Question 1

1. Suppose $X_1, \ldots, X_n$ are independent $Normal(\mu, \sigma^2)$ rvs. Denote by $\overline{X}$ and s^2 the sample mean and sample variance statistics.

   (a) Derive the distribution of the rv
   $$T_1 = \frac{\overline{X} - \mu}{s/\sqrt{n}}$$
   You may use without proof results from handouts concerning $\overline{X}$ and s^2, but must present details of the derivation for T_1. 4 Marks

$X_1, X_2, \cdots X_n$ are i.i.d $N(\mu, \sigma^2)$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$T_1 = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{\bar{x}-\mu}{\sqrt{s^2/n}}$$

$$= \frac{\bar{x}-\mu}{\sqrt{(n-1)\,s^2\sigma^2/n(n-1)\sigma^2}}$$

$$= \frac{\bar{x}-\mu}{\sqrt{\frac{(n-1)s^2}{\sigma^2} * \frac{\sigma^2}{n(n-1)}}}$$

$$= \frac{(\bar{x}-\mu)/\sigma\sqrt{n}}{\sqrt{\frac{(n-1)s^2}{\sigma^2}/n-1}}$$

· According to the MGF of X_i : $M_{X_i}(t) = E[e^{x_i t}] = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$

Then $\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$ then $M_{\bar{y}}(t) = E[e^{(\frac{x_1}{n}+\frac{x_2}{n}+\cdots+\frac{x_n}{n})t}]$

$$= E[e^{\frac{x_1}{n}t} \cdot e^{\frac{x_2}{n}t} \cdots e^{\frac{x_n}{n}t}]$$

Since i.i.d $\longrightarrow$

$$= E[e^{\frac{x_i}{n}t} \cdot e^{\frac{x_i}{n}t} \cdots e^{\frac{x_i}{n}t}]$$

$$= \exp\{n(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2})\}$$

According to the central limit theorem, we have that :

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

For $\frac{(n-1)s^2}{\sigma^2} = \frac{\frac{n-1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2} \sim \chi^2_{n-1}$

Now let $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ and $\chi = \chi_{n-1}^2$

$$T_1 = \frac{z}{\sqrt{\chi / n-1}}$$

then let $N = n-1$

$$f_{T_1}(t) = \int_0^\infty \frac{1}{2^{N/2}\Gamma(N/2)} x^{\frac{N}{2}-1} e^{-\frac{x}{2}} * \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t^2 \frac{x}{N})} \quad dx$$

$$= \frac{1}{2^{N/2}\Gamma(N/2)\sqrt{2\pi N}} \int_0^\infty x^{\frac{N}{2}-1} e^{-\frac{1}{2}(x + \frac{t^2 x}{N})} \quad dx$$

Now let $u = \frac{x}{2}(1+\frac{t^2}{N})$ then

$$f_{T_1}(t) = \frac{(\frac{1}{2} + \frac{t^2}{2N})^{-(N+1)/2}}{2^{N/2}\Gamma(N/2)\sqrt{2\pi N}} \int_0^\infty u^{\frac{N+1}{2}-1} e^{-u} \ du$$

$$= \frac{(1+\frac{t^2}{N})^{-(N+1)/2}}{\Gamma(N/2)\sqrt{2\pi N}} \int_0^\infty u^{\frac{N+1}{2}-1} e^{-u} \ du$$

Now $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$

$$= \frac{\Gamma(N+1/2)}{\Gamma(\frac{N}{2})\sqrt{\pi N}} * \left(1+\frac{t^2}{N}\right)^{-(N+1)/2}$$

then $$f_{T_1}(t) = \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})\sqrt{\pi(n-1)}} * \left(1+\frac{t^2}{n-1}\right)^{-n/2}$$

which is a student t distribution

(b) By first considering its form for fixed finite n, derive the *limiting distribution* of s^2, that is, the probability distribution of s^2 as $n \longrightarrow \infty$. 2 Marks

Hint: consider the expectation and variance of s^2.

Let $X = \chi^2_{n-1}$ then $M_X(t) = (1-2t)^{-\frac{n-1}{2}}$

$X = \frac{(n-1)s^2}{\sigma^2}$ then $M_{s^2}(t) = E[e^{s^2 t}]$

$$= E[e^{\frac{\sigma^2 X}{n-1} t}]$$

$$= M_X\left(\frac{\sigma^2}{n-1} t\right)$$

$$= \left(1 - 2\frac{\sigma^2}{n-1} t\right)^{-\frac{n-1}{2}}$$

$$= \left(\frac{n-1-2\sigma^2 t}{n-1}\right)^{-\frac{n-1}{2}}$$

$$= \left[\frac{(n-1)/2\sigma^2 - t}{(n-1)/2\sigma^2}\right]^{-\frac{n-1}{2}}$$

$$= \left[\frac{\frac{n-1}{2\sigma^2}}{\frac{n-1}{2\sigma^2} - t}\right]^{\frac{n-1}{2}}$$

Then we have that s^2 is Gamma $\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$

Then $E[s^2] = \frac{\alpha}{\beta} = \sigma^2$ and $Var[s^2] = \frac{\alpha}{\beta^2} = \frac{n-1}{2} * \frac{4\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}$

As $n \to \infty$, we have that $Var[s^2] \to 0$, so the probability distribution becomes:

$$F_{s^2}(x) = \begin{cases} 1 & \text{if } x \geq \sigma^2 \\ 0 & \text{if } x < \sigma^2 \end{cases}$$

(c) Derive the limiting distribution of T_1 as $n \longrightarrow \infty$.

$$T_1 = \frac{\bar{X} - \mu}{S/\sqrt{n}} \quad \text{where} \quad S \rightarrow \sigma \quad \text{as } n \rightarrow \infty$$

then we have that

$$T_1 = \frac{\bar{X} - \mu}{S/\sqrt{n}} \xrightarrow{n \to \infty} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

According to the central limit theorem :

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

So the limiting distribution of T_1 is standard normal distribution

Question 2

2. Suppose that for positive integers n_1 and n_2, rvs $V_1 \sim \chi^2_{n_1}$ and $V_2 \sim \chi^2_{n_2}$ are independent.

(a) Derive using multivariate transformation techniques the distribution of

$$T_2 = \frac{V_1/n_1}{V_2/n_2}.$$

Show full details of the calculation.

Let $U = V_2$ and $T_2 = \frac{V_1/n_1}{V_2/n_2}$ then $(V_1, V_2) \longmapsto (U, T_2)$ is one to one.

$V_2 = U$ and $V_1 = \frac{T_2 V_2 n_1}{n_2}$

Then

$\frac{\partial V_1}{\partial U} = 0$ $\quad \frac{\partial V_2}{\partial U} = 1$

$\frac{\partial V_1}{\partial T_2} = \frac{V_2 n_1}{n_2}$ $\quad \frac{\partial V_2}{\partial T_2} = 0$ $\quad \Rightarrow$

$$J = \begin{vmatrix} \frac{\partial V_1}{\partial U} & \frac{\partial V_1}{\partial T_2} \\ \frac{\partial V_2}{\partial U} & \frac{\partial V_2}{\partial T_2} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{V_2 n_1}{n_2} & 0 \end{vmatrix}$$

Since V_1 and V_2 are independent then

$$f_{V_1,V_2}(v_1, v_2) = f_{V_1}(v_1) \cdot f_{V_2}(v_2)$$
$$= \frac{1}{2^{n_1/2}\Gamma(n_1/2)} V_1^{\frac{n_1}{2}-1} e^{-\frac{V_1}{2}} * \frac{1}{2^{n_2/2}\Gamma(n_2/2)} V_2^{\frac{n_2}{2}-1} e^{-\frac{V_2}{2}}$$

$$f_{U,T_2} = f_{V_1,V_2}(v_1, v_2) \cdot \det|J|$$
$$= f_{V_1,V_2}(v_1, V_2) * \frac{V_2 n_1}{n_2}$$
$$= \frac{1}{2^{n_1+n_2/2}\Gamma(n_1/2)\Gamma(n_2/2)} \left(\frac{T_2 u n_1}{n_2}\right)^{\frac{n_1}{2}-1} e^{-\frac{1}{2}\frac{T_2 u n_1}{n_2}} u^{\frac{n_2}{2}-1} e^{-\frac{u}{2}} * \frac{u n_1}{n_2}$$
$$= \frac{\left(\frac{T_2 n_1}{n_2}\right)^{\frac{n_1}{2}-1}\left(\frac{n_1}{n_2}\right)}{2^{n_1+n_2/2}\Gamma(n_1/2)\Gamma(n_2/2)} e^{-\frac{1}{2}\frac{T_2 u n_1}{n_2}} e^{-\frac{u}{2}} u^{\frac{n_1}{2}-1} u \cdot u^{\frac{n_2}{2}-1}$$

then $\quad f_{T_2}(t_2) = \int_0^\infty f_{U,T_2}(u, t_2)\, du$

$$f_{T_2}(t_2) = \int_0^\infty f_{V,T_2}(u, t_2)\, du$$

$$= \frac{t_2^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{2^{n_1+n_2/2} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty u^{\frac{n_1+n_2}{2}-1} * e^{-\frac{1}{2}u\left(1+\frac{t_2 n_1}{n_2}\right)}\, du$$

Let $\chi = \frac{1}{2}u\left(1+\frac{t_2 n_1}{n_2}\right) = \frac{u}{2}\,\frac{n_2+t_2 n_1}{n_2} \Rightarrow u = \frac{2n_2\chi}{n_2+t_2 n_1}$

$$f_{T_2}(t_2) = \frac{t_2^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{2^{n_1+n_2/2} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \cdot \frac{2n_2}{n_2+t_2 n_1} \cdot \left(\frac{2n_2}{n_2+t_2 n_1}\right)^{\frac{n_1+n_2}{2}-1} \int_0^\infty \chi^{\frac{n_1+n_2}{2}-1} e^{-\chi}\, d\chi$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cancel{2}^{-\frac{n_1+n_2}{2}} t_2^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \cdot \cancel{2}^{\frac{n_1+n_2}{2}} \cdot \left(\frac{n_2}{n_2+n_1 t_2}\right)^{\frac{n_1+n_2}{2}}$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} t_2^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \cdot \left(1+\frac{n_1}{n_2}t_2\right)^{-\frac{n_1+n_2}{2}}$$

which is the F- distribution where $T_2 \sim F(n_1, n_2)$

(b) Identify the limiting distribution (as defined in Q1) of T_2 as $n_2 \longrightarrow \infty$.

$T_2 = \frac{V_1/n_1}{V_2/n_2}$. Since V_2 is $\chi^2_{n_2}$ then let $z_1, z_2, \cdots z_n$ be i.i.d χ^2_1

then $V_2 = z_1 + z_2 + \cdots + z_n$ and $\frac{V_2}{n_2} = \frac{z_1+z_2+\cdots z_n}{n_2} = \bar{z}$

According to the weak law large number, as $n_2 \to \infty$ $\bar{z} \to E[z_i] = 1$

Then

$$T_2 = \frac{V_1/n_1}{V_2/n_2} \xrightarrow{n\to\infty} \frac{V_1/n_1}{1} = \frac{V_1}{n_1}$$

then the limiting distribution is :

$$f_{T_2}(t) = f_{V_1}(v_1)\frac{dv_1}{dT_2} = \frac{1}{2^{n_1/2}\Gamma(n_1/2)} V_1^{n_1/2-1} e^{-V_1/2} * n_1$$

$$= \frac{1}{2^{n_1/2}\Gamma(n_1/2)} (n_1 t)^{n_1/2-1} e^{-n_1 t/2} * n_1$$

and $F_{T_2}(t) = \int_0^t f_{T_2}(t)\, dt$

Question 3

3. Suppose that X is a continuous random variable with cdf

$$F_X(x) = \mathbb{1}_{(0,\infty)}(x)\left(\frac{x^2}{1+x^2}\right)^n$$

where n is a positive integer.

(a) Derive, for fixed $x \in \mathbb{R}$, $P_X[X > x]$ 1 Mark

$$P[X > x] = 1 - P[X \leq x] = 1 - F_X(x)$$

$$= 1 - \mathbb{1}_{(0,\infty)}(x)\left(\frac{x^2}{1+x^2}\right)^n$$

(b) Describe, for fixed $x \in \mathbb{R}$, the behaviour of $P_X[X > x]$ as $n \longrightarrow \infty$.

$$\because \quad P[X > x] = 1 - \mathbb{1}_{(0,\infty)}(x)\left(\frac{x^2}{1+x^2}\right)^n$$

$$\because \quad \frac{x^2}{1+x^2} < 1 \qquad \therefore \quad \left(\frac{x^2}{1+x^2}\right) \to 0 \qquad \text{for } x > 0$$

As $n \to \infty$ if $x > 0$ then $P[X \leq x] \to 0$

if $x \leq 0$ then $P[X \leq x] \to 0$

Then $F_X(x) \to 0$ as $n \to \infty$ for any fixed x

$\to$ $F_X(x)$ is no longer a distribution function but $F_X(x)$ converges to 0

so

$$P[X \geq x] = 1 - F_X(x) \quad \text{converges to} \quad 0$$

(c) Describe, for fixed $y \in \mathbb{R}$, the behaviour of $P_Y[Y > y]$ as $n \longrightarrow \infty$ if Y is the random variable defined by

$$Y = \frac{X}{\sqrt{n}}.$$

Since $Y = \frac{X}{\sqrt{n}}$

$$\begin{aligned}
P[Y \le y] &= P\left[\frac{X}{\sqrt{n}} \le y\right] \\
&= P[X \le \sqrt{n}y] \\
&= \mathbb{1}_{(0,\infty)}(y)\left(\frac{ny^2}{1+ny^2}\right)^n \\
&= \mathbb{1}_{(0,\infty)}(y)\left(1 - \frac{1}{1+ny^2}\right)^n \\
&= \mathbb{1}_{(0,\infty)}(y)\left(1 - \frac{1}{1+ny^2}\right)^{(1+ny^2)\frac{n}{1+ny^2}}
\end{aligned}$$

Then as $n \to \infty$

$$\begin{aligned}
P[Y \le y] &= \mathbb{1}_{(0,\infty)}(y)\, e^{-\frac{n}{1+ny^2}} \\
&= \mathbb{1}_{(0,\infty)}(y)\, e^{-\frac{1}{y^2}}
\end{aligned}$$

Since $\lim_{y \to \infty} P[Y \le y] = 1$ $\quad \lim_{y \to -\infty} P[Y \le y] = 0$

for $y_1 \le y_2$ $\quad P[Y \le y_1] \le P[Y \le y_2]$ and $P[Y \le y]$ is right continuity

So this is converge to a distribution function $F_Y(y) = \mathbb{1}_{(0,\infty)}(y)\, e^{-\frac{1}{y^2}}$

$\Rightarrow$

$P[Y > y]$ converges to $1 - \mathbb{1}_{(0,\infty)}(y)\, e^{-\frac{1}{y^2}}$