# Math 0450: Homework 1

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[Ex. 1.2.2] Show that there is no rational number $r$ satisfying $2^r = 3$.
*Proof.* The above statement can be rewritten as

$$\forall r \in \mathbf{Q} : \qquad \neg(2^r = 3) \tag{1}$$

This is equivalent to

$$\neg\exists r \in \mathbf{Q} : \qquad 2^r = 3 \tag{2}$$

Suppose there exists an $r \in \mathbf{Q}$ that satisfies the above statement (2), i.e. $2^r = 3$. By the definition of $\mathbf{Q}$, $r = \frac{p}{q}$, where $p, q \in \mathbf{Z}$ and $q \neq 0$.
Plugging $r = \frac{p}{q}$ into (2) gives you

$$2^{\frac{p}{q}} = 3 \tag{3}$$

We can rewrite this as:

$$\log_2 2^{\frac{p}{q}} = \log_2 3 \iff \frac{p}{q} = \log_2 3 \tag{4}$$

$$\iff \frac{p}{q} = \frac{\log_x 3}{\log_x 2} \tag{5}$$

where $x \in \mathbf{R}$. Since 3 is not a power of 2, $\log_2 3$ is not rational. So we rewrite equation (4) as (5), and now need to find an $x$ that is both a root of 3 and 2. That is:

$$(p = \log_x 3) \wedge (q = \log_x 2) \tag{6}$$

So $x$ needs to be an integer that is both a root of 3 and a root of 2. But because no such $x$ exists, statement (5) is false, and we have reached a contradiction with our initial supposition that the statement is true. Therefore there is no rational number $r$ satisfying $2^r = 3$.

[Ex. 1.2.3 (a-b)] Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$ are all sets containing an infinite number of elements, then the intersection $\cap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\cap_{n=1}^{\infty} A_n$ is finite and nonempty.

*Solution.*

(a) False statement.
*Proof*. Consider the following collection of sets with infinite elements that satisfies the nested relation in (a):

$$
\begin{aligned}
A_1 &= \mathbf{N} = \{1, 2, 3, \ldots\} \\
A_2 &= \{2, 3, 4, \ldots\} \\
A_3 &= \{3, 4, 5, \ldots\}
\end{aligned}
$$

In general, for each $n \in \mathbf{N}$, let

$$A_n = \{n, n+1, n+2, \ldots\}$$

Suppose there exists some natural number $m$ that satisfies $m \in \cap_{n=1}^{\infty} A_n$. This implies that $m \in A_n$ for every $A_n$ in this collection of sets. But consider the set $A_{m+1} = \{m+1, m+2, m+3, \ldots\}$. Clearly, $m$ does not exist in $A_m$, and thus the infinite intersection $\cap_{n=1}^{\infty} A_n$ that includes $A_{m+1}$ would be empty and not infinite. Therefore, by contradiction, the statement in (a) is false.

(b) True statement.
*Proof*. (**Nested Interval Property**). Consider the collection of finite, nonempty sets of real numbers as being a collection of closed intervals $I_n = [a_n, b_n]$. This means that there exists $x \in \mathbf{R}$ such that $a_n \leq x \leq b_n$. To show that the intersection $\cap_{n=1}^{\infty} A_n$ is finite and nonempty, we need to find a single $x \in I_n$ for every $n \in \mathbf{N}$. Hence, consider the set

$$A = \{a_n : n \in \mathbf{N}\}$$

that is made up of the left bounds of each $I_n$ interval. Since each $I_{n+1} = [a_{n+1}, b_{n+1}]$ interval is nested within every $I_n = [a_n, b_n]$ interval, $b_n$ serves as an upper bound of set $A$. Because the interval $I_n$ gets smaller and smaller, we set $x \in [a_n, b_n]$ as the least upper bound of the set $A$. So $a_n \leq x$, and because each $b_n$ is an upper bound of $A$ but not the least upper bound, $x \leq b_n$. Therefore it has to be the case that $x$ is found in any smallest interval $I_n$, and hence it follows that $x \in \cap_{n=1}^{\infty} A_n$, and the intersection is not empty.

[Ex. 1.2.4] Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every $A_i$ has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\cup_{i=1}^{\infty} A_i = \mathbf{N}$.

*Solution*. For each $n \in \mathbf{N}$, let

$$P = \{2, 3, 5, 7, \ldots, p_n\}$$

denote the infinite set of prime numbers. Then consider the following collection of sets:

$$A_1 = \{2, 4, 6, 8, \ldots\} \cup \{1\}$$
$$A_2 = \{3, 9, 15, 21, \ldots\}$$
$$A_3 = \{5, 25, 35, \ldots\}$$

In general, for each $n \in \mathbf{N}$, we define

$$A_n = \{n \in \mathbf{N} \mid \text{smallest prime factor of } n \text{ is } p_n\}$$

Therefore, each $A_n$ is disjoint with every other set, i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$. By definition of $A_n$, and special definition of $A_1$, condition $\cup_{i=1}^{\infty} A_i = \mathbf{N}$ is satisfied.

[Ex. 1.2.8] Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

*Solution.* First define *1-1* and *onto.* A function $f : A \to B$ is *1-1* if $a_1 \neq a_2$ in $A$ implies that $f(a_1) \neq f(a_2)$ in $B$. A function $f : A \to B$ is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

(a) For each $n \in \mathbf{N}$,
$$f(n) = 2n$$

(b) For each $n \in \mathbf{N}$,
$$f(n) = \lfloor \frac{n}{2} \rfloor$$

(c) For each $n \in \mathbf{N}$ and each $z \in \mathbf{Z}$,

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ z & \text{if } n \text{ is odd} \\ -z & \text{if } n \text{ is even} \end{cases}$$