1 Part I

1. Use set builder notation to give a description of each of these sets

a. $\{0,3,6,9,12\}$.

$\because x \in \mathbb{N}$

$0/3=0$ $3/3=1$ $6/3=2$ $9/3=3$ $12/3=4$

$\therefore \{x \in \mathbb{N} \mid x$ is a multiple of 3 and $x \le 12\}$.

b. $\{-3,-2,-1,0,1,2,3\}$.

$\because x \in \mathbb{Z}$ and between -3 and 3

$\therefore \{x \in \mathbb{Z} \mid -3 \le x \le 3\}$.

c. $\{m,n,o,p\}$

$\because$ m, n, o, p are the letters in alphabet from m to p

$\therefore \{x \mid x$ is a letter in the alphabet from m to p$\}$.

2. Determine whether each of these pairs of sets are equal

a. $\{1,3,3,3,5,5,5,5,5\}$, $\{5,3,1\}$.

Yes, because $1=1$, $3=3$, $5=5$, and the order doesn't matter in this case.

b. $\{\{1\}\}$, $\{1,\{1\}\}$.

No, because $\{\{1\}\}$ has one element, $\{1\}$. However, $\{1,\{1\}\}$ has two elements, 1 and $\{1\}$.

c. $\emptyset$, $\{\emptyset\}$

No, because $\emptyset$ represents no element, but $\{\emptyset\}$ has one element, that is $\emptyset$.

3. Determine whether these statements are true or false:

a. $\emptyset \in \{\emptyset\}$

True, because $\emptyset$ is the element of set $\{\emptyset\}$

b. $\emptyset \in \{\emptyset, \{\emptyset\}\}$

True, because $\emptyset$ and $\{\emptyset\}$ are both the elements of set $\{\emptyset, \{\emptyset\}\}$.

c. $\{\emptyset\} \in \{\emptyset\}$

False, because only $\emptyset$ is the element of set $\{\emptyset\}$, $\{\emptyset\}$ is not

d. $\{\emptyset\} \in \{\{\emptyset\}\}$

True, because $\{\emptyset\}$ is the element of set $\{\{\emptyset\}\}$.

e. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

True, because $\emptyset$ and $\{\emptyset\}$ are both the elements of set $\{\emptyset, \{\emptyset\}\}$.
Thus, $\{\emptyset\}$ is the proper subset of $\{\emptyset, \{\emptyset\}\}$.

f. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

True, because $\{\emptyset\}$ is the element of set $\{\emptyset, \{\emptyset\}\}$. Thus, $\{\{\emptyset\}\}$ is the proper subset of $\{\emptyset, \{\emptyset\}\}$

g. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

False, because only $\{\emptyset\}$ and $\{\emptyset\}$ are the elements of set $\{\{\emptyset\}, \{\emptyset\}\}$, and $\{\emptyset\} = \{\emptyset\}$. Thus $\{\emptyset\}$ is the only element of the set $\{\emptyset, \{\emptyset\}\}$, like the set $\{\{\emptyset\}\}$. Therefore $\{\{\emptyset\}\}$ should be $=$ or $\subseteq \{\{\emptyset\}, \{\emptyset\}\}$, but not $\subset$.

4. Determine whether these statement are true or false:

a. $x \in \{x\}$

True, because x is the element of set $\{x\}$.

b. $\{x\} \subseteq \{x\}$

True, because $\{x\} = \{x\}$.

c. $\{x\} \in \{x\}$

False, because $\{x\}$ is the subset of $\{x\}$ not an element, and they are also equal.

d. $\{x\} \in \{\{x\}\}$.

True, because $\{x\}$ is the element of set $\{\{x\}\}$

e. $\emptyset \subseteq \{x\}$

True, because the empty set is a subset of any other set

f. $\emptyset \in \{x\}$

False, because $\emptyset$ is not an element, and x is the only element of the set $\{x\}$.

5. Find two sets A and B such that $A \in B$ and $A \subseteq B$

$A = \emptyset$, $B = \{\emptyset\}$

Because $\emptyset \in \{\emptyset\}$, and $\emptyset$ is a subset of any other set.

6. What is the cardinality of each of these sets?

a. $\emptyset$ 0

b. $\{\emptyset\}$ 1

c. $\{\emptyset, \{\emptyset\}\}$. 2

d. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. 3

7. Find the power set of each of these sets, where a and b are distinct elements

a. $\{a\}$. $\{\emptyset, \{a\}\}$

b. $\{a, b\}$. $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

c. $\{\emptyset, \{\emptyset\}\}$. $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

2 Part II

1. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

Assume $a \in A$, $b \in B$.

$\because A \subseteq B$ and $a \in A$

$\therefore a \in B$

$\because B \subseteq C$ and $a \in B$

$\therefore a \in C$ $\quad \therefore A \subseteq C$.

2. Prove that $P(A) \subseteq P(B)$ if and only if $A \subseteq B$.

Assume $a \in A$

if $A \subseteq B$, then $a \in B$, $\{a\} \in P(B)$

$P(A) \subseteq P(B)$

if $A \nsubseteq B$, then $a \notin B$, $\{a\} \notin P(B)$

$P(A) \nsubseteq P(B)$

3. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

Assume $a \in A$, $b \in B$

$\because A \subseteq C$ and $a \in A$

$\therefore a \in C$

$\because B \subseteq D$ and $b \in B$

$\therefore b \in D$

$A \times B = ab$

$\because a \in C$, $b \in D$

$\therefore ab \in C \times D$

$\therefore A \times B \subseteq C \times D$

4. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$

method ①: The cartesian products of the sets is the set of all ordered pairs. For whether the first or the second set is

empty set, there is no ordered pairs. Therefore,
$\emptyset \times A = A \times \emptyset = \emptyset$.

②: If $A=\emptyset$ or $B=\emptyset$, then $|A\times B| = |A| \times |B| = 0$.

5. This exercise presents Russell's paradox. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$

a). Show the assumption that S is a member of S leads to a contradiction

Suppose S is a member of S
$\forall x$ in S, x is a member of x
$\therefore x \in x$
$\therefore$ lead to a contradiction

b). Show the assumption that S is not a member of S leads to a contradiction

$\because$ S is not a member of S
$\therefore S \notin S$
but a set stands for the elements present in it
$\therefore S \in S$
$\therefore$ lead to a contradiction

By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have

3. Part III

1. Let A and B be sets. Show that

a. $(A \cap B) \subseteq A$

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

$\forall x (x \in A) \rightarrow$ for $A \cap B$

$\therefore (A \cap B) \subseteq A$

b. $A \subseteq (A \cup B)$

$A \cup B = \{x \mid x \in A \vee x \in B\}$

$A = \{x \mid x \in A\}$.

$\because \{x \mid x \in A\} \subseteq \{x \mid x \in A \vee x \in B\}$.

$\therefore A \subseteq (A \cup B)$

c. $A - B \subseteq A$

$A - B = \{x \mid x \in A \wedge x \notin B\}$.

$\because \{x \mid x \in A \wedge x \notin B\} \subseteq \{x \mid x \in A\}$

$\therefore A - B \subseteq A$

d. $A \cap (B - A) = \emptyset$

$B - A = \{x \mid x \in B \wedge x \notin A\}$.

$\nexists x (x \in A) \rightarrow$ for $B - A$.

$\therefore \nexists x (x \in A \wedge x \in B - A)$.

$\therefore A \cap (B - A) = \emptyset$

e. $A \cup (B - A) = A \cup B$

$B - A = \{x \mid x \in B \wedge x \notin A\}$

$A = \{x \mid x \in A\}$.

$\therefore A \cup (B - A) = \{x \mid x \in B \wedge x \notin A \vee x \in A\}. = \{x \mid x \in B \vee x \in A\}$.

$\because A \cup B = \{x \mid x \in A \vee x \in B\}$.

$\therefore A \cup (B - A) = A \cup B$.

2. Show that if A and B are sets with $A \subseteq B$, then

a. $A \cup B = B$

$A \cup B = \{x \mid x \in A \vee x \in B\}$

$\because A \subseteq B$

$\therefore A \cup B = \{x \mid x \in B\}$.

$\because B = \{x \mid x \in B\}$.

$\therefore A \cup B = B$.

b. $A \cap B = A$

$A \cap B = \{x \mid x \in A \wedge x \in B\}$.

$\because A \subseteq B$

$\therefore B - A \nsubseteq \{x \mid x \in A \wedge x \in B\}$.

$B - A \nsubseteq A \cap B$

$\therefore A \cap B = A$

3. Can you conclude that $A = B$ if A, B, and C are sets such that (prove it or find a counterexample)

a. $A \cup C = B \cup C$? No

if $A \subseteq C$ and $B \subseteq C$,

then $A \cup C = C$

$B \cup C = C$

$A \cup C = B \cup C$

b. $A \cap C = B \cap C$? No

if $C \subseteq A$ and $C \subseteq B$,

then $A \cap C = C$

$B \cap C = C$

$A \cap C = B \cap C$.

c. $A \cup C = B \cup C$ and $A \cap C = B \cap C$? Yes

Assume $x \in A$

if $x \in C$, then $x \in A \cap C$

$\because A \cap C = B \cap C$

$\therefore x \in B \cap C \quad \therefore x \in B$

if $x \notin C$, then $x \in A \cup C$

$\because A \cup C = B \cup C$

$\therefore x \in B \cup C \quad \therefore x \in B$

$\therefore A \subseteq B$

Assume $x \in B$

if $x \in C$, then $x \in B \cap C$

$\because B \cap C = A \cap C$

$\therefore x \in A \cap C \quad \therefore x \in A$

if $x \notin C$, then $x \in B \cup C$

$\because B \cup C = A \cup C$

$\therefore x \in A \cup C \quad \therefore x \in A$

$\therefore B \subseteq A$

$\because A \subseteq B \land B \subseteq A$

$\therefore A = B$.

4. Show that if A and B are sets, then

a. $A \oplus B = B \oplus A$

$A \oplus B = (A-B) \cup (B-A)$

$B \oplus A = (B-A) \cup (A-B)$

$\therefore A \oplus B = B \oplus A$.

b. $(A \oplus B) \oplus B = A$

According to associative laws, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

$\therefore (A \oplus B) \oplus B = A \oplus (B \oplus B) = A \oplus \emptyset = A$

$\therefore (A \oplus B) \oplus B = A$