Question 1

1. Suppose that X and Y are positive, independent continuous random variables with cdfs F_X and F_Y. Show that

$$P[X < Y] = \int_0^1 F_X\left(F_Y^{-1}(t)\right) dt.$$

where F_Y^{-1} is the inverse function for the 1-1 function F_Y.

Hint: Sketch the region in the positive quadrant corresponding to the required probability. Recall that

$$F_X(x) = \int_0^x f_X(t)\, dt.$$

6 Marks

$$P[X < Y] = P[X < y \mid Y]\, P[Y]$$

$$= \int_{-\infty}^{\infty} P[X<y \mid Y=y] \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} F_{X|Y}[y \mid Y] \cdot f_Y(y)\, dy$$

Since X and Y are independent: $F_{X|Y}[x \mid y] = F_X[x]$

$$= \int_{-\infty}^{\infty} F_X(y) \cdot f_Y(y)\, dy$$

$$= \int_0^1 F_X(y) \cdot dF_Y(y)$$

$$= \int_0^1 F_X\left(F_Y^{-1}(F_Y(y))\right) dF_Y(y)$$

$$= \int_0^1 F_X\left(F_Y^{-1}[t]\right) dt$$

Question 2

2. Suppose that Z_1 and Z_2 are independent random variables each having an $Exponential(1)$ distribution. Find the joint pdf of random variables Y_1 and Y_2 defined by

$$Y_1 = \frac{Z_1}{Z_1 + Z_2} \qquad Y_2 = Z_1 + Z_2.$$

5 Marks

Are Y_1 and Y_2 independent ? Justify your answer. 1 Mark

Since $(Z_1, Z_2) \longrightarrow (Y_1, Y_2)$ is one to one :

- when $Z_1 = z_1$, $Z_2 = z_2$, there is a unique value for both Y_1 and Y_2
- when $Y_1 = y_1 = \frac{z_1}{y_2}$, $Y_2 = y_2$ there is a unique pair of corresponding value (z_1, z_2)

$\because Y_1 = \frac{Z_1}{Z_1 + Z_2} = \frac{Z_1}{Y_2} \Rightarrow Z_1 = Y_1 Y_2$

$\because Y_2 = Z_1 + Z_2 \Rightarrow Z_2 = Y_2 - Z_1 = Y_2 - Y_1 Y_2$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \cdot \det \begin{vmatrix} \frac{\partial Z_1}{\partial Y_1} & \frac{\partial Z_1}{\partial Y_2} \\ \frac{\partial Z_2}{\partial Y_1} & \frac{\partial Z_2}{\partial Y_2} \end{vmatrix}$$

$$= f_{X_1}(x_1) \cdot f_{X_2}(x) \cdot \det \begin{vmatrix} Y_2 & Y_1 \\ -Y_2 & 1 - Y_1 \end{vmatrix}$$

$$= e^{-Z_1} \cdot e^{-Z_2} \cdot (Y_2 - Y_2 Y_1 + Y_1 Y_2)$$

$$= e^{-Y_2} \cdot Y_2$$

$$f_{Y_1}(y_1) = \int_0^\infty f_{Y_1, Y_2}(y_1, y_2)\, dy_2 = \int_0^\infty e^{-Y_2} \cdot Y_2\, dy_2 = \left[-e^{-Y_2} \cdot Y_2 - \int -e^{-Y_2}\, dy_2\right]_0^\infty$$

$$= \left[-Y_2 \cdot e^{-Y_2} - e^{-Y_2}\right]_0^\infty = 1$$

- Since $Y_1 = \frac{Z_1}{Z_1 + Z_2}$, $Z_1, Z_2 \in (0, \infty)$, we know that $Y_1 \in (0, 1)$:

$$f_{Y_2}(y_2) = \int_0^1 f_{Y_1, Y_2}(y_1, y_2)\, dy_1 = \int_0^1 Y_2 \cdot e^{-Y_2}\, dy_1 = Y_2 \cdot e^{-Y_2} [Y_1]_0^1 = Y_2 \cdot e^{-Y_2}$$

- We can get that

$f_{Y_1, Y_2}(y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) \Rightarrow Y_1$ and Y_2 are independent.

Question 3

3. Consider the distribution for continuous random variable X with pdf specified via the two dimensional parameter $\theta = (\psi, \gamma)$ as

$$f_X(x;\psi,\gamma) = \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi\gamma x^3}}\exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

for $\psi, \gamma > 0$ and

(a) Is this a location-scale family distribution ? Justify your answer. 2 Marks

Let $z = ax + b$ then $x = \frac{1}{a}z - \frac{1}{a}b$

$$f_Z(z) = f_X(x) \cdot \frac{dx}{dy}$$

$$= \frac{1}{a} f_X(x)$$

$$= \frac{1}{a} \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2x\gamma}\right\}$$

$$= \frac{1}{a} \mathbb{1}_{(0,\infty)}\left(\frac{z-b}{a}\right) \sqrt{\frac{1}{2\pi\gamma(\frac{1}{a}z-\frac{b}{a})^3}} \exp\left\{-\frac{1}{2}\psi^2\gamma\left(\frac{1}{a}z-\frac{b}{a}\right) + \psi - \frac{1}{2\gamma(\frac{1}{a}z-\frac{1}{a}b)}\right\}$$

$$= \frac{1}{a} \mathbb{1}_{(b,\infty)}(z) \cdot \sqrt{\frac{1}{2\pi\gamma(\frac{z-b}{a})^3}} \exp\left\{-\frac{1}{2}\psi^2\gamma\left(\frac{z-b}{a}\right) + \psi - \frac{1}{2\gamma\frac{z-b}{a}}\right\}$$

$$= \frac{1}{a} \mathbb{1}_{(b,\infty)}(z) \cdot \sqrt{\frac{1}{2\pi\left(\frac{\sqrt[3]{\gamma}}{a}z - \frac{b}{a}\sqrt[3]{\gamma}\right)^3}} \exp\left\{-\frac{1}{2}\left(\psi\sqrt{\gamma\frac{z-b}{a}} - \frac{1}{\sqrt{\gamma\cdot\frac{z-b}{a}}}\right)\right\}$$

$$= \frac{1}{a} \mathbb{1}_{(b,\infty)}(z) \cdot \sqrt{\frac{1}{2\pi\left(\frac{\sqrt[3]{\gamma}}{a}z - \frac{b}{a}\sqrt[3]{\gamma}\right)^3}} \exp\left\{-\frac{1}{2}\left(\sqrt{\frac{\psi^2\gamma}{a}z - \frac{b}{a}\psi^2} - \frac{1}{\sqrt{\frac{\gamma}{a}z - \frac{\gamma b}{a}}}\right)\right\}$$

then this is not a location-scale family, since $f_Z(z)$ depends on parameter b and ψ.

(b) Is this an Exponential Family distribution ? Justify your answer. 2 Marks

$$f_X(x;\psi,\gamma) = \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi\gamma x^3}}\ \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

$$= \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi x^3}} * \gamma^{\frac{1}{2}}\ \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

$$= \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi x^3}} * e^{\ln\gamma^{\frac{1}{2}}} * \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

$$= \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi x^3}}\ \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x} + \ln\gamma^{\frac{1}{2}}\right\}$$

$$= h(x)\ \exp\left\{\{c(\theta)\}^T T(x) - A(\theta)\right\}$$

then

$$h(x) = \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi x^3}} \in \mathbb{R}^+ \cup \{0\}$$

$$\{c(\theta)\}^T = [-\frac{1}{2}\psi^2\gamma,\ -\frac{1}{2\gamma}] \in \mathbb{R}^3$$

$$T(x) = [x,\ \frac{1}{x}] \in \mathbb{R}^3$$

$$A(\theta) = \ln\gamma^{\frac{1}{2}} + \psi = \frac{1}{2}\ln\gamma + \psi \in \mathbb{R}$$

$\Rightarrow$ This is an Exponential family distribution.

(c) For this model, the result concerning the expected score holds, that is

$$\mathbb{E}_X[\mathbf{S}(X;\theta)] = \mathbf{0} \qquad (2 \times 1)$$

where

$$\mathbf{S}(x;\theta) = \begin{pmatrix} S_1(x;\theta) \\ S_2(x;\theta) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial \psi} \log\{f_X(x;\psi,\gamma)\} \\ \dfrac{\partial}{\partial \gamma} \log\{f_X(x;\psi,\gamma)\} \end{pmatrix}$$

Using this result, find $\mathbb{E}_X[X]$ and $\mathbb{E}_X[1/X]$ 4 Marks

$$\frac{\partial}{\partial \psi} \log\{f_X(x;\psi,\gamma)\} = \frac{\partial}{\partial \psi} \log\left\{ \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} \exp\left(-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right) \right\}$$

$$= \frac{\partial}{\partial \psi} \left[\log \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} + \left(-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right) \right\}$$

$$= \frac{\partial}{\partial \psi} \left(-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right)$$

$$= -\psi\gamma x + 1$$

$$\frac{\partial}{\partial \gamma} \log\{f_X(x;\psi,\gamma)\} = \frac{\partial}{\partial \gamma} \log\left\{ \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} \exp\left(-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right) \right\}$$

$$= \frac{\partial}{\partial \gamma} \left\{ \log \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} + \left(-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right) \right\}$$

$$= \frac{\partial}{\partial \gamma} \left\{ \log (2\pi x^3 \gamma)^{-\frac{1}{2}} + \left(-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right) \right\}$$

$$= \frac{1}{(2\pi x^3\gamma)^{-\frac{1}{2}}} * \left(-\frac{1}{2}\right) (2\pi x^3\gamma)^{-\frac{3}{2}} * 2\pi x^3 + \left(-\frac{1}{2}\psi^2 x + \frac{1}{2x\gamma^2}\right)$$

$$= (2\pi x^3\gamma)^{-1} * 2\pi x^3 * \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\psi^2 x + \frac{1}{2x\gamma^2}\right)$$

$$= \left(-\frac{1}{2\gamma}\right) - \frac{1}{2}\psi^2 x + \frac{1}{2x\gamma^2}$$

then $E[S_1(x;\theta)] = E\left[\frac{\partial}{\partial \psi} \log\{f_X(x;\psi,\gamma)\}\right]$

$$= \int_0^\infty (-\psi\gamma x + 1) f_X(x)\, dx$$

$$= -\psi\gamma E_X[X] + 1 = 0$$

$$E_X[X] = \frac{1}{\psi\gamma}$$

$$E[\,S_2(x;\theta)\,] = E\left[\frac{\partial}{\partial \gamma}\log\{f_X(x;\psi,\gamma)\}\right]$$

$$= \int_0^\infty \left(-\frac{1}{2\gamma} - \frac{1}{2}\psi^2 x + \frac{1}{2x\gamma^2}\right) f_X(x)\,dx$$

$$= -\frac{1}{2\gamma} - \frac{1}{2}\psi^2 E_X[X] + \frac{1}{2\gamma^2}E_X\left[\frac{1}{X}\right] = 0$$

$$-\frac{1}{2\gamma} - \frac{1}{2}\psi^2 \times \frac{1}{\psi\gamma} + \frac{1}{2\gamma^2}E_X\left[\frac{1}{X}\right] = 0$$

$$\frac{1}{2\gamma} + \frac{1}{2}\frac{\psi}{\gamma} = \frac{1}{2\gamma^2}E_X\left[\frac{1}{X}\right]$$

$$E_X\left[\frac{1}{X}\right] = \gamma + \psi\gamma$$