MATH 0450: HOMEWORK 5

TEOH ZHIXIANG

Problem 1. Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by $f(a, b) = (a + b)(a + b + 1)/2 + b$ is bijective.

Proof. To prove bijection we will attempt to find an inverse function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, well-defined for all $n \in \mathbb{N}$. This means that

$$\begin{aligned} n &= \frac{(a+b)(a+b+1)}{2} + b \\ \Leftrightarrow 2n - 2b &= (a+b)(a+b+1) \end{aligned}$$

for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. In other words there exists an $m \in \mathbb{N}$ such that $2n - 2b = m(m + 1)$, and we need to find this $b \in \mathbb{R}$ in terms of this m. Define

$$\begin{aligned} a_m &= m(m+1) \\ &= m^2 + m \end{aligned}$$

for $m \in \mathbb{N}$. Then $a_{m-1} = (m - 1)m = m^2 - m < m^2 + m = a_m$, since $m \geq 1$. $a_m - a_{m-1} = (m^2 + m) - (m^2 - m) = 2m > 0$. So $\{a_m\}$ is a strictly increasing sequence, and $a_m > a_{m-1}$ for all $m \in \mathbb{N}$. Each a_m represents an even natural number, so $\{a_m\}$ represents a strictly increasing sequence of even natural numbers. Given any $k \in \mathbb{N}$, $\exists m \in \mathbb{N}$ such that $a_{m-1} \leq k < a_m$. In other words, all natural numbers k are either even numbers or odd numbers sandwiched between two even numbers, and this is true. Now let $n \in \mathbb{N}$, then

$$a_{m-1} \leq 2n < a_m$$

for some $m \in \mathbb{N}$. Then let

$$b = \frac{2n - a_{m-1}}{2} = \frac{2n - (m-1)m}{2} < \frac{a_m - a_{m-1}}{2} = m$$

knowing $a_{m-1} = m^2 - m \geq 0$. Let

$$a = (m - 1) - b.$$

With this, $f(a, b) = n$ as follows:

$$\begin{aligned} f(a,b) &= \frac{(a+b)(a+b+1)}{2} + b \\ &= \frac{(m-1-b+b)(m-1-b+b+1)}{2} + \frac{2n-(m-1)m}{2} \\ &= \frac{\cancel{(m-1)m}}{2} + \frac{2n\cancel{-(m-1)m}}{2} \\ &= n, \end{aligned}$$

and we know f is onto.

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Now, to check f injective, let $f(a', b') = n$, where $a' \neq a$ and $b' \neq b$. Then

$$\begin{aligned} \frac{(a'+b')(a'+b'+1)}{2} + b' &= n \\ \Rightarrow (a'+b')(a'+b'+1) &= 2n - 2b' \end{aligned}$$

Let $a' + b' = l$. Then

$$\begin{aligned} l(l+1) &= a_l \\ &= 2n - 2b' \end{aligned}$$

Note that $2b$ is the smallest natural number we have to subtract from $2n$ to get an element in $\{a_m\}$, for all $m \in \mathbb{N}$. So

$$2b' \geq 2b \Rightarrow b' \geq b.$$

Observe that

$$\begin{aligned} a_{m-1} - a_{m-2} &= (m-1)m - (m-1)(m-2) \\ &= (m-m+2)(m-1) \\ &= 2(m-1), \\ a_{m-2} - a_{m-3} &= 2(m-2) \\ &\vdots \\ a_{l+1} - a_l &= 2(l+1). \end{aligned}$$

where each difference is a difference of natural numbers and thus is positive. Since $2b'$ is the natural number we subtract from $2n$ to give us a_l,

$$\begin{aligned} 2b' &= 2(m-1) + 2(m-2) + \cdots + 2(l+1) + 2b \\ &\geq 2(m-1) + 2b \\ b' &\geq m - 1 + b \\ \Rightarrow -b' &\leq -(m-1) - b \end{aligned}$$

We know $a_l \leq a_m \leq a_{m-1} \Rightarrow l \leq m-1$. From this, the previous result, and $a' + b' = l$ as above, we have

$$\begin{aligned} a' &= l - b' \\ &\leq l - (m-1) - b \\ &\leq \cancel{m-1} \cancel{-(m-1)} - b = -b \\ \Rightarrow a' &\leq -b. \end{aligned}$$

If $b' = b$, then $a' = a$. So we may assume $b' > b$. Then $a_l < a_{m-1} \Rightarrow l < m-1$, but this contradicts the earlier statement above that $l \leq m-1$. And so we have shown that there is no (a', b') that maps onto the same n as (a, b). So f injective. Therefore f is bijective. $\square$