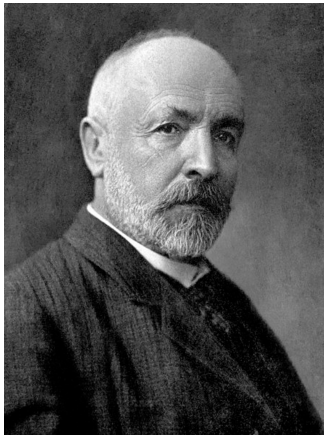
# DEFINITIONS: SETS AND ELEMENTS



- A *set* is an **unordered** collection of objects.
  - Naïve set theory:
  - Georg Cantor (1845-1918) father of set theory
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that $a$ is an element of the set $A$.
- If $a$ is not a member of $A$, write $a \notin A$
- Uppercase letters used for sets
  - Lowercase letters for elements
  - ZyBook Participation Activity 1.1.1

# SOME IMPORTANT SETS

- $\mathbb{N}$ = natural numbers = { 0, 1, 2, 3, 4, … }
- $\mathbb{Z}$ = Integers= { …,-4,-3,-2,-1,0, 1, 2, 3, 4, … }
- $\mathbb{Z}^{+}$ = Positive Integers = {1,2,3,4,…}
- $\mathbb{Q}$ = rational numbers = $\{p/q : p \in \mathbb{Z}, q \in \mathbb{Z}^{+}\}$
- $\mathbb{R}$ = real numbers = any decimal number of arbitrary precision
- Irrational numbers = $\{ x: x \in \mathbb{R}$ and $x \notin \mathbb{Q} \}$
  - (e.g., pi, e,)
- $\emptyset = \{\}$ is the empty set

# INTERVAL NOTATION

$$[a,b] = \{x \mid a \le x \le b\}$$

$$[a,b) = \{x \mid a \le x < b\}$$

$$(a,b] = \{x \mid a < x \le b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

*closed interval* [a,b]

*open interval* (a,b)

(caution: $\infty$ can never have a square bracket)

# SUBSETS

**Definition**: The set $A$ is a *subset* of $B$, if and only if every element of $A$ is also an element of $B$.

- The notation $A \subseteq B$ is used to indicate that $A$ is a subset of the set $B$.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
- ZyBook Exercises 1.1

# SHOWING A SET IS OR IS NOT A SUBSET OF ANOTHER SET

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if $x$ belongs to $A$, then **x** also belongs to $B$:

$$\forall x(x \in A \rightarrow x \in B)$$

- **Showing that A is not a Subset of B**: $A \nsubseteq B$

$$\neg\forall x(x \in A \rightarrow x \in B)$$

$$\exists x\neg(x \in A \rightarrow x \in B)$$

$$\exists x(x \in A \wedge x \notin B)$$

- To show that $A$ is not a subset of $B$, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an $x$ is a counterexample to the claim that $x \in A$ implies $x \in B$.)

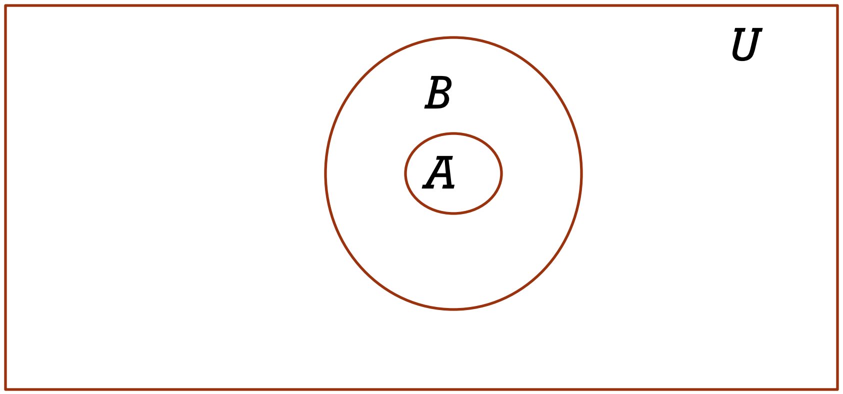
# PROPER SUBSETS

**Definition**: If $A \subseteq B$, but $A \neq B$, then we say $A$ is a *proper subset* of $B$, denoted by $A \subset B$. i.e.,:

$A \subset B$ ***iff***

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

Venn Diagram



# SET EQUALITY

**Definition**: Two sets are *equal* **if and only if (iff)** they have the same elements; (Definitions are "iff" even if they only say "if")

If they have the same elements, then they are equal,

If they are equal, then they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if
  - i.e., $$\forall x(x \in A \leftrightarrow x \in B)$$

$$\forall x[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

  - **We write *A* = *B* if *A* and *B* are equal sets.**

$\{1,3,5\} = \{3, 5, 1\}$

$\{1,5,5,5,3,3,1\} = \{1,3,5\}$

# ANOTHER LOOK AT EQUALITY OF SETS

- Recall that two sets $A$ and $B$ are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \to x \in B) \land (x \in B \to x \in A)]$$

- This is equivalent to

$A \subseteq B$ and $B \subseteq A$

# POWER SETS

**Definition**: The set of all subsets of a set $A$, denoted $\mathcal{P}(A)$, is called the *power set* of $A$.

**Example**: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has $n$ elements, then the cardinality of the power set is $2^n$.

- ZyBook Exercises 1.2

# SUMMARY OF SET OPERATIONS

| Operation | Notation | Description |
|---|---|---|
| Intersection | $A \cap B$ | $\{ x : x \in A \text{ and } x \in B \}$ |
| Union | $A \cup B$ | $\{ x : x \in A \text{ or } x \in B \text{ or both} \}$ |
| Difference | $A - B$ | $\{ x : x \in A \text{ and } x \notin B \}$ |
| Symmetric difference | $A \oplus B$ | $\{ x : x \in A - B \text{ or } x \in B - A \}$ |
| Complement | $\overline{A}$ | $\{ x : x \notin A \}$ |

ZyBook Exercises 1.5

# SET IDENTITIES (ZYBOOK EXERCISE 1.6)

| Name | Identities | |
|---|---|---|
| Idempotent laws | $A \cup A = A$ | $A \cap A = A$ |
| Associative laws | $(A \cup B) \cup C = A \cup (B \cup C)$ | $(A \cap B) \cap C = A \cap (B \cap C)$ |
| Commutative laws | $A \cup B = B \cup A$ | $A \cap B = B \cap A$ |
| Distributive laws | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ |
| Identity laws | $A \cup \varnothing = A$ | $A \cap U = A$ |
| Domination laws | $A \cap \varnothing = \varnothing$ | $A \cup U = U$ |
| Double Complement law | $\overline{\overline{A}} = A$ | |
| Complement laws | $A \cap \overline{A} = \varnothing$<br>$\overline{U} = \varnothing$ | $A \cup \overline{A} = U$<br>$\overline{\varnothing} = U$ |
| De Morgan's laws | $\overline{A \cup B} = \overline{A} \cap \overline{B}$ | $\overline{A \cap B} = \overline{A} \cup \overline{B}$ |
| Absorption laws | $A \cup (A \cap B) = A$ | $A \cap (A \cup B) = A$ |

# *N*-TUPLES

( 11, 12 ) ***ordered pair***

(  ,  ,  ) a 3-tuple

(  ,  ,  , 11, Leo ) a 5-tuple

- As opposed to sets, repetition and ordering do matter with *n*-tuples.
  - (11, 11, 11, 12, 13) ≠ ( 11, 12, 13 )
  - (  ,  ,  ) ≠ (  ,  ,  )

# CARTESIAN PRODUCT

**Definition**: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered $n$-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where $a_i$ belongs to $A_i$ for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

**Example**: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

**Solution:** $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

**ZyBook Exercises 1.3 (1.3.1-1.3.3)**

# Set Partitions

Two sets, A and B, are said to be ***disjoint*** if their intersection is empty ($A \cap B = \varnothing$). A sequence of sets, $A_1, A_2, \ldots, A_n$, is ***pairwise disjoint*** if every pair of distinct sets in the sequence is disjoint (i.e., $A_i \cap A_j = \varnothing$ for any i and j in the range from 1 through n where $i \neq j$).

A ***partition*** of a non-empty set A is a collection of non-empty subsets of A such that each element of A is in exactly one of the subsets. $A_1, A_2, \ldots, A_n$ is a partition for a non-empty set A if all of the following conditions hold:

- For all i, $A_i \subseteq A$.
- For all i, $A_i \neq \varnothing$
- $A_1, A_2, \ldots, A_n$ are pairwise disjoint.
- $A = A_1 \cup A_2 \cup \ldots \cup A_n$

# PROPOSITIONAL LOGIC

- Constructing Propositions
  - Propositional Variables: $p$, $q$, $r$, $s$, …
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

# Different Ways of Expressing

$$P \rightarrow Q$$

- if $p$, then $q$
- if $p$, $q$
- q **unless** $\neg p$
- $q$ **if** $p$
- $q$ **whenever** $p$
- $q$ **follows from** $p$
- $p$ **implies** $q$
- $p$ **only if** $q$
- $q$ **when** $p$
- $p$ **is sufficient for** $q$
- $q$ **is necessary for** $p$
- **a necessary condition for** $p$ **is** $q$
- **a sufficient condition for** $q$ **is** $p$
- *See also page 44 of Book of Proof and Table 2.3.2 in ZyBook*

ZyBook Challenge Activity 2.3.1

# EQUIVALENT STATEMENTS FOR $P \rightarrow Q$

- If $x > 2$ then $x^2>4$
- $x^2>4$ if $x >2$
- $x > 2$ only if $x^2>4$
- $x > 2$ is sufficient for $x^2>4$
- $x^2>4$ is necessary for $x>2$

- You will find any of the above statements in many math books when introducing theorems
  - You need to memorize the above equivalent statements to understand what the theorem is saying.

# PROVING $P \to Q$ IS FALSE: $P \wedge \neg Q$

- Sometimes you want to prove a $p \to q$ assertion is false
- Recall this is a promise that if p is true, then q is true
- So I need to find an example where p is true and q is false:
  - $p \wedge \neg q$
- If $x^2>4$ then $x>2$?
  - False!
    - Consider $x = -3$.
    - $x^2=9$ so p is true, and $x<2$ so q is false.

# CONVERSE AND INVERSE OF $P \to Q$

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg\, p \to \neg\, q$ is the **inverse** of $p \to q$

**Example**: Find the converse, and inverse of "if you are a bird, then you have wings."

**Solution:**

**converse**: If you have wings, then you are a bird

**inverse**: If you are not a bird, then you do not have wings.

- They are not equivalent to $p \to q$
  - If you are a bird you have wings, (assume this is true)
    - If you have wings then you are a bird (not necessarily true)
    - If you are not a bird, then you do not have wings (not necessarily true)

# CONVERSE, CONTRAPOSITIVE, INVERSE

| Proposition: | $p \rightarrow q$ | Ex: If it is raining today, the game will be cancelled. |
|---|---|---|
| **Converse:** | $q \rightarrow p$ | If the game is cancelled, it is raining today. |
| **Contrapositive:** | $\neg q \rightarrow \neg p$ | If the game is not cancelled, then it is not raining today. |
| **Inverse:** | $\neg p \rightarrow \neg q$ | If it is not raining today, the game will not be cancelled. |

Only the contrapositive is equivalent to the original statement

# $\neg Q \rightarrow \neg P$ IS THE SAME AS $P \rightarrow Q$

- Two propositions are ***equivalent*** if they always have the same truth value.
- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.

  **Solution:**

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | F | T | F | F |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

You can also show the equivalence using "proofs": ZyBook 2.5

# BICONDITIONAL

- If $p$ and $q$ are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "$p$ **if and only if** $q$." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

- If $p$ denotes "I am at home." and $q$ denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

# EXPRESSING THE BICONDITIONAL

- Some alternative ways "$p$ if and only if $q$" is expressed in English:

  - **$p$ is necessary and sufficient for $q$**
  - **if $p$ then $q$ , and conversely**
  - **$p$ iff $q$**

# LOGICALLY EQUIVALENT

- Two compound propositions p and q are logically equivalent iff $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where $p$ and $q$ are compound propositions.
- Two compound propositions $p$ and $q$ are equivalent if and only if the columns in a truth table giving their truth values agree.
- T

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \rightarrow q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

# LAWS OF PROPOSITIONAL LOGIC

| | | |
|---|---|---|
| Idempotent laws: | $p \vee p \equiv p$ | $p \wedge p \equiv p$ |
| Associative laws: | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ |
| Commutative laws: | $p \vee q \equiv q \vee p$ | $p \wedge q \equiv q \wedge p$ |
| Distributive laws: | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ |
| Identity laws: | $p \vee F \equiv p$ | $p \wedge T \equiv p$ |
| Domination laws: | $p \wedge F \equiv F$ | $p \vee T \equiv T$ |
| Double negation law: | $\neg\neg p \equiv p$ | |
| Complement laws: | $p \wedge \neg p \equiv F$<br>$\neg T \equiv F$ | $p \vee \neg p \equiv T$<br>$\neg F \equiv T$ |
| De Morgan's laws: | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ |
| Absorption laws: | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| Conditional identities: | $p \rightarrow q \equiv \neg p \vee q$ | $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ |

# UNIVERSAL QUANTIFIER $\forall$

- $\forall x\, P(x)$ is read as "For all $x$, P($x$)" or "For every $x$, P($x$)"

  **Examples**:

  1) If $P(x)$ denotes "$x > 0$" and $U$ is the integers, then $\forall x\, P(x)$ is false.
  2) If $P(x)$ denotes "$x > 0$" and $U$ is the positive integers, then $\forall x\, P(x)$ is true.
  3) If $P(x)$ denotes "$x$ is even" and $U$ is the integers, then $\forall x\, P(x)$ is false.

# EXISTENTIAL QUANTIFIER $\exists$

- $\exists x\, P(x)$ is read as "For some $x$, P($x$)", or as "There is an $x$ such that P($x$)," or "For at least one $x$, P($x$)."

**Examples:**

1. If $P(x)$ denotes "$x > 0$" and $U$ is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and $U$ is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "$x$ is even" and $U$ is the integers, then $\exists x\, P(x)$ is true.

# UNIQUENESS QUANTIFIER $\exists!$

- $\exists! x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> $x$ in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique $x$ such that $P(x)$."
  - "There is one and only one $x$ such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique $x$ such that $P(x)$ can be expressed as:

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x))$$

Table 2.8.1: Summary of De Morgan's laws for quantified statements.

| | | |
|---|---|---|
| $\neg\forall x\, P(x)$ | $\equiv$ | $\exists x\, \neg P(x)$ |
| $\neg\exists x\, P(x)$ | $\equiv$ | $\forall x\, \neg P(x)$ |

# QUANTIFICATIONS OF TWO VARIABLES

| Statement | When True? | When False |
|---|---|---|
| $\forall x \forall y P(x,y)$<br>$\forall y \forall x P(x,y)$ | *P*(*x*,*y*) is true for every pair *x*,*y*. | There is a pair *x*, *y* for which *P*(*x*,*y*) is false. |
| $\forall x \exists y P(x,y)$ | For every *x* there is a *y* for which *P*(*x*,*y*) is true. | There is an x such that *P*(*x*,*y*) is false for every *y*. |
| $\exists x \forall y P(x,y)$ | There is an *x* for which *P*(*x*,*y*) is true for every *y*. | For every *x* there is a y for which *P*(x,y) is false. |
| $\exists x \exists y P(x,y)$<br>$\exists y \exists x P(x,y)$ | There is a pair *x*, *y* for which *P*(*x*,*y*) is true. | *P*(x,y) is false for every pair *x*,*y* |

ZyBook 2.9.1-2.9.4

Table 2.9.2: De Morgan's laws for nested quantified statements.

| |
|---|
| $\neg\forall x\, \forall y\, P(x, y) \equiv \exists x\, \exists y\, \neg P(x, y)$ |
| $\neg\forall x\, \exists y\, P(x, y) \equiv \exists x\, \forall y\, \neg P(x, y)$ |
| $\neg\exists x\, \forall y\, P(x, y) \equiv \forall x\, \exists y\, \neg P(x, y)$ |
| $\neg\exists x\, \exists y\, P(x, y) \equiv \forall x\, \forall y\, \neg P(x, y)$ |

# THEOREMS

- A theorem is any inference obtained from:
  - a set of premises: $\Gamma$
  - axioms or previously proved Theorems and
  - the rules of inference
- **Notation (Hilbert system):** **(ZyBook Notation):**

$$\Gamma \vdash \alpha$$

$$\vdash \alpha \qquad \text{if } \Gamma \text{ is empty}$$

$$\frac{\Gamma}{\alpha}$$

$\vdash$ is called the (single) "Turnstile" symbol

# EXAMPLE: HILBERT SYSTEM

- Small set of rules of inference and axioms from where all "theorems" can be derived
- Lukasiewicz Axioms and only Modus Ponens
- 3 fixed Axioms (all truth tables are Tautologies) doc cam:

$$\text{A1} \quad \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$\text{A2} \quad (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$$

$$\text{A3} \quad (\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha)$$

- Modus Ponens
  - $p, p \rightarrow q \vdash q$
- *Goal: Use only axioms and modus ponens (and previous results) to prove all theorems of propositional logic*

# IT IS VERY IMPORTANT TO SPELL OUT THE REASON FOR EACH STEP

**Theorem 2** $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$

**Proof**

| | |
|---|---|
| $1. \beta \to \gamma$ | $P$ |
| $2. (\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$ | $A1$ |
| $3. \alpha \to (\beta \to \gamma)$ | $MP1,2$ |
| $4. (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ | $A2$ |
| $5. (\alpha \to \beta) \to (\alpha \to \gamma)$ | $MP3,4$ |
| $6. \alpha \to \beta$ | $P$ |
| $7. \alpha \to \gamma$ | $MP5,6$ |

# PROOF BY CONTRADICTION

**Meta – Theorem** Let $\Gamma$ be a set of premises

$$\Gamma, \neg\alpha \vdash \beta, \neg\beta \implies \Gamma \vdash \alpha$$

**Meta – Corollary**

$$(a)\Gamma, \alpha \vdash \beta, \neg\beta \implies \Gamma \vdash \neg\alpha$$

$$(b)\Gamma, \neg\alpha \vdash \alpha \implies \Gamma \vdash \alpha$$

# USING PROOF BY CONTRADICTION

**Theorem 6** $\alpha \to \beta, \neg\beta \vdash \neg\alpha$( Modus Tollens)

**Proof** Assume $\alpha \to \beta, \neg\beta, \neg\neg\alpha$

$$1.\alpha \to \beta \qquad P$$

$$2.\neg\beta \qquad P$$

$$3.\neg\neg\alpha \qquad P$$

$$4.\alpha \qquad Thm4, 3$$

$$5.\beta \qquad MP1, 4$$

You get a contradiction with lines 2 and 5

**Axioms, Rules, and Theorems that can be used to justify Proofs**

**You only need them when a question has this symbol: $\vdash$**
**Also if the question has the symbol $\vdash$ you also do not need anything else from the other pages**

| | | |
|---|---|---|
| A1 | $\alpha \to (\beta \to \alpha)$ | |
| A2 | $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ | |
| A3 | $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$ | |
| A4 | $(\alpha \wedge \beta) \to \alpha$ | |
| A5 | $(\alpha \wedge \beta) \to \beta$ | |
| A6 | $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma))$ | |
| A7 | $\alpha \to (\alpha \vee \beta)$ | |
| A8 | $\beta \to (\alpha \vee \beta)$ | |
| A9 | $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$ | |
| Rule 1 | $\alpha \vdash \beta \to \alpha$ | |
| Rule 2 | $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ | |
| Rule 3 | $\neg\alpha \to \neg\beta \vdash \beta \to \alpha$ | Contrapositive |
| Rule 4 | $\alpha \wedge \beta \vdash \alpha$ | Simplification |
| Rule 5 | $\alpha \wedge \beta \vdash \beta$ | Simplification |
| Rule 6 | $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma))$ | |
| Rule 7 | $\alpha \vdash \alpha \vee \beta$ | Addition |
| Rule 8 | $\beta \vdash \alpha \vee \beta$ | Addition |
| Rule 9 | $\alpha \to \gamma, \beta \to \gamma \vdash (\alpha \vee \beta) \to \gamma$ | |
| Theorem 1 | $\vdash \alpha \to \alpha$ | |
| Theorem 2 | $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$ | Hypothetical Syllogism |
| Theorem 3 | $\neg\alpha \vdash \alpha \to \beta$ | |
| Theorem 4 | $\neg\neg\alpha \vdash \alpha$ | |
| Theorem 5 | $\alpha, \neg\alpha \vdash \beta$ | |
| Theorem 6 | $\alpha \to \beta, \neg\beta \vdash \neg\alpha$ | Moduls Tollens |
| Theorem 7 | $\alpha \to \beta, \neg\alpha \to \beta \vdash \beta$ | |
| Theorem 8 | $\alpha, \neg\beta \vdash \neg(\alpha \to \beta)$ | |
| Theorem 9 | $\alpha \to \gamma, \beta \to \gamma \vdash (\alpha \vee \beta) \to \gamma$ | Conjunction |
| Theorem 10 | $\alpha, \beta \vdash \alpha \wedge \beta$ | |
| Theorem 11 | $\alpha \vee \beta, \neg\alpha \vdash \beta$ | Disjunctive Syllogism |
| Theorem 12 | $\vdash \alpha \vee \neg\alpha$ | |
| Theorem 13 | $\neg\alpha, \neg\beta \vdash \neg(\alpha \vee \beta)$ | De Morgan's Law |
| Theorem 14 | $\neg\alpha \vdash \neg(\alpha \wedge \beta)$ | |
| Theorem 15 | $\neg\beta \vdash \neg(\alpha \wedge \beta)$ | |