ANNOUNCEMENTS

- Homework 4 shorter, is out.
  - Due date: June 5 (midnight) on Canvas
- Quiz 8: 5/28
  - Logicomix: 6. Incompleteness, and material from lectures on 5/21 and 5/26
- Exam 2: June 2. Canvas quiz during class time. Covering all month of May.
  - Logical Reasoning, Proofs, Functions, Cardinality of infinite sets, Relations
  - Short review today, but mostly topics from homework, quizzes, and lectures.
- Final classes:
  - 5/28: Proofs by induction
  - 6/4: Turing Machines
  - Short Review of material
- Final Exam: June 9 12-1:30pm. All topics from the class.

MATHEMATICAL INDUCTION

MOTIVATION: PROVING PROPERTIES ABOUT FORMULAS

- *Is it true that for all positive integers* $n < 2^n$
  - It works for 1
    - 1<2
  - It works for 2
    - 2<4
  - Is it true for the rest of integers?
- Is it true that for all sets S with n elements, S has 2^n subsets.
  - It works for |S|=1
    - Let the element be r
    - The subsets are the empty set, and {r}
  - It works for |S|=2
  - Is it true for all sizes "n"

INDUCTION

- Mathematical induction is the application of an inference rule that can be used to prove that formulas like the ones in the previous slide are true for all integers.

- Rule of inference:
- Assume the domain of k and n is the set of all positive integers
- $(P(1) \wedge \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$,
  - Or "equivalently"
- Theorem (Mathematical Induction)
  - $P(1)\,,\ \forall k\,(P(k) \rightarrow P(k+1)) \vdash \forall n\, P(n)$,

PROOF OF THE PREVIOUS THEOREM

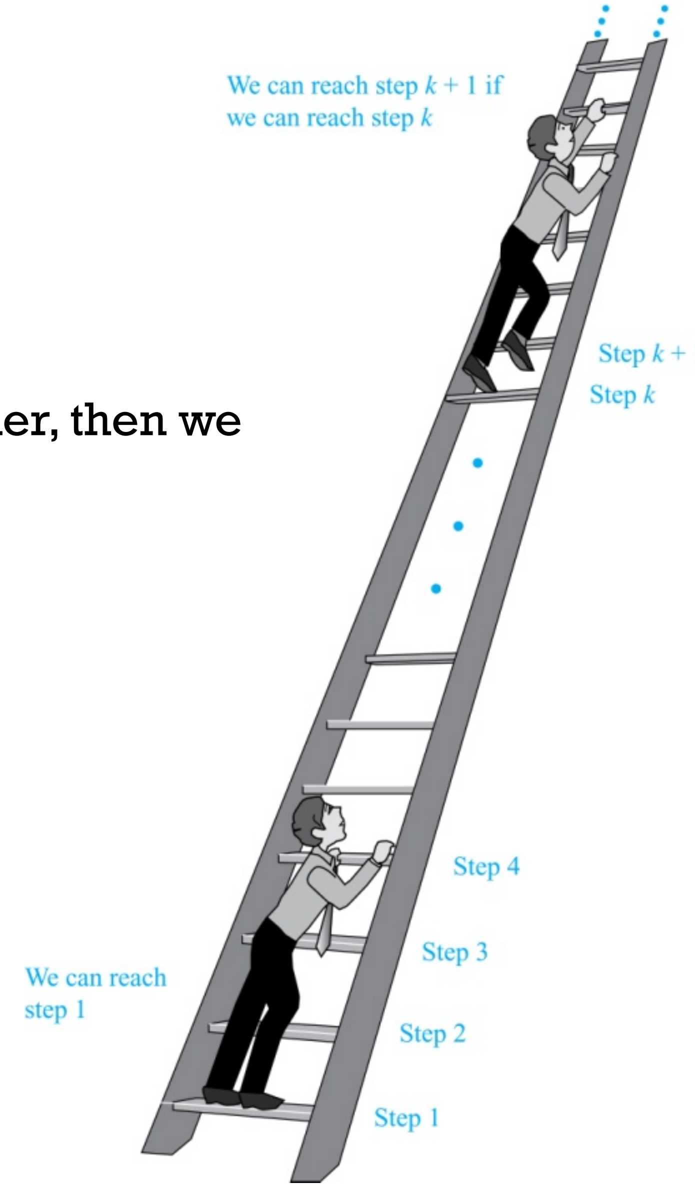
- Suppose that $P(1)$ holds and $P(k) \rightarrow P(k+1)$ is true for all positive integers k.
- Assume by contradiction there is at least one positive integer n for which P(n) is false. Then the set S of positive integers for which P(n) is false is nonempty.
- By the well-ordering property, S has a least element, say m.
- We know that m can not be 1 since $P(1)$ holds.
- Since m is positive and greater than 1, $m - 1$ must be a positive integer. Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true.
- But then, since the conditional $P(k) \rightarrow P(k+1)$ for every positive integer k holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
- Hence, $P(n)$ must be true for every positive integer n.

ZyBooks 7.4.1

CLIMBING AN INFINITE LADDER

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.



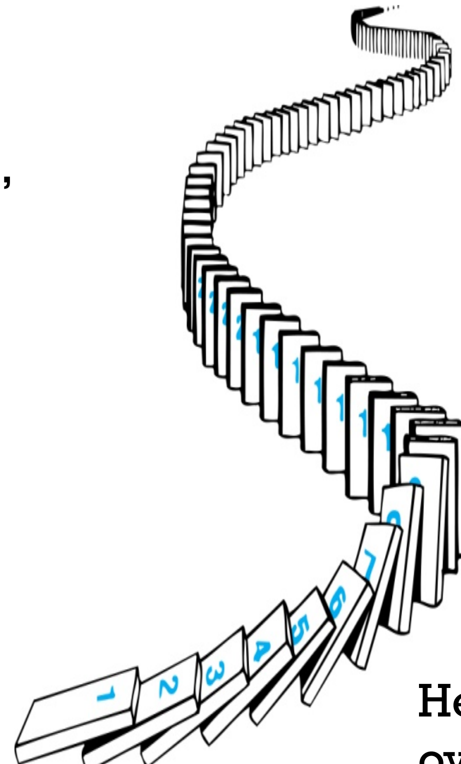
From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.

REMEMBERING HOW MATHEMATICAL INDUCTION WORKS

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

PRINCIPLE OF MATHEMATICAL INDUCTION

Principle of Mathematical Induction:

To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that must $P(k+1)$ be true.

Climbing an Infinite Ladder Example:

- BASIS STEP: By (1), we can reach rung 1.
- INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k+1$.

Hence, $P(k) \rightarrow P(k+1)$ is true for all positive integers k. We can reach every rung on the ladder.

GENERAL INDUCTION

- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point *b* where *b* is an integer.

- The rule of inference assumes the domain of k and n is the set of all integers greater than or equal to b
- $P(\mathrm{b})\ ,\ \forall k\,(P(k) \rightarrow P(k+1)) \vdash \forall n\, P(n),$

CLARIFICATION

- In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers!
- We show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.

PROVING A SUMMATION FORMULA BY MATHEMATICAL INDUCTION

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

Example: Show that: $$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Solution:

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$. Premise
- INDUCTIVE STEP: Assume true for $P(k)$.

The inductive hypothesis is $$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$ Premise

Under this assumption,

$$\sum_{j=1}^{k+1} j = 1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

By Premise inductive hypothesis

$$= \frac{k(k+1) + 2(k+1)}{2}$$

By algebra

$$= \frac{(k+1)(k+2)}{2}$$

See if conclusion is true

CONJECTURING AND PROVING A FORMULA

Example: Conjecture and prove correct a formula for the sum of the first n positive odd integers. Then prove your conjecture.

Solution: We have $1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$, $1 + 3 + 5 + 7 + 9 = 25$.

- We can conjecture that the sum of the first n positive odd integers is n^2,

$$\boxed{1 + 3 + 5 + \cdots + (2n - 1) = n^2}$$

- We prove the conjecture is proved correct with mathematical induction.
- BASIS STEP: $P(1)$ is true since $1^2 = 1$.
- INDUCTIVE STEP: $P(k) \rightarrow P(k + 1)$ for every positive integer k.

Assume the inductive hypothesis holds and then show that $P(k + 1)$ holds has well.

$$\boxed{\textbf{Inductive Hypothesis: } 1 + 3 + 5 + \cdots + (2k - 1) = k^2}$$

Premise

- So, assuming $P(k)$, it follows that:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) \quad (\textit{by the inductive hypothesis}) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

See if conclusion is true

By Premise

- Hence, we have shown that $P(k + 1)$ follows from $P(k)$. Therefore the sum of the first n positive odd integers is n^2.

PROVING INEQUALITIES

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

- BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.
- Must show that $P(k + 1)$ holds. Since by the inductive hypothesis,
- $k < 2^k$, it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Algebra on the Premise
Slightly different from the previous proofs

Therefore $n < 2^n$ holds for all positive integers n.

PROVING INEQUALITIES

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

- BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&< 2 \cdot k! \qquad (\textit{by the inductive hypothesis}) \\
&< (k + 1)k! \qquad \text{Because k > 3} \\
&= (k + 1)!
\end{aligned}
$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

STRONG INDUCTION

- *Strong Induction*: To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:
  - *Basis Step*: Verify that the proposition $P(1)$ is true.
  - *Inductive Step*: Show the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ holds for all positive integers k.

Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

STRONG INDUCTION AND THE INFINITE LADDER

Strong induction tells us that we can reach all rungs if:

1. We can reach the first rung of the ladder.
2. For every integer k, if we can reach the first k rungs, then we can reach the $(k + 1)$st rung.

To conclude that we can reach every rung by strong induction:

- BASIS STEP: $P(1)$ holds
- INDUCTIVE STEP: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for an arbitrary integer k, and show that $P(k + 1)$ must also hold.

We will have then shown by strong induction that for every positive integer n, $P(n)$ holds, i.e., we can reach the nth rung of the ladder.