

Lecture 5

Frequency Domain Filtering (1)

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Outline

- Why Fourier Transform in DIP
- Introduction (recap) of Fourier Transform (CTFT, DTFT, DFT)
- Discrete Fourier Transform and its Inverse (1D & 2D)
- Sampling Theorem & Aliasing (1D & 2D)
- Properties of 2D DFT and IDFT
- 2D DFT implementation

Jean Baptiste Joseph Fourier (1768-1830)

□ Had a crazy idea (1807)

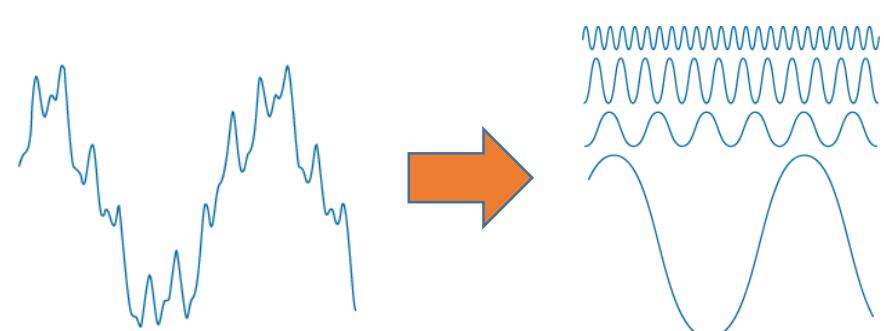
- Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies

□ Who didn't believe it?

- Lagrange, Laplace, Poisson...
- Not translated into English until 1878!

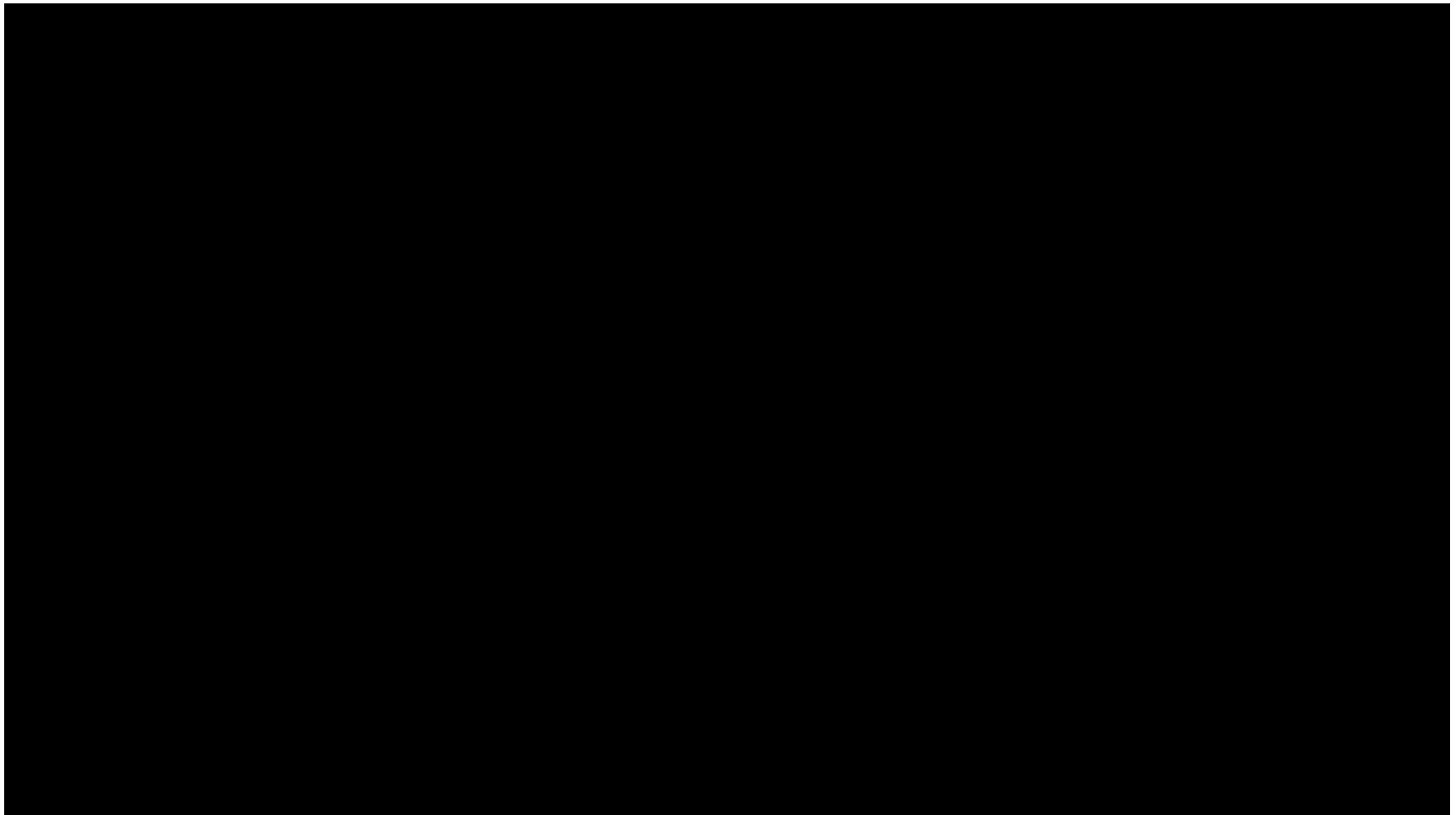
□ But it's true!

- We call it Fourier Series



Fourier Series

<https://www.youtube.com/watch?v=cUD1gMAl6W4>



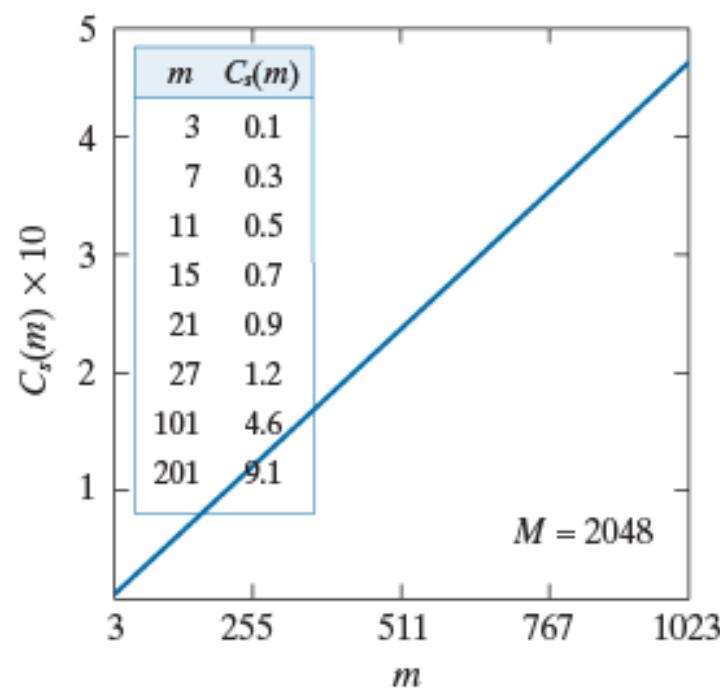
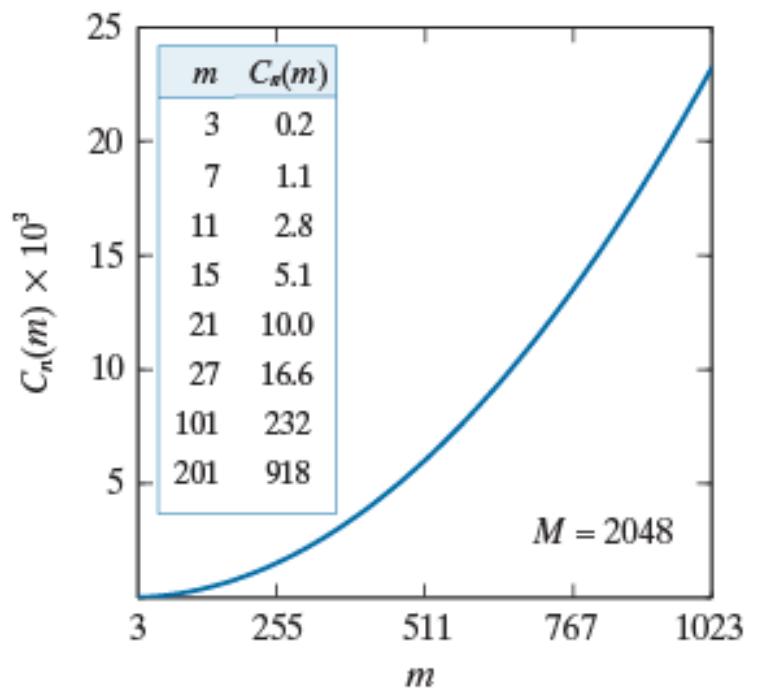
Why Fourier Transform in DIP

- Exposing image features not visible in spatial domain
- Image filtering in frequency domain (sometimes even faster!)
- Image compression or encoding
- Image reconstruction or recovery
- Classification
- ...
- *Or simply to provide another perspective to look at the image*



Why Fourier Transform in DIP

□ The importance of FFT



a b

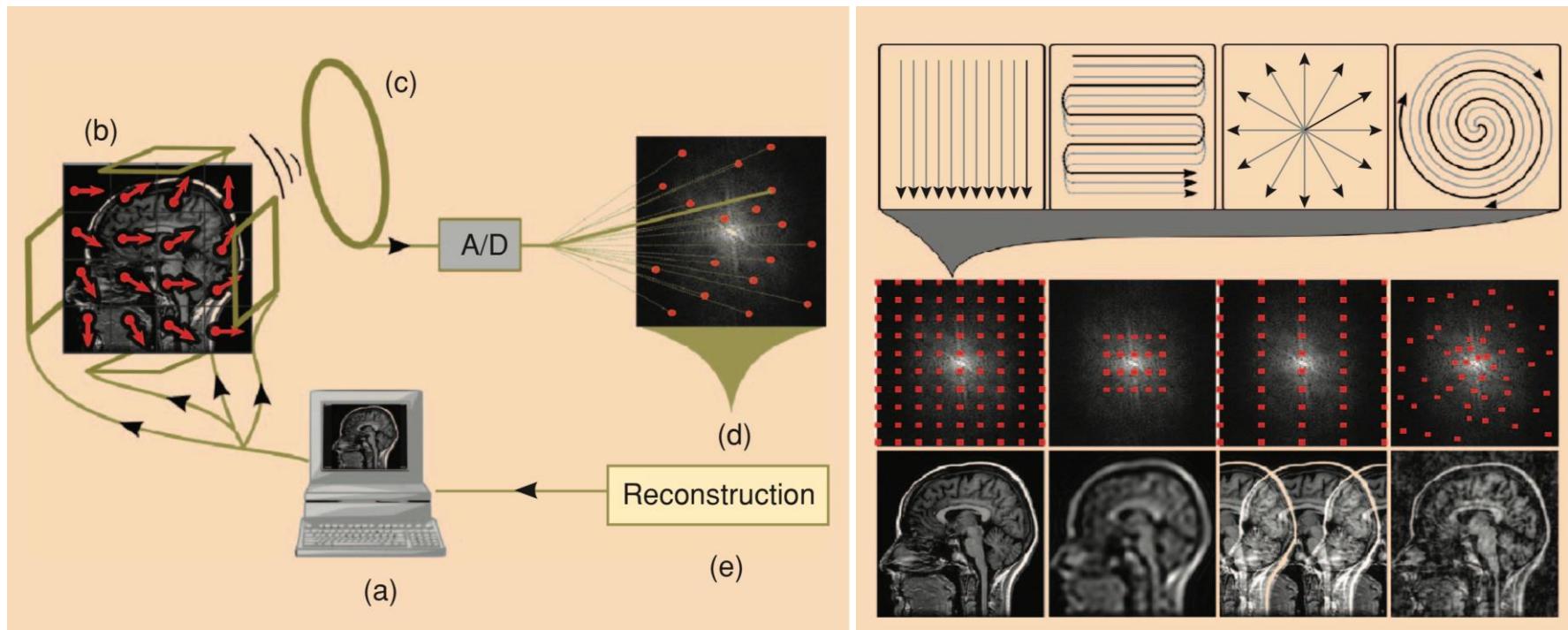
FIGURE 4.2

(a) Computational advantage of the FFT over non-separable spatial kernels.
(b) Advantage over separable kernels. The numbers for $C_s(m)$ in the inset tables are not to be multiplied by the factors of 10 shown for the curves.



Why Fourier Transform in DIP

□ A great example: Fourier Transform in MRI



The Nyquist criterion sets the required k-space coverage, which can be achieved using various sampling trajectories.

Introduction (recap) of Fourier Transform

□ Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)]$$

□ Continuous-time Fourier transform (CTFT)

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

□ Discrete-time Fourier Transform (DTFT)

$$F(\omega) = \sum_{n=1}^{\infty} x[n] e^{-j\omega n}, \omega \in [0, 2\pi], \omega = 2\pi/T$$

□ Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}, (k = 0, 1, 2, \dots, N-1)$$

Four Types of Fourier Transform

	Time Domain Non-Periodic	Time Domain Periodic
Frequency Domain Non-Periodic	CTFT (Both domains are continuous)	Fourier Series (Time domain continuous, Frequency domain discrete)
Frequency Domain Periodic	DTFT (Time domain discrete, Frequency domain continuous)	DFT (Both domains are discrete, and finite duration)

DFT and IDFT

DFT:
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$
$$= X(e^{j\omega}) \Big|_{\omega=2\pi k/N}, \quad k = 0, 1, \dots, N - 1$$

Using: $W_N = e^{-j2\pi/N}$, we can rewrite the DFT as

IDFT:
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad k = 0, 1, \dots, N - 1.$$

2D DFT and IDFT

➤ 2D Discrete Fourier Transform (DFT)

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

➤ 2D Inverse Discrete Fourier Transform (IDFT)

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

- $f(x, y)$: M*N input image
- (x, y) : spatial variables, ($x = 0, 1, 2, \dots, M - 1$; $y = 0, 1, 2, \dots, N - 1$)
- (u, v) : frequency variables, defines the continuous frequency domain ($u = 0, 1, 2, \dots, M - 1$; $v = 0, 1, 2, \dots, N - 1$).
- Examples of basis function.



Separability

2D DFT to 1D DFT

$$\begin{aligned} F(u, v) &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})} = \sum_{x=0}^{M-1} e^{-j2\pi\frac{ux}{M}} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\frac{vy}{N}} \\ &= \mathcal{F}_x\{\mathcal{F}_y\{f(x, y)\}\} \end{aligned}$$

Calculate IDFT by DFT

$$\begin{aligned} f(x, y) &= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(\frac{ux}{M} + \frac{vy}{N})} \\ MNf^*(x, y) &= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})} \end{aligned}$$



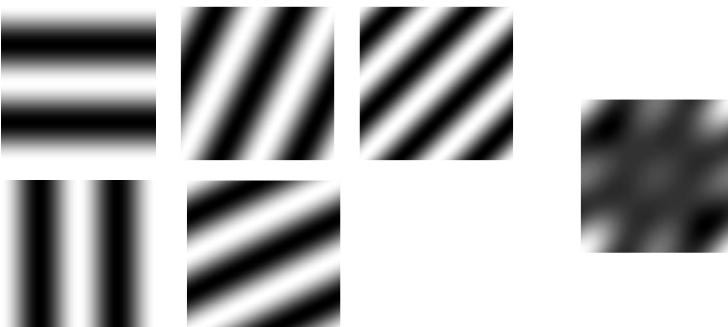
Basis Functions in 2D DFT

➤ [0,0]: Constant

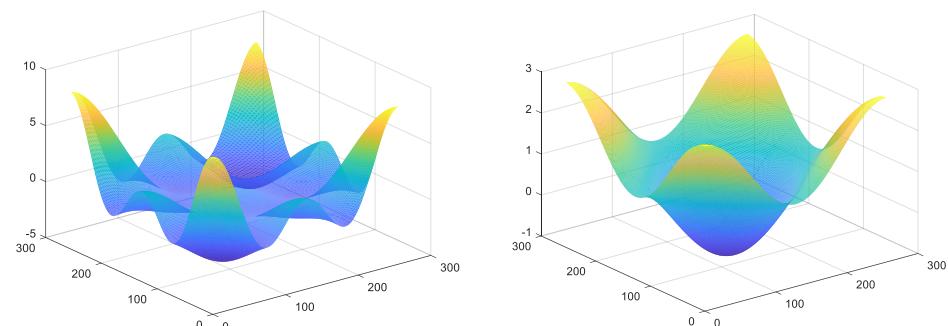
➤ [0,1] [1,0] [1,1]



➤ [0,2] [2,0] [1,2] [2,1] [2,2]



```
function im = bf(m, n)
N = 256;
[x, y] = meshgrid(0:(N-1), 0:(N-1));
im = real(exp(-j*2*pi*(m*x/N + n*y/N)));
if (m==0) && (n==0)
    im = round(im);
end
figure; imshow(im, []);
```



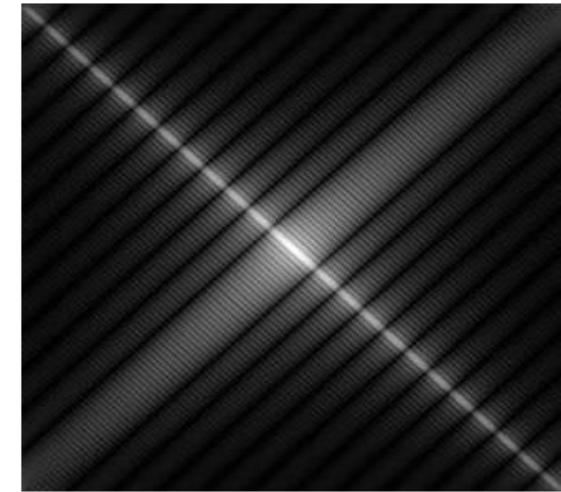
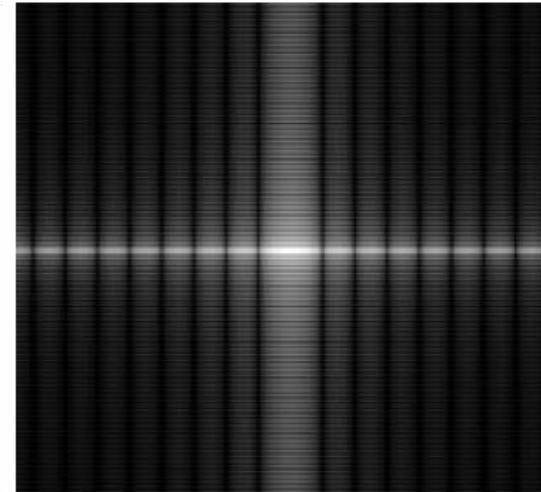
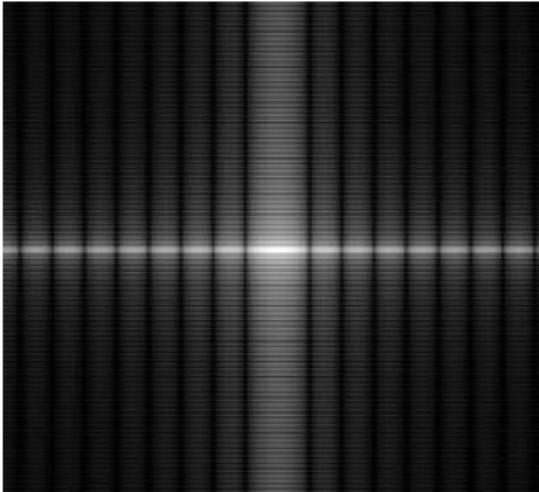
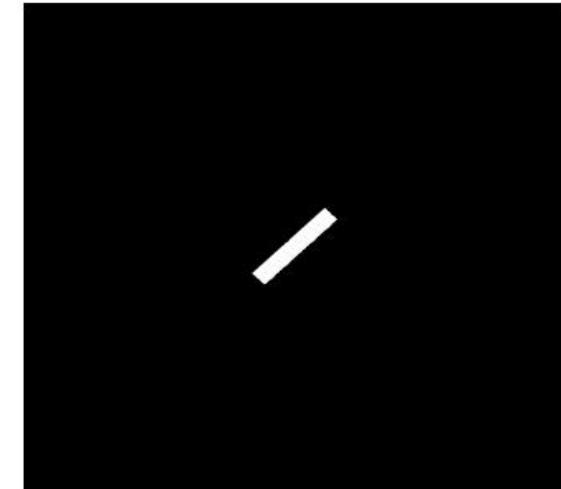
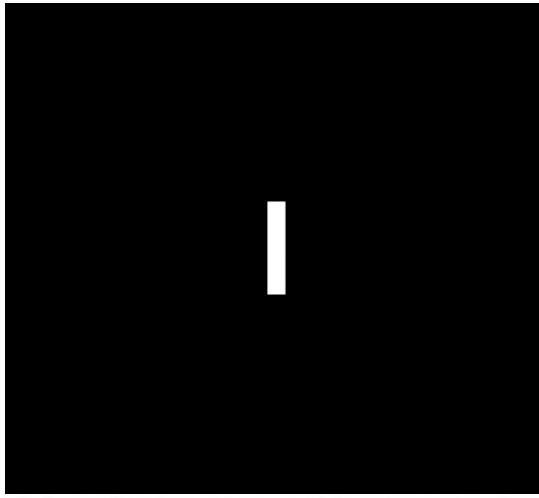
Spectrum and Phase angle

2D DFT in polar form: $F(u, v) = |F(u, v)|e^{-j\Phi(u, v)}$, then

- Fourier spectrum (频谱) : $|F(u, v)| = [R^2(u, v) + I^2(u, v)]^{\frac{1}{2}}$
- Phase angle (相角) : $\Phi(u, v) = \arctan \frac{I(u, v)}{R(u, v)}$
- Power spectrum(功率谱): $P(u, v) = |F(u, v)|^2$
- DC component(直流分量): $F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = MN \overline{f(x, y)}$

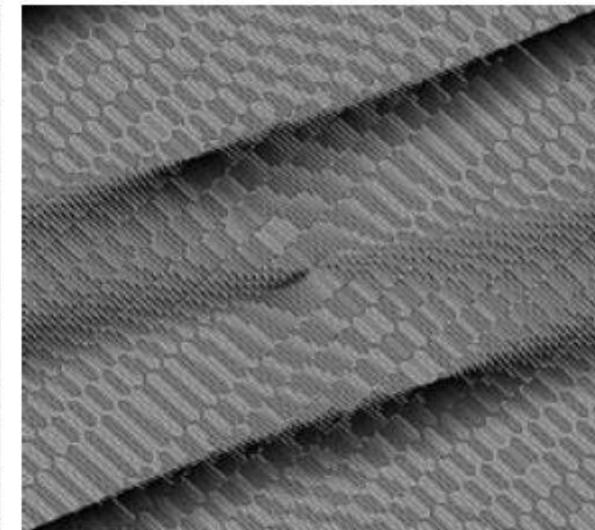
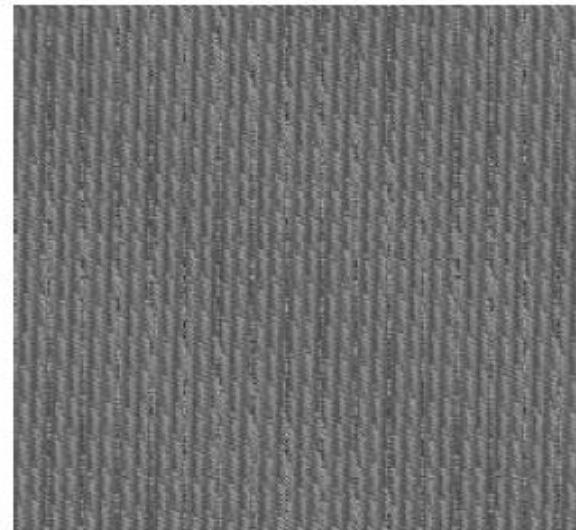
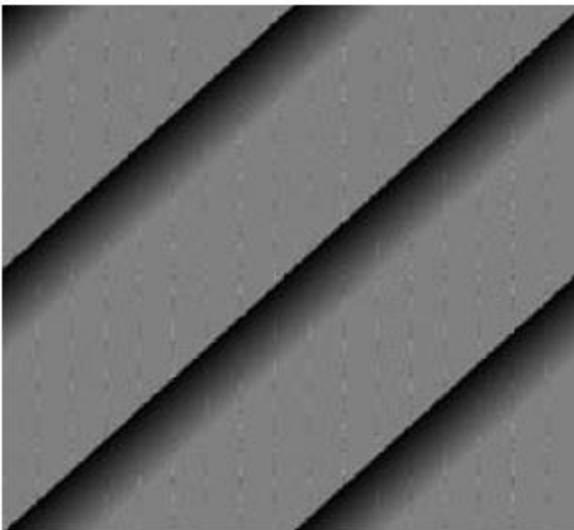
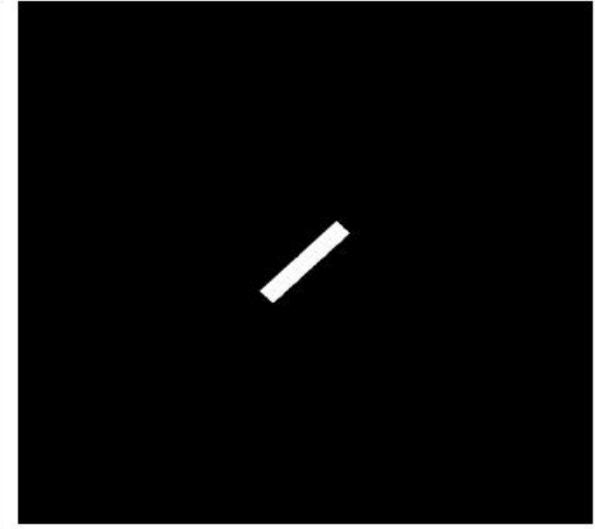
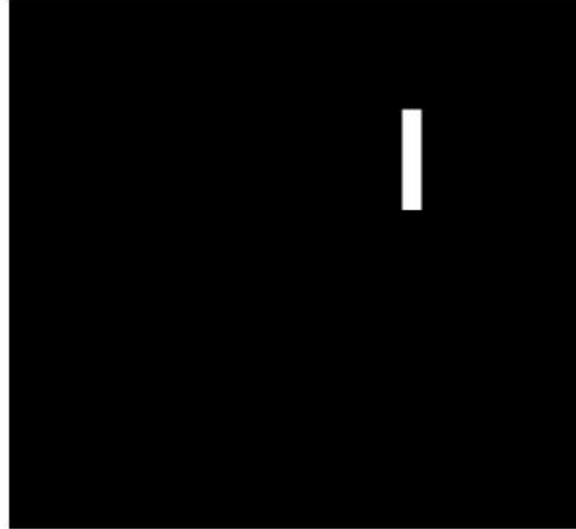
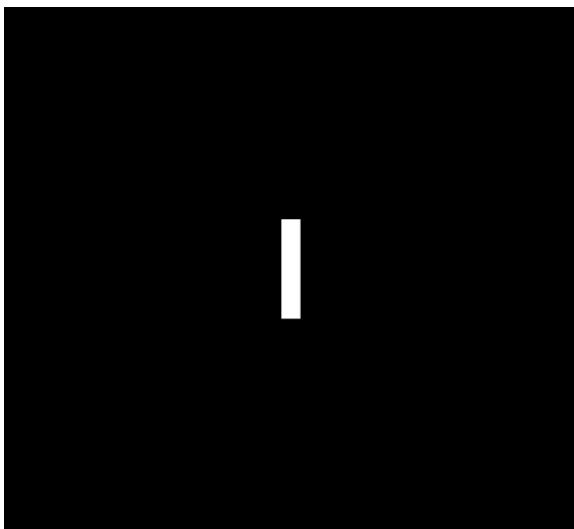


Fourier Spectrum (频谱)



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Phase angle (相角)

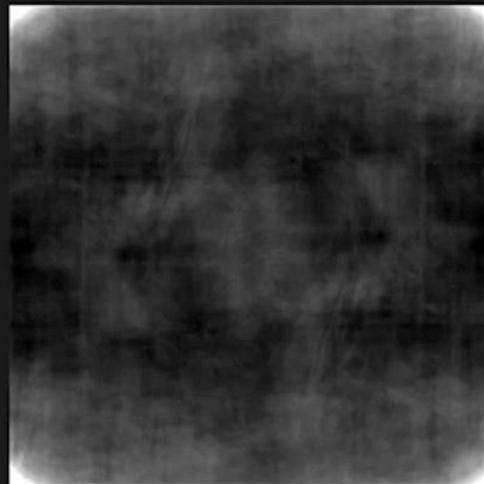


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The Importance of Phase



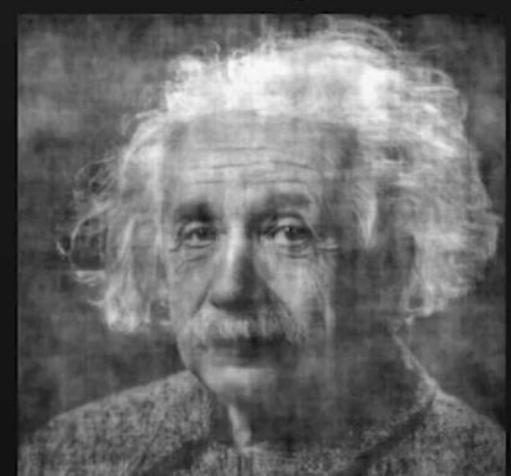
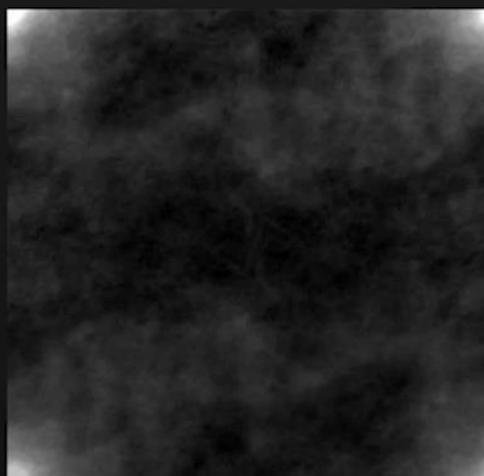
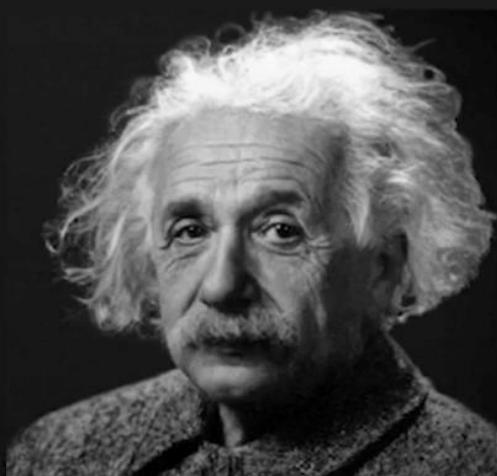
Original Image



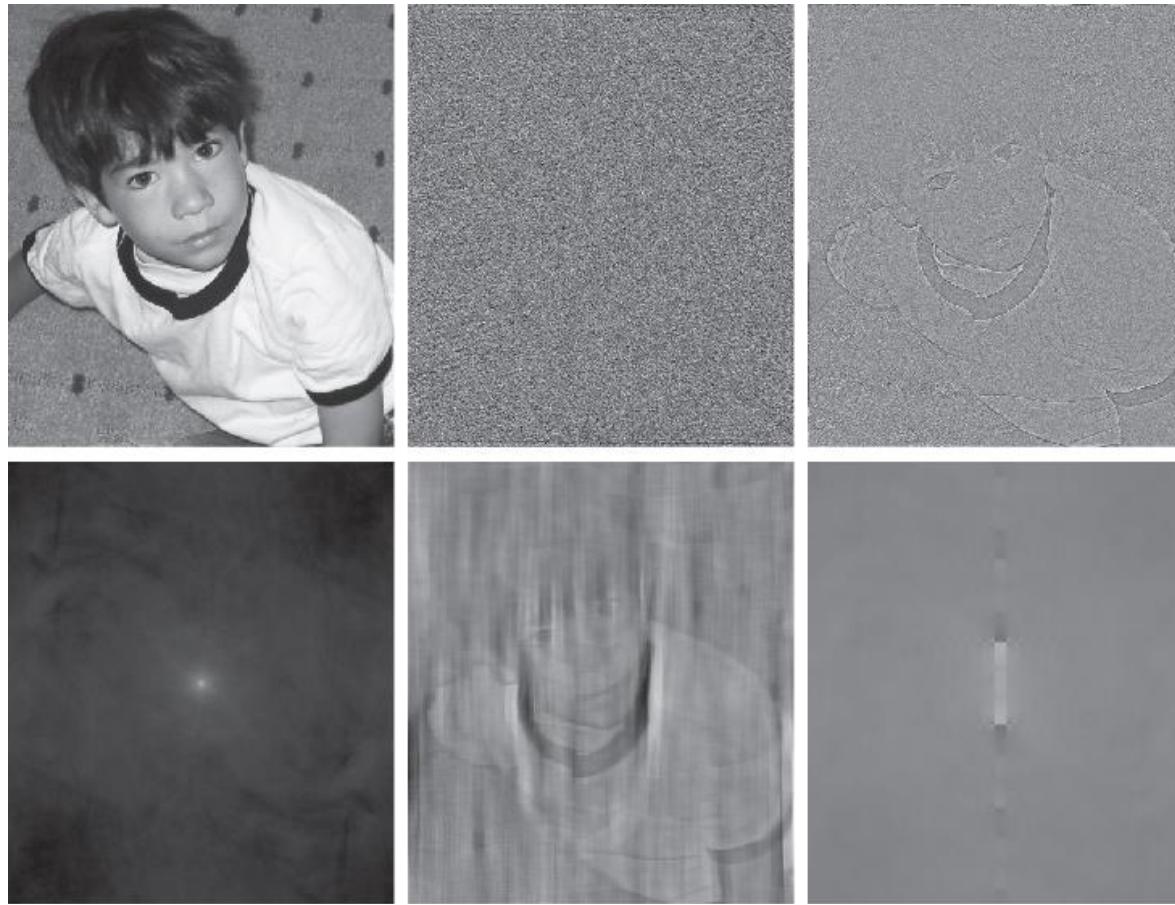
Magnitude Preserved,
Phase Set to Zero



Phase Preserved,
Magnitude Set to Average
of Natural Images



Spectrum and Phase angle



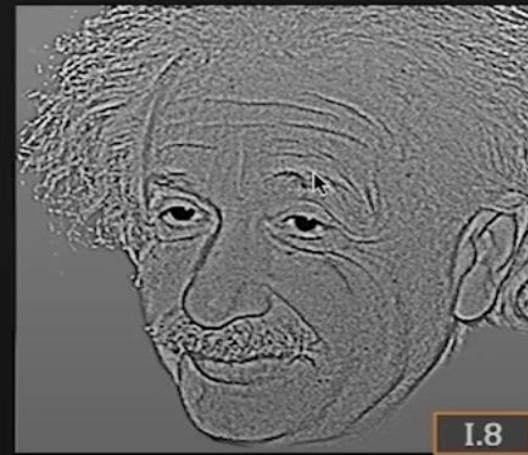
a b c
d e f

FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

Monroe or Einstein?



Low Freq Only



I.8

High Freq Only



Hybrid (Sum) Image



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Coding task 1

1. Load image vallay-house2.jpeg.
2. `fft_im = fft2(im);`
3. Try to show the magnitude of DFT.

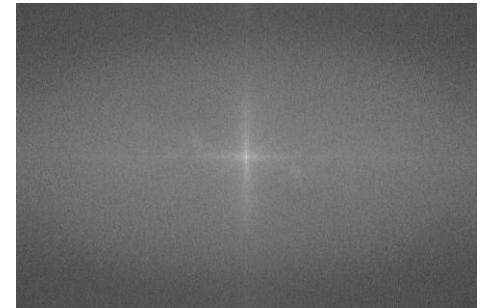
$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$



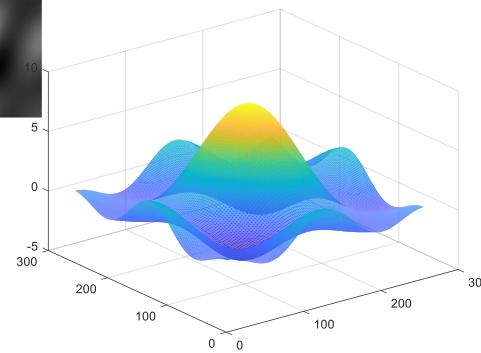
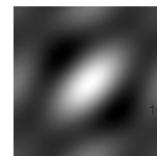
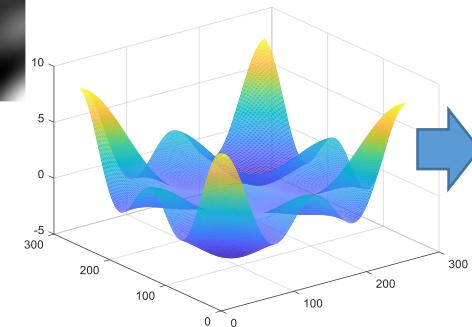
Coding task 1



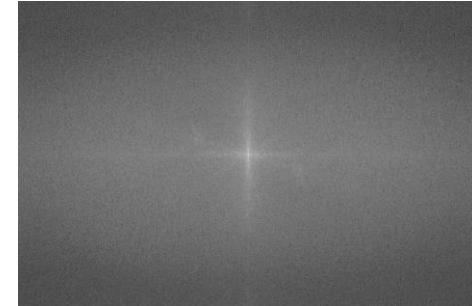
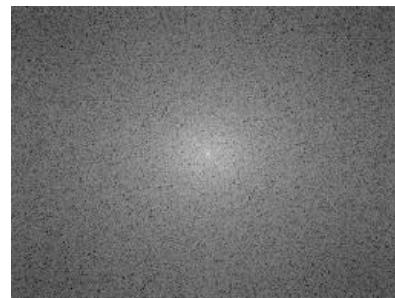
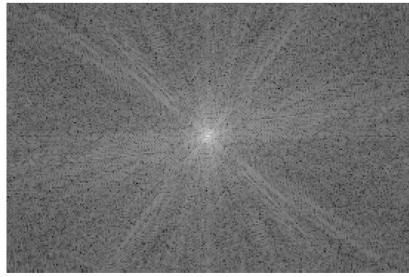
- Load image vallay-house2.jpeg.
 - $\text{fft_im} = \text{fft2(im);}$
- Try to show the magnitude of DFT.
 - $\text{fft_im_shifted} = \text{fftshift(fft_im);}$
 - $\text{im_recover} = \text{ifft2(fft_im);}$
- Try to show the inverse DFT image.



Visualization of DFT

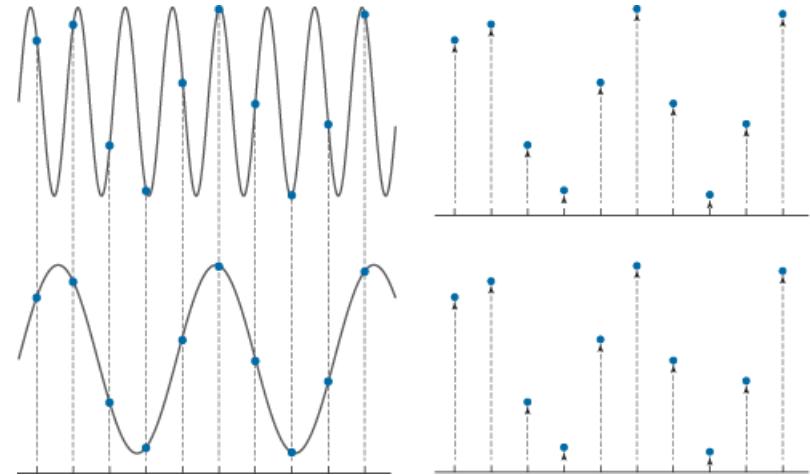
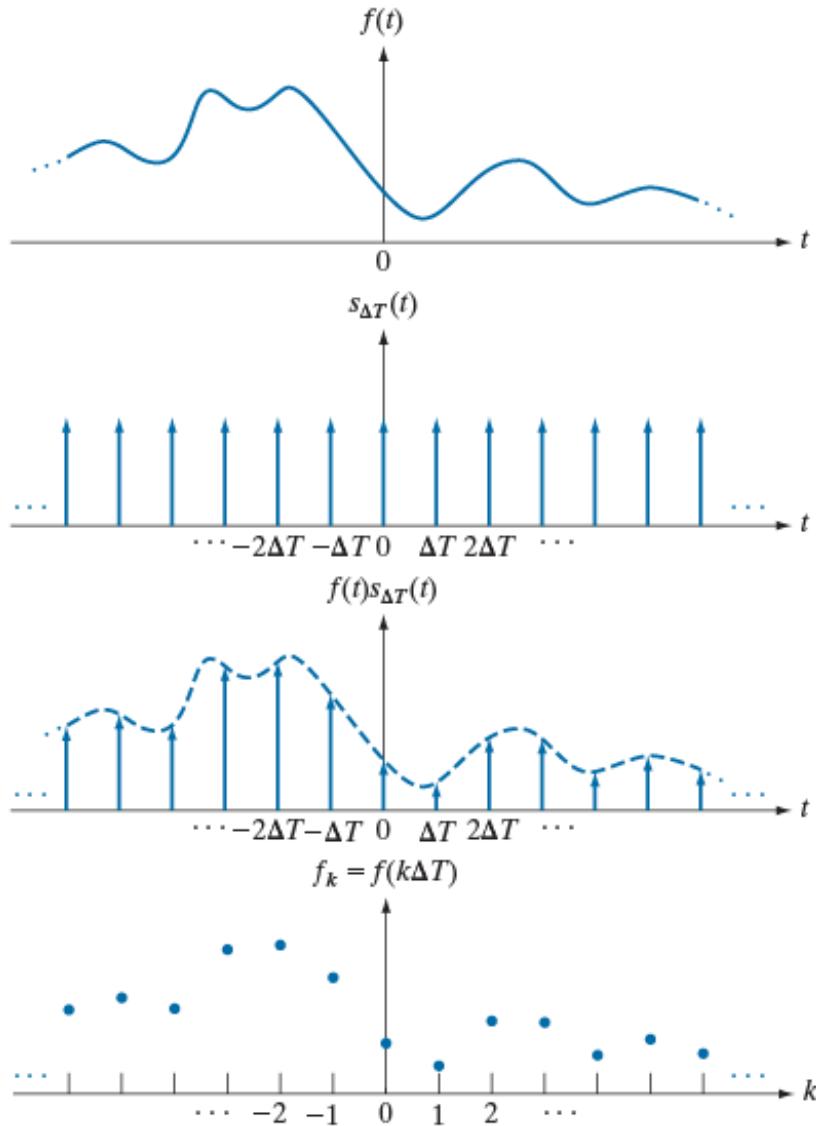


Visualization of DFT



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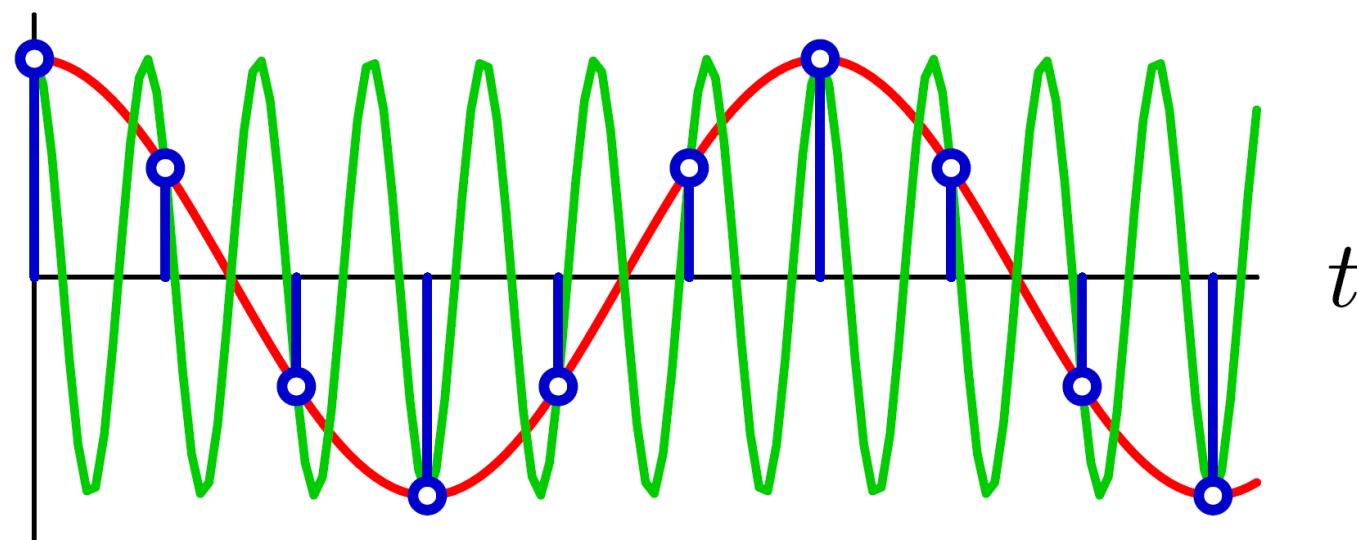
1D Sampling



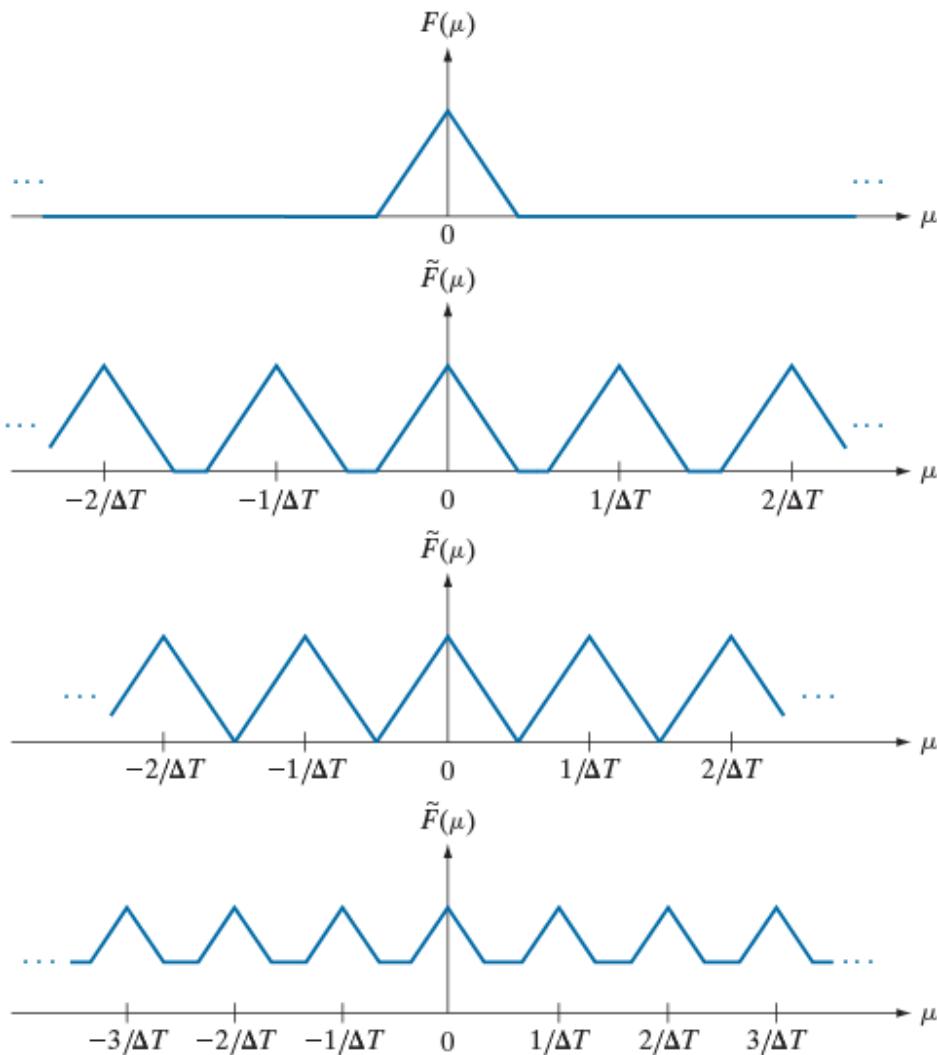
Aliasing

$\cos \frac{7\pi}{3}n?$

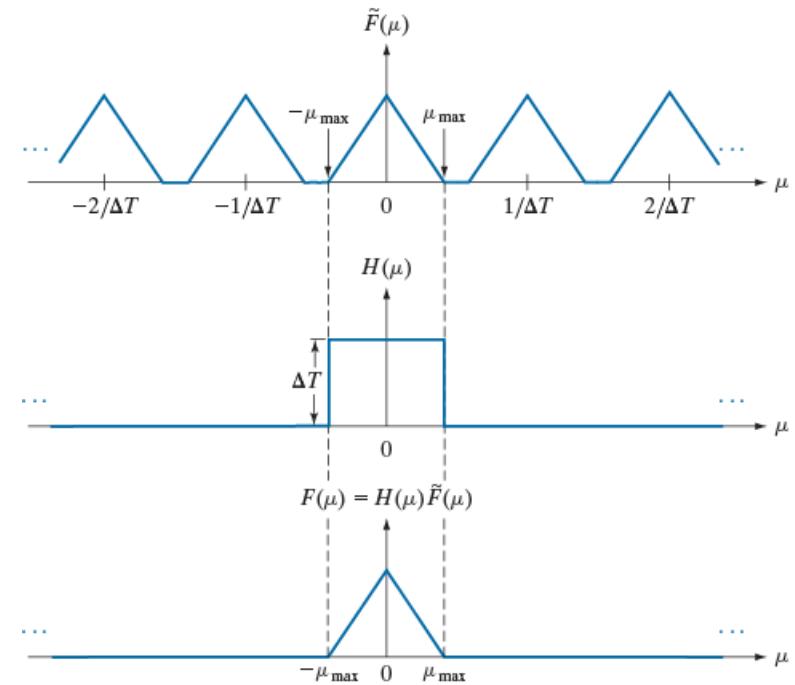
$\cos \frac{\pi}{3}n?$



1D Sampling Theorem



Non-aliasing v.s. Aliasing



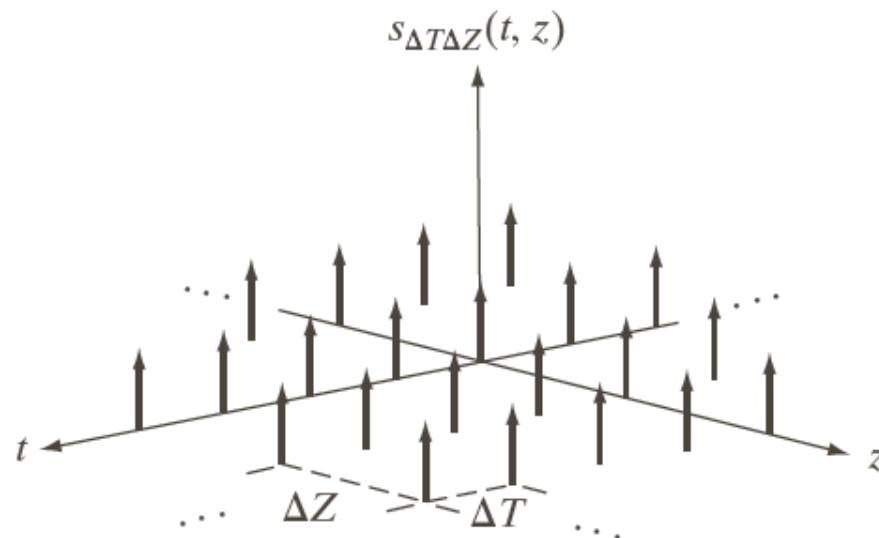
Recovery of the signal



2D Sampling

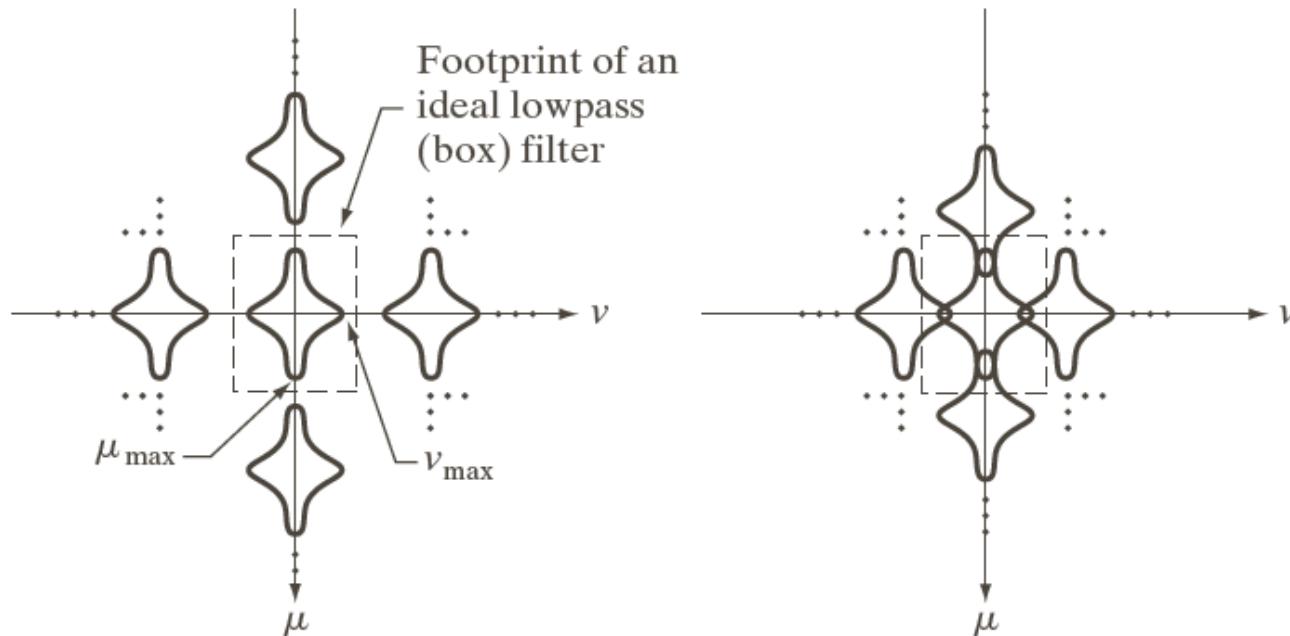
2D Sampling function (二维取样函数)

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



2D Sampling Theorem

- $f(t, z)$ is band-limited (带限函数) if $F(\mu, \nu) = 0$, $|\mu| \geq \mu_{\max}$ and $|\nu| \geq \nu_{\max}$
- The sampling rate: $\frac{1}{\Delta T} > 2\mu_{\max}$, $\frac{1}{\Delta Z} > 2\nu_{\max}$



Spatial Aliasing (空间混淆)



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Properties of 2D DFT and IDFT

- Translation (平移)
- Periodicity (周期性)
- Rotation (旋转)
- Separability (可分性)
- Symmetry (对称性)
- 2D Convolution theorem (卷积定理)



Translation (平移)

Translation

$$f(x, y)e^{j2\pi(\frac{u_0x}{M} + \frac{v_0y}{N})} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(\frac{ux_0}{M} + \frac{vy_0}{N})}$$

When $u_0 = \frac{M}{2}, v_0 = \frac{N}{2}$

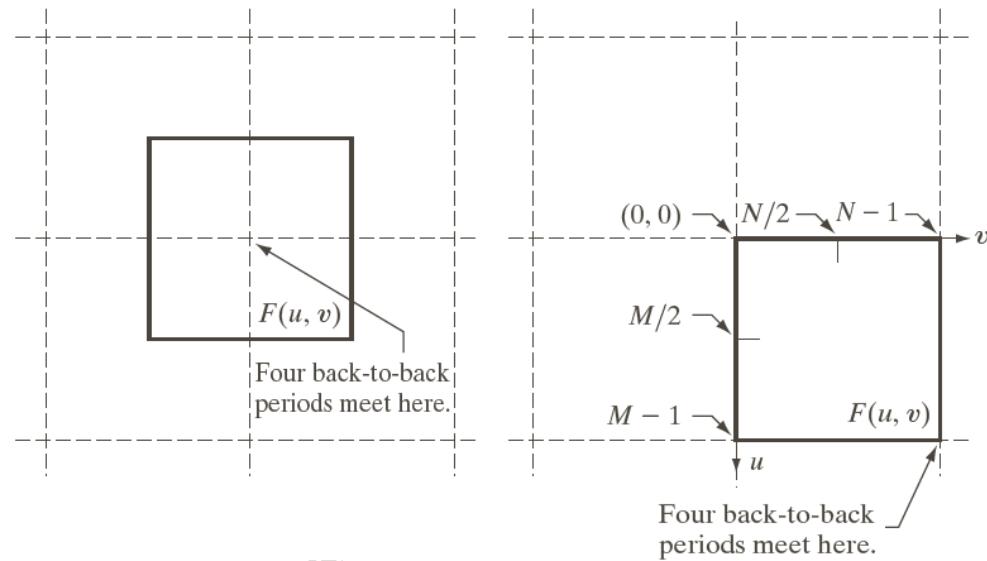
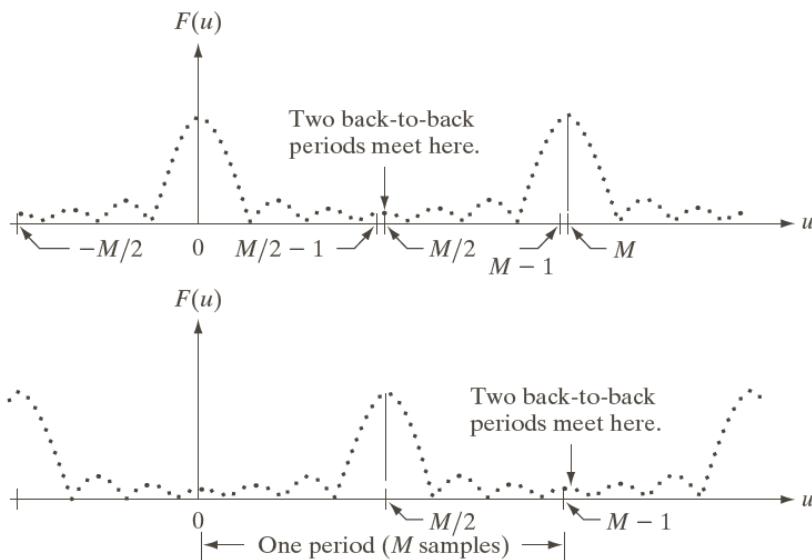
$$F\left(u - \frac{M}{2}, v - \frac{N}{2}\right) \Leftrightarrow f(x, y)e^{j\pi(x+y)} = f(x, y)(-1)^{(x+y)}$$



Periodicity (周期性)

- $f(x, y) = f(x + k_1 M, y) = f(x, y + k_2 N) = f(x + k_1 M, y + k_2 N)$
- $F(u, v) = F(u + k_1 M, v) = F(u, v + k_2 N) = F(u + k_1 M, v + k_2 N)$

Where k_1 and k_2 are integers

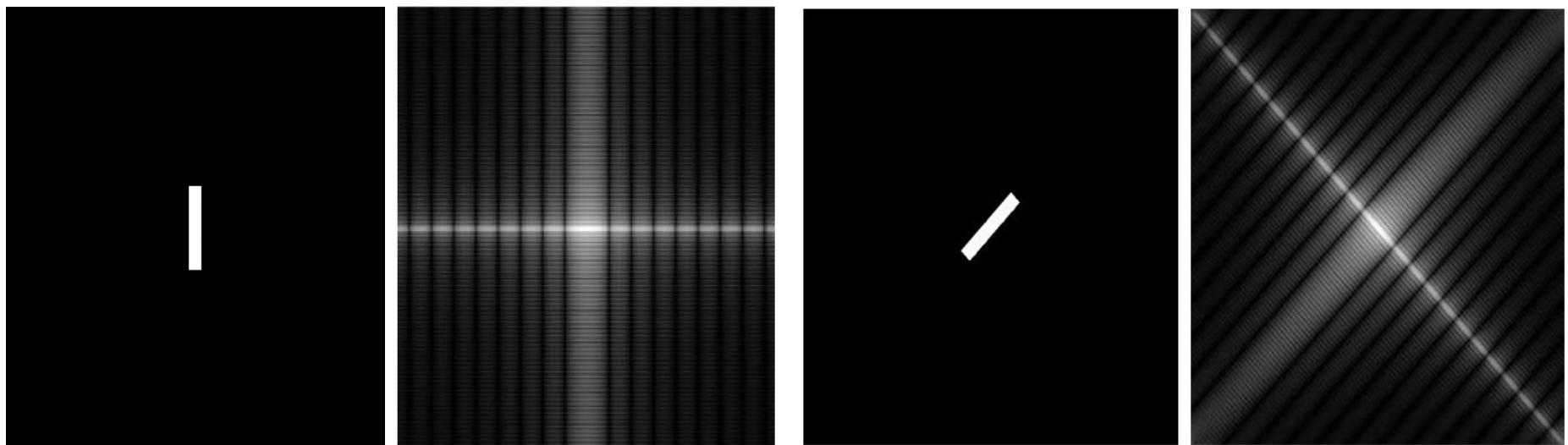


Rotation (旋转)

□ Rotation

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

Where $x = r\cos\theta$, $y = r\sin\theta$, $u = \omega\cos\varphi$, $v = \omega\sin\varphi$



Symmetry (对称性)

➤ Even Function (偶函数)

$$w_e(x, y) = w_e(-x, -y) \quad w_e(x, y) = w_e(M - x, N - y)$$

➤ Odd Function (奇函数)

$$w_o(x, y) = -w_o(-x, -y) \quad w_o(x, y) = -w_o(M - x, N - y)$$

➤ Conjugate symmetric (共轭对称)

$$F^*(u, v) = F(-u, -v) \quad F^*(u, v) = F(M - u, N - v)$$

➤ Conjugate antisymmetric (共轭反对称)

$$F^*(u, v) = -F(-u, -v) \quad F^*(u, v) = -F(M - u, N - v)$$



2D Discrete Fourier Transform (DFT)

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

Q1:

$$f(x, y) \quad \Leftrightarrow \quad F(u, v)$$

$$f(x, y) e^{j2\pi(\frac{u_0x}{M} + \frac{v_0y}{N})} \quad \Leftrightarrow \quad ?$$

Q2: Given that $f(x, y)$ is pure imaginary

Prove that $F^*(u, v) = -F(-u, -v)$



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Symmetry (对称性)

	Spatial Domain [†]	Frequency Domain [†]
1)	$f(x, y)$ real	$\Leftrightarrow F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow F(u, v)$ complex and odd



2D Convolution theorem (卷积定理)

➤ Convolution theorem

$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

or $f(x, y) \circledast h(x, y) \Leftrightarrow \frac{1}{MN}F(u, v) \star H(u, v)$

➤ Zero padding (零填充)

$$f_p(x, y) = \begin{cases} f(x, y), & 0 \leq x \leq A - 1, 0 \leq y \leq B - 1 \\ 0, & A \leq x \leq P, B \leq y \leq Q \end{cases}$$

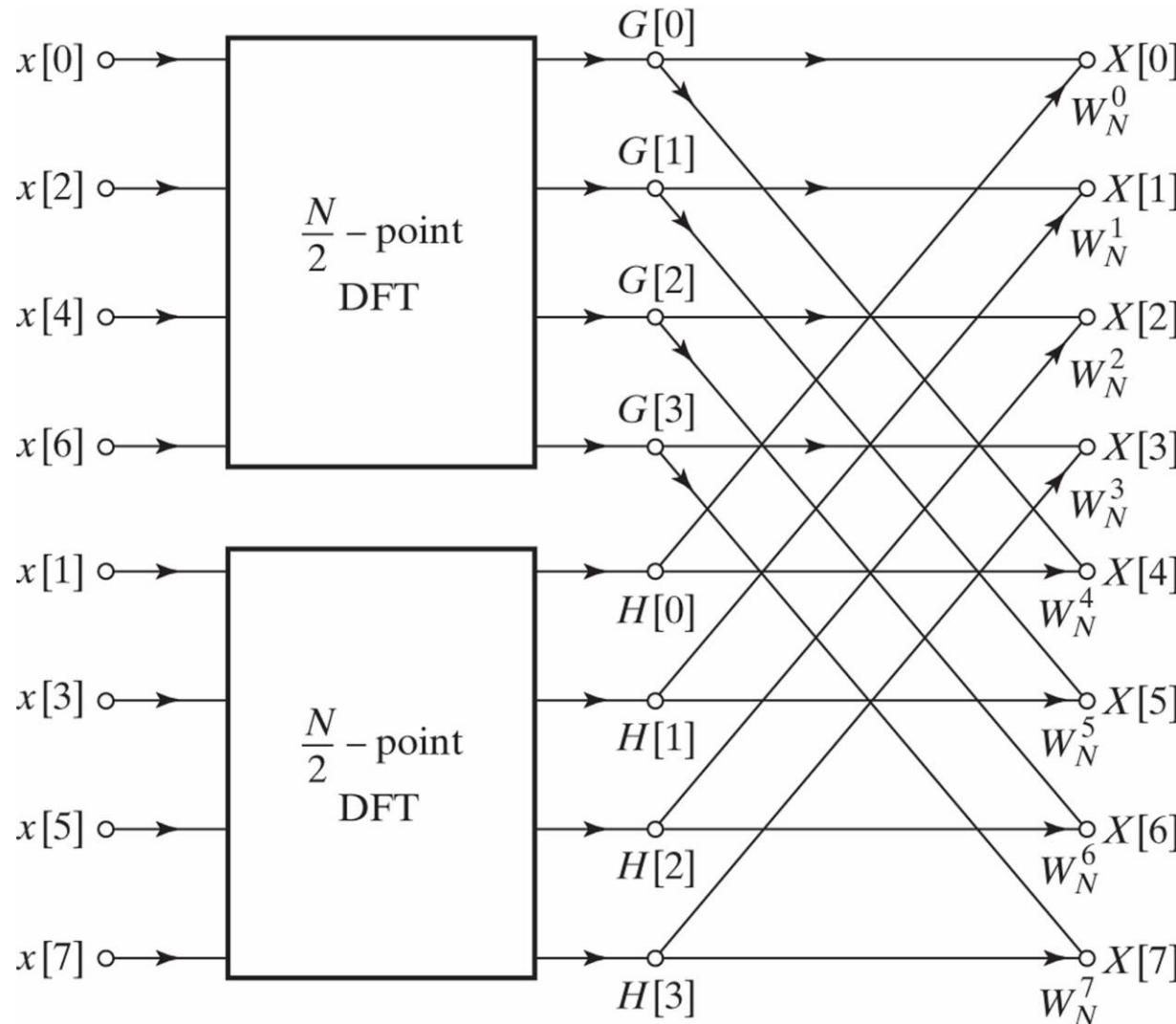
$$h_p(x, y) = \begin{cases} h(x, y), & 0 \leq x \leq C - 1, 0 \leq y \leq D - 1 \\ 0, & C \leq x \leq P, D \leq y \leq Q \end{cases}$$

Where $f(x, y)$: $A \times B$ image; $h(x, y)$: $C \times D$ image; $P \geq A + C - 1$; $Q \geq B + D - 1$

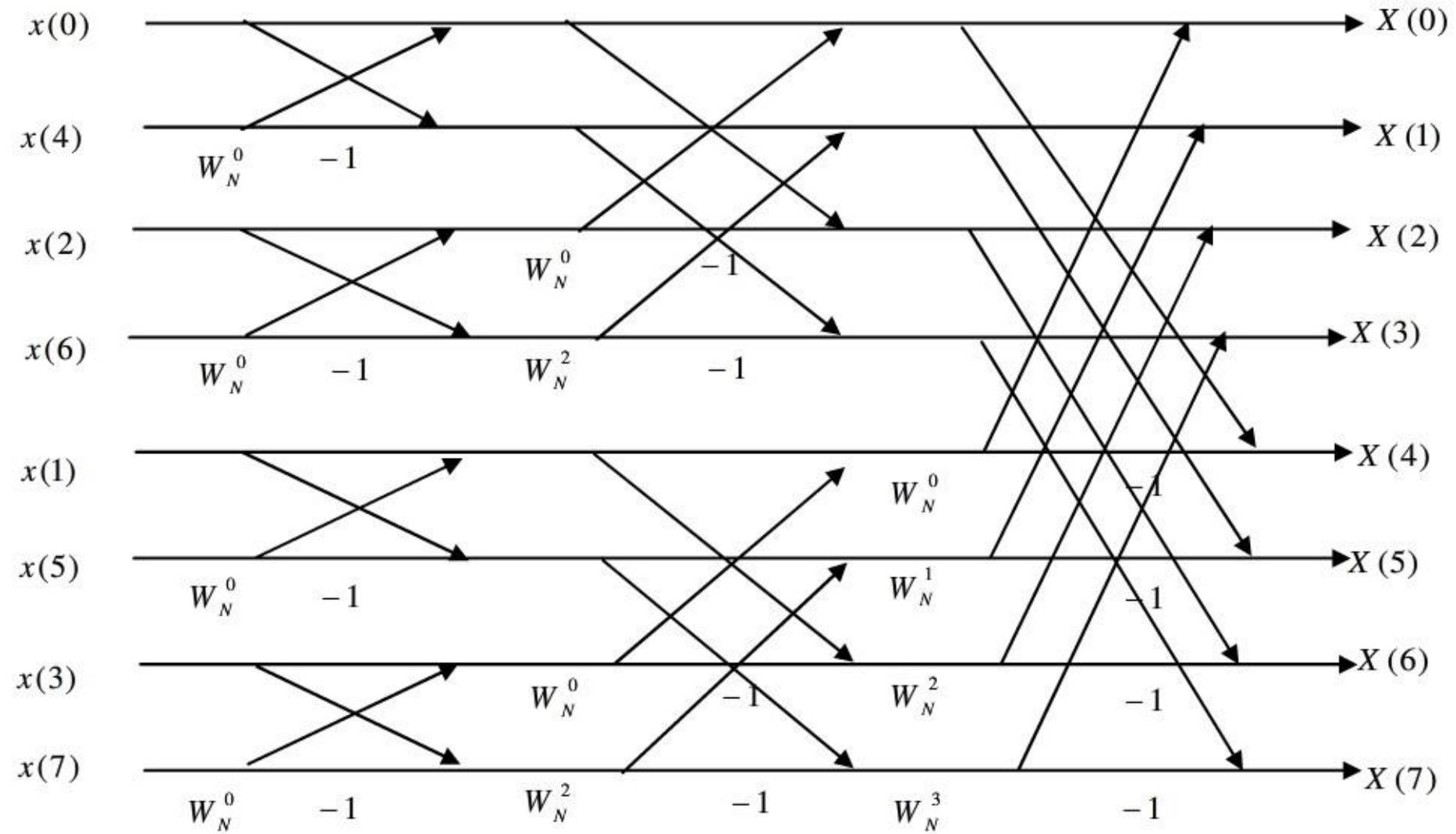
2D DFT implementation (FFT)

- **Fast Fourier transform (FFT):** an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT).
- The basic idea is to **decompose** successively the N-point DFT into smaller-size DFTs.
- Improvement of speed $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log_2(N))$
- There are many FFT algorithms!
- A classical one is Cooley-Tukey FFT (published in 1965; for more generalized complex FFT)

2D DFT implementation (FFT)



2D DFT implementation (FFT)



Appendix 1: DFT definition and expressions

TABLE 4.3
Summary of DFT definitions and corresponding expressions.

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M+vy/N)}$
3) Spectrum	$ F(u, v) = [R^2(u, v) + I^2(u, v)]^{1/2} \quad R = \text{Real}(F); I = \text{Imag}(F)$
4) Phase angle	$\phi(u, v) = \tan^{-1} \left[\frac{I(u, v)}{R(u, v)} \right]$
5) Polar representation	$F(u, v) = F(u, v) e^{j\phi(u, v)}$
6) Power spectrum	$P(u, v) = F(u, v) ^2$
7) Average value	$\bar{f} = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1 M, v) = F(u, v + k_2 N)$ $= F(u + k_1, v + k_2 N)$ $f(x, y) = f(x + k_1 M, y) = f(x, y + k_2 N)$ $= f(x + k_1 M, y + k_2 N)$
9) Convolution	$(f \star h)(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
10) Correlation	$(f \star\! \star h)(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.
12) Obtaining the IDFT using a DFT algorithm	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M+vy/N)}$ <p>This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.</p>

Appendix 2: DFT properties

TABLE 4.4

Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the continuous expressions.

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$a f_1(x,y) + b f_2(x,y) \Leftrightarrow a F_1(u,v) + b F_2(u,v)$
3) Translation (general)	$f(x,y) e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u,v) e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5) Rotation	$f(r,\theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(y/x) \quad \omega = \sqrt{u^2 + v^2} \quad \varphi = \tan^{-1}(v/u)$
6) Convolution theorem [†]	$f \star h(x,y) \Leftrightarrow (F \star H)(u,v)$ $(f \star h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$
7) Correlation theorem [†]	$(f \diamond h)(x,y) \Leftrightarrow (F^* \star H)(u,v)$ $(f^* \star h)(x,y) \Leftrightarrow (1/MN)[(F \diamond H)(u,v)]$
8) Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9) Rectangle	$\text{rect}[a,b] \Leftrightarrow ab \frac{\sin(\pi ua)}{(\pi ua)} \frac{\sin(\pi vb)}{(\pi vb)} e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \frac{jMN}{2} [\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)]$
11) Cosine	$\cos(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \frac{1}{2} [\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)]$
12) Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\frac{\partial}{\partial t}\right)^m \left(\frac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m (j2\pi\nu)^n F(\mu, \nu)$ $\frac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu, \nu); \quad \frac{\partial^n f(t,z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu, \nu)$
13) Gaussian	$A 2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow A e^{-(\mu^2 + \nu^2)/2\sigma^2} \quad (A \text{ is a constant})$

[†] Assumes that $f(x,y)$ and $h(x,y)$ have been properly padded. Convolution is associative, commutative, and distributive. Correlation is distributive (see Table 3.5). The products are elementwise products (see Section 2.6).