WE HAVE COVERED THE MOST IMPORTANT ASPECTS OF CSE 16

- Understanding mathematical notation
- Understanding logic
- Understanding proofs
  - Direct proofs
  - Proof by contradiction
- The rest is just "practice" of the above with new mathematical definitions
  - So far, we have practiced with basic arithmetic, but now:
  - Functions
  - Relations
  - Algorithms
  - Sequences

MOTIVATING QUESTION

- What is the cardinality of the natural numbers?
- What is the cardinality of the integers?
- What is the cardinality of the real numbers?
- Is any of them larger than the others?

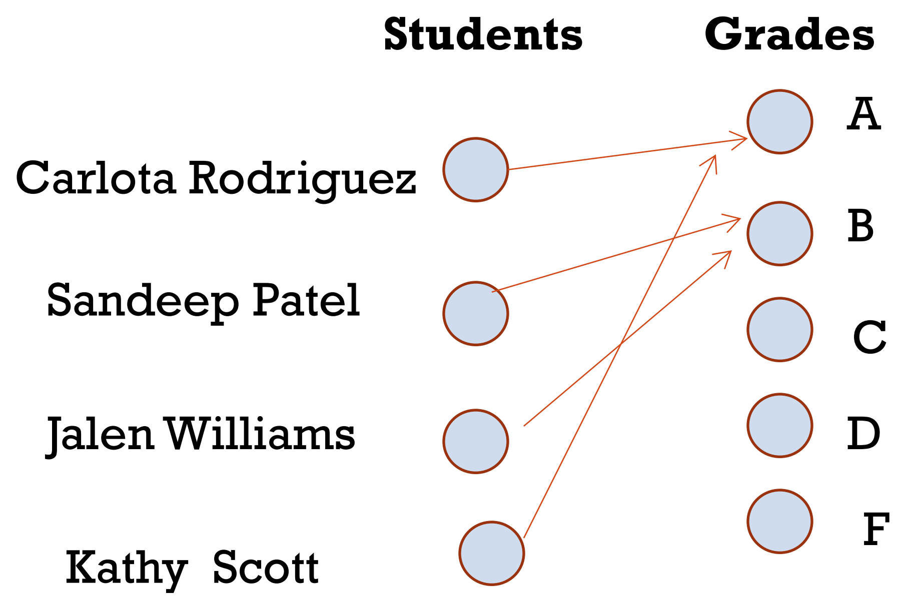
- Poll 1: is the sent of integers larger than the set of natural numbers?
  - 62% said yes! The real answer to this question is NO!
- Poll 2: is the set of real numbers larger than the set of integers?
  - 69% said yes! The real answer to this question is YES!

FUNCTIONS

FUNCTIONS

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



FUNCTIONS

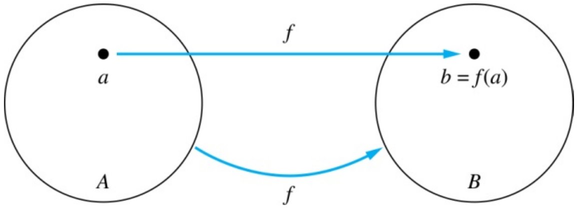
- A function $f: A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2) \in f] \rightarrow y_1 = y_2]$$

DEFINITIONS



Given a function $f: A \rightarrow B$:

- We say f *maps* A to B *or*

  f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain, target* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

REPRESENTING FUNCTIONS

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number.

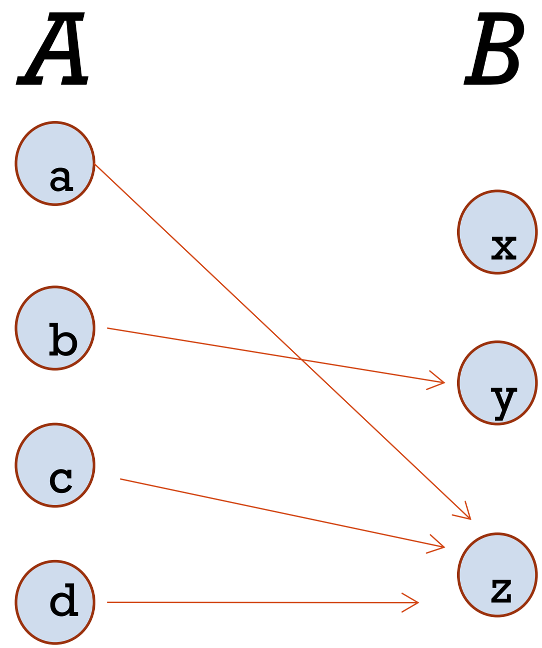
QUESTION ON FUNCTIONS AND SETS

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}



QUESTIONS

$f(a) = ?$ z

The image of d is ? z

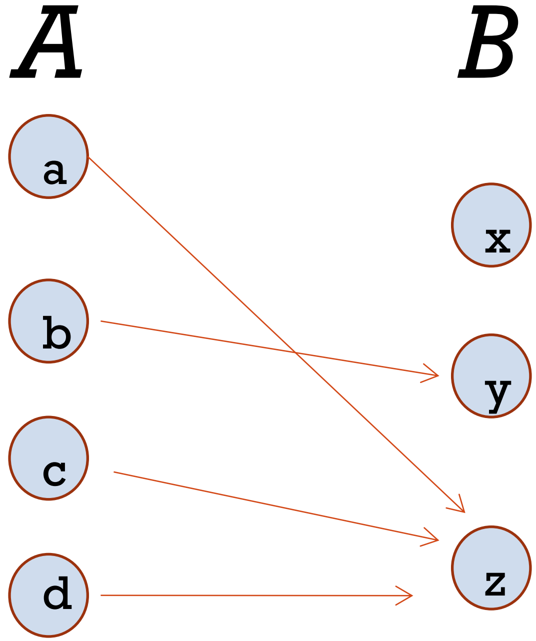
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A) = ?$ {y,z}

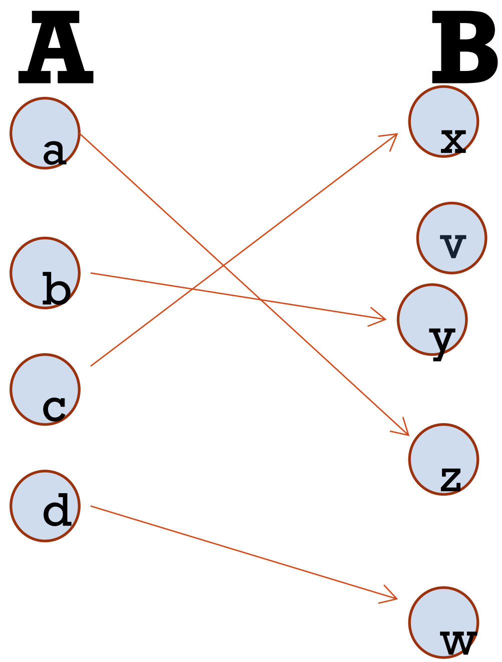
The preimage(s) of z is (are) ? {a,c,d}



DEFINITION OF ONE-TO-ONE FUNCTION

Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

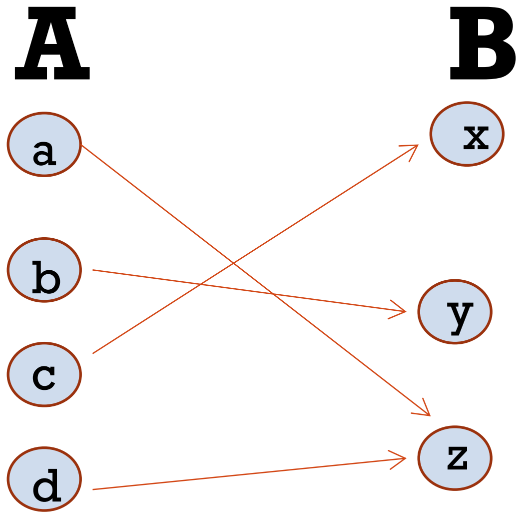
$$\forall a \forall b \left[f(a) = f(b) \rightarrow a = b \right]$$



DEFINITION OF AN ONTO FUNCTION

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is *onto*.

$$\forall b \; \exists a \quad f(a) = b$$



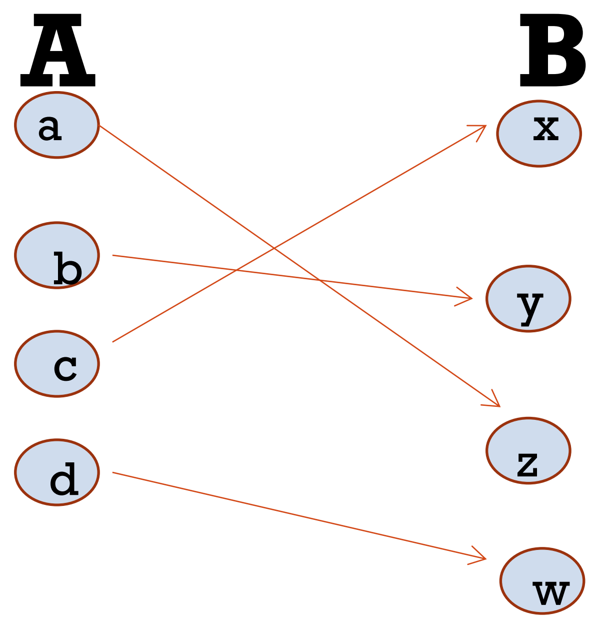
BIJECTIONS

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one (injective) and onto (surjective).

$$\forall a \forall b \left[f(a) = f(b) \rightarrow a = b \right]$$

and

$$\forall b \; \exists a \quad f(a) = b$$



ZyBooks 5.1-5.3

SHOWING THAT F IS (OR IS NOT) ONE-TO-ONE (INJECTIVE) OR ONTO (SURJECTIVE)

One-to-one aka Injective iff $\forall a \forall b \, [f(a) = f(b) \rightarrow a = b]$

Onto aka Surjective iff $\forall b \; \exists a \quad f(a) = b$

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

SHOWING THAT f IS ONE-TO-ONE OR ONTO

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

RECALL THAT CARDINALITY IS DEFINED WITH BIJECTIONS

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |\mathrm{B}|,$$

if and only if there is a bijection from A to B.

- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

CARDINALITY IS DEFINED WITH BIJECTIONS

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called *countable*. A set that is not countable is *uncountable*.
- The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

SHOWING THAT A SET IS COUNTABLE

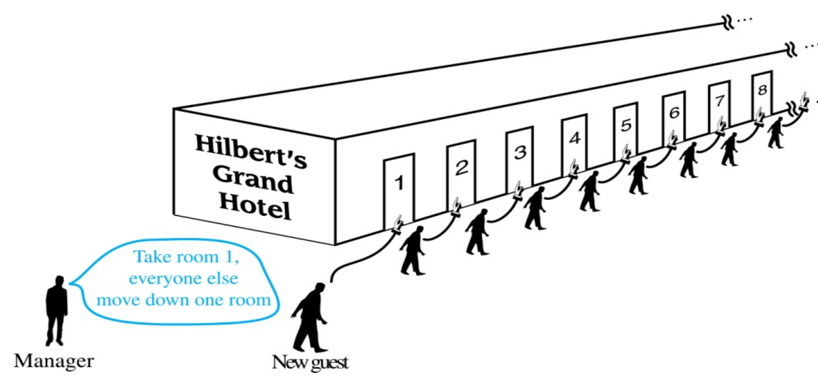
- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

HILBERT'S GRAND HOTEL

David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

SHOWING THAT A SET IS COUNTABLE

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots \end{array}$$

Then f is a bijection from $\mathbf{Z}^+$ to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

SHOWING THAT A SET IS COUNTABLE

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

$0, 1, -1, 2, -2, 3, -3$,……….

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$

THE POSITIVE RATIONAL NUMBERS ARE COUNTABLE

- **Definition**: A *rational number* can be expressed as the ratio of two integers *p* and *q* such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. $\rightarrow$

THE POSITIVE RATIONAL NUMBERS ARE COUNTABLE

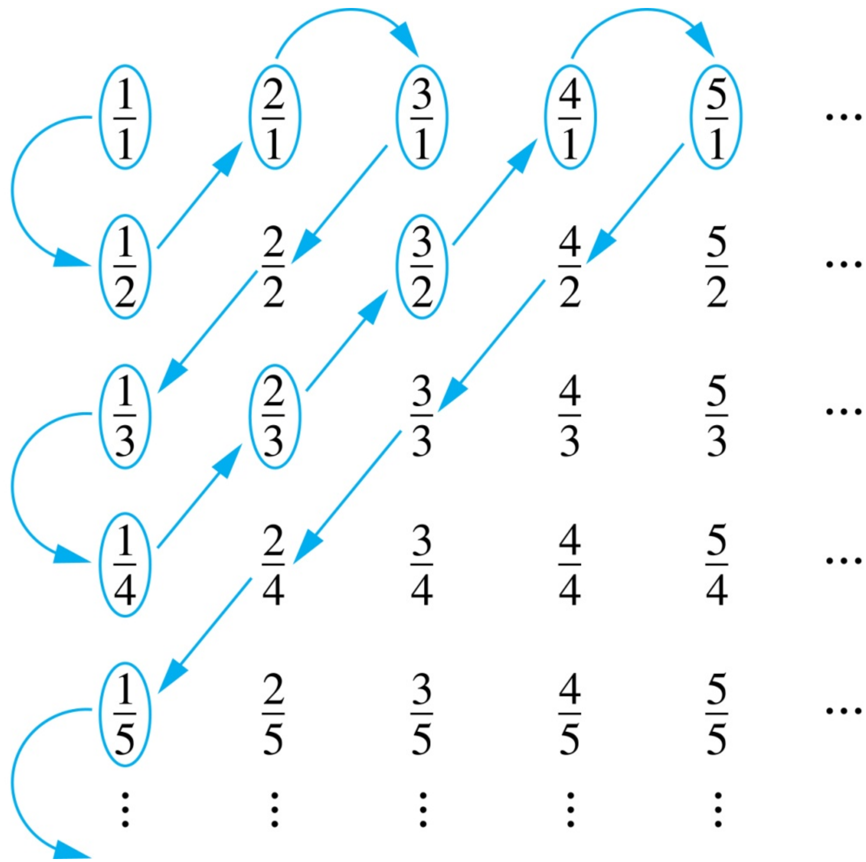
Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

First row $q = 1$.
Second row $q = 2$.
etc.

Terms not circled are not listed because they repeat previously listed terms



1, ½, 2, 3, 1/3,1/4, 2/3, ...

STRINGS

Example 4: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from **N** to S and hence it is a countably infinite set.

THE SET OF ALL JAVA PROGRAMS IS COUNTABLE.

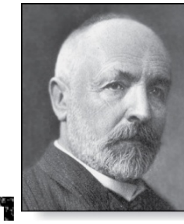
Example 5: Show that the set of all Java programs is countable.

Solution: Let *S* be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable. ◀

Georg Cantor
(1845-1918)



THE REAL NUMBERS ARE UNCOUNTABLE

Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagonalization argument and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\vdots \end{aligned}$$

4. Form a new real number with the decimal expansion

$$r = .r_1 r_2 r_3 r_4 \ldots$$

where $r_i = 3 \text{ if } d_{ii} \neq 3$ and $r_i = 4 \text{ if } d_{ii} = 3$

5. r is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its ith position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.