CSE 16 Applied Discrete Mathematics

Homework 3

1. Use any result in page 36 of the cheat sheet (except Rule 9, which is what we are trying to prove) to prove the following:

   $a \rightarrow g,\ b \rightarrow g\ \vdash (a \vee b) \rightarrow g$ (Hint: you need to use Axiom 9.)

2. Use any result in page 36 of the cheat sheet (except Theorem 10, which is what we are trying to prove) to complete the following proof:

   $a, b\ \vdash a \wedge b$

   Proof:

   1. $(a \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow (a \wedge b)))$ Axiom 6
   2. $b \rightarrow (a \rightarrow b)$ Axiom 1
   3. $a \rightarrow a$ Theorem 1

3. Use any result in page 36 of the cheat sheet (except Theorem 11, which is what we are trying to prove) to complete the following proof:

   $a \vee b, \neg a \vdash b$

   Proof:

   1. $b \rightarrow b$ Theorem 1

4. Use any result in page 36 of the cheat sheet to complete the proof the following theorem (except Rule 4 and Rule 5):

   $\vdash \neg(\neg a \wedge a)$

   Assume by contradiction $\neg a \wedge a$. (this is by Meta-Corollary (a) in Cheat Sheet 35)

   Proof:

   1. $(\neg a \wedge a) \rightarrow a$ Axiom 5

5. Use any result in page 36 of the cheat sheet (except Theorem 13) to prove the following theorem by contradiction:

   $\neg a, \neg b \vdash \neg(a \lor b)$

   Assume by contradiction $a \lor b$

6. Use any result in page 36 of the cheat sheet (except Theorem 14) to prove the following theorem by contradiction:

   $\neg a \vdash \neg(a \land b)$

   Assume by contradiction $a \land b$.

7. Use any result in page 36 of the cheat sheet to prove the following theorem by contradiction:

   $a \lor (a \land b) \vdash a$

   Assume by contradiction $\neg a$

8. (Exercise 2.12.5) Proving arguments in English are valid or invalid.

   Give the form of each argument by translating each sentence to a proposition in logic. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

(a) If I get a job then I will buy a new car and a new house.
I won't buy a new house.
____
∴ I will not get a job.

p: I get a job
q: I buy a new car
r: I buy a new house

(d) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.
____
∴ I will not buy a new car.

p: I buy a new car
r: I buy a new house
q: I get a job

9. (Exercise 2.13.4) Determine and prove whether an argument is valid or invalid.

   (c) $\forall x\,(P(x) \land Q(x)) \vdash \forall x\,Q(x) \land \forall x\,P(x)$

10. (Exercise 2.13.5) Determine and prove whether an argument in English is valid or invalid.

    Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.

    The domain for each problem is the set of students in a class.

    (e) Every student who missed class or got a detention did not get an A.
    Penelope is a student in the class.
    Penelope got an A.
    ____
    Penelope did not get a detention.

11. (Exercise 4.2.3) Find a counterexample.

    Find a counterexample to show that each of the statements is false.

    (d) Every positive integer can be expressed as the sum of the squares of two integers.
    (e) The multiplicative inverse of a real number x, is a real number y such that xy = 1. Every real number has a multiplicative inverse.

12. (Exercise 4.2.5) Proving existential statements.

    (h) For every pair of real numbers, x and y, there exists a real number z such that x - z = z - y.
    (i) Show that there is an integer n such that n^2-1 is prime.

13. (Exercise 4.3.1) Fill in the words to form a complete proof.

    Theorem: The sum of the squares of any two consecutive integers is odd.
    `x^2 + (x+1)^2 = x^2 + (x^2 + 2x + 1) = 2x^2 + 2x +1 = 2(x^2 +x) + 1`

14. (Exercise 4.3.3) Find the mistake in the proof - odd and even numbers.

    Theorem: If n and m are odd integers, then $n^2 + m^2$ is even

For each "proof" of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

The assumption about k and j are not stated, they are both integers. Should be: "Since, by assumption, n is an odd integer, then n = 2k+1, for some integer k. Since, by assumption, m is an odd integer, then m = 2j+1, for some integer j."

(b) Proof.

Since, by assumption, n is an odd integer, then n = 2k+1. Since, by assumption, m is an odd integer, then m = 2j+1. Plugging in 2k+1 for n and 2j+1 for m into the expression $n^2 + m^2$ gives.

`n^2 + m^2 = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 2(2k^2 + 2k + 2j^2 + 2j + 1)`

Since k and j are integers, `2k^2 + 2k + 2j^2 + 2j + 1` is also an integer. Since $n^2 + m^2$ is equal to two times an integer, then $n^2 + m^2$ is an even integer. ■

15. (Exercise 4.4.1) Proving statements about odd and even integers with direct proofs.

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as 2k+1, where k is an integer. An even integer is an integer that can be expressed as 2k, where k is an integer.

Assume x is an odd integer and y is also an odd integer. We will show that the sum of x and y is an even integer.

Prove each of the following statements using a direct proof.

(b) The sum of two odd integers is an even integer.
(d) The product of two odd integers is an odd integer.
(f) If x is an even integer and y is an odd integer, then 3x + 2y is even.
(h) The negative of an odd integer is also odd.
(i) If x is an even integer then $(-1)^x = 1$.

16. (Exercise 4.4.4) Showing a statement is true or false by direct proof or counterexample.

Determine whether the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.

(a) If x and y are even integers, then x+y is an even integer.
(b) If x+y is an even integer, then x and y are both even integers

17. (Exercise 4.5.1) Proof by contrapositive of statements about odd and even integers.

Prove each statement by contrapositive

(a) For every integer n, if n^2 is an odd, then n is odd.

(b) For every integer n, if n^3 is even, then n is even.

18. **(Exercise 4.5.5) Proving statements using a direct proof or by contrapositive.**

    Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other.

    (e) If n and m are integers such that n^2+m^2 is odd, then m is odd or n is odd.

19. **(Exercise 4.6.5) Proof by contrapositive vs. proof by contradiction.**

    For each statement, write what would be assumed and what would be proven in a proof by contrapositive of the statement. Then write what would be assumed and what would be proven in a proof by contradiction of the statement.

    (b) For all integers n, if n^2 is odd, then n is also odd.

20. **(Exercise 4.6.6) Proofs by contradiction.**

    Give a proof for each statement.

    (b) If a person buys at least 400 cups of coffee in a year, then there is at least one day in which the person has bought at least two cups of coffee.
    (g) For all even integers n, n^2 is a multiple of 4.
    (i) If the product of two positive real numbers is larger than 400, then at least one of the two numbers is larger than 20.
    (j) If the sum of two positive real numbers is larger than 400, then at least one of the two numbers is larger than 200.

21. **(Exercise 4.7.2) Proofs by cases - even/odd integers and divisibility.**

    Prove each statement.

    (c) If integers x and y where $x<y$ are consecutive, then they have opposite parity.