MATH 0450: HOMEWORK 6

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Problem 1. (Ex. 2.2.8*) For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero-heavy?
(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.
(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample?
(d) From the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if ...

Proof.

(a) Take $M = 2$. Because in between any two 1s is a 0, for all $N \in \mathbb{N}$ there is an n satisfying $N \leq n \leq N + 2$ where $x_n = 0$.
(b) Yes. Assume for contradiction a zero-heavy sequence (x_n) contains a finite number of zeros. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \neq 0$. But this means that $x_n \neq 0$ for any n satisfying $N \leq n \leq N + M$, which contradicts our initial assumption that the sequence is zero-heavy.
(c) No. Consider the sequence given by

$$x_n = \begin{cases} 0 & \text{if } \sqrt{n} \in \mathbb{N} \\ n \in \mathbb{N} & \text{otherwise} \end{cases}$$

Because $\mathbb{N}$ infinite, and there is an infinite number of perfect squares of natural numbers, (x_n) contains an infinite number of zeros. Consider $M = 3$. Note that if $N = 1$, (x_n) is conveniently zero-heavy, but if $N = 5$, then there is no n in $5 \leq n \leq 5 + 3 = 8$ such that $x_n = 0$. So (x_n) not necessarily zero-heavy.
(d) A sequence is *not* zero-heavy if for all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

□

Problem 2. (Ex. 2.3.5) Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Proof. We need to prove both directions.

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($\Rightarrow$) Assume (z_n) converges to a limit L. By definition, $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$, $|z_n - L| < \epsilon$. Define

$$x_n = z_{2n-1},$$

and

$$y_n = z_{2n}.$$

Then $n \geq N \Rightarrow 2n - 1 \geq n \geq N$ and $2n > n \geq N$, for all $n \in \mathbb{N}$. But this means

$$|z_{2n-1} - L| = |x_n - L| < \epsilon,$$

$\lim x_n = L$, and

$$|z_{2n} - L| = |y_n - L| < \epsilon.$$

$\lim y_n = L = \lim x_n$.

($\Leftarrow$) Assume $(x_n), (y_n)$ convergent with $\lim x_n = \lim y_n = L$. Then by definition $\forall \epsilon > 0, \exists N_1 \in \mathbb{N} : \forall n \geq N_1$,

$$|x_n - L| < \epsilon,$$

and $\exists N_2 \in \mathbb{N} : \forall n \geq N_2$,

$$|y_n - L| < \epsilon.$$

But $\exists N = \max\{2N_1 - 1, 2N_2\}$ such that $\forall n \geq N$,

$$|z_n - L| < \epsilon.$$

Therefore (z_n) converges to L, meaning it is convergent. □

Problem 3. (Ex. 2.3.7) Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;
(b) sequences (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;
(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;
(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;
(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Proof.

(a) Consider $x_n = n$, $y_n = -n$, for all $n \in \mathbb{N}$. Using the theorem that all convergent sequences are bounded, because x_n and y_n are respectively $\mathbb{N}$ and $-\mathbb{N}$, they are unbounded, and therefore not convergent $\Rightarrow$ divergent. But note that their sum $x_n + y_n = n + (-n) = 0$, for all $n \in \mathbb{N}$. So $(x_n + y_n) \to 0$.
(b) Request impossible. If (x_n) and $(x_n + y_n)$ converge, then $(y_n) = ((x_n + y_n) - x_n)$ must converge as well, by the Algebraic Limit Theorem. This is because by Algebraic Limit Theorem (i), take $c = -1$, and $\lim -x_n = \lim -1 \cdot x_n = -\lim x_n$. In other words $(-x_n)$ converges. Then by (ii), $\lim((x_n + y_n) - x_n) = \lim y_n = \lim(x_n + y_n) + \lim(-x_n)$, and so (y_n) must converge as well.
(c) Consider the known convergent sequence $b_n = \frac{1}{n} \to 0$, where $b_n \neq 0$ for all $n \in \mathbb{N}$. Then $\frac{1}{b_n} = \frac{1}{1/n} = n$ diverges, as discussed in part (a).
(d) Request impossible. (b_n) convergent $\Rightarrow$ (b_n) bounded. Because $(a_n - b_n)$ bounded, it follows that $(a_n) = (a_n - b_n + b_n)$ bounded as well. Consider the triangle inequality

$$|a_n| = |(a_n - b_n) + b_n| \leq |a_n - b_n| + |b_n|.$$

$|a_n - b_n|$ and $|b_n|$ are each less than or equal to some real number, and it has been shown above that $|a_n|$ is bounded by the summation of these individual bounds.

(e) Consider $a_n = 0$, for all $n \in \mathbb{N}$, and (b_n) a divergent sequence given by $b_n = n$, for all $n \in \mathbb{N}$. Here, $(a_n b_n) \to 0 \cdot \lim b_n = 0$ and $(a_n) \to 0$, but (b_n) is divergent and does not converge, as exhibited in part (a).

□

Problem 4. (Ex. 2.3.9 (a-b))

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

Proof.

(a) (a_n) bounded $\Rightarrow |a_n| \leq M$ for some positive $M \in \mathbb{R}$.

$$|a_n b_n| = |a_n||b_n| = M|b_n|.$$

Since $\lim b_n = 0$, $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$,

$$|b_n - 0| < \epsilon,$$

Fix $\epsilon_0 = \frac{\epsilon}{M} > 0$. Then

$$|a_n b_n| = M|b_n| < M\epsilon_0 = \cancel{M}\frac{\epsilon}{\cancel{M}} = \epsilon.$$

And we have shown $\lim(a_n b_n) = 0$, by definition of limit.

We are not allowed to use Algebraic Limit Theorem because (a_n) given is bounded but not necessarily convergent, but Algebraic Limit Theorem only applies if both (a_n) and (b_n) are convergent.

(b) No. Consider (a_n) a bounded but divergent sequence, defined by

$$a_n = (-1)^n,$$

for all $n \in \mathbb{N}$. If $(b_n) \to b \neq 0$, then $(a_n b_n) = ((-b)^n)$ divergent. But now consider (a_n) bounded, convergent, defined by

$$a_n = \frac{1}{n}.$$

Then $(a_n b_n) = (\frac{b}{n})$ convergent. Therefore depending on (a_n), the resultant $(a_n b_n)$ can be either convergent or divergent, therefore we cannot conclude anything if we assume (b_n) converges to a nonzero limit b.

□

Problem 5. (Ex. 2.3.10) Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$, for all $n \in \mathbb{N}$, then $(b_n) \to b$.

Proof.

(a) False. Consider $a_n = b_n = n$, for all $n \in \mathbb{N}$. Then $\lim(a_n - b_n) = \lim(n - n) = \lim(0) = 0$, but the sequences $a_n = b_n = n$, for all $n \in \mathbb{N}$, are divergent, and hence don't converge to limits.

(b) True. By definition, $(b_n) \to b \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$,

$$|b_n - b| < \epsilon.$$

Separately, we have

$$\begin{aligned} |b_n| - |b| &\leq |b_n - b| \\ |b| - |b_n| &\leq |b - b_n| \end{aligned}$$

which gives us

$$\Rightarrow ||b_n| - |b|| \leq |b_n - b| < \epsilon$$

which proves that $|b_n| \to |b|$ follows from $b_n \to b$, which is exactly what we want.

(c) True. We have $(a_n) \to a$, which means that $\forall \epsilon_1 > 0, \exists N_1 \in \mathbb{N} : \forall n \geq N_1$,

$$|a_n - a| < \epsilon_1.$$

We also have $(b_n - a_n) \to 0$, which means that $\forall \epsilon_2 > 0, \exists N_2 \in \mathbb{N} : \forall n \geq N_2$,

$$|b_n - a_n - 0| < \epsilon_2.$$

Fix $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$, arbitrary. Then, applying triangle inequality, and choosing $N = \max\{N_1, N_2\}$, if $n \geq N$,

$$\begin{aligned} |b_n - a| &\leq |b_n - a_n| + |a_n - a| \\ &< \epsilon_1 + \epsilon_2 \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

(it was right about this moment I realized I could've just cited Algebraic Limit Theorem (ii), lol) So, we have shown $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$, $|b_n - a| < \epsilon$. Therefore $(b_n) \to a$.

(d) True. We have $(a_n) \to 0$, which means that $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$,

$$|a_n - 0| = |a_n| < \epsilon.$$

We also have that $(b_n - b)$ is bounded, as in $|b_n - b| \leq a_n$. So we know

$$\begin{aligned} 0 \leq |b_n - b| &\leq a_n \\ \Rightarrow 0 = \lim 0 \leq \lim |b_n - b| &\leq \lim a_n = 0. \end{aligned}$$

By Squeeze Theorem, we have $|b_n - b| \to 0$. This means that $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$,

$$||b_n - b| - 0| = |b_n - b| < \epsilon.$$

By definition, $(b_n) \to b$.

□

Problem 6. (Ex. 2.3.11*) (Cesaro Means).

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages
$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$
also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Proof.

(a) Let $(x_n) \to x$. This means $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N$,
$$|x_n - x| < \epsilon.$$
Now consider
$$\begin{aligned} y_n - x &= \frac{x_1 + x_2 + \cdots + x_n}{n} - x \\ &= \frac{x_1 + x_2 + \cdots + x_n - nx}{n} \\ \Rightarrow |y_n - x| &= |\frac{(x_1 - x) + (x_2 - x) + \cdots + (x_n - x)}{n}| \\ &\leq \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_{N-1} - x| + \cdots + |x_N - x| + \cdots + |x_n - x|}{n} \\ &< \frac{M(N-1) + \epsilon(n - N + 1)}{n} \end{aligned}$$
since there are $(n - N + 1)$ large terms of x_n above where $n \geq N$, and each $|x_n| - x < \epsilon$, So $|x_N - x| + \cdots + |x_n - x| < \epsilon(n - N + 1)$. Also (x_n) convergent implies (x_n) bounded. This means for all smaller terms of x_n where $n < N$, from 1 to $N - 1$, $|x_n| < M$, where $M \in \mathbb{R}$, positive.

We pick $\epsilon = M > 0$. We have
$$\begin{aligned} |y_n - x| &< \frac{M(N-1) + \epsilon(n - N + 1)}{n} \\ &= \frac{\epsilon(N-1) + \epsilon(n - N + 1)}{n} \\ &= \frac{\cancel{n}\epsilon}{\cancel{n}} \\ &= \epsilon, \end{aligned}$$
and by limit definition, $\lim y_n = x = \lim x_n$.

(b) Consider $x_n = (-1)^n$ for all $n \in \mathbb{N}$, and correspondingly
$$0 = -\frac{1}{n} \leq y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{(-1) + 1 + \cdots + (-1)^n}{n} \leq 0$$
$\lim(-\frac{1}{n}) = 0$, $\lim \frac{(-1)+1+\cdots+(-1)^n}{n} = 0$, therefore $\lim y_n = 0$. (y_n) converges, but (x_n) diverges.

$\square$

Problem 7. (Ex. 2.4.3)

(a) Show that
$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$
converges and find the limit.

(b) Does the sequence
$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$
converge? If so, find the limit.

Proof.

(a) Define the above sequence as follows:
$$\begin{aligned} a_1 &= \sqrt{2}, \\ a_2 &= \sqrt{2+\sqrt{2}} \\ &= \sqrt{2+a_1}, \\ &\vdots \\ a_{n+1} &= \sqrt{2+a_n}. \end{aligned}$$
for all $n \in \mathbb{N}$. Consider the following representation:
$$\begin{aligned} x &= \sqrt{2+\sqrt{2+\sqrt{2+}}} \\ \Rightarrow x^2 &= 2+x \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2 \text{ or } \cancel{x=-1} \end{aligned}$$
where $x \neq -1$ since $x > 0$. We will prove this bound by induction. For initial check $a_1 = \sqrt{2} \leq 2$, straightforward. Assume $a_n \leq 2$ for some $n \in \mathbb{N}$. Then
$$\begin{aligned} a_{n+1} &= \sqrt{2+a_n} \\ &\leq \sqrt{2+2} = 2. \end{aligned}$$
So $a_n \leq 2$ for all $n \in \mathbb{N}$, and (a_n) bounded above by 2. Note a_n is an increasing sequence, since $a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2-a_n)(1+a_n) > 0$, for all $n \in \mathbb{N}$. By Monotone Convergence Theorem, (a_n) increasing and bouunded $\Rightarrow$ (a_n) converges.

To evaluate $\lim a_n$, we use Lemma Thm 2.5.2 [Subsequences converge to same limit as original sequence], and set $\lim a_{n+1} = \lim a_n = L$. Then by Algebraic Limit Theorem we know
$$(\lim a_{n+1})^2 = L^2 = \lim a_{n+1}^2 = \lim(2+a_n) = 2 + \lim a_n = 2 + L.$$
Therefore
$$L^2 - L - 2 = 0.$$
And we obtain a similar result as above when used to obtain upper bound for a_n. We get $\lim a_n = L = 2$.

(b) Yes, sequence converges. Consider

$$\begin{aligned} a_1 &= \sqrt{2}, \\ a_2 &= \sqrt{2\sqrt{2}} \\ &= \sqrt{2a_1}, \\ &\vdots \\ a_{n+1} &= \sqrt{2a_n}. \end{aligned}$$

Inductively we can prove (a_n) increasing: initial check $a_2 = \sqrt{2} \cdot \sqrt{\sqrt{2}} = a_1\sqrt{\sqrt{2}} > a_1$. Assume $a_n > a_{n-1}$ for some $n \in \mathbb{N}$. Then $\sqrt{2a_{n-1}} > a_{n-1}$. We have

$$a_{n+1} = \sqrt{2a_n} = \sqrt{2} \cdot \sqrt{2a_{n-1}} > \sqrt{2} \cdot a_{n-1} > \sqrt{2a_{n-1}} = a_n.$$

We can also prove (a_n) bounded, inductively. By observation we guess that (a_n) bounded above by 2. We check this by induction: initial check $a_1 = \sqrt{2} < 2$, valid. Assume $a_n < 2$ for some $n \in \mathbb{N}$, and we know $a_n > 0$ for all n. Then

$$a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2.$$

So $a_n < 2$ for all $n \in \mathbb{N}$. By Monotone Convergence Theorem, (a_n) converges.

We want to find $\lim a_n$. By Lemma [Subsequences preserve limits], similar to in part (a), we set $\lim a_{n+1} = \lim a_n = L$. Then we have

$$(\lim a_{n+1})^2 = L^2 = \lim a_{n+1}^2 = \lim(\sqrt{2a_n})^2 = \lim 2a_n = 2 \cdot \lim a_n = 2L.$$

Now

$$\begin{aligned} L^{\cancel{2}} &= 2\cancel{L} \\ L &= 2 \end{aligned}$$

since $a_n > 0 \Rightarrow L > 0$. We have shown $\lim a_n = L = 2$.

$\square$