ANNOUNCEMENTS

- Homework 1 is out on Canvas:
  - Due date: April 17 (midnight) on Canvas
    - Should be submitted as a pdf file
    - Show how you got an answer (just giving an answer without explanation will cost you points). You do not need to give a "formal proof" to each point (or use only math notation), but you need to provide enough details so the grader can see your logical reasoning.
    - Venn diagrams are NOT good arguments for justifying answers.
- Quiz 2: April 16, last 10 minutes of class.
  - Material: "Pembroke Lodge" chapter from Logicomix and lectures 4 and 5 (today's lecture is lecture 5).
  - You will only get 2 attempts.
- Exam 1: May 5. Canvas quiz during class time. Covering all month of April

LECTURE SUMMARY

- Final Set Operations
- More on Set Cardinality
- Set Identities
- Proving Identities
- Russell's paradox and axiomatic set theory
- Introduction to logic

GENERALIZED UNIONS AND INTERSECTIONS

- Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

  These are well defined, since union and intersection are associative.

- For $i = 1,2,\ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

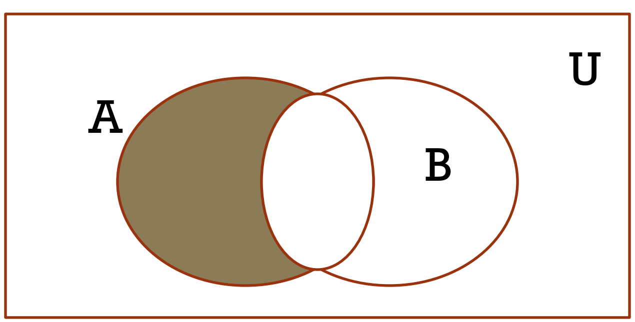
$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

DIFFERENCE

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in \mathrm{A} \wedge x \notin B\} = A \cap \overline{B}$$



Venn Diagram for $A - B$

SYMMETRIC DIFFERENCE

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

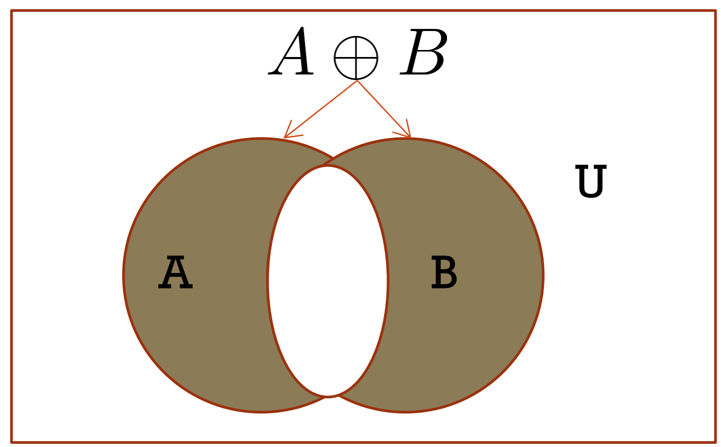
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

- **Solution**: $\{1,2,3,6,7,8\}$

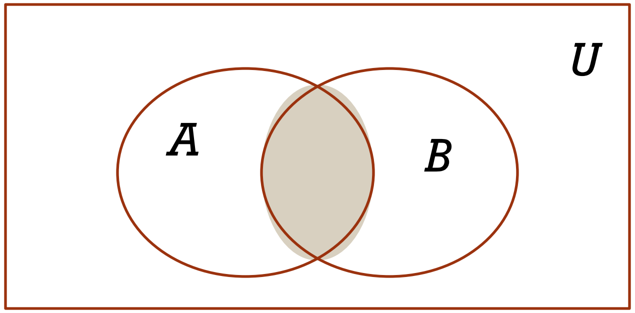


Venn Diagram

THE CARDINALITY OF THE UNION OF TWO SETS

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let A be the math majors in your class and B be the CSE majors. To count the number of students who are either math majors or CSE majors, add the number of math majors and the number of CSE majors, and subtract the number of joint CSE/math majors.

SUMMARY OF SET OPERATIONS

Operation	Notation	Description
Intersection	$A \cap B$	$\{ x : x \in A \text{ and } x \in B \}$
Union	$A \cup B$	$\{ x : x \in A \text{ or } x \in B \text{ or both} \}$
Difference	$A - B$	$\{ x : x \in A \text{ and } x \notin B \}$
Symmetric difference	$A \oplus B$	$\{ x : x \in A - B \text{ or } x \in B - A \}$
Complement	$\overline{A}$	$\{ x : x \notin A \}$

ZyBook Exercises 1.5

REVIEW QUESTIONS

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$

   Solution: $\{4,5\}$
3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$
4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$
5. $A - B$

   Solution: $\{1,2,3\}$
6. $B - A$

   Solution: $\{6,7,8\}$

SET IDENTITIES (ZYBOOK EXERCISE 1.6)

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
Domination laws	$A \cap \varnothing = \varnothing$	$A \cup U = U$
Double Complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \varnothing$ $\overline{U} = \varnothing$	$A \cup \overline{A} = U$ $\overline{\varnothing} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

PROOF OF SECOND DE MORGAN LAW

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Zybook 1.6 exercise shows proof by "equivalence", here we specifically prove 1) and 2)

PROOF OF SECOND DE MORGAN LAW

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$$
\begin{array}{ll}
x \in \overline{A \cap B} & \text{by assumption} \\
x \notin A \cap B & \text{defn. of complement} \\
\neg((x \in A) \wedge (x \in B)) & \text{defn. of intersection} \\
\neg(x \in A) \vee \neg(x \in B) & \text{1st De Morgan Law for Prop Logic} \\
x \notin A \vee x \notin B & \text{defn. of negation} \\
x \in \overline{A} \vee x \in \overline{B} & \text{defn. of complement} \\
x \in \overline{A} \cup \overline{B} & \text{defn. of union}
\end{array}
$$

Continued on next slide →

PROOF OF SECOND DE MORGAN LAW

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

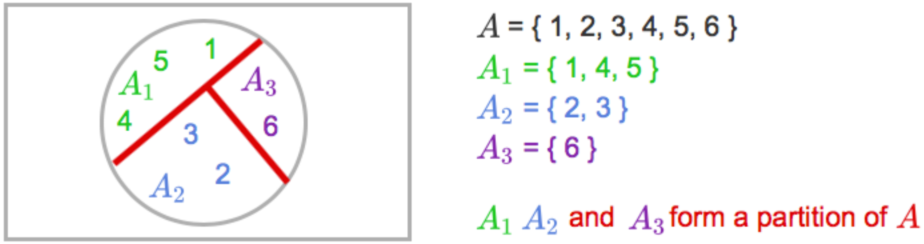
BOOLEAN ALGEBRA

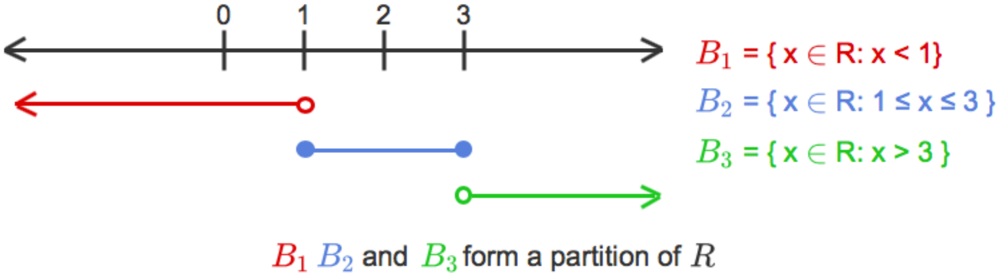
- Set theory and propositional calculus (our next topic) are both instances of an algebraic system called a *Boolean Algebra*.
  - We will see more details on Section 3 of the ZyBook
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set *U*. All sets are assumed to be subsets of *U*.

▪ $A \cap B$	▪ $A \wedge B$
▪ $A \cup B$	▪ $A \vee B$
▪ $\bar{A}$	▪ $\neg A$

PARTITIONS

- Two sets A and B are disjoint if their intersection is empty:
  - $(A \cap B = \emptyset)$
- A partition of a non-empty set A is a collection of non-empty subsets of A such that each element of A is in exactly one of the subsets.





- ZyBook 1.7

FINAL THOUGHTS ON SETS

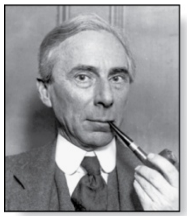
MATHEMATICAL PARADOXES

- Paradoxes are statements that result in an inconsistency
- φ and $\neg\varphi$ are derived from the statement

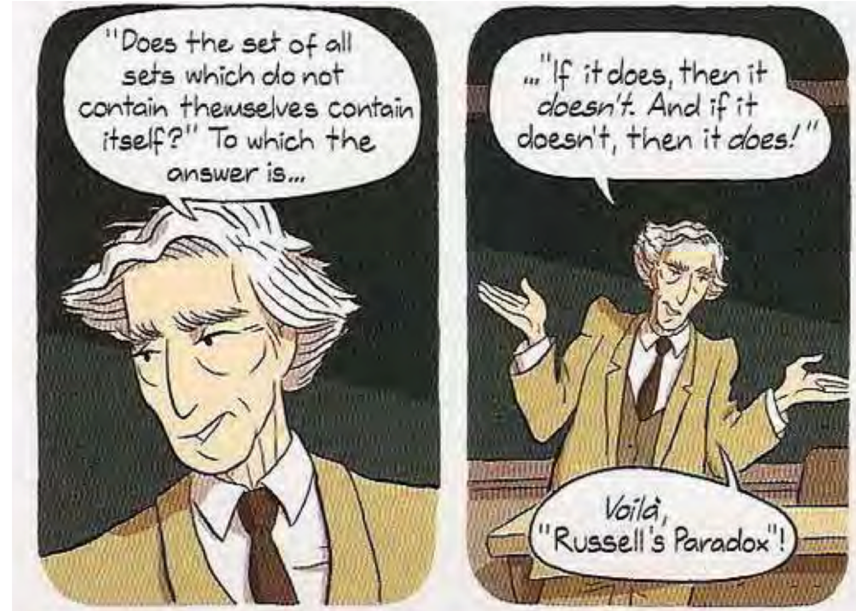
- I am now lying to you
  - If I am now lying = φ, then the opposite must be true
    - Therefore I am now telling the truth = $\neg\varphi$
  - If I am telling the truth = $\neg\varphi$, then I am lying = φ

- If a mathematical theory is inconsistent, then everything statement is true
  - We want our systems to be consistent!

RUSSELL'S PARADOX



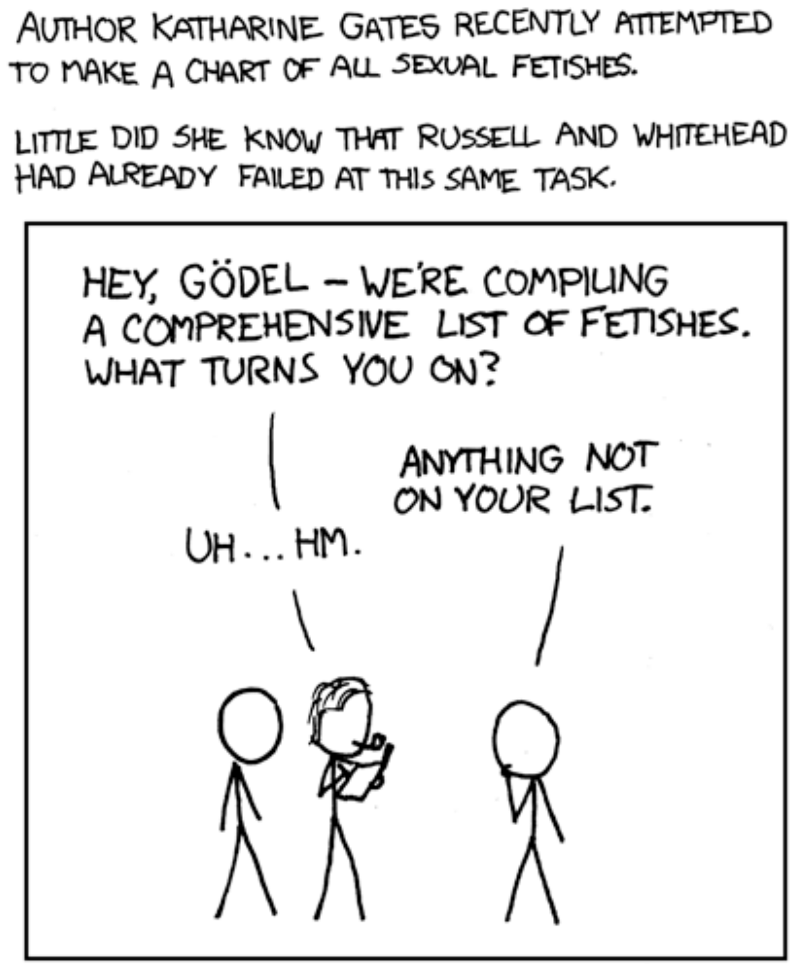
Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner



- Let *S* be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is *S* a member of itself?"
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?" (see Logicomix)

Pages 32 and 33 of "Book of Proof"

SECONDARY GOAL OF TODAYS' CLASS: UNDERSTAND JOKES ABOUT RUSSELL'S PARADOX

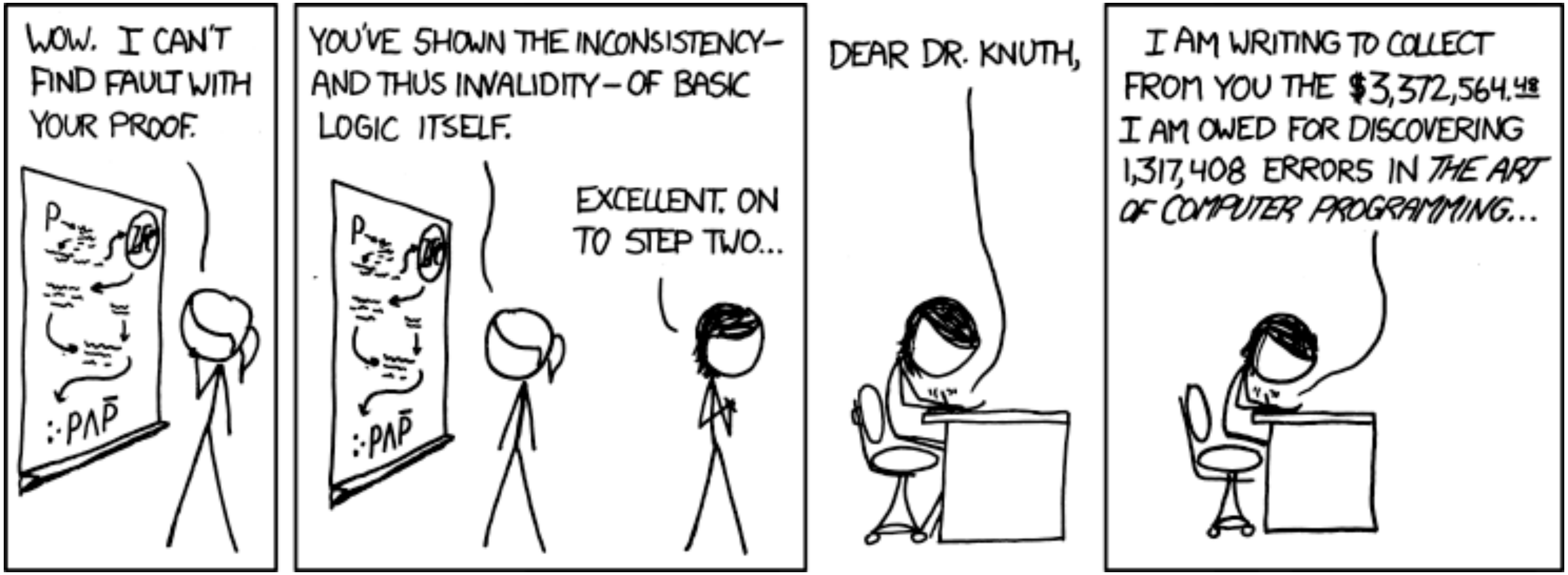


Xkcd-468

AXIOMATIC SET THEORY

- We can develop a set theory without paradoxes
- Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC)
  - Prevents most paradoxes
  - But it limits what can be a set (sets are no longer defined as a "collection of elements")

Xkcd-982



Xkcd-816

NEXT TOPIC: LOGIC

- Book of Proof:
  - Chapter 2: Logic
- ZyBook:
  - Section 2: Logic
  - Section 3: Boolean Algebra

LOGIC

Some images from Tracy Larrabee, McGraw-Hill Education (Discrete Mathematics and its Applications), Stanford Introduction to Logic, and Introducing Logic A Graphic guide

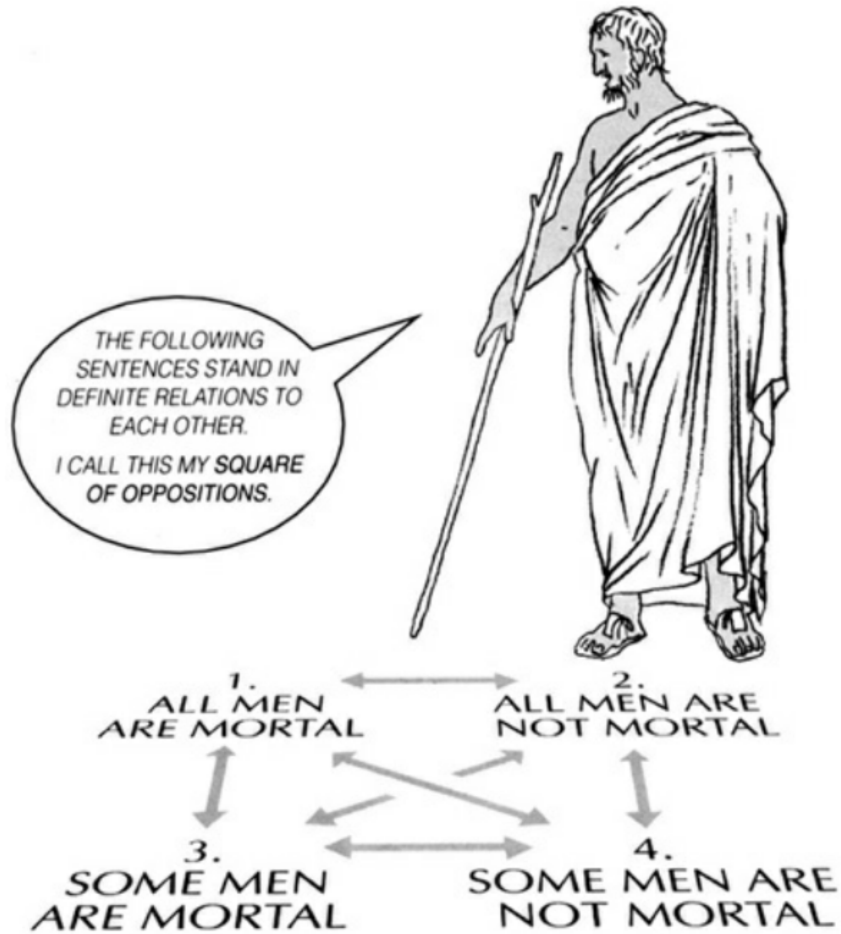
LOGIC

- Given a premise (that is true) we want to reach a valid conclusion
  - We do this through an argument
- Logic is the study of truth-preserving arguments

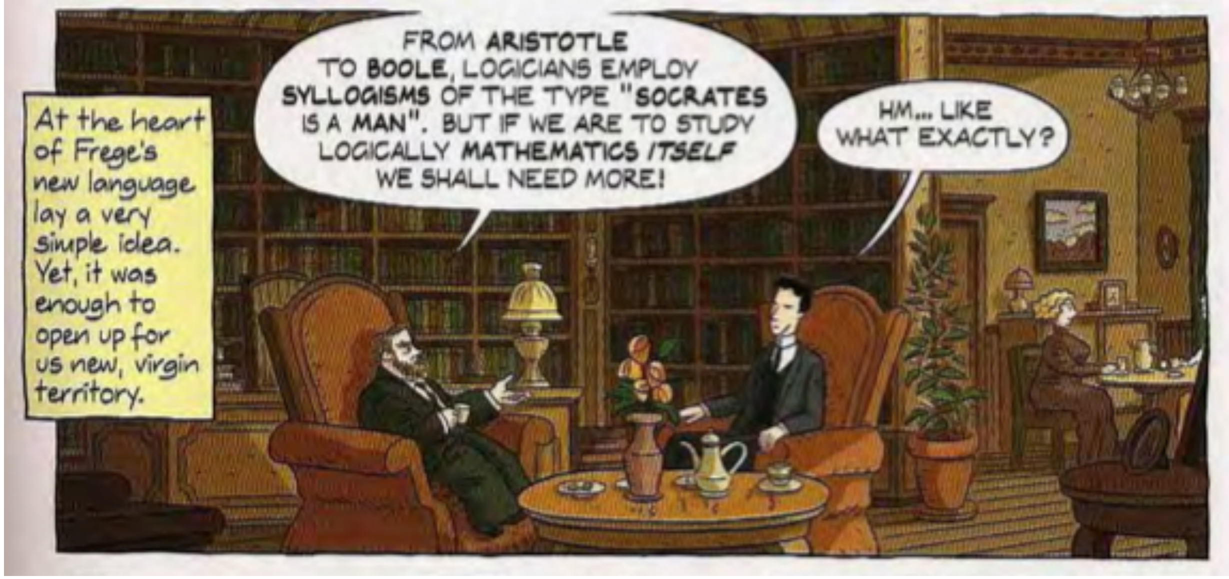
ORIGINS

- Logic is one of the oldest intellectual disciplines in human history, dating back to Aristotle
- Subject of the sentence: objects we talk about
- Predicate: what we say about the subject
- Syllogism: subject of the first statement is predicate of the second, and third statement composed of remaining terms

1. All men are mortal
2. Socrates is a man
3. Socrates is mortal



PROPOSITIONAL CALCULUS



- Modern logic began in 1879 with the publication of Gottlob Frege's Begriffsschrift.
- A proposition is a statement that is either true or false:
  - Proposition:
    - There are an infinite number of prime numbers
  - Not a proposition:
    - Have a nice day
- Reasoning of propositions with connectives
  - If… then… and… not
- If you are a bird, then you have wings ($p \to q$)
  - You cannot, be a bird and not have wings $\neg(p \land \neg q)$
  - If you do not have wings, then you are not a bird ($\neg q \to \neg p$)

COMMENT FROM CLASS

- Bats have wings are they are not birds
  - Does this contradict the previous slide?
    - If you are a bird, then you have wings $p \to q$
- Answer:
  - No! That is not a contradiction.
  - You assumed the last slide said $q \to p$
    - If you have wings, then you are a bird. That statement is clearly false: counterexample: Bats
- To find a contradiction to the original statement, I need to find p and not q
  - $p \wedge \neg q$
  - Moas are birds, and they do not have wings. Therefore the original proposition is false?
    - (I need to check if Moas have wings, they appear extinct, so the following seems true: if you are a bird in 2020, then you have wings)

IS THIS ARGUMENT CORRECT?

"Then you should say what you mean," the March Hare went on.

"I do," Alice hastily replied: "at least I mean what I say –that's the same thing, you know."

ALICE IN WONDERLAND

- 'Not the same thing a bit!' said the Hatter. 'You might just as well say that "I see what I eat" is the same thing as "I eat what I see"!'
- 'You might just as well say,' added the March Hare, 'that "I like what I get" is the same thing as "I get what I like"!'
- 'You might just as well say,' added the Dormouse, who seemed to be talking in his sleep, 'that "I breathe when I sleep" is the same thing as "I sleep when I breathe"!'

- Alice seems to believe that the sentence $p \Rightarrow q$ is the same as $q \Rightarrow p$; and this is a logical error, as made clear by the comments of the Mad Hatter, the March Hare, and the Dormouse. Clearly, Alice needs to sign up for Introduction to Logic.

PREDICATE CALCULUS

- Rusell reintroduced Aristotle's distinction between subject and predicate
- Some cats do not have fur: ∃ **cat** P(cat)
  - Predicate P(), do not have fur
- All men are mortal: ∀ **men** Q(men)
  - Predicate Q(), are mortal
- It cannot be that all cats have fur
- It cannot be that there is a man that is immortal

IS THIS ARGUMENT CORRECT?

- All x are y
- Some y are z
- Therefore, some x are z

- E.g.,
- All Toyotas are Japanese cars
- Some Japanese cars are made in America
- Therefore, some Toyotas are made in America

UNSOUND RULE OF INFERENCE

- All Toyotas are cars
- Some cars are Fiats
- Therefore, some Toyotas are Fiats

PROOFS: TRUTH-PRESERVING ARGUMENTS

- A conclusion is provable for a set of premises if and only if there is a finite sequence of sentences in which every element is either a premise or the result of applying a sound rule of inference to earlier members in the sequence
- Constructing these sound inferences is the main goal of the next part of the course (and the most important part)

INFORMAL DESCRIPTION OF LOGIC IN MATH

- Axioms:
  - Statement or Propositions that you assume to be true
  - Ideally truths that are self-evident
- Rules of inference:
  - Take these axioms and produce new results such as
  - Lemma: Statement that is used to prove other statements
  - Theorem: A new statement that is "important"
  - Corollary: A special case of a theorem that is popular/useful
- You can take any previously proved Lemma, Theorem, or Corollary (along with axioms) and continue proving new results with rules of inference

PROPOSITIONAL LOGIC

PROPOSITIONAL LOGIC SUMMARY

- The Language of Propositions
  - Connectives
  - Truth Values
- Applications
  - Translating English Sentences
  - System Specifications
  - Logic Puzzles
  - Logic Circuits
- Logical Equivalences
  - Important Equivalences
  - Showing Equivalence
  - Satisfiability

SECTION SUMMARY

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth Tables

PROPOSITIONS

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Trenton is the capital of New Jersey.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

PROPOSITIONAL LOGIC

- Constructing Propositions
  - Propositional Variables: p, q, r, s, …
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

COMPOUND PROPOSITIONS: NEGATION

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- **Example**: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

CONJUNCTION

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

DISJUNCTION

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

THE CONNECTIVE OR IN ENGLISH

- In English "or" has two distinct meanings.
  - "Inclusive Or" - In the sentence "Students who have taken CS202 or Math120 may take this class," we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
  - "Exclusive Or" - When reading the sentence "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F