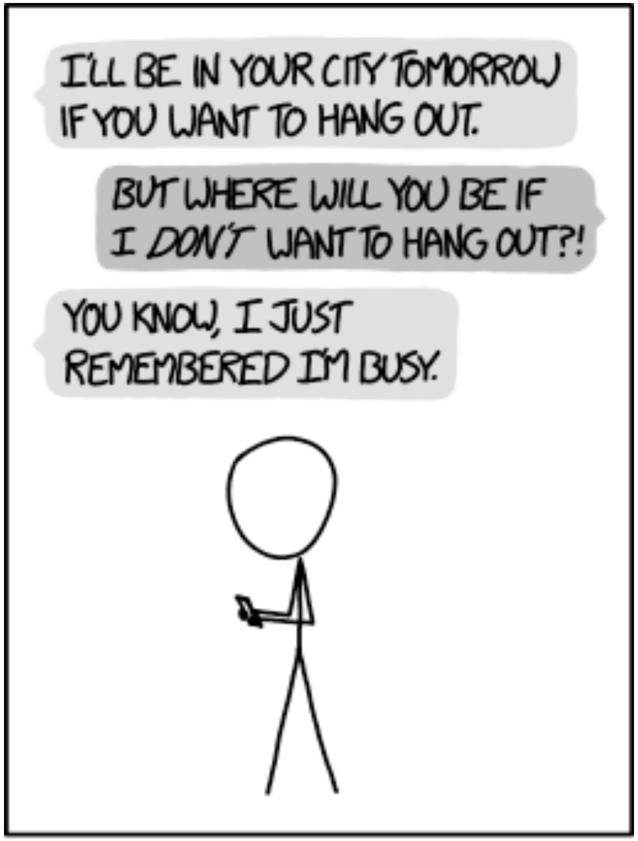
EQUIVALENT STATEMENTS FOR $P \to Q$

- If $x > 2$ then $x^2>4$
- $x^2>4$ if $x >2$
- $x > 2$ only if $x^2>4$
- $x > 2$ is sufficient for $x^2>4$
- $x^2>4$ is necessary for $x>2$

- You will find any of the above statements in many math books when introducing theorems
  - You need to memorize the above equivalent statements to understand what the theorem is saying.

COMMON MISTAKE FOR $P \to Q$

- If x > 2 then $x^2>4$
  - $(p \to q)$
  - Does this mean that If $x^2>4$ then x>2?
  - i.e., $(q \to p)$?
- If you are a bird, you have wings $p \to q$
  - In class someone said the above is false because bats have wings and they are not birds
    - This assumes I said if you have wings, you are a bird $q \to p$
      - *But that is now what I said!*
- If you make this mistake, you will fail this class
  - $(p \to q)$
  - *That does not mean that if you don't make this mistake, you will pass the class*



WHY I TRY NOT TO BE PEDANTIC ABOUT CONDITIONALS.

Xkcd1652:
Cueball does not assume the converse

BICONDITIONAL

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p **if and only if** q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

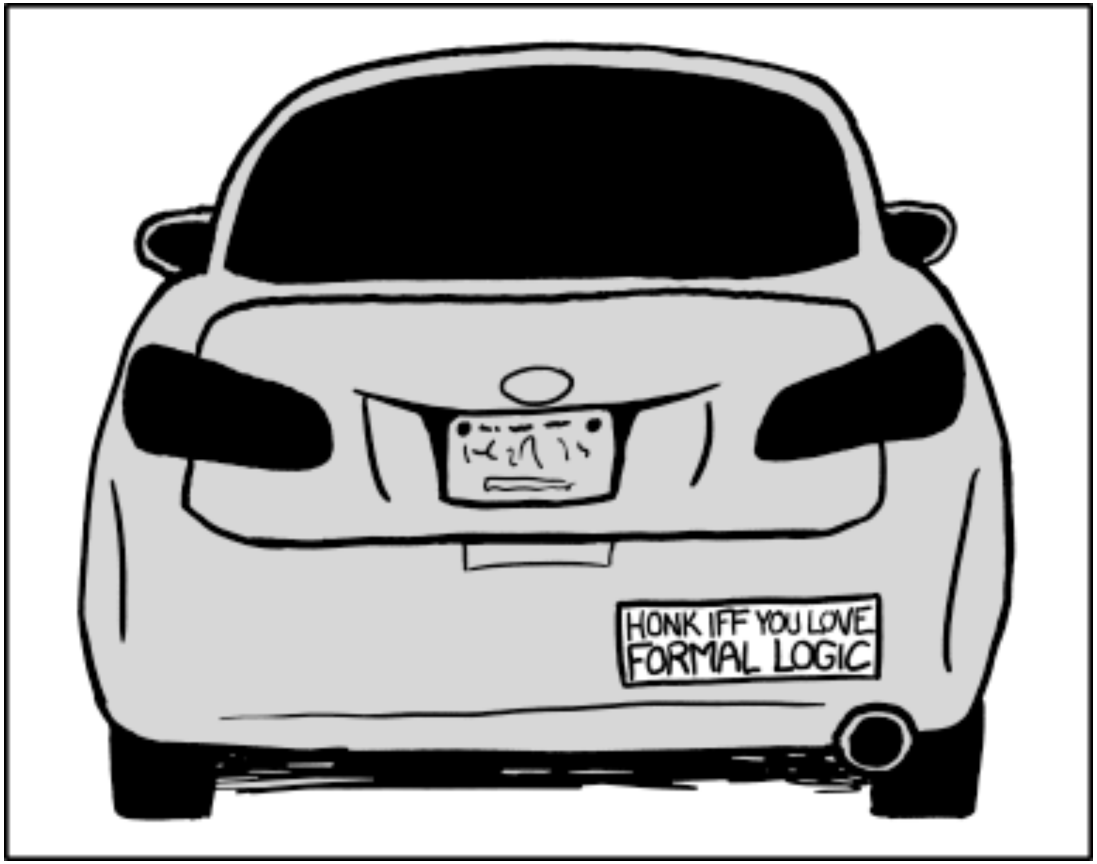
- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

EXPRESSING THE BICONDITIONAL

- Some alternative ways "*p* if and only if *q*" is expressed in English:

  - ***p* is necessary and sufficient for *q***
  - **if *p* then *q* , and conversely**
  - ***p* iff *q***

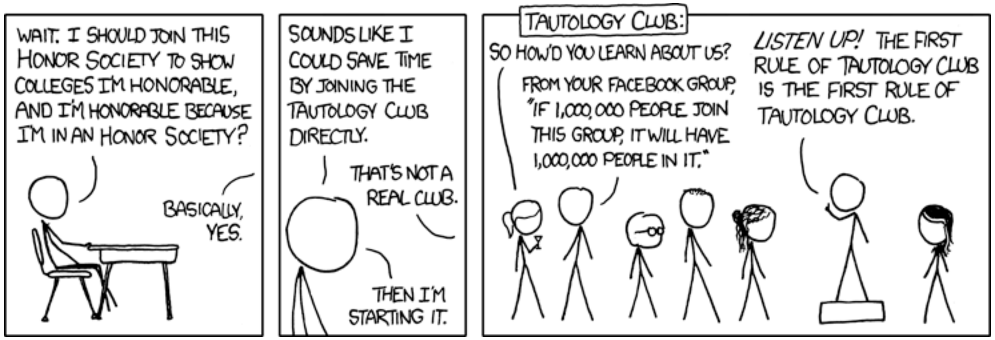
CAN YOU HONK IF HE STOPPED FOR A PEDESTRIAN?



- Note that this implies you should NOT honk solely because I stopped for a pedestrian and you're behind me.
  - If you honk, then you love formal logic
  - If you love formal logic you honk

TAUTOLOGIES AND CONTRADICTIONS

- A tautology is a proposition which is always true.
  - Example: $p \lor \neg p$
- A *contradiction* is a proposition which is always false.
  - Example: $p \land \neg p$



p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

CONVERSE AND INVERSE OF $P \to Q$

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg\, p \to \neg\, q$ is the **inverse** of $p \to q$

Example: Find the converse, and inverse of "if you are a bird, then you have wings."

Solution:

converse: If you have wings, then you are a bird

inverse: If you are not a bird, then you do not have wings.

- They are not equivalent to $p \to q$
  - If you are a bird you have wings, (assume this is true)
    - If you have wings then you are a bird (not necessarily true)
    - If you are not a bird, then you do not have wings (not necessarily true)

USING A TRUTH TABLE TO SHOW NON-EQUIVALENCE

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution: (the inverse and converse are equivalent between themselves, but they are both different to the original statement)

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Contrapositive $P \rightarrow Q$

- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
- **Example**: Find the contrapositive of "It raining is a sufficient condition for my not going to town."
  - **contrapositive**: If I go to town, then it is not raining.
- Assume the following statement is true:
  - If you are a bird, then you have wings.
  - Contrapositive: if you do not have wings, then you are not a bird
  - Is the contrapositive true?

LOGICALLY EQUIVALENT

- Two compound propositions p and q are logically equivalent iff $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- T

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

DE MORGAN'S LAWS

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

LAWS OF PROPOSITIONAL LOGIC

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\neg Q \rightarrow \neg P$ IS THE SAME AS $P \rightarrow Q$

- Two propositions are ***equivalent*** if they always have the same truth value.
- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

You can also show the equivalence using "proofs": ZyBook 2.5

CONVERSE, CONTRAPOSITIVE, INVERSE

Proposition:	$p \rightarrow q$	Ex: If it is raining today, the game will be cancelled.
Converse:	$q \rightarrow p$	If the game is cancelled, it is raining today.
Contrapositive:	$\neg q \rightarrow \neg p$	If the game is not cancelled, then it is not raining today.
Inverse:	$\neg p \rightarrow \neg q$	If it is not raining today, the game will not be cancelled.

Only the contrapositive is equivalent to the original statement

PREDICATES AND QUANTIFIERS

ZyBook 2.6-2.10

SECTION SUMMARY

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic
- Logic Programming

PROPOSITIONAL LOGIC

- Premises:
- *If Jack knows Jill, then Jill knows Jack.*
- *Jack knows Jill.*

- Conclusion:
- *Is it the case that Jill knows Jack?*

PROBLEM

- Premises:
- *If one person knows another, then the second person knows the first.*
- *Jack knows Jill.*

- Conclusion:
- *Is it the case that Jill knows Jack?*

- How do we represent the first premise in a way that allows us to derive the desired conclusion?

PREDICATE LOGIC

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x), M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

PROPOSITIONAL FUNCTIONS

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).
- The statement $P(x)$ is said to be the value of the propositional function P at x.
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  $P(-3)$ is false.

  $P(0)$ is false.

  $P(3)$ is true.

- Often the domain is denoted by U. So in this example U is the integers.

EXAMPLES OF PROPOSITIONAL FUNCTIONS

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(x, 3, z)

  Solution: Not a Proposition

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(x, 3, z)

  Solution: Not a Proposition

COMPOUND EXPRESSIONS

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(3) \rightarrow \neg P(-1)$ **Solution**: T
- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

QUANTIFIERS

Charles Peirce
(1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier,* **"**For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

UNIVERSAL QUANTIFIER $\forall$

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

  Examples:

  1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
  2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
  3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall\, x\, P(x)$ is false.

EXISTENTIAL QUANTIFIER $\exists$

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

UNIQUENESS QUANTIFIER $\exists!$

- $\exists! x\ P(x)$ means that $P(x)$ is true for <u>one and only one</u> x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\ P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\ P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that $P(x)$ can be expressed as:

$$\exists x\ (P(x) \land \forall y\ (P(y) \to y = x))$$

BOUND AND FREE VARIABLES

- An occurrence of a variable is **bound** if and only if it lies in the scope of a quantifier of that variable. Otherwise, it is **free**.
- $\exists y.q(x,y)$
- In this example, x is free and y is bound.

THINKING ABOUT QUANTIFIERS

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Zybook Challenge Activity 2.7.1

PROPERTIES OF QUANTIFIERS

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- Examples:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  1. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  1. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$",

  then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

PRECEDENCE OF QUANTIFIERS

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean
- $\forall x\, (P(x) \vee Q(x))$.

NEGATING QUANTIFIED STATEMENTS

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg(P(a_1) \land P(a_2) \land \ldots \land P(a_n)) \equiv \neg P(a_1) \lor \neg P(a_2) \lor \ldots \lor \neg P(a_n)$$

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

$$\neg(P(a_1) \lor P(a_2) \lor \ldots \lor P(a_n)) \equiv \neg P(a_1) \land \neg P(a_2) \land \ldots \land \neg P(a_n)$$

ZyBook 2.8

NEGATING QUANTIFIED EXPRESSIONS

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

NEGATING QUANTIFIED EXPRESSIONS

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent