Appendix

Proof. **(Proof of Theorem 1)** *Line 2 iterates through each edge in $\mathcal{E}(\mathcal{G})$ once, therefore, the total time complexity is $O(m)$. $\mathcal{G}_q$ takes $O(n+m)$ space, therefore, the total space complexity is $O(n+m)$.*

Proof. **(Proof of Theorem 2)** *Lines 1-2 initialize the $\mathcal{A}(u)$ and $\mathcal{D}(u)$ for each vertex $u \in \mathcal{V}(\mathcal{G})$, which takes $O(n)$ time. Line 6 scans each edge in $\mathcal{E}(\mathcal{G})$ once, which takes $O(m)$ time. Therefore, the total time complexity is $O(n+m)$.*

Throughout the algorithm, we maintain $\mathcal{A}(u)$, $\mathcal{D}(u)$ and pointer in $N_{out}(u,\mathcal{G})$ (resp. $N_{in}(u,\mathcal{G})$) for each vertex $u \in \mathcal{V}(\mathcal{G})$; during the traversal, in Line 9, at most one copy of each vertex is in Q, resulting in a space complexity of $O(n)$.

Proof. **(Proof of Theorem 3)** *Lines 4 initialize $TCV_\tau(s,u)$ for each $\tau \in \mathcal{T}_{in}(u,\mathcal{G}_q)$ of each vertex $u \in \mathcal{V}(\mathcal{G}_q) \setminus \{s,t\}$, which takes $O(m)$ time. Lines 5-6 take $O(n)$ time to initialize the* completed *indicator and the pointer of each vertex $u \in \mathcal{V}(\mathcal{G}_q) \setminus \{s,t\}$. For each edge in $\mathcal{E}(\mathcal{G}_q)$, Lines 8-15 take $O(1)$ time, Lines 17 and 19 take $O(\theta)$ time as the number of vertices in each entry of $TCV.(s,\cdot)$ is bounded by $\theta-1$. Therefore, the total time complexity is $O(n+\theta\cdot m)$.*

There are $O(m)$ entries in $TCV.(s,\cdot)$ and $TCV.(\cdot,s)$, and each entry has a length bounded by $\theta-1$. Throughout the algorithm, we maintain the completed *indicator and the pointer for each vertex $u \in \mathcal{V}(\mathcal{G}_q)$. Therefore, the total space complexity is $O(n+\theta\cdot m)$.*

Proof. **(Proof of Theorem 4)** *The pointer initialization in Line 1 takes $O(n)$ time and the pointer operations in Lines 8-10 take $O(m)$ time in total. Line 3 iterates through each edge in $\mathcal{E}(\mathcal{G}_q)$ once and the intersection operation in Line 17 is performed at most m times. Each intersection operation takes $O(\theta)$ time as the length of each entry in $TCV.(s,\cdot)$ and $TCV.(\cdot,s)$ is bounded by $\theta-1$. Therefore, the total time complexity is $O(n+\theta\cdot m)$. $\mathcal{G}_t$ takes $O(n+m)$ space, therefore, the total space complexity is $O(n+m)$.*

Proof. **(Proof of Theorem 5)** *Line 2 initializes a* verified *indicator for each edge, therefore takes $O(m)$ time. Lines 3-5 can be implemented as a traversal in $\mathcal{G}_t$ from s (resp. t), which takes $O(m)$ time. For each unverified edge $e(u,v,\tau)$, the Bidirectional DFS in Line 9 has a depth of $\theta-1$ at most with a time complexity bounded by $O(d'^{\theta-1})$, Lines 11-18 takes $O(d'\cdot\theta)$ time to verify edges in the batch of paths. Therefore the total time complexity is $O(m\cdot d'^{\theta-1})$.*

$tspG$ takes $O(n+m)$ space and the verified indicator for all edges takes $O(m)$ space. The space for stacks $\mathcal{S}_v$ and $\mathcal{S}_e$ is in $O(n)$. Thus, the total space complexity is $O(n+m)$.

Proof. **(Proof of Lemma 3)** ***Sufficiency***. *If there exists a temporal simple path $p^*_{[\tau_b,\tau_e]}(s,t)$ through $e(u,v,\tau)$, there exist two temporal simple paths $p^*_{[\tau_b,\tau_i]}(s,u)$,$p^*_{[\tau_j,\tau_e]}(v,t)$ s.t. $\tau_i < \tau < \tau_j$, and $\mathcal{V}(p^*_{[\tau_b,\tau_i]}(s,u)) \cap \mathcal{V}(p^*_{[\tau_j,\tau_e]}(v,t)) = \emptyset$. Based on the definition of time-stream common vertices, we have $TCV_{\tau_i}(s,u) \subseteq \mathcal{V}(p^*_{[\tau_b,\tau_i]}(s,u))$ and $TCV_{\tau_j}(v,t) \subseteq \mathcal{V}(p^*_{[\tau_j,\tau_e]}(v,t))$, thus, $TCV_{\tau_i}(s,u) \cap TCV_{\tau_j}(v,t) = \emptyset$.*

Necessity. *Consider $e(c,f,4) \in \mathcal{E}(\mathcal{G}_q)$ in Fig. 3(c) as an counterexample. There only exist $\tau_i = 3$ and $\tau_j = 5$ satisfying $\tau_i < 4 < \tau_j$, and we have $TCV_3(s,c) \cap TCV_5(f,t) = \emptyset$ as $TCV_3(s,c) = \{b,c\}$ and $TCV_5(f,t) = \{f\}$. However, there does not exist a temporal simple path $p^*_{[2,7]}(s,t)$ through $e(c,f,4)$. Therefore, the necessity is not established.*

Proof. **(Proof of Lemma 5)** *To proof Lemma 5, we first proof $\mathcal{P}^*_{[\tau_b,\tau_l]}(s,u) = \mathcal{P}^*_{[\tau_b,\tau]}(s,u)$ by contradiction. Since $\tau_l \le \tau$, $\mathcal{P}^*_{[\tau_b,\tau_l]}(s,u) \subseteq \mathcal{P}^*_{[\tau_b,\tau]}(s,u)$. Suppose $\mathcal{P}^*_{[\tau_b,\tau_l]}(s,u) \neq \mathcal{P}^*_{[\tau_b,\tau]}(s,u)$, this implies that there exists $p^*_{[\tau_b,\tau]}(s,u) \in \mathcal{P}^*_{[\tau_b,\tau]}(s,u)$ such that $p^*_{[\tau_b,\tau]}(s,u) \notin \mathcal{P}^*_{[\tau_b,\tau_l]}(s,u)$. It is evident that such $p^*_{[\tau_b,\tau]}(s,u)$ contains an in-coming edge $e(v,u,\tau')$ of u where $\tau_l < \tau' \le \tau$, which contradicts $\tau_l = \max\{\tau_i | \tau_i \in \mathcal{T}_{in}(u,\mathcal{G}_q), \tau_i \le \tau\}$. In conclusion, $\mathcal{P}^*_{[\tau_b,\tau_l]}(s,u) = \mathcal{P}^*_{[\tau_b,\tau]}(s,u)$, and we have $TCV_\tau(s,u) = TCV_{\tau_l}(s,u)$ following the definition of the time-stream common vertices. We omit the proof for $TCV_\tau(u,t) = TCV_{\tau_r}(u,t)$ as it follows a similar approach.*

Proof. **(Proof of Lemma 6)** *For each temporal path $p \in \mathcal{P}_{[\tau_b,\tau]}(s,u)$, there exists a corresponding temporal simple path $p^* \in \mathcal{P}^*_{[\tau_b,\tau]}(s,u)$ such that the path p^* contains all edges in the path p except those forming cycles. Then, we have $\mathcal{V}(p^*) \subseteq \mathcal{V}(p)$ to derive that $\mathcal{V}(p^*) \cap \mathcal{V}(p) = \mathcal{V}(p^*)$. Thus, $TCV_\tau(s,u) = \bigcap_{p^* \in \mathcal{P}^*_{[\tau_b,\tau]}(s,u)\ s.t.\ t \notin \mathcal{V}(p^*)} \mathcal{V}(p^*) \setminus \{s\}$ $= \bigcap_{p \in \mathcal{P}_{[\tau_b,\tau]}(s,u)\ s.t.\ t \notin \mathcal{V}(p)} \mathcal{V}(p) \setminus \{s\}$. The proof for $TCV_\tau(u,t)$ follows the same approach.*

Proof. **(Proof of Lemma 8)** *If $TCV_{\tau_l}(s,u) \cap TCV_{\tau_r}(v,t) \neq \emptyset$, i.e., there exits w such that $w \in TCV_{\tau_l}(s,u)$ and $w \in TCV_{\tau_r}(v,t)$, then based on the definition of time-stream common vertices, $\forall\ p^* \in \mathcal{P}^*_{[\tau_b,\tau_l]}(s,u)$ s.t. $t \notin \mathcal{V}(p^*)$, $w \in p^*$. Since $\forall\ \tau_b \le \tau_i < \tau_l$, we have $\mathcal{P}^*_{[\tau_b,\tau_i]}(s,u) \subseteq \mathcal{P}^*_{[\tau_b,\tau_l]}(s,u)$, thus, $\forall\ p^*_i \in \mathcal{P}^*_{[\tau_b,\tau_i]}(s,u)$ s.t. $t \notin \mathcal{V}(p^*_i)$, $w \in p^*_i$, that is, $w \in TCV_{\tau_i}(s,u)$. Similarly, $\forall\ \tau_r < \tau_j \le \tau_e$, $w \in TCV_{\tau_j}(v,t)$. Therefore, $TCV_{\tau_i}(s,u) \cap TCV_{\tau_j}(v,t) \neq \emptyset$.*

Proof. **(Proof of Lemma 9)** ($\Rightarrow$) *If an edge $e(u,v,\tau) \in \mathcal{E}(\mathcal{G}_q)$, where $u \neq s$ and $v \neq t$, satisfies condition i), based on Lemma 3 and Lemma 8, such an edge will not be excluded as an unpromising edge. Therefore, it should be included in $\mathcal{G}_t$. If an edge $e(u,v,\tau) \in \mathcal{E}(\mathcal{G}_q)$, where $u = s$ or $v = t$, based on the condition ii) of Lemma 2, we can conclude that $e(u,v,\tau)$ belongs to $tspG$. Since $\mathcal{G}_t$ is the upper-bound graph of $tspG$, $e(u,v,\tau)$ must belong to $\mathcal{G}_t$.*

($\Leftarrow$) *If an edge $e(u,v,\tau)$ belongs to $\mathcal{G}_t$, there does not exist a vertex w appearing in all $p^*_{[\tau_b,\tau_i]}(s,u)$ and $p^*_{[\tau_j,\tau_e]}(v,t)$ where $\tau_i < \tau < \tau_j$. Based on the Definiton 5, it is easy to derive that $e(u,v,\tau)$ satisfies condition i) s.t. $u \neq s$ and $v \neq t$. Since $e(u,v,\tau) \in \mathcal{G}_t$ must belong to $\mathcal{G}_q$, and we have discussed $w \neq s$ and $w \neq t$ for each edge in $\mathcal{G}_q$, therefore u can be the source vertex s or v can be the target vertex t.*

Proof. **(Proof of Lemma 10)** *Let $\tau_l = \max\{\tau'' | \tau'' \in \mathcal{T}_{in}(u,\mathcal{G}_q) \wedge \tau_b \le \tau'' < \tau\}$, $\tau_r = \min\{\tau'' | \tau'' \in \mathcal{T}_{out}(v,\mathcal{G}_q) \wedge \tau < \tau'' \le \tau_e\}$. For condition i), if there exists an edge $e(s,u,\tau') \in \mathcal{E}(\mathcal{G}_t)$ such that $\tau_b \le \tau' \le \tau_l < \tau$, then we have a temporal simple* *path* $p^*_{[\tau_b,\tau_l]}(s,u) = \langle e(s,u,\tau')\rangle$, *therefore* $TCV_{\tau_l}(s,u) = \{u\}$. *Since* $e(u,v,\tau) \in \mathcal{G}_t$, *according to Lemma 9,* $TCV_{\tau_l}(s,u) \cap TCV_{\tau_r}(v,t) = \emptyset$, *that is,* $u \notin TCV_{\tau_r}(v,t)$. *Then, based on the definition of* $TCV_{\tau_r}(v,t)$, *there exists a temporal simple path* $p^*_{[\tau_r,\tau_e]}(v,t)$ *that does not pass through* s *and* u. *Therefore, a temporal simple path* $p^*_{[\tau_b,\tau_e]}(s,t)$ *that includes* $e(u,v,\tau)$ *can be formed. We omit the proof for condition* $ii)$ *as it is similar to that of condition* $i)$.