APPENDIX A
PROOF OF SECTION IV

A. Auxiliary Lemmas and Propositions

Lemma 3. *We recall $\alpha^* = \arg\max_{\alpha\in(0,1)} D_\alpha(\varphi_0\|\varphi_1)$ and Y has density function φ_{α^*}, then $\sum_{j=1}^n \mathbb{E}[Z_j] = 0$.*

Proof. By definition of Y, we have

$$\mathbb{E}\Big[\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big] = \int_\Omega \varphi_{\alpha^*}\log\frac{\varphi_0(y)}{\varphi_1(y)}d\mu(y) = e^{-D_{\alpha^*}(\varphi_0\|\varphi_1)}\int_\Omega \varphi_0^{1-\alpha_*}\varphi_1^{\alpha_*}\log\frac{\varphi_1}{\varphi_0}d\mu.$$

We omit the variable y in the integral in the rest of the proof. Recall that we assume the Kullback–Leibler divergence $D_{\mathrm{KL}}(\varphi_0\|\varphi_1)$ and $D_{\mathrm{KL}}(\varphi_1\|\varphi_0)$ exist, so for $\alpha\in(0,1)$, $\left|\varphi_0^{1-\alpha}\varphi_1^\alpha\log\frac{\varphi_0}{\varphi_1}\right| \le \left|(\varphi_0+\varphi_1)\log\frac{\varphi_0}{\varphi_1}\right|$ is integrable. By mean value theorem and dominated convergence theorem,

$$\int_\Omega \varphi_0^{1-\alpha}\varphi_1^\alpha\log\frac{\varphi_1}{\varphi_0}d\mu = \int_\Omega \frac{d}{d\alpha}\varphi_0^{1-\alpha}\varphi_1^\alpha d\mu = \frac{d}{d\alpha}\int_\Omega \varphi_0^{1-\alpha}\varphi_1^\alpha d\mu.$$

Since $\alpha\mapsto -\varphi_0(x)^{1-\alpha}\varphi_1(x)^\alpha$ is convex for $x\in\Omega$, $\alpha\mapsto -\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha d\mu$ is also convex, and it is indeed strictly convex if $\varphi_0\ne\varphi_1$ on a set with nonzero measure. Therefore, $D_\alpha(\varphi_0\|\varphi_1)$ achieves maximum if and only if $\frac{d}{d\alpha}\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha d\mu = 0$, which is true if we evaluate at $\alpha=\alpha^*$. Hence $\mathbb{E}[\log l(Y)] = 0$. □

Proposition 1. *Let Φ be the cumulative distribution function of standard normal distribution, then for $x>0$,*

$$\frac{1}{x}-\frac{1}{x^3} \le \sqrt{2\pi}e^{x^2/2}\Phi(-x) \le \frac{1}{x}. \tag{16}$$

and

$$\sqrt{2\pi}e^{x^2/2}\Big(\Phi(x)-\frac{1}{2}\Big) \ge x+\frac{x^3}{3}. \tag{17}$$

In particular,

$$\Phi(x) \ge \frac{1}{2}+\frac{1}{\sqrt{2\pi}}\Big(x-\frac{2x^3}{3}\Big). \tag{18}$$

Proof. For $x>0$, we have divergent series expanded at ∞:

$$\sqrt{2\pi}e^{x^2/2}\Phi(-x) = \frac{1}{x}+\sum_{i=1}^\infty \frac{(-1)^i(2i-1)!}{2^{i-1}(i-1)!}\frac{1}{x^{2i+1}} = \frac{1}{x}-\frac{1}{x^3}+\frac{3}{x^5}-\frac{15}{x^7}+\dots$$

which implies (16). We also have the power series expanded at 0. Let $n!! = 1\cdot3\cdot\dots\cdot n$ for odd integer n, we have

$$\sqrt{2\pi}e^{x^2/2}\Big(\Phi(x)-\frac{1}{2}\Big) = \sum_{i=0}^\infty\frac{x^{2i+1}}{(2i+1)!!} = x+\frac{x^3}{3}+\frac{x^5}{3\cdot5}+\frac{x^7}{3\cdot5\cdot7}+\dots$$

which implies (17). Then we expand $e^{-x^2/2}\Big(x+\frac{x^3}{3}\Big)$ and have

$$e^{-x^2/2}\Big(x+\frac{x^3}{3}\Big) = x-\frac{2x^3}{3}+\frac{7x^5}{30}-\dots$$

Then we obtain (18). □

Lemma 4. *Let $g_\alpha(x)=\exp(\min(\alpha x,(\alpha-1)x))$. Suppose $Z\sim\mathcal{N}(0,\sigma^2)$, then*

$$\frac{1}{\sqrt{2\pi}\sigma\alpha(1-\alpha)}-\frac{1}{\sqrt{2\pi}\sigma^3\alpha^3(1-\alpha)^3} \le \mathbb{E}[g_\alpha(Z)] \le \frac{1}{\sqrt{2\pi}\sigma\alpha(1-\alpha)}.$$

Proof. We have

$$\begin{aligned}\mathbb{E}[g_\alpha(Z)] &= \mathbb{E}[\exp(\min(\alpha Z,(\alpha-1)Z))]\\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^0 e^{\alpha x}e^{-\frac{x^2}{2\sigma^2}}dx+\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{(\alpha-1)x}e^{-\frac{x^2}{2\sigma^2}}dx\\ &= \frac{e^{\sigma^2\alpha^2/2}}{\sqrt{2\pi}\sigma}\int_{-\infty}^0 e^{-\frac{(x-\sigma^2\alpha)^2}{2\sigma^2}}dx+\frac{e^{\sigma^2(1-\alpha)^2/2}}{\sqrt{2\pi}\sigma}\int_0^\infty e^{-\frac{(x-\sigma^2(1-\alpha))^2}{2\sigma^2}}dx\end{aligned}$$

$$= \exp\Big(\frac{\sigma^2\alpha^2}{2}\Big)\Phi(-\sigma\alpha) + \exp\Big(\frac{\sigma^2(1-\alpha)^2}{2}\Big)\Phi(-\sigma(1-\alpha))$$

. By (16), we have

$$\mathbb{E}[g_\alpha(Z)] \le \frac{1}{\sqrt{2\pi}\sigma\alpha} + \frac{1}{\sqrt{2\pi}\sigma(1-\alpha)} = \frac{1}{\sqrt{2\pi}\sigma\alpha(1-\alpha)},$$

and

$$\begin{aligned}\mathbb{E}[g_\alpha(Z)] &\ge \frac{1}{\sqrt{2\pi}\sigma\alpha} - \frac{1}{\sqrt{2\pi}\sigma^3\alpha^3} + \frac{1}{\sqrt{2\pi}\sigma(1-\alpha)} - \frac{1}{\sqrt{2\pi}\sigma^3(1-\alpha)^3} \\ &\ge \frac{1}{\sqrt{2\pi}\sigma\alpha(1-\alpha)} - \frac{1}{\sqrt{2\pi}\sigma^3\alpha^3(1-\alpha)^3}.\end{aligned}$$

This completes the proof. □

Theorem 3 (Berry-Esseen theorem)**.** *Let $\{Z_i\}_{i=1}^n$ be independent random variables with zero means and $\mathbb{E}[\sum_{i=1}^n Z_i^2] = \sigma^2$. Let F be the distribution function of $\sum_{i=1}^n Z_i/\sigma$, then there exists an absolute constant $C_0 \le 0.56$ such that for every $x \in \mathbb{R}$,*

$$|F(x) - \Phi(x)| \le \frac{C_0 \sum_{i=1}^n \mathbb{E}[|Z_i|^3]}{\sigma^3}. \tag{19}$$

Proof. The proof is given in [23]. □

B. Proof of Theorem 1

We recall that $\varphi_\alpha = \varphi_0^{1-\alpha}\varphi_1^\alpha e^{D_\alpha(\varphi_0\|\varphi_1)}$. In this proof, we briefly write $\int_\Omega \min(\varphi_0(x), \varphi_1(x))d\mu(x)$ as $\int_\Omega \min(\varphi_0, \varphi_1)d\mu$. We have

$$\begin{aligned}\int_\Omega \min(\varphi_0, \varphi_1)d\mu &= \int_\Omega \varphi_0^{1-\alpha}\varphi_1^\alpha \min\Big(\Big(\frac{\varphi_0}{\varphi_1}\Big)^\alpha, \Big(\frac{\varphi_1}{\varphi_0}\Big)^{1-\alpha}\Big)d\mu \\ &= e^{-D_\alpha(\varphi_0\|\varphi_1)}\int_\Omega \varphi_\alpha \exp\Big(\min\Big(\log\frac{\varphi_0}{\varphi_1}, (\alpha-1)\log\frac{\varphi_0}{\varphi_1}\Big)\Big)d\mu \\ &= e^{-D_\alpha(\varphi_0\|\varphi_1)}\mathbb{E}\Big[g_\alpha\Big(\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big)\Big] \\ &= e^{-D_\alpha(\varphi_0\|\varphi_1)}\mathbb{E}\Big[g_\alpha\Big(\sum_{j=1}^n Z_j\Big)\Big],\end{aligned}$$

where $g_\alpha(x) = \exp(\min(\alpha x, (\alpha-1)x))$ from Lemma 3 and Y and Z_j have been defined in Section IV. Now it suffices to show that

$$\mathbb{E}\Big[g_\alpha\Big(\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big)\Big] \asymp \frac{1}{\sqrt{n}\bar{\sigma}_n\alpha(1-\alpha)}$$

when $\alpha = \alpha^*$. In this case, we have $\mathbb{E}[\sum_{j=1}^n Z_j] = 0$ by Lemma 3. Let us define $\sigma^2 = \mathbb{E}[\sum_{j=1}^n Z_j^2] = -D''_{\alpha^*}(\varphi_0\|\varphi_1)$. By assumption $\sum_{j=1}^n \mathbb{E}|Z_j|^3 \le C_1 n\bar{\sigma}_n^2$, the distribution function F of $\sum_{j=1}^n Z_j/\sigma$ satisfies for $x \in \mathbb{R}$, $|F(x) - \Phi(x)| \le \frac{C}{\sigma}$ where $C := 0.56C_1$. Recall that $\log l(Y) = \sum_{j=1}^n Z_j$, we have

$$\begin{aligned}\mathbb{E}\Big[g_\alpha\Big(\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big)\Big] &= \int_0^1 \mathbb{P}\Big(g_\alpha\Big(\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big) > x\Big)dx \\ &= \int_0^1 \mathbb{P}\Big(\frac{\log x}{\alpha\sigma} < \frac{1}{\sigma}\log\frac{\varphi_0(Y)}{\varphi_1(Y)} < \frac{\log x}{(\alpha-1)\sigma}\Big)dx \\ &= \int_0^1 F\Big(\frac{\log x}{(\alpha-1)\sigma}\Big) - F\Big(\frac{\log x}{\alpha\sigma}\Big)dx \\ &\le \int_0^1 \Phi\Big(\frac{\log x}{(\alpha-1)\sigma}\Big) - \Phi\Big(\frac{\log x}{\alpha\sigma}\Big)dx + \frac{2C}{\sigma} \\ &= \frac{1}{\sqrt{2\pi}\sigma\alpha(1-\alpha)} + \frac{2C}{\sigma} \le \frac{1 + C/2}{\sigma\alpha(1-\alpha)}.\end{aligned}$$

This completes the proof of upper bound. On the other hand, for any $t \in [0,1]$,

$$\begin{aligned}\mathbb{E}\Big[g_\alpha\Big(\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big)\Big] &= \int_0^1 F\Big(\frac{\log x}{(\alpha-1)\sigma}\Big) - F\Big(\frac{\log x}{\alpha\sigma}\Big)dx \\ &\geq \int_0^t \Phi\Big(\frac{\log x}{(\alpha-1)\sigma}\Big) - \Phi\Big(\frac{\log x}{\alpha\sigma}\Big) - \frac{2C}{\sigma}dx \\ &= \int_0^t \Phi\Big(\frac{\log x}{(\alpha-1)\sigma}\Big) - \Phi\Big(\frac{\log x}{\alpha\sigma}\Big)dx - \frac{2tC}{\sigma}.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}&\int_0^t \Phi\Big(\frac{\log x}{\sigma(\alpha-1)}\Big) - \Phi\Big(\frac{\log x}{\sigma\alpha}\Big)dx = \int_0^t \frac{1}{\sqrt{2\pi}}\int_{\frac{\log x}{\sigma\alpha}}^{\frac{\log x}{\sigma(\alpha-1)}} e^{-y^2/2}dydx \\ &= \frac{1}{\sqrt{2\pi}}\Big[\int_{-\infty}^{\frac{\log t}{\sigma\alpha}} e^{\sigma\alpha y-\frac{y^2}{2}}dy + \int_{\frac{\log t}{\sigma(\alpha-1)}}^{\infty} e^{\sigma(\alpha-1)x-\frac{y^2}{2}}dy\Big] + t\Big[\Phi\Big(\frac{\log t}{\sigma(\alpha-1)}\Big) - \Phi\Big(\frac{\log t}{\sigma\alpha}\Big)\Big].\end{aligned} \tag{20}$$

The first integral in the last line can be evaluated as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log t}{\sigma\alpha}} e^{\sigma\alpha y-\frac{y^2}{2}}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log t}{\sigma\alpha}} e^{\frac{\sigma^2\alpha^2}{2}-\frac{(y+\sigma\alpha)^2}{2}}dy = e^{\frac{\sigma^2\alpha^2}{2}}\Phi\Big(\frac{\log t}{\sigma\alpha} - \sigma\alpha\Big).$$

For the first integral, we similarly have

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{\log t}{\sigma(\alpha-1)}}^{\infty} e^{\sigma(\alpha-1)x-\frac{y^2}{2}}dy = e^{\frac{\sigma^2(1-\alpha)^2}{2}}\Phi\Big(\frac{\log t}{\sigma(1-\alpha)} - \sigma(1-\alpha)\Big).$$

Assuming $\sigma\alpha(1-\alpha) \geq \sqrt{2\pi}C \vee 2$ and letting $t = \exp[-2\sqrt{2\pi}C(1-\alpha)\alpha]$, using $\alpha(1-\alpha) \leq 1/4$, we have

$$-\frac{\log t}{\sigma\alpha} \leq \frac{2\sqrt{2\pi}C\alpha(1-\alpha)}{\sqrt{2\pi}C} \leq \frac{1}{2} \quad \text{and} \quad \sigma\alpha - \frac{\log t}{\sigma\alpha} \leq \sigma\alpha + \frac{1}{2} \leq \frac{5\sigma\alpha}{4}.$$

By (16) and the fact that the function $\frac{1}{x} - \frac{1}{x^3}$ is decreasing on $[\sqrt{3},\infty]$, we have

$$\begin{aligned}e^{\frac{\sigma^2\alpha^2}{2}}\Phi\Big(\frac{\log t}{\sigma\alpha} - \sigma\alpha\Big) &\geq \frac{1}{\sqrt{2\pi}}\exp\Big(\frac{\sigma^2\alpha^2 - \big(\frac{\log t}{\sigma\alpha} - \sigma\alpha\big)^2}{2}\Big)\Big[\frac{1}{\sigma\alpha - \frac{\log t}{\sigma\alpha}} - \frac{1}{\big(\sigma\alpha - \frac{\log t}{\sigma\alpha}\big)^3}\Big] \\ &\geq \frac{1}{\sqrt{2\pi}}\exp\Big(\log t - \frac{1}{2}\Big(\frac{\log t}{\sigma\alpha}\Big)^2\Big)\Big[\frac{4}{5\sigma\alpha} - \frac{64}{125\sigma^3\alpha^3}\Big] \\ &\geq \frac{1}{\sqrt{2\pi}}\exp(-2\sqrt{2\pi}C(1-\alpha)\alpha - 1/8)\Big[\frac{4}{5\sigma\alpha} - \frac{64}{125(4\sigma\alpha)}\Big] \\ &\geq \frac{\exp(-2\sqrt{2\pi}C)}{5\sigma\alpha}.\end{aligned}$$

Similarly, we have

$$e^{\frac{\sigma^2(1-\alpha)^2}{2}}\Phi\Big(\frac{\log t}{\sigma(1-\alpha)} - \sigma(1-\alpha)\Big) \geq \frac{\exp(-2\sqrt{2\pi}C)}{5\sigma(1-\alpha)}.$$

Hence the integral in (20) has lower bound

$$\frac{1}{\sqrt{2\pi}}\Big[\int_{-\infty}^{\frac{\log t}{\sigma\alpha}} e^{\sigma\alpha y-\frac{y^2}{2}}dy + \int_{\frac{\log t}{\sigma(\alpha-1)}}^{\infty} e^{\sigma(\alpha-1)x-\frac{y^2}{2}}dy\Big] \geq \frac{\exp(-2\sqrt{2\pi}C)}{5\sigma\alpha} + \frac{\exp(-2\sqrt{2\pi}C)}{5\sigma(1-\alpha)} = \frac{\exp(-2\sqrt{2\pi}C)}{5\sigma\alpha(1-\alpha)}. \tag{21}$$

Now we consider another term $\Phi\Big(\frac{\log t}{\sigma(\alpha-1)}\Big)$ in (20). By (18), we have

$$\begin{aligned}\Phi\Big(\frac{\log t}{\sigma(\alpha-1)}\Big) &\geq \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\Big(\frac{\log t}{\sigma(\alpha-1)} - \frac{2}{3}\Big(\frac{\log t}{\sigma(\alpha-1)}\Big)^3\Big) \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\Big(\frac{-2\sqrt{2\pi}C\alpha(1-\alpha)}{\sigma(\alpha-1)} - \frac{2}{3}\Big(\frac{-2\sqrt{2\pi}C\alpha(1-\alpha)}{\sigma(\alpha-1)}\Big)^3\Big)\end{aligned}$$

$$= \frac{1}{2} + \frac{2C\alpha}{\sigma} - \frac{32\pi C^3\alpha^3}{3\sigma^3}$$

and similarly the term $-\Phi\Big(\frac{\log t}{\sigma\alpha}\Big)$ has lower bound

$$-\Phi\Big(\frac{\log t}{\sigma\alpha}\Big) = \Phi\Big(-\frac{\log t}{\sigma\alpha}\Big) - 1 \geq \frac{1}{2} + \frac{2C(1-\alpha)}{\sigma} - \frac{32\pi C^3(1-\alpha)^3}{3\sigma^3} - 1.$$

Therefore,

$$t\Big[\Phi\Big(\frac{\log t}{(\alpha-1)\sigma}\Big) - \Phi\Big(\frac{\log t}{\alpha\sigma}\Big)\Big] - \frac{2tC}{\sigma} \geq \frac{2tC}{\sigma} - \frac{32\pi C^3\alpha^3}{3\sigma^3} - \frac{32\pi C^3(1-\alpha)^3}{3\sigma^3} - \frac{2tC}{\sigma} \geq -\frac{32\pi C^3}{3\sigma^3}.$$

Assuming $\sigma\alpha(1-\alpha) \geq 2C^{3/2}\exp(\sqrt{2\pi}C)$, then $\sigma^2\alpha^2(1-\alpha)^2 \geq 4C^3\exp(2\sqrt{2\pi}C)$, so we have

$$\frac{32\pi C^3}{3\sigma^3} \leq \frac{32\pi C^3\alpha^3(1-\alpha)^3}{3\sigma^3\alpha^3(1-\alpha)^3} \leq \frac{\pi C^3}{6\sigma^3\alpha^3(1-\alpha)^3} \leq \frac{\pi\exp(-2\sqrt{2\pi}C)}{24\sigma\alpha(1-\alpha)} \leq \frac{\exp(-2\sqrt{2\pi}C)}{6\sigma\alpha(1-\alpha)}. \tag{22}$$

We combine (21) and (22) and have,

$$\mathbb{E}\Big[g_\alpha\Big(\log\frac{\varphi_0(Y)}{\varphi_1(Y)}\Big)\Big] \geq \frac{\exp(-2\sqrt{2\pi}C)}{5\sigma\alpha(1-\alpha)} - \frac{\exp(-2\sqrt{2\pi}C)}{6\sigma\alpha(1-\alpha)} = \frac{\exp(-2\sqrt{2\pi}C)}{30\sigma\alpha(1-\alpha)}.$$

C. *Proof of Corollary 1*

It is easy to check that

$$D_\alpha(\tilde\varphi_0^{\otimes n}\|\tilde\varphi_1^{\otimes n}) = -\log\int(\tilde\varphi_0^{\otimes n})^{1-\alpha}(\tilde\varphi_0^{\otimes n})^{1-\alpha} = -\log\prod_{j=1}^n\int\tilde\varphi_0^{1-\alpha}\tilde\varphi_0^{\alpha} = -n\log\int\tilde\varphi_0^{1-\alpha}\tilde\varphi_0^{\alpha} = nD_\alpha(\tilde\varphi_0\|\tilde\varphi_1).$$

Now it is sufficient to check the conditions of Theorem 1 and apply the theorem. For fixed $\tilde\varphi_0$ and $\tilde\varphi_1$, $D''_{\alpha^*}(\tilde\varphi_0\|\tilde\varphi_1)$, $\mathbb{E}[|Z_j|^3]$ and α^* in Theorem 1 are constants, so $\alpha^*(1-\alpha^*) \asymp 1$

$$\sum_{j=1}^n\mathbb{E}[|Z_j|^3] = n\mathbb{E}[|Z_1|^3] \lesssim -nD''_{\alpha^*}(\tilde\varphi_0\|\tilde\varphi_1) = -D''_{\alpha^*}(\tilde\varphi_0^{\otimes n}\|\tilde\varphi_1^{\otimes n}).$$

Finally, $-nD''_{\alpha^*}(\tilde\varphi_0\|\tilde\varphi_1)$ is large enough as n increases. Therefore, all assumptions in Theorem 1 are satisfied and the proof is complete.

D. *Proof of* (12)

We have

$$\frac{d^2}{d\alpha^2}\log\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha d\mu = \frac{d}{d\alpha}\frac{\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha\log\frac{\varphi_1}{\varphi_0}d\mu}{\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha d\mu}$$
$$= \frac{e^{-D_\alpha(\varphi_0\|\varphi_1)}\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha\big(\log\frac{\varphi_1}{\varphi_0}\big)^2d\mu - \big(\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha\log\frac{\varphi_1}{\varphi_0}d\mu\big)^2}{e^{-2D_\alpha(\varphi_0\|\varphi_1)}}.$$

By Lemma 3, evaluating at α^*, we have $\int_\Omega\varphi_0^{1-\alpha^*}\varphi_1^{\alpha^*}\log\frac{\varphi_1}{\varphi_0}d\mu = 0$. Thus

$$\frac{d^2}{d\alpha^2}\log\int_\Omega\varphi_0^{1-\alpha}\varphi_1^\alpha d\mu\Big|_{\alpha=\alpha^*} = e^{D_{\alpha^*}(\varphi_0\|\varphi_1)}\int_\Omega\varphi_0^{1-\alpha^*}\varphi_1^{\alpha^*}\big(\log\frac{\varphi_1}{\varphi_0}\big)^2d\mu = \sum_{j=1}^n\mathbb{E}[Z_j^2].$$

By Lemma 3, $\sum_{j=1}^n\mathbb{E}[Z_j] = 0$, so $\sum_{j=1}^n\mathrm{E}[Z_j^2] = \sum_{j=1}^n\mathrm{var}[Z_j]$.

APPENDIX B
PROOF OF SECTION V

A. *Proof of Lemma 1*

With help of this representation, we have for any projection operator E_1,

$$\mathrm{Tr}(\rho_0E_1) = \mathrm{Tr}\left(\left(\sum_{i=1}^n\lambda_i|x_i\rangle\langle x_i|\right)E_1\right) = \sum_{i=1}^d\lambda_i\langle x_i|E_1x_i\rangle = \sum_{i=1}^d\lambda_i\|E_1x_i\|^2 = \sum_{i=1}^d\lambda_i\sum_{j=1}^d|\langle E_1x_i|y_j\rangle|^2.$$

where the last equality is by Parseval's identity. In the same manner,

$$\mathrm{Tr}(\rho_1(I-E_1)) = \sum_{i=1}^{d}\gamma_i\sum_{j=1}^{d}|\langle (I-E_1)x_i|y_j\rangle|^2.$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\left(\mathrm{Tr}(\rho_0E_1)+\mathrm{Tr}(\rho_1(1-E_1))\right) &= \frac{1}{2}\sum_{i,j=1}^{d}\left(\lambda_i|E_1\langle x_i|y_j\rangle|^2+\gamma_j|(I-E_1)\langle x_i|y_j\rangle|^2\right)\\ &\geq \frac{1}{2}\sum_{i,j=1}^{d}\min(\lambda_i,\gamma_j)\left(\frac{|E_1\langle x_i|y_j\rangle+(I-E_1)\langle x_i|y_j\rangle|^2}{2}\right)\\ &\geq \frac{1}{4}\sum_{i,j=1}^{d}\min\{\lambda_i,\gamma_j\}|\langle x_i|y_j\rangle|^2.\end{aligned}$$

B. Proof of Lemma 2

By independence of X_k's, we have

$$\mathbb{P}((X_1,\ldots,X_n)=((i_1,j_1),\ldots,(i_n,j_n)) = \prod_{k=1}^{n}A^{(k)}_{i_kj_k}.$$

Hence the lemma follows from the definition of tensor product.

C. Proof of Corollary 2

We observe that

$$\mathrm{Tr}((\tilde\rho_0^{\otimes n})^{1-\alpha}(\tilde\rho_1^{\otimes n})^{\alpha}) = \mathrm{Tr}((\tilde\rho_0^{1-\alpha}\tilde\rho_1^{\alpha})^{\otimes n}) = \mathrm{Tr}(\rho_0^{1-\alpha}\rho_1^{\alpha})^{n}.$$

Now it is sufficient to check the conditions of Theorem 2 . For fixed $\tilde\rho 0$ and $\tilde\rho_1$, $D''_{\alpha^*}(\tilde\rho_0\|\tilde\rho_1)$, $\mathbb{E}[|Z_j|^3]$ and α^* in Theorem 2 are constants, so $\alpha^*(1-\alpha^*)\asymp 1$

$$\sum_{j=1}^{n}\mathbb{E}[|Z_j|^3] = n\mathbb{E}[|Z_1|^3] \lesssim -nD''_{\alpha^*}(\tilde\varphi_0\|\tilde\varphi_1) = -D''_{\alpha^*}(\tilde\varphi_0^{\otimes n}\|\tilde\varphi_1^{\otimes n}).$$

Finally, $-nD''_{\alpha^*}(\tilde\varphi_0\|\tilde\varphi_1)$ is large enough as n increases. Therefore, all assumptions in Theorem 1 are satisfied and the proof is complete.