1. $a\to g,\ b\to g \vdash (a\vee b)\to g$

Proof:

1	$a\to g\to((b\to g)\to((a\vee b)\to g))$.	A9.
2	$a\to g$.	Premise.
3	$(b\to g)\to((a\vee b)\to g)$.	MP1,2.
4	$b\to g$.	Premise.
5	$(a\vee b)\to g$.	MP3,4

$\blacksquare$

2. $a, b \vdash a\wedge b$.

Proof:

1	$(a\to a)\to((a\to b)\to(a\to(a\wedge b)))$.	A6
2	$b\to(a\to b)$	A1
3	$a\to a$	Theorem1.
4	$(a\to a)\to((a\to b)\to(a\to(a\wedge b)))$.	Rule6.
5	$(a\to b)\to(a\to(a\wedge b))$.	MP3,4
6	b	Premise
7	$a\to b$	MP2,6
8	$a\to(a\wedge b)$	MP5,7
9.	a.	Premise
10.	$a\wedge b$	MP8,9.

$\blacksquare$

3. $a\vee b, \neg a \vdash b$

Proof:

1	$b\to b$	Theorem 1
2	$(b\to b)\to((a\to b)\to((b\vee a)\to b))$.	A9
3	$(a\to b)\to((b\vee a)\to b)$.	MP1,2
4	$\neg a$	Premise
5	$a\to b$	Theorem3,4

→ (next page)

6	$(a \vee b) \rightarrow b$	MP3,5
7	$a \vee b$	Premise
8	b	MP6,7

■

4. $\vdash \neg(\neg a \wedge a)$

Proof:

1	$(\neg a \wedge a) \rightarrow a$	A5
2	$\neg a \wedge a$	Premise
3	a	MP1,2
4	$\neg(\neg a \wedge a)$.	Theorem 14,3

■

5. $\neg a, \neg b \vdash \neg(a \vee b)$

Proof:

1	$\neg a$	Premise
2	$\neg b$	premise
3	$a \vee b$	premise
4	b	Theorem 11.1,3

■

6. $\neg a \vdash \neg(a \wedge b)$

Proof:

1	$\neg a$	Premise
2	$(a \wedge b) \rightarrow a$	A4
3	$a \wedge b$	Premise
4	a.	MP2,3.

■

7. $a \vee (a \wedge b) \vdash a$

Proof:

1	$\neg a$	premise
2	$a \vee (a \wedge b)$	premise
3	$\neg(a \wedge b)$.	Theorem 14,1
4.	$\neg(a \vee (a \wedge b))$	Theorem 13.1,3 ■

Proof:

8. a). $p \to q \wedge r$

$\neg r$

$\therefore \neg p$ valid.

p: I get a job.

q: I buy a new car

r: I buy a new house

1	$p \to q \wedge r$	Premise.
2	$\neg p \vee (q \wedge r)$	Conditional identity
3	$(\neg p \vee q) \wedge (\neg p \vee r)$	distributive law
4	$\neg p \vee r$	Rule 4 Simplification.
5	$\neg r$	Premise
6	$\neg p$	Theorem 11, 4, 5 Disjunctive Syllogism. ∎

proof:

d). $p \wedge r \to q$

$\neg q$

r

$\therefore \neg p.$ valid.

p: I buy a new car

q: I buy a new house.

r: I get a job.

1	$p \wedge r \to q$	Premise
2	$\neg(p \wedge r) \vee q$	conditional identity
3	$(\neg p \vee \neg r) \vee q$	De Morgan's Law.
4	$\neg q$	Premise
5	$\neg p \vee \neg r$	Theorem 11, 3, 4 Disjunctive Syllogism.
6	r	Premise
7.	$\neg p$	Theorem 11, 5, 6 Disjunctive Syllogism

∎

9. c). $\forall x(P(x) \wedge Q(x)) \vdash \forall x Q(x) \wedge \forall x P(x)$

Proof:

1	c is an arbitrary element of the domain.	element definition.
2	$P(c) \wedge Q(c)$	Premise
3	$Q(c)$	Rule 4 Simplification, 2
4	$\forall x Q(x)$	Universal generalization
5	$P(c)$	Rule 4 Simplification, 2
6	$\forall x P(x)$	universal generalization
7	$\forall x Q(x) \wedge \forall x P(x)$	Theorem 10, 3, 5 Conjunction

∎

10.e).

$\forall x((P(x) \vee Q(x)) \rightarrow \neg B(x))$.

Penelope is a student in the class

B(Penelope).

$\therefore \neg Q$(Penelope). valid.

P(x): x missed class.

Q(x): x got a detention.

B(x): x got an A.

Proof:

1	Penelope is a student in the class	Premise
2	$\forall x((P(x) \vee Q(x)) \rightarrow \neg B(x))$.	Premise
3.	(P(Penelope) $\vee$ Q(penelope)) $\rightarrow \neg$B(penelope).	UI 1, 2.
4	$\neg$(P(penelope) $\vee$ Q(penelope)) $\vee \neg$B(penelope)	conditional identity
5	($\neg$P(penelope) $\wedge \neg$Q(penelope)) $\vee \neg$B(penelope)	De Morgan's law.
6	B(Penelope)	Premise
7.	$\neg$P(Penelope) $\wedge \neg$Q(Penelope)	Theorem 11. 5, 6. Disjunctive Syllogism
8.	$\neg$Q(Penelope)	Rule 5, 7 Simplification

■

11. d) 3

m and n is two positive integers.

$m^2+n^2=3$.

$(m^2=1, n^2=2)$ or $(m^2=2, n^2=1)$.

n doesn't exist

e). 0.

x and y are two real numbers.

when $x = 0$.

$0 \cdot y = 0 \neq 1$

12. h).

Proof: x and y are two real numbers

Let $x + y = 2z$.

$$z = \frac{x+y}{2}.$$

Since the sum of two odd numbers is even number,

and the sum of two even numbers is also even number:

$(2k+1) + (2k+1) = 2(2k+1) = 4k+2 \rightarrow$ even number.

$2m + 2m = 4m \rightarrow$ even number.

$(k, m \in \mathbb{R})$

$x+y$ is always even number (is divisible by 2).

Therefore, there exists an real number z that is equal to $\frac{x+y}{2}$ for every pair of x and y.

∎

13.

Proof: Assume the two consecutive integers are x and $x+1$

We will show that the sum of the squares of these two integers is odd.

The sum of the squares of x and $x+1$ is:

$$x^2 + (x+1)^2 = x^2 + (x^2+2x+1) = 2x^2+2x+1 = 2(x^2+x)+1$$

Since x is a integers, x^2+x is also an integer.

Therefore, $2(x^2+x)+1$ is odd

∎

14.b).

The assumption about k and j are not stated, they are both integers. Should be: "Since, by assumption, n is an odd integer, then $n=2k+1$, for some integer k. Since, by assumption, m is an odd integer, then $m=2j+1$, for some integer j."

15.b).

Proof: Assume x is an odd integer and y is also an odd integer. We will show that the sum of x and y is an even integer. Since x is odd, $x=2k+1$ for some integer k. Since y is also odd, $y=2j+1$ for some integer j. Then:

$$\begin{aligned} x+y &= (2k+1)+(2j+1) \\ &= 2k+2j+2 \\ &= 2(k+j+1). \end{aligned}$$

Since k and j are both integers, then $k+j+1$ is also an integer. Since $x+y=2C$, where $C=k+j+1$ is an integer, then $x+y$ is even ■

Proof: d). Assume x is an odd integer and y is also an odd integer. We will show that the product of x and y is an odd integer. Since x is odd, $x=2k+1$ for some integer k. Since y is also odd, $y=2j+1$ for some integer j. Then:

$$xy=(2k+1)(2j+1)=4kj+2k+2j+1=2(2kj+k+j)+1$$

Since k and j are both integers, then $2kj+k+j$ is also an integer. Since $xy=2c+1$, where $c=2kj+k+j$ is an integer, then $x+y$ is odd. ■

Proof: f). Assume x is an even integer and y is an odd integer. We will show that $3x+2y$ is even.
Since x is even, $x=2k$ for some integer k. Since y is odd, $y=2j+1$ for some integer j. Then:

$$3x+2y=3(2k)+2(2j+1)$$
$$=6k+4j+2$$
$$=2(3k+2j+1)$$

Since k and j are both integers, then $3k+2j+1$ is also an integer. Since $3x+2y=2c$, where $c=3k+2j+1$ is an integer, then $3x+2y$ is even. ■

Proof: h). Assume x is an odd integer. We will show that $-x$ is also odd.
Since x is odd, $x=2k+1$ for some integer k. Then:

$$-x=-(2k+1)=-2k-1=2(-k)-1$$

Since k is integer, then $-k$ is also an integer.
Since $-x=2c-1$, where $c=-k$ is an integer, then $-x$ is odd. ■

Proof: i). Assume that x is an even number. We will show that $(-1)^x=1$
Since x is even, $x=2k$ for some integer k. Then:

$$(-1)^x=(-1)^{2k}=((-1)^2)^k=1^k$$

Since k is an integer, $1^k=1$. Therefore, $(-1)^x=1$.
■

Proof: 16.a). Assume x is an even integer, and y is also an even integer. We will show that $x+y$ is an even integer.
Since x is even, $x=2k$ for some integer k. Since y is also even, $y=2j$ for some integer j. Then:

→ (next page)

$$x+y=(2k)+(2j)$$
$$=2k+2j$$
$$=2(k+j)$$

Since k and j are both integers, then $k+j$ is also an integer. Since $x+y=2c$, where $c=k+j$ is an integer, then $x+y$ is even. So the statement is true. ■

b). The statement is false, because the sum of two odd integers can also be even. For example, $3+5=8$.

Proof: Assume x is an odd integer, and y is also an odd integer. We will show that $x+y$ is even.

Since x is odd, $x=2k+1$ for some integer k. Since y is also odd, $y=2j+1$ for some integer j. Then:

$$x+y=(2k+1)+(2j+1)$$
$$=2k+1+2j+1$$
$$=2k+2j+2$$
$$=2(k+j+1).$$

Since k and j are both integers, then $k+j+1$ is also an integer. Since $x+y=2c$, where $c=k+j+1$ is an integer, then $x+y$ is even, which is a counterexample of the statement. Therefore, the statement is false.

■

Proof: 17.a). Assume n is an even integer. We will show that n^2 is even.

Since n is even, $n = 2k$ for some integer k. Then:

$$n^2 = (2k)^2$$
$$= 4k^2$$
$$= 2(2k^2)$$

Since k is an integer, then $2k^2$ is also an integer.
Since $n^2 = 2c$, where $c = 2k^2$ is an integer, then n^2 is even ■

Proof: b). Assume n is an odd integer. We will show that n^3 is odd.

Since n is odd, $n = 2k+1$ for some integer k. Then:

$$n^3 = (2k+1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^3 + 6k^2 + 3k) + 1$$

Since k is integer, then $4k^3 + 6k^2 + 3k$ is also an integer.
Since $n^3 = 2c+1$, where $c = 4k^3 + 6k^2 + 3k$ is an integer, then n^3 is odd. ■

(proof by contrapositive).

Proof: 18.e). Assume m is an even integer, and n is also an even integer. We will show that $n^2 + m^2$ is even.

Since m is even, $m = 2k$ for some integer k. Since n is also even, $n = 2j$ for some integer j. Then:

$$n^2 + m^2 = (2j)^2 + (2k)^2$$
$$= 4j^2 + 4k^2$$
$$= 2(2j^2 + 2k^2).$$

Since k and j are both integers, then $2j^2 + 2k^2$ is also an integer. Since $n^2 + m^2 = 2c$, where $c = 2j^2 + 2k^2$ is an integer,

→ (next page)

then n^2+m^2 is even.

■

19. b).

Proof by contrapositive: Assume n is an even integer and show that n^2 is even.

Proof by contradiction: Assume n is an odd integer and n^2 is even. Prove a contradiction. One possible contradiction would be to show that n^2 must be odd, which contradicts the assumption that n^2 is even.

20. b). Proof:

Proof by contradiction. Suppose that the person bought less than two cups of coffee everyday and he/she at least buys 400 cups of coffee in a year. If the person bought less than two coffee per day, the most cups of coffee he/she bought in a day is one cup. There are 365 days in a year, then the total number of cups of coffee the person bought in a year is at most $1 \cdot 365 = 365$, which contradicts the fact that he/she buys at least 400 cups of coffee in a year. ■

g). Proof:

Assume n is a even integer and n^2 is not a multiple of 4. Since n is even, $n=2k$ for some integer k. Then:

$$n^2=(2k)^2=4k^2.$$

$$\frac{4k^2}{4}=k^2$$

→ (next page)

Since k is an integer, then k^2 is also an integer.
Since $n^2 = 4k^2$ and $\frac{4k^2}{4} = k^2$, where k^2 is a integer, then n^2 can be divided by 4. Therefore, n^2 is a multiple of 4, which contradicts with the assumption that n^2 is not a multiple of 4. ■

Proof: Proof by contradiction:

i). Assume the product of two positive real numbers, x and y, is larger than 400 and $0 \le x \le 20$; $0 \le y \le 20$. Then:

$$0 \cdot 0 \le xy \le 20 \cdot 20$$
$$0 \le xy \le 400$$

The product of x and y is less than 400, but not larger than 400, which contradicts with the assumption that the product of x and y is larger than 400.
Therefore, the assumption that there exists such integers x and y is false. ■

Proof: Proof by contradiction:

ii). Assume the sum of two positive real numbers, x and y, is larger than 400, and $0 \le x \le 200$, $0 \le y \le 200$. Then:

$$0 + 0 \le x + y \le 200 + 200$$
$$0 \le x + y \le 400$$

The sum of x and y is less than 400, but not larger than 400, which contradicts with the assumption that the sum of x and y is larger than 400.
Therefore, the assumption that there exists such integers x and y is false. ■

4.c). Proof:

Assume integers x and y are consecutive and $x<y$. We will show that they have opposite parity.

Since integers x and y are consecutive and $x<y$, $x=y-1$ or $y=x+1$

Case 1: x is an odd integer, then $x=2k+1$ for some integer k. Then: $y=x+1=(2k+1)+1=2k+2=2(k+1)$.

Since k is an integer, then $k+1$ is also an integer.

Since $y=2c$, where $c=k+1$ is an integer, then y is even.

Therefore, x and y have opposite parity.

case 2: x is an even integer, then $x=2j$ for some integer k. Then: $y=x+1=2j+1$

since j is and integer and $y=2j+1$, then y is odd.

Therefore, x and y have opposite parity.

∎