MATH 0450: HOMEWORK 9

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Instructions: Submit solutions for 5 of the 8 problems below (not counting extra credit problems).

Problem 1. (Ex. 4.2.7) Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$.

Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Proof. We want to show $\lim_{x\to c} g(x)f(x) = 0$, which means for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|g(x)f(x) - 0| < \epsilon$. Fix $\epsilon_0 = \epsilon/M$. By definition of functional limit, $\lim_{x\to c} g(x) = 0$ means that for this ϵ_0 there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies

$$|g(x) - 0| < \epsilon_0 = \epsilon/M.$$

Then because $|f(x)| \leq M$, where $M > 0$, for all $x \in A$,

$$|g(x)f(x) - 0| = |g(x)f(x)| < \epsilon_0 \cdot M = \frac{\epsilon}{M} \cdot M = \epsilon,$$

which is what we wanted to show.

□

Problem 2. (Ex. 4.2.11) (Squeeze Theorem). Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$ as well.

Proof. Given $f(x) \leq g(x) \leq h(x)$ for all x in a common domain A, we can rewrite as

$$f(x) - L \leq g(x) - L \leq h(x) - L.$$

Using $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$, whenever $|x - c| < \delta$, we want to show $|g(x) - L| < \epsilon$ as well, where $\epsilon > 0$, arbitrary. Now to take absolute value on the individual terms in $f(x) - L \leq g(x) - L \leq h(x) - L$, we examine cases (varying the signs of the terms) and resolve it to

$$|g(x) - L| \leq |f(x) - L| < \epsilon$$

or

$$|g(x) - L| \leq |h(x) - L| < \epsilon$$

or both. Whichever the case we have shown $|g(x) - L| < \epsilon$ whenever $|x - c| < \delta$, where $\delta > 0$ as chosen for $f(x)$ and $h(x)$.

□

Problem 3. (Ex. 4.3.3)

(a) Supply a proof for Theorem 4.3.9 using the ϵ-δ characterization of continuity.

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Theorem 1 (4.3.9 Composition of Continuous Functions). *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2(iii)).

Theorem 2 (4.3.2(iii) Characterizations of Continuity). *For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.*

Proof. (a) Fix $\epsilon > 0$, arbitrary. Given that g is defined on A and continuous at $f(c) \in B$, for this ϵ there exists a $\delta_1 > 0$ such that for all $y \in B$,

$$|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \epsilon.$$

Because f is continuous at $c \in A$, for an $\epsilon_0 = \delta_1 > 0$, there exists a δ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon_0 = \delta_1.$$

Combining the two statements gives us for this arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1 \Rightarrow |g(f(x)) - g(f(c))| < \epsilon,$$

which by definition means that $g \circ f$ is continuous at c.

(b) Because f is continuous at $c \in A$, for all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$. Similarly, g is continuous at $f(c) \in B$ so for all $f(x_n) \to f(c)$ (with $f(x_n) \in B$), $g(f(x_n)) \to g(f(c))$.

Combining the two statements, we get for all $(x_n) \to c$ (with $x_n \in A$),

$$g(f(x_n)) \to g(f(c)),$$

which by sequential characterization of continuity means $g \circ f$ is continuous at c.

∎

Problem 4. (Ex. 4.3.9) Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Proof. To prove K is closed, it is enough to prove K contains all its limit points, by following Lemma.

Lemma 3 (Definition 3.2.7). *A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.*

Let x_0 be an arbitrary limit point of K. By definition of limit point, there exists a sequence $(x_n) \in K$ such that $(x_n) \to x_0$. Since $x_n \in K$ for all $n \in \mathbb{N}$, we have

$$h(x_n) = 0$$

for all $n \in \mathbb{N}$. Now, h is continuous, so

$$(x_n) \to x_0 \Rightarrow (h(x_n)) \to h(x_0).$$

Since $(h(x_n))$ is the constant sequence with each $h(x_n) = 0$, it converges to 0. So $h(x_0) = 0$, which implies that $x_0 \in K$. Therefore K contains all its limit points, and thus is closed.

∎

Problem 5. (Ex. 4.3.11*) (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(c)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.
(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.
(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.
(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in *b*.

Proof. (a) Fix $\epsilon > 0$, arbitrary. For this ϵ, set $\delta = \epsilon/c$. Now, whenever $|x - y| < \delta = \epsilon/c$, we have

$$|f(x) - f(y)| \leq c|x - y| < c \cdot \frac{\epsilon}{c} = \epsilon$$

for all $x, y \in \mathbb{R}$, with $0 < c < 1$, and so f continuous on $\mathbb{R}$.

(b) In general $y_{n+1} = f(y_n)$. For any two elements $f^n(y_1), f^m(y_1)$ of the sequence, with $n > m$,

$$\begin{aligned}|f^n(y_1) - f^m(y_1)| \leq c|f^{n-1}(y_1) - f^{m-1}(y_1)| &\leq c^2|f^{n-2}(y_1) - f^{m-2}(y_1)| \\ &\leq \cdots \\ &\leq c^m|f^{n-m}(y_1) - y_1|.\end{aligned}$$

Given f defined on all of $\mathbb{R}$, $f^{n-m}(y_1)$ is defined and so there exists some $N \in \mathbb{N}$ such that for all $m \geq N$,

$$c^m|f^{n-m}(y_1) - y_1| \leq \epsilon$$

for all $\epsilon > 0$. Thus the sequence (y_n) as defined is Cauchy. Hence we may let $y = \lim y_n$.

(c) By $y = \lim y_n$ as defined in (b), notice that

$$f(y) = f(\lim y_n) = f(\lim f^{n-1}(y_1)) = f(\lim f^{n-2}(y_1))$$

since limit of sequence not dependent on first (finitely many) term y_1. Furthermore, by f continuous we have that

$$f(\lim f^{n-2}(y_1)) = \lim f(f^{n-2}(y_1)) = \lim f^{n-1}(y_1) = \lim y_n = y$$

again by $y = \lim y_n$. So $f(y) = y$. Assume for contradiction y is not unique in its fixed point property, i.e. there exists an $x \in \mathbb{R}$ such that $f(x) = x$ (x is a fixed point). Then

$$|f(x) - f(y)| = |x - y| \leq c|x - y|$$

which implies that $c \geq 1$. But this contradicts our initial fundamental assumption that $0 < c < 1$. So our assumption that $f(x) = x$ is false, and so y is the only fixed point.

(d) Assume $x \in \mathbb{R}$, arbitrary. Because y is a fixed point ($f(y) = y$), observe that

$$\begin{aligned}|f^n(x) - y| = |f^n(x) - f^n(y)| &\leq c|f^{n-1}(x) - f^{n-1}(y)| \\ &\leq \cdots \\ &\leq c^n|x - y|\end{aligned}$$

for any $n \in \mathbb{N}$. Because $0 < c < 1$, $(c^n|x-y|)$ converges to 0, $(|f^n(x) - y|)$ also converges to 0, i.e. $\lim |f^n(x) - y| = 0$. In other words

$$\lim f^n(x) = y$$

and the proof is complete.

$\square$

Problem 6. (Ex. 4.3.12*) Let $F \subseteq \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x-a| : a \in F\}$. Show that g is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Proof. Fix $\epsilon > 0, c \in \mathbb{R}$, arbitrary. Let $\delta = \epsilon$. Our aim is to show that $|f(x)-g(c)| \leq |x-c| < \delta = \epsilon$. For $a \in F$, notice that

$$|x-a| \leq |x-c| + |c-a|$$

and

$$\begin{aligned} |c-a| &\leq |c-x| + |x-a| = |x-c| + |x-a| \\ \Rightarrow -|x-c| + |c-a| &\leq |x-a|. \end{aligned}$$

Combining the two inequalities we have

$$|c-a| - |x-c| \leq |x-a| \leq |c-a| + |x-c|.$$

Taking inf,

$$\inf\{|c-a| - |x-c| : a \in F\} \leq \inf\{|x-a| : a \in F\} \leq \inf\{|c-a| + |x-c|\}$$

which is equivalent to

$$\inf\{|c-a| : a \in F\} - |x-c| \leq \inf\{|x-a| : a \in F\} \leq \inf\{|c-a| : a \in F\} + |x-c|$$

By definition of g, we can resolve to above to

$$g(c) - |x-c| \leq g(x) \leq g(c) + |x-c|$$

which in other words means

$$|g(x) - g(c)| \leq |x-c| < \delta = \epsilon$$

and we have shown g continuous on $\mathbb{R}$ since the original $c \in \mathbb{R}$ we picked is arbitrary.

Given F closed, we prove for all $x \notin F$, $g(x) \neq 0$. Suppose for contradiction that $g(x) = 0$. Then there exists a sequence $(a_n) \subseteq F$ such that $|x - a_n| = |a_n - x| \to \inf\{|x-a| : a \in F\} = g(x) = 0$. This implies $(a_n) \to x$, a limit point outside F. But this contradicts assumption that F closed and so contains all its limit points. Thus our assumption that $g(x) = 0$ is false, i.e. $g(x) \neq 0$ for all $x \notin F$.

$\square$

Problem 7. (Ex. 4.3.14*)

(a) Let F be a closed set. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that the set of points where f fails to be continuous is precisely F. (The concept of the interior of a set, discussed in Exercise 3.2.14, may be useful.)
(b) Now consider an open set O. Construct a function $g : \mathbb{R} \to \mathbb{R}$ whose set of discontinuous points is precisely O. (For this problem, the function in Exercise 4.3.12 may be useful.)

Lemma 4 (Function in Exercise 4.3.12)**.**

$$g(x) = \inf\{|x-a| : a \in F\},$$

where $F \subseteq \mathbb{R}$ is a nonempty closed set, g continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Proof. (a) Consider the function f defined by

$$f(x) = \begin{cases} D(x) & \text{for } x \in [a,b] \subseteq F \\ x & \text{if } x \notin F \\ 0 & \text{if } x \in F \end{cases}$$

for some $a, b \in \mathbb{R}$, where $D(x)$ is defined as the Dirichlet's function, i.e.

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(b) We use the function in Exercise 4.3.12

Lemma 5 (Function in Exercise 4.3.12)**.**

$$g(x) = \inf\{|x-a| : a \in F\},$$

where $F \subseteq \mathbb{R}$ is a nonempty closed set, g continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

With this idea, O open implies $\mathbb{R} \setminus O = O^c$ closed; let function g be defined as

$$g(x) = \begin{cases} \inf\{|x-a| : a \in \mathbb{R} \setminus O\} & \text{if } x \notin O \\ D(x) & \text{if } x \in O \end{cases}.$$

□

Problem 8. (Ex. 4.4.5) Assume that g is defined on an open interval (a,c) and it is known to be uniformly continuous on $(a,b]$ and $[b,c)$, where $a < b < c$. Prove that g is uniformly continuous on (a,c).

Proof. Fix $\epsilon_0 = \epsilon/2 > 0$, arbitrary. Because g is uniformly continuous on $(a,b]$, there exists a $\delta_1 > 0$ such that for all $p, q \in (a,b]$,

$$|p-q| < \delta_1 \Rightarrow |g(p) - g(q)| < \epsilon_0.$$

Similarly, because f is uniformly continuous on $[b,c)$, for this same ϵ, there exists a $\delta_2 > 0$ such that for all $r, s \in [b,c)$,

$$|r-s| < \delta_2 \Rightarrow |g(r) - g(s)| < \epsilon_0.$$

We want to show there exists a $\delta > 0$ such that for all $x, y \in (a,c)$, $|x-y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. As the most general (worst case), assume $x \in (a,b]$ and $y \in [b,c)$. Then

$$|x-y| < \delta \Rightarrow |g(x) - g(y)| \leq |g(x) - g(b)| + |g(b) - g(y)| < \epsilon_0 + \epsilon_0 = 2\frac{\epsilon}{2} = \epsilon$$

which by definition means g is uniformly continuous on (a,c).

□

Problem 9. (Ex. 4.5.3) A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property (Definition 4.5.3), then f continuous on $[a,b]$.

Definition 6 (4.5.3 Intermediate Value Property)**.** *A function f has the intermediate value property on an interval $[a,b]$ if for all $x<y$ in $[a,b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x,y)$ where $f(c)=L$.*

Proof. Fix $\epsilon > 0$. Our aim is to show that there exists $\delta > 0$ such that for $c \in [a,b]$, $|x-c| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$. First pick $c \in (a,b)$ such that $a < c < b$. By the intermediate value property, we can find a point $d \in (a,c)$, satisfying $a < d < c$, where $f(d) = L_1 = f(c) - \epsilon$. Let $\delta_1 = c - d$. Given f strictly increasing, for all $x \in [d,c]$ such that $|x-c| < c-d = \delta_1$, we have

$$f(c) - \epsilon = f(d) \leq f(x) \leq f(c).$$

Similarly, by the intermediate value property of f we can pick another point $e \in (c,b)$ satisfying $c < e < b$, where $f(e) = L_2 = f(c) + \epsilon$. Let $\delta_2 = e - c$. Given f strictly increasing, for all $x \in [c,e]$ such that $|x-c| < e-c = \delta_2$, we have

$$f(c) \leq f(x) \leq f(e) = f(c) + \epsilon$$

Next we consider the case when $c = a$. For our particular ϵ, for $x > a = c$, we can find a $\delta_3 > 0$ such that whenever $|x-c| < \delta_3$ we have that $|f(x)-f(c)| < \epsilon$. Similarly for $c = b$, for $x < b = c$ we can find a $\delta_4 > 0$ such that whenever $|x-c| < \delta_4$ we have that $|f(x)-f(c)| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. For all $\epsilon > 0$, $|x-c| < \delta$ implies

$$|f(x) - f(c)| < \epsilon,$$

and we have shown f continuous at c.

□

Problem 10. (Ex. 4.5.4) Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x \in A : f(x) = f(y) \text{ for some } y \neq x \text{ and } y \in A\}.$$

Show F is either empty or uncountable.

Proof. If F empty, we are done. If F not empty, because g is continuous on interval $F \subseteq A$, then by definition of F there exist at least two points $x, y \in F$. So $g(x) = g(y)$ on this interval F. Now there are two cases: either g is constant on the interval $[x,y]$ or g not constant on this interval. If g is constant on $[x,y]$, $[x,y] \subseteq F$ and therefore F is uncountable.

If g not constant on this interval, then because g continuous on this compact set $[x,y] \subseteq \mathbb{R}$, by Extreme Value Theorem it attains a minimum and maximum value, $g(x_{\text{min}})$ and $g(x_{\text{max}})$ respectively. Because g not constant on this interval, either $x_{\text{max}} \in (x,y)$ or $x_{\text{min}} \in (x,y)$. Assume the former. Then consider the interval of range $[g(x), g(x_{\text{max}})]$. By the Intermediate Value Theorem applied to $g : [x, x_{\text{max}}] \to \mathbb{R}$ and $g : [x_{\text{max}}, y] \to \mathbb{R}$, g obtains every value in the interval $[g(x), g(x_{\text{max}})]$ at least twice, which proves g fails to be one-to-one on the interval $[x,y] = [x, x_{\text{max}}] \cup [x_{\text{max}}, y] \subseteq F$ which is uncountable. We have proven F is either empty or, if it's not empty, uncountable. □

Problem 11. (Ex. 4.5.8) (Inverse functions). If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$.

Show that if f is continuous on an interval $[a,b]$ and one-to-one, then f^{-1} is also continuous.

Proof. We begin by proving the following for our continuous one-to-one function $f : [a,b] \to \mathbb{R}$.

Theorem 7. *A function f is monotone if it is one-to-one.*

f one-to-one means that for $x, y \in [a, b]$, $f(x) = f(y) \Rightarrow x = y$. For contradiction assume f is not monotone, i.e. there exists $x < y < z$ in $[a, b]$ for which ($f(x) < f(y)$ and $f(y) > f(z)$), or ($f(x) > f(y)$ and $f(y) < f(z)$). Suppose $f(y) > f(x)$ and $f(y) > f(z)$ (the other case is proved similarly). Because f continuous, by intermediate value theorem, there exist $x_1 \in [x, y)$ and $x_2 \in (y, z]$ such that $f(x_1) = f(x_2)$. Since $x_1 < y < x_2$, $x_1 \neq x_2$ and this contradicts our assumption that f is one-to-one. By contradiction we have proven f is monotone.

We know f is monotone. Suppose, without loss of generality, that f is strictly increasing. Fix $\epsilon > 0$, arbitrary. Let $y_1 \in f(A)$. Let $x_1 = f^{-1}(y_1) \in (a, b)$. There exist $c, d \in [a, b]$ with

$$x_1 - \epsilon < c < x_1 < d < x_1 + \epsilon.$$

Then

$$f(c) < f(x_1) < f(d),$$

that is,

$$f(c) < y_1 < f(d).$$

Set $\delta = \min\{y_1 - f(c), f(d) - y_1\}$. We pick a $y \in f(A)$ such that $|y - y_1| < \delta$. For this y, we see that

$$|y - y_1| < \delta \Rightarrow y_1 - \delta < y < y_1 + \delta$$

and so

$$f(c) < y < f(d).$$

Then taking f^{-1},

$$c < f^{-1}(y) < d.$$

With this, because $x_1 = f^{-1}(y_1)$ and $c - \epsilon < x_1 < d + \epsilon$, it follows that

$$|f^{-1}(y) - f^{-1}(y_1)| < \epsilon,$$

and we have proven for $\epsilon > 0$, there exists a $\delta > 0$ such that $|y - y_1| < \delta \Rightarrow |f^{-1}(y) - f^{-1}(y_1)| < \epsilon$, which is exactly the statement that f^{-1} is continuous on (a, b).

Now we just have to prove f^{-1} continuous on endpoints a and b. Consider case when $x_1 = f^{-1}(y_1) = a$. For $\epsilon > 0$ we have that

$$x_1 < e < x_1 + \epsilon$$

for some $e > a = x_1$. Then $f(x_1) = y_1 < f(e) \Leftrightarrow x_1 = f^{-1}(y_1) < e$. Set $\delta = f(e) - y_1$. For $y \in f(A)$ such that $|y - y_1| < \delta$, $|f^{-1}(y) - f^{-1}(y_1)| < \epsilon$ because $|e - x_1| = |x_1 - e| < \epsilon$. A very similar argument follows for case when $x_1 = b$.

$\square$

Extra practice

Problem 12. (Ex. 4.2.6**) Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.
(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.
(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3|f(x) - 2|^2 = 3(L - 2)^2$.
(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f.)

Proof. (a) True. A particular $\delta > 0$ suitable for a particular $\epsilon > 0$ means that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$, where c is a limit point of domain. Let the smaller positive δ be δ' such that $\delta > \delta' > 0$. Then by definition of functional limit, this δ' is also a suitable response to ϵ since $0 < |x - c| < \delta' < \delta$ for $|f(x) - L| < \epsilon$ to still hold.

(b) False. The idea is that the limit of a limit point a in the domain of f can be different from the image $f(a)$ of a. Consider Thomae's Function as a counterexample:

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For this function note that $\lim_{x \to 1} t(x) = 0 = L \neq 1 = t(1)$.

(c) True. Define a constant function $g(x) = 2$ on the domain of f. We observe that $\lim_{x \to a} g(x) = 2$, so we have

$$\begin{aligned} \lim_{x \to c} |f(x) - 2| &= \begin{cases} f(x) - 2 = f(x) - g(x) = L - 2 & \text{if positive} \\ -(f(x) - 2) = -1 \cdot (f(x) - g(x)) = -(L - 2) & \text{if negative} \end{cases} \\ &= |L - 2| \end{aligned}$$

by Algebraic Limit Theorem for Functional Limits. Then

$$\lim_{x \to c} 3|f(x) - 2|^2 = \lim_{x \to c} 3 \cdot |f(x) - 2| \cdot |f(x) - 2| = 3|L - 2|^2.$$

(d) False. Consider the function f defined by

$$f(x) = x - a$$

on the domain $\mathbb{R} \setminus \{a\}$, and the function g defined by

$$g(x) = \frac{1}{x - a}$$

on the same domain. Then

$$\lim_{x \to a} f(x) = 0,$$

but

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} (x - a) \cdot \frac{1}{x - a} = 1 \neq 0.$$

□

Problem 13. (Ex. 4.3.2**) To gain a deeper understanding of the relationship between ϵ and δ in the definition of continuity, let's explore some modest variations of Definition 4.3.1. In all of these, let f be a function defined on all of $\mathbb{R}$.

(a) Let's say f is *onetinuous* at c if for all $\epsilon > 0$ we can choose $\delta = 1$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuous on all of $\mathbb{R}$.

(b) Let's say f is *equaltinuous* at c if for all $\epsilon > 0$ we can choose $\delta = \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on $\mathbb{R}$ that is nowhere onetinuous, or explain why there is no such function.

(c) Let's say f is *lesstinuous* at c if for all $\epsilon > 0$ we can choose $0 < \delta < \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is lesstinuous on $\mathbb{R}$ that is nowhere equaltinuous, or explain why there is no such function.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

Proof. (a) Consider the function f defined by

$$f(x) = 0.$$

Let $c \in \mathbb{R}$, arbitrary. Observe that for all $\epsilon > 0$, $|f(x) - f(c)| = 0 < \epsilon$ whenever $|x - c| < 1 = \delta$.

(b) Consider the function f defined by

$$f(x) = x.$$

Let $c \in \mathbb{R}$, arbitrary. Observe that for all $\epsilon > 0$, $|f(x) - f(c)| = |x - c| < \delta = \epsilon$ whenever $1 \leq |x - c| < \delta$, but for $|x - c| < 1$ if we pick $\epsilon = 2$ then $|f(x) - f(c)| > 2 = \epsilon$ so f nowhere onetinuous.

(c) Consider the function f defined by

$$f(x) = 2x.$$

Let $c \in \mathbb{R}$, arbitrary. Observe that for all $\epsilon > 0$, $|f(x) - f(c)| = |2x - 2c| = 2|x - c|$. We can set $\delta = \epsilon/2$, which gives us

$$|f(x) - f(c)| = 2|x - c| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

whenever $|x - c| < \delta = \epsilon/2$, where $0 < \delta < \epsilon$. If $\delta = \epsilon$ then if we pick $\epsilon = 2$, $|f(x) - f(c)| = 2|x - c| < 4 > \epsilon$ whenever $|x - c| < 2 = \delta = \epsilon$; so by contradiction f is nowhere equaltinuous.

(d) Yes, every lesstinuous function is continuous. From definition of lesstinuous function, $\delta > 0$ where $\delta < \epsilon$, and whenever $|x - c| > \delta$, $|f(x) - f(c)| < \epsilon$ which is exactly the definition of continuity.

Yes, every continuous function is lesstinuous. Definition of continuity is that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. If $\delta \geq \epsilon$, we can always choose $\delta' < \epsilon$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta' < \epsilon < \delta$.

□

Problem 14. (Ex. 4.3.4**) Assume f and g are defined on all of $\mathbb{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$.

(a) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.

(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Proof. (a) Let $f(x) = 0$ constant function, and g defined by

$$g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then

$$\lim_{x \to 0 = q} g(x) = 0 = r$$

but

$$\lim_{x \to 0} g(f(x)) = \lim_{x \to 0} g(0 = p) = \lim_{x \to 0} 1 = 1 \neq 0 = r$$

(b) Assuming f and g are continuous, the conditions of Theorem 4.3.9 Composition of Continuous Functions are fulfilled, thus by the same theorem $g \circ f$ is continuous at p and so $\lim_{x\to p} g(f(x)) = r$ follows by

$$\lim_{x\to p} g(f(x)) = g(\lim_{x\to p} f(x)) = g(f(p)) = g(q) = r.$$

Theorem 8 (4.3.9 Composition of continuous Functions)**.** *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f = g(f(x))$ is defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

(c) No, result doesn't hold if we only assume f is continuous, and that was the result shown in (a).

If we only assume g is continuous,

$$\lim_{x\to q} g(x) = g(q) = r.$$

But we know $\lim_{x\to p} f(x) = q$, so

$$\lim_{x\to p} g(f(x)) = \lim_{f(x)\to q} g(f(x)) = g(q) = r.$$

□

Problem 15. (Ex. 4.3.8**) Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.
(b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.
(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Proof. (a) True. Assume for contradiction $g(1) < 0$. Pick $\epsilon = |g(1)| > 0$. Then we observe that for all x such that $|x - 1| < \delta$, $|g(x) - g(1)| \geq |g(1)| = \epsilon$, which contradicts fundamental assumption that g is continuous. So $g(1) \geq 0$.

(b) True. By Density of $\mathbb{Q}$ in $\mathbb{R}$ we know that there exists $r \in \mathbb{Q}$ such that $r \in V_\delta(x)$ for some $\delta > 0$, for all $x \in \mathbb{R}$. For all $\epsilon > 0$, there exists such a δ-neighborhood such that whenever, $|x - r| < \delta \Leftrightarrow r \in V_\delta(x)$, $|g(x) - g(r)| < \epsilon$ implies $g(x) = g(r) = 0$.

(c) True. By continuity of g, for all $\epsilon > 0$, whenever $|x - x_0| < \delta$ we have that

$$|g(x) - g(x_0)| < \epsilon.$$

Picking $\epsilon = |g(x_0)|$, notice that for the above to hold, it must be that

$$g(x) > 0,$$

i.e. $g(x)$ strictly positive. Lastly note that the length $|x - x_0| < \delta$ is uncountable in $\mathbb{R}$ for all $x \in \mathbb{R}$.

□

Problem 16. (Ex. 4.4.2**)

(a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

(b) Is $g(x) = \sqrt{x^2+1}$ uniformly continuous on $(0,1)$?
(c) Is $h(x) = x\sin(1/x)$ uniformly continuous on $(0,1)$?

Proof. (a) No. Assume $f(x) = 1/x$ continuous. Observe that given $\epsilon > 0$

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{x-y}{xy}| < \epsilon$$

whenever $|x-y| < \delta$. This means that

$$|xy| \geq \frac{\delta}{\epsilon} \Rightarrow \delta \leq \epsilon|xy|$$

which shows that f not uniformly continuous on $(0,1)$.

Alternatively, define two sequences (x_n) and (y_n) by $x_n = 1/n$ and $y_n = 1/2n$. Then for $\epsilon_0 = 1$,

$$|x_n - y_n| = |\frac{1}{n} - \frac{1}{2n}| = |\frac{1}{2n}| \to 0,$$

but

$$|f(x_n) - f(y_n)| = |n - 2n| = |n| \geq 1 = \epsilon_0$$

so by Theorem 4.4.5 f is not uniformly continuous on $(0,1)$.

(b) Yes. Let $x, y \in (0,1)$, then

$$\begin{aligned}
|g(x) - g(y)| &= |\frac{(\sqrt{x^2+1} - \sqrt{y^2+1})(\sqrt{x^2+1} + \sqrt{y^2+1})}{\sqrt{x^2+1} + \sqrt{y^2+1}}| \\
&= |\frac{x^2+1-y^2-1}{\sqrt{x^2+1} + \sqrt{y^2+1}}| \\
&= \frac{|x^2-y^2|}{\sqrt{x^2+1} + \sqrt{y^2+1}} \\
&\leq |x-y|\frac{|x+y|}{2} \\
&< |x-y|
\end{aligned}$$

since $\sqrt{x^2+1} \geq 1$ and $\sqrt{y^2+1} \geq 1$ given $x, y \in (0,1)$, and $|x+y|/2 < 1$. Set $\delta = \epsilon$, and we have that

$$|g(x) - g(y)| < |x-y| < \delta = \epsilon$$

for all $\epsilon > 0$ whenever $|x-y| < \delta$. Therefore g is uniformly continuous on $(0,1)$.

(c) Yes. Consider a modified $h'(x)$ defined by

$$h'(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We will prove $h(x)$ uniformly continuous by first proving that $h'(x)$ is continuous on the compact set $[0,1]$ and consequently is uniformly continuous on $[0,1]$, and the open interval $(0,1)$, and use this to show $h(x)$ uniformly continuous on $(0,1)$.

Given $\epsilon > 0$, for some $c \neq 0$ we can estimate

$$\begin{aligned}
|h(x) - h(c)| &= |x\sin(\frac{1}{x}) - c\sin(\frac{1}{c})| \\
&\leq |x\sin(\frac{1}{x}) - x\sin(\frac{1}{c})| + |x\sin(\frac{1}{c}) - c\sin(\frac{1}{c})| \\
&= |x||\sin(\frac{1}{x}) - \sin(\frac{1}{c})| + |x-c||\sin(\frac{1}{c})| \\
&= |x||2\sin(\frac{1/x - 1/c}{2})cos(\frac{1/x - 1/c}{2})| + |x-c||\sin(\frac{1}{c})| \\
&= 2|x||\sin(\frac{c-x}{2cx})cos(\frac{x+c}{2cx})| + |x-c||\sin(\frac{1}{c})| \\
&= 2|x||\sin(\frac{c-x}{2cx})| + |x-c||\sin(\frac{1}{c})| \\
&= 2|x||\frac{c-x}{2cx}| + |x-c||\sin(\frac{1}{c})| \\
&= |x||\frac{c-x}{cx}| + |x-c||\sin(\frac{1}{c})| \\
&\leq |\frac{x-c}{c}| + |\frac{x-c}{c}| \\
&= 2|\frac{x-c}{c}|.
\end{aligned}$$

We can find $\delta = |c|\epsilon/2$ such that

$$|f(x) - f(c)| \leq 2|\frac{x-c}{c}| < \cancel{2}\frac{|c|\epsilon/\cancel{2}}{|c|} < \epsilon$$

whenever $|x-a| < \delta$. For $h'(0) = 0$ see Example 4.3.6. So we have shown $h'(x)$ continuous on $[0,1]$, by Theorem 4.4.7 Uniform Continuity on Compact Sets, $h'(x)$ is uniformly continuous on $[0,1]$, meaning $h'(x)$ also uniformly continuous on $(0,1)$. Given $h'(x)$ is just $h(x)$ extended to be defined on 0, it follows that naturally $h(x)$ is also uniformly continuous on $(0,1)$.

□

Problem 17. (Ex. 4.4.4**) Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a,b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a,b]$ (meaning $1/f$ has bounded range).
(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.
(c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

Proof. (a) True. f is continuous on the compact set $[a,b]$, so by Theorem 4.4.2 Extreme Value Theorem, we know f attains a maximum and minimum value. In particular, there exists $x_0, x_1 \in [a,b]$ such that

$$f(x_0) \leq f(x) \leq f(x_1)$$

for all $x \in [a,b]$. Given $f(x) > 0$ for all $a \leq x \leq b$, $f(x_0) > 0$,

$$\frac{1}{f(x_1)} \leq \frac{1}{f(x)} \leq \frac{1}{f(x_0)}$$

and we have shown $1/f$ is bounded on $[a,b]$.

(b) True. Assume for contradiction that $f(A)$ is unbounded. This means that for all $a \in A$, $|f(a)| \geq N$ for some $N \in \mathbb{N}$. In particular, for the sequence $(a_n) \subseteq A$, $|f(a_n)| \geq N$. Then because A bounded, we know by Bolzano-Weierstrass Theorem that every bounded sequence contains a convergent subsequence, i.e. for some $(a_n) \subseteq A$ there exists a convergent subsequence (a_{n_k}). Then we know $(a_{n_{k+1}})$ also converges to the same limit as (a_{n_k}) because the limit of a convergent sequence is not dependent on first finitely many terms. So with this, we have
$$|a_{n_k} - a_{n_{k+1}}| \to 0.$$
But
$$|f(a_{n_k} - f(a_{n_{k+1}})| \geq |f(a_{n_k})| - |f(a_{n_{k+1}})| \geq N_1 - N_2 \in \mathbb{R},$$
and by Theorem 4.4.5 Sequential Criterion for Absence of Uniform Continuity, f fails to be uniformly continuous on A; which contradicts the stipulation that f is uniformly continuous on A. So our assumption was false, and therefore $f(A)$ is bounded.

(c) False. Consider the counterexample given by Dirichlet's function:
$$d(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$
that is defined on $\mathbb{R}$, with $f(K) = [0,1]$ whenever K compact (closed and bounded), but f evidently not continuous on $\mathbb{R}$.

□

Problem 18. (Ex. 4.4.6**) Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;
(b) A uniformly continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;
(c) A continuous function $f : [0,\infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Proof. (a) Consider the function f given by
$$f(x) = \frac{1}{x}.$$
Let $x_n = 1/n$ for all $n \in \mathbb{N}$, then
$$f(x_n) = \frac{1}{1/n} = n$$
is an unbounded sequence and therefore not convergent, i.e. not Cauchy.

(b) Impossible request. Given f uniformly continuous, we know for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in (0,1)$,
$$|x - y| < \delta \Rightarrow |f(x) = f(y)| < \epsilon.$$
Given $(x_n) \subseteq (0,1)$ Cauchy, by definition for all $\delta > 0$ there exist $N \in \mathbb{N}$ such that for all $n, m \geq N$,
$$|x_n - x_m| < \delta \Rightarrow |f(x_n) - f(y_n)| < \epsilon.$$

(c) Impossible request. Note that the domain $[0, \infty)$ is closed (complement $(-\infty, 0)$ is open), and so it contains all its limit points. So, for (x_n) convergent (implied by Cauchy criterion) with $(x_n) \to x \in [0, \infty)$, given f continuous,

$$f(x_n) \to f(x)$$

which means $f(x_n)$ Cauchy.

□

Problem 19. (Ex. 4.4.8**) Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on [0,1] with range (0,1).
(b) A continuous function defined on (0,1) with range [0,1].
(c) A continuous function defined on (0,1] with range (0,1).

Proof. (a) Impossible request. $[0, 1]$ is compact. By Theorem 4.4.1 Preservation of Compact Sets, $f([0, 1])$ (range of f) is compact as well, and so cannot be $(0, 1)$.
(b) Consider the function f given by

$$f(x) = |\cos(2\pi x)|$$

that is continuous on domain $(0, 1)$, given $\cos(x)$ continuous on $\mathbb{R}$, with range $[0, 1]$.
(c)

□

Problem 20. (Ex. 4.5.2**) Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.
(b) A continuous function defined on a closed interval with range equal to an open interval.
(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.
(d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

Proof. (a) Consider the function $f : (0, 1) \to [0, 1]$ given by

$$f(x) = |\cos(2\pi x)|$$

for $0 < x < 1$, that is continuous on domain $(0, 1)$, given $\cos(x)$ is continuous on $\mathbb{R}$, with range $[0, 1]$.
(b) Impossible request. A closed interval is closed and bounded, and therefore, by Heine-Borel Theorem, compact. By Preservation of Compact Sets, a continuous function defined on a compact set has range that is compact as well, but open interval is not closed therefore not compact.
(c) Consider the function $f : (-\pi/2, \pi/2)$ given by

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ \tan(x) & \text{if } x \geq 0 \end{cases}$$

and observe that for the restricted open interval domain $(-\pi/2, \pi/2) \subseteq \mathbb{R}$, the range $f((-\pi/2, \pi/2)) = [0, \infty)$ is an unbounded closed set (since $[0, \infty)^c = (-\infty, 0)$ is open) different from $\mathbb{R}$.

(d) Impossible request. $\mathbb{R}$ is a connected set, by definition, so by Preservation of Connected Sets, a continuous function on $\mathbb{R}$ must have a range that is connected as well, but $\mathbb{Q}$ not connected since (letting $A = (\ \mathbb{Q} \cap (-\infty, \sqrt{2})\ )$, $B = (\ \mathbb{Q} \cap (\sqrt{2}, \infty)\ )$),

$$\overline{A} = (-\infty, \sqrt{2}],$$

and

$$\overline{B} = [\sqrt{2}, \infty)$$

but

$$\overline{A} \cap B = \emptyset = A \cap \overline{B}$$

and we have

$$\begin{cases} \mathbb{Q} = A\ \cup\ B \\ A\ \neq\ \emptyset\ \neq\ B \\ \overline{A} \cap B\ =\ \emptyset\ =\ A \cap \overline{B} \end{cases} .$$

We have proven $\mathbb{Q}$ not connected, and since g continuous on compact set $\mathbb{R}$, such a continuous function with range equal to $\mathbb{Q}$ (disconnected set) does not exist.

$\square$

Problem 21. (Ex. 4.5.7**) Let f be a continuous function on the closed interval [0,1] with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

Proof. If either $f(0) = 0$ or $f(1) = 1$, we are done.

Assume $f(0) \neq 0$ and $f(1) \neq 1$. Because range is also contained in $[0, 1]$,

$$f(0) > 0 \Rightarrow f(0) - 0 > 0$$

and

$$f(1) < 1 \Rightarrow f(1) - 1 < 0.$$

We define $g : [0, 1] \to [-1, 1]$ as $g(x) = f(x) - x$ which is continuous. Note that $-1 \leq f(x) - x \leq 1$. By the Intermediate Value Theorem, g continuous on $[0, 1]$ so

$$[g(1), g(0)] \subseteq g([0, 1]).$$

Since

$$g(1) = f(1) - 1 < 0 < f(0) - 0 < g(0),$$

we have $0 \in [g(1), g(0)] \subseteq g([0, 1])$. So there exists $x \in [0, 1]$ such that $g(x) = 0$, i.e. $f(x) - x = 0 \Rightarrow f(x) = x$.

$\square$