NANYANG TECHNOLOGICAL UNIVERSITY
SINGAPORE

Module 5: Interpolation

Learning Objectives

- Lagrange interpolation
- Newton's divided differences
- Bezier spline
- Cubic B-spline interpolation
- Applications in font representations

Sources

- Textbook (Chapter 3: Interpolation)

- Interpolation

  https://en.wikipedia.org/wiki/Interpolation
- Metafont (for Tex, Latex)

  https://en.wikipedia.org/wiki/Metafont
- TrueType (for Mac, Windows)

  https://en.wikipedia.org/wiki/TrueTypet
- PostScript (for Adobe)

  https://en.wikipedia.org/wiki/PostScript_fonts#Type_1

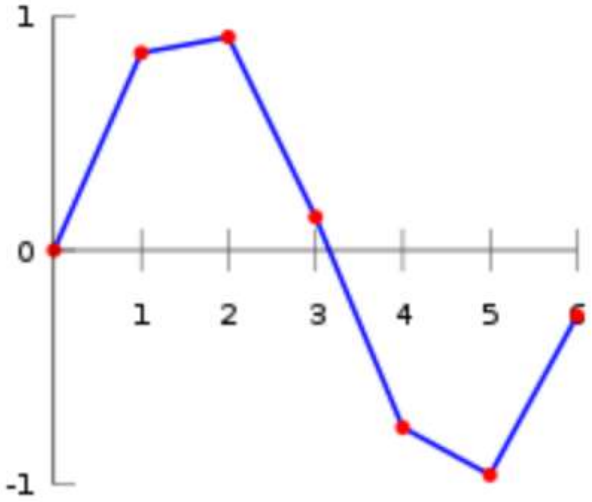
Outline

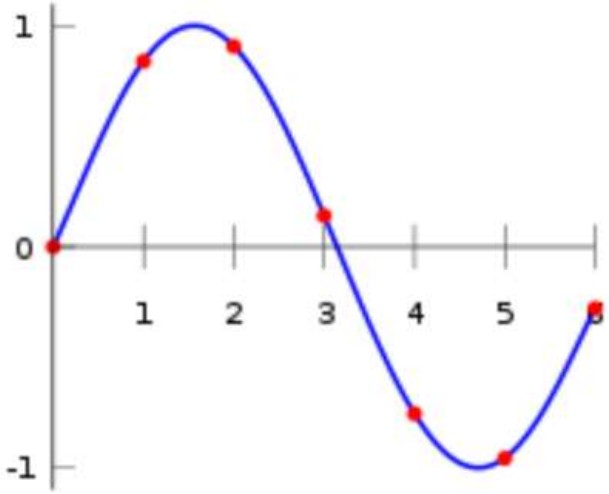
Introduction

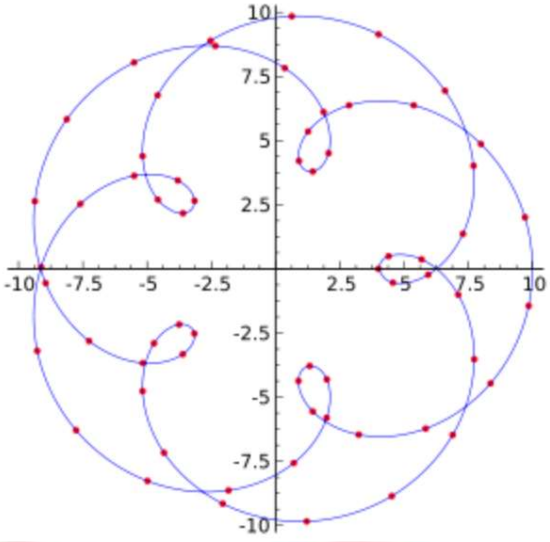
- Problem: given a set of data pairs (t_i, P_i), i =1,2,…,n, where
  - t_i, are scalar numbers, called parameter values;
  - P_i, are scalar or vectors;

  find a function f(t) such that $f(t_i) = P_i$ for each $1 \le i \le n$.

  Such a function f(t) is said to **interpolate** the data pairs.







Motivation

- Interpolation is a fundamental numerical technique.
- Wide applications
  - Shipbuilding, aircraft industries (Renault, Citroen, Boeing), automobile industries (General Motors)
  - Computer typesetting
  - Digital signal processing
  - Data compression
  - …

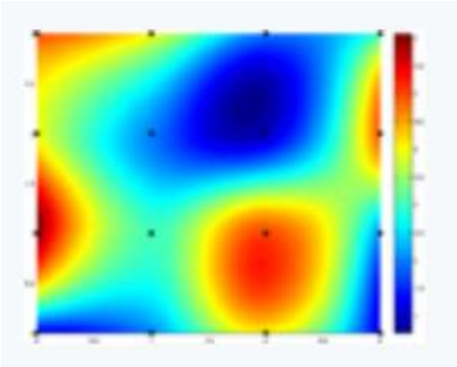
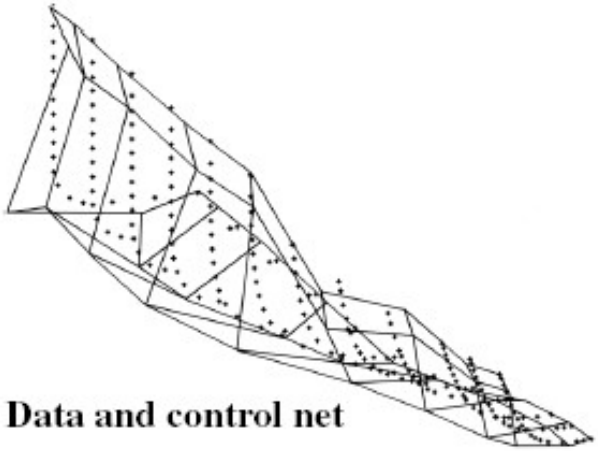
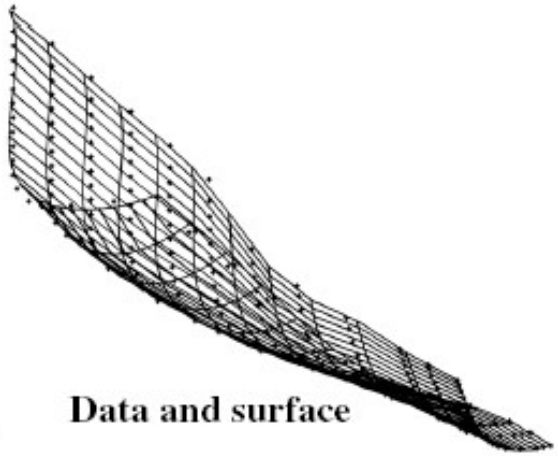


Image bicubic interpolation



Data and control net



Data and surface

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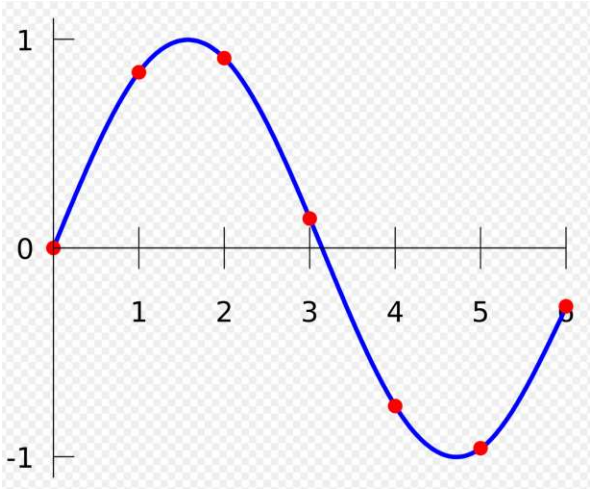
Our focus

- Polynomial interpolation
  - Interpolation function is a polynomial.
  - Eg: $f(x) = -0.0001521x^6 - 0.003130x^5 + 0.07321x^4 - 0.3577x^3 + 0.2255x^2 + 0.9038x.$



- Spline interpolation
  - Interpolation function is a spline.
  - Eg:

$$f(x) = \begin{cases} -0.1522x^3 + 0.9937x, & \text{if } x \in [0,1], \\ -0.01258x^3 - 0.4189x^2 + 1.4126x - 0.1396, & \text{if } x \in [1,2], \\ 0.1403x^3 - 1.3359x^2 + 3.2467x - 1.3623, & \text{if } x \in [2,3], \\ 0.1579x^3 - 1.4945x^2 + 3.7225x - 1.8381, & \text{if } x \in [3,4], \\ 0.05375x^3 - 0.2450x^2 - 1.2756x + 4.8259, & \text{if } x \in [4,5], \\ -0.1871x^3 + 3.3673x^2 - 19.3370x + 34.9282, & \text{if } x \in [5,6]. \end{cases}$$

Outline

Lagrange interpolation: degree 1

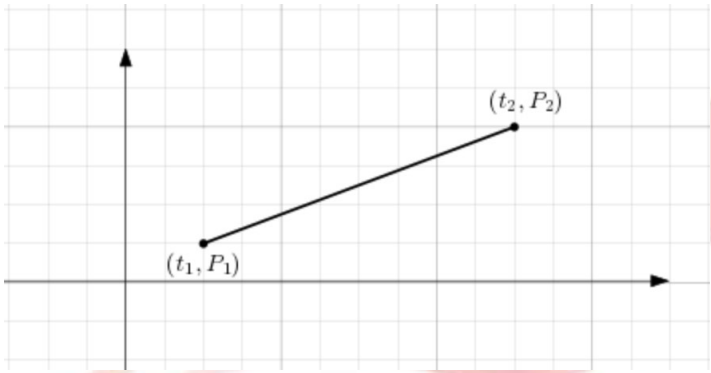
- **Lagrange interpolation** is an ***explicit*** formula, for writing down a polynomial of degree (n-1) that interpolates n data points.
- Degree 1 case

Given 2 data points (t_1, P_1) and (t_2, P_2), the polynomial

$$f_1(t) = \frac{t - t_2}{t_1 - t_2} P_1 + \frac{t - t_1}{t_2 - t_1} P_2$$

is the Lagrange interpolating polynomial.

That is, $f_1(t_1) = P_1, \quad f_1(t_2) = P_2.$



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Lagrange interpolation: degree 2

- Degree 2 case

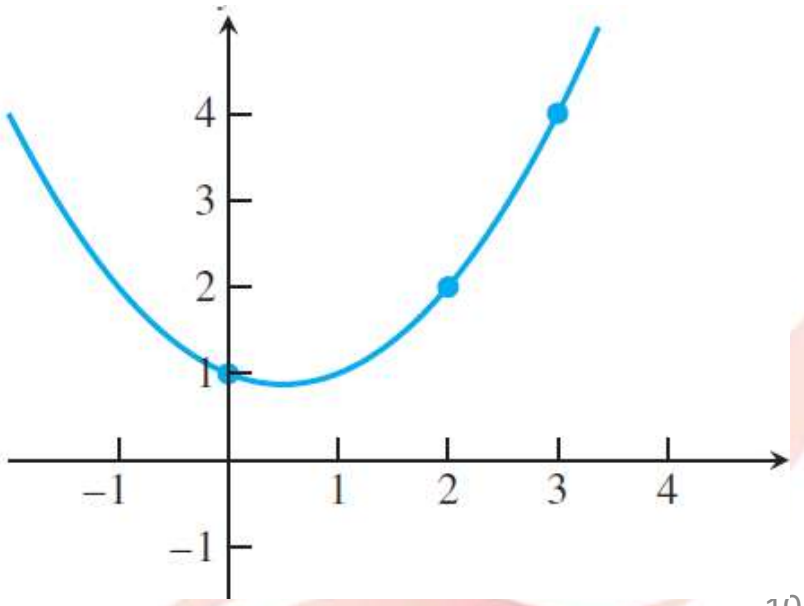
Given 3 data points $(t_1, P_1), (t_2, P_2)$ and (t_3, P_3), the polynomial

$$f_2(t) = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}P_1 + \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}P_2 + \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}P_3$$

is the Lagrange interpolating polynomial.

That is, $f_2(t_1) = P_1, \quad f_2(t_2) = P_2, \quad f_2(t_3) = P_3.$

  - Q for you: With reference to the figure, construct the degree 2 Lagrange interpolating polynomial for the 3 data points (1,1), (2,2) and (3,4).



Lagrange interpolation: general

- Degree (n-1) case
  - For each k between 1 and n, define the degree $n-1$ polynomial:

$$L_k(t) = \frac{(t-t_1)\cdots(t-t_{k-1})(t-t_{k+1})\cdots(t-t_n)}{(t_k-t_1)\cdots(t_k-t_{k-1})(t_k-t_{k+1})\cdots(t_k-t_n)}.$$

  - $L_k(t)$ has the following properties:
    - $L_k(t_k) = 1$ and $L_k(t_j) = 0$ for $j \neq k$.
    - $\sum_{k=1}^{n} L_k(t) \equiv 1$ for any t.
  - Define the degree $n-1$ polynomial:

$$f_{n-1}(t) = L_1(t)P_1 + L_2(t)P_2 + \cdots + L_n(t)P_n.$$

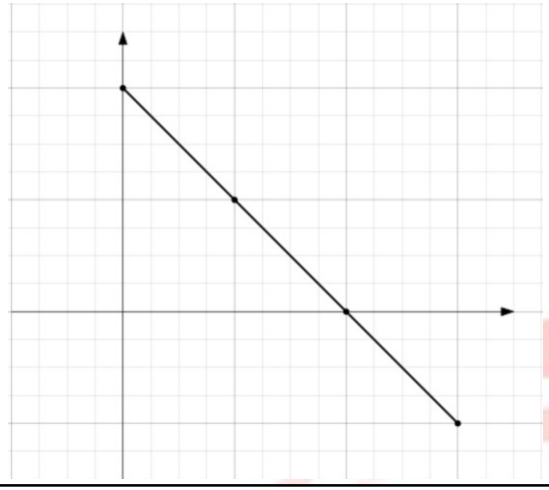
    It is the Lagrange interpolating polynomial for n data points $(t_1, P_1), (t_2, P_2), \cdots, (t_n, P_n)$.
  - It is easy to verify that $f_{n-1}(t_j) = P_j$.

Lagrange interpolation: example

- Example: Find the polynomial of degree 3 or less that interpolates the points (0,2), (1,1), (2,0) and (3,-1).

We treat the first coordinate as parameter t and the second coordinate as P. Then the Lagrange polynomial is as follows:

$$\begin{aligned} f_3(t) &= \frac{(t-1)(t-2)(t-3)}{(0-1)(0-2)(0-3)}2 + \frac{(t-0)(t-2)(t-3)}{(1-0)(1-2)(1-3)}1 \\ &\quad + \frac{(t-0)(t-1)(t-3)}{(2-0)(2-1)(2-3)}0 + \frac{(t-0)(t-1)(t-2)}{(3-0)(3-1)(3-2)}(-1) \\ &= \frac{1}{-3}(t^3-6t^2+11t-6) + \frac{1}{2}(t^3-5t^2+6t) + \frac{1}{-6}(t^3-3t^2+2t) \\ &= -t+2 \end{aligned}$$



Discussions

- Uniqueness (Main Theorem of Polynomial Interpolation)

Theorem. Let $(t_1, P_1), \cdots, (t_n, P_n)$ be n data pairs with *distinct* t_i. Then there exists one and only one polynomial $f(t)$ of degree $n-1$ or less that satisfies $f(t_i) = P_i$ for $i = 1, \cdots, n$.

- Advantage of Lagrange interpolation
  - explicit, constructive, intuitive
- Disadvantage of Lagrange interpolation
  - computationally complex
  - not progressive

Outline

Newton's divided differences

- Newton's divided differences provide a *more manageable* and *progressive* form for polynomial interpolation.

- Degree 0 case

  Given 1 data pair (t_1, P_1), we let $f_0(t) = P_1$.
  $f_0(t)$ is a constant function satisfying $f_0(t_1) = P_1$.

- Degree 1 case

  Given 2 data points (t_1, P_1) and (t_2, P_2), we let

$$f_1(t) = P_1 + \frac{P_2 - P_1}{t_2 - t_1}(t - t_1).$$

  Then, $f_1(t_1) = P_1, \quad f_1(t_2) = P_2$.

Newton's divided differences

- Degree 2 case

Given 3 data points (t_1, P_1), (t_2, P_2) and (t_3, P_3), we let

$$f_2(t) = P_1 + \frac{P_2 - P_1}{t_2 - t_1}(t - t_1) + \frac{\frac{P_3 - P_2}{t_3 - t_2} - \frac{P_2 - P_1}{t_2 - t_1}}{t_3 - t_1}(t - t_1)(t - t_2).$$

Then, $f_2(t_1) = P_1, \quad f_2(t_2) = P_2, \quad f_2(t_3) = P_3.$

Proof. Expand $f_2(t)$ and check the coefficients of P_1, P_2 and P_3:

$$P_3 : \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

$$P_2 : \frac{t-t_1}{t_2-t_1} - \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)} - \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_2-t_1)} = \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}$$

$$P_1 : 1 - \frac{t-t_1}{t_2-t_1} + \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_2-t_1)} = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}$$

Newton's divided differences

- General

Introduce the divided differences:

$$f[t_k] = P_k$$

$$f[t_k, t_{k+1}] = \frac{f[t_{k+1}] - f[t_k]}{t_{k+1} - t_k}$$

$$f[t_k, t_{k+1}, t_{k+2}] = \frac{f[t_{k+1}, t_{k+2}] - f[t_k, t_{k+1}]}{t_{k+2} - t_k}$$

and so on.

The **Newton's divided difference** formula is

$$\begin{aligned} f_{n-1}(t) &= f[t_1] + f[t_1, t_2](t - t_1) \\ &+ f[t_1, t_2, t_3](t - t_1)(t - t_2) \\ &+ f[t_1, t_2, t_3, t_4](t - t_1)(t - t_2)(t - t_3) \\ &+ \cdots \\ &+ f[t_1, t_2, \cdots, t_n](t - t_1)(t - t_2) \cdots (t - t_{n-1}). \end{aligned}$$

Newton's divided differences

- Newton form

Let $N_i(t) = \prod_{j=1}^{i-1}(t-t_j), i = 1, \cdots, n$, which we call the *Newton basis polynomials*. Then the interpolating polynomial of degree $n-1$ or less in Newton form is

$$f_{n-1}(t) = \sum_{i=1}^{n} B_i N_i(t)$$

where the coefficients $B_i = f[t_1, \cdots, t_i]$ are the $(i-1)$-th order *divided differences* of the interpolation data P_i with respect to the parameter values t_i, which result from the recursive definition:

$$f[t_i] = P_i,$$

$$f[t_i, \cdots, t_{i+j}] = \frac{f[t_{i+1}, \cdots, t_{i+j}] - f[t_i, \cdots, t_{i+j-1}]}{t_{i+j} - t_i},$$

$$i = 1, \cdots, n-j; \quad j = 1, \cdots, n-1$$

Newton's divided differences

- Diagram for computing the divided differences

$$
\begin{array}{l|llll}
t_1 & f[t_1] & & & \\
t_2 & f[t_2] & f[t_1,t_2] & & \\
t_3 & f[t_3] & f[t_2,t_3] & f[t_1,t_2,t_3] & \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
t_n & f[t_n] & f[t_{n-1},t_n] & f[t_{n-2},t_{n-1},t_n] & \cdots \quad f[t_1,\cdots,t_n]
\end{array}
$$

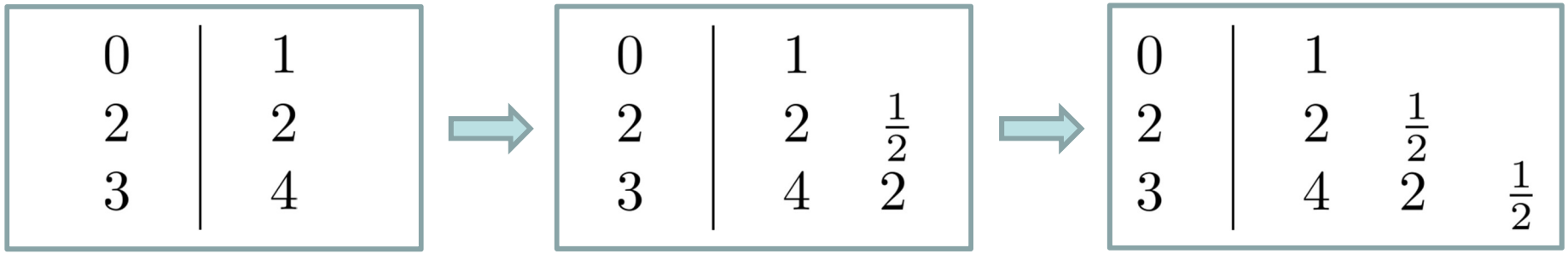
- Algorithm

Once the coefficients B_i have been obtained, the evaluation of the interpolating polynomial at t can be done with $O(n)$ operations via a Horner-like scheme:

$P := B_n$
for i from $(n-1)$ to 1 by (-1) do
    $P := (t - t_i)P + B_i$
return P.

Newton's divided differences

- Example: use Newton's divided differences to find the interpolating polynomial passing through the points (0,1), (2,2) and (3,4).



Therefore, the degree 2 interpolating polynomial is

$$f_2(x) = 1 + \frac{1}{2}x + \frac{1}{2}x(x-2) = \frac{1}{2}x^2 - \frac{1}{2}x + 1.$$

Interpolation error

Theorem. Assume that $g(t)$ is a C^n continuous function and $f_{n-1}(t)$ is the (degree $n-1$ or less) interpolating polynomial passing through the n points $(t_1, g(t_1)), \cdots, (t_n, g(t_n))$. The interpolation error is

$$g(t) - f_{n-1}(t) = \frac{(t-t_1)(t-t_2)\cdots(t-t_n)}{n!} g^{(n)}(c)$$

where c lies between the smallest and largest of the numbers $t, t_1, \cdots, t_n$.

Outline of the proof:

- For any u, construct the degree n or less polynomial interpolating $(t_1, g(t_1)), \cdots, (t_n, g(t_n))$, and $(u, g(u))$:

  $$f_n(t) = f_{n-1}(t) + g[t_1, \cdots, t_n, u](t-t_1)\cdots(t-t_n).$$

  Hence $g(u) = f_n(u) = f_{n-1}(u) + g[t_1, \cdots, t_n, u](u-t_1)\cdots(u-t_n)$.

- $f_n(t)$ matches $g(t)$ at at least $n+1$ points. Therefore there exsits a c such that $f_n^{(n)}(c) = g^{(n)}(c)$.

- $g^{(n)}(c) = g[t_1, \cdots, t_n, u]n!$

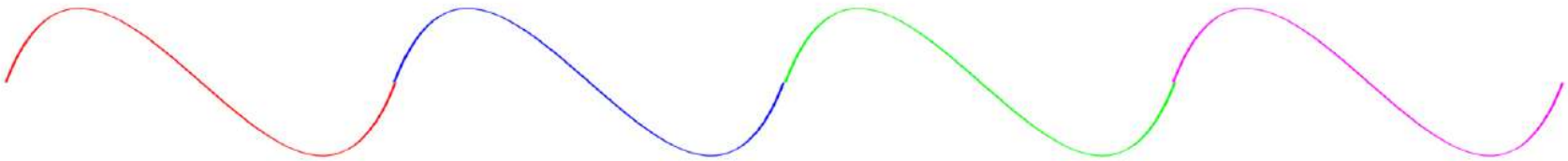
Outline

Background

- Designing complex shapes with a *single* polynomial or Bezier curve is not recommended. This is because it would require choosing a large degree n, which is bad for 2 reasons.
  - the $O(n^2)$ evaluation with the de Casteljau algorithm becomes slow;
  - the local control property is not very effective for large n.
- A common approach is to fit several low-degree Bezier curves together to form a *Bezier spline*. This is actually what common programs like *Adobe Illustrator* and *PowerPoint* do, and what is used for the definition of ***TrueType*** fonts.

Bezier splines

- Construction of geometrically complex curves
- Bezier splines
  - Join several Bezier curves of low degrees
  - Guarantee continuous join from one curve to the next



Bezier curve revisited

- A degree n Bezier curve with control points $P_0, \cdots, P_n$ is defined by

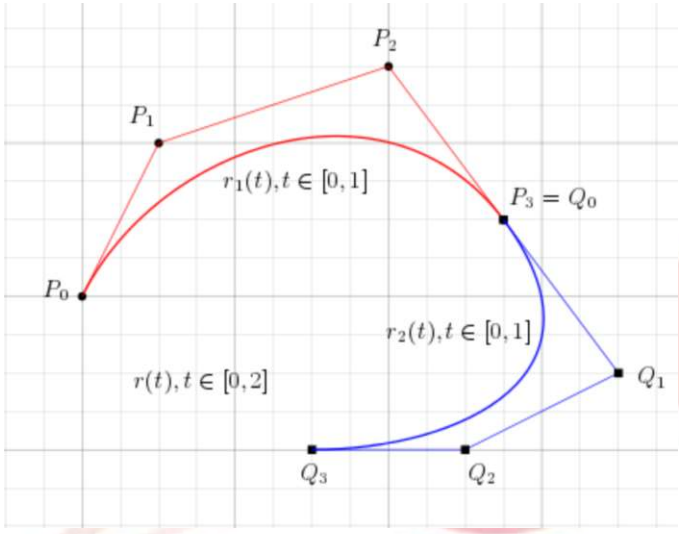
$$r(t) = \sum_{i=0}^{n} P_i B_i^n(t), \quad t \in [0,1]$$

- A degree n Bezier curve with control points $P_0, \cdots, P_n$ over $[a,b]$ is

$$r(t) = \sum_{i=0}^{n} P_i B_i^n\left(\frac{t-a}{b-a}\right), \quad t \in [a,b]$$

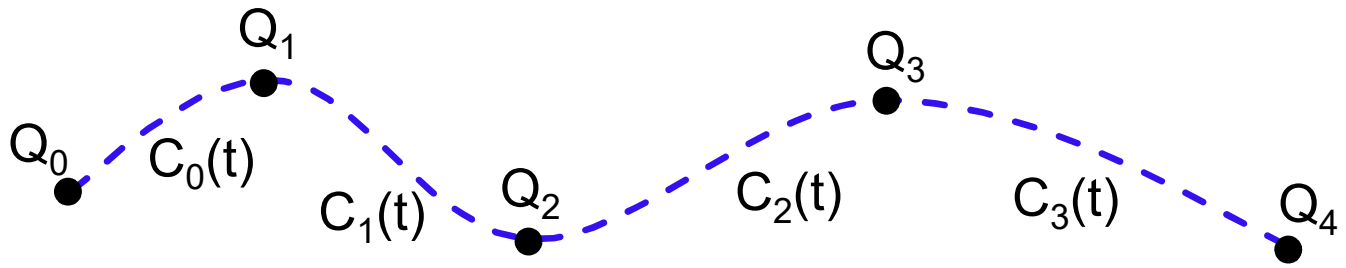
- Two Bezier curves $r_1(t) = \sum_{i=0}^{n} P_i B_i^n(t)$ and $r_2(t) = \sum_{i=0}^{n} Q_i B_i^n(t), \quad t \in [0,1]$ can be combined into a Bezier spline $r(t),\ t \in [0,2]$:

$$r(t) = \begin{cases} \sum_{i=0}^{n} P_i B_i^n(t), & t \in [0,1] \\ \sum_{i=0}^{n} Q_i B_i^n(t-1), & t \in [1,2] \end{cases}$$



C^1 cubic Bezier splines

Problem: Given a sequence of points $Q_0, Q_1, \ldots, Q_n$, we construct n cubic Bezier curves. The two adjacent curves are connected continuously. The i^{th} Bezier curve $C_i(t)$ is the segment from Q_i to Q_{i+1}.

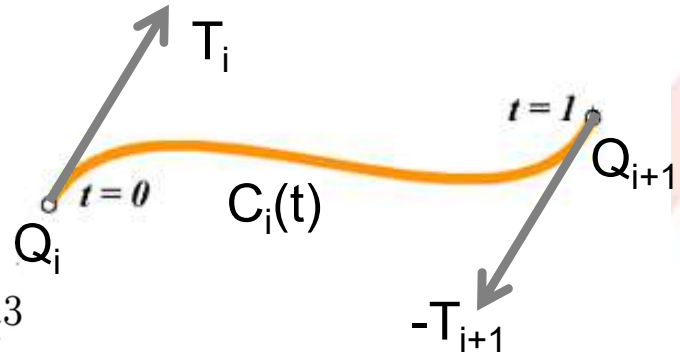


Construction: *To define the i^{th} Bezier curve $C_i(t)$, we have to find the four Bezier points P_{i0}, P_{i1}, P_{i2}, and P_{i3}. We assign two tangent vectors T_i and T_{i+1} to P_{i0} and P_{i3}, respectively. Then*

$$P_{i0} = Q_i, \quad P_{i1} = Q_i + \tfrac{1}{3}T_i$$

$$P_{i3} = Q_{i+1}, \quad P_{i2} = Q_{i+1} - \tfrac{1}{3}T_{i+1}$$

$$C_i(t) = P_{i0}(1-t)^3 + P_{i1}3(1-t)^2t + P_{i2}3(1-t)t^2 + P_{i3}t^3$$



Strategies for tangents

- Choice of tangents T_i (i = 1,2, …, n-1)
  - FMILL tangents: $T_i = \frac{Q_{i+1} - Q_{i-1}}{2}$

    This guarantees that the Bezier spline is C^1 continuous.

    This Bezier spline is also called **Catmull-Rom spline**.

- Conditions at the end points Q_0 and Q_n.
  - Natural end condition:

$$C_0''(0) = C_{n-1}''(1) = 0$$

$$\text{which gives } P_{01} = \frac{Q_0 + P_{02}}{2} \text{ and } P_{(n-1)2} = \frac{P_{(n-1)}1 + Q_n}{2}.$$

  - Simple endpoint condition:

$$C_0'(0) = C_{n-1}'(1) = 0, \quad \text{giving } P_{01} = Q_0, \quad P_{(n-1)2} = Q_n.$$

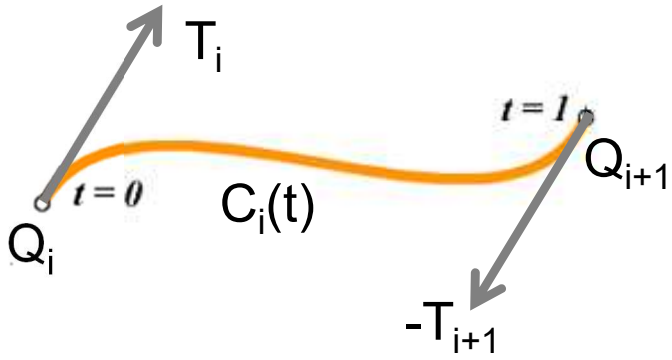
Extension of Catmull-Rom spline

Construction: *To define the i^{th} Bezier curve $C_i(t)$, we have to find the four Bezier points P_{i0},P_{i1}, P_{i2},and P_{i3}. We assign two tangent vectors T_i and T_{i+1} to P_{i0} and P_{i3}, respectively. Then*

$$P_{i0} = Q_i, \qquad P_{i1} = Q_i + \frac{1}{3}\lambda_i \frac{T_i}{|T_i|}$$

$$P_{i3} = Q_{i+1}, \quad P_{i2} = Q_{i+1} - \frac{1}{3}\lambda_i \frac{T_{i+1}}{|T_{i+1}|}$$

with $\lambda_i = |Q_{i+1} - Q_i|$.



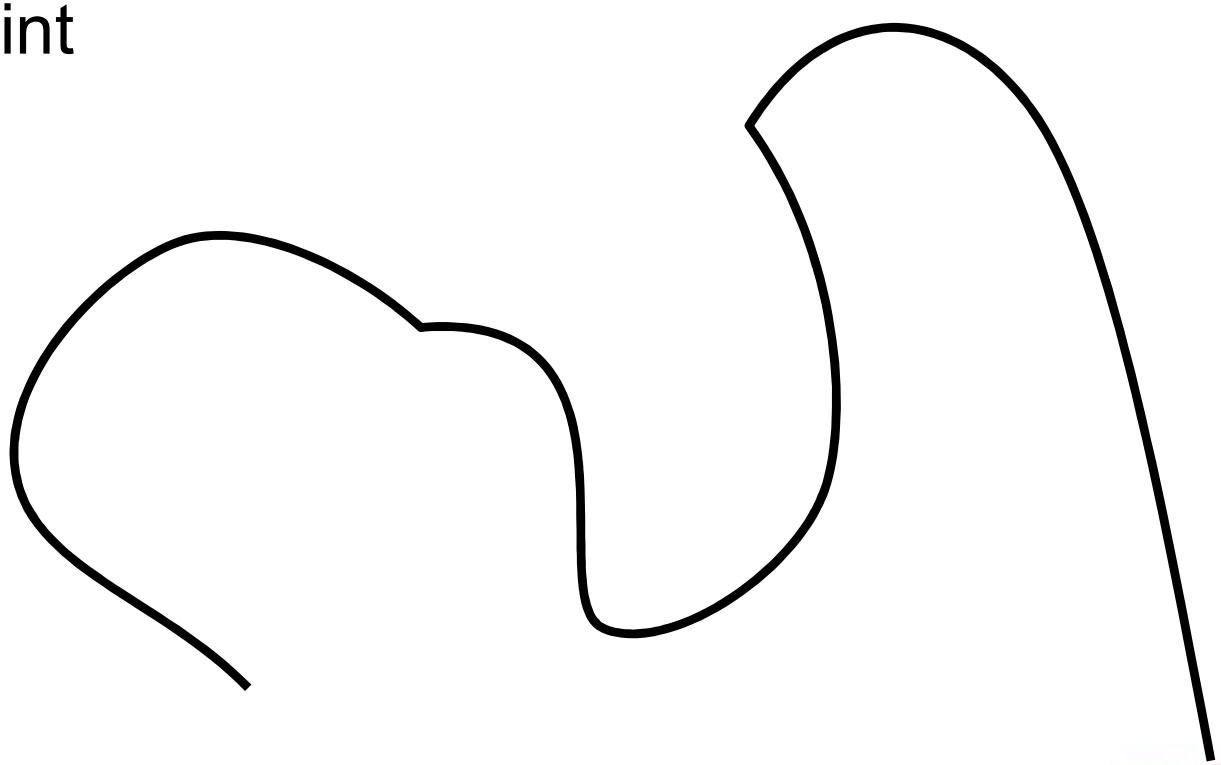
Then the Bezier curve is expressed as:

$$C_i(t) = P_{i0}(1-t)^3 + P_{i1}3(1-t)^2 t + P_{i2}3(1-t)t^2 + P_{i3}t^3$$

Q for you: Are $C_i(t)$ and $C_{i+1}(t)$ now still C^1 continuous? Are the two adjacent Bezier curve segments connected smoothly?

Example: freeform curve design

- PowerPoint or MS-Word: AutoShapes
  - Smooth point
  - Straight point
  - Corner point

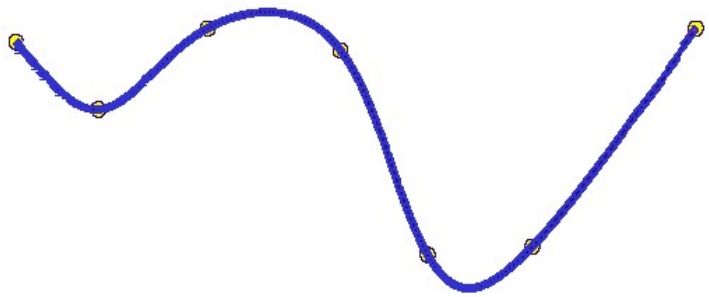


Outline

C^2 cubic B-spline interpolation

- Problem: Given n+1 data points $\{D_i \mid i=0,1, \ldots,n\}$, find a cubic B-spline curve that interpolates them.

- This involves two basic steps:
  - Step 1. Parameterize the data points: assign appropriate parameter values to data point D_i
  - Step 2. find the B-spline representation.



Step 1. Parameterization

- **Uniform parameterization**: If there are n+1 data points $\{D_i\}$, i=0,…,n, then n+1 evenly spaced parameter values t_i within [0,1] are chosen for $\{D_i\}$. That is,

$$\begin{cases} t_0 &= 0 \\ t_i &= i/n \quad \text{for } 1 \le i \le n-1 \\ t_n &= 1 \end{cases}$$

Uniform parameterization may result in unwanted wiggles, which is known as *Runge phenomenon* (refer to Chapter 3.2.3).

- **Chord length parameterization**: The parameter values are chosen based on the spatial distribution of the data points.

$$\begin{cases} t_0 &= 0 \\ t_i &= \frac{\sum_{k=1}^{i} |D_k - D_{k-1}|}{\sum_{k=1}^{n} |D_k - D_{k-1}|} \quad \text{for } 1 \le i \le n-1 \\ t_n &= 1 \end{cases}$$

Step 2. Find the B-spline curve

- Once the parameterization for the data points is done, we now need to determine the B-spline curve, which requires to specify the *degree*, *knots*, and the *de Boor points*.
- The idea is to create a cubic B-spline curve $r(t)$ with n segments whose parameter intervals are $[t_0,t_1]$, $[t_1,t_2]$, …, $[t_{n-1},t_n]$, respectively, satisfying $r(t_i) = D_i$.
  - Set degree = 3
  - Set knots = $\{t_0, t_0, t_0, t_0, t_1,t_2,\ldots,t_{n-2},t_{n-1}, t_n, t_n, t_n, t_n\}$
  - What remains is to determine the (n+3) de Boor points $P_0, P_1, \ldots, P_{n+2}$. This is done by building a set of equations.

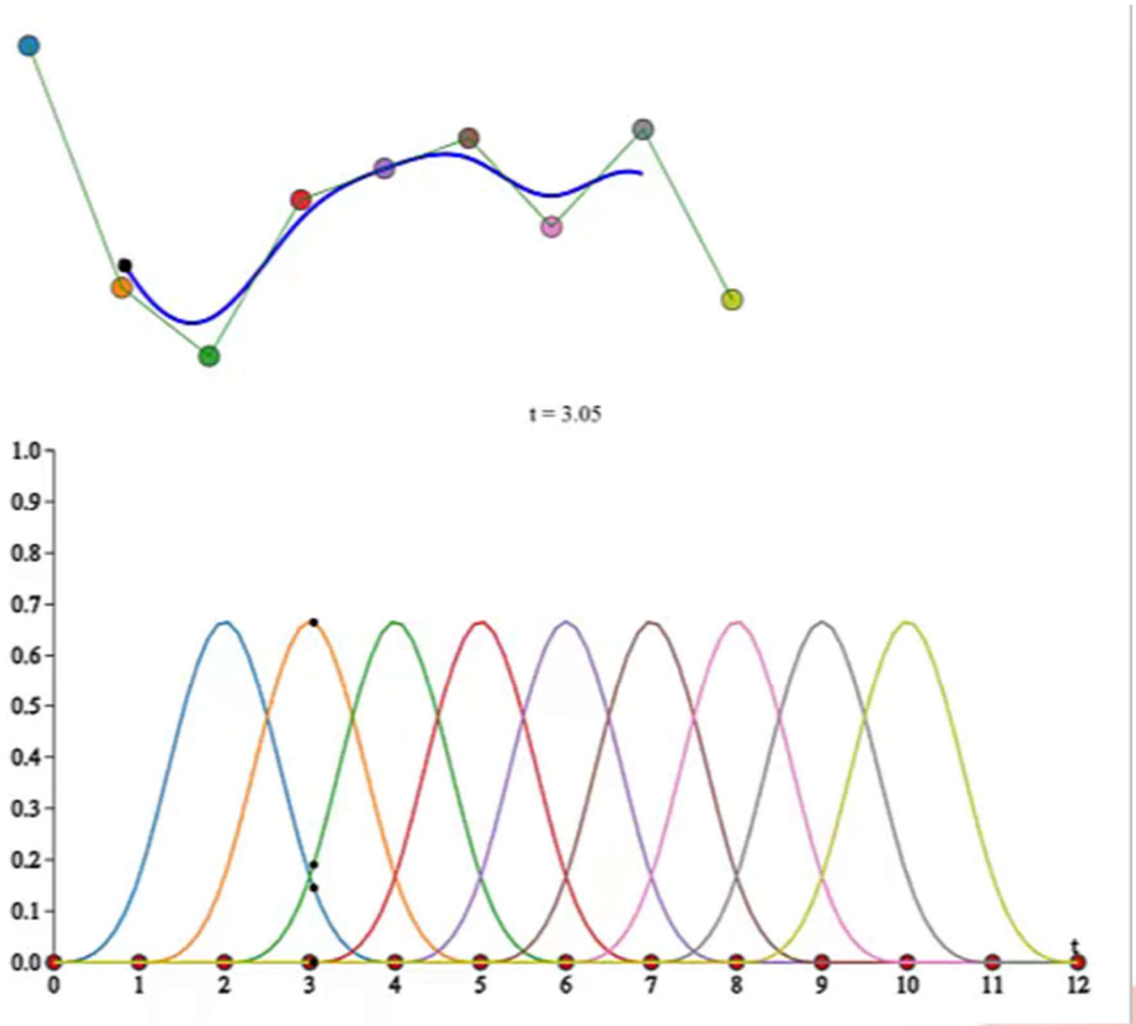
Cubic B-spline basis functions revisited

- Cubic B-spline basis

$$N_i^3(u) = \begin{cases} \dfrac{(u-u_i)^3}{(u_{i+1}-u_i)(u_{i+2}-u_i)(u_{i+3}-u_i)}, & u \in [u_i, u_{i+1}) \\ \dfrac{(u-u_i)^2(u_{i+2}-u)}{(u_{i+2}-u_{i+1})(u_{i+3}-u_i)(u_{i+2}-u_i)} + \dfrac{(u_{i+3}-u)(u-u_i)(u-u_{i+1})}{(u_{i+2}-u_{i+1})(u_{i+3}-u_{i+1})(u_{i+3}-u_i)} + \dfrac{(u_{i+4}-u)(u-u_{i+1})^2}{(u_{i+2}-u_{i+1})(u_{i+4}-u_{i+1})(u_{i+3}-u_{i+1})}, & u \in [u_{i+1}, u_{i+2}) \\ \dfrac{(u-u_i)(u_{i+3}-u)^2}{(u_{i+3}-u_{i+2})(u_{i+3}-u_{i+1})(u_{i+3}-u_i)} + \dfrac{(u_{i+4}-u)(u_{i+3}-u)(u-u_{i+1})}{(u_{i+3}-u_{i+2})(u_{i+4}-u_{i+1})(u_{i+3}-u_{i+1})} + \dfrac{(u_{i+4}-u)^2(u-u_{i+2})}{(u_{i+3}-u_{i+2})(u_{i+4}-u_{i+2})(u_{i+4}-u_{i+1})}, & u \in [u_{i+2}, u_{i+3}) \\ \dfrac{(u_{i+4}-u)^3}{(u_{i+4}-u_{i+3})(u_{i+4}-u_{i+2})(u_{i+4}-u_{i+1})}, & u \in [u_{i+3}, u_{i+4}) \end{cases}$$

Cubic B-spline basis functions revisited

At knot u_i, only 3 basis functions $N^3_{i-3}(u), N^3_{i-2}(u), N^3_{i-1}(u)$ do not vanish.



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Step 2. Find the B-spline curve (cont)

- The interpolation requirement gives n+1 equations:

$$\text{For } k = 0, 1, \cdots, n,$$

$$r(t_k) = \sum_{i=0}^{n+2} P_i N_i^3(t_k) = D_k, \text{ which is}$$

$$N_k^3(t_k)P_k + N_{k+1}^3(t_k)P_{k+1} + N_{k+2}^3(t_k)P_{k+2} = D_k$$

- Endpoint conditions
  - Natural conditions: r”(u)=0 at t_0 (=u_3) and t_n (=u_{n+3}).

    This gives another two equations:

$$\frac{d^2 N_0^3(t_0)}{du^2} P_0 + \frac{d^2 N_1^3(t_0)}{du^2} P_1 + \frac{d^2 N_2^3(t_0)}{du^2} P_2 = 0$$

$$\frac{d^2 N_n^3(t_n)}{du^2} P_n + \frac{d^2 N_{n+1}^3(t_n)}{du^2} P_{n+1} + \frac{d^2 N_{n+2}^3(t_n)}{du^2} P_{n+2} = 0$$

Step 2. Find the B-spline curve (cont)

- Thus we obtain (n+3) linear equations in (n+3) variables P_0, P_1, … P_{n+2}:

$$\begin{bmatrix} \frac{d^2N_0^3(t_0)}{du^2} & \frac{d^2N_1^3(t_0)}{du^2} & \frac{d^2N_2^3(t_0)}{du^2} & \cdots & 0 & 0 & 0 \\ N_0^3(t_0) & N_1^3(t_0) & N_2^3(t_0) & \cdots & 0 & 0 & 0 \\ 0 & N_1^3(t_1) & N_2^3(t_1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N_n^3(t_{n-1}) & N_{n+1}^3(t_{n-1}) & 0 \\ 0 & 0 & 0 & \cdots & N_n^3(t_n) & N_{n+1}^3(t_n) & N_{n+2}^3(t_n) \\ 0 & 0 & 0 & \cdots & \frac{d^2N_n^3(t_n)}{du^2} & \frac{d^2N_{n+1}^3(t_n)}{du^2} & \frac{d^2N_{n+2}(t_n)}{du^2} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_n \\ P_{n+1} \\ P_{n+2} \end{bmatrix} = \begin{bmatrix} 0 \\ D_0 \\ D_1 \\ \vdots \\ D_{n-1} \\ D_n \\ 0 \end{bmatrix}$$

Step 2. Find the B-spline curve (cont)

- Since we choose multiple knots at t_0 and t_n, the first and last control points P_0,P_{n+2} interpolate D_0,D_{n+2}. It is easy to verify that $N_0^3(t_0) = N_{n+2}^3(t_n) = 1, N_1^3(t_0) = N_2^3(t_0) = N_n^3(t_n) = N_{n+1}^3(t_n) = 0.$ Swapping the 1st and 2nd rows, the (n+2)-th and (n+3)-th rows gives

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{d^2N_0^3(t_0)}{du^2} & \frac{d^2N_1^3(t_0)}{du^2} & \frac{d^2N_2^3(t_0)}{du^2} & \cdots & 0 & 0 & 0 \\ 0 & N_1^3(t_1) & N_2^3(t_1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N_n^3(t_{n-1}) & N_{n+1}^3(t_{n-1}) & 0 \\ 0 & 0 & 0 & \cdots & \frac{d^2N_n^3(t_n)}{du^2} & \frac{d^2N_{n+1}^3(t_n)}{du^2} & \frac{d^2N_{n+2}(t_n)}{du^2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_n \\ P_{n+1} \\ P_{n+2} \end{bmatrix} = \begin{bmatrix} D_0 \\ 0 \\ D_1 \\ \vdots \\ D_{n-1} \\ 0 \\ D_n \end{bmatrix}$$

- Note that the coefficient matrix is a tri-diagonal matrix. This linear system can be solved very efficiently (see Module 2. Linear Systems or refer to https://en.wikipedia.org/wiki/Tridiagonal_matrix).

Algorithm

Input: n+1 data points D_0, …, D_n

Output: A cubic B-spline curve that goes through all D_i in the *given* order

Algorithm:

computing a set of n+1 parameters t_0, ..., t_n;
set knot vector T;
for i := 0 to n do
  for j := 0 to n+2 do
    Evaluate $N_j^3(t_i)$ into row i+2 and column j+1 of matrix [N];
for i := 0 to n do
  Place data point D_i on row i+2 of matrix [D];
add the boundary conditions into the second row and the (last-1) row

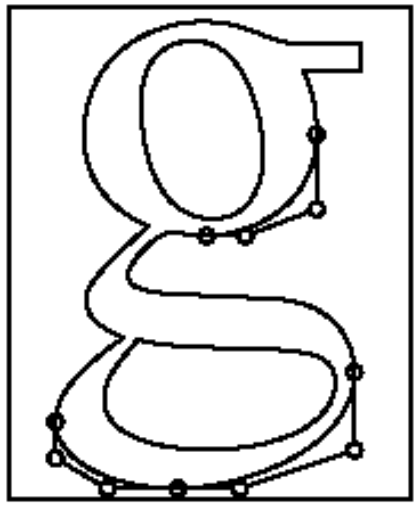
Use an efficient linear system solver to solve for [P] from [N]·[P] = [D].
Row i of [P] is control point P_{i-1}; Control points P_0, ..., P_{n+2}, and knot vector T determine an interpolating cubic B-spline curve.

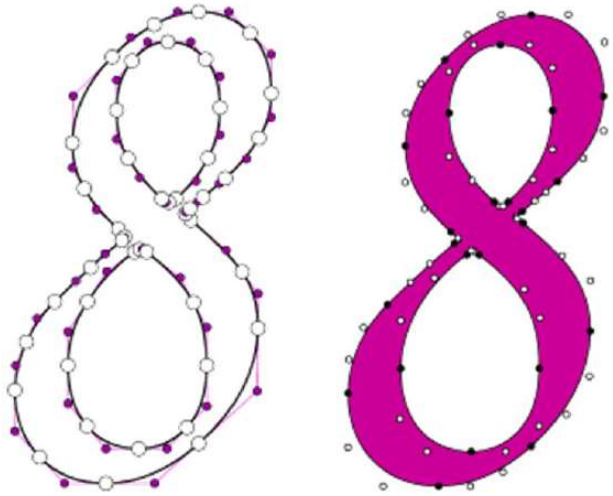
Outline

Applications in font definition

- A common use for Bezier curves is in font definition.
  - Type 1: cubic Bezier curves (PostScript, Adobe)
  - TrueType: quadratic Bezier curves (Mac, Windows)
  - MetaFont: cubic Bezier curves (D Knuth, Tex)

- Adobe illustrator
  - Cubic Bezier curves
- PowerPoint
  - Cubic Bezier curves





Truetype Type 1

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Fonts from Bezier curves

- Modern fonts are built directly from Bezier curves, which were a fundamental part of the PostScript language from its start in the 1980s.

- The PostScript commands have been migrated to the PDF format.

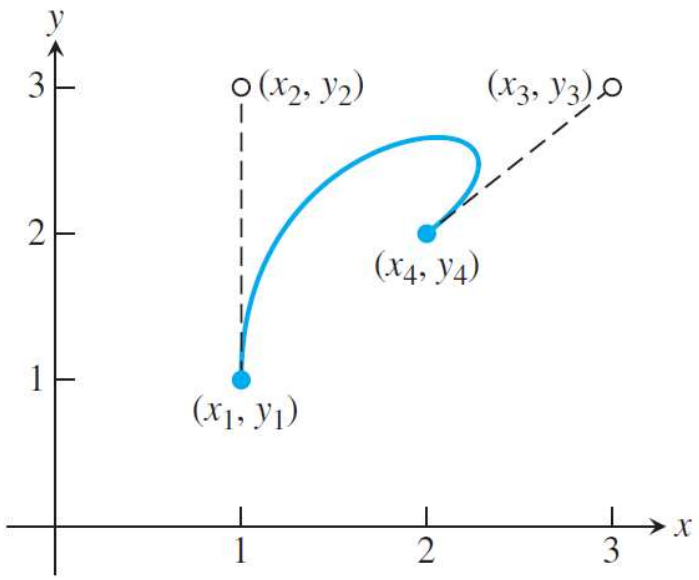
A complete PDF file

```
%PDF-1.7
1 0 obj
<<
/Length 2 0 R
>>
stream
100 100 m
100 300 300 300 200 200 c
S
endstream
endobj
2 0 obj
1000
endobj
4 0 obj
<<
/Type /Page
/Parent 5 0 R
/Contents 1 0 R
>>
endobj
5 0 obj
<<
/Kids [4 0 R]
/Count 1
/Type /Pages
/MediaBox [0 0 612 792]
>>
endobj
```

```
3 0 obj
<<
/Pages 5 0 R
/Type /Catalog
>>
endobj
xref
0 6
0000000000 65535 f
0000000100 00000 n
0000000200 00000 n
0000000500 00000 n
0000000300 00000 n
0000000400 00000 n
trailer
<<
/Size 6
/Root 3 0 R
>>
startxref
1000
%%EOF
```

The lines between **stream** and **endstream** identify the Bezier curve.

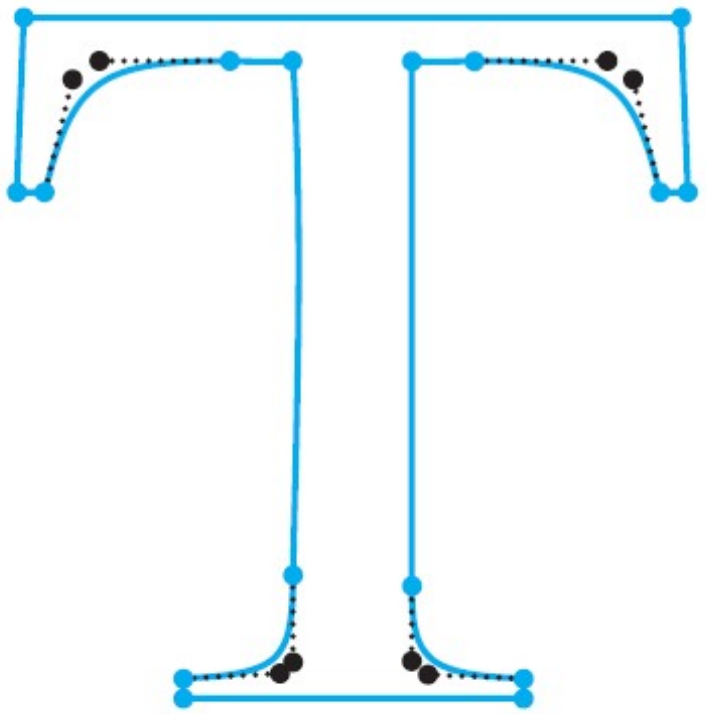
- m: set the current point
- c: accept another 3 points for a curve
- S: draw the curve



Example: Times-Roman T made with Bezier splines

A Bezier spline consists of 16 Bezier curves.

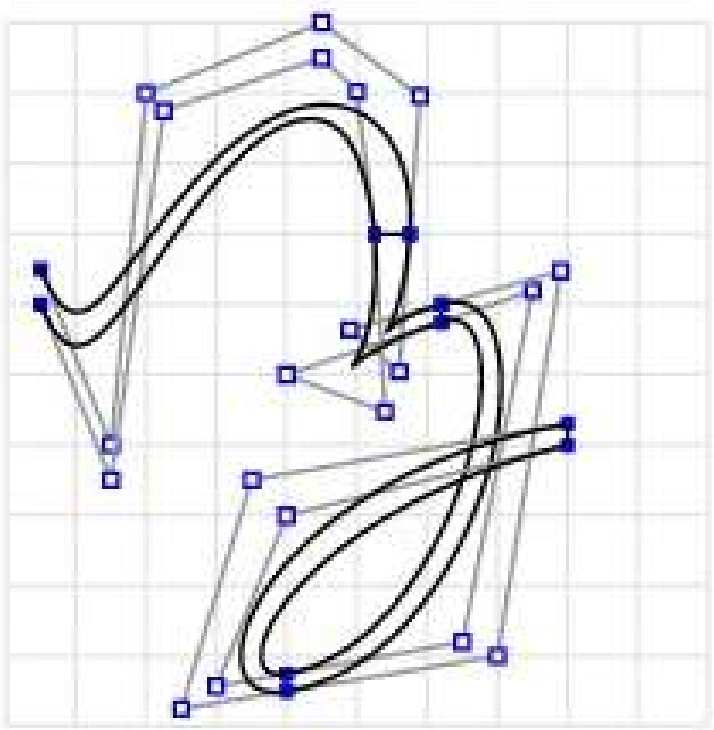
```
237 620 237 620 237 120 237 120;
237 120 237  35 226  24 143  19;
143  19 143  19 143   0 143   0;
143   0 143   0 435   0 435   0;
435   0 435   0 435  19 435  19;
435  19 353  23 339  36 339 109;
339 109 339 108 339 620 339 620;
339 620 339 620 393 620 393 620;
393 620 507 620 529 602 552 492;
552 492 552 492 576 492 576 492;
576 492 576 492 570 662 570 662;
570 662 570 662   6 662   6 662;
  6 662   6 662   0 492   0 492;
  0 492   0 492  24 492  24 492;
 24 492  48 602  71 620 183 620;
183 620 183 620 237 620 237 620;
```

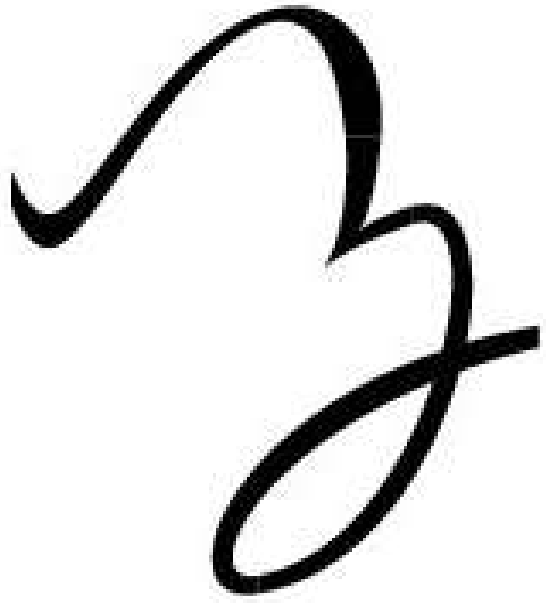


To create a PDF file, one needs to add commands:

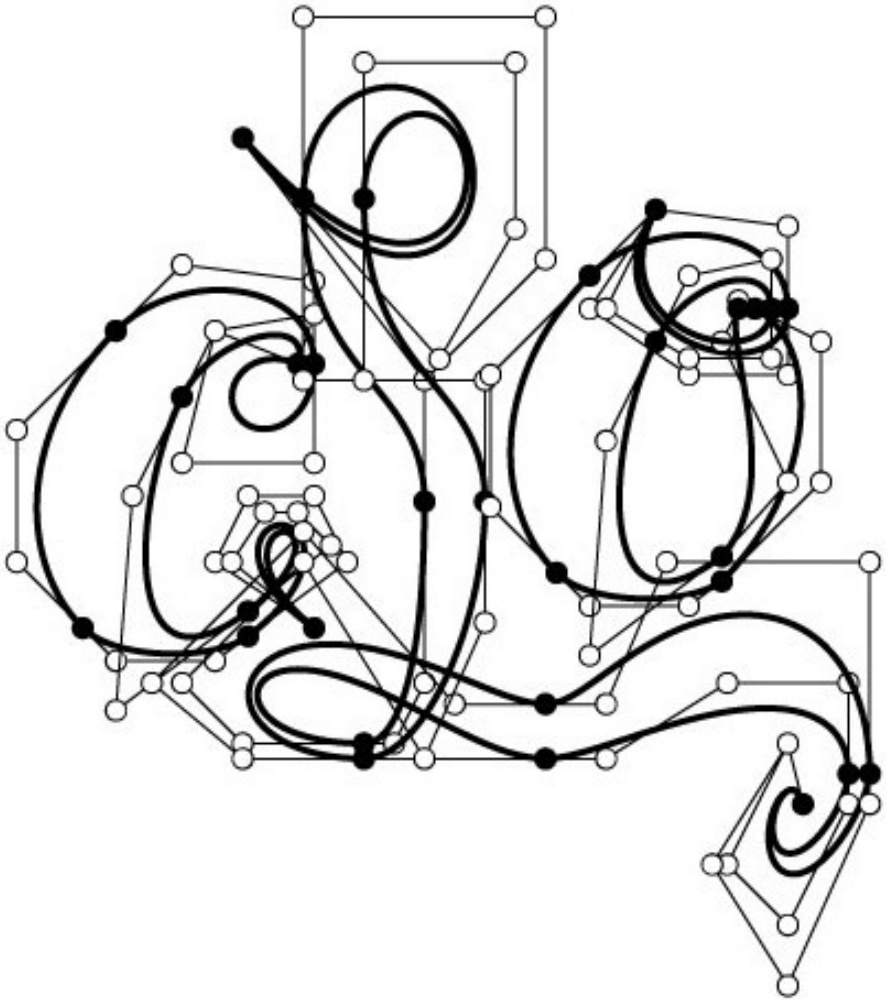
237 620 m
237 620 237 120 237 120 c
…

Example: font design

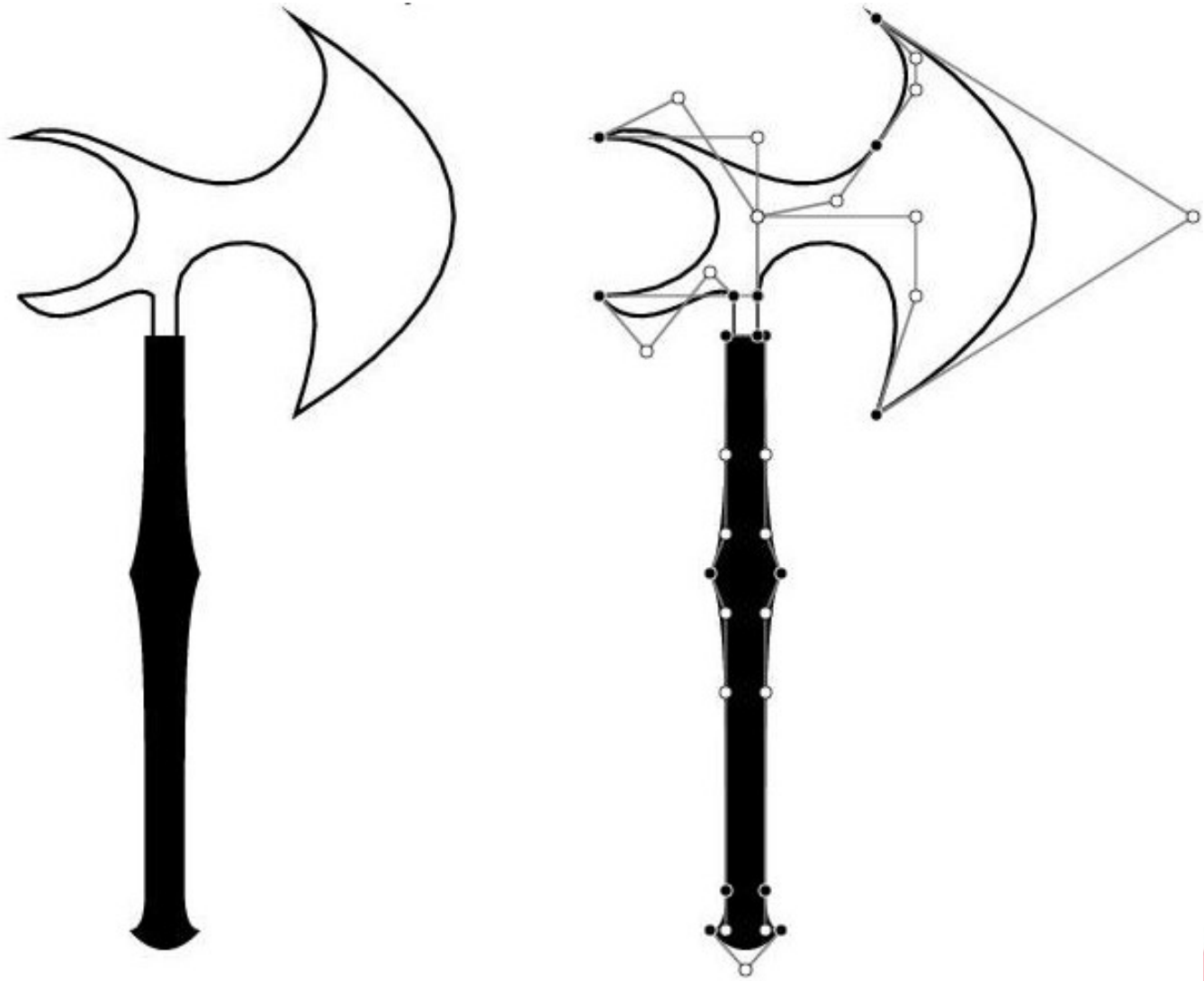




Example: font design



Example: font design



Outline

Homework

If we define the polygon (blue) with vertices

$$P_0 = \begin{pmatrix} 200 \\ 200 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 200 \\ 500 \end{pmatrix},$$

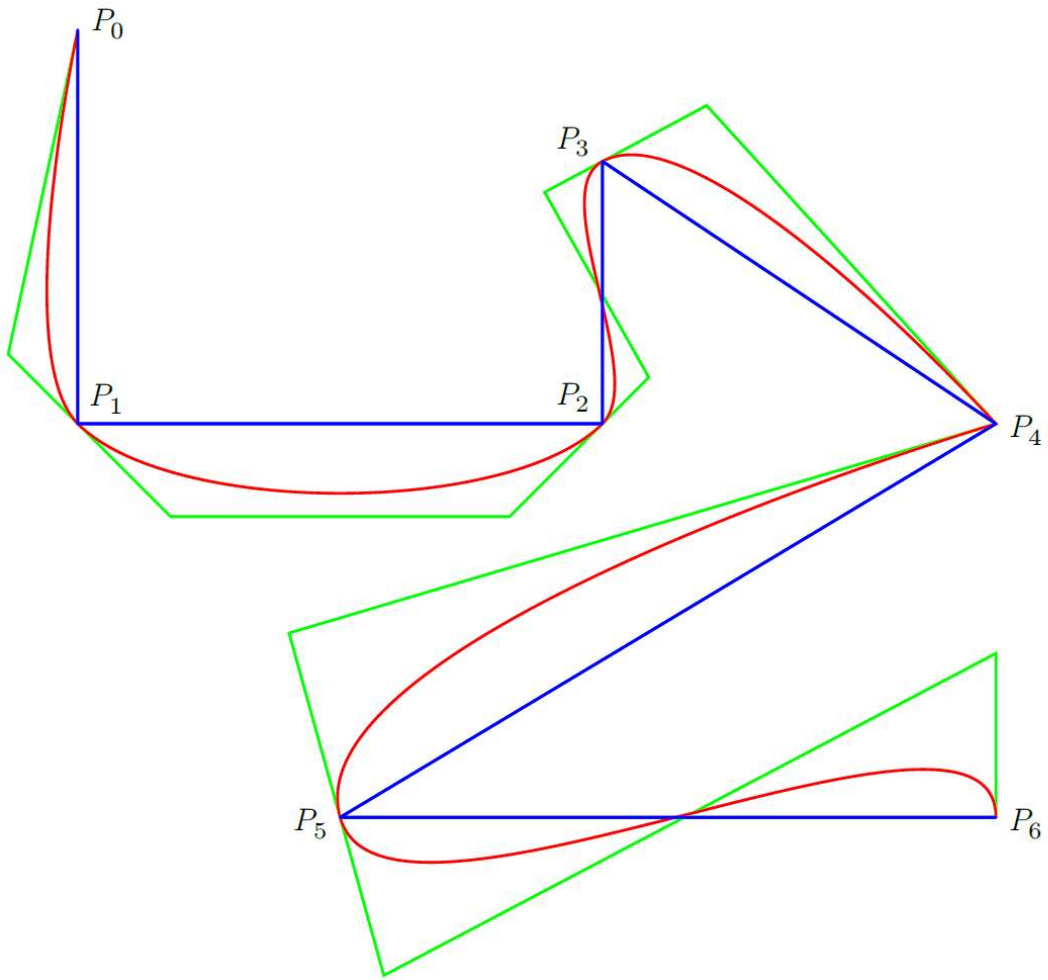
$$P_2 = \begin{pmatrix} 600 \\ 500 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 600 \\ 300 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 900 \\ 500 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 400 \\ 800 \end{pmatrix},$$

$$P_6 = \begin{pmatrix} 900 \\ 800 \end{pmatrix}$$



Homework (cont)

in *Adobe Illustrator* and convert the points P_1, P_2, P_3, P_5, P_6 from *corner* to *smooth*, then we get a cubic Bézier spline (red) with control points (green)

$$C_{0,0} = \begin{pmatrix} 200 \\ 200 \end{pmatrix}, \quad C_{0,1} = \begin{pmatrix} 200 \\ 200 \end{pmatrix}, \quad C_{0,2} = \begin{pmatrix} 146.97 \\ 446.97 \end{pmatrix}, \quad C_{0,3} = \begin{pmatrix} 200 \\ 500 \end{pmatrix},$$

$$C_{1,0} = \begin{pmatrix} 200 \\ 500 \end{pmatrix}, \quad C_{1,1} = \begin{pmatrix} 270.71 \\ 570.71 \end{pmatrix}, \quad C_{1,2} = \begin{pmatrix} 529.29 \\ 570.71 \end{pmatrix}, \quad C_{1,3} = \begin{pmatrix} 600 \\ 500 \end{pmatrix},$$

$$C_{2,0} = \begin{pmatrix} 600 \\ 500 \end{pmatrix}, \quad C_{2,1} = \begin{pmatrix} 635.36 \\ 464.64 \end{pmatrix}, \quad C_{2,2} = \begin{pmatrix} 555.92 \\ 323.59 \end{pmatrix}, \quad C_{2,3} = \begin{pmatrix} 600 \\ 300 \end{pmatrix},$$

$$C_{3,0} = \begin{pmatrix} 600 \\ 300 \end{pmatrix}, \quad C_{3,1} = \begin{pmatrix} 679.47 \\ 257.47 \end{pmatrix}, \quad C_{3,2} = \begin{pmatrix} 900 \\ 500 \end{pmatrix}, \quad C_{3,3} = \begin{pmatrix} 900 \\ 500 \end{pmatrix},$$

$$C_{4,0} = \begin{pmatrix} 900 \\ 500 \end{pmatrix}, \quad C_{4,1} = \begin{pmatrix} 900 \\ 500 \end{pmatrix}, \quad C_{4,2} = \begin{pmatrix} 361.09 \\ 659.52 \end{pmatrix}, \quad C_{4,3} = \begin{pmatrix} 400 \\ 800 \end{pmatrix},$$

$$C_{5,0} = \begin{pmatrix} 400 \\ 800 \end{pmatrix}, \quad C_{5,1} = \begin{pmatrix} 433.37 \\ 920.46 \end{pmatrix}, \quad C_{5,2} = \begin{pmatrix} 900 \\ 675 \end{pmatrix}, \quad C_{5,3} = \begin{pmatrix} 900 \\ 800 \end{pmatrix}.$$

Figure out which strategy *Adobe* has adopted for defining the tangents at the points that are converted to *smooth* and how they handle the situation at points that remain *corners*.

Describe the strategy and how you found it in detail, either with words or formulas.

Outline

Summary

- Polynomial interpolation
  - Explicit: Lagrange interpolation
  - Progressive: Newton's divided differences
- Spline interpolation
  - Bezier splines
  - Cubic B-splines
- Application in font representation

End