ANNOUNCEMENTS

- Homework 2 is out on Canvas:
  - Due date: May 1 (midnight) on Canvas
    - Should be submitted as a pdf file
    - It is not necessary to justify all your answers. Please use your judgement.
- Quiz 3: Today, last 10 minutes of class.
  - Material: "The Sorcerer's Apprentice" chapter from Logicomix and lectures 6 and 7 (today's lecture is lecture 8).
  - You will get 2 attempts.
- Exam 1: May 5. Canvas quiz during class time. Covering all month of April

NEGATING QUANTIFIED STATEMENTS

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\equiv \qquad \equiv$$

$$\neg(P(a_1) \land P(a_2) \land \ldots \land P(a_n)) \equiv \neg P(a_1) \lor \neg P(a_2) \lor \ldots \lor \neg P(a_n)$$

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

$$\equiv \qquad \equiv$$

$$\neg(P(a_1) \lor P(a_2) \lor \ldots \lor P(a_n)) \equiv \neg P(a_1) \land \neg P(a_2) \land \ldots \land \neg P(a_n)$$

ZyBook 2.8.1-2.8.3

NEGATING QUANTIFIED EXPRESSIONS

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

NEGATING QUANTIFIED EXPRESSIONS

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

NESTED QUANTIFIERS

ZyBook 2.9 and 2.10

NESTED QUANTIFIERS

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

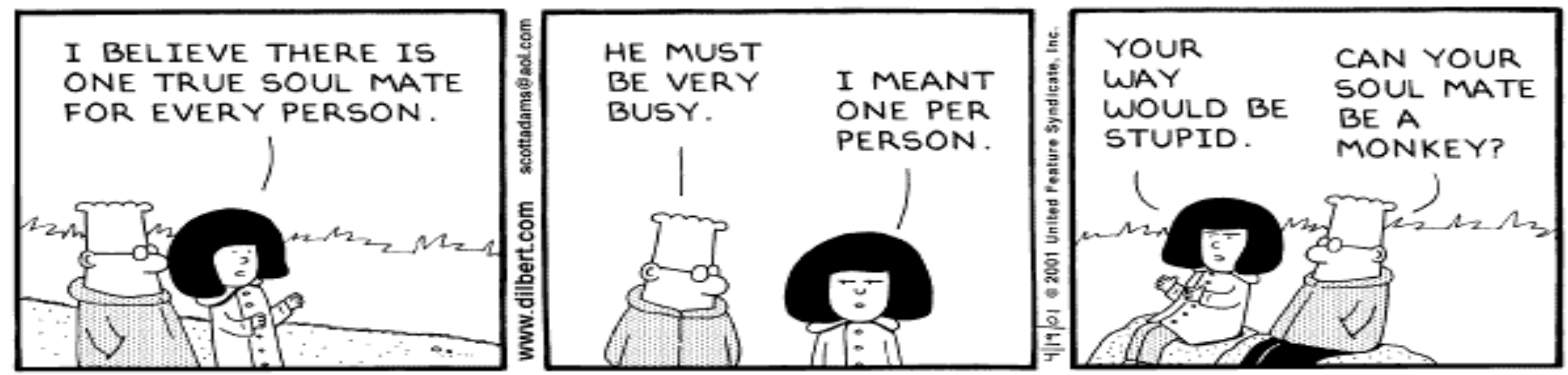
  Example: "Every real number has an inverse" is

  $\forall x \exists y(x + y = 0)$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:
  $\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is
  $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

$\exists Y \forall X$ VS $\forall X \exists Y$



P(x,y) = y is a soul mate of x

$\exists y \forall x \; P(x,y)$

VS

$\forall x \exists y P \; P(x,y)$

VS

$\forall x \exists! y P \; P(x,y)$

ORDER OF QUANTIFIERS

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

QUESTIONS ON ORDER OF QUANTIFIERS

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y \ P(x,y)$

   Answer: True

4. $\exists x \ \exists \ y \ P(x,y)$

   Answer: True

QUESTIONS ON ORDER OF QUANTIFIERS

Example 2: Let U be the real numbers,

Define $P(x,y)$: $x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: False

3. $\exists x \forall y\ P(x,y)$

   Answer: False

4. $\exists x\ \exists\ y\ P(x,y)$

   Answer: True

QUANTIFICATIONS OF TWO VARIABLES

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

ZyBook 2.9.1-2.9.4

NEGATING NESTED QUANTIFIERS

$\neg\forall x\, \forall y\, P(x, y) \equiv \exists x\, \exists y\, \neg P(x, y)$
$\neg\forall x\, \exists y\, P(x, y) \equiv \exists x\, \forall y\, \neg P(x, y)$
$\neg\exists x\, \forall y\, P(x, y) \equiv \forall x\, \exists y\, \neg P(x, y)$
$\neg\exists x\, \exists y\, P(x, y) \equiv \forall x\, \forall y\, \neg P(x, y)$

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$
2. $\forall w \neg \forall a \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

ZyBook 2.9.5-2.9.6

TRANSLATING NESTED QUANTIFIERS INTO ENGLISH

Example 1: Translate the statement

$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

where C(x) is "*x* has a computer," and $F(x,y)$ is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists \mathbf{x} \ \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

TRANSLATING MATHEMATICAL STATEMENTS INTO PREDICATE LOGIC

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

   "For all positive integers x and y, $x + y$ is positive."

3. The result is:

   $$\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

TRANSLATING ENGLISH INTO LOGICAL EXPRESSIONS EXAMPLE

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

   $$\exists w\, \forall a\, \exists f\, (P(w,f) \land Q(f,a))$$

EXAMPLE: CALCULUS

2 **Precise Definition of a Limit** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

CALCULUS IN LOGIC

Example: Use quantifiers to express the definition of the limit of *a* real-valued function *f(x)* of a real variable *x* at a point *a* in its domain.

Solution: Recall the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is "For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$."

Using quantifiers:

$$\forall \epsilon \exists \delta \forall x (0 < | x - a | < \delta \rightarrow | f(x) - L | < \epsilon)$$

Where the domain for the variables ε and δ consists of all positive real numbers and the domain for *x* consists of all real numbers.

RETURN TO CALCULUS AND LOGIC

Example : Recall the logical expression developed in the calculus example 2 slides back. $\lim_{x \to a} f(x)$

Use quantifiers and predicates to express that $\lim_{x \to a} f(x) \neq L$ does not exist.

1. We need to say that for all real numbers *L*,

$$\neg \forall \epsilon \exists \delta \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$

2. The result from the previous example can be negated to yield:

$$\neg \forall \epsilon \exists \delta \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$

3. Now we can repeatedly apply the rules for negating quantified expressions:

$$\equiv \exists \epsilon \neg \exists \delta \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$
$$\equiv \exists \epsilon \forall \delta \neg \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$
$$\equiv \exists \epsilon \forall \delta \exists x \neg (0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$
$$\equiv \exists \epsilon \forall \delta \exists x (0 < | x - a | < \delta \wedge | f(x) - L | \geq \epsilon)$$

The last step uses the equivalence $\neg(p \to q) \equiv p \wedge \neg q$

CALCULUS IN PREDICATE LOGIC

4. **Therefore, to say that** $\lim_{x \to a} f(x)$ **does not exist means that for all real numbers** ***L***, $\lim_{x \to a} f(x) \neq L$ **can be expressed as:**

$$\forall L \exists \epsilon \forall \delta \exists x \; (0 < |\, x - a \,| < \delta \wedge |\, f(x) - L \,| \geq \epsilon)$$

Remember that ε and δ range over all positive real numbers and ***x*** over all real numbers.

5. **Translating back into English we have, for every real number L, there is a real number** ε > 0, such that for every real number δ > 0, there exists a real number ***x*** such that

5. $0 < |\, x - a \,| < \delta$ and $|f(x) - L\,| \geq \varepsilon$.

QUESTIONS ON TRANSLATION FROM ENGLISH

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\, \forall y\, (B(\mathrm{x,y}) \rightarrow S(\mathrm{x,y}))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\, \forall y\, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\, \exists y\, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\, \forall x\, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\, \exists y\, L(x,y)$

Example 6: "Everyone loves themselves"

Solution: $\forall x\, L(x,x)$

SOME QUESTIONS ABOUT QUANTIFIERS

- Can you switch the order of quantifiers? $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
  - Is this a valid equivalence?

    Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.
  - Is this a valid equivalence? $\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$

    Solution: No! The left and the right side may have different truth values for some propositional functions for *P*. Try "x + y = 0" for *P(x,y)* with *U* being the integers. The order in which the values of *x* and *y* are picked does matter.

- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

    Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.
  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

    Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.