Lecture 7
Image restoration & reconstruction

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Outline

- **Model of Image Degradation/Restoration Process**
- **Noise Reduction** $\eta(x, y)$
  - Noise Models
  - Spatial domain/Frequency domain Filtering
- **Image Restoration** $H(x, y)$
  - Degradation Function
  - Inverse Filtering
  - Wiener Filtering
- **Image reconstruction (e.g., CT)**

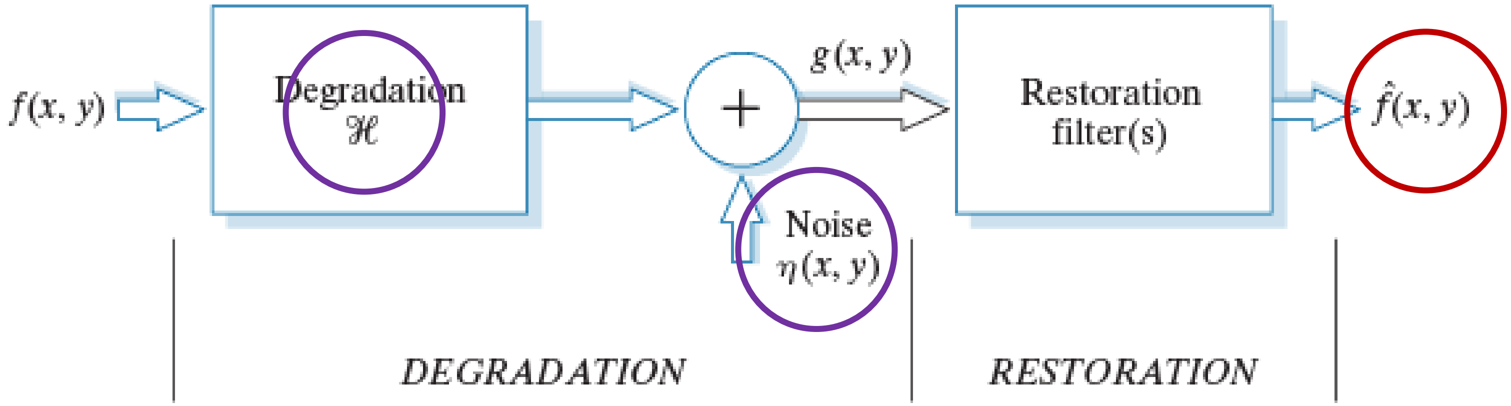
Something we have already known

Median filter

Unsharp filter

Image restoration v.s. Image enhancement

Model of Image Degradation (图像退化模型)



Spatial domain: $g(x,y) = h(x,y) \star f(x,y) + \eta(x,y)$

Frequency domain: $G(u,v) = H(u,v)F(u,v) + N(u,v)$

***H* is a linear, position-invariant process**

$g(x,y)$: a degraded image $f(x,y)$: input image

$h(x,y)$: degradation function $\eta(x,y)$: additive noise term

Task for restoration: to find out $\hat{f}(x,y)$, estimation of original $f(x,y)$

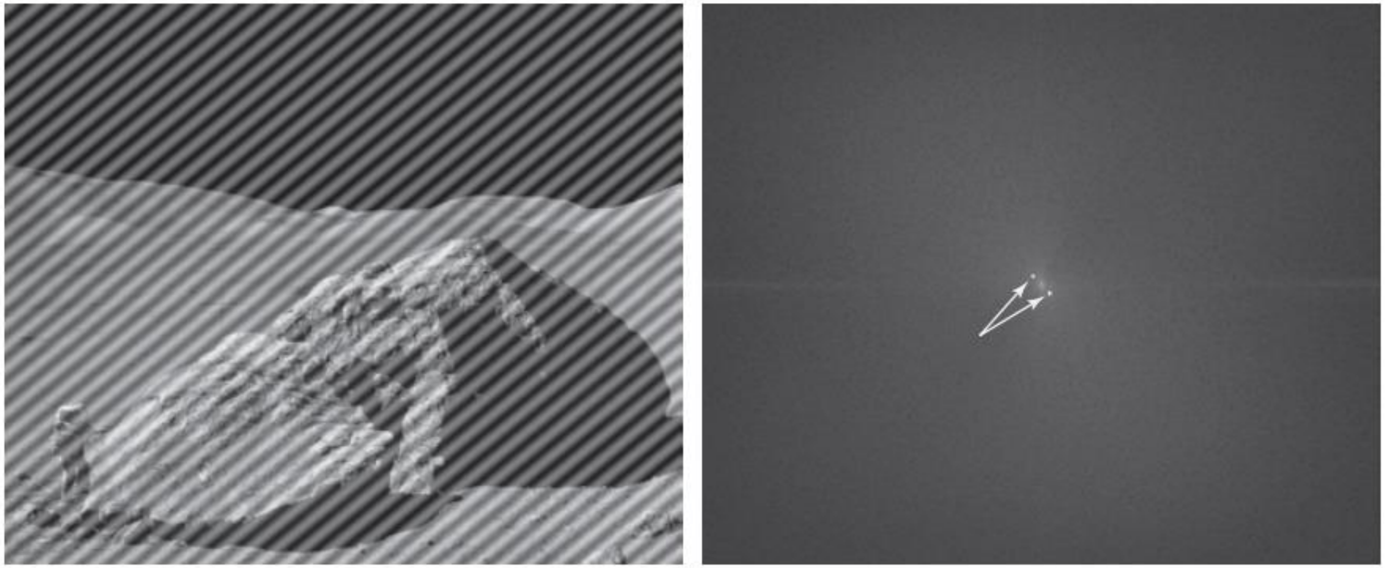
Restoration with only noise

- For easier case: when ***H*** is identity (no blur). Degraded image contain only additive noise.
  - Spatial domain: $g(x,y) = h(x,y) \star f(x,y) + \eta(x,y)$
  - Frequency domain: $G(u,v) = H(u,v)F(u,v) + N(u,v)$
- Noise is often described by Probability Density Function (PDF).
- Noise may be due to:
  - Non-ideal sensor elements.
  - Environmental conditions (light level, temperature…).
  - Corruption during transmission/ compression.

Properties of Noise

- **Spatial properties:** parameters that define spatial characteristics of noise
- **Frequency properties:** frequency content of noise
  - e.g., Periodic noise
- **Spatial filtering:**
  - **Independent of spatial coordinates**
  - **Uncorrelated with respect to the image itself**
  - **Noise we can deal with: NON-IID Noise (IID: independently identically distributed)**

Periodic Noise



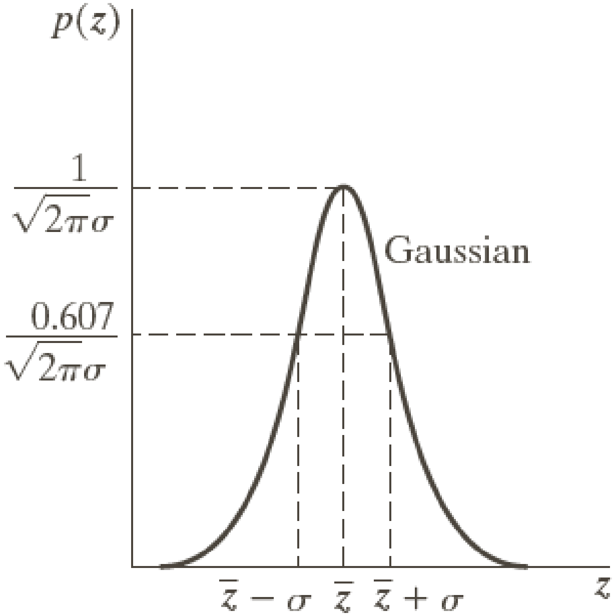
a b

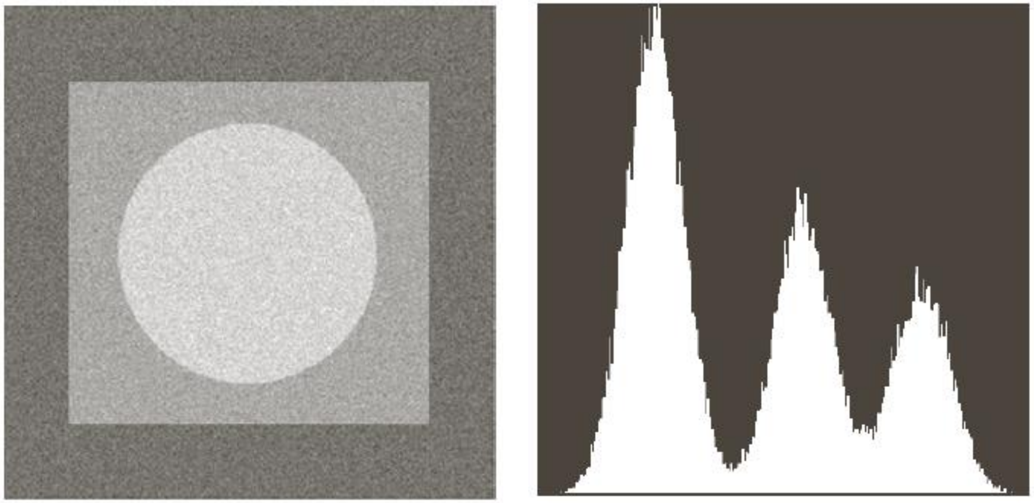
FIGURE 5.5
(a) Image corrupted by additive sinusoidal noise. (b) Spectrum showing two conjugate impulses caused by the sine wave. (Original image courtesy of NASA.)

Gaussian Noise (高斯噪声)

Gaussian Noise(高斯噪声): $p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\bar{z})^2}{2\sigma^2}}$

$\bar{z}$: mean (average) σ: standard deviation σ^2: variance





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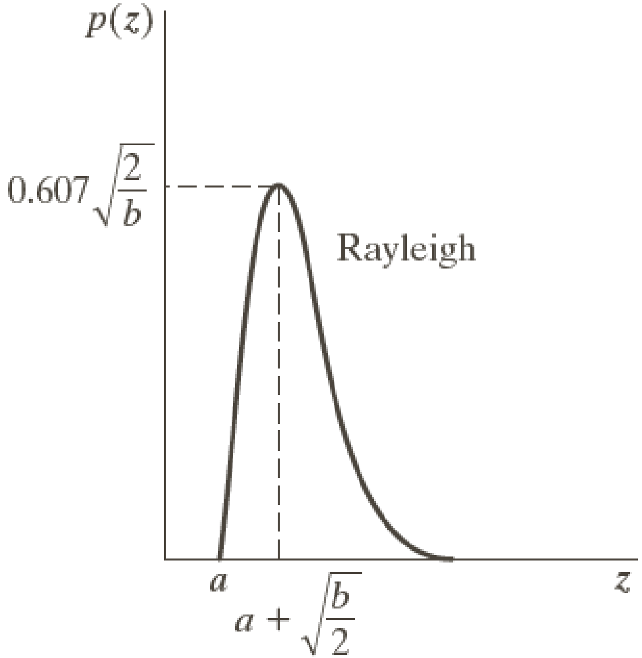
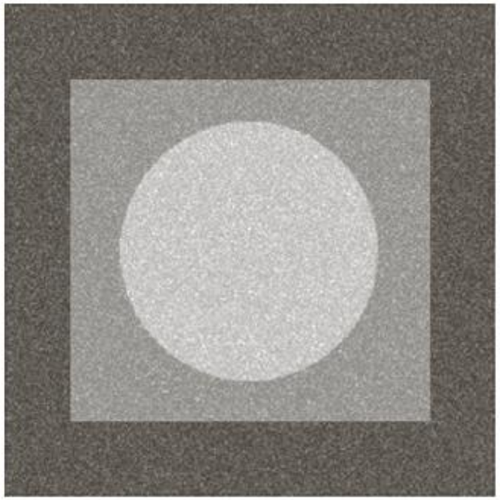
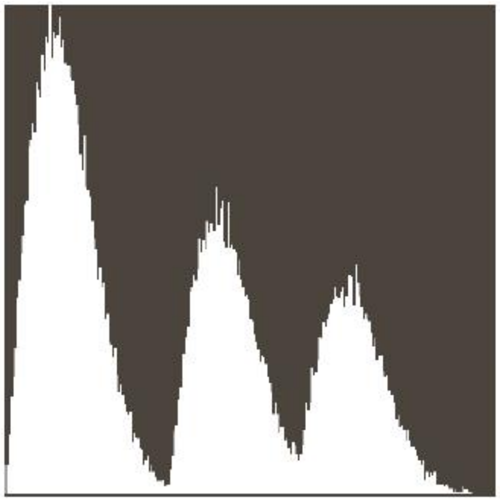
Rayleigh Noise（瑞利噪声）

Rayleigh Noise（瑞利噪声）： $p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-\frac{(z-a)^2}{b}}, & \text{for } z \geq a \\ 0, & \text{for } z < a \end{cases}$

$$\bar{z} = a + \sqrt{\pi b/4} \qquad \sigma^2 = \frac{b(4-\pi)}{4}$$



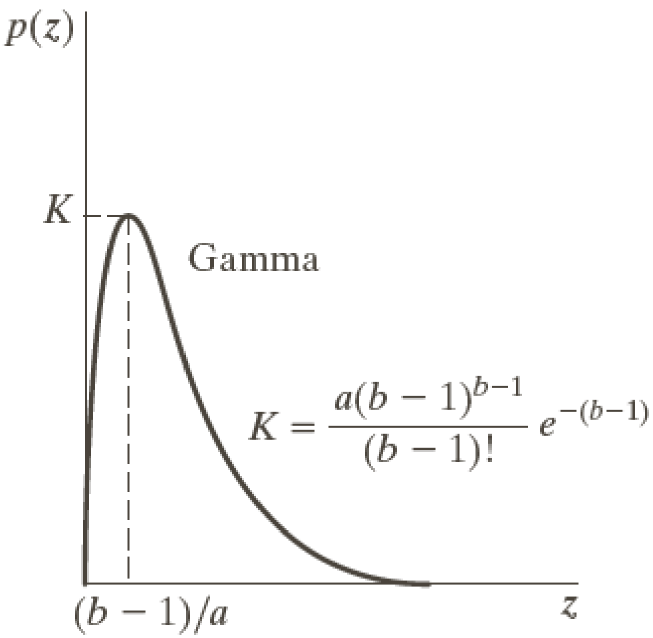


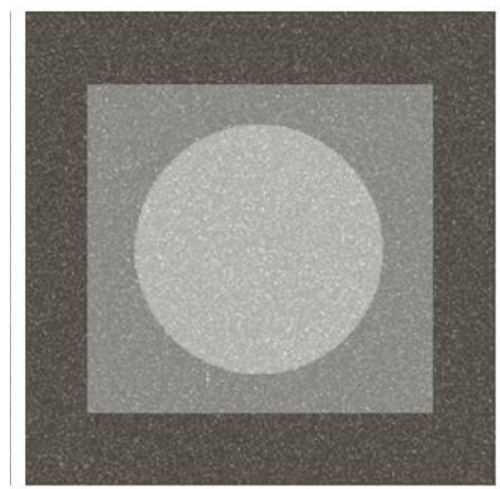


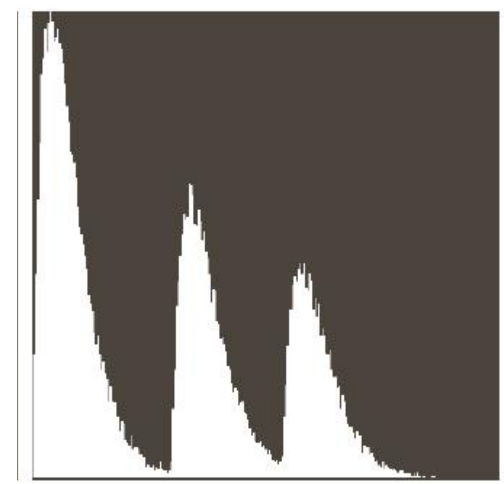
Erlang (gamma) Noise
(爱尔兰/伽马噪声)

Erlang (gamma) Noise（爱尔兰/伽马噪声）： $p(z) = \begin{cases} \dfrac{a^b z^{b-1}}{(b-1)!} e^{-az}, & \text{for } z \geq 0 \\ 0, & \text{for } z < 0 \end{cases}$

$$\bar{z} = \frac{b}{a} \qquad \sigma^2 = \frac{b}{a^2}$$







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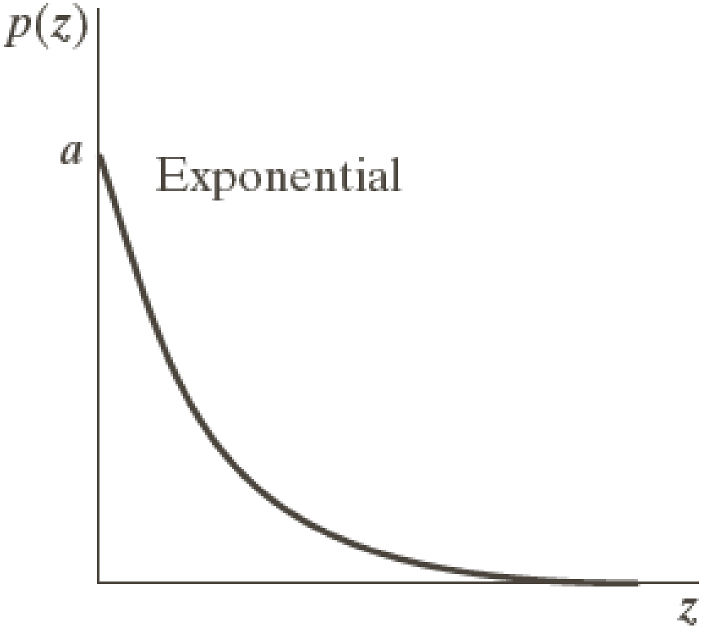
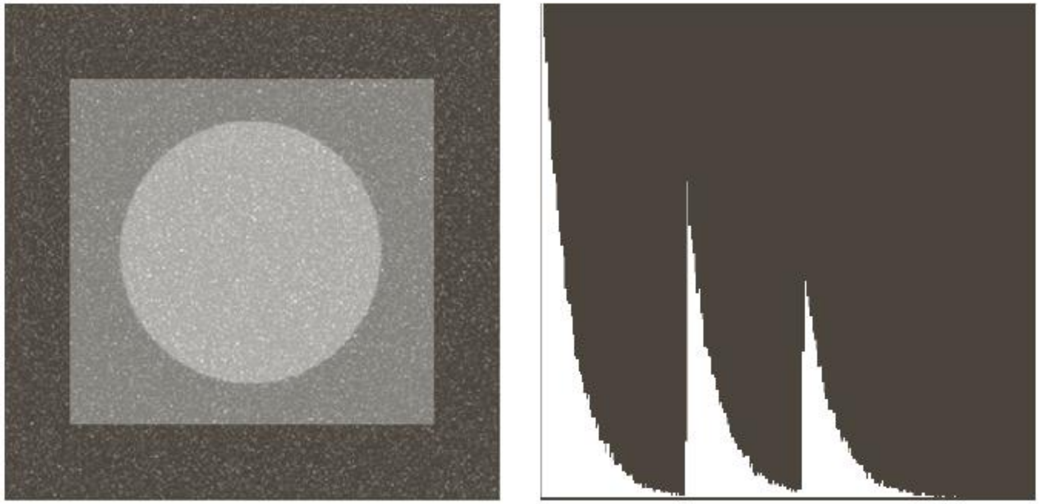
Exponential Noise（指数噪声）

Exponential Noise（指数噪声）： $p(z)=\begin{cases} ae^{-az}, \text{for } z \geq 0 \\ 0, \quad\quad \text{for } z < 0 \end{cases}$

$$\bar{z}=\frac{1}{a} \qquad \sigma^2=\frac{1}{a^2}$$





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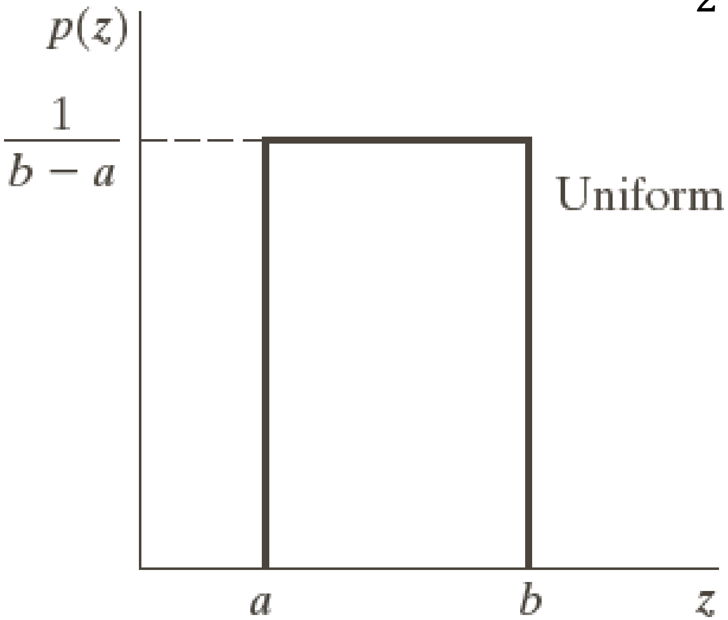
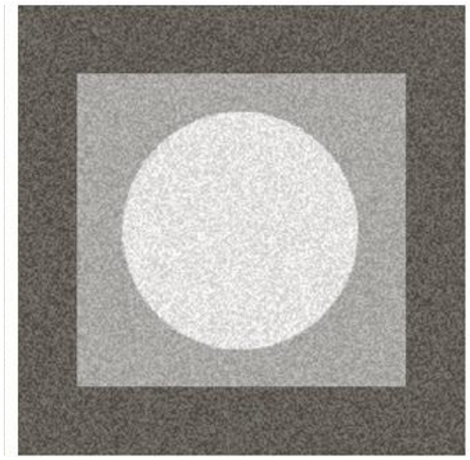
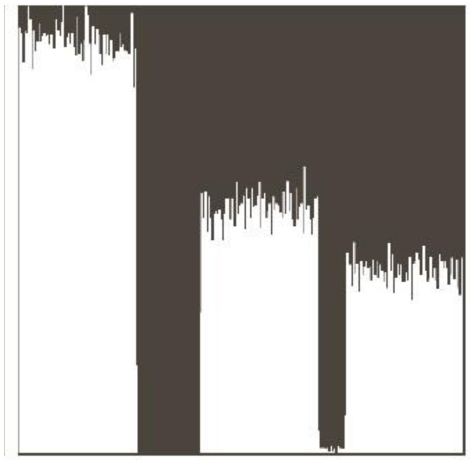
Uniform Noise（均匀噪声）

Uniform Noise（均匀噪声）：
$$p(z) = \begin{cases} \frac{1}{b-a}, \text{for } a \le z \le b \\ 0, \quad \text{otherwise} \end{cases}$$

$$\bar{z} = \frac{a+b}{2} \qquad \sigma^2 = \frac{(b-a)^2}{12}$$



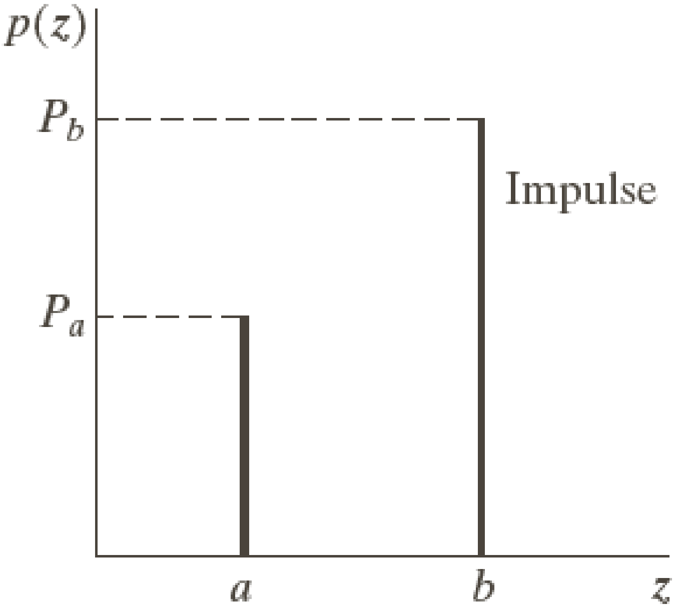


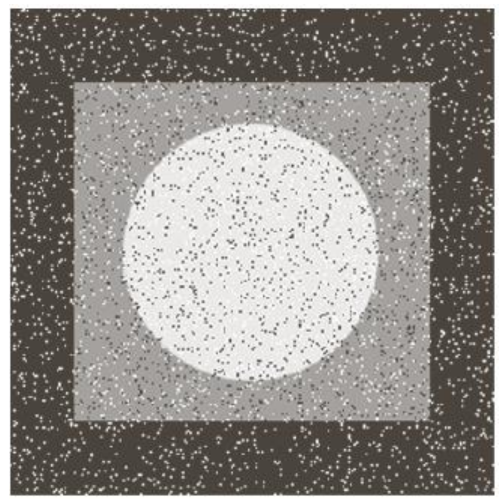


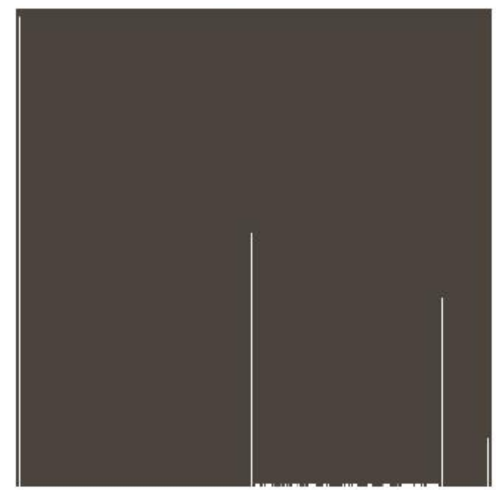
Impulse (salt-and-pepper) Noise (脉冲/椒盐噪声)

Impulse (salt-and-pepper) Noise（脉冲/椒盐噪声）:

$$
p(z) = \begin{cases} P_a, & \text{for } z = a \\ P_b, & \text{for } z = b \\ 1 - P_a - P_b, & \text{otherwise} \end{cases}
$$







Different noise PDFs

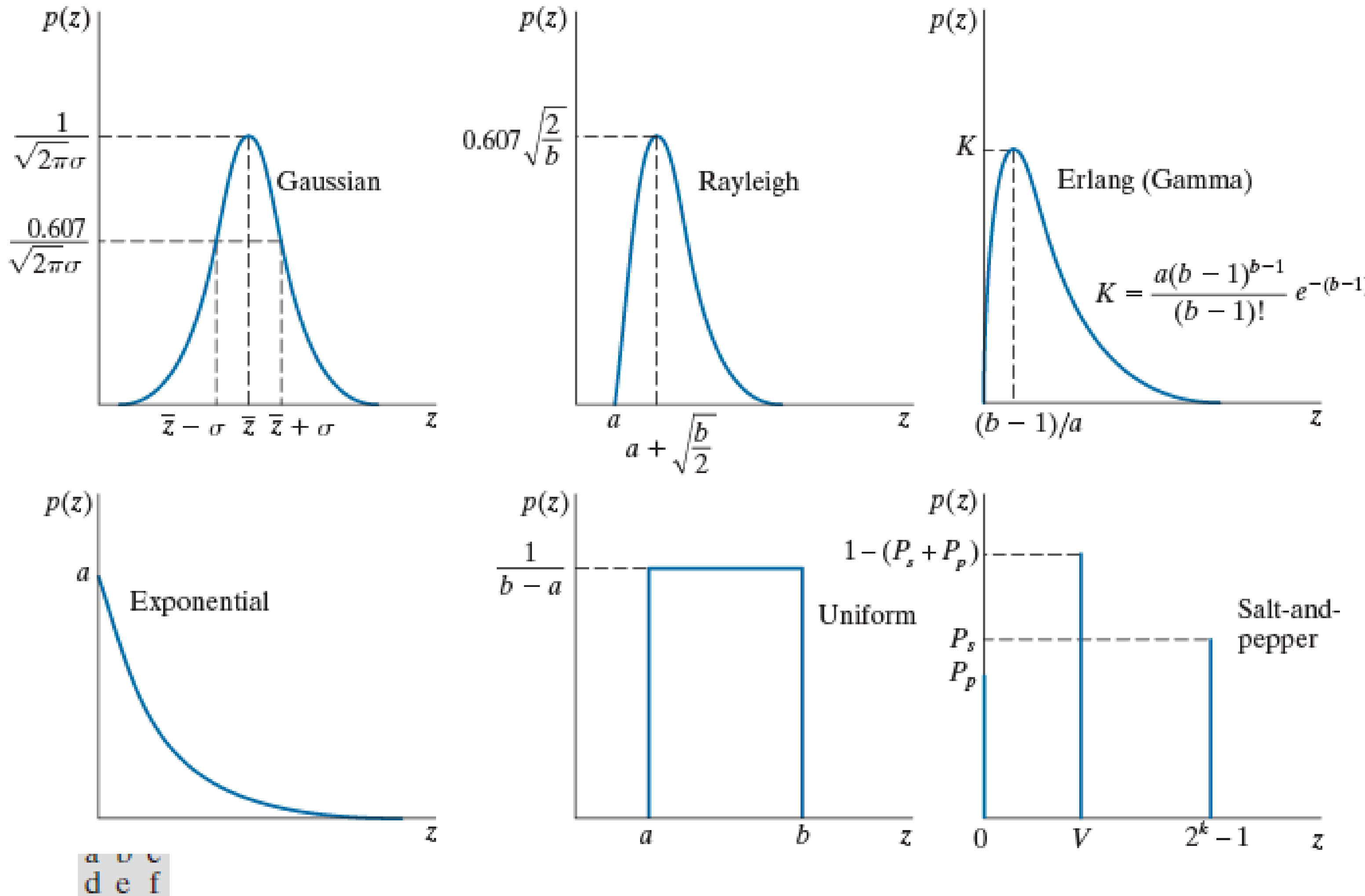


FIGURE 5.2 Some important probability density functions.

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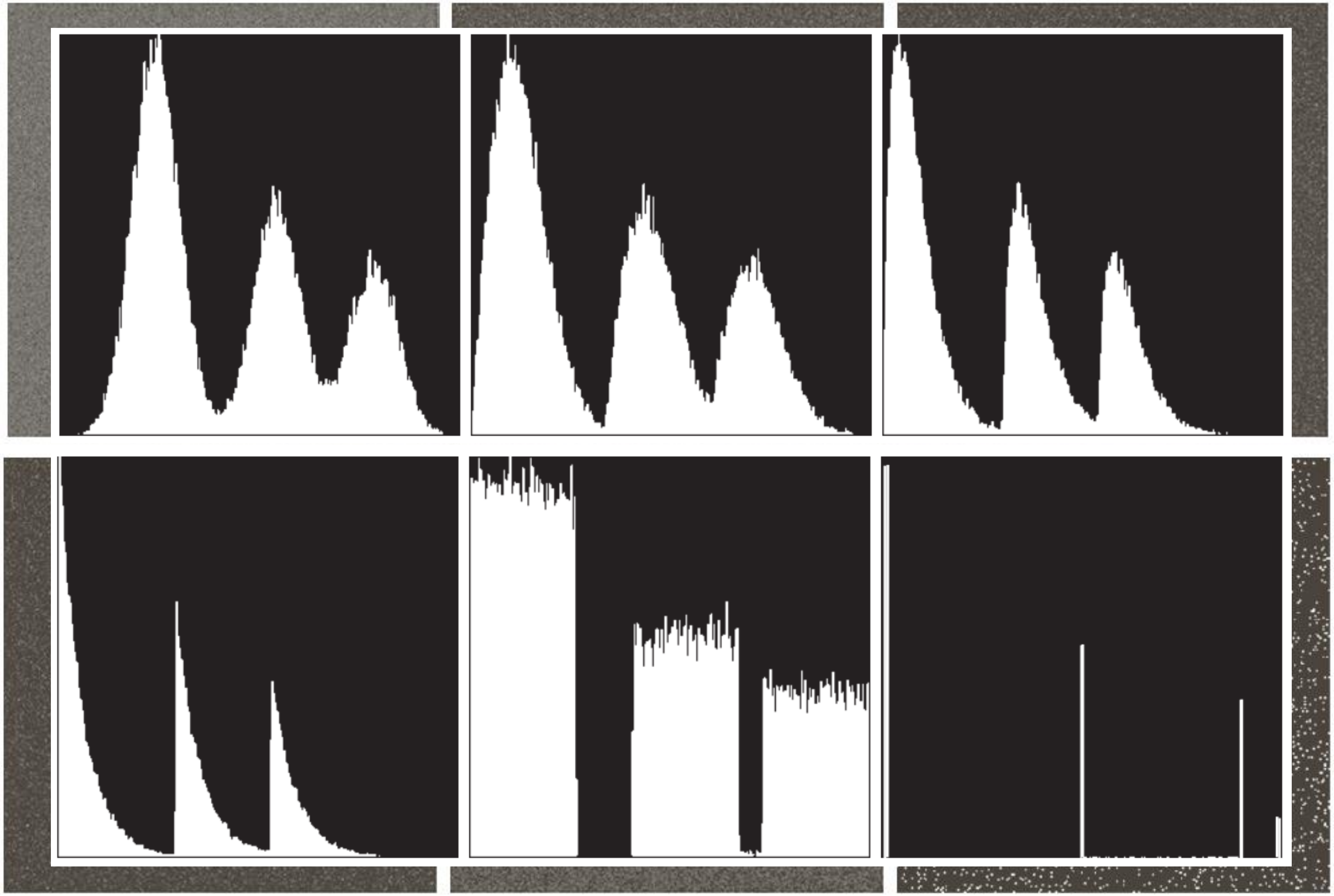
Images with different noise



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Estimating noise parameters

- How?
  - From sensor specifications
  - Test with a flat board
  - Read a small patch
- Basic parameters needed for estimating a PDF?
  - Mean and variance

$$\bar{z} = \sum_{i=0}^{L-1} z_i p(z_i) \qquad \sigma^2 = \sum_{i=0}^{L-1} (z_i - \bar{z})^2 p(z_i)$$

Spatial Filtering

- **Mean Filters (均值滤波器)**
  - Arithmetic mean filter (算术均值滤波器)
  - Geometric mean filter (几何均值滤波器)
  - Harmonic mean filter (谐波均值滤波器)
  - Contraharmonic mean filter (逆谐波均值滤波器)
- **Order-statistic Filters (统计排序滤波器)**
  - Median filter (中值滤波器)
  - Max and Min filter (最大值和最小值滤波器)
  - Midpoint filter (中点滤波器)
  - Alpha-trimmed mean filter (修正的阿尔法均值滤波器)
- **Adaptive Filters (自适应滤波器)**
  - Adaptive local noise reduction filter (自适应局部降噪滤波器)
  - Adaptive median filter (自适应中值滤波器)

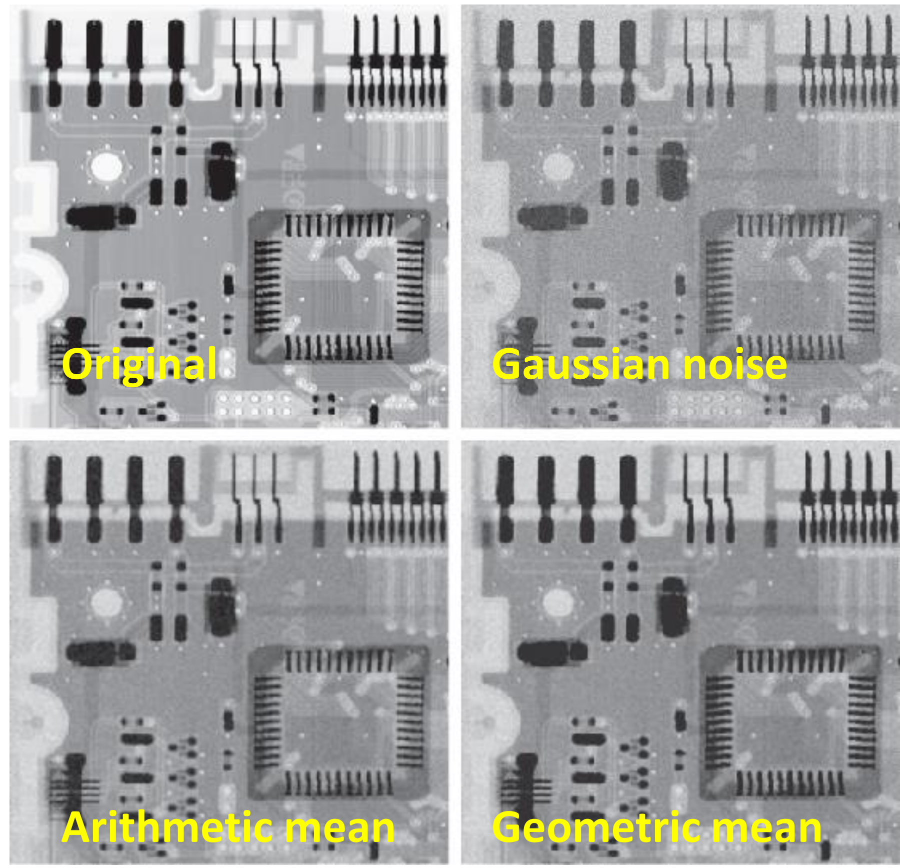
Mean Filters (均值滤波器)

- **Arithmetic mean filter (算术均值滤波器):**

$$\hat{f}(x,y) = \frac{1}{mn} \sum_{(s,t)\in S_{xy}} g(s,t)$$

- **Geometric mean filter (几何均值滤波器):**

$$\hat{f}(x,y) = \left(\prod_{(s,t)\in S_{xy}} g(s,t) \right)^{\frac{1}{mn}}$$



a b
c d

FIGURE 5.7 (a) X-ray image of circuit board. (b) Image corrupted by additive Gaussian noise. (c) Result of filtering with an arithmetic mean filter of size 3×3. (d) Result of filtering with a geometric mean filter of the same size. (Original image courtesy of Mr. Joseph E. Pascente, Lixi, Inc.)

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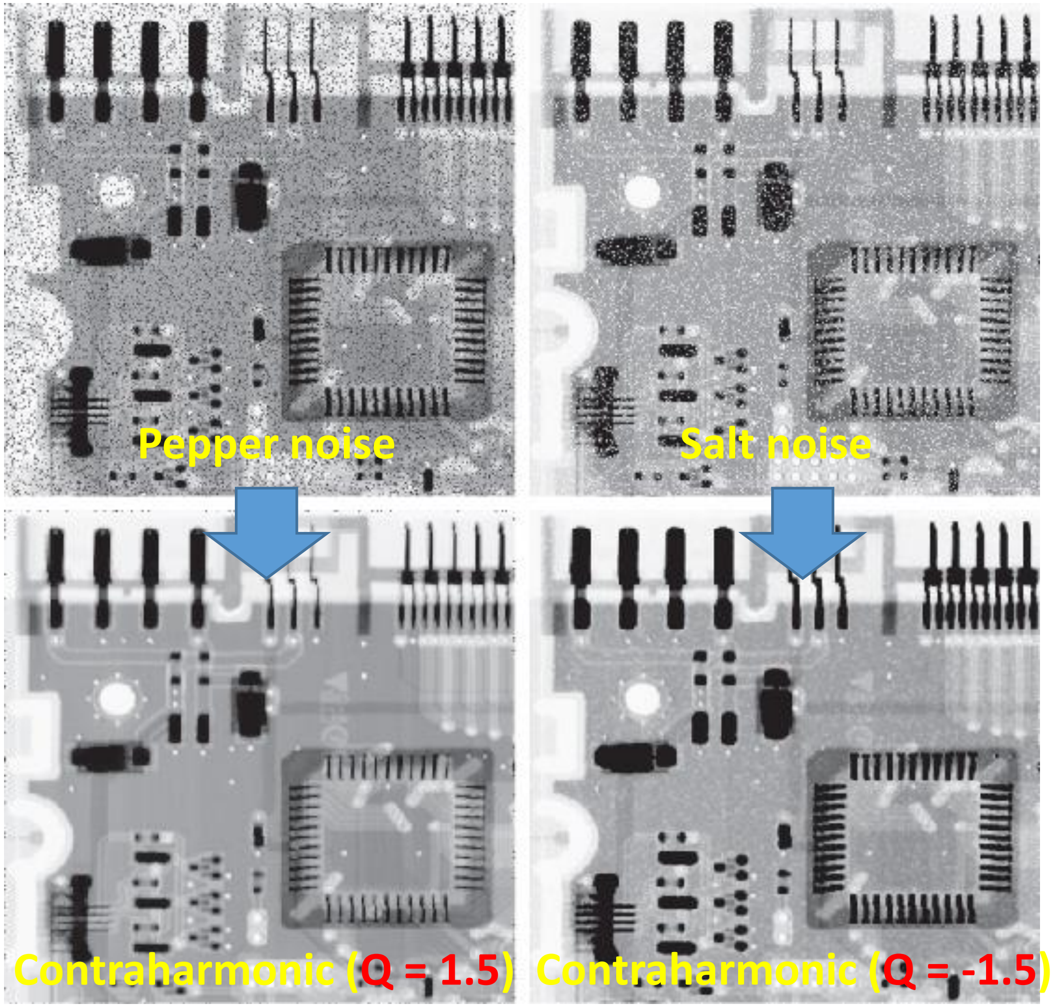
Mean Filters (均值滤波器)

- **Harmonic mean filter (谐波均值滤波器):**

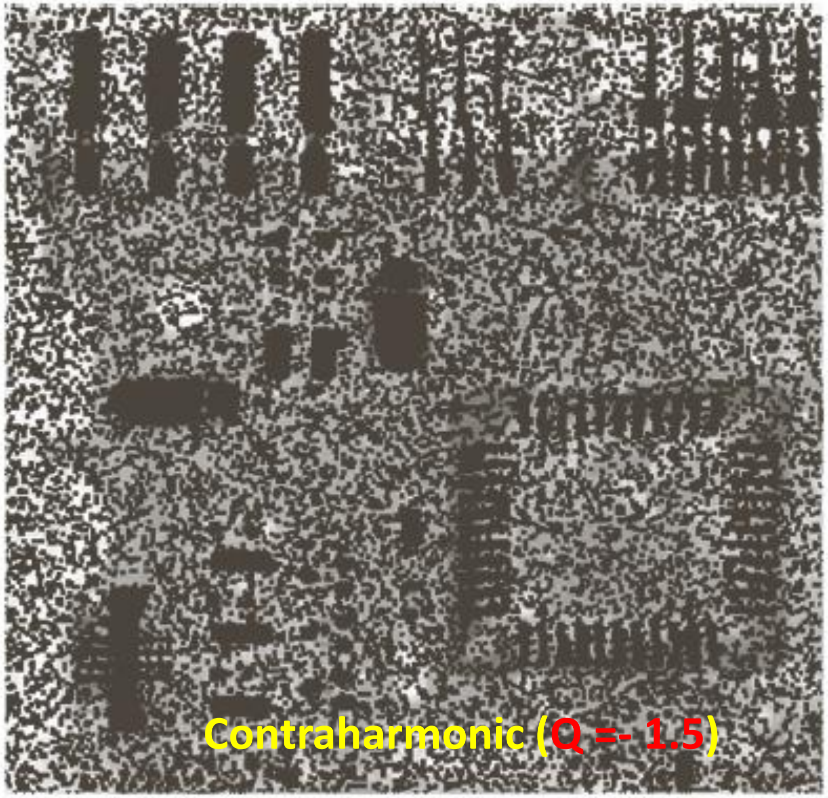
$$\hat{f}(x,y) = \frac{mn}{\sum_{(s,t)\in S_{xy}} \frac{1}{g(s,t)}}$$

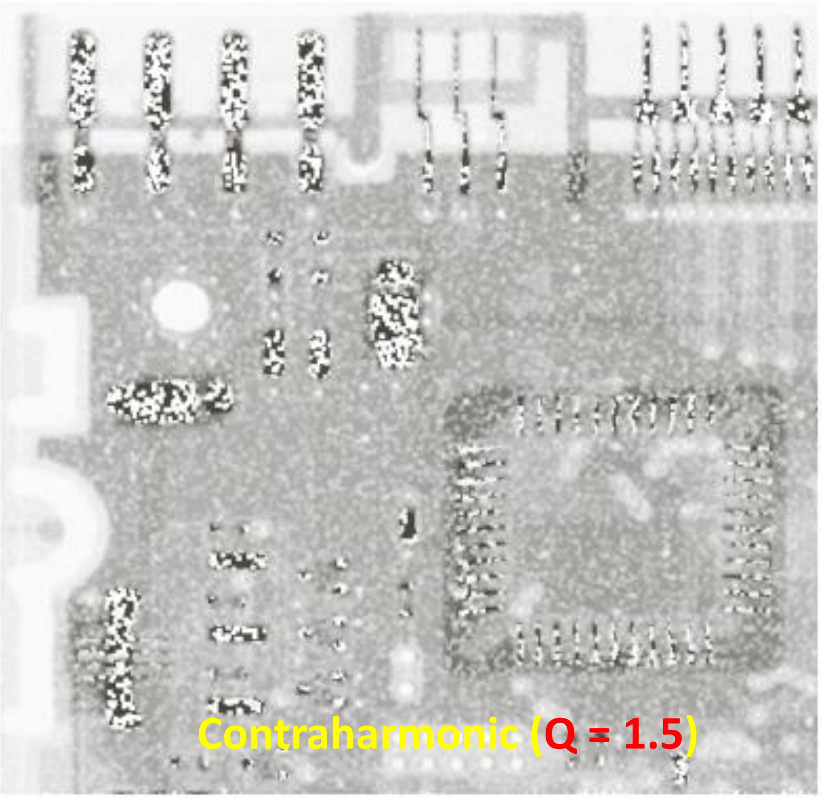
- **Contraharmonic mean filter (逆谐波均值滤波器):**

$$\hat{f}(x,y) = \frac{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q+1}}{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q}}$$



Contraharmonic mean filter (wrong value of Q)



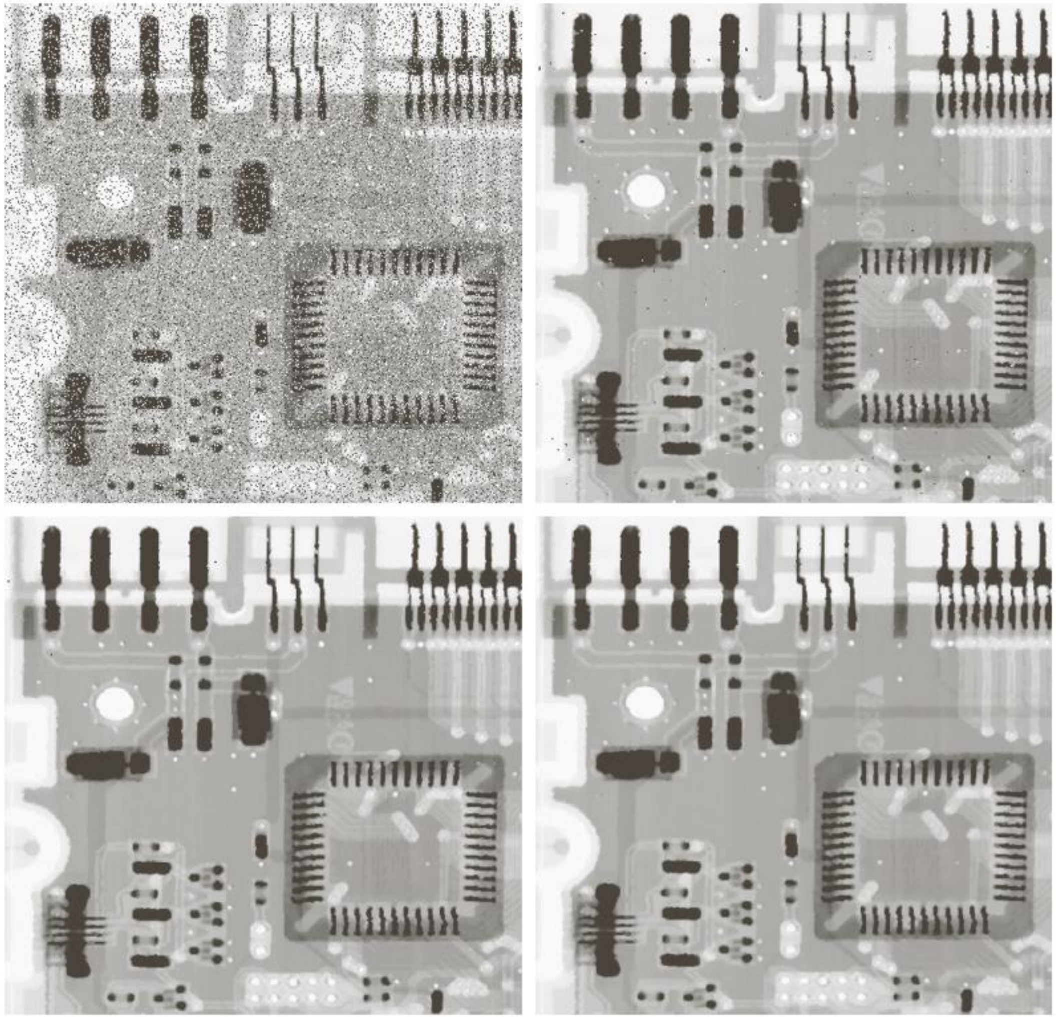


Order-statistic Filters (统计排序滤波器)

- **Median filter (中值滤波器):**

  $\hat{f}(x, y)$

  $= \underset{(s,t)\in S_{xy}}{\text{median}}\{g(s, t)\}$



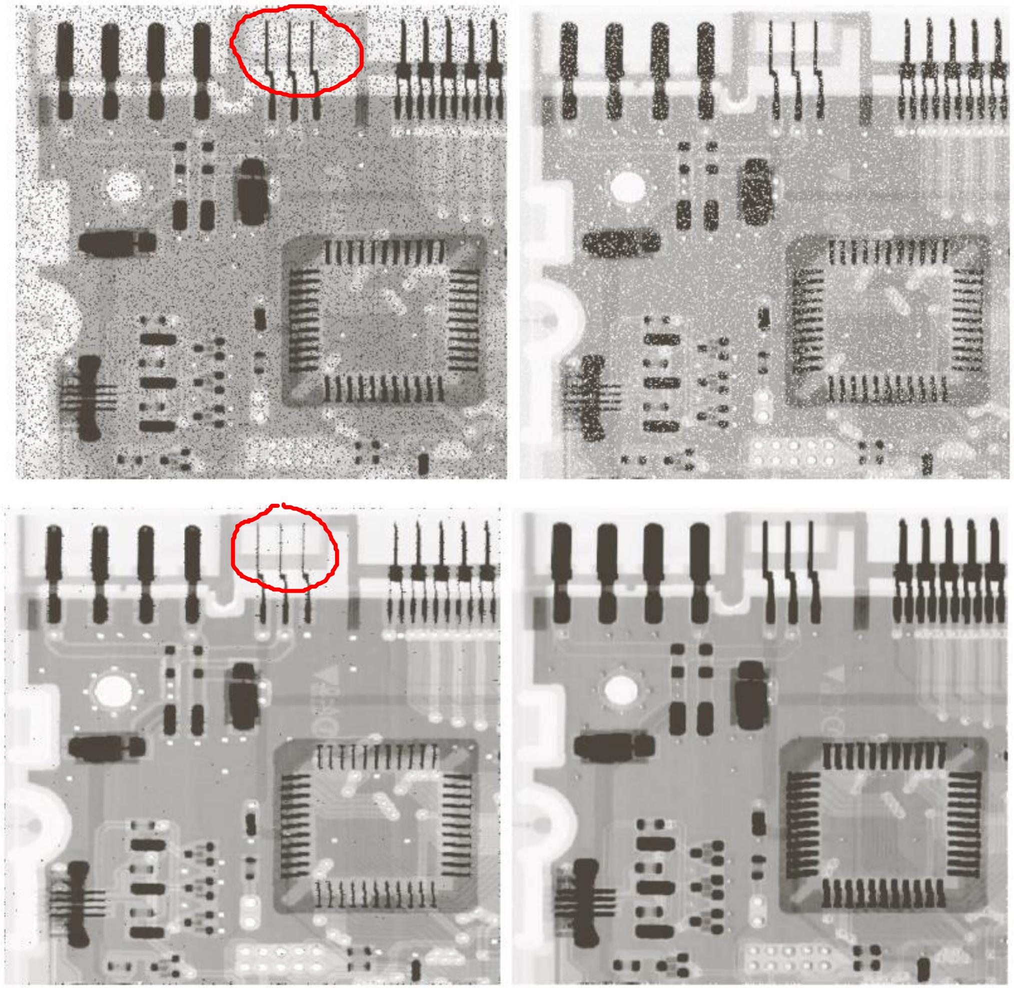
Order-statistic Filters (统计排序滤波器)

- **Max filter**
  (最大值滤波器)

$$\hat{f}(x,y) = \max_{(s,t)\in S_{xy}} \{g(s,t)\}$$

- **Min filter**
  (最小值滤波器)

$$\hat{f}(x,y) = \min_{(s,t)\in S_{xy}} \{g(s,t)\}$$



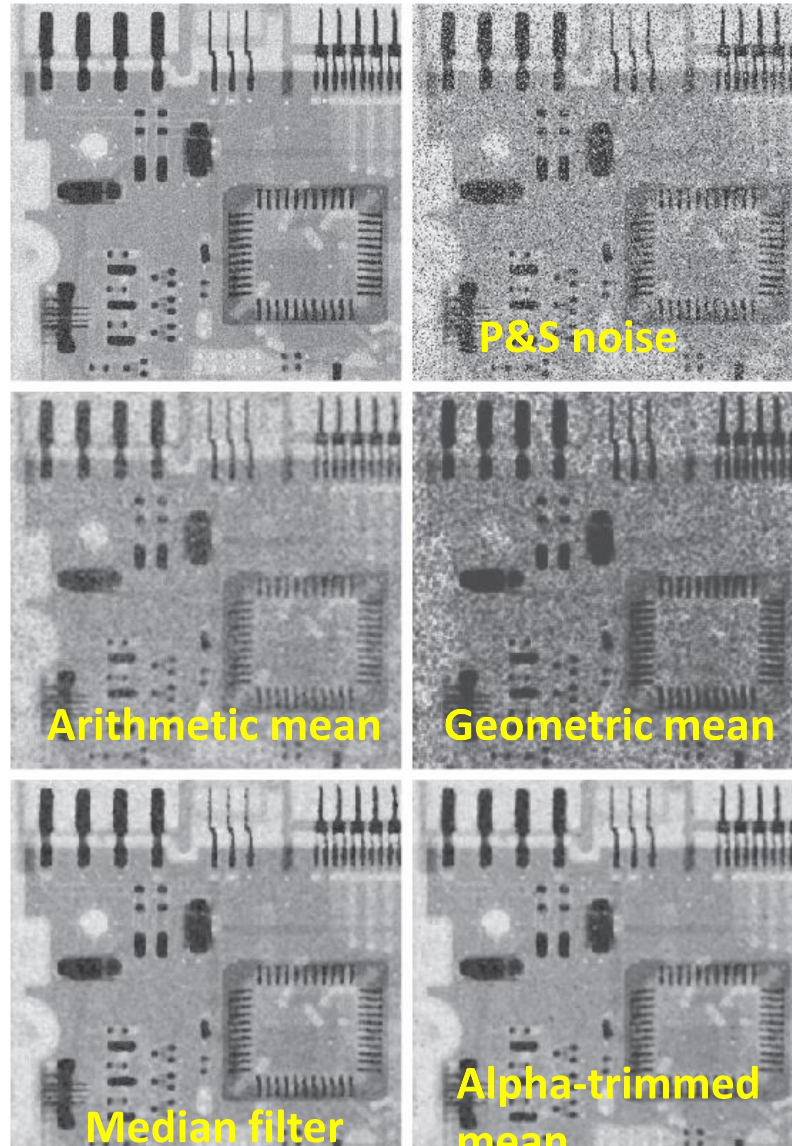
Order-statistic Filters (统计排序滤波器)

- **Midpoint filter (中点滤波器)**

$$\hat{f}(x,y) = \frac{1}{2}\left[\max_{(s,t)\in S_{xy}}\{g(s,t)\} + \min_{(s,t)\in S_{xy}}\{g(s,t)\}\right]$$

- **Alpha-trimmed mean filter (修正的阿尔法均值滤波器)**

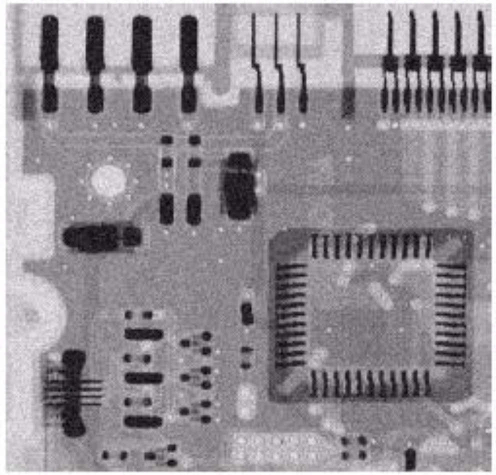
$$\hat{f}(x,y) = \frac{1}{mn-d}\sum_{(s,t)\in S_{xy}} g_r(s,t)$$

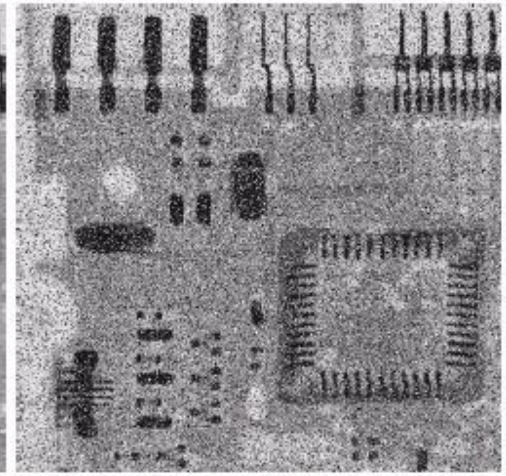


a b
c d
e f

FIGURE 5.12 (a) Image corrupted by additive uniform noise. (b) Image additionally corrupted by additive salt-and-pepper noise. (c)-(f) Image (b) filtered with a 5×5: (c) arithmetic mean filter; (d) geometric mean filter; (e) median filter; (f) alpha-trimmed mean filter, with $d = 6$.

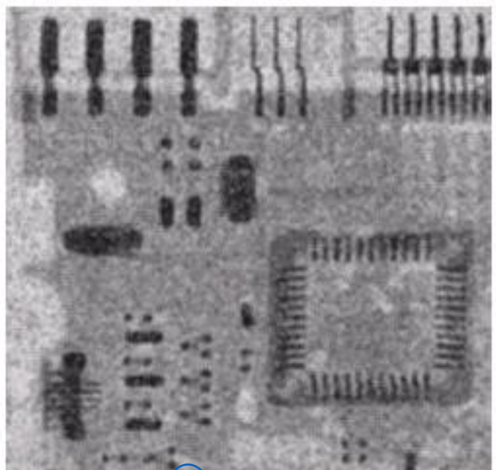
Uniform noise

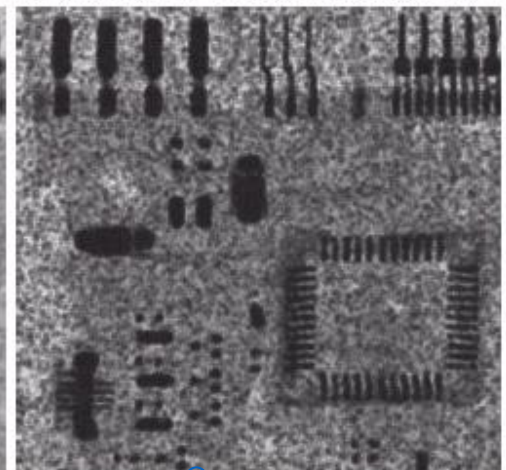




Left +
Bipolar Noise
$P_a = 0.1$
$P_b = 0.1$

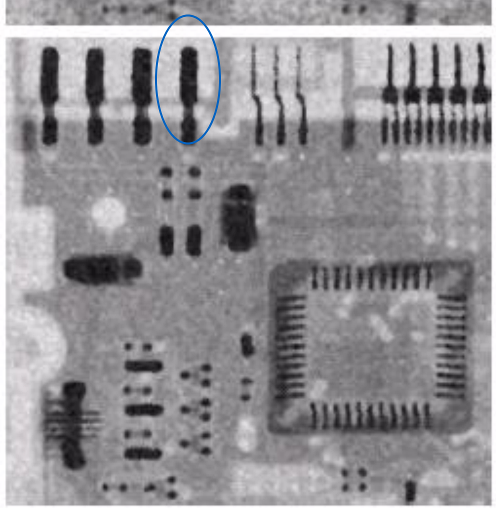
5x5
Arith. Mean
filter

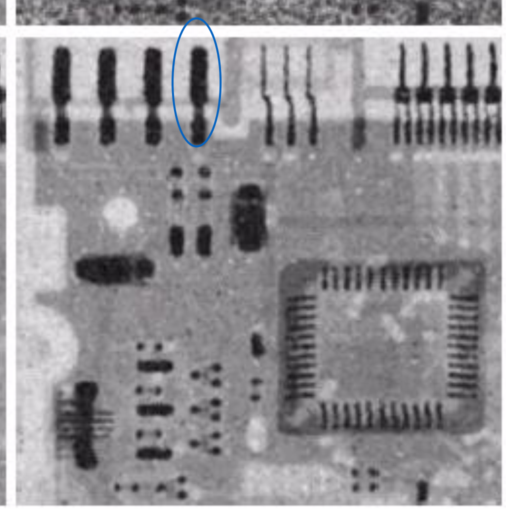




5x5
Geometric
mean

5x5
Median
filter





5x5
Alpha-trim.
Filter
d=5

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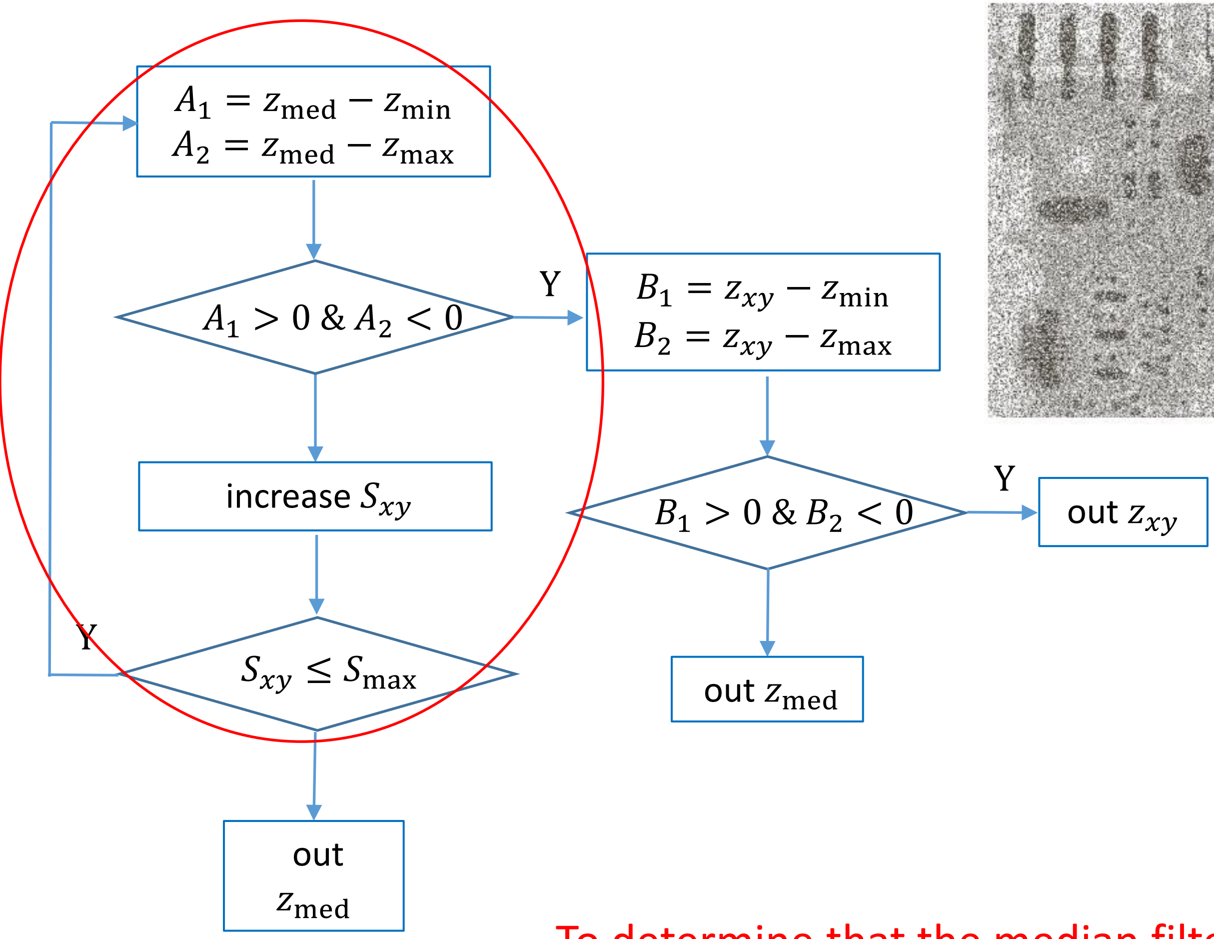
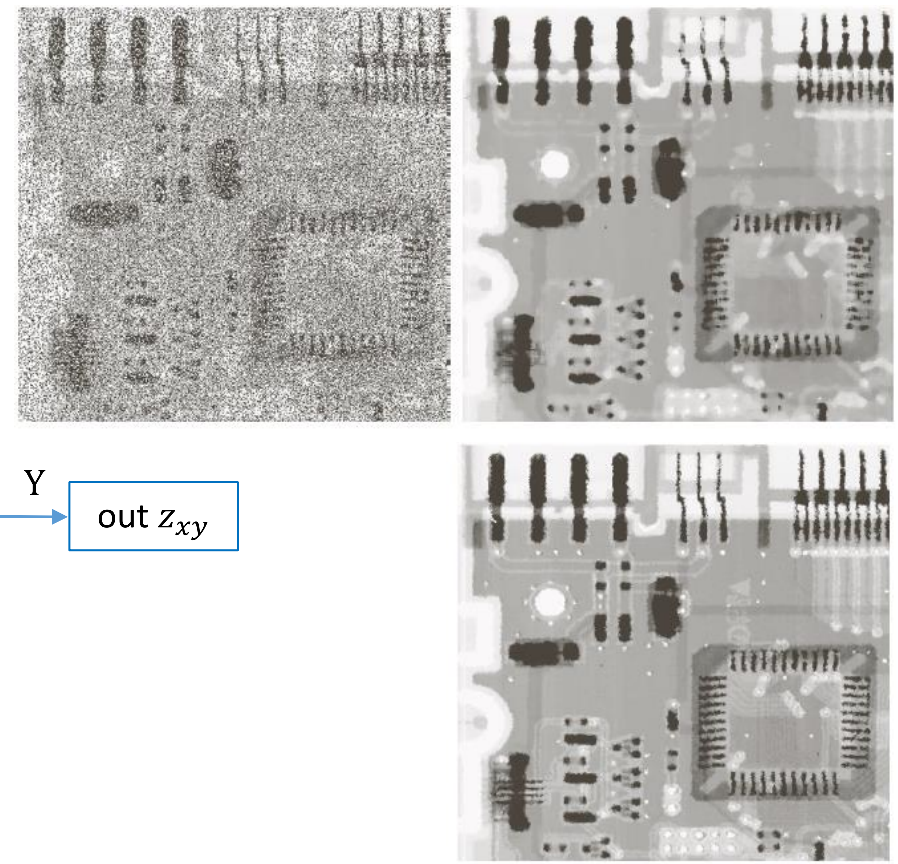
Adaptive Filters (自适应滤波器)

❑ Adaptive Median filter (自适应中值滤波器):





To determine that the median filter output z_{med} is not a impulse

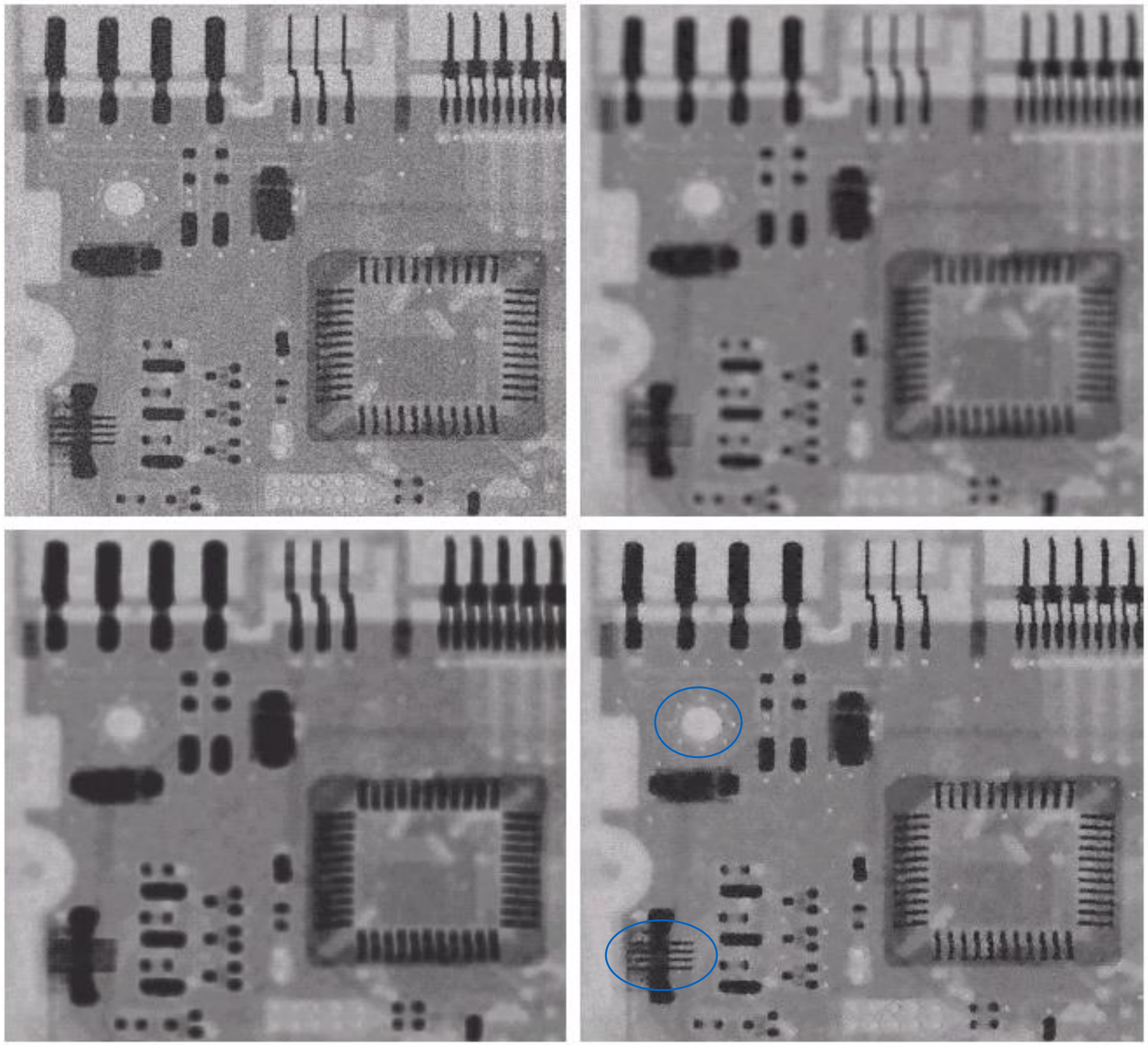
Adaptive, Local Noise-reduction filter

- **Adaptive depending on noise characteristics in a local window**
- **Suppose we know**
  - $\hat{I}(\mathrm{x,y})$: **degraded image**
  - σ_y^2: **noise variance across entire image**
  - $\widehat{\mu_L}(\mathrm{x,y})$: **local mean around (x,y)**
  - $\hat{\sigma}_L(\mathrm{x,y})$: **local variance around (x,y)**
- **Then we can use:** $\hat{\hat{I}}(\mathrm{x,y}) = \hat{I}(x,y) - \frac{\sigma_y^2}{\hat{\sigma}_L{}^2}\left(\hat{I}(x,y) - \widehat{\mu_L}\right)$
  - **If** $\sigma_y^2 = 0$ **,then** $\hat{\hat{I}}(\mathrm{x,y}) = \hat{I}(x,y)$
  - **If** $\hat{\sigma}_L{}^2 \gg \sigma_y^2$**, then** $\hat{\hat{I}}(\mathrm{x,y}) = \hat{I}(x,y)$ **Why good? A high local variance means an edge. Edge preserved.**
  - **If** $\hat{\sigma}_L{}^2 \approx \sigma_y^2$**, then** $\hat{\hat{I}}(\mathrm{x,y}) \approx \widehat{\mu_L}(\mathrm{x,y})$

Adaptive, Local Noise-reduction filter

Gaussian noise

Arith. mean 7x7



Geometric mean 7x7

adaptive

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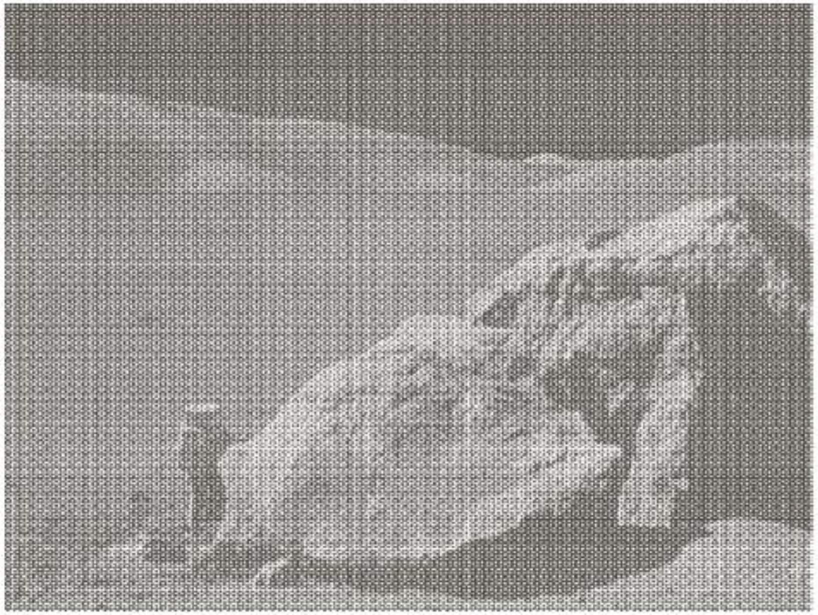
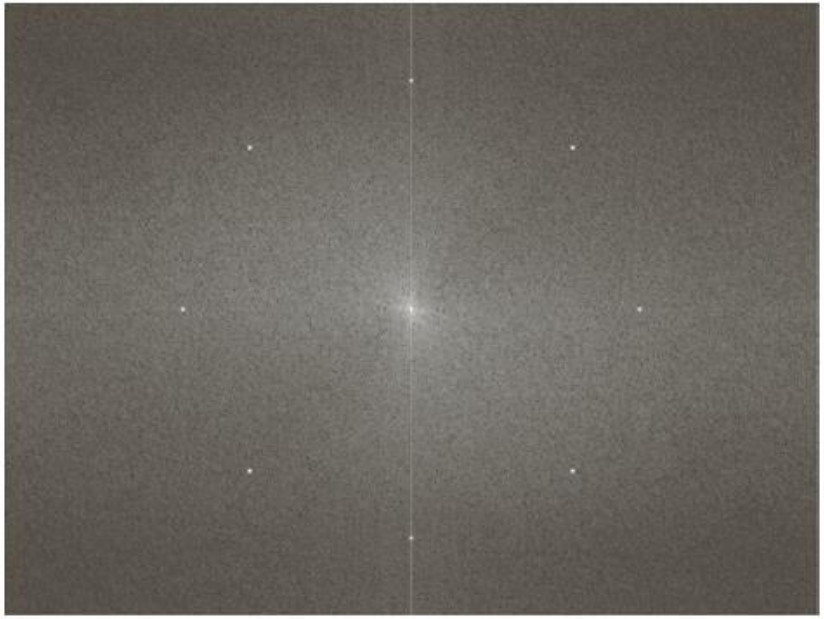
Periodic Noise





Solution: Bandpass filter & Notch filter

Periodic noise reduction

- Pure sine wave
  - Appear as a pair of impulse (conjugate) in the frequency domain

$$\begin{cases} f(x,y) = A\sin(u_0 x + v_0 y) \\ F(u,v) = -j\dfrac{A}{2}\left[\delta(u-\dfrac{u_0}{2\pi}, v-\dfrac{v_0}{2\pi}) - \delta(u+\dfrac{u_0}{2\pi}, v+\dfrac{v_0}{2\pi})\right] \end{cases}$$

Frequency Domain Filtering

- Mainly for periodic noise
  - Bandreject Filters (带阻滤波器)
  - Bandpass Filters (带通滤波器)
  - Notch Filters (陷波滤波器)
  - Optimum Notch Filters (最佳陷波滤波器)

Bandreject Filters

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

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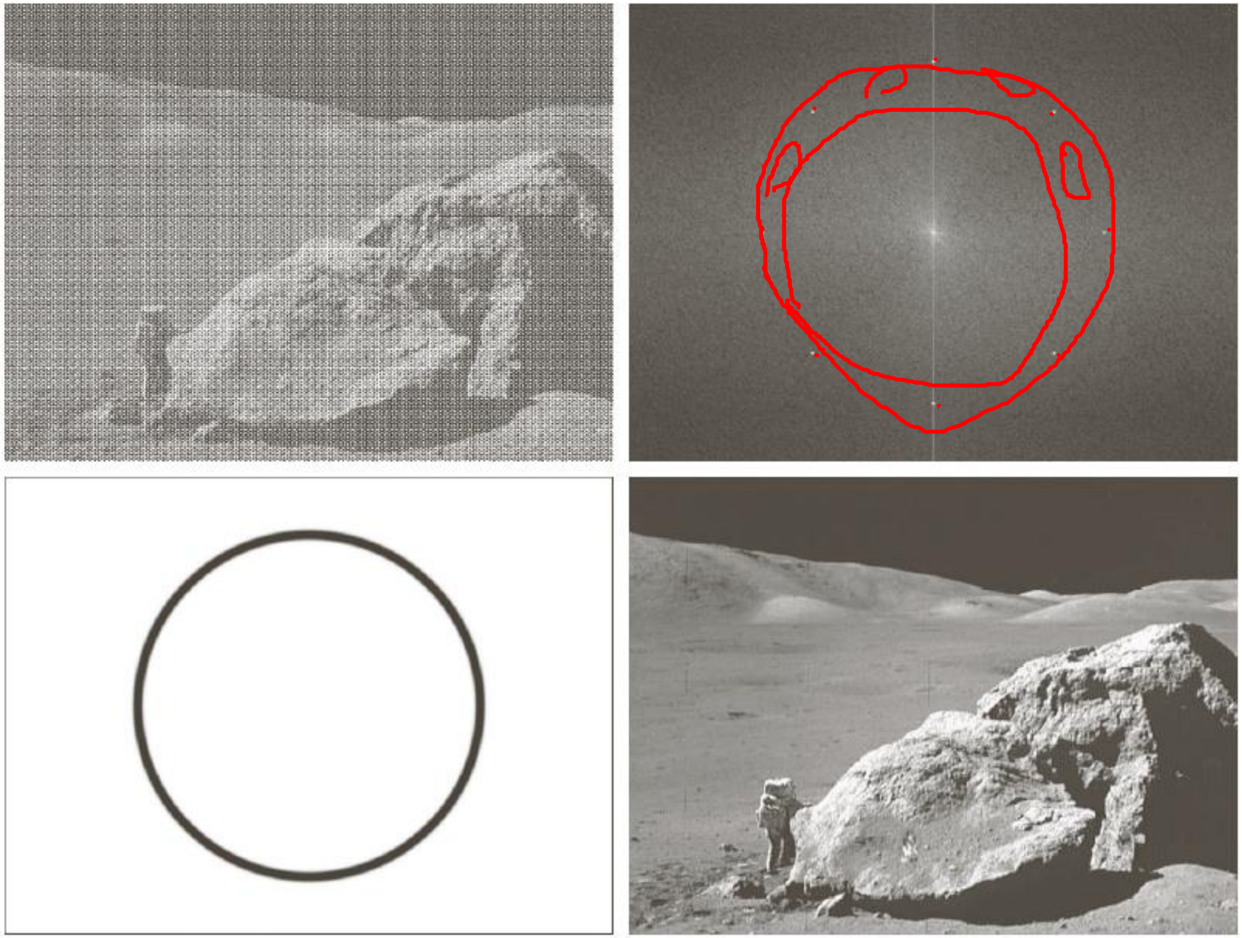
Bandreject Filters



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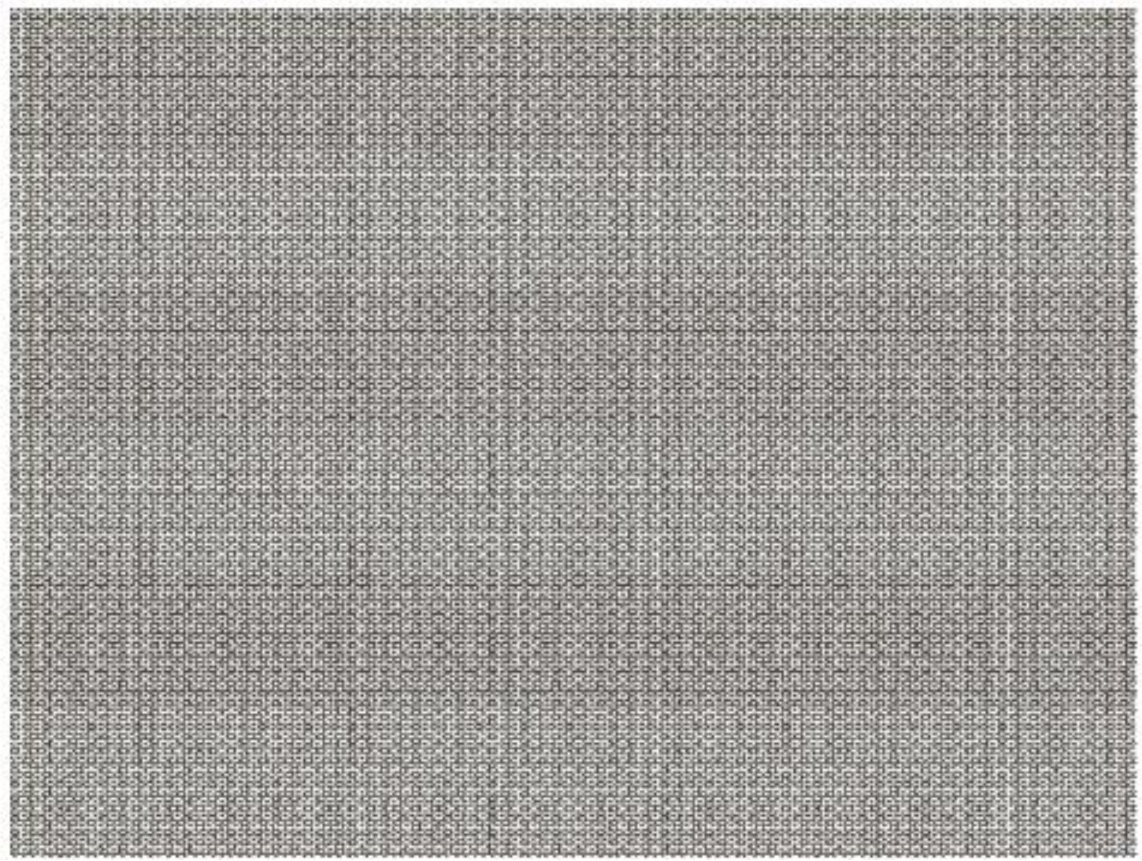
Bandpass Filters

$$H_{\text{BP}}(u,v) = 1 - H_{\text{BR}}(u,v)$$



Notch Filters

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v)H_{-k}(u,v)$$

$$H_{\mathrm{NP}}(u,v) = 1 - H_{\mathrm{NR}}(u,v)$$

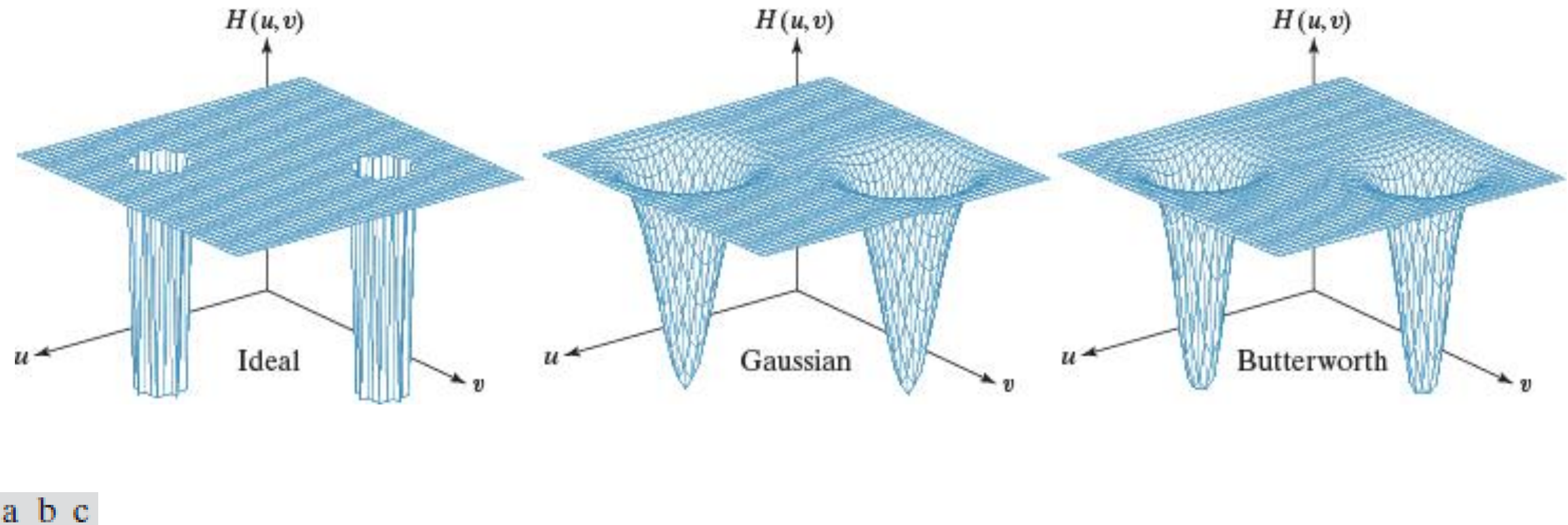
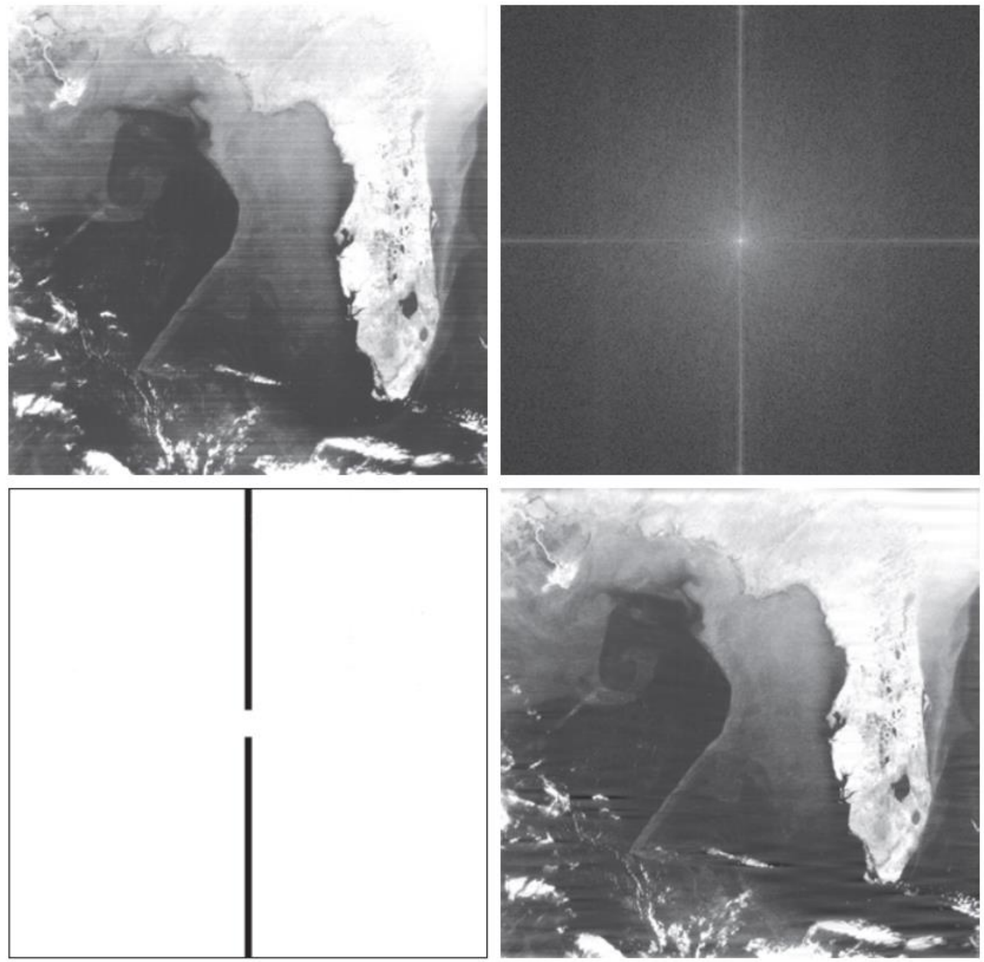


FIGURE 5.15 Perspective plots of (a) ideal, (b) Gaussian, and (c) Butterworth notch reject filter transfer functions.

Notch Filters



a b
c d

FIGURE 5.18
(a) Satellite image of Florida and the Gulf of Mexico. (Note horizontal sensor scan lines.) (b) Spectrum of (a). (c) Notch reject filter transfer function. (The thin black border is not part of the data.) (d) Filtered image. (Original image courtesy of NOAA.)

Optimum Notch Filters (最佳陷波滤波器)

Noise pattern in spatial domain

$$\eta(x,y) = \mathcal{F}^{-1}\{H_{\mathrm{NP}}(u,v)G(u,v)\}$$

Obtain estimate of $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$

$$\hat{f}(x,y) = g(x,y) - w(x,y)\eta(x,y)$$

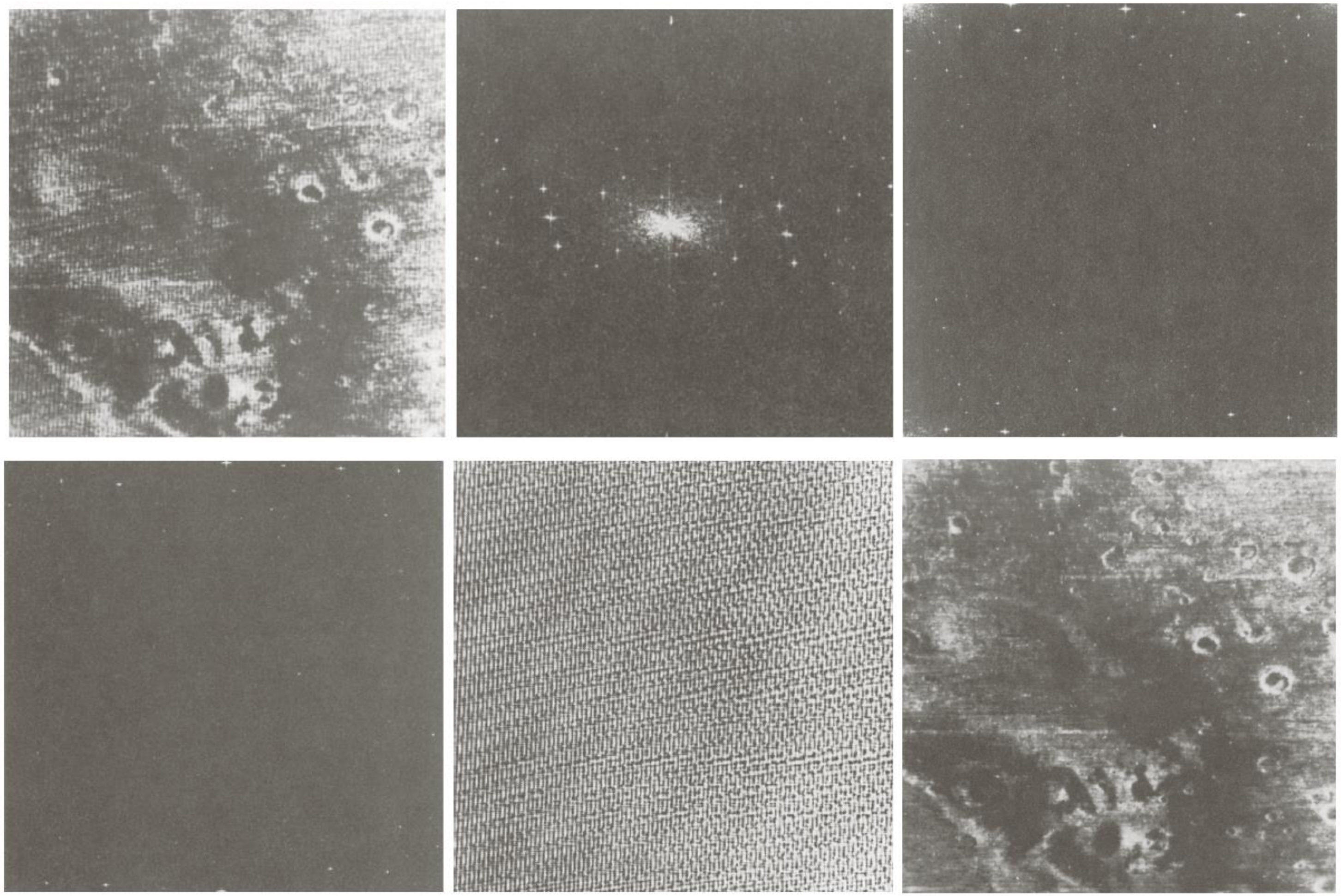
Estimate variance of $\hat{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y})$

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}\left[\hat{f}(x+s,y+t) - \bar{\hat{f}}(x,y)\right]^2$$

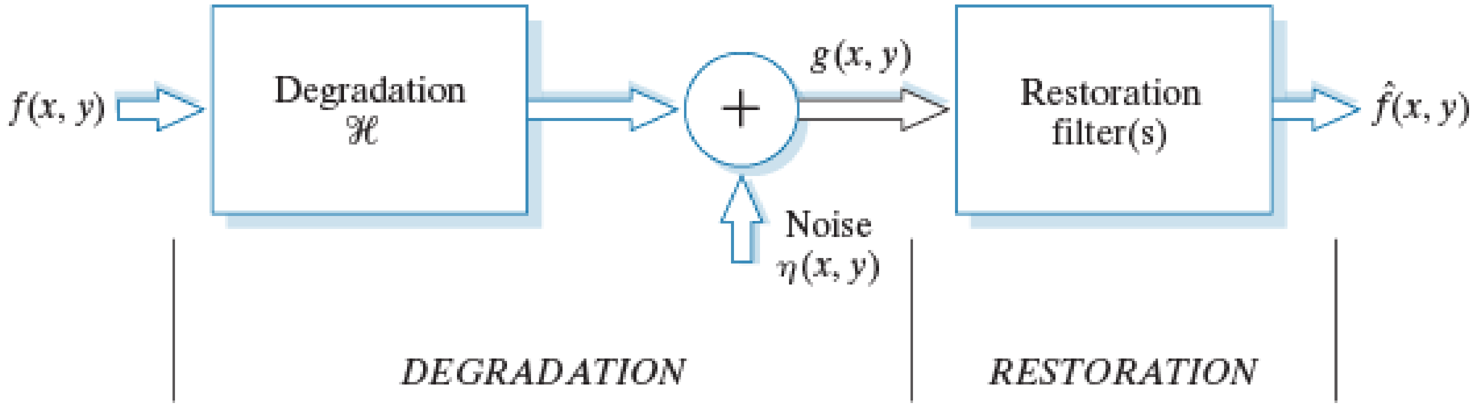
Minimize $\sigma^2(x,y)$, and solve $w(x,y)$

$$\frac{\partial \sigma^2(x,y)}{\partial w(x,y)} = 0 \quad \Rightarrow \quad w(x,y) = \frac{\overline{g(x,y)\eta(x,y)} - \bar{g}(x,y)\bar{\eta}(x,y)}{\overline{\eta^2}(x,y) - \bar{\eta}^2(x,y)}$$

Optimum Notch Filters
(最佳陷波滤波器)



Model of Image Degradation (图像退化模型)



Spatial domain: $g(x,y) = h(x,y) \star f(x,y) + \eta(x,y)$

Frequency domain: $G(u,v) = H(u,v)F(u,v) + N(u,v)$

***H* is a linear, position-invariant process**

$g(x,y)$: a degraded image $\quad f(x,y)$: input image

$h(x,y)$: degradation function $\quad \eta(x,y)$: additive noise term

Task for restoration: to find out $\hat{f}(x,y)$, estimation of original $f(x,y)$

Linear, position-invariant degradation system

Properties of the degradation function H

- Linear system
  - $H[af_1(x,y)+bf_2(x,y)]=aH[f_1(x,y)]+bH[f_2(x,y)]$
- Position(space)-invariant system
  - $H[f(x,y)]=g(x,y)$
  - $\Leftrightarrow$ $H[f(x-a, y-b)]=g(x-a, y-b)$
- c.f. 1-D signal
  - LTI (linear time-invariant system)

Linear, position-invariant degradation system

- Linear system theory is ready
- Non-linear, position-dependent system
  - May be general and more accurate
  - Difficult to solve computationally
- Image restoration: find H(u,v) and apply inverse process
- Image deconvolution

Estimating the degradation function

- Estimation by Image observation
- Estimation by experimentation
- Estimation by modeling

Estimation by image observation

- Take a window in the image
  - Sample structure
  - Strong signal content
- Estimate the original image in the window

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

known

estimate

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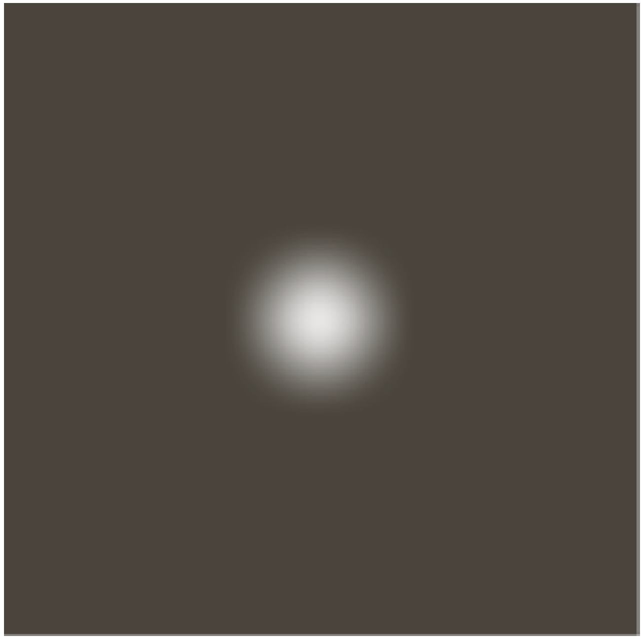
Estimation by experimentation

- If the image acquisition system is ready
- Obtain the impulse response

$$H(u,v) = \frac{G(u,v)}{A}$$

$G(u,v)$:the observed image $\quad$ A: a constant strength of the impulse



Estimation by modeling

- Ex. Atmospheric model $H(u,v) = e^{-k(u^2+v^2)^{5/6}}$

original

k=0.0025

k=0.001

k=0.00025

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Estimation by modeling (2)

- Derive a **mathematical model**
- Ex. Motion of image

$$g(x,y)=\int_0^T f(x-x_0(t),y-y_0(t))dt$$

Planar motion

$$x_0 = at/T$$
$$y_0 = bt/T$$

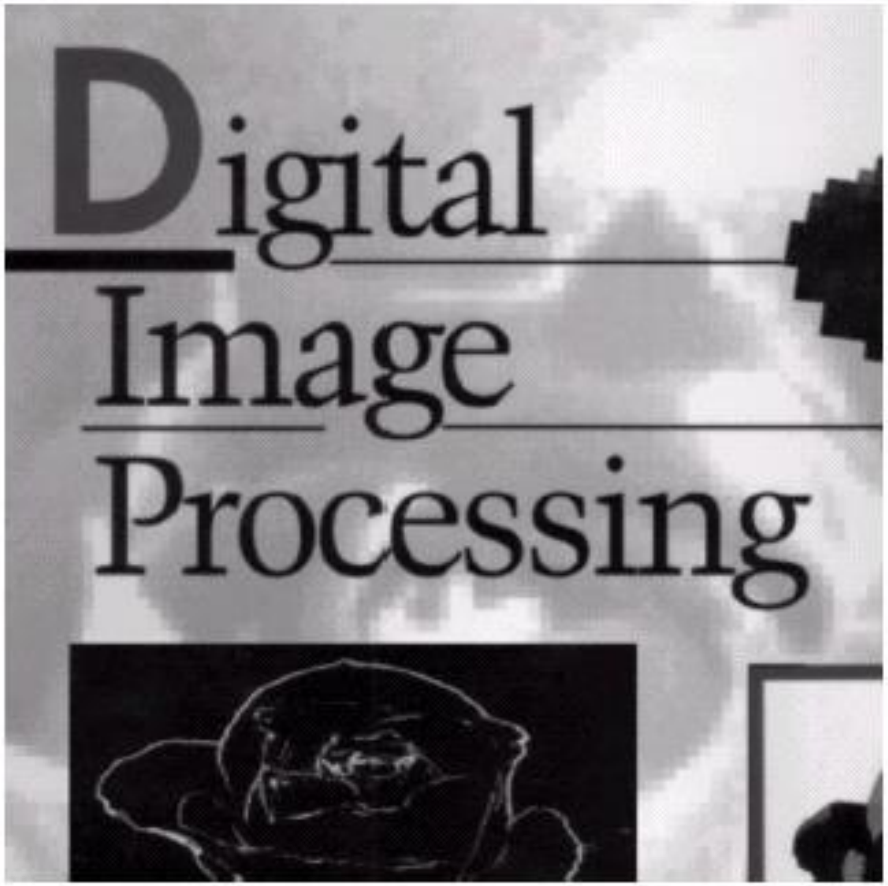
Fourier transform

$$G(u,v)=F(u,v)\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$H(u,v)=\frac{T}{\pi(ua+vb)}\sin[\pi(ua+vb)]e^{-j\pi(ua+vb)}$$

Estimation by modeling: example

Original



Apply motion model

Inverse filtering

- With the estimated degradation function H(u,v)

Unknown noise

G(u,v)=F(u,v)H(u,v)+N(u,v)

$$=> \quad \hat{F}(u,v) = \frac{G(u,v)}{H(u,v)} = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

Estimate of original image

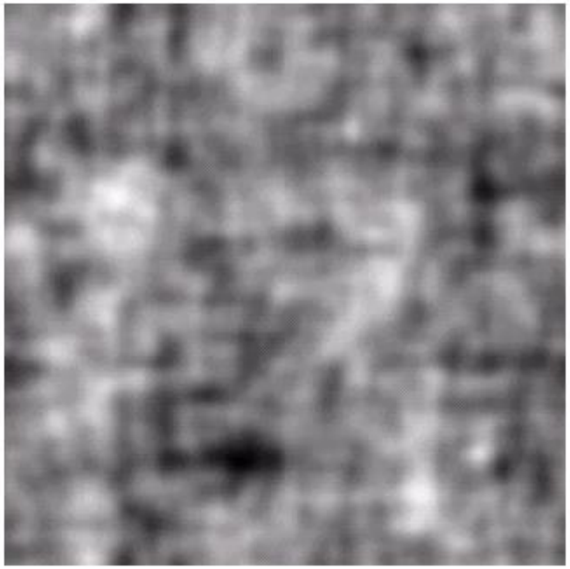
Problem: 0 or small values

Sol: limit the frequency around the origin

Inverse filtering

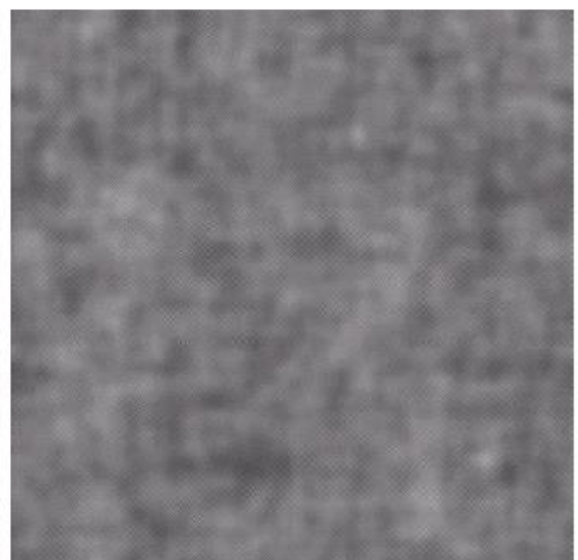
Full inverse filter for

k=0.0025



Cut Outside 40%

Cut Outside 70%



Cut Outside 85%

Wiener Filtering（维纳滤波）

- Expected value of mean square error (最小均方误差滤波器)

$$e^2 = E\left\{\left(I(x,y) - \hat{I}(\mathrm{x},\mathrm{y})\right)^2\right\}$$

- The estimate of f in frequency domain

$$\hat{F}(u,\mathrm{v}) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}\right] G(u,\mathrm{v}) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}\right] G(u,\mathrm{v})$$

- $S_f(u,v) = |I(u,v)|^2 \;\; S_n(u,v) = |N(u,v)|^2$； if no noise, get the inverse filter.

- We may be able to estimate $S_n(u,v)$, but we don't know $S_f(u,v)$. So instead we use:

$$\hat{F}(u,\mathrm{v}) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + K}\right] G(u,\mathrm{v})$$

Where K denote a tuning parameter.

https://en.wikipedia.org/wiki/Wiener_filter

Wiener filter derivation (fast)

Aim is to find filter which minimizes

$$\mathcal{E} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(f(x,y) - \hat{f}(x,y)\right)^2 dxdy$$

$$\boxed{\begin{aligned}\mathcal{E} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y) - \hat{f}(x,y)\right|^2 dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v) - \hat{F}(u,v)\right|^2 dudv\end{aligned}}$$

Why?

$$\hat{F} = WG = WHF + WN$$

G: observed, H: PSF, N: noise

$$F - \hat{F} = (1 - WH)F - WN$$

$$\begin{aligned}\mathcal{E} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(1 - WH)F - WN|^2\, dudv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{|(1 - WH)F|^2 + |WN|^2\}\, dudv\end{aligned}$$

since $f(x,y)$ and $\eta(x,y)$ uncorrelated

- Note, integrand is sum of two squares

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Wiener filter derivation (fast)

Minimize integral if integrand minimum for all *(u,v)*

NB $\frac{\partial}{\partial z}(zz^*) = 2z^*$

$$\frac{\partial}{\partial z} \rightarrow 2\left(-(1 - W^* H^*) H |F|^2 + W^* |N|^2\right) = 0$$

$$W^* = \frac{H|F|^2}{|H|^2|F|^2 + |N|^2}$$

$$\boxed{W = \frac{H^*}{|H|^2 + |N|^2/|F|^2}}$$

H: PSF filter

N^2/F^2: noise to signal ratio (NSR)

Note: filter is defined in the Fourier domain

Wirtinger derivatives

Basic rules for this *Wirtinger* calculus are given in the next theorem.

Theorem 3.0.2: *For the Wirtinger derivatives, the common rules for differentiation known from real-valued analysis concerning the sum, product, and composition of two functions hold as well. In particular,*

$$\frac{\partial}{\partial z} z^* = \frac{\partial}{\partial z^*} z = 0,$$

which means that z^* can be regarded as a constant when differentiating with respect to z, as well as z can be regarded constant when differentiating with respect to z^*.

Proof: With $z = x + \mathrm{j}y$ and $z^* = x - \mathrm{j}y$, Theorem 3.0.2 follows immediately from (3.8). $\square$

Examples:

- $\frac{\partial}{\partial z} |z|^2 = \frac{\partial}{\partial z} (zz^*) = z^*$
- $\frac{\partial}{\partial z} \exp\left(-|z|^2\right) = \frac{\partial}{\partial z} \exp\left(-zz^*\right) = -z^* \exp\left(|z|^2\right)$

https://mediatum.ub.tum.de/doc/631019/631019.pdf

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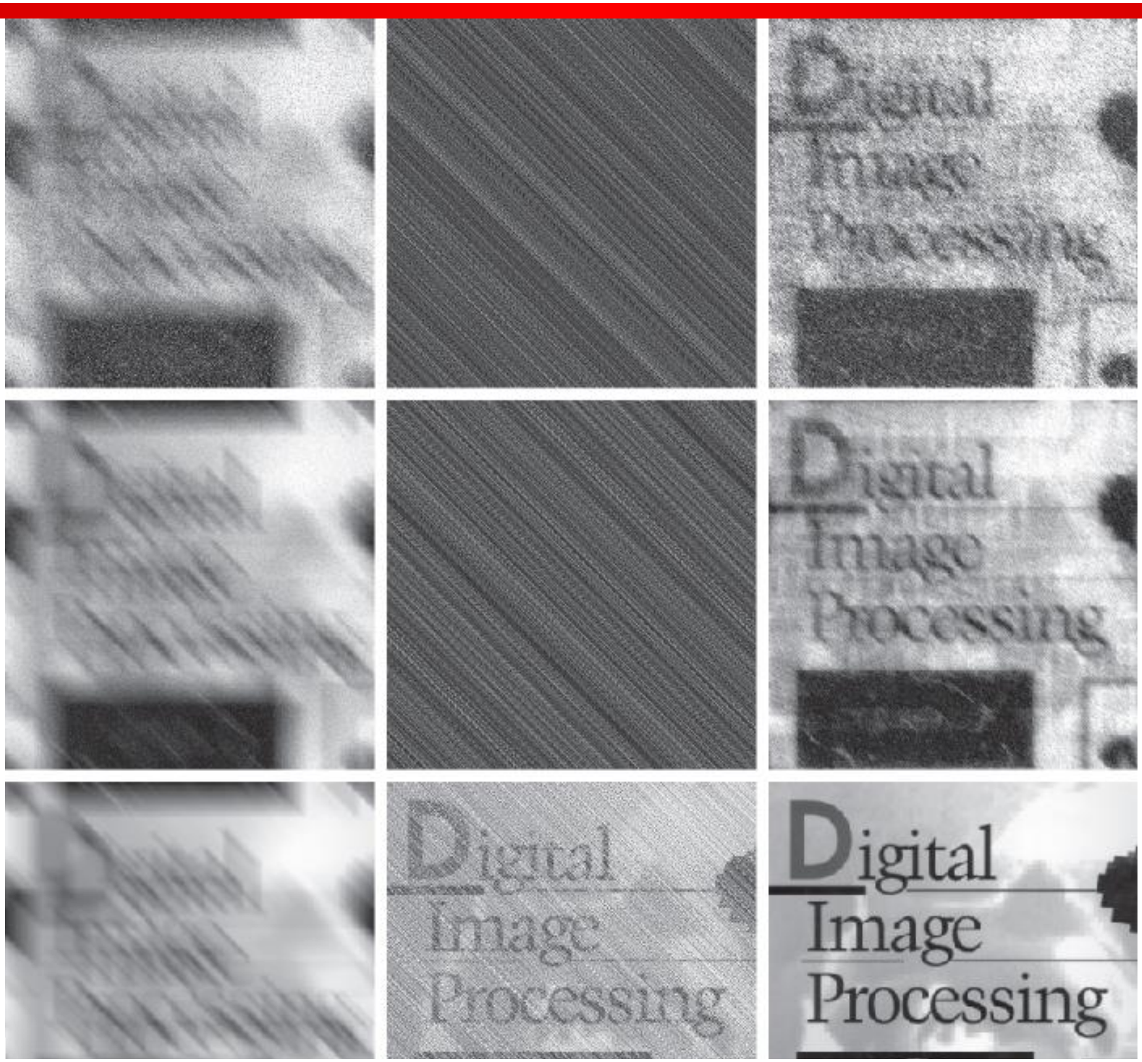
Wiener Filtering

a b c

FIGURE 5.28 Comparison of inverse and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.

Tuning K in Wiener Filtering



a b c
d e f
g h i

FIGURE 5.29 (a) 8-bit image corrupted by motion blur and additive noise. (b) Result of inverse filtering. (c) Result of Wiener filtering. (d)–(f) Same sequence, but with noise variance one order of magnitude less. (g)–(i) Same sequence, but noise variance reduced by five orders of magnitude from (a). Note in (h) how the deblurred image is quite visible through a "curtain" of noise.

Matlab Wiener Filter

```
% blurring
PSF = fspecial('motion',21,11);
Idouble = im2double(Ioriginal);
blurred = imfilter(Idouble,PSF,'conv','circula

% wiener filtering w/o noise
wnr1 = deconvwnr(blurred,PSF);

% wiener filtering with estimated NSR
estimated_nsr = noise_var/var(Idouble(:));
wnr3 = deconvwnr(blurred_noisy, PSF, estimated_nsr);
```

Blurred Image

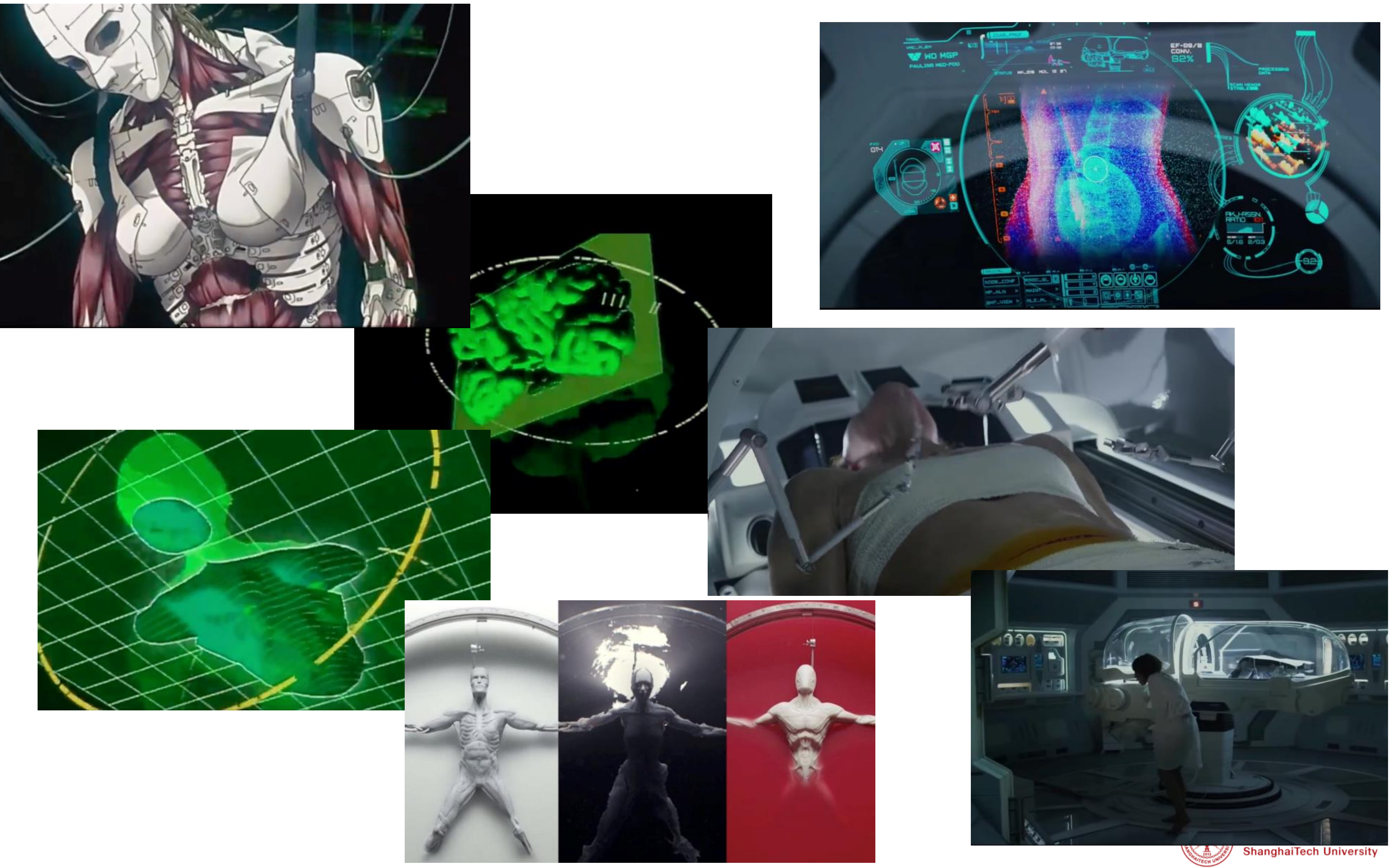
Restored Blurred Image

Restoration of Blurred Quantized Image (Estimated NSR)

Image reconstruction-outline

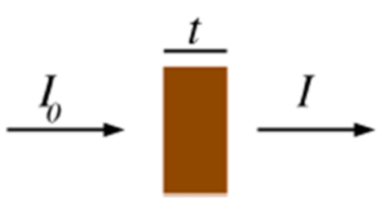
- The reconstruction problem
- Principles of Computed Tomography (CT)
- The Radon transform
- The Fourier-slice theorem
- Reconstruction by filtered back-projections

Image reconstruction & Sci-Fi

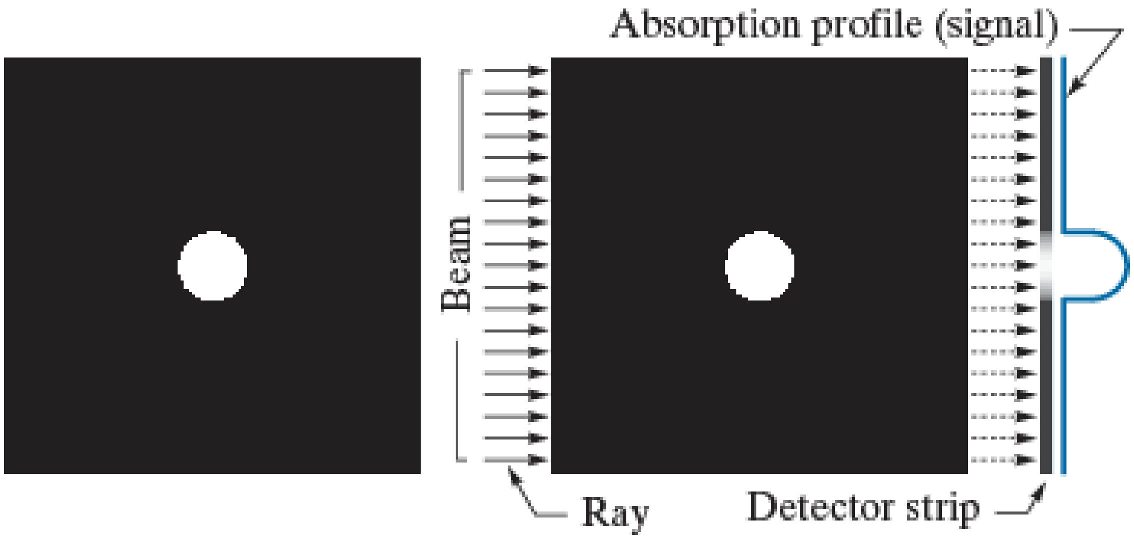


Reconstruction problem

- Consider a single object on a uniform background (suppose that this is a cross section of 3D region of a human body). Background represents soft, uniform tissue and the object is also uniform but with higher absorption characteristics.
- A beam of X-rays is emitted and part of it is absorbed by the object. The energy of absorption is detected by a set of detectors. The collected information is the absorption signal.

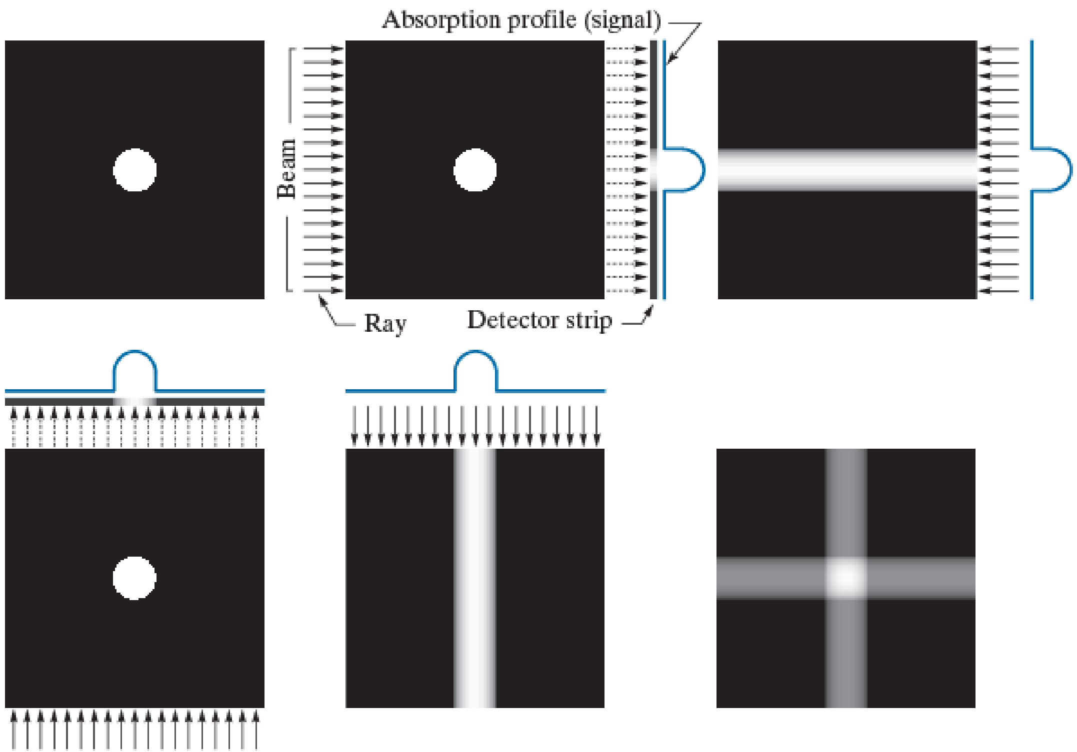


$$I(t) = I_0\, e^{-\mu t}$$



A simple way of reconstruction

- A simple way to recover the object is to back-project the 1D signal across the direction the beam came. This simply means to duplicate the signal across the 1D beam.



a b c
d e f

FIGURE 5.32
(a) Flat region with a single object. (b) Parallel beam, detector strip, and profile of sensed 1-D absorption signal. (c) Result of back-projecting the absorption profile. (d) Beam and detectors rotated by 90°. (e) Backprojection. (f) The sum of (c) and (e), intensity-scaled. The intensity where the backprojections intersect is twice the intensity of the individual back-projections.

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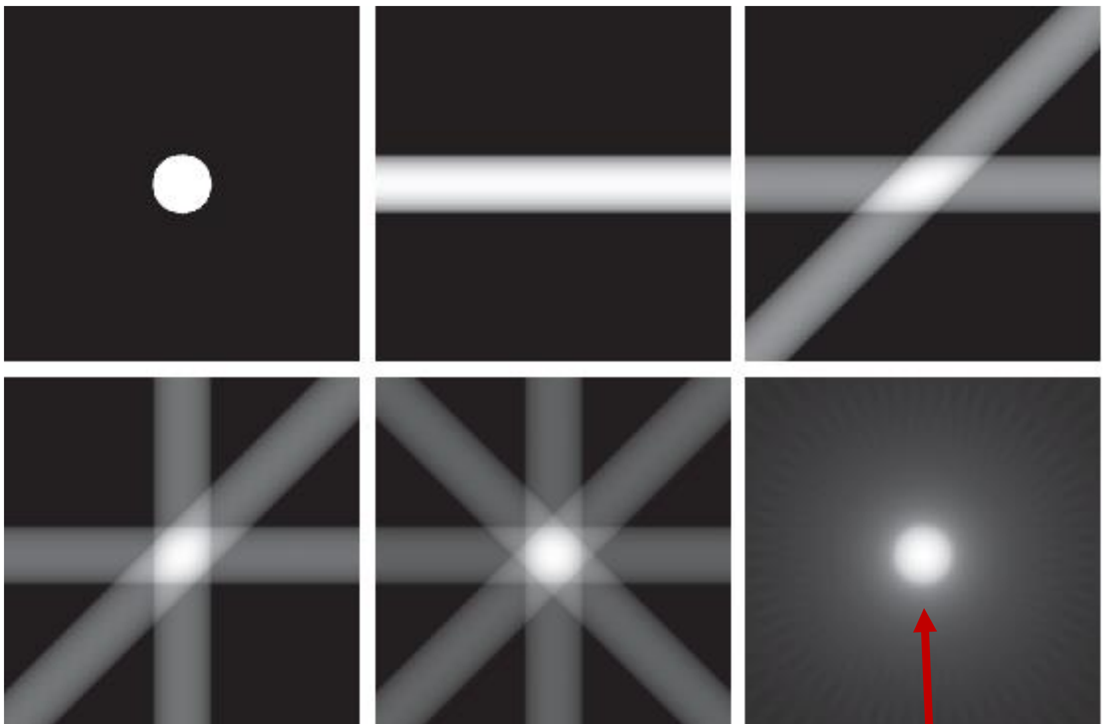
A simple way of reconstruction

- With more projection data



a b c
d e f

FIGURE 5.33 (a) Same as Fig. 5.32(a). (b)-(e) Reconstruction using 1, 2, 3, and 4 backprojections 45° apart. (f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).

But the image is highly blurred!

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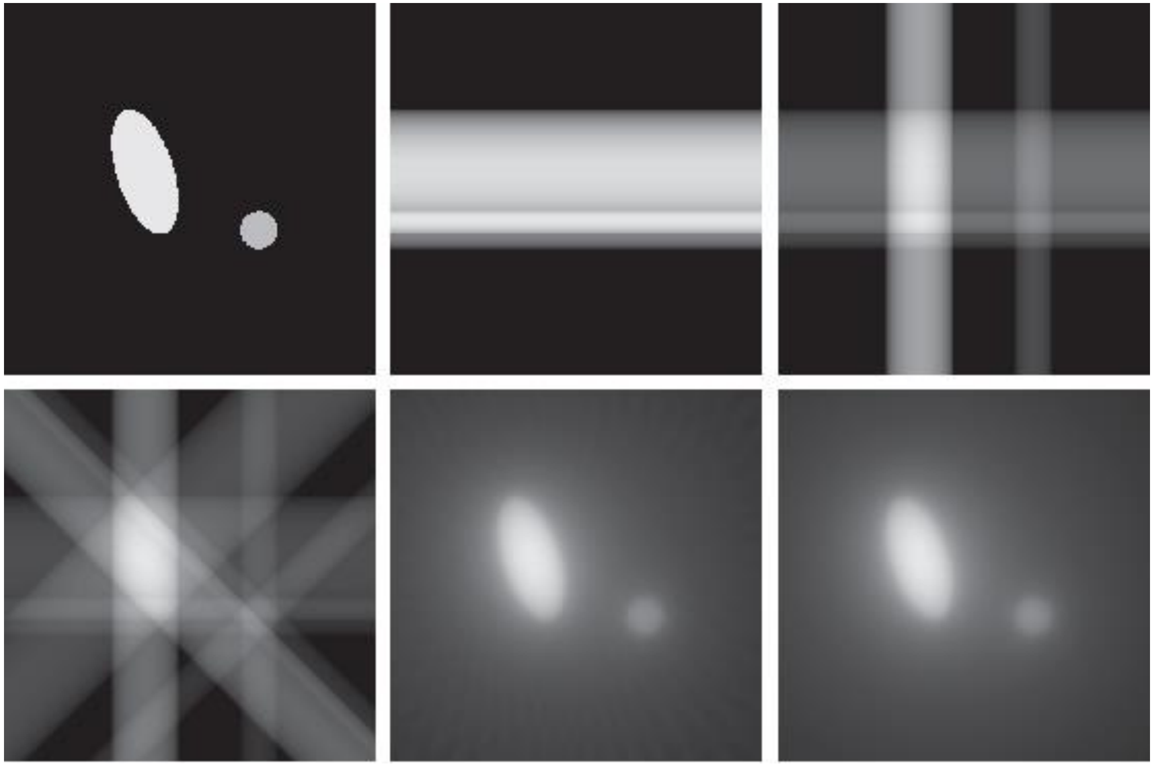
A simple way of reconstruction

- With more complicated target



a b c
d e f

FIGURE 5.34 (a) Two objects with different absorption characteristics. (b)–(d) Reconstruction using 1, 2, and 4 backprojections, 45° apart. (e) Reconstruction with 32 backprojections, 5.625° apart. (f) Reconstruction with 64 backprojections, 2.8125° apart.

Computed Tomography (CT)

- The goal of CT is to obtain a 3D representation of the internal structure of an object by X-raying it from many different directions. Imagine the traditional chest X-ray obtained by different directions. The image is the 2D equivalent of a line projections. Back-projecting the image would result in a 3D volume of the chest cavity.
- **Theory** developed in 1917 by J. Radon.
- **Application** developed in 1964 by A. M. Cormack and G. N. Hounsfield independently. They shared the Nobel prize in Medicine in 1979.

Johann Radon

Radon in c. 1920

Born	16 December 1887 Děčín, Bohemia, Austria-Hungary
Died	25 May 1956 (aged 68) Vienna, Austria
Nationality	Austrian

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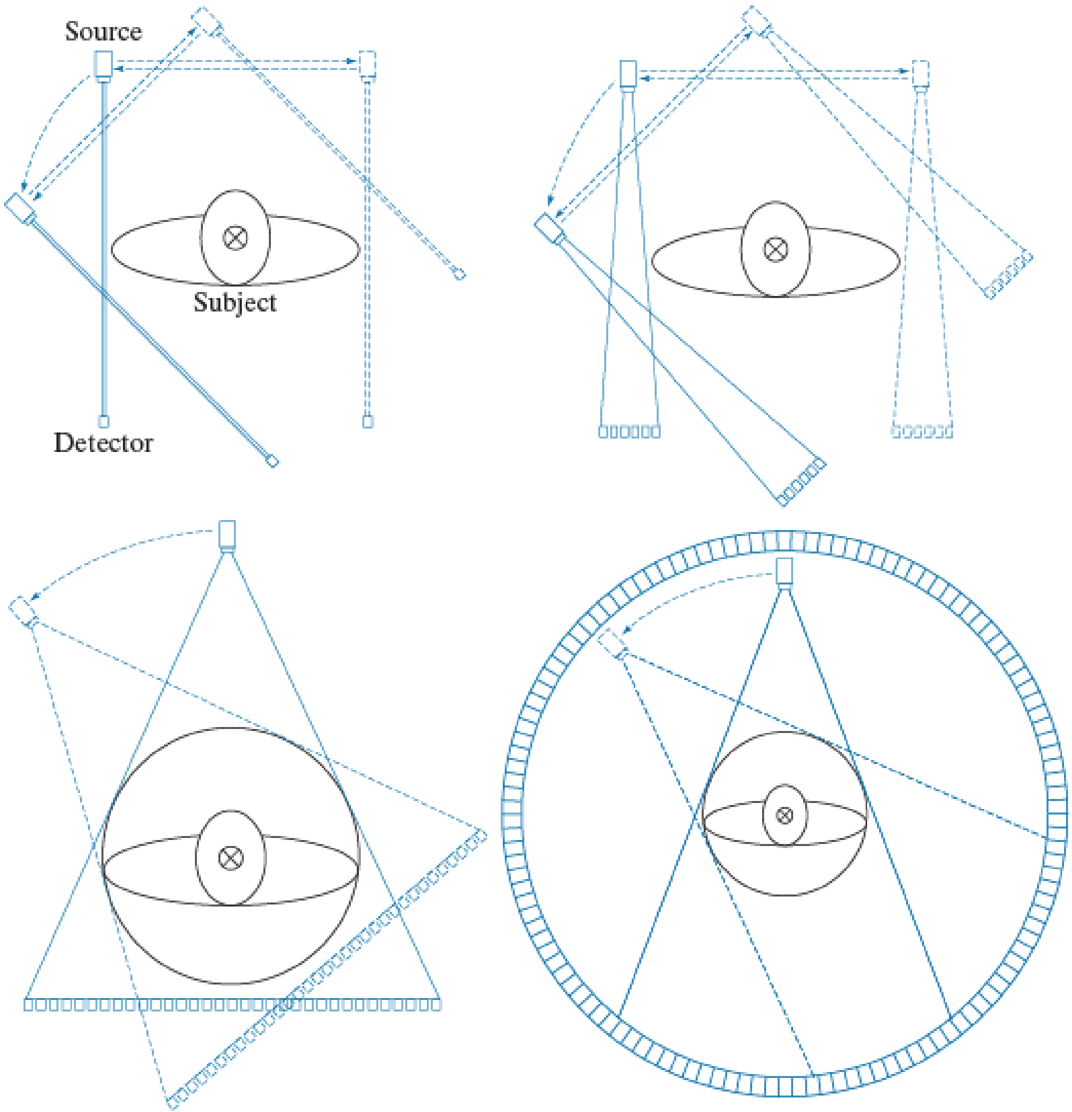
Computed Tomography (CT)

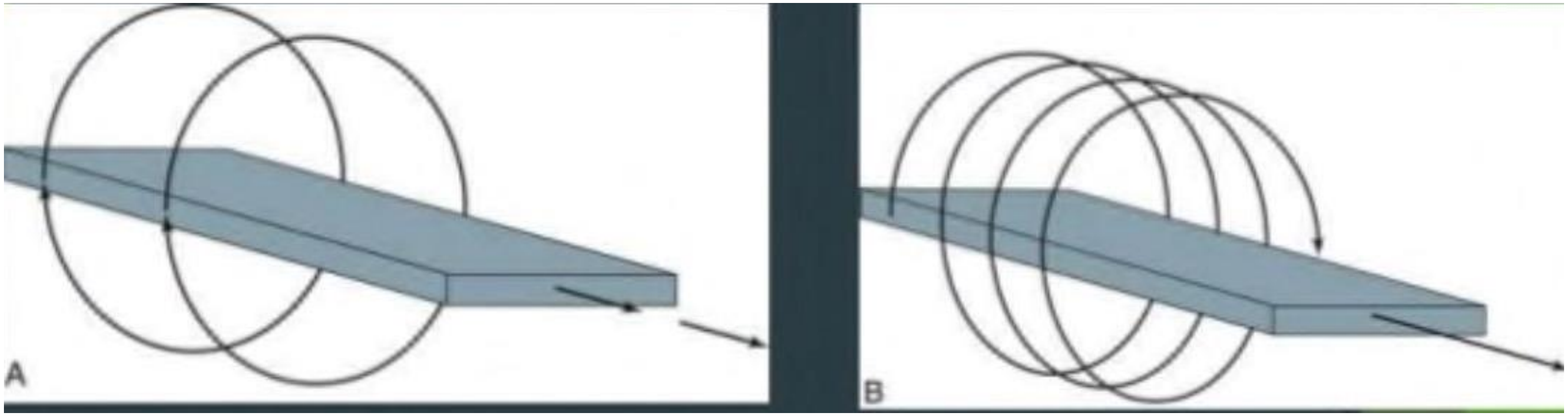


a b
c d

FIGURE 5.35
Four generations of CT scanners. The dotted arrow lines indicate incremental linear motion. The dotted arrow arcs indicate incremental rotation. The cross-mark on the subject's head indicates linear motion perpendicular to the plane of the paper. The double arrows in (a) and (b) indicate that the source/detector unit is translated and then brought back into its original position.

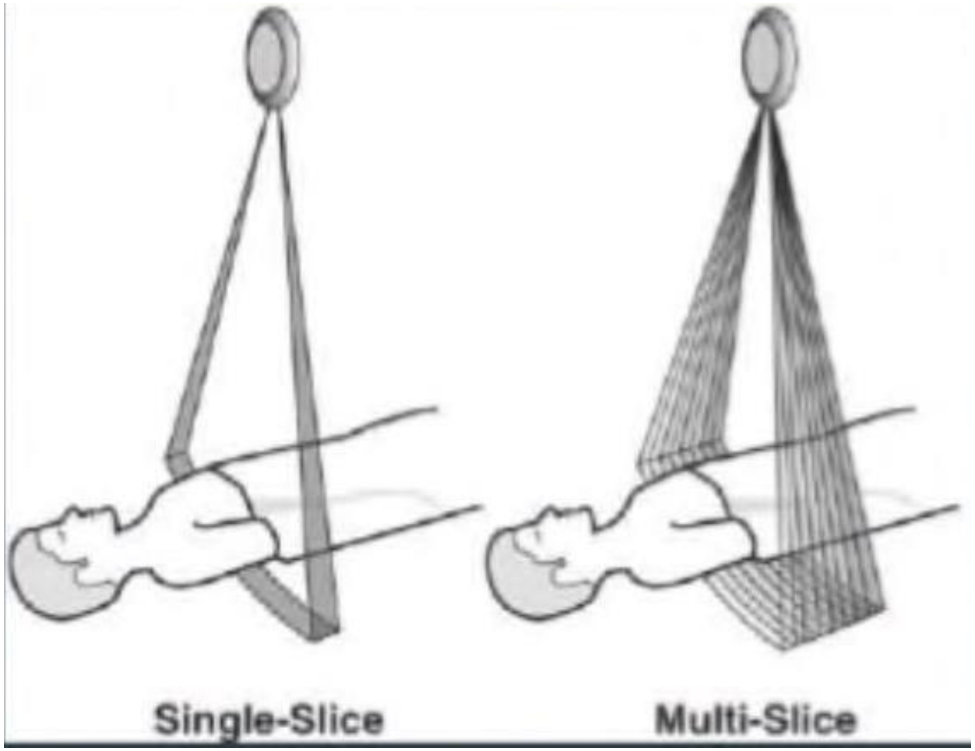
Helical CT

- The x-ray machine scans the body in a spiral path. This allows more images to be made in a shorter time than with older CT methods.
- Also known as spiral CT



Slice-by-slice CT Helical CT

Multi-slice CT



The Radon transform

- A straight line in Cartesian coordinates may be described by its slope-intercept form:

$$y = ax + b$$

- or by its normal representation:

$$x\cos\theta + y\sin\theta = \rho$$

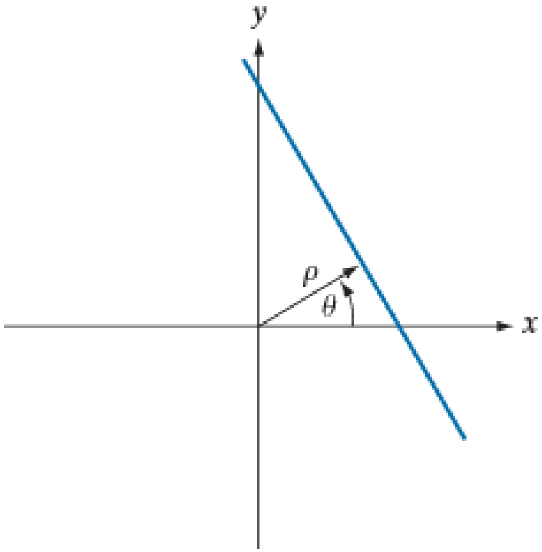
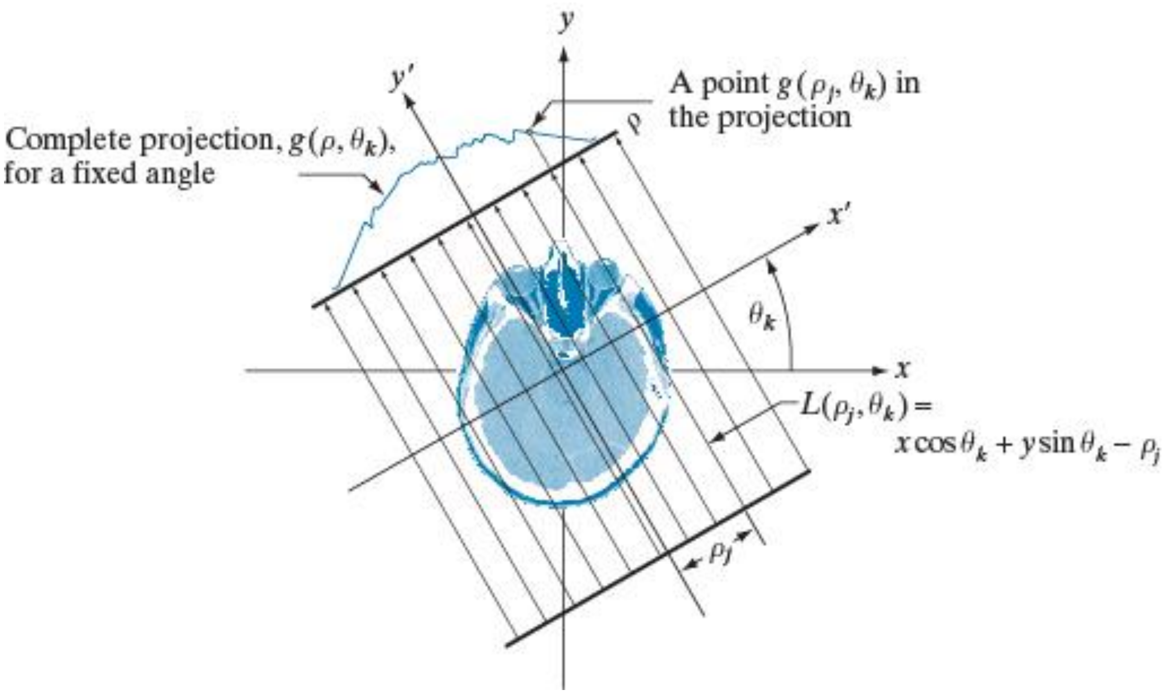


FIGURE 5.36 Normal representation of a line.

The Radon transform

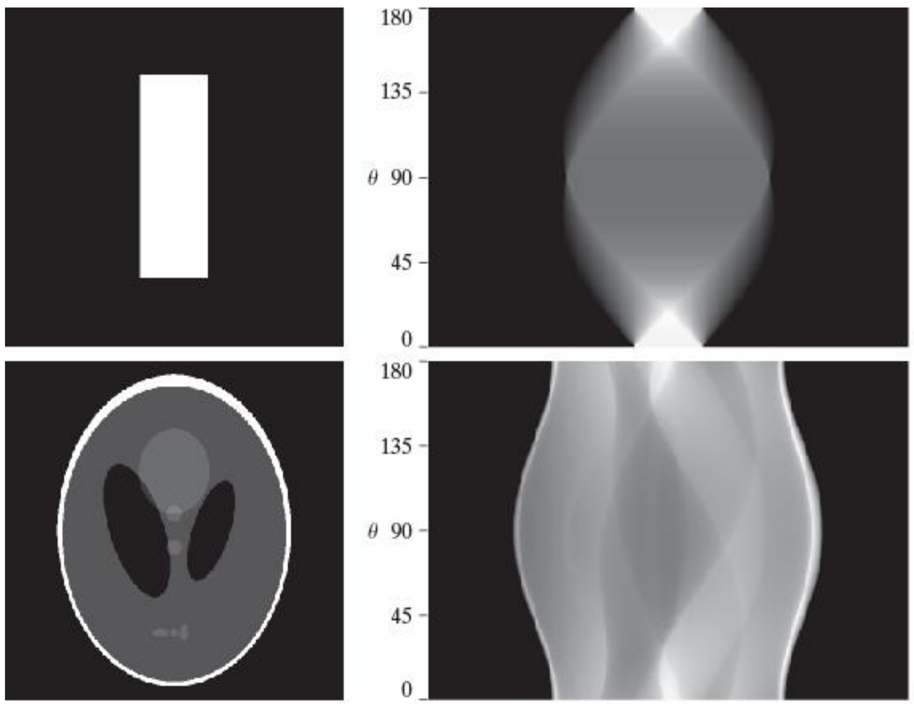


Continuous form: $$g(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

Discrete form: $$g(\rho,\theta)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)$$

Radon transform and Sinogram

- Radon transform $g(\rho, \theta)$ is displayed as an image with ρ and θ as rectilinear coordinates



a b
c d

FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. (Note that the horizontal axis of the sinograms are values of ρ.) Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

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Why called Sinogram

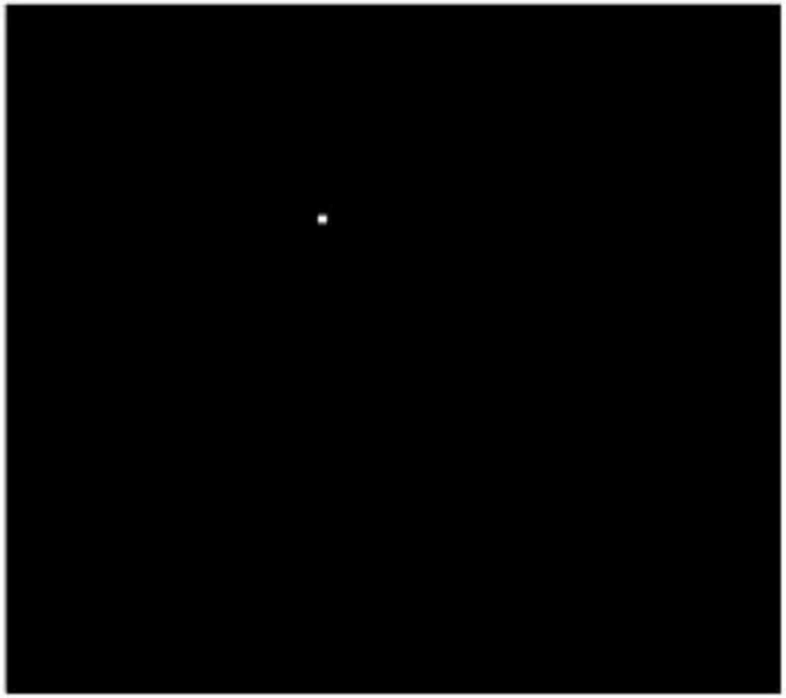
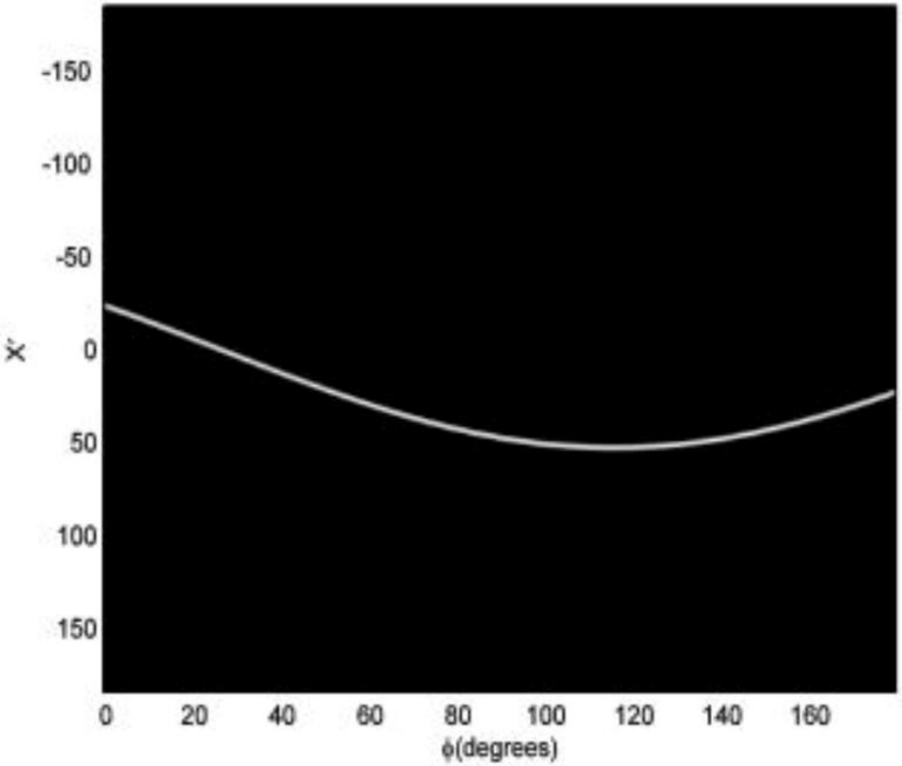


Image of a single point



The Radon transform

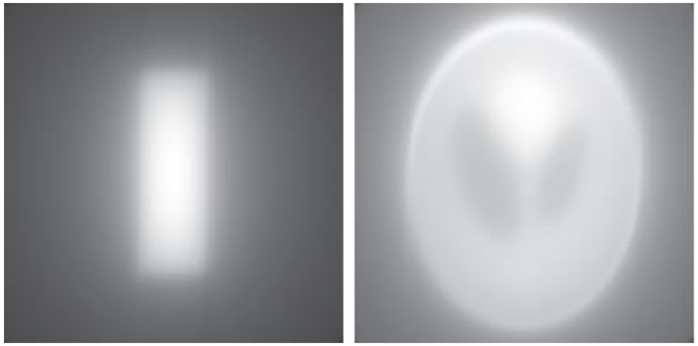
Backprojection

- For a fixed projection angle θ_k

$$f_{\theta_k}(x,y) = g(\rho,\theta_k) = g(x\cos\theta_k + y\sin\theta_k)$$

- Integrating over all the backprojected images

$$f(x,y) = \int_0^{\pi} f_\theta(x,y)\,d\theta$$



a b

FIGURE 5.40 Backprojections of the sinograms in Fig. 5.39.

The Fourier-Slice Theorem

- The Fourier-slice theorem or the central slice theorem relates the 1D Fourier transform of a projection with the 2D Fourier transform of the region of the image from which the projection was obtained. It is the basis of image reconstruction methods.

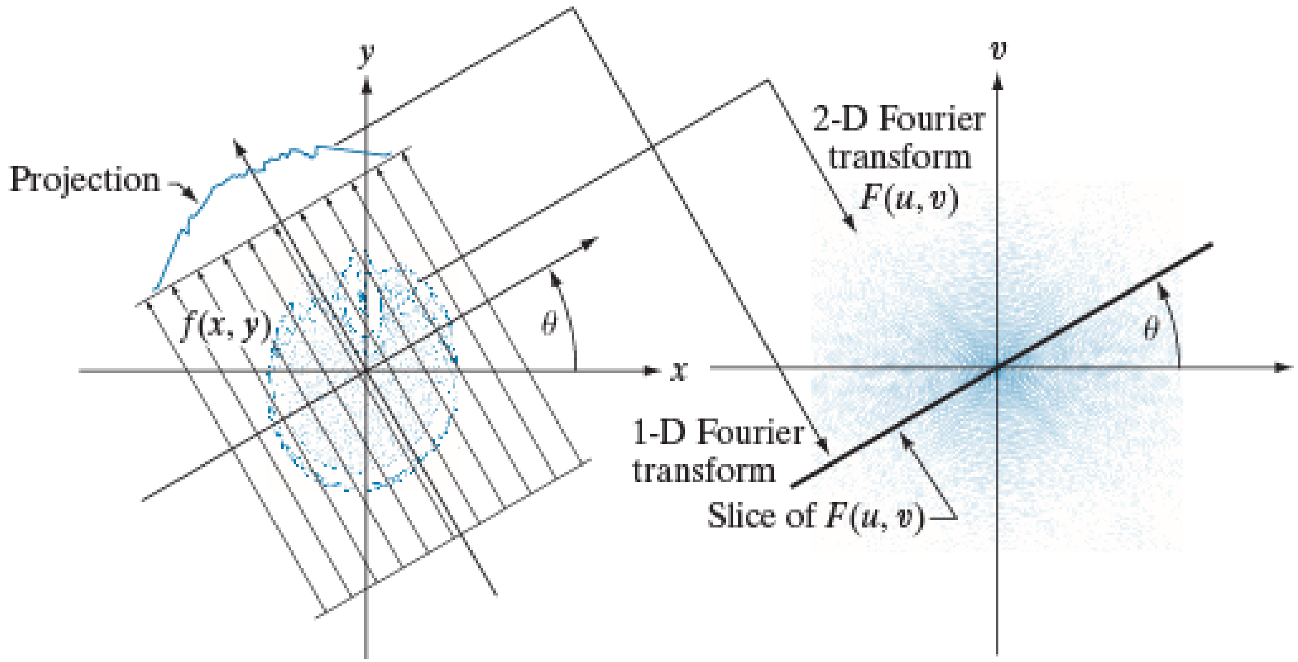


FIGURE 5.41 Illustration of the Fourier-slice theorem. The 1-D Fourier transform of a projection is a slice of the 2-D Fourier transform of the region from which the projection was obtained. Note the correspondence of the angle θ in the two figures.

The Fourier-Slice Theorem

$$G(\omega,\theta) = \int_{-\infty}^{\infty} g(\rho,\theta) e^{-j2\pi\omega\rho} d\rho$$

wave number $\omega^2 = \mathrm{u}^2 + v^2$

$$G(\omega,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho) e^{-j2\pi\omega\rho} dx\, dy\, d\rho$$

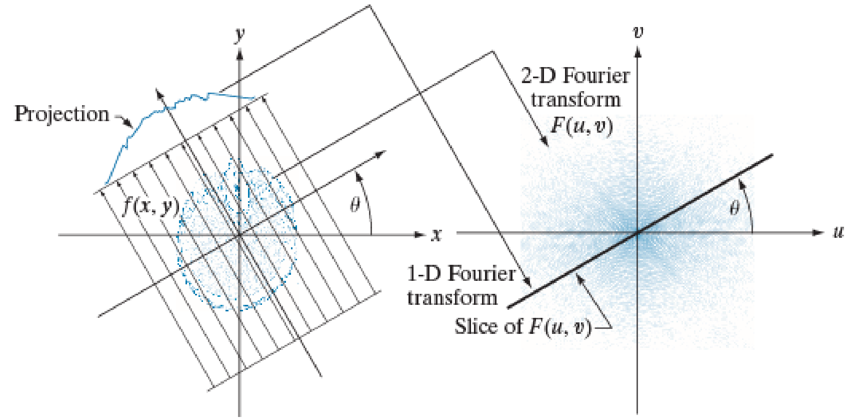
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \left[\int_{-\infty}^{\infty} \delta(x\cos\theta + y\sin\theta - \rho) e^{-j2\pi\omega\rho} d\rho\right] dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\omega(x\cos\theta + y\sin\theta)} dx\, dy$$

$$G(\omega,\theta) = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\, dy\right]_{u=\omega\cos\theta;\, v=\omega\sin\theta}$$

$$G(\omega,\theta) = [F(u,v)]_{u=\omega\cos\theta;\, v=\omega\sin\theta}$$

$$= F(\omega\cos\theta\,, \omega\sin\theta)$$



The Fourier-Slice Theorem

- The (1D) Fourier transform of a projection is a slice of the 2D Fourier transform of the region from which the projection was obtained.

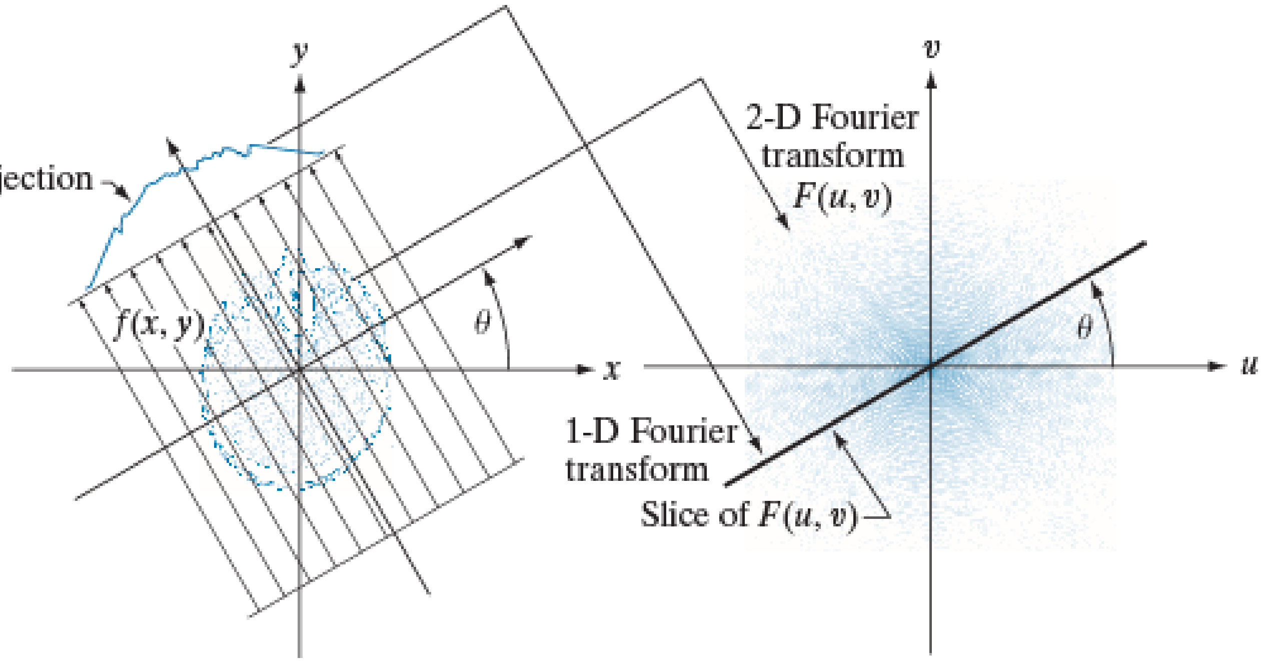
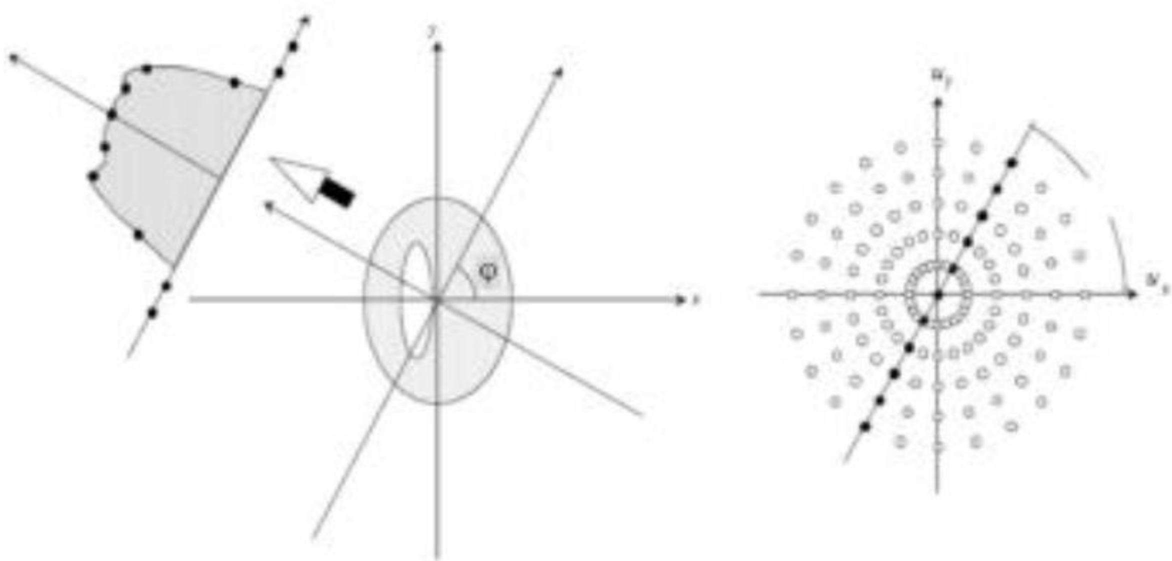


FIGURE 5.41 Illustration of the Fourier-slice theorem. The 1-D Fourier transform of a projection is a slice of the 2-D Fourier transform of the region from which the projection was obtained. Note the correspondence of the angle θ in the two figures.

The Fourier-Slice Theorem

- We could obtain f(x,y) by evaluating the F.T. of every projection and inverting them. However, this procedure needs irregular interpolation which introduces inaccuracies.



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Reconstruction by filtered back-projections (FBP)

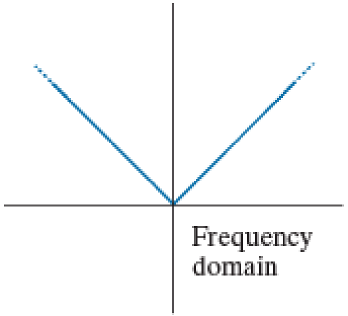
- The inverse F.T. of F(u,v):

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega d\omega d\theta$$

$$= \int_{0}^{\pi}\left[\int_{-\infty}^{+\infty}|\omega|G(\omega,\theta)e^{j2\pi\omega\rho}d\omega\right]_{\rho=x\cos\theta+y\sin\theta} d\theta$$

Ramp filter $|\omega|$



- Back-projection with convolution

$$f(x,y) = \int_{0}^{\pi}[s(\rho)\otimes g(\rho,\theta)]_{\rho=x\cos\theta+y\sin\theta}d\theta$$

Where $s(\rho) = IFT(|\omega|)$, $g(\rho,\theta) = IFT(G(\omega,\theta))$

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- The complete back-projection is obtained as follows:
  1. Compute the 1-D Fourier transform of each projection.
  2. Multiply each Fourier transform by the filter function $|\omega|$ (multiplied by a suitable window, e.g. Hamming).
  3. Obtain the inverse 1-D Fourier transform of each resulting filtered transform.
  4. Back-project and integrate all the 1-D inverse transforms from step 3.

Reconstruction by FBP

$$= \int_0^{\pi} \left[\int_{-\infty}^{+\infty} \boxed{|\omega| G(\omega, \theta)} e^{j2\pi\omega\rho} d\omega \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

What's the problem here?

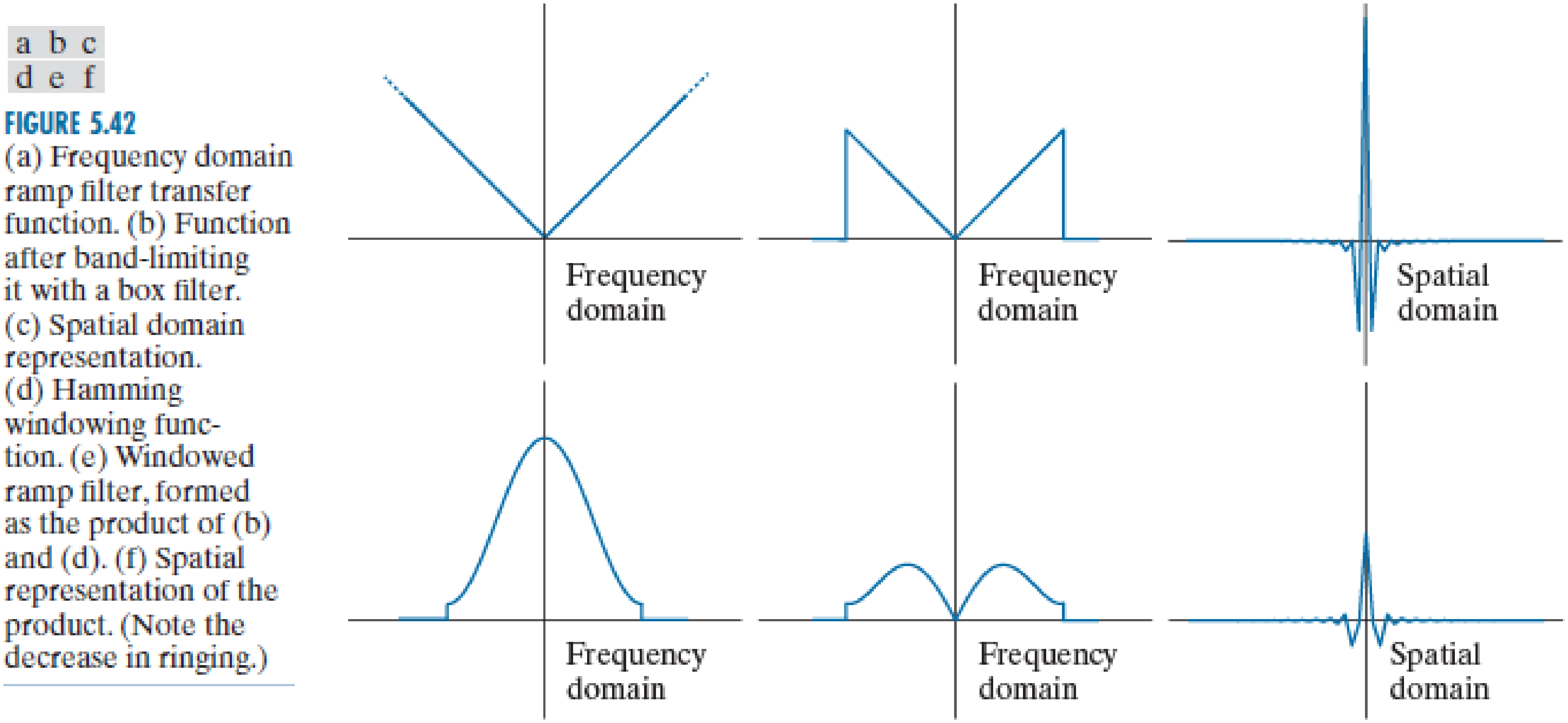

FIGURE 5.42 (a) Frequency domain ramp filter transfer function. (b) Function after band-limiting it with a box filter. (c) Spatial domain representation. (d) Hamming windowing function. (e) Windowed ramp filter, formed as the product of (b) and (d). (f) Spatial representation of the product. (Note the decrease in ringing.)

Reconstruction by FBP

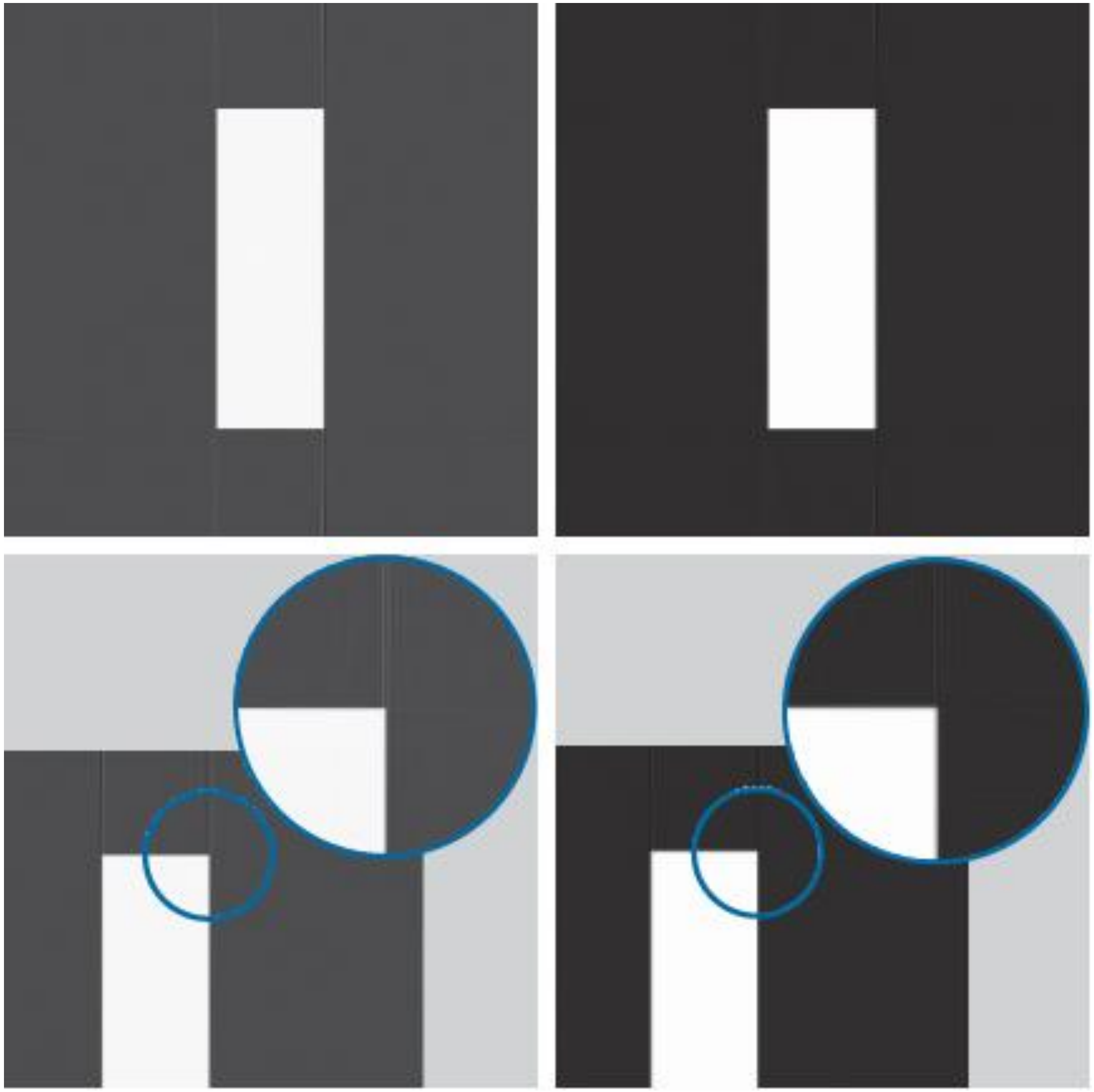

FIGURE 5.43 Filtered backprojections of the rectangle using (a) a ramp filter, and (b) a Hamming windowed ramp filter. The second row shows zoomed details of the images in the first row. Compare with Fig. 5.40(a).

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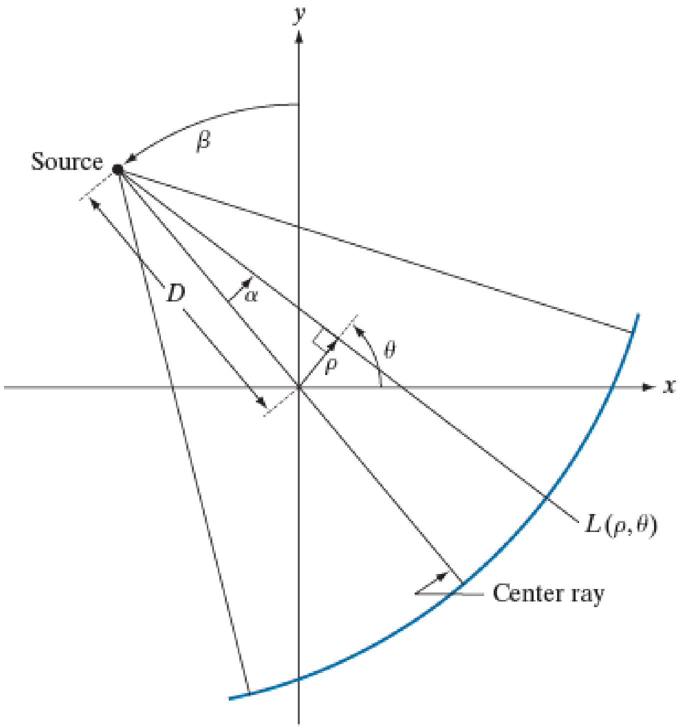
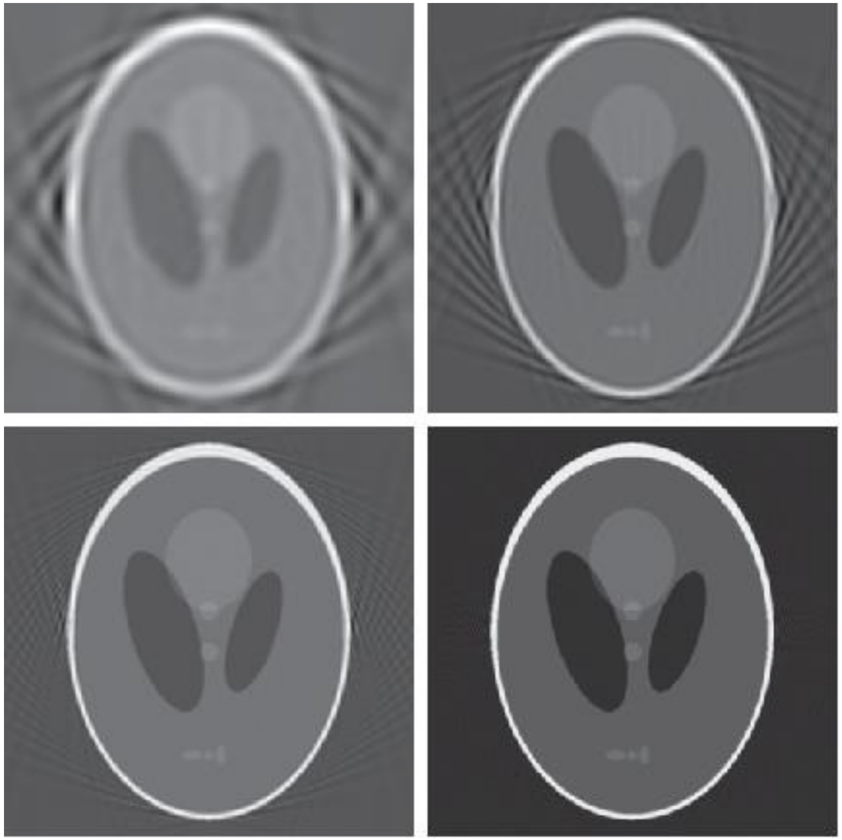
a b

FIGURE 5.44 Filtered backprojections of the head phantom using (a) a ramp filter, and (b) a Hamming windowed ramp filter. Compare with Fig. 5.40(b)

Fan-Beam based FBP





a b
c d

FIGURE 5.49
Reconstruction of the head phantom image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.44(b).

Take home message

- CT imaging reconstruction is based on accumulating back- projections data directive, while the reconstructed images are blurred.
- The 2D Fourier transform of an image for reconstruction can be obtained by accumulating the Fourier transform of the projections at different angles (Fourier-Slice Theorem).
- Filtered back-projection can mitigate the blurring effects.