ANNOUNCEMENTS

- Homework 4 shorter, will be out tonight or tomorrow:
  - Due date: June 5 (midnight) on Canvas
- Quiz 8: 5/28
  - Logicomix: 6. Incompleteness, and material from lectures on 5/21 and 5/26
- Exam 2: June 2. Canvas quiz during class time. Covering all month of May.
  - Logical Reasoning, Proofs, Functions, Cardinality of infinite sets,
    - Relations / Inductions ? (maybe)
  - Short review next class, but mostly topics from homework, quizzes, and lectures.
- Final classes: 5/28 and 6/4
  - Short Review of material
  - Proofs by Induction
  - Computation (High-level ideas; Turing Machines, Impossible Problems)
- Final Exam: June 9 12-1:30pm. All topics from the class.

RECALL MOTIVATING QUESTION

- What is the cardinality of the natural numbers?
- What is the cardinality of the integers?
- What is the cardinality of the real numbers?
- Is any of them larger than the others?

- Poll 1: is the sent of integers larger than the set of natural numbers?
  - 62% said yes! The real answer to this question is NO!
- Poll 2: is the set of real numbers larger than the set of integers?
  - 69% said yes! The real answer to this question is YES!

THE SET OF RATIONAL NUMBERS HAS SAME CARDINALITY AS THE INTEGERS?

- We need to find a bijection between the natural numbers (or the positive integers) and the rational numbers.
- The above is equivalent as saying that we only need to find a sequence that we know will include all the rational numbers
- $r_1, r_2, r_3, \ldots$
- Why?
  - Because that is a bijection between the positive integers and the rational numbers, where
    - f(i) = r_i
- *We can also have a sequence that looks like a matrix:*
  - $r_{11}, r_{12}, r_{13}, \ldots\ r_{21}, r_{22}, r_{23}, \ldots\ r_{31}, r_{32}, \ldots$

THE RATIONAL NUMBERS ARE COUNTABLE

- **Definition**: A *rational number* can be expressed as the ratio of two integers *p* and *q* such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

The rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done for the positive rational numbers.

You can find the proof for all the rational numbers in "Book of Proof" Third Edition, Theorem 14.4

THE POSITIVE RATIONAL NUMBERS ARE COUNTABLE

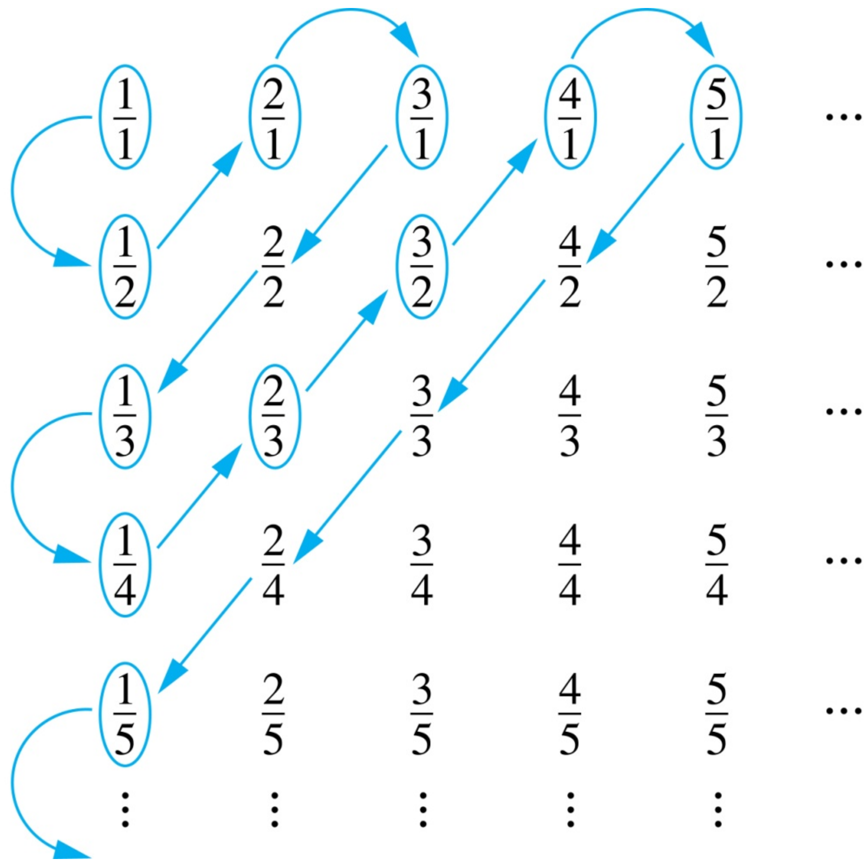
Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

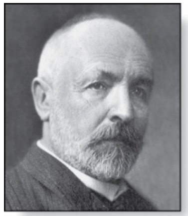
First row $q = 1$.
Second row $q = 2$.
etc.

Terms not circled are not listed because they repeat previously listed terms



1, ½, 2, 3, 1/3,1/4, 2/3, ….

Georg Cantor
(1845-1918)



$$|\mathbb{R}| \neq \aleph_0 = |\mathbb{Z}| = |\mathbb{N}|$$

The set of real numbers is uncountable.

Solution: The method is called the Cantor diagonalization argument and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\vdots \end{aligned}$$

4. Form a new real number with the decimal expansion

$$r = .r_1r_2r_3r_4\ldots$$

where $r_i = 3 \text{ if } d_{ii} \neq 3$ and $r_i = 4 \text{ if } d_{ii} = 3$

5. r is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its ith position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

Pages 271-272 of "Book of Proof"

THE CONTINUUM HYPOTHESIS (P 289 BOOK OF PROOF, 3RD EDITION)

- Cantor proved that $|\mathbb{R}| \neq \aleph_0$
- In fact, the following two facts are also true:
  - $|\mathbb{R}| = \mathcal{P}(\mathbb{N})$
  - $\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \ldots.$
- Continuun hypothesis:
  - Is there a cardinality in between $|\mathbb{N}|$ and $|\mathcal{P}(\mathbb{N})|$?
  - Continuun hypothesis: $\aleph_1 = |\mathbb{R}|$
- Logicomix
  - One of Hilbert's Problems of his speech in Chapter 3
  - Gödel proved that there are statements that cannot be proven or disproven (Chapter 6: Incompleteness)
- Gödel and later Cohen proved that the continuum hypothesis cannot be proved

SUMMARY AND NEXT TOPICS ABOUT FUNCTIONS

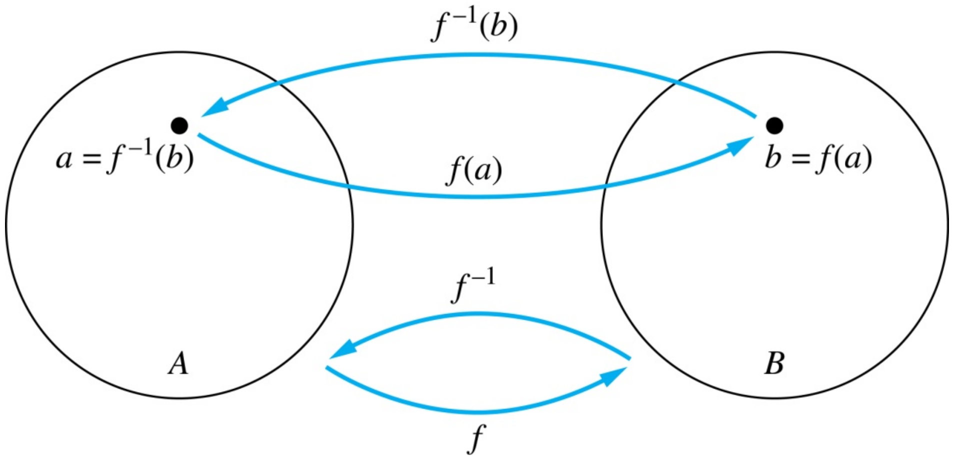
- We have seen the following definitions:
  - Function: a subset of AxB such that $\forall x \exists ! y (x, y) \in f$
  - One-to-one: $\forall a \forall b \left[f(a) = f(b) \rightarrow a = b \right]$
  - Onto: $\forall b \; \exists a \quad f(a) = b$
  - Bijection: both one-to-one and onto
  - Using the above definitions to order the cardinality of sets
    - Especially, infinite sets
- Now two new definitions:
  - Inverse
  - Composition

INVERSE FUNCTIONS

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as

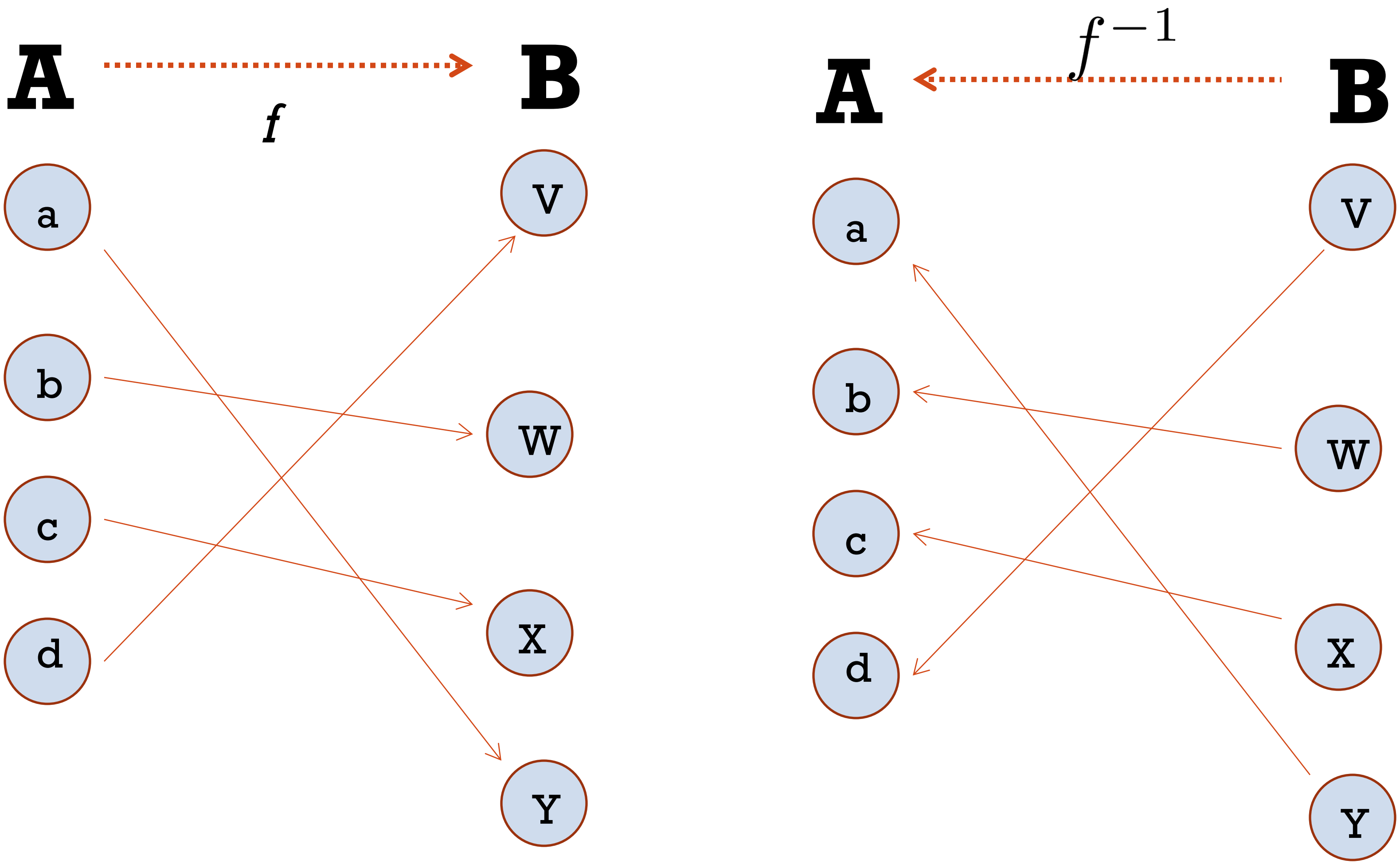
$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless f is a bijection. Why?



ZyBooks 5.4

INVERSE FUNCTIONS



QUESTIONS

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so, what is its inverse?

Solution: The function f is invertible because it is a bijection. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

QUESTIONS

Example 2: Let $f\colon \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a bijection. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

QUESTIONS

Example 3: Let *f*: **R** → **R** be such that $f(x) = x^2$.

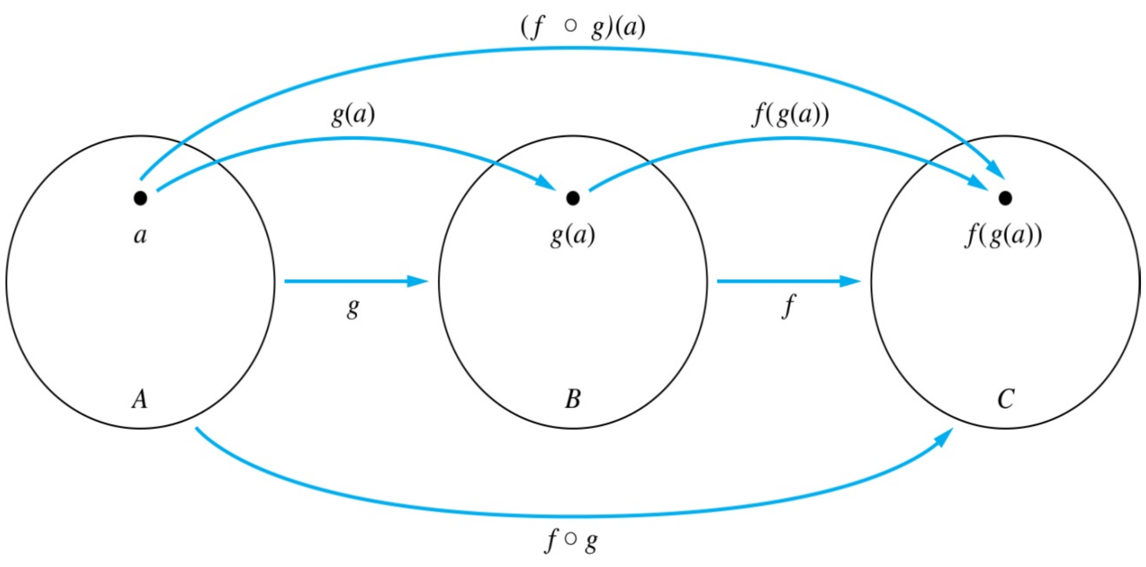
Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one .

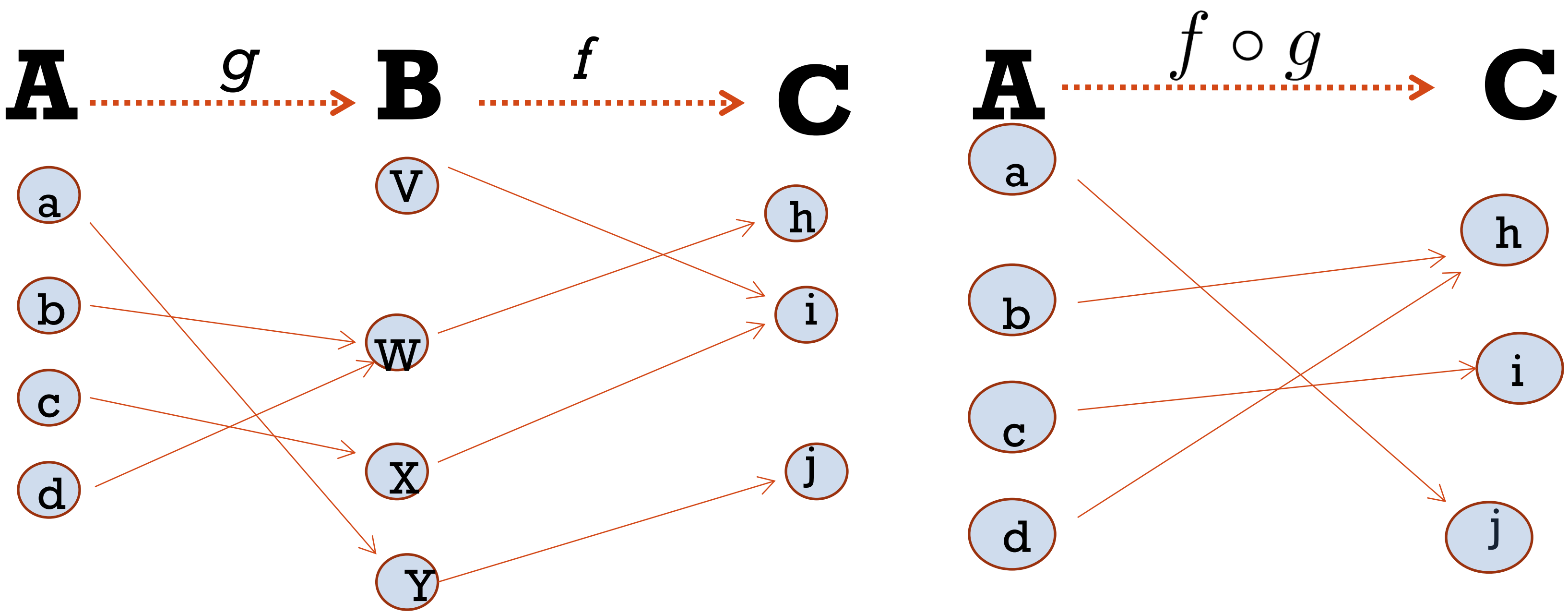
COMPOSITION

- **Definition**: Let *f*: *B* → *C*, *g*: *A* → *B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f(g(x))$$



COMPOSITION



ZyBooks 5.5

COMPOSITION IS NOT COMMUTATIVE

Example : If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

COMPOSITION QUESTIONS

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f{\circ}g$ is defined by

$f{\circ}g\ (a) = f(g(a)) = f(b) = 2.$

$f{\circ}g\ (b) = f(g(b)) = f(c) = 1.$

$f{\circ}g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g{\circ}f$ is not defined, because the range of f is not a subset of the domain of g.

COMPOSITION QUESTIONS

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of *f* and *g*, and the composition of *g* and *f* ?

Solution:

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

RELATIONS

BINARY RELATIONS

Examples: “>“, “=”, “≤”, “⊆”

Definition: A *binary relation R* from a set *A* to a set *B* is a subset

$R \subseteq A \times B.$

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a), (2, b)\}$ is a relation from *A* to *B*.

- Recall that a function is a subset of AxB defined by

$$\forall x \exists ! y (x, y) \in f$$

Relations are more general than functions. A function is a relation where exactly one element of *B* is related to each element of *A*.

RELATION DEFINITIONS

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \longrightarrow (x,x) \in R]$$

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y\ [(x,y) \in R \longrightarrow (y,x) \in R]$$

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y\ [(x,y) \in R \land (y,x) \in R \longrightarrow x = y]$$

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y\ \forall z[(x,y) \in R \land (y,z) \in \mathrm{R} \longrightarrow (x,z) \in R\ ]$$

ZyBooks 6.1 & 6.2