MATH 0450: HOMEWORK 2

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Use *mathematical induction* to prove the following statements.

Problem 1. Let x be a real number, $x \neq 1$. Then, for any $n \in \mathbb{N}$ we have

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Proof. Let $P(n)$ be defined as

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x},$$

for $n \in \mathbb{N}$. Define the set N as $N = \{n \in \mathbb{N} \mid P(n)\}$. To prove the initial base case, we take

$$\begin{aligned} P(1) : 1 + x^1 &= \frac{1 - x^{1+1}}{1 - x} \\ 1 + x &= \frac{1 - x^2}{1 - x} \\ &= \frac{\cancel{(1 - x)}(1 + x)}{\cancel{1 - x}} \quad \text{because } x \neq 1 \Rightarrow 1 - x \neq 0 \\ &= 1 + x \end{aligned}$$

Therefore $1 \in N$. Now we want to prove the statement $[P(n) \Rightarrow P(n+1)]$. If $P(n)$ is false, the statement is vacuously true. Now assume $P(n)$ is true for some $n \in \mathbb{N}$, that is

$$P(n) : 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \tag{1}$$

for some $n \in N$. Then to prove $P(n+1)$ true:

$$\begin{aligned} 1 + x + x^2 + \cdots + x^{n+1} &= 1 + x + x^2 + \cdots + x^n + x^{n+1} \\ &= \frac{1 - x^{n+1}}{1 - x} + x^{n+1} \quad \text{by induction step 1} \\ &= \frac{1 - x^{n+1}}{1 - x} + \frac{x^{n+1} - x^{n+2}}{1 - x} \\ &= \frac{1 - \cancel{x^{n+1}} + \cancel{x^{n+1}} - x^{n+2}}{1 - x} \\ &= \frac{1 - x^{n+2}}{1 - x}. \end{aligned}$$

Therefore $n + 1 \in N$, and this proves $[P(n) \Rightarrow P(n+1)]$ is true. Hence $N = \mathbb{N}$, and $P(n)$ true. $\square$

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Problem 2. The number of diagonals in a convex polygon with $n \geq 3$ sides equals $n(n-3)/2$.

Proof. Let $P(n)$ be defined as the statement

$$P(n): \quad \text{number of diagonals in a convex polygon with } n \text{ sides} = \frac{n(n-3)}{2}$$

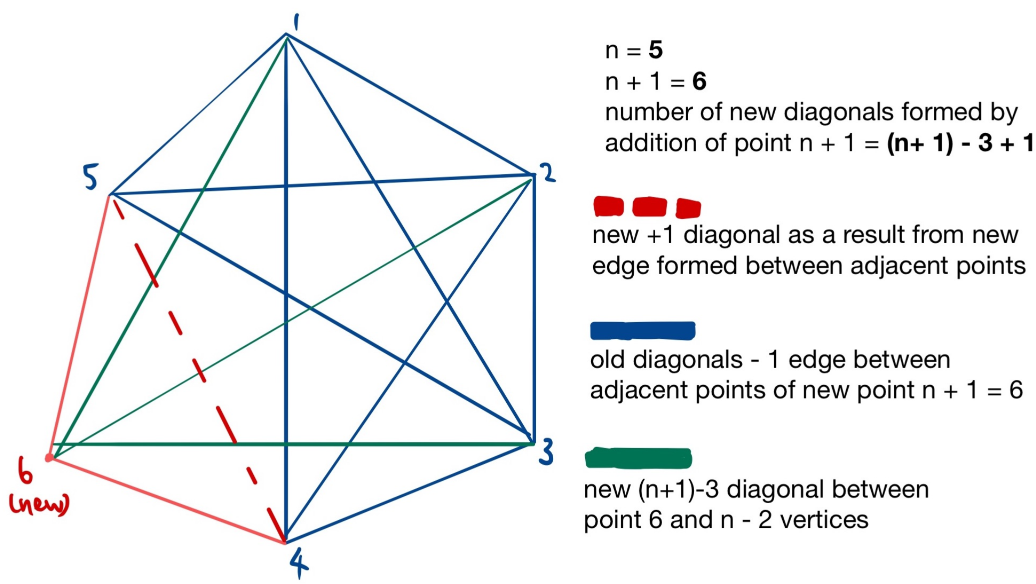
for $n \in \mathbb{N} \setminus \{1, 2\}$. Define the set N as $N = \{n \in \mathbb{N} \setminus \{1, 2\} \mid P(n)\}$. To prove the initial base case, we take

$$\begin{aligned} P(3): \quad \text{number of diagonals in a convex polygon with 3 sides} &= \frac{3(3-3)}{2} \\ &= \frac{3 * 0}{2} \\ &= 0 \end{aligned}$$

Therefore $3 \in N$. Now we want to prove the statement $[P(n) \Rightarrow P(n+1)]$. If $P(n)$ is false, the statement is vacuously true. Now assume $P(n)$ is true for some $n \in \mathbb{N}$, that is

$$\text{number of diagonals in a convex polygon with } n \text{ sides} = \frac{n(n-3)}{2}$$

is true for some convex polygon with $n \in \mathbb{N} \setminus \{1, 2\}$ sides. Consider an $n + 1^{\text{th}}$ vertex added to this n-sided polygon that preserves the convex polygon nature of the resulting polygon. This point has to be somewhere outside the convex polygon. The picture below demonstrates this, for $n = 5$.



Therefore by the combinatorial sum rule, this new point creates $((n+1)-3)+1$ new diagonals; $(n+1)-3$ diagonals between the new $n+1^{th}$ point with the existing $(n+1)-3$ points that excludes three points, namely the $n+1^{th}$ point itself and its two adjacent points, and 1 additional diagonal that is the edge between its two adjacent points. As a result we have

$$\begin{aligned}\text{number of diagonals in a convex polygon with } n+1 \text{ sides} &= \frac{n(n-3)}{2} + ((n+1)-3)+1 \\ &\quad \text{(using induction step } P(n) \\ &\quad \text{and number of new diagonals)} \\ &= \frac{n(n-3)}{2} + n - 1 \\ &= \frac{n(n-3)}{2} + \frac{2(n-1)}{2} \\ &= \frac{n^2-3n+2n-2}{2} \\ &= \frac{n^2-n-2}{2} \\ &= \frac{(n+1)(n-2)}{2} \\ &= \frac{(n+1)((n+1)-3)}{2}\end{aligned}$$

which shows $n+1 \in N$, and this proves $[P(n) \Rightarrow P(n+1)]$ is true. Hence $N = \mathbb{N} \setminus \{1,2\}$, and $P(n)$ true. □

Problem 3 (Prime factorization). Any natural number greater than 1 can be written as a product of prime numbers and/or 1.

Proof. Define the set of prime numbers $\mathbb{P}$ as

$$\mathbb{P} = \{p \in \mathbb{N} \mid p \text{ is prime}\}$$

Let $P(n)$ be defined as the statement

$$P(n): \quad n = a * b,$$

for $n \in \mathbb{N} \setminus \{1\}$, $a, b \in \mathbb{P} \cup \{1\}$ or $[(a \text{ product of primes}) \vee (b \text{ product of primes})]$.

Define the set N as $N = \{n \in \mathbb{N} \setminus \{1\} \mid P(n)\}$. To prove the initial base case, we take

$$P(2): \quad 2 = 2 * 1,$$

where $2 \in \mathbb{P} \cup \{1\}$, and so $2 \in N$. Next we want to prove the statement $[(P(2) \wedge P(3) \wedge P(4) \wedge P(5) \wedge \cdots \wedge P(n)) \Rightarrow P(n+1)]$ (Principle of Strong Induction). If the antecedent is false, the statement is vacuously true. Now assume $P(2) \wedge P(3) \wedge P(4) \wedge P(5) \wedge \cdots \wedge P(n)$ is true, that is that $P(n)$ is true for all natural numbers $1 < k \leq n$. Then we want to prove $P(n+1)$ true. Consider two cases: if $n+1 \in \mathbb{P}$ then $n+1 = (n+1) * 1$, and $P(n+1)$ is true. If $n+1 \notin \mathbb{P}$, then because $n+1 > 2$ we have

$$n+1 = u * v,$$

where $1 < u, v < n+1 \in \mathbb{N}$. Note that above we assumed $P(2) \wedge P(3) \wedge P(4) \wedge P(5) \wedge \cdots \wedge P(n)$ is true. Therefore any $u, v < n+1 \in N$ are individually primes or products of primes; consequently $n+1 = u * v$ is also product of primes. $n+1 \in N$, and thus $P(n+1)$ is true. Since we have proven strong inductive step $[(P(2) \wedge P(3) \wedge P(4) \wedge P(5) \wedge \cdots \wedge P(n)) \Rightarrow P(n+1)]$, we have proven

$P(n) \in N$: all natural numbers greater than 1 can be written as primes or products of primes and/or 1. □

Problem 4 (Binary expansion). Any $n \in \mathbb{N}$ can be written as

$$n = c_k 2^k + c_{k-1} 2^{k-1} + \cdots + c_0 2^0,$$

for some $k \in \mathbb{N}_*$ and $c_i \in \{0, 1\}$, $0 \leq i \leq k$.

Proof. Let $P(n)$ be the statement that $n \in \mathbb{N}$ can be written as

$$n = c_k 2^k + c_{k-1} 2^{k-1} + \cdots + c_0 2^0,$$

for some $k \in \mathbb{N}_*$ and $c_i \in \{0, 1\}$, $0 \leq i \leq k$. Define the set N as $N = \{n \in \mathbb{N} \mid P(n)\}$. First we show $P(0)$ true:

$$0 = 0 \cdot 2^0$$

and so $0 \in N$, $P(0)$ true. Next we assume that $P(n)$ true for all $n \geq 0$ (Principle of Strong Induction). We need to show $P(n+1)$ true as a result. Consider two cases: $n+1$ even and $n+1$ odd. If $n+1$ even, then $\frac{n+1}{2} \leq n \in \mathbb{N}$, and so by inductive step

$$\frac{n+1}{2} = c_k 2^k + c_{k-1} 2^{k-1} + \cdots + c_0 2^0.$$

Consequently

$$\begin{aligned} n + 1 &= c_k 2^{k+\mathbf{1}} + c_{k-1} 2^{k-1+\mathbf{1}} + \cdots + c_0 2^{0+\mathbf{1}} \\ &= c_k 2^{k+1} + c_{k-1} 2^k + \cdots + c_0 2^1 + 0 \cdot 2^0. \end{aligned}$$

If $n+1$ odd, then $n \in \mathbb{N}$ must be even, i.e.

$$\begin{aligned} \frac{n}{2} &= c_k 2^k + c_{k-1} 2^{k-1} + \cdots + c_0 2^0 \\ n &= c_k 2^{k+1} + c_{k-1} 2^k + \cdots + c_0 2^1 \end{aligned}$$

and therefore we can surely also represent $n+1$ as a binary representation:

$$n + 1 = c_k 2^{k+1} + c_{k-1} 2^k + \cdots + c_0 2^1 + 1 \cdot 2^0,$$

proving $n + 1 \in N$. And so we have proven the binary expansion theorem for all $n \in \mathbb{N}$. □

Problem 5 (Cauchy-Schwarz inequality). For any $n \in \mathbb{N}$ and real numbers a_i, b_i, $1 \leq i \leq n$ we have

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2} (b_1^2 + b_2^2 + \cdots + b_n^2)^{1/2}.$$

Proof. Proof by induction. Let $P(n)$ be defined as

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2} (b_1^2 + b_2^2 + \cdots + b_n^2)^{1/2} 7,$$

for $n \in \mathbb{N}$ and real numbers and real numbers a_i, b_i, where $1 \leq i \leq n$. We shall prove two base cases $P(1)$ and $P(2)$, then prove the inductive step $P(n) \Rightarrow P(n+1)$. Define the set N as $N = \{n \in \mathbb{N} \mid P(n)\}$.

Base cases:

$$\begin{aligned} P(1): \quad |a_1b_1| &\leq (a_1^2)^{1/2}(b_1^2)^{1/2} \\ (a_1^2)^{1/2}(b_1^2)^{1/2} &= (a_1^2b_1^2)^{1/2} \\ &= (a_1b_1)^{2(1/2)} \\ &= +a_1b_1 \\ &= |a_1b_1| \\ &\geq |a_1b_1| \end{aligned}$$

so $1 \in N$. And

$$\begin{aligned} P(2): \quad |a_1b_1 + a_2b_2| &\leq (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2} \\ &\text{(square both sides)} \\ (a_1b_1 + a_2b_2)^2 &\leq (a_1^2 + a_2^2)(b_1^2 + b_2^2) \\ a_1^2b_1^2 + a_2^2b_2^2 + 2a_1b_1a_2b_2 &\leq a_1^2b_1^2 + a_2^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2 \\ &\text{to show:} \\ 2a_1b_1a_2b_2 &\leq a_1^2b_2^2 + a_2^2b_1^2 \\ 0 &\leq a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_1a_2b_2 \\ 0 &\leq (a_1b_2 - a_2b_1)^2 \end{aligned}$$

so $2 \in N$. Next we assume $P(n)$ true for some $n \in \mathbb{N}$. Then to prove $P(n+1) : |a_1b_1 + a_2b_2 + \cdots + a_{n+1}b_{n+1}| \leq (a_1^2 + a_2^2 + \cdots + a_{n+1}^2)^{1/2}(b_1^2 + b_2^2 + \cdots + b_{n+1}^2)^{1/2}$ true:

$$\begin{aligned} |a_1b_1 + a_2b_2 + \cdots + a_{n+1}b_{n+1}| &= |(a_1b_1 + a_2b_2 + \cdots + a_nb_n) + a_{n+1}b_{n+1}| \\ &\leq |(a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}(b_1^2 + b_2^2 + \cdots + b_n^2)^{1/2} + a_{n+1}b_{n+1}| \\ &\text{let } (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2} = A \text{ and } (b_1^2 + b_2^2 + \cdots + b_n^2)^{1/2} = B \\ &\overbrace{\leq} |AB + a_{n+1}b_{n+1}| \\ &\text{by } P(2) \\ &\overbrace{\leq} (A^2 + a_{n+1}^2)^{1/2}(B^2 + b_{n+1}^2)^{1/2} \\ &= (a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2)^{1/2}(b_1^2 + b_2^2 + \cdots + b_n^2 + b_{n+1}^2)^{1/2} \end{aligned}$$

and so $n + 1 \in N$. Therefore by Principle of Mathematical Induction, $N = \mathbb{N}$, and $P(n) :=$ Cauchy-Schwarz Inequality is true for all $n \in \mathbb{N}$. $\square$