Math 0450: Homework 1 Revised Solutions

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[Ex. 1.2.8] Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

Solution.

(a) For each $n \in \mathbf{N}$,

$$f(n) = 2n$$

Each n in the domain $\mathbf{N}$ maps to a unique value $2n$, so the function is *1-1*. But the value $f(n) = 2n = 1$ has no pre-image in $\mathbf{N}$ and therefore the function is not *onto*.

(b) For each $n \in \mathbf{N}$,

$$f(n) = \lfloor \frac{n}{2} \rfloor$$

The floor function $f(n) = \lfloor \frac{n}{2} \rfloor$ is defined as the smallest integer greater than or equal to $\frac{n}{2}$. Every element $f(x)$ is an image under f of some element $n \in \mathbf{N}$, more specifically the elements $2f(x)$ and $2f(x)+1$, so the function is *onto*. But because two distinct elements of the domain of f can map to the same $f(x)$ value, for example $f(2) = \lfloor \frac{2}{2} \rfloor = 1 = f(3) = \lfloor \frac{3}{2} \rfloor$, the function $f(n)$ is not *1-1*.

(c) $\mathbf{N}$ does not contain $\{0\}$. For $n \in \mathbf{N}$,

$$f(n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd and } n \geq 3 \\ 0 & \text{if } n = 1 \\ -\lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is even} \end{cases}$$

Both $\mathbf{N}$ and $\mathbf{Z}$ are infinite sets. $f(n)$ maps each odd natural number greater than 1 to the positive floor function $\lfloor \frac{n}{2} \rfloor$ so that each positive integer has a pre-image. 0 is defined to be mapped from the natural number 1, and each even number in $\mathbf{N}$ maps to the negative floor function $-\lfloor \frac{n}{2} \rfloor$, such that the overall function is *onto*. Every element in the domain $\mathbf{N}$ maps to a unique element in the co-domain $\mathbf{Z}$, so the function is *1-1*.