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Mathematical foundation of Liutex theory

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Abstract: Liutex is a mathematical definition of vortex, which is called the third generation of vortex definition and identification. This paper introduces the mathematical foundation of the Liutex theoretical system including differences in definition and operations between tensor/vector and matrix. The right version of velocity gradient tensor matrix is given to correct the old version which has been widely distributed by many mathematics and fluid dynamics textbooks. A unique velocity gradient principal matrix is provided. The mathematical foundation for Liutex definition is given. The coordinate rotation (Q- and P- rotation) for principal coordinate system and principal matrix is derived, which is the key issue of the new fluid kinematics. The divergence of velocity gradient tensor is given in different forms which may be beneficial in developing new governing equations for fluid dynamics.

Key words: Liutex, mathematical foundation, tensor, matrix, velocity gradient, fluid dynamics

Chaoqun Liu's Short Vita

Dr. Chaoqun Liu received both BS (1968) and MS (1981) from Tsinghua University, Beijing, China, and Ph. D. (1989) from University of Colorado at Denver, USA. He is currently the Tenured and Distinguished Professor and the Director of Center for Numerical Simulation and Modeling at University of Texas at Arlington, Arlington, Texas, USA. He has worked on high order direct numerical simulation (DNS) and large eddy simulation (LES) for flow transition and turbulence for 32 years since 1990. As PI, he has been awarded by NSF, NASA, US Air Force, and US Navy with 51 federal research grants of over 5.95×10^6 US dollars in the United States. He has published 14 professional books, 141 journal papers and 170 conference papers and book chapters. He is the founder and principal contributor of Liutex and the third generation of vortex definition and identification methods including the Omega, Liutex/Rortex, Modified Liutex-Omega, Liutex-Core-Line methods, Objective Liutex, RS vorticity decomposition and UTA R-NR velocity gradient tensor decomposition. He is also the founder of Liutex based new fluid kinematics.

Introduction

Vortex is ubiquitous in universe such as tornado, hurricane, airplane tip vortex, and even star vortex in Galaxy. Vortices are also building blocks, muscles, and sinews of turbulent flows[1]. A vortex is intuitively recognized as a rotational/swirling motion of fluids, but until recently had no rigorous mathematical definition. In 1858, Helmholtz first defined vortex as tubes composed of so-called vortex filaments[2], which are infinitesimal vorticity tubes. We call the vorticity tube definition as the first generation of vortex definition and identification, or G1. G1 has been accepted by almost all fluid dynamics textbooks for more than 164 years. In classical vortex dynamics (see Lamb[3], 1932), vortex is usually associated with vorticity which has a rigorous mathematical definition (curl of velocity). Nitsche[4] pointed out in

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Encyclopedia[5] "A vortex is commonly associated with the rotational motion of fluid around a common centerline. It is defined by the vorticity in the fluid, which measures the rate of local fluid rotation." Wu et al.[6] pointed out in their vortex dynamics book that vortex is "a connected fluid region with high concentration of vorticity compared with its surrounding." Wu[7] further gave a clearer definition in Science China (2018) "in summary, we have: Definition 1. A vortex is a vorticity tube surrounded by irrotational fluid", which is consistent to his books and research papers and consistent to countless fluid dynamics textbooks. However, in 1989, Robinson et al.[8] pointed out that "the association between regions of strong vorticity and actual vortices can be rather weak in the turbulent boundary layer, especially in the near wall region." Wang et al.[9] in 2016 obtained a similar result that the magnitude of vorticity inside a Lambda vortex can be substantially smaller than the surrounding near the solid wall in a flat plate transitional boundary layer. Although G1 has been accepted by the fluid dynamics community and almost all textbooks for over a century, we can find many immediate counterexamples. For example, in the laminar boundary layer, where the vorticity (shear) is very large near the wall, no rotation (no vortex) exists. Same thing happens in laminar channel flow which has an analytic solution, $u(y)=4y(1-y)$, has very large vorticity near the wall, but no vortex or flow rotation can be found. These counterexamples, experiments and DNS results lead to a clear conclusion: Vortex cannot be represented by vorticity. Although both are vectors, they are different vectors. To solve these contradictions, many vortex criteria methods have been developed during the past 4 decades. More popular methods are represented by the Q [10], Δ [11], λ_2 [12], λ_{ci} [13] criteria methods. These methods have achieved part of success in vortex identification. These vortex criteria methods are classified as second generation of vortex identification or G2 by Liu et al.[14]. However, they have several critical drawbacks. First, they are all scalars which have no rotation axis directions, but vortex is a vector. It is hard or impossible to use a scalar to represent a vector. Second, like vorticity, these criteria methods are all shear contaminated in different degrees. However, shear cannot be recognized as part of fluid rotation or vortex. Third, they are all dependent and very sensitive on threshold selection. For a same data set made by same experiment or DNS computation, these G2 methods will give completely different vortex structures. If the threshold is large, their visualizations show "vortex breakdown to turbulence". However, if the threshold is reduced, all vortices will be connected, and the physical conclusion will be "there is no vortex breakdown". Physics that if turbulence is generated by vortex breakdown or there is no vortex breakdown is not determined by experiment or DNS computations, but by the threshold and vortex criteria method selections. These G1 and G2 vortex definition and identification are bottlenecks for turbulence research. According to G1, vortex is vorticity tube. Because $\nabla\cdot(\nabla\times\boldsymbol{v})=0$ or vorticity's divergence is always zero everywhere, there is no vorticity tube (vortex) breakdown in the world, which directly and clearly opposes "turbulence is generated by vortex breakdown". G2 shows completely different vortex structure by different thresholds, which leads to no answer if we have hairpin or no hairpin vortex in turbulent flow. Facing these controversial questions[15], Liu et al.[14, 16-17] at University of Texas at Arlington (UTA) discovered and developed Liutex and Third Generation of Vortex Definition and Identification or G3. Liutex is a vector with both direction and magnitude. The direction is the local rotation axis, and the magnitude is twice the local angular speed of the rigid rotation. Liu[17] further found the local rotation axis is the eigenvector of the velocity gradient matrix. After over 160 years effort since Helmholtz (1858), the unique and rigorous mathematical definition of fluid rotation or vortex is discovered and defined, which is called Liutex.

The basic idea of Liutex is to define a local rotation axis, in which the velocity increment is stretching or compression only, i.e., $\mathrm{d}\boldsymbol{v}=\mathrm{grad}\boldsymbol{v}\cdot\boldsymbol{r}=\lambda\boldsymbol{r}$ which is the real eigenvector of the matrix form of $\mathrm{grad}\boldsymbol{v}$. In this paper, $\mathrm{grad}\boldsymbol{v}$ is used to represent the velocity gradient tensor rather than widely used $\nabla\boldsymbol{v}$ because it will be discussed that whether $\nabla\boldsymbol{v}$ or $(\nabla\boldsymbol{v})^{\mathrm{T}}$ is the correct velocity gradient tensor in this paper. In fact, Liutex is the rigid rotation part extracted from fluid motion to represent vortex, which is a mathematically rigorous tool suitable for vortex characterization[18] and uniquely defined. The location of the natural vortex rotation axis is then the local maxima of Liutex (not vorticity). Although the similar ideas of decomposition of vorticity tensor to a pure rotation and anti-symmetric shear have been given by[19-20], they did not find a vector definition for vortex/Liutex. Several vortex identification methods have been developed based on the Liutex definition by the UTA Team and their collaborators[21-22]. Examples of these methods are modified Liutex-Omega and Liutex-Core-Line. The former is threshold insensitive, and the latter is threshold-free. These methods have been shown to accurately visualize vortical structures in turbulent flows[23-24]. Guo et al.[25] compared Liutex with experimental results. Yu et al.[26-28] showed that G2 are contaminated by shear or stretching but Liutex not. Cuissa et al.[29] indicated that Liutex is an innovative and highly reliable mathematical criterion.

In Ref. [30], Liutex is said to be "the most reliable criterion for the extraction of physical information from vortical flows". Liutex also shows the potential to be applied in fluid dynamics[31-35]. Wang et al.[36] advised to consider six critical issues to define a natural vortex core including: (1) Rotation axis, (2) Absolute rotation strength, (3) Relative rotation strength, (4) Vortex core center, (5) Vortex tube, (6) Vortex boundary. In fact, G1 cannot answer any above questions and G2 may provide roughly the vortex boundary except for Q- criteria which is inconsistent to Δ criteria. However, Liutex and G3 can answer all the above questions which is considered as the touch stone of vortex definition and identification. The answers[36] are (1) Local Liutex maxima, (2) Liutex, (3) Liutex-Omega, (4) Liutex-core-Line, (5) Liutex tube, (6) Liutex > 0, respectively.

Liutex is a physical quantity which can be used to measure the fluid rotation strength. Liutex related methods are called the Third Generation of Vortex Definition and Identification. Since vortex is omnipresent in universe, a mathematical definition of vortex or Liutex will play a critical role in scientific research. It is projected that the Liutex theory will play an important role in the investigations of the vortex dynamics in hydrodynamics, aerodynamics, thermodynamics, oceanography, meteorology, metallurgy, civil engineering, astronomy, biology, etc. and in the research of the generation, sustenance, modeling and controlling of turbulence. Apparently, as a rigorous mathematical definition of vortex is given, a new era will be open for quantified vortex and turbulence research.

There is still a question why it took 160 years to discover Liutex to replace vorticity? The authors found there are many misunderstandings in tensor and matrix definitions and operations in classical fluid dynamics textbooks. This paper tries to give the mathematical foundation for discovering Liutex and clarify the misunderstandings in classical fluid kinematics and fluid dynamics.

This paper is organized as follows: Section 2 explains the difference in definitions and operations between tensor/vector and matrix. Section 3 discusses Hamilton operators, and which one is the correct version of velocity gradient tensor matrix. In section 4, a unique velocity gradient principal matrix is provided. Section 5 describes the mathematical foundation for Liutex definition. Section 6 introduces the coordinate system rotation for principal coordinate system[37] and principal matrix. In Section 7, the divergence of velocity gradient tensor is given in different forms which may be beneficial in developing new governing equations for fluid dynamics. The last chapter will give some conclusions.

1. Vector, tensor and matrix

Vector and tensor are physical quantities which are in general unique. To show vectors and second order tensor, people usually use matrix to represent vector/tensor, which is in general dependent on the coordinate system.

Figure 1 shows that a unique vector $\boldsymbol{V}$ can be written as $\boldsymbol{V}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ in (a) and $\boldsymbol{V}=\begin{bmatrix}\sqrt{2}\\0\\0\end{bmatrix}$ in (b). In fact, same vector can be expressed by countless 3×1 matrices depending on which coordinate system is selected. The vector/tensor operations are also different from matrix operations. Table 1 gives the difference of definition and operations between vector/tensor and matrix.

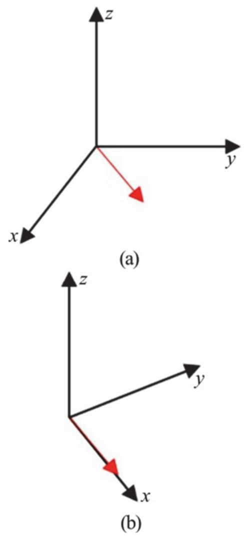

Fig. 1 A vector in different coordinate system

Table 1 Difference in definition and operations between vector/tensor and matrices

	Vector/tensor	Matrix
Meaning	Physics	Mathematics
Features	Objective/Galilean invariant	Dependent on coordinate systems
-	Unique	Infinity
Operations	Dot, cross, dyadic	Addition, subtraction, multiplication, inversion, transpose

1.1 *Correspondence between vector/second-order tensor and matrices*

Let the vector be $\boldsymbol{v}=a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3$. Easy to know

$$\boldsymbol{v}=a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3=[\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3]\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \tag{1}$$

If the choice of basis is obvious, the default, $[\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$, can be omitted and the matrix corresponding to $\boldsymbol{v}$ can simply be follows.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2}$$

Consider a second order tensor $\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$. Easy to know

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} \boldsymbol{e}_1^{\mathrm{T}} \\ \boldsymbol{e}_2^{\mathrm{T}} \\ \boldsymbol{e}_3^{\mathrm{T}} \end{bmatrix} \tag{3}$$

If the choice of basis is obvious, the $[\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] = I$ and $\begin{bmatrix} \boldsymbol{e}_1^{\mathrm{T}} \\ \boldsymbol{e}_2^{\mathrm{T}} \\ \boldsymbol{e}_3^{\mathrm{T}} \end{bmatrix} = I$ can be omitted and the matrix corresponding to $\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ can simply be

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{4}$$

$(\cdot)_M$ is used to express the corresponding matrix of the vector/second-order tensor, e.g.

$$(\boldsymbol{v})_M = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or simply } \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{5}$$

1.2 *Vector dot product vector*

$$\boldsymbol{a} \cdot \boldsymbol{b} = (\boldsymbol{a})_M^{\mathrm{T}} (\boldsymbol{b})_M \tag{6}$$

Proof

$$\text{LHS} = \boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} a_i b_i \tag{7}$$

$$\text{RHS} = (\boldsymbol{a})_M^{\mathrm{T}} (\boldsymbol{b})_M = [a_1 \quad a_2 \quad a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \sum_{i=1}^{3} a_i b_i = \text{LHS} \tag{8}$$

In Eq. (6), the left-hand side is a dot product of two vectors. When we use the matrix multiplication in the right-hand side, we must use $(\boldsymbol{a})_M^{\mathrm{T}}$ to replace $(\boldsymbol{a})_M$. It is easy to understand for dot product of two vectors.

1.3 *Second-order tensor dot product vector*

$$\boldsymbol{A} \cdot \boldsymbol{b} = (\boldsymbol{A})_M (\boldsymbol{b})_M \tag{9}$$

Proof

$$\text{LHS} = \boldsymbol{A} \cdot \boldsymbol{b} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\right) \cdot \left(\sum_{i=1}^{3} b_i\boldsymbol{e}_i\right) = \sum_{i=1}^{3} a_{1i}b_i\boldsymbol{e}_1 + \sum_{i=1}^{3} a_{2i}b_i\boldsymbol{e}_2 + \sum_{i=1}^{3} a_{3i}b_i\boldsymbol{e}_3 = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3]\begin{bmatrix} \sum_{i=1}^{3} a_{1i}b_i \\ \sum_{i=1}^{3} a_{2i}b_i \\ \sum_{i=1}^{3} a_{3i}b_i \end{bmatrix} \tag{10}$$

$$\text{RHS} = (\boldsymbol{A})_M (\boldsymbol{b})_M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{3} a_{1i}b_i \\ \sum_{i=1}^{3} a_{2i}b_i \\ \sum_{i=1}^{3} a_{3i}b_i \end{bmatrix} = \text{LHS} \tag{11}$$

1.4 *Vector dot product with second-order tensor*

$$\boldsymbol{b} \cdot \boldsymbol{A} = [(\boldsymbol{b})_M^{\mathrm{T}} (\boldsymbol{A})_M]^{\mathrm{T}} \tag{12}$$

Proof

$$\text{LHS} = \boldsymbol{b} \cdot \boldsymbol{A} = \left(\sum_{i=1}^{3} b_i\boldsymbol{e}_i\right) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\right) = \sum_{i=1}^{3} a_{i1}b_i\boldsymbol{e}_1 + \sum_{i=1}^{3} a_{i2}b_i\boldsymbol{e}_2 + \sum_{i=1}^{3} a_{i3}b_i\boldsymbol{e}_3 = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3]\begin{bmatrix} \sum_{i=1}^{3} a_{i1}b_i \\ \sum_{i=1}^{3} a_{i2}b_i \\ \sum_{i=1}^{3} a_{i3}b_i \end{bmatrix} \tag{13}$$

$$\text{RHS} = [(\boldsymbol{b})_M^{\mathrm{T}} (\boldsymbol{A})_M]^{\mathrm{T}} = \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)^{\mathrm{T}} = \begin{bmatrix} \sum_{i=1}^{3} a_{i1}b_i \\ \sum_{i=1}^{3} a_{i2}b_i \\ \sum_{i=1}^{3} a_{i3}b_i \end{bmatrix} = \text{LHS} \tag{14}$$

1.5 *Dyadic operator*

$$\boldsymbol{ab} = \boldsymbol{a} \otimes \boldsymbol{b} = (\boldsymbol{a})_M (\boldsymbol{b})_M^{\mathrm{T}} \tag{15}$$

Proof

$$\mathrm{LHS} = \boldsymbol{ab} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \boldsymbol{e}_i \boldsymbol{e}_j = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] \cdot$$

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1^{\mathrm{T}} \\ \boldsymbol{e}_2^{\mathrm{T}} \\ \boldsymbol{e}_3^{\mathrm{T}} \end{bmatrix} \tag{16}$$

$$\mathrm{RHS} = (\boldsymbol{a})_M (\boldsymbol{b})_M^{\mathrm{T}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}^{\mathrm{T}} =$$

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \mathrm{LHS} \tag{17}$$

We cannot arbitrarily add or drop the dyadic sign $\otimes$ which requires a transpose on the right vector. We cannot add or drop the "dot" either.

2. Hamilton operators

Hamilton operator is

$$\nabla = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \text{ or simply } \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \tag{18}$$

If we consider Hamilton operator like a vector, the vector/tensor operation can be simplified a lot.

2.1 *Divergence*

$$\nabla \cdot \boldsymbol{v} = (\nabla)_M^{\mathrm{T}} (\boldsymbol{v})_M = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{19}$$

2.2 *Cross product*

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{20}$$

2.3 *Dyadic*

$$\nabla \boldsymbol{v} = (\nabla)_M (\boldsymbol{v})_M^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} [u \quad v \quad w] = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z} \end{bmatrix} \tag{21}$$

Note that, the term containing "u" is placed as the first column (not first row). And here $\nabla \boldsymbol{v}$ is the result after dyadic operation, but it is not stated that $\nabla \boldsymbol{v}$ is the velocity gradient tensor. What the velocity gradient tensor is will be discussed in Section 3.

3. Velocity gradient tensor matrix

People usually believe that ∇f and $\nabla \boldsymbol{v}$ are equivalent to "the gradient of f" and "the gradient of $\boldsymbol{v}$". Apparently, this statement is true for the scalar f because it will not face a "transpose" problem for a vector, but it does face for a tensor. We have the following two candidates for the velocity gradient tensor.

$$\nabla \boldsymbol{v} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_j}{\partial e_i} \boldsymbol{e}_i \boldsymbol{e}_j \tag{22}$$

$$(\nabla \boldsymbol{v})^{\mathrm{T}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial e_j} \boldsymbol{e}_i \boldsymbol{e}_j \tag{23}$$

whose matrix forms are follows:

$$(\nabla \boldsymbol{v})_M = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z} \end{bmatrix} \tag{24}$$

$$(\nabla \boldsymbol{v})_M^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix} \tag{25}$$

These two forms can both be found in fluid dynamics textbooks, e.g., Eq. (24) is used in Ref. [38] and Eq. (25) is used in Ref. [39]. The question is which one is the correct? Or are the two forms both correct?

The correct velocity gradient tensor can be determined by its following feature, i.e.

$$\mathrm{d}\boldsymbol{v} = \mathrm{grad}\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] \begin{bmatrix} \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z \\ \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y + \frac{\partial v}{\partial z}\mathrm{d}z \\ \frac{\partial w}{\partial x}\mathrm{d}x + \frac{\partial w}{\partial y}\mathrm{d}y + \frac{\partial w}{\partial z}\mathrm{d}z \end{bmatrix} \tag{26}$$

and

$$\nabla \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial e_i} \boldsymbol{e}_i \boldsymbol{e}_j \right) \cdot (\mathrm{d}x\boldsymbol{e}_1 + \mathrm{d}y\boldsymbol{e}_2 + \mathrm{d}z\boldsymbol{e}_3) =$$

$$[\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] \begin{bmatrix} \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial x}\mathrm{d}y + \frac{\partial w}{\partial x}\mathrm{d}z \\ \frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y + \frac{\partial w}{\partial y}\mathrm{d}z \\ \frac{\partial u}{\partial z}\mathrm{d}x + \frac{\partial v}{\partial z}\mathrm{d}y + \frac{\partial w}{\partial z}\mathrm{d}z \end{bmatrix} \neq \mathrm{d}\boldsymbol{v} \tag{27}$$

$$(\nabla \boldsymbol{v})^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{r} = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_i}{\partial e_j} \boldsymbol{e}_i \boldsymbol{e}_j \right) \cdot (\mathrm{d}x\boldsymbol{e}_1 + \mathrm{d}y\boldsymbol{e}_2 + \mathrm{d}z\boldsymbol{e}_3) =$$

$$[\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] \begin{bmatrix} \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z \\ \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y + \frac{\partial v}{\partial z}\mathrm{d}z \\ \frac{\partial w}{\partial x}\mathrm{d}x + \frac{\partial w}{\partial y}\mathrm{d}y + \frac{\partial w}{\partial z}\mathrm{d}z \end{bmatrix} = \mathrm{d}\boldsymbol{v} \tag{28}$$

So, $(\nabla \boldsymbol{v})^{\mathrm{T}}$ should be the correct velocity gradient tensor. Then the definition of velocity gradient tensor can be uniquely given.

Definition 1 The velocity gradient tensor is $(\nabla \boldsymbol{v})^{\mathrm{T}} = \sum_{i=1}^{3} \sum_{j=1}^{3} (\partial u_i / \partial e_j) \boldsymbol{e}_i \boldsymbol{e}_j$, and its matrix form is $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$.

Admittedly it is true that ∇f is equivalent to "the gradient of f for a scalar, but things become more complicated for a vector. Due to the inertia of thinking, some people believe that $\nabla \boldsymbol{v}$ is equivalent to "the gradient of $\boldsymbol{v}$" for a vector as well. However, after checking the condition Eq. (26), only $(\nabla \boldsymbol{v})^{\mathrm{T}}$ is the correct one for representing the velocity gradient tensor.

In the Tensor Analysis textbook[40] , it also introduces the "left gradient" and "right gradient". If "left" or "right" is not emphasized, it is "right gradient" by default. Then, $\nabla \boldsymbol{v}$ can be viewed as the "left gradient" tensor of the velocity. And the increment of velocity should be evaluated by the following formula.

$$\mathrm{d}\boldsymbol{v} = \mathrm{d}\boldsymbol{r} \cdot \nabla \boldsymbol{v} \tag{29}$$

However, people's habit is using the right gradient and calculating the right eigenvector of a matrix. For a unique definition and due to people's habit, using "right gradient" as the velocity gradient tensor is more appropriate.

4. Liutex definition

4.1 *Eigenvalue and eigenvector*

Let us take an example

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \tag{30}$$

whose characteristic equation is

$$p(\lambda) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ 4 & -1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(-1-\lambda) -$$

$$(1-\lambda)(-2)(4) = -\lambda^3 + 3\lambda^2 - 7\lambda + 5 = 0 \tag{31}$$

The eigenvalues of A are 1, $1-2i$ and $1+2i$ (two conjugate complex eigenvalues). There must be one real eigenvector

$$\begin{bmatrix} 3 & -2 & 0 \\ 4 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{32}$$

Thus, the real eigenvector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. However, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not the eigenvector of A^{T} because

$$\begin{bmatrix} 3 & 4 & 2 \\ -2 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \neq 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{33}$$

So, from the example, the eigenvector of one matrix is in general not the eigenvector of the transpose of the same matrix.

4.2 *Local rotation axis*

Definition 2 A local fluid rotation axis is defined as a vector that can only have stretching (compression) along its length.

$$\mathrm{d}\boldsymbol{v} = \mathrm{grad}\boldsymbol{v} \cdot \boldsymbol{r} = (\nabla \boldsymbol{v})_M^{\mathrm{T}} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \lambda \boldsymbol{r} \tag{34}$$

which shows that there is only stretching or compression in the $\boldsymbol{r}$ direction. The rotation axis $\boldsymbol{r}$ is eigenvector of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$. Here we limited $\boldsymbol{r}$ by the condition of $\boldsymbol{\omega} \cdot \boldsymbol{r} > 0$ and $\|\boldsymbol{r}\|_2 = 1$ where $\boldsymbol{\omega}$ is the vorticity. Obviously, the eigenvector of $(\nabla \boldsymbol{v})_M$ is not the rotation axis as can be seen from Section 4.1.

4.3 *Liutex definition*

Definition 3 Liutex is defined as

$$\boldsymbol{R} = R\boldsymbol{r} = \left[\boldsymbol{\omega} \cdot \boldsymbol{r} - \sqrt{(\boldsymbol{\omega} \cdot \boldsymbol{r})^2 - 4\lambda_{ci}^2}\right]\boldsymbol{r} \quad \text{and} \quad \boldsymbol{\omega} \cdot \boldsymbol{r} > 0 \tag{35}$$

where $\boldsymbol{r}$ is the Liutex direction vector (real eigenvector of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$, $\boldsymbol{\omega}$ is the vorticity vector and λ_{ci} is the imaginary part of the complex eigenvalue of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$.

Since the Liutex direction vector is the real eigenvector of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$, but not $(\nabla \boldsymbol{v})_M$, which may be the reason why it took 160 years to find Liutex that the velocity gradient tensor matrix is mistakenly written as

$$(\nabla \boldsymbol{v})_M = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{36}$$

while the correct velocity gradient tensor should be Eq. (25).

Lamb-Oseen vortex model is used to compare Liutex and other vortex identification methods. Lamb-Oseen vortex model has the following analytical velocity field in polar coordinates:

$$v_\theta = v_{\max}\left(1 + \frac{1}{2\alpha}\right)\frac{r_{\max}}{r}\left[1 - \exp\left(-\alpha\frac{r^2}{r_{\max}^2}\right)\right], \quad v_r = 0 \tag{37}$$

where $\alpha = 1.256$ and it is set that $v_{\max} = 1$ and $r_{\max} = 0.3$. The comparison between Q- criterion, vorticity and Liutex is shown in Fig. 2.

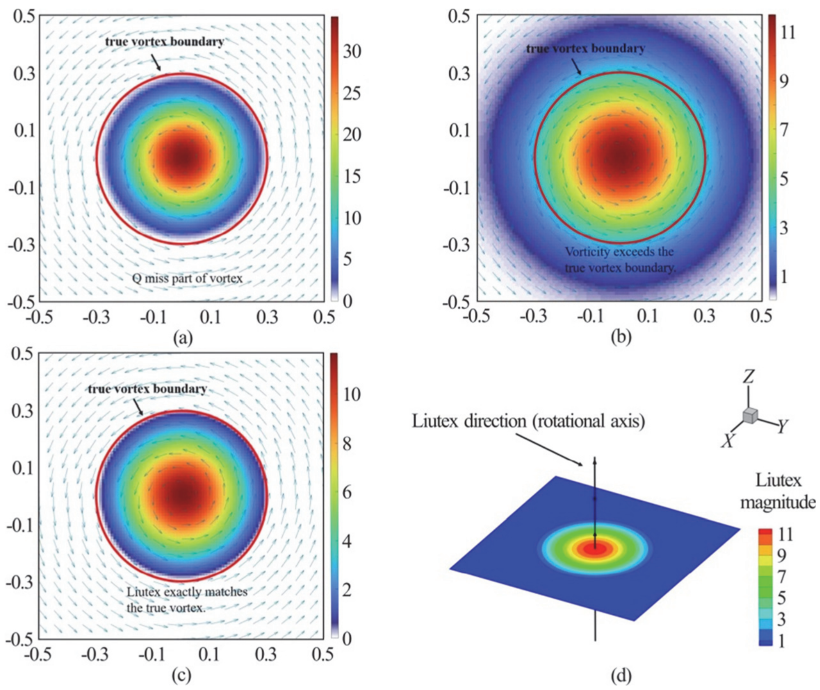

Fig. 2 Lamb-Oseen vortex detected by (a) Q- criterion (b) Vorticity and (c) Liutex. Arrows represent the velocity directions. The red circle exhibits the true boundary of Lamb-Oseen vortex. (d) Rotational axis shown by Liutex direction

In Fig. 2(a) Q-criterion miss some vortex region and in Fig. 2(b) vortex region predicted by vorticity obviously exceeds the true vortex boundary. It clearly shows that Liutex is the only method that exactly estimates the boundary of Lamb-Oseen vortex. The result is similar to what Cuissa got in Ref. [29].

5. Principal coordinate system and principal matrix

For rotational point, we would like to create a unique matrix which we call principal matrix for the velocity gradient tensor. To do this, we must find a principal coordinate system, which needs coordinate rotations. We call them Q- and P-rotations as shown in Fig. 3.

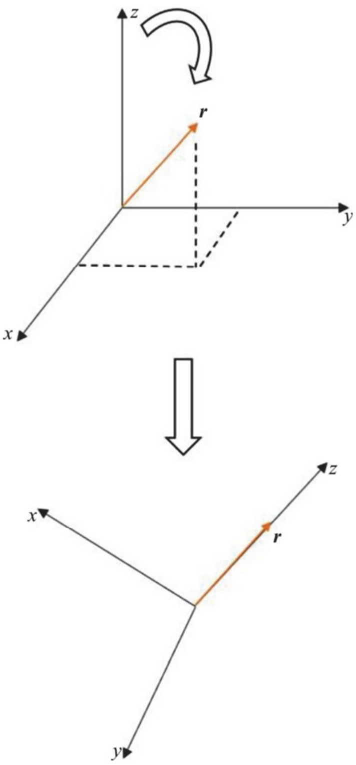

Fig. 3 Q-rotation

5.1 Q-rotation

The Q-rotation matrix can rotate the coordinate system to make the Z-axis $(\boldsymbol{Z})$ parallel to $\boldsymbol{r}$, i.e.

$$Q^{\mathrm{T}}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}=\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ r_x & r_y & r_z \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{38}$$

where $\boldsymbol{Q}_1=(Q_{11},Q_{12},Q_{13})^{\mathrm{T}}$, $\boldsymbol{Q}_2=(Q_{21},Q_{22}.Q_{23})^{\mathrm{T}}$ and $\boldsymbol{r}$ are three orthogonal coordinate vectors, which makes Q an orthogonal matrix. In this new coordinate system after the Q-rotation, the velocity gradient tensor becomes

$$(\nabla \boldsymbol{V})_M^{\mathrm{T}}=Q^{\mathrm{T}}(\nabla \boldsymbol{v})_M^{\mathrm{T}}Q=\begin{bmatrix} \dfrac{\partial U}{\partial X} & \dfrac{\partial U}{\partial Y} & 0 \\ \dfrac{\partial V}{\partial X} & \dfrac{\partial V}{\partial Y} & 0 \\ \dfrac{\partial W}{\partial X} & \dfrac{\partial W}{\partial Y} & \dfrac{\partial W}{\partial Z} \end{bmatrix}=$$

$$\begin{bmatrix} \dfrac{\partial U}{\partial X} & \dfrac{\partial U}{\partial Y} & 0 \\ \dfrac{\partial V}{\partial X} & \dfrac{\partial V}{\partial Y} & 0 \\ \dfrac{\partial W}{\partial X} & \dfrac{\partial W}{\partial Y} & \lambda_r \end{bmatrix} \tag{39}$$

Definition 4 Q^{T} is defined as a rotation matrix to rotate the z-axis to be parallel to $\boldsymbol{r}$, where $Q=[\boldsymbol{Q}_1 \quad \boldsymbol{Q}_2 \quad \boldsymbol{Q}_3]=\begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}$ and $\boldsymbol{Q}_1 \perp \boldsymbol{Q}_2$, $\boldsymbol{Q}_1 \perp \boldsymbol{Q}_3$ and $\boldsymbol{Q}_2 \perp \boldsymbol{Q}_3$.

Theorem 1 If the third column of rotation Q is $\boldsymbol{r}$, at least one Q can be given by $Q=\begin{bmatrix} Q_{11} & Q_{21} & r_x \\ Q_{12} & Q_{22} & r_y \\ Q_{13} & Q_{23} & r_z \end{bmatrix}=\begin{bmatrix} 0 & ar_z & r_x \\ r_z & r_z & r_y \\ -r_y & -r_y-ar_x & r_z \end{bmatrix}$, where $\boldsymbol{r}$ is the real eigenvector of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$, $a=-(r_y^2+r_z^2)/r_x r_y$, assuming $r_x \neq 0$ and $r_y \neq 0$.

Proof

$$\boldsymbol{Q}_1 \cdot \boldsymbol{Q}_3=[0 \quad r_z \quad -r_y]\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}=0 \tag{40}$$

$$\boldsymbol{Q}_2 \cdot \boldsymbol{Q}_3=[ar_z \quad r_z \quad -r_y-ar_x]\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}=ar_z r_x+r_z r_y+$$

$$(-r_y-ar_x)r_z=ar_z r_x+r_z r_y-r_y r_z-ar_x r_z=0 \tag{41}$$

$$\boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2=[ar_z \quad r_z \quad -r_y-ar_x]\begin{bmatrix} 0 \\ r_z \\ -r_y \end{bmatrix}=r_z^2+r_y^2+ar_x r_y=$$

$$r_z^2 + r_y^2 - \frac{r_y^2 + r_z^2}{r_x r_y} r_x r_y = 0 \tag{42}$$

Therefore, $\boldsymbol{Q}_1 \perp \boldsymbol{Q}_2$, $\boldsymbol{Q}_1 \perp \boldsymbol{Q}_3$, $\boldsymbol{Q}_2 \perp \boldsymbol{Q}_3$ and $Q^{\mathrm{T}} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Assume $b = \sqrt{r_y^2 + r_z^2}$, $c = \sqrt{(a^2+1)r_z^2 + (-r_y - ar_x)^2}$, the normalized Q-rotation matrix can be written as

$$Q = \begin{bmatrix} 0 & \frac{ar_z}{c} & r_x \\ \frac{r_z}{b} & \frac{r_z}{c} & r_y \\ -\frac{r_y}{b} & \frac{-r_y - ar_x}{c} & r_z \end{bmatrix} \tag{43}$$

5.2 *P-rotation*

After Q-rotation, the Z-direction in the new coordinate system is parallel to Liutex or the eigenvector of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$. However, the new coordinate system is not unique, and then the new matrix is not unique.

Definition 5 The matrix rotated from $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$ in any coordinate system such that it has two zeros on the upper left corner and first two diagonal elements equal is called principal matrix of $(\nabla \boldsymbol{v})_M^{\mathrm{T}}$.

A P-rotation in 2-D X-Y plane can be used to obtain the principal matrix as shown in Fig. 4. Let $\begin{bmatrix} x \\ y \end{bmatrix}$ refers to the coordinates in the original coordinate system, $\begin{bmatrix} x \\ y \end{bmatrix}$ refer to the coordinates in the new (rotated) coordinate system. Their relation can be expressed as $\begin{bmatrix} X \\ Y \end{bmatrix} = P \begin{bmatrix} x \\ y \end{bmatrix}$, where $P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix which is orthogonal, namely, $PP^{\mathrm{T}} = P^{\mathrm{T}}P = I$ where I is the identity matrix.

5.3 *Velocity gradient tensor in the after-rotation coordinate system*

If a rotation matrix P is used to rotate the xy-frame to XY-frame, the velocity gradient tensor in the XY-frame $\nabla \boldsymbol{V}$ is related to the velocity gradient tensor in the xy-frame $\nabla \boldsymbol{v}$ through the following expression

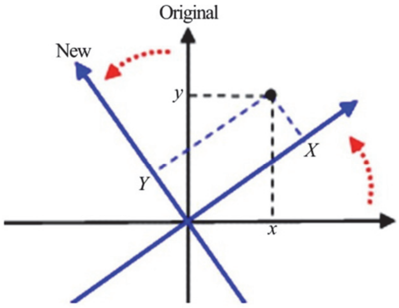

Fig. 4 2-D coordinate rotation

$$(\nabla \boldsymbol{V})_M^{\mathrm{T}} = P^{-1} (\nabla \boldsymbol{v})_M^{\mathrm{T}} P = P^{\mathrm{T}} (\nabla \boldsymbol{v})_M^{\mathrm{T}} P \tag{44}$$

equivalent

$$(\nabla \boldsymbol{V})_M^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial V}{\partial Y} \\ \frac{\partial U}{\partial X} & \frac{\partial V}{\partial Y} \end{bmatrix} \tag{45}$$

For 3-D after Q-rotation, we only need a 2-D rotation in the X-Y plane:

$$P^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

$$P^{\mathrm{T}} (\nabla \boldsymbol{V})_M^{\mathrm{T}} P = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \lambda_r \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial U}{\partial X}\cos\theta - \frac{\partial U}{\partial Y}\sin\theta & \frac{\partial U}{\partial X}\sin\theta + \frac{\partial U}{\partial Y}\cos\theta & 0 \\ \frac{\partial V}{\partial X}\cos\theta - \frac{\partial V}{\partial Y}\sin\theta & \frac{\partial V}{\partial X}\sin\theta + \frac{\partial V}{\partial Y}\cos\theta & 0 \\ \frac{\partial W}{\partial X}\cos\theta - \frac{\partial W}{\partial Y}\sin\theta & \frac{\partial W}{\partial X}\sin\theta + \frac{\partial W}{\partial Y}\cos\theta & \lambda_r \end{bmatrix} =$$

$$(\nabla \boldsymbol{V}_\theta)_M^{\mathrm{T}} \tag{47}$$

To satisfy **Definition 5** we require the first two diagonal elements equals to each other, which gives one equation:

$$\cos\theta\left(\frac{\partial U}{\partial X}\cos\theta - \frac{\partial U}{\partial Y}\sin\theta\right) -$$

$$\sin\theta\left(\frac{\partial V}{\partial X}\cos\theta - \frac{\partial V}{\partial Y}\sin\theta\right) =$$

$$\sin\theta\left(\frac{\partial U}{\partial X}\sin\theta + \frac{\partial U}{\partial Y}\cos\theta\right) +$$

$$\cos\theta\left(\frac{\partial V}{\partial X}\sin\theta + \frac{\partial V}{\partial Y}\cos\theta\right) \tag{48}$$

$$\left(\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y}\right)\cos 2\theta - \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)\sin 2\theta = 0 \tag{49}$$

$$\tan 2\theta = \frac{\dfrac{\partial U}{\partial X} - \dfrac{\partial V}{\partial Y}}{\dfrac{\partial U}{\partial Y} + \dfrac{\partial V}{\partial X}} \tag{50}$$

To find a unique θ, we limit $\theta \le \pi/2$.

5.4 *The principal matrix of* $(\nabla \boldsymbol{V})_M^{\mathrm{T}}$

After the Q- and P-rotation, we will have a unique matrix

$$\begin{bmatrix} a & b & 0 \\ c & a & 0 \\ \dfrac{\partial \overline{W}}{\partial \overline{X}} & \dfrac{\partial \overline{W}}{\partial \overline{Y}} & \lambda_r \end{bmatrix} \tag{51}$$

Because the vorticity and eigenvalues are Galilean invariant

$$p(\lambda) = \begin{vmatrix} a-\lambda & b & 0 \\ c & a-\lambda & 0 \\ \dfrac{\partial \overline{W}}{\partial \overline{X}} & \dfrac{\partial \overline{W}}{\partial \overline{Y}} & \lambda_r \end{vmatrix} = (\lambda_r - \lambda)\cdot$$

$$[(a-\lambda)(a-\lambda) - bc] = 0 \tag{52}$$

$$(a-\lambda)^2 = bc < 0 \tag{53}$$

Eq. (53) is required for point with rotation where $bc < 0$.

Assume the complex eigenvalues are $\lambda_{cr} + \mathrm{i}\lambda_{ci}$

$$a = \lambda_{cr} \tag{54}$$

$$bc = -\lambda_{ci}^2 \tag{55}$$

Due to the Galilean invariant of vorticity

$$c - b = \omega_{\overline{Z}} = \boldsymbol{\omega}\cdot\boldsymbol{r} \tag{56}$$

Assume $|b| < |c|$ and solve the above equations, we obtain follows:

$$b(b + \omega_{\overline{Z}}) = -\lambda_{ci}^2 \tag{57}$$

$$b^2 + \omega_{\overline{Z}} b + \lambda_{ci}^2 = 0 \tag{58}$$

$$b = \frac{1}{2}\left[-\boldsymbol{\omega}\cdot\boldsymbol{r} - \sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2 - 4\lambda_{ci}^2}\right] = -\frac{1}{2}R \tag{59}$$

where R is the magnitude of Liutex.

$$c = \boldsymbol{\omega}\cdot\boldsymbol{r} - \frac{1}{2}R \tag{60}$$

Similarly, it is easy to find $\partial\overline{W}/\partial\overline{X} = -\omega_{\overline{Y}}$ and $\partial\overline{W}/\partial\overline{Y} = \omega_{\overline{X}}$. Note that the $\overline{X}$, $\overline{Y}$ in $-\omega_{\overline{Y}}$ and $\omega_{\overline{X}}$ formulas are capital $\overline{X}$ and $\overline{Y}$.

Finally, we obtain the unique principal matrix for the velocity gradient tensor

$$(\nabla \boldsymbol{V})_M^{\mathrm{T}} = \begin{bmatrix} \lambda_{cr} & -\dfrac{1}{2}R & 0 \\ \omega_{\overline{Z}} - \dfrac{1}{2}R & \lambda_{cr} & 0 \\ -\omega_{\overline{Y}} & \omega_{\overline{X}} & \lambda_r \end{bmatrix} \tag{61}$$

Therefore, although the principal coordinate system and the principal matrix are obtained by Q- and P-rotations, the final principal matrix is determined by the vorticity and eigenvalues, which are Galilean invariant and independent of coordinate systems. However, the Q- and P-rotation are needed when we go back from the principal coordinate system to the original xyz-coordinate systems.

The unique matrix of velocity gradient tensor will naturally be decomposed to three parts: rotation, stretching and shear, which may lead to new governing equations for fluid dynamics (See Ref. [41]). The principal decomposition can be written as

$$(\nabla \boldsymbol{V})_M^{\mathrm{T}} = \begin{bmatrix} \lambda_{cr} & -\frac{1}{2}R & 0 \\ \frac{1}{2}R+\varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}R & 0 \\ \frac{1}{2}R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \varepsilon & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} = \boldsymbol{R} + \boldsymbol{SC} + \boldsymbol{S} \tag{62}$$

Or $(\nabla \boldsymbol{V})^{\mathrm{T}}$ = Rotation (Liutex) + Stretching (Compression) + Shear. It is unique and physical meaning is very clear which is completely different from traditional Cauchy-Stokes decomposition (Helmholtz velocity decomposition).

6. Divergence of a tensor

To derive the fluid dynamics equation, we need to use the divergence of a tensor. If we limit the vector to be column matrix, we must pay attention not to use row vector.

6.1 *Divergence of* $\nabla \boldsymbol{v}$

$$\nabla \cdot \nabla \boldsymbol{v} = \left(\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \right)^{\mathrm{T}} =$$

$$\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix} = \begin{bmatrix} \nabla^2 u \\ \nabla^2 v \\ \nabla^2 w \end{bmatrix} \tag{63}$$

Note that $\nabla \boldsymbol{v}$ is not the velocity gradient tensor.

6.2 *Divergence of velocity gradient tensor* $(\nabla \boldsymbol{v})^{\mathrm{T}}$

Theorem 2 The divergence of velocity gradient, $\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} = \nabla(\nabla \cdot \boldsymbol{v})$, and specifically, $\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} \equiv \boldsymbol{0}$ for incompressible flow.

Proof

$$\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} = \left(\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \right)^{\mathrm{T}} =$$

$$\begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\ \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(\nabla \cdot \boldsymbol{v}) \\ \frac{\partial}{\partial y}(\nabla \cdot \boldsymbol{v}) \\ \frac{\partial}{\partial z}(\nabla \cdot \boldsymbol{v}) \end{bmatrix} \tag{64}$$

For incompressible flow, $\nabla \cdot \boldsymbol{v} = 0$, then Eq. (64) becomes

$$\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{65}$$

6.3 *Different formulas for governing equations of fluid dynamics*

We only discuss incompressible flow here and the compressible flow is similar. Since $\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} \equiv \boldsymbol{0}$,

$$\nabla \cdot \nabla \boldsymbol{v} = \nabla \cdot \nabla \boldsymbol{v} + k \nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} \tag{66}$$

If $k = 1$

$$\nabla \cdot \nabla \boldsymbol{v} = \nabla \cdot \nabla \boldsymbol{v} + \nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} = 2\nabla \cdot \left\{ \frac{1}{2}[(\nabla \boldsymbol{v})^{\mathrm{T}} + \nabla \boldsymbol{v}] \right\} \tag{67}$$

$1/2[(\nabla \boldsymbol{v})^{\mathrm{T}} + \nabla \boldsymbol{v}]$ is called symmetric strain tensor which has 6 independent elements. That is the one which is used by Navier-Stokes equations.

If $k = -1$

$$\nabla \cdot \nabla \boldsymbol{v} = \nabla \cdot \nabla \boldsymbol{v} - \nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} = -2\nabla \cdot \left\{ \frac{1}{2}[(\nabla \boldsymbol{v})^{\mathrm{T}} - \nabla \boldsymbol{v}] \right\} \tag{68}$$

$1/2[(\nabla \boldsymbol{v})^{\mathrm{T}} - \nabla \boldsymbol{v}]$ is called anti-symmetric vorticity tensor which has 3 independent elements only. That is the one which is used by our new fluid dynamics governing equations[42]. The Navier-Stokes equation for incompressible flow is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{v}) = \boldsymbol{f} - \nabla p + \nabla \cdot \{\mu[(\nabla \boldsymbol{v})^{\mathrm{T}} + \nabla \boldsymbol{v}]\} \tag{69}$$

Our new governing equation is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{v}) = \boldsymbol{f} - \nabla p - \nabla \{\mu[(\nabla \boldsymbol{v})^{\mathrm{T}} - \nabla \boldsymbol{v}]\} \tag{70}$$

The two equations are equivalent in mathematics but different in physics, which has been discussed by Liu et al. in 2021[42]. The Navier-Stokes equation with Stokes assumption shows that stress is proportional to strain, and both are symmetric. Our new governing equations shows both stresses and strains are anti-symmetric. In fact, both strain and stress cannot be either symmetric or ant-symmetric and they are general. However, two equations will get same numerical results since $\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} \equiv \boldsymbol{0}$ or divergence of velocity gradient tensor is zero for incompressible flow.

7. Conclusions

Based on the above discussion, we can make some conclusions:

(1) Although we can use matrix to show vectors and second order tensors, they are different.

(2) The velocity gradient tensor is $(\nabla \boldsymbol{v})^{\mathrm{T}}$ but not $\nabla \boldsymbol{v}$, which is different from many fluid dynamics textbooks.

(3) The tensor/vector has dot, cross, dyadic operations, but matrix only has addition, subtraction, multiplication, inversion, and transpose. Tensor/vector operations and matrix operations are different.

(4) We cannot simply drop "dot" or "dyadic" when we transfer tensor/vector operations to matrix operations. The corresponding relations between tensor/vector operations and matrix operations are presented in Section 1.

(5) For vortex point, the rotation axis or Liutex is the eigenvector of $(\nabla \boldsymbol{v})^{\mathrm{T}}$ but not $\nabla \boldsymbol{v}$.

(6) The velocity gradient tensor is incorrectly given by some fluid dynamics textbooks as both matrix and matrix transpose have same eigenvalues but different eigenvectors. This may be the reason why it took so long time (160 years) to find the vortex vector or Liutex.

(7) We want to get a unique matrix to represent velocity gradient tensor, which is called principal matrix of a rotating tensor. This can be done by Q-and P-rotations. The final unique tensor is Galilean invariant, which is uniquely determined by the eigenvalues and eigenvectors of $(\nabla \boldsymbol{V})_M^{\mathrm{T}}$.

(8) We should use $(\nabla \boldsymbol{v})^{\mathrm{T}}$ to find the viscous force, but we can use either single symmetric strain or single anti-symmetric vorticity to find the viscous force as $\nabla \cdot (\nabla \boldsymbol{v})^{\mathrm{T}} \equiv \boldsymbol{0}$ for incompressible flow.

These conclusions are mathematic foundations of the Liutex theory. However, this mathematical foundation may also help understand new fluid kinematics, which may be beneficial to turbulence research and new fluid dynamics.

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Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent: Informed consent was obtained from all individual participants included in the study.

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