MATH 0450: HOMEWORK 4

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Problem 1. (Ex. 1.3.2) Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.
(b) A finite set that contains its infimum but not its supremum.
(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

Solution.

(a) Consider the singleton set $B = \{1\}$. Here, $\inf \{1\} = 1 = \sup \{1\}$. So $\inf B = \sup B$, and because $\inf B \not> \sup B$, $\inf B \geq \sup B$ is trivially true and so the statement $\inf B \geq \sup B$ is true for this set B.
(b) Impossible. A finite set always contains both its infimum and supremum, which are respectively its minimum and maximum elements.
(c) Consider the set defined by:
$$A = \{q \in \mathbb{Q} \mid 0 < q \leq \frac{1}{n}\} \Leftrightarrow q \in (0, \frac{1}{n}]$$
$\sup A = 1 \in A$, if we pick $n = 1$, but $\inf A = 0 \notin A$, by Archimedean Property, as observable in the half-closed interval above.

□

Problem 2. (Ex. 1.3.8) Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbb{N}$ with $m < n\}$.
(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
(c) $\{n/(3n + 1) : n \in \mathbb{N}\}$.
(d) $\{m/(m + n) : m, n \in \mathbb{N}\}$.

Solution.

(a) Let $A = \{m/n : m, n \in \mathbb{N}$ with $m < n\}$. $\sup A = 1$, $\inf A = 0$. Given $m < n$, we pick, without loss of generality, $m = n - 1$:
$$\frac{m}{n} = \frac{n-1}{n} = 1 - \frac{1}{n} < 1.$$
So $\sup A = 1$, since $\inf \{\frac{1}{n}\} = 0$ for all $n \in \mathbb{N}$, by Archimedean property. To find $\inf A$ we set $m = 1$ in an attempt to make the numerator as small as possible:
$$\frac{m}{n} = \frac{1}{n} > 0.$$

Date: February 14, 2020.

But as we see from above, Archimedean Property leads us to find inf $A = 0$.

(b) Let $A = \{(-1)^m/n : m, n \in \mathbb{N}\}$. sup $A = 1$, inf $A = -1$. To see this we first pick m to be any element in the subset of even natural numbers. Then

$$\frac{(-1)^m}{n} = \frac{1}{n} < 1$$

for all $n \in \mathbb{N}$, as in part (a), due to the Archimedean Property with real number $\epsilon = 1$. So sup $A = 1$. A similar argument follows by picking $m \in$ subset of odd natural numbers, which gives the result $\frac{(-1)^m}{n} = -\frac{1}{n} > -1$. The last result is just a flip of the inequality above for $\frac{1}{n}$.

(c) Let $A = \{n/(3n+1) : n \in \mathbb{N}\}$. sup $A = \frac{1}{3}$, inf $A = \frac{1}{4}$. First we rewrite $\frac{n}{3n+1}$ as follows:

$$\frac{n}{3n+1} = \frac{1}{3+\frac{1}{n}} < \frac{1}{3}.$$

An attempt to find the smallest $\frac{1}{n} = \inf\{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$ to obtain the smallest denominator for a largest overall fraction yields the above result. Likewise note that an attempt to find the largest $\frac{1}{n} = \sup\{\frac{1}{n} \mid n \in \mathbb{N}\} = 1$ yields

$$\frac{n}{3n+1} = \frac{1}{3+\frac{1}{n}} > \frac{1}{3+1} = \frac{1}{4}.$$

(d) Let $A = \{m/(m+n) : m, n \in \mathbb{N}\}$. sup $A = 1$, inf $A = 0$. Rewrite $\frac{m}{m+n}$ as follows:

$$\frac{m}{m+n} = \frac{1}{1+\frac{n}{m}}.$$

Then we notice that

$$0 < \frac{1}{1+\frac{n}{m}} < \frac{1}{1+0} = 1$$

by picking $n, m \in \mathbb{N}$ without loss of generality. To see why $\frac{1}{1+\frac{n}{m}} > 0$: consider $n > m$ such that denominator $1 + \frac{n}{m}$ is largest. Then by Archimedean Property $\frac{1}{\epsilon} > 0$ where $\epsilon \in \mathbb{R}$, $\epsilon > 0$. The derivation of sup $A = 1$ follows simply by picking $n = 1$ and finding smallest $\frac{1}{m}$.

□

Problem 3. (Ex. 1.3.9)

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Proof.

(a) Given sup $A <$ sup B. Let ($s =$ sup A and $t =$ sup B) $\Leftrightarrow s < t \Leftrightarrow t - s > 0$.

[Lemma]: Let B be a set bounded above by a $t \in \mathbb{R}$. $t =$ sup B if and only if for any real number $\epsilon > 0$, there exists a $b_1 \in B$ such that $b_1 > t - \epsilon$.

By definition of supremum, we know for all $a \in A$, $a \leq s$, and for all $b \in B$, $b \leq t$. Using above Lemma, pick $\epsilon = t - s > 0$. Then we have $b_1 > t - (t - s) = s$, and so $b_1 >$ sup $A \Rightarrow b_1 > a \Rightarrow b_1 \geq a$ for all $a \in A$.

(b) Consider

$$A = \{r \in \mathbb{R} \mid r^2 \leq 2\},$$
$$B = \{q \in \mathbb{Q} \mid q^2 < 2\}.$$

Note sup A = sup $B = 2$, but $\max(A) = 2$ whereas $\max(B) = \emptyset$. So interestingly this might or might not work depending on which set we pick. If we pick set A, then we find that it is true that there exists an $a \in A =$ sup $A = \max(A)$ that is an upper bound for B, since sup $B = \max(A) \notin B$. However if we pick set B, then we find that it is not true because $\max(A)$ is an upper bound of B.

□

Problem 4. (Ex. 1.3.11) Decide if the following statements about suprema and infimum are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then sup $A \leq$ sup B.
(b) If sup $A <$ inf B for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then sup $A <$ inf B.

Proof.

(a) True. A and B are nonempty and bounded, so by Axiom of Completeness there exists a sup A and sup B. If $A \subseteq B$, then by definition of subset every element $a \in A$ is also an element of B. That is, for all $a \in A$, $a \in B$. Then consider two cases:

$$(1) : \text{sup } A \in A, \text{and}$$
$$(2) : \text{sup } A \notin A.$$

If (1), then sup $A \in B$ by subset relation, and so sup $B \geq b$ for all $b \in B$ also satisfies sup $B \geq$ sup A. If (2), remember that $A \subseteq B$ and so sup $A \not> $ sup B because that would mean sup $A \notin B$, and so we will come to a contradiction with the original assumption $A \subseteq B$. That leaves us with sup $A \leq$ sup B, and the proof is complete.

(b) True. Let $s =$ sup A and $t =$ inf B. By definitions of supremum and infimum, $a \leq s$ for all $a \in A$, and $t \leq b$ for all $b \in B$. If sup $A <$ inf B for sets A and B, then $s < t$. Overall we have $a \leq s < t \leq b$, for all $a \in A$ and $b \in B$.

[Variation of Lemma 1.3.8]: Let B be a set bounded below by a $t \in \mathbb{R}$. $t =$ inf B if and only if for any real number $\epsilon > 0$, there exists a $b_1 \in B$ such that $b_1 < t + \epsilon \Leftrightarrow b_1 - \epsilon < t$.

[Lemma 1.3.8]: Let A be a set bounded above by a $s \in \mathbb{R}$. $s =$ sup B if and only if for any real number $\epsilon > 0$, there exists a $a_1 \in A$ such that $a_1 > s - \epsilon \Leftrightarrow a_1 + \epsilon > s$.

We need to show there exists a $c \in \mathbb{R}$ such that $s < c < t \Leftrightarrow a < c < b$. Pick $\epsilon = \frac{b_1 - a_1}{2}$. Then we have

$$s < a_1 + \epsilon = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}$$

and

$$b_1 - \epsilon = b_1 - \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2} < t.$$

Let $c \in \mathbb{R}$ be $\frac{a_1+b_1}{2}$. Then we have shown $s < c < t$, and therefore $a < c < b$.

(c) False. Consider the sets
$$A = \{q \in \mathbb{Q} \mid q^2 < 2\}$$
and
$$B = \{q \in \mathbb{Q} \mid q^2 > 2\}$$
There is a $c \in \mathbb{R}$ satisfying $c^2 = 2$ that satisfies the inequality relation $a < c < b$ for all $a \in A$ and $b \in B$, by definitions of A and B, but $\sup A = \inf B = c \Rightarrow \sup A \not< \inf B$.

□

Problem 5. (Ex. 1.4.8) Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.
(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)
(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Proof. (a) Let
$$A = \mathbb{Q} \cap (0, 1)$$
and
$$B = (\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1).$$
$\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, so $A \cap B = \emptyset$. $\sup A = \sup B = 1$, and $1 \notin A$, $1 \notin B$.

(b) Consider
$$J_n = \{x \in \mathbb{R} \mid -\frac{1}{n} < x < \frac{1}{n} \text{ for } n \in \mathbb{N}\} = (-\frac{1}{n}, \frac{1}{n}).$$
sequence of J_n nested open intervals, since every $J_{n+1} = (-\frac{1}{n+1}, \frac{1}{n+1}) \subseteq (-\frac{1}{n}, \frac{1}{n}) = J_n$. $\bigcap_{n=1}^{\infty} J_n = 0$, finite and nonempty, because there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all real numbers $\epsilon > 0$, by Archimedean Property, so $0 \in J_n$ for all $n \in \mathbb{N}$.

(c) Consider the sequence of nested unbounded closed intervals given by (as prompted in question):
$$L_n = \{x \in \mathbb{R} \mid n \geq x \text{ for } n \in \mathbb{N}\} = [n, \infty).$$
Note [Unboundedness of $\mathbb{N}$]: For all real numbers $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $n > \epsilon$.

Suppose for contradiction that there exists an element $m \in \bigcap_{n=1}^{\infty} L_n$. Then the element m belongs to L_n for all $n \in \mathbb{N}$. So $m \geq n$ for all $n \in \mathbb{N}$, in other words meaning m is an upper bound of $\mathbb{N}$. But this contradicts the unboundedness theorem of $\mathbb{N}$.

(d) Impossible. Suppose for contradiction that that there is such a sequence of closed bounded intervals I_n. That means that there exists two sets in this sequence I_j and I_k such that $I_j \cap I_k = \emptyset$, for some $j, k \in \mathbb{N}$. Now assume, without loss of generality, that $j < k$. Then $\bigcap_{n=1}^{k} I_n = \emptyset$, contradicting property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$.

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