MATH 0450: HOMEWORK 5

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Problem 1. Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by $f(a, b) = (a + b)(a + b + 1)/2 + b$ is bijective.

Proof. To prove bijection we will attempt to find an inverse function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, well-defined for all $n \in \mathbb{N}$. This means that

$$\begin{aligned} n &= \frac{(a+b)(a+b+1)}{2} + b \\ \Leftrightarrow 2n - 2b &= (a+b)(a+b+1) \end{aligned}$$

for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. In other words there exists an $m \in \mathbb{N}$ such that $2n - 2b = m(m + 1)$, and we need to find this $b \in \mathbb{R}$ in terms of this m. Define

$$\begin{aligned} a_m &= m(m+1) \\ &= m^2 + m \end{aligned}$$

for $m \in \mathbb{N}$. Then $a_{m-1} = (m - 1)m = m^2 - m < m^2 + m = a_m$, since $m \geq 1$. $a_m - a_{m-1} = (m^2 + m) - (m^2 - m) = 2m > 0$. So $\{a_m\}$ is a strictly increasing sequence, and $a_m > a_{m-1}$ for all $m \in \mathbb{N}$. Each a_m represents an even natural number, so $\{a_m\}$ represents a strictly increasing sequence of even natural numbers. Given any $k \in \mathbb{N}$, $\exists m \in \mathbb{N}$ such that $a_{m-1} \leq k < a_m$. In other words, all natural numbers k are either even numbers or odd numbers sandwiched between two even numbers, and this is true. Now let $n \in \mathbb{N}$, then

$$a_{m-1} \leq 2n < a_m$$

for some $m \in \mathbb{N}$. Then let

$$b = \frac{2n - a_{m-1}}{2} = \frac{2n - (m-1)m}{2} < \frac{a_m - a_{m-1}}{2} = m$$

knowing $a_{m-1} = m^2 - m \geq 0$. Let

$$a = (m - 1) - b.$$

With this, $f(a, b) = n$ as follows:

$$\begin{aligned} f(a,b) &= \frac{(a+b)(a+b+1)}{2} + b \\ &= \frac{(m-1-b+b)(m-1-b+b+1)}{2} + \frac{2n-(m-1)m}{2} \\ &= \frac{\cancel{(m-1)m}}{2} + \frac{2n \cancel{-(m-1)m}}{2} \\ &= n. \end{aligned}$$

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We define an inverse function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ as $g(n) = (a, b)$, where a and b are defined from n as above. We see that $f(g(n)) = f((a, b)) = n$, and $g(f(a, b)) = g(n) = (a, b)$. Therefore f and g are mutually inverse, and f is bijective. □

Problem 2. (Ex. 1.5.1) Finish the proof for Theorem 1.5.7: If $A \subseteq B$ and B is countable, then A is either countable or finite.

Proof. If A is finite, we are done. B is countable. Thus there exists a bijective function $f : \mathbb{N} \to B$ which is 1–1 and onto. Let A be an infinite subset of B. Note $A \neq \emptyset$ since empty sets are finite sets.

We want to define a $g : \mathbb{N} \to A$. Let $n_1 = \min\{m \in \mathbb{N} : f(m) \in A\}$, and set $g(1) = f(n_1)$. Assume

$$n_m = \min\{m \in \mathbb{N} \setminus \cup_{i=1}^{m-1} n_i : f(m) \in A\}.$$

well-defined for all $n \in \mathbb{N}$. Then

$$n_{m+1} = \min\{m \in (\mathbb{N} \setminus \cup_{i=1}^{m-1} n_i) \setminus n_n : f(m) \in A\}$$

with $(\mathbb{N} \setminus \cup_{i=1}^{m-1} n_i) \setminus n_m$ nonempty subset of infinite $\mathbb{N}$. Note that by well-ordering principle there exists a minimal element $n_{m+1} \in \mathbb{N}$ that maps to an element $f(n_{m+1}) \in A \subseteq B$, since B infinite. Because f is onto, and $A \subseteq B$, every element $a \in A \subseteq B$ has a preimage n_m as defined above. Because f is 1–1, we know that every n_m maps to a distinct image under A, that is $\forall n_m \in \mathbb{N}$, $\exists! f(n_m) \in A : g(m) = f(n_m)$.

Hence we define $g : \mathbb{N} \to A$ as

$$g(m) = f(n_m)$$

which is bijective since f bijective. Therefore A is countable. □

Problem 3. (Ex. 1.5.2) Use the following outline (as specified in the textbook) to supply proofs for the statements in Theorem 1.5.8.

Proof. Two statements in Theorem 1.5.8:

(1) If $A_1, A_2, \dots A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \dots \cup A_m$ is countable.
(2) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(a) Proof by induction. We first prove the statement for two countable sets, that is $A_1 \cup A_2$ countable if A_1, A_2 countable. First replace A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. Note that the union $A_1 \cup B_2 = A_1 \cup A_2$, but crucially the sets A_1 and B_2 are disjoint. Because A_1 countable, there exists a bijective function $f : \mathbb{N} \to A_1$.

Consider two cases: B_2 finite and B_2 infinite. If B_2 finite, $\exists n \in \mathbb{N} : B_2 = \{b_k : \forall k \in \mathbb{N}, k \leq n\}$ and we define a function $g : \mathbb{N} \to B_2$ by

$$g(n) = b_n$$

and

$$g(n + m) = f(m)$$

for all $m \in \mathbb{N} : f(m) \in A_1$. So g is a 1–1 function from $\mathbb{N}$ to $A_1 \cup B_2 = A_1 \cup A_2$, and $A_1 \cup A_2$ countable.

[Lemma 1.5.7] If $A \subseteq B$ and B is countable, then A is either countable or finite.

If B_2 infinite, $B_2 \subseteq A_2$, and by Lemma 1.5.7, B_2 is countable. Here, we attempt to partition the infinite set of $\mathbb{N}$ to use as inputs for our two bijective functions $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to B_2$ to produce an overall bijective function $h : \mathbb{N} \to A_1 \cup A_2$. We have

$$h(n) = \begin{cases} f((\frac{n+1}{2})) & \text{if } n \text{ odd} \Leftrightarrow n = 2n-1, \forall n \in \mathbb{N} \\ g(\frac{n}{2}) & \text{if } n \text{ even} \Leftrightarrow n = 2n, \forall n \in \mathbb{N} \end{cases}$$

which is bijective. Therefore $A_1 \cup A_2$ countable.

Next assume $A_1 \cup A_2 \cup A_3 \cup \cdots A_m$ is countable for some $m \in \mathbb{N}$. Then

$$A_1 \cup A_2 \cup A_3 \cup \cdots A_m \cup A_{m+1} = \overbrace{(A_1 \cup A_2 \cup A_3 \cup \cdots A_m)}^{\text{countable}} \cup \overbrace{A_{m+1}}^{\text{countable}}$$

Since we have shown $A_1 \cup A_2 \cup A_3 \cup \cdots A_m$ countable $\Rightarrow A_1 \cup A_2 \cup A_3 \cup \cdots A_m \cup A_{m+1}$ countable, by Principle of Mathematical Induction, $A_1 \cup A_2 \cup A_3 \cup \cdots A_m$ countable for all $m \in \mathbb{N}$.

(b) $\bigcup_{n=1}^{\infty} A_n = \lim_{N\to\infty} \cup_{n=1}^{N} A_n$. Induction from part (i) cannot be used to evaluate limits to infinity. The principle of mathematical induction states that given a proposition $P(n)$, if $P(1)$ true and $P(n) \Rightarrow P(n+1)$ true, for some $n \in \mathbb{N}$, then $P(n)$ true for all $n \in \mathbb{N}$. Here, the proposition $P(n)$ is $\bigcup_{n=1}^{N} A_n$, where each A_n countable, is countable for N number of A_n sets. By induction in the first part we have shown $P(n)$ true for all $n \in \mathbb{N}$. But the statement of the limit to infinity of this union is not in the proposition $P(n)$, therefore induction cannot be used to prove part (ii) from part(i).

(c) Define

$$A_1 = A_1,$$
$$A_2 = A_2 \setminus A_1,$$

and

$$A_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

We will assume, without loss of generality, that all A_n are nonempty, countably infinite sets. This definition of A_n is valid because the value of the infinite union is preserved. Each A_n and A_{n+1} are pairwise disjoint. Arranging $\mathbb{N}$ into a two-dimensional array, define

$$\begin{aligned} B_1 &= \{1, 3, 6, \ldots\}, \\ B_2 &= \{2, 5, 9, \ldots\}, \\ B_3 &= \{4, 8, 13, \ldots\}, \\ &\vdots \\ B_n &= \{k \in \mathbb{N} : k \text{ belongs to the } n^{th} \text{ row of the 2-d array}\} \end{aligned}$$

It can be seen that each B_n is countable, by the columns of the two-dimensional $\mathbb{N}$ array, $B_n \sim \mathbb{N}$ and there exists bijective function $g_n : \mathbb{N} \to B_n$ for all B_n. From above definition of A_n we see that each A_n is countably infinite, and therefore there exists a bijective function $f_n : \mathbb{N} \to A_n$ for all A_n.

Next we want to define a bijective function $h : B_n \to A_n$. Denote each element of B_n as b_{n_i} for $i \in \mathbb{N}$, and each element of A_n as a_{n_i} for $i \in \mathbb{N}$. Then consider the function

$$h(b_{n_i}) = a_{n_i}$$

which is 1–1 and onto. It is 1–1 because each b_{n_i} and a_{n_i} is unique, and onto because each a_{n_i} has a preimage b_{n_i}, as shown above. So we have

$$h : B \to A$$

where $B = \bigcup_{n=1}^{\infty} B_n = \mathbb{N}$ by definition and $A = \bigcup_{n=1}^{\infty} A_n$. In other words

$$h : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$$

and we have shown $\bigcup_{n=1}^{\infty} A_n$ is countable.

□

Problem 4. (Ex. 1.5.4)

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).
(b) Show that an unbounded interval like $(a, \infty) = \{x \ : \ x > a\}$ has the same cardinality as $\mathbb{R}$ as well.
(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.

Proof.

(a) We know the open real interval $(-1, 1) \sim \mathbb{R}$. This means that there is a bijective function $f : (-1, 1) \to \mathbb{R}$, and $(-1, 1)$ has the same cardinality as $\mathbb{R}$. We will attempt to find a bijective function from the open interval (a, b) to $(-1, 1)$. This would mean (a, b) has same cardinality as $(-1, 1)$ and consequently same cardinality as $\mathbb{R}$, therefore we would have proven $(a, b) \sim \mathbb{R}$.

Define $g : (a, b) \to (-1, 1)$ by

$$\begin{aligned} g(x) &= \underbrace{(\sup(-1,1) - \inf(-1,1))}_{1-(-1)=2} \cdot \frac{x - \overbrace{\inf(a,b)}^{a}}{\underbrace{\sup(a,b) - \inf(a,b)}_{b-a}} - \frac{\overbrace{\sup(-1,1) - \inf(-1,1)}^{1-(-1)=2}}{2} \\ &= 2 \cdot \frac{x-a}{b-a} - 1. \end{aligned}$$

This is valid because $(a, b) \Rightarrow a < b \Rightarrow b - a > 0$. We need to check g is 1–1 and onto. To check 1–1 we pick any two arbitrary preimages x_1, x_2 and show that $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$:

$$\begin{aligned} g(x_1) &= 2 \cdot \frac{x_1 - a}{b - a} - 1 \\ &= 2 \cdot \frac{x_2 - a}{b - a} - 1 \\ &= g(x_2) \\ \Leftrightarrow \cancel{2} \cdot \frac{x_1 \cancel{- a}}{\cancel{b - a}} \cancel{- 1} &= \cancel{2} \cdot \frac{x_2 \cancel{- a}}{\cancel{b - a}} \cancel{- 1} \\ \Leftrightarrow x_1 &= x_2 \end{aligned}$$

So g is 1–1. To check onto we show that for all elements $y \in (-1,1)$ there exists a $x \in (a,b)$ such that $g(x) = y$. Pick an arbitary $y \in (-1,1)$. We see that

$$\begin{aligned}
&-1 < y < 1 \\
g(x) &= 2 \cdot \frac{x-a}{b-a} - 1 \\
&= \frac{2x-a-b}{b-a} \\
&= \frac{2x-(b-a)-2a}{b-a} \\
&= -1 + \frac{2(x-a)}{b-a} \\
&< -1 + 2 \\
&= 1
\end{aligned}$$

This is valid because $x < b \Rightarrow x - a < b - a$. Also note

$$-1 < -1 + \frac{2(x-a)}{b-a}$$

because $x > a \Rightarrow x - a > 0$. Therefore g is onto. We have proven g 1–1 and onto, therefore we have proven g bijective, and $(a,b) \sim (-1,1) \sim \mathbb{R}$.

(b) To show that $(a,\infty) \sim \mathbb{R}$, we just need to show $(a,\infty) \sim (0,1)$ because we know $(0,1) \sim \mathbb{R}$.
Define a function $f : (a,\infty) \to (0,1)$ by

$$f(x) = \frac{1}{1 + \underbrace{x-a}_{\lim_{x\to\infty} x-a=\infty}}$$

We need to check that f is bijective, i.e. 1–1 and onto. To check 1–1 we pick any two arbitrary preimages $x_1, x_2 \in (a,\infty)$ and show that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$:

$$\begin{aligned}
f(x_1) &= \frac{1}{1+x_1-a} \\
&= \frac{1}{1+x_2-a} \\
&= f(x_2) \\
\Leftrightarrow \frac{1}{1+x_1-a} &= \frac{1}{1+x_2-a} \\
\cancel{1} + x_2 \cancel{-a} &= \cancel{1} + x_1 \cancel{-a} \\
\Leftrightarrow x_1 &= x_2
\end{aligned}$$

This is valid because $x_1, x_2 > a \Rightarrow (x_1 - a > 0$ and $x_2 - a > 0)$. To check onto we show that for all elements $y \in (b,c)$ there exists a $x \in (a,\infty)$ such that $f(x) = y$. Pick an arbitary

$y \in (0,1)$. We see that

$$\begin{aligned} y &= f(x) \\ &= \frac{1}{1+x-a} \\ \frac{1}{y} &= 1+x-a \\ x &= \frac{1}{y} - 1 + a \\ &> 1 - 1 + a \\ &= a \end{aligned}$$

Note that $x = 1/y - 1 + a$ is unbounded because $0 < y < 1 \Rightarrow 1/y > n$ for any $n \in \mathbb{N}$, by Archimedean Property and Density of $\mathbb{Q}$ in $\mathbb{R}$. Therefore $x = 1/y - 1 + a$ gets larger and larger as y gets closer and closer to 0, and therefore we have shown $x \in (a, \infty)$ exists for any $y \in (0,1)$. So f onto. Therefore f bijective, and $(a,\infty) \sim (0,1) \sim \mathbb{R}$.

(c) Consider the function $f : [0,1) \to (0,1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n}, \forall n \in \mathbb{N} : n \geq 2 \\ x & \text{if } x \neq \frac{1}{n} \end{cases}$$

To check f 1–1 we pick any two arbitrary preimages $x_1, x_2 \in (a, \infty)$ and show that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$:

$$\begin{aligned} f(x_1) &= \begin{cases} \frac{1}{2} & \text{if } x_1 = 0 \\ \frac{1}{n+1} & \text{if } x_1 = \frac{1}{n}, \forall n \in \mathbb{N} : n \geq 2 \\ x_1 & \text{if } x_1 \neq \frac{1}{n} \end{cases} \\ &= \begin{cases} \frac{1}{2} & \text{if } x_2 = 0 \\ \frac{1}{n+1} & \text{if } x_2 = \frac{1}{n}, \forall n \in \mathbb{N} : n \geq 2 \\ x_2 & \text{if } x_2 \neq \frac{1}{n} \end{cases} \\ &= f(x_2) \end{aligned}$$

3 cases:
(1) $f(x_1) = f(x_2) = \frac{1}{2}$. Then $x_1 = 0 = x_2$.
(2) $f(x_1) = f(x_2) = \frac{1}{n+1}$. Then $x_1 = \frac{1}{n} = x_2$.
(3) $f(x_1) = x_1 = x_2 = f(x_2)$. Then $x_1 = x_2$.
Therefore f 1–1. To show f onto we pick an arbitrary $y \in (0,1)$, and show that there exists a preimage $x \in [0,1)$ that maps to y under f.

Again, 3 cases:
(1) $y = \frac{1}{2} \Rightarrow x = 0 \in [0,1)$.
(2) $y = \frac{1}{n} \Rightarrow x = \frac{1}{n}, \forall n \in \mathbb{N} : n \geq 2$. This $x \in [0,1)$ since $\frac{1}{n} < 1$ for all $n \in \mathbb{N}$.
(3) $y = x$. This covers all other $y \in (0,1)$ that doesn't fall under the first two cases.
So f onto. Therefore f bijective, and $[0,1) \sim (0,1)$.

$\square$

Problem 5. (Ex. 1.5.6 (b)) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Proof. No such collection exists.

[Lemma] Density of $\mathbb{Q}$ in $\mathbb{R}$. For any open interval $(a, b) \in \mathbb{R}$, there exists a $q \in \mathbb{Q}$ such that $a < q < b$.

[Lemma] $\mathbb{Q}$ is countable. That is, there exists a function $f : \mathbb{Q} \to \mathbb{N} \Leftrightarrow \mathbb{Q} \sim \mathbb{N}$.

Because these open intervals are disjoint, no two open intervals in this collection share the same rational number q, and each open interval contains a distinct $q \in \mathbb{Q}$. Because $\mathbb{Q} \sim \mathbb{N}$, and $\mathbb{Q}$ countable, if we define each open interval (a, b) in $\mathbb{R}$ by their distinct $q \in (a, b)$, we find that there is a bijection $f : \bigsqcup_{i=1}^{\infty} (a_i, b_i) \to \mathbb{Q}$. Any collection of disjoint open intervals in $\mathbb{R}$ corresponds to a countable collection of rational numbers $q \in \mathbb{R}$, therefore there is no such uncountable collection of disjoint open intervals. $\square$