CSE 16 Applied Discrete Mathematics

Homework 2

1. Truth values for conditional statements in English. (Zybooks 2.3.1)

Which of the following conditional statements are true and why?

(a) If February has 30 days, then 7 is an odd number.
(b) If January has 31 days, then 7 is an even number.
(c) If 7 is an odd number, then February does not have 30 days.
(d) If 7 is an even number, then January has exactly 28 days.

2. The inverse, converse, and contrapositive of conditional sentences in English. (Zybooks 2.3.2)

Give the inverse, contrapositive, and converse for each of the following statements:

(a) If she finished her homework, then she went to the party.
(b) If he trained for the race, then he finished the race.
(c) If the patient took the medicine, then she had side effects.
(d) If it was sunny, then the game was held.
(e) If it snowed last night, then school will be cancelled.

3. Truth values for the inverse, contrapositive, and converse of a conditional statement (Zybooks 2.3.3)

State the inverse, contrapositive, and converse of each conditional statement. Then indicate whether the inverse, contrapositive, and converse are true.

(a) If 3 is a prime number then 5 is an even number.
(b) If $7 < 5$, then $5 < 3$.
(c) If 5 is a negative number, then 3 is a positive number.

4. Truth tables for logical expressions with conditional operations. (Zybooks 2.3.4)

Give a truth table for each expression

(a) $(\neg p \wedge q) \rightarrow p$
(b) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
(c) $(p \vee q) \leftrightarrow (q \rightarrow \neg p)$
(d) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
(e) $(p \vee q) \leftrightarrow (q \wedge p)$

5. Expressing conditional statements in English using logic. (Zybooks 2.3.5)

Define the following propositions:

c: I will return to college.
j: I will get a job.

Translate the following English sentences into logical expressions using the definitions above:

(a) Not getting a job is a sufficient condition for me to return to college.
(b) If I return to college, then I won't get a job.
(c) I am not getting a job, but I am still not returning to college.
(d) I will return to college only if I won't get a job.
(e) There's no way I am returning to college.
(f) I will get a job and return to college.

6. Expressing English sentences in if-then form. (Zybooks 2.3.6)

Give an English sentence in the form "If...then...." that is equivalent to each sentence.

(a) Maintaining a B average is sufficient for Joe to be eligible for the honors program.
(b) Maintaining a B average is necessary for Joe to be eligible for the honors program.
(c) Rajiv can go on the roller coaster only if he is at least four feet tall.
(d) Rajiv can go on the roller coaster if he is at least four feet tall.

7. Expressing conditional statements in English using logic. (Zybooks 2.3.7)

Define the following propositions:

s: a person is a senior
y: a person is at least 17 years of age
p: a person is allowed to park in the school parking lot

Express each of the following English sentences with a logical expression:

(a) A person is allowed to park in the school parking lot only if they are a senior and at least seventeen years of age.
(b) A person can park in the school parking lot if they are a senior or at least seventeen years of age.
(c) Being 17 years of age is a necessary condition for being able to park in the school parking lot.
(d) A person can park in the school parking lot if and only if the person is a senior and at least 17 years of age.

(e) Being able to park in the school parking lot implies that the person is either a senior or at least 17 years old.

8. Translating logical expressions into English. (Zybooks 2.3.8)

Define the following propositions:

w: the roads were wet
a: there was an accident
h: traffic was heavy

Express each of the logical expressions as an English sentence:

(a) $w \rightarrow h$
(b) $w \wedge a$
(c) $\neg(a \wedge h)$
(d) $h \rightarrow (a \vee w)$
(e) $w \wedge \neg h$

9. Translating English propositions into logical expressions. (Zybooks 2.3.9)

Translating English propositions into logical expressions.

- y: the applicant is at least eighteen years old
- p: the applicant has parental permission
- c: the applicant can enroll in the course.

(a) The applicant is not eighteen years old but does have parental permission.
(b) If the applicant is at least eighteen years old or has parental permission, then the applicant can enroll in the course
(c) The applicant can enroll in the course only if the applicant has parental permission.
(d) Having parental permission is a necessary condition for enrolling in the course.

10. Determining if a truth value of a compound expression is known given a partial truth assignment. (Zybooks 2.3.10)

The variable p is true, q is false, and the truth value for variable r is unknown. Indicate whether the truth value of each logical expression is true, false, or unknown.

(a) $p \rightarrow (q \wedge r)$
(b) $(p \vee r) \rightarrow r$
(c) $(p \vee r) \leftrightarrow (q \wedge r)$

(d) $(p \wedge r) \leftrightarrow (q \wedge r)$
(e) $p \rightarrow (r \vee q)$
(f) $(p \wedge q) \rightarrow r$

11. Applying De Morgan's laws. (Zybooks 2.4.6)

Translate each English sentence into a logical expression using the propositional variables defined below. Then negate the entire logical expression using parentheses and the negation operation. Apply De Morgan's law to the resulting expression and translate the final logical expression back into English.

- p: the applicant has written permission from his parents
- e: the applicant is at least 18 years old
- s: the applicant is at least 16 years old

(a) The applicant has written permission from his parents and is at least 16 years old.
(b) The applicant has written permission from his parents or is at least 18 years old.

12. Label the steps in a proof of logical equivalence. (Zybooks 2.5.1)

Below are several proofs showing that two logical expressions are logically equivalent. Label the steps in each proof with the law used to obtain each proposition from the previous proposition. The first line in the proof does not have a label.

(a)

$(p \rightarrow q) \wedge (q \vee p)$
$(\neg p \vee q) \wedge (q \vee p)$
$(q \vee \neg p) \wedge (q \vee p)$
$q \vee (\neg p \wedge p)$
$q \vee (p \wedge \neg p)$
$q \vee F$
q

(b)

$(\neg p \vee q) \rightarrow (p \wedge q)$
$\neg(\neg p \vee q) \vee (p \wedge q)$
$(\neg\neg p \wedge \neg q) \vee (p \wedge q)$
$(p \wedge \neg q) \vee (p \wedge q)$
$p \wedge (\neg q \vee q)$
$p \wedge T$
p

(c)

$r \vee (\neg r \rightarrow p)$
$r \vee (\neg\neg r \vee p)$
$r \vee (r \vee p)$
$(r \vee r) \vee p$
$r \vee p$

13. Using the laws of logic to prove logical equivalence. (Zybooks 2.5.2)

Use the laws of propositional logic to prove the following:

(a) $\neg p \to \neg q \equiv q \to p$

(b) $p \wedge (\neg p \to q) \equiv p$

(c) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

(d) $\neg p \to (q \to r) \equiv q \to (p \vee r)$

(e) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

(f) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

(g) $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \equiv p \wedge \neg r$

(h) $p \leftrightarrow (p \wedge r) \equiv \neg p \vee r$

(i) $(p \wedge q) \to r \equiv (p \wedge \neg r) \to \neg q$

14. Logical relationships between the inverse, converse, and contrapositive. (Zybooks 2.5.4)

Use the laws of propositional logic to prove each of the following assertions. Start by defining a generic conditional statement $p \to q$, and then restate the assertion as the equivalence or non-equivalence of two propositions using p and q. Finally prove that the two propositions are equivalent or non-equivalent.

For example, the statement: "A conditional statement is not logically equivalent to its converse" is proven by showing that that $p \to q$ is not logically equivalent to $q \to p$.

(a) A conditional statement is not logically equivalent to its converse.

(b) A conditional statement is not logically equivalent to its inverse.

(c) A conditional statement is logically equivalent to its contrapositive.

(d) The converse and inverse of a conditional statement are logically equivalent.

15. Logical equivalence of two mathematical statements. (Zybooks 2.5.5)

Show that the two sentences below are logically equivalent. Express each pair of sentences using a logical expression. Then prove whether the two expressions are logically equivalent. Note: you can assume that x and y are real numbers, so if x is not irrational, then x is rational, and if x is not rational, then x is an irrational number.

- If x is a rational number and y is an irrational number then x-y is an irrational number.
- If x is a rational number and x-y is a rational number then y is a rational number.

16. Truth values for quantified statements about integers. (Zybooks 2.6.2)

In this problem, the domain is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

(a) $\exists x\ (x + x = 1)$

(b) $\exists x\ (x + 2 = 1)$

(c) $\forall x\ (x^2 - x \neq 1)$

(d) $\forall x\ (x^2 - x \neq 0)$

(e) $\forall x\ (x^2 > 0)$

(f) $\exists x\ (x^2 > 0)$

17. Translating mathematical statements in English into logical expressions. (Zybooks 2.6.3)
Consider the following statements in English. Write a logical expression with the same meaning. The domain of discourse is the set of all real numbers.

(a) There is a number whose cube is equal to 2.
(b) The square of every number is at least 0.
(c) There is a number that is equal to its square.
(d) Every number is less than or equal to its square.

18. Translating quantified statements from English to logic (Zybooks 2.7.4)
In the following question, the domain of discourse is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

S(x): x was sick yesterday
W(x): x went to work yesterday
V(x): x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

(a) At least one person was sick yesterday.
(b) Everyone was well and went to work yesterday.
(c) Everyone who was sick yesterday did not go to work.
(d) Yesterday someone was sick and went to work.
(e) Everyone who did not go to work yesterday was sick.
(f) Everyone who missed work was sick or on vacation (or both).
(g) Someone who missed work was neither sick nor on vacation.
(h) Each person missed work only if they were sick or on vacation (or both).
(i) Ingrid was sick yesterday but she went to work anyway.
(j) Someone other than Ingrid was sick yesterday. (Note for this question, you will need the expression (x≠Ingrid).)
(k) Everyone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression (x ≠ Ingrid).)

19. Translating quantified statements from logic to English. (Zybooks 2.7.6)
In the following question, the domain of discourse is the set of employees of a company. Define the following predicates:

A(x): x is on the board of directors. (Note: members of the board of directors are also employees.)
E(x): x earns more than $100,000
W(x): x works more than 60 hours per week

Translate the following logical expressions into English:

(a) $\forall x\, (A(x) \rightarrow E(x))$
(b) $\exists x\, (E(x) \land \neg W(x))$
(c) $\forall x\, (W(x) \rightarrow E(x))$
(d) $\exists x\, (\neg A(x) \land E(x))$
(e) $\forall x\, (E(x) \rightarrow (A(x) \lor W(x)))$

(f) $\exists x\, (A(x) \land \neg E(x) \land W(x))$

20. Applying De Morgan's law for quantified statements to logical expressions. (Zybooks 2.8.1)
Apply De Morgan's law to each expression to obtain an equivalent expression in which each negation sign applies directly to a predicate. (i.e., $\exists x\, (\neg P(x) \lor \neg Q(x))$ is an acceptable final answer, but not $\neg\exists x\, P(x)$ or $\exists x\, \neg(P(x) \land Q(x))$).

(a) $\neg\exists x\, P(x)$

(b) $\neg\exists x\, (P(x) \lor Q(x))$

(c) $\neg\forall x\, (P(x) \land Q(x))$

(d) $\neg\forall x\, (P(x) \land (Q(x) \lor R(x)))$

21. Applying De Morgan's law for quantified statements to English statements. (Zybooks 2.8.2)
In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo

D(x): x was given the medication

M(x): x had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.
Sample question: Some patient was given the placebo and the medication.

$\exists x\, (P(x) \land D(x))$

Negation: $\neg\exists x\, (P(x) \land D(x))$

Applying De Morgan's law: $\forall x\, (\neg P(x) \lor \neg D(x))$

English: Every patient was either not given the placebo or not given the medication (or both).

(a) Every patient was given the medication.

(b) Every patient was given the medication or the placebo or both.

(c) There is a patient who took the medication and had migraines.

(d) Every patient who took the placebo had migraines. (Hint: you will need to apply the conditional identity, $p \to q \equiv \neg p \lor q$.)

(e) There is a patient who had migraines and was given the placebo.

22. Applying De Morgan's law for quantified statements to English statements. (Zybooks 2.8.3)
In the following question, the domain of discourse is a set of students who show up for a test. Define the following predicates:

P(x): x showed up with a pencil

C(x): x showed up with a calculator

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.
Sample question: Every student showed up with a calculator.

$\forall x\, C(x)$

Negation: $\neg\forall x\, C(x)$

Applying De Morgan's law: $\exists x\, \neg C(x)$

English: Some student showed up without a calculator.

(a) At least one of the students showed up with a pencil.

(b) Every student showed up with a pencil or a calculator (or both).

(c) Every student who showed up with a calculator also had a pencil.

(d) There is a student who showed up with both a pencil and a calculator.

(e) Some student showed up with a pencil or a calculator.

(f) Every student showed up with a pencil and a calculator.

23. Using De Morgan's law for quantified statements to prove logical equivalence. (Zybooks 2.8.4)

Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a) $\neg\forall x\, (P(x) \land \neg Q(x)) \equiv \exists x\, (\neg P(x) \lor Q(x))$

(b) $\neg\forall x\, (\neg P(x) \to Q(x)) \equiv \exists x\, (\neg P(x) \land \neg Q(x))$

(c) $\neg\exists x\, (\neg P(x) \lor (Q(x) \land \neg R(x))) \equiv \forall x\, (P(x) \land (\neg Q(x) \lor R(x)))$

24. Truth values for mathematical expressions with nested quantifiers. (Zybooks 2.9.3)

Determine the truth value of each expression below. The domain is the set of all real numbers.

(a) $\forall x\exists y\, (xy > 0)$

(b) $\exists x\forall y\, (xy = 0)$

(c) $\forall x\forall y\exists z\, (z = (x - y)/3)$

(d) $\forall x\exists y\forall z\, (z = (x - y)/3)$

(e) $\forall x\forall y\, (xy = yx)$

(f) $\exists x\exists y\exists z\, (x^2 + y^2 = z^2)$

(g) $\forall x\exists y\, (y^2 = x)$

(h) $\forall x\exists y\, (x < 0 \lor y^2 = x)$

(i) $\exists x\, \exists y\, (x^2 = y^2 \land x \neq y)$

(j) $\exists x\, \exists y\, (x^2 = y^2 \land |x| \neq |y|)$

(k) $\forall x\, \forall y\, (x^2 \neq y^2 \lor |x| = |y|)$

25. De Morgan's law and nested quantifiers. (Zybooks 2.9.4)

Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositional logic.

(a) $\forall x\, \exists y\, \exists z\, P(y, x, z)$

(b) $\forall x\, \exists y\, (P(x, y) \land Q(x, y))$

(c) $\exists x\, \forall y\, (P(x, y) \to Q(x, y))$

(d) $\exists x\, \forall y\, (P(x, y) \leftrightarrow P(y, x))$

(e) $\exists x\, \exists y\, P(x, y) \land \forall x\, \forall y\, Q(x, y)$

26. Applying De Morgan's law to English statements with nested quantifiers. (Zybooks 2.9.5)

The domain for variables x and y is a group of people. The predicate F(x, y) is true if and only if x is a friend of y. For the purposes of this problem, assume that for any person x and person y, either x is a friend of y or x is an enemy of y. Therefore, ¬F(x, y) means that x is an enemy of y.

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

(a) Everyone is a friend of everyone.
(b) Someone is a friend of someone.
(c) Someone is a friend of everyone.
(d) Everyone is a friend of someone.

27. Truth values for mathematical statements with nested quantifiers. (Zybooks 2.10.4)

The domain for all variables in the expressions below is the set of real numbers. Determine whether each statement is true or false. Justify your answer.

(a) $\forall x \exists y (x + y = 0)$
(b) $\exists x \forall y (x + y = 0)$
(c) $\exists x \forall y (xy = y)$
(d) $\exists x \exists y ((x2 = y2) \wedge (x \neq y))$
(e) $\forall x \forall y \exists z (z = (x + y)/2)$
(f) $\forall x \exists y \forall z (z = (x + y)/2)$

28. Mathematical statements into logical statements with nested quantifiers. (Zybooks 2.10.4)

Translate each of the following English statements into logical expressions. The domain of discourse is the set of all real numbers.

(a) There are two numbers whose ratio is less than 1.
(b) The reciprocal of every positive number is also positive.
(c) There are two numbers whose sum is equal to their product.
(d) The ratio of every two positive numbers is also positive.
(e) The reciprocal of every positive number less than one is greater than one.
(f) There is no smallest number.
(g) Every number other than 0 has a multiplicative inverse.
(h) Every number other than 0 has a unique multiplicative inverse.