## Question 1

1. Suppose that $X$ has a finite mixture distribution with cdf

$$F_X(x) = \sum_{k=1}^{K} \omega_k F_k(x) \qquad x \in \mathbb{R}$$

where $K$ is a positive integer, $F_1, \ldots, F_K$ are distinct cdfs, and $\omega_1, \ldots, \omega_K$ satisfy

$$0 < \omega_k < 1 \quad \text{for all } k \qquad \sum_{k=1}^{K} \omega_k = 1.$$

Find the characteristic function (cf) for $X$ in terms of the cfs corresponding to $F_1, \ldots, F_K$. 4 Marks

Let the cfs corresponding to $F_1, F_2, \ldots F_K$ be $\varphi_1(t), \ldots \varphi_K(t)$

then

$$\varphi_X(t) = E[e^{itx}]$$

$$= \int_{-\infty}^{\infty} e^{itx} \, dF_X(x)$$

$$= \int_{-\infty}^{\infty} e^{itx} \, d\left(\sum_{k=1}^{K} w_k F_k(x)\right)$$

$$= \int_{-\infty}^{\infty} e^{itx} \frac{d\left[\sum_{k=1}^{K} w_k F_k(x)\right]}{dx} dx$$

$$= \int_{-\infty}^{\infty} e^{itx} \left[w_1 f_{x_1}(x) + \cdots w_K f_{x_K}(x)\right] dx$$

$$= \int_{-\infty}^{\infty} w_1 e^{itx} \, dF_1(x) + \cdots \int_{-\infty}^{\infty} w_K e^{itx} \, dF_K(x)$$

$$= w_1 \varphi_1(x) + w_2 \varphi_2(x) + \cdots\cdots + w_K \varphi_K(x)$$

$$= \sum_{k=1}^{K} w_k \varphi_k(x)$$

Question 2

2. A sufficient condition for a distribution defined on $\mathbb{R}$ to be (absolutely) continuous is that its cf $\varphi(t)$ satisfies

$$\int_{-\infty}^{\infty} |\varphi(t)|\, dt < \infty.$$

where $|\varphi(t)|$ is the modulus of the complex-valued quantity $\varphi(t)$. By finding a suitable counterexample, show that this is not a necessary condition for (absolute) continuity. That is, find an (absolutely) continuous distribution with cf $\varphi(t)$ for which

$$\int_{-\infty}^{\infty} |\varphi(t)|\, dt = \infty.$$

4 Marks

for chi-square $\chi_1^2$ : $\varphi_X(t) = (1-2it)^{-\frac{1}{2}}$

$$|\varphi_X(t)| = \sqrt{\varphi_X(t)\cdot\varphi_X(-t)}$$

$$= \sqrt{(1-2it)^{-\frac{1}{2}}\cdot(1+2it)^{-\frac{1}{2}}}$$

$$= \left[\frac{1}{\sqrt{1-2it}}\cdot\frac{1}{\sqrt{1+2it}}\right]^{\frac{1}{2}}$$

$$= \left(\frac{1}{\sqrt{(1-2it)(1+2it)}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{\sqrt{1+4t^2}}\right)^{\frac{1}{2}}$$

$$\int_0^\infty \left(\frac{1}{\sqrt{1+4t^2}}\right)^{\frac{1}{2}} dt > \int_0^\infty \left(\frac{1}{\sqrt{1+4t^2+2t}}\right)^{\frac{1}{2}} dt$$

$$= \int_0^\infty \left(\frac{1}{\sqrt{(2t+1)^2}}\right)^{\frac{1}{2}} dt = \int_0^\infty \left(\frac{1}{2t+1}\right)^{\frac{1}{2}} dt$$

$$> \int_0^\infty \frac{1}{2t+1}\, dt$$

$$= \left[\frac{1}{2}\log(2t+1)\right]_0^\infty = \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |\varphi_X(t)|\, dt = \infty.$$

and $\chi_1^2$ is absolutely continuous since $P(x\in A) = \int_A dF_X(x)$.

## Question 3

3. Suppose that cf $\varphi_X(t)$ takes the form

$$\varphi_X(t) = \frac{1}{2}\left(\cos(t) + \cos(\pi t)\right).$$

(a) Is the distribution of $X$ (absolutely) continuous ? Justify your answer. 2 Marks

$$\varphi_X(t) = \frac{1}{2}(\cos t + \cos(\pi t))$$

$$= \frac{1}{2}\left(\frac{e^{it}+e^{-it}}{2} + \frac{e^{i\pi t}+e^{-i\pi t}}{2}\right)$$

$$= \frac{1}{4}e^{it} + \frac{1}{4}e^{-it} + \frac{1}{4}e^{i\pi t} + \frac{1}{4}e^{-i\pi t}$$

$$(*) = \frac{1}{4}\sum_{x=-\infty}^{\infty} e^{itx} f_1(x) + \frac{1}{4}\sum_{k=-\infty}^{\infty} e^{itx} f_2(x) + \frac{1}{4}\sum_{k=-\infty}^{\infty} e^{itx} f_3(x) + \frac{1}{4}\sum_{k=-\infty}^{\infty} e^{itx} f_4(x)$$

where

$$f_1(x) = \begin{cases} 1, & x=1 \\ 0, & \text{otherwise} \end{cases} \qquad f_2(x) = \begin{cases} 1, & x=-1 \\ 0, & \text{otherwize} \end{cases}$$

$$f_3(x) = \begin{cases} 1, & x=\pi \\ 0, & \text{otherwise} \end{cases} \qquad f_4(x) = \begin{cases} 1, & x=-\pi \\ 0, & \text{othermise} \end{cases}$$

$$(*) = \frac{1}{4}\sum_{x=-\infty}^{\infty} e^{itx}\left[f_1(x)+f_2(x)+f_3(x)+f_4(x)\right]$$

$$= \sum_{x=-\infty}^{\infty} e^{itx} \times \frac{1}{4}\left[f_1(x)+f_2(x)+f_3(x)+f_4(x)\right]$$

then the pmf of $x$ : $f_X(x) = \frac{1}{4}\left[f_1(x)+f_2(x)+f_3(x)+f_4(x)\right]$

where $f_X(-\pi) = f_X(-1) = f_X(1) = f_X(\pi) = \frac{1}{4}$ and $\sum_{x=-\infty}^{\infty} f_X(x) = 1.$

$\Rightarrow$

$X$ is not absolutely continuous.

(b) Comment on the finiteness or existence of $\mathbb{E}_X[X^r]$ for $r \geq 1$.

$$\varphi_X(t) = \frac{1}{2}\left( \cos(t) + \cos(\pi t) \right)$$

then
$$\varphi_X(t) = \sum_{x=-\infty}^{\infty} e^{itx} f_X(x)$$
$$= \sum_{x=-\infty}^{\infty} \left( \sum_{j=0}^{\infty} \frac{(itx)^j}{j!} \right) f_X(x)$$
$$= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{x=-\infty}^{\infty} x^j \cdot f_X(x)$$
$$= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} E_X[X^j]$$

then
$$\frac{d^j}{dt^j} \varphi_X(t) \Big|_{t=0} = E_X[X^j] \cdot (i)^j$$
$$E_X[X^j] = \varphi_X^{(j)}(0) \cdot (i)^{-j}$$

when $j$ is even : $E_X[X^j] = -\varphi_X^{(j)}(0) \leq 1 < \infty$

$j$ is odd : $E_X[X^j] = (i)^{-j} * 0 = 0$

$\Rightarrow$ $E_X[X^r]$ always exists.

Question 4

4. Compute

(a) The first four cumulants of the $Poisson(\lambda)$ distribution.

$$M_X(t) = e^{\lambda(e^t-1)}$$

$$K_X(t) = \log M_X(t) = \lambda(e^t-1) = \lambda e^t - \lambda$$

$$K_{X1} = \frac{d}{dt}\{K_X(t)\}_{t=0} = \lambda e^t = \lambda$$

$$K_{X2} = \frac{d^2}{dt^2}\{K_X(t)\}_{t=0} = \lambda e^t = \lambda$$

$$K_{X3} = \frac{d^3}{dt^3}\{K_X(t)\}_{t=0} = \lambda e^t = \lambda$$

$$K_{X4} = \frac{d^4}{dt^4}\{K_X(t)\}_{t=4} = \lambda e^t = \lambda$$

(b) The first three cumulants of the $Normal(\mu, \sigma^2)$ distribution.

$$M_X(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

$$K_X(t) = \log M_X(t) = t\mu + \frac{1}{2}\sigma^2 t^2$$

$$K_{X1} = \frac{d}{dt}\{K_X(t)\}_{t=0} = \mu + \sigma^2 t = \mu$$

$$K_{X2} = \frac{d^2}{dt^2}\{K_X(t)\}_{t=0} = \sigma^2$$

$$K_{X3} = \frac{d^3}{dt^3}\{K_X(t)\}_{t=0} = 0$$

## Question 5

5. Consider the function

$$\varphi(t) = \frac{1}{(1+2t^2+t^4)}.$$

Assess whether this function is a valid cf, and if it is valid, describe in as much detail as possible the distribution to which it corresponds. 4 Marks

$$\varphi(t) = \frac{1}{1+2t^2+t^4} = \frac{1}{(t^2+1)^2} = \frac{1}{[(1+it)(1-it)]^2}$$

Let $X_1$, $X_2$, $X_3$ and $X_4$ be independent Exponential (1) then:

$$\varphi_{X_1}(t) = \varphi_{X_2}(t) = \varphi_{X_3}(t) = \varphi_{X_4}(t) = (1-it)^{-1}$$

$$X = X_1 + X_2 - X_3 - X_4$$

$$\varphi_X(t) = \varphi_{X_1}(t)\,\varphi_{X_2}(t)\,\varphi_{X_3}(-t)\,\varphi_{X_4}(-t)$$

$$= (1-it)^{-1}(1-it)^{-1}(1+it)^{-1}(1+it)^{-1}$$

$$= \frac{1}{(t^2+1)^2}$$

$$= \varphi(t)$$

$\Rightarrow$ this function is a valid cf.

Let $Y_1 = X_1 - X_3$ and $Y_2 = X_2 - X_4$ then

$$P[Y_1 \le y_1] = P[X_1 - X_3 \le y_1]$$

- When $y_1 \geq 0$: $P[X_1 \le X_3 + y_1 \mid X_3] \cdot P[X_3]$

$$= \int_0^\infty \int_0^{x_3+y_1} f_{X_1}(x_1)\, f_{X_3}(x_3)\, dx_1 \cdot dx_3$$

$$= [-\tfrac{1}{2}e^{-y_1} + 1]$$

- when $y_1 < 0$: $P[X_3 - X_1 \geq -y]$

$$= \int_0^\infty \int_{-y+x_1}^\infty f_{X_3}(x_3)\, f_{X_1}(x_1)\, dx_1\, dx_3$$

$$= \frac{1}{2}e^{y_1}$$

then $f_{Y_1}(y_1) = \frac{1}{2}e^{-|y_1|}$

Since $Y_1$ and $Y_2$ are i.i.d then $f_{Y_2}(y_2) = \frac{1}{2}e^{-|y_2|}$

· when $x \geqslant 0$ :

$$f_X(x) = f_{Y_1,Y_2}(y_1, y_2) = \int_{-\infty}^{\infty} f_{Y_1}(y_1)\, f_{Y_2}(x-y_1)\, dy_1$$

$$(*) = \int_{-\infty}^{\infty} \frac{1}{2}e^{-|y_1|} \cdot \frac{1}{2}e^{-|x-y_1|}\, dy_1$$

There are totally 3 siuations :

$y_1 \in (-\infty, 0)$ , $y_1 \in [0, x)$ , $y_1 \in [x, \infty)$

then

$$(*) = \int_{-\infty}^{0} \frac{1}{2}e^{y_1} \cdot \frac{1}{2}e^{y_1-x}\, dy_1 + \int_0^x \frac{1}{2}e^{-y_1} \cdot \frac{1}{2}e^{y_1-x}\, dy_1 + \int_x^{\infty} \frac{1}{2}e^{-y_1} \cdot \frac{1}{2}e^{x-y_1}\, dy_1$$

$$= \int_{-\infty}^{0} \frac{1}{4}e^{2y_1-x}\, dy_1 + \int_0^x \frac{1}{4}e^{-x}\, dy_1 + \int_x^{\infty} \frac{1}{4}e^{x-2y_1}\, dy_1$$

$$= \left[\frac{1}{8}e^{2y_1-x}\right]_{-\infty}^{0} + \left[\frac{1}{4}e^{-x}y_1\right]_0^x + \left[-\frac{1}{8}e^{x-2y_1}\right]_x^{\infty}$$

$$= \frac{1}{8}\left[e^{-x}\right] + \frac{1}{4}e^{-x} \cdot x - \frac{1}{8}\left[-e^{-x}\right]$$

$$= \frac{1}{4}e^{-x} + \frac{1}{4}xe^{-x}$$

· when $x < 0$ :

$$f_X(x) = f_{Y_1,Y_2}(y_1, y_2) = \int_{-\infty}^{\infty} f_{Y_1}(y_1)\, f_{Y_2}(x-y_1)\, dy_1$$

$$(**) = \int_{-\infty}^{\infty} \frac{1}{2}e^{-|y_1|} \cdot \frac{1}{2}e^{-|x-y_1|}\, dy_1$$

There are totally 3 situations :

$y_1 \in (-\infty, x)$ ; $y_1 \in [x, 0)$ $y_1 \in [0, \infty)$ then

$$(**) = \int_{-\infty}^{x} \frac{1}{2}e^{y_1} \cdot \frac{1}{2}e^{y_1-x}\, dy_1 + \int_x^0 \frac{1}{2}e^{y_1} \cdot \frac{1}{2}e^{x-y_1}\, dy_1 + \int_0^{\infty} \frac{1}{2}e^{-y_1} \cdot \frac{1}{2}e^{x-y_1}\, dy_1$$

$$= \int_{-\infty}^{x} \frac{1}{4}e^{2y_1-x}\, dy_1 + \int_x^0 \frac{1}{4}e^{x}\, dy_1 + \int_0^{\infty} \frac{1}{4}e^{x-2y_1}\, dy_1$$

$$= \frac{1}{4}\left[\frac{1}{2}e^{2y_1-x}\right]_{-\infty}^{x} + \frac{1}{4}\left[e^{x}y_1\right]_x^0 + \frac{1}{4}\left[-\frac{1}{2}e^{x-2y_1}\right]_0^{\infty}$$

$$= \frac{1}{8}\left[e^{x}\right] + \frac{1}{4}\left(-xe^{x}\right] - \frac{1}{8}\left[-e^{x}\right]$$

$$= \frac{1}{8}e^{x} - \frac{1}{4}xe^{x} + \frac{1}{8}e^{x}$$

$$= \frac{1}{4}e^{x} - \frac{1}{4}xe^{x}$$

- $$f_X(x) = \begin{cases} \frac{1}{4}e^{x} - \frac{1}{4}xe^{x} & , x<0 \\ \frac{1}{4}e^{-x} + \frac{1}{4}xe^{-x} & , x \geq 0 \end{cases}$$

— when $X < 0$ :

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{4}e^{x} - \frac{1}{4}xe^{x} \ dx$$

$$= \left[\frac{1}{4}e^{x} - \frac{1}{4}e^{x}(x-1)\right]_{-\infty}^{x}$$

$$= \frac{1}{4}e^{x} - \frac{1}{4}e^{x}(x-1)$$

$$= \frac{1}{4}e^{x} - \frac{1}{4}e^{x}\cdot x + \frac{1}{4}e^{x} = \frac{1}{2}e^{x} - \frac{1}{4}x\cdot e^{x}$$

— when $x > 0$ :

$$F_X(x) = \int_{-\infty}^{0} \frac{1}{4}e^{x} - \frac{1}{4}xe^{x} \ dx + \int_{0}^{x} \frac{1}{4}e^{-x} + \frac{1}{4}xe^{-x} \ dx$$

$$= \frac{1}{2} + \left[-\frac{1}{4}e^{-x} - \frac{1}{4}e^{-x}(x+1)\right]_{0}^{x}$$

$$= \frac{1}{2} + \left[-\frac{1}{4}e^{-x} - \frac{1}{4}e^{-x}(x+1) + \frac{1}{4} + \frac{1}{4}\right]$$

$$= 1 - \frac{1}{4}e^{-x} - \frac{1}{4}xe^{-x} - \frac{1}{4}e^{-x}$$

$$= 1 - \frac{1}{4}x\cdot e^{-x} - \frac{1}{2}e^{-x}$$