Assignment 3

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MATH 447

Question 1

4.19 Let $T = \min\{n : Z_n = 0\}$ be the time of extinction for a branching process. Show that $P(T = n) = G_n(0) - G_{n-1}(0)$, for $n \geq 1$.

$$G_n(0) = \sum_{k=0}^{\infty} 0^k * Pr(Z_n = k)$$

$$= 0^0 \cdot Pr(Z_n = 0) = Pr(Z_n = 0)$$

$$G_{n-1}(0) = 0^0 \cdot Pr(Z_{n-1} = 0) = Pr(Z_{n-1} = 0)$$

$$P(T = n) = Pr(Z_n = 0, Z_{n-1} > 0)$$

$$\Rightarrow G_n(0) - G_{n-1}(0) = Pr(Z_n = 0) - Pr(Z_{n-1} = 0)$$

$$= Pr(Z_n = 0 \mid Z_{n-1} = 0) Pr(Z_{n-1} = 0) + Pr(Z_n = 0 \mid Z_{n-1} > 0) Pr(Z_{n-1} > 0) - Pr(Z_{n-1} = 0)$$

$$= 1 * Pr(Z_{n-1} = 0) + Pr(Z_n = 0 \mid Z_{n-1} > 0) Pr(Z_{n-1} > 0) - Pr(Z_{n-1} = 0)$$

$$= Pr(Z_n = 0 \mid Z_{n-1} > 0) Pr(Z_{n-1} > 0)$$

$$= Pr(Z_n = 0, Z_{n-1} > 0) = P(T = n)$$

Question 2

4.25 *Total progeny, continued*. Let $\phi_n(s) = E\left(s^{T_n}\right)$ be the probability generating function of T_n, as defined in Exercise 4.24.

(a) Show that ϕ_n satisfies the recurrence relation

$$\phi_n(s) = sG(\phi_{n-1}(s)), \quad \text{for } n = 1, 2, \ldots,$$

$SE[(Y_i)^{Z_1}])$

where $G(s)$ is the pgf of the offspring distribution. Hint: Condition on Z_1 and use Exercise 4.16(a).

$SG(Y_i)$

(b) From (a), argue that

$E[S^T] = S\,G(E[S^T])$ $\quad E[S^{T_{n-1}}]$

$E[S^{T_{n-1}}] \overset{\lim_{n\to\infty}}{=} E[S^{Y_i}]$

$$\phi(s) = sG(\phi(s)),$$

E $\quad E[S^{T_{n-1}+Z_n}]$

$Z_{n-1}]$ $\quad 0 < \mu < 1 \quad \mu > 1 \quad E[S^{-}\ \mu Z_{n-1}$

where $\phi(s)$ is the pgf of the total progeny T. $\quad E[X_i] > 1 \quad = SG(E[S^{T_{n-1}}])$

(c) Use (b) to find the mean of T in the subcritical case.

(a) $T_n = Z_0 + Z_1 + Z_2 + \cdots + Z_n$

Assume that $Z_1 = z_1$ and each individual in z_1 has total progeny $= Y_i$

then we have

$T_{n-1} = Y_i$ Since each individual in z_1 can be considered as the beginning of a brunching process

then $\phi_n(s) = E[S^{T_n}]$

$= E[E[S^{Z_0+Z_1+\cdots Z_n} \mid Z_1 = z_1]]$

$= S^{Z_0} E[E[S^{Z_1+Z_2+\cdots Z_n} \mid Z_1 = z_1]]$

$= S\, E[E(S^{Y_1+Y_2+\cdots+Y_{z_1}})]$

Since all the children in Z_1 is independent of each other

$\Rightarrow\ = S * E[E(S^{T_{n-1}} * S^{T_{n-1}} \cdots * S^{T_{n-1}})]$

$= S * E_{Z_1}[(E[S^{T_{n-1}}])^{Z_1}]$

$= S * E_{Z_1}[(\phi_{n-1}(s))^{Z_1}]$

$= S * G[\phi_{n-1}(s)]$

(b) Since $T = \lim_{n\to\infty} T_n = \lim_{n\to\infty}(Z_0 + Z_1 + \cdots + Z_n)$

$\phi(s) = \lim_{n\to\infty} \phi_n(s)$

$= \lim_{n\to\infty} S * E[E(S^{Y_1} * S^{Y_2} \cdots * S^{Y_n})]$

$= \lim_{n\to\infty} S * E[E(S^{T_{n-1}} * S^{T_{n-1}} \cdots * S^{T_{n-1}})]$

As $n\to\infty$ $\because$ Y_i is the total progeny starting from Z_1

As $n \to \infty$, $\because Y_i$ is the total progeny starting from z_1 then

$$Y_i = T_{n-1}$$

$$T = \lim_{n\to\infty} Y_i = \lim_{n\to\infty} T_{n-1}$$

$$\Rightarrow \phi(s) = s\,E(E[s^{T_{n-1}} * \cdots s^{T_{n-1}}])$$
$$= sE(E[s^T * \cdots\cdots s^T])$$
$$= sE_{z_1}((E[s^T])^{z_1})$$
$$= s\,G_X(\phi(s))$$

(c) Let $\mu = E[X_i] < 1$ & $\phi(s) = sG(\phi(s))$

$\Rightarrow \phi(s) = E[s^T] = sG(E[s^T])$

$$E[T] = G_T'(1) = \phi'(s)\big|_{s=1}$$

$$\because \quad \phi(s) = sG(\phi(s))$$
$$\phi'(s) = G'(\phi(s)) + sG(\phi(s))$$
$$\phi'(s)\big|_{s=1} = G'(\phi(s))\big|_{s=1} + sG(\phi(s))\big|_{s=1}$$
$$\underbrace{\phi'(s)\big|_{s=1} = \sum_{x_i=0}^{\infty} P_r(X=x_i) * x_i * [\phi(s)]^{x_i-1} * \phi'(s)\big|_{s=1} + sG(\phi(s))\big|_{s=1}} \quad (*)$$

$\because \phi(s) = E[s^T] = \sum_{k=0}^{\infty} s^k * P_r(T=k)$ when $s=1$

$$\Rightarrow \phi(1) = \sum_{k=0}^{\infty} P_r(T=k) = 1$$

$$\Rightarrow (*): \quad E[T] = \sum_{x_i=0}^{\infty} P_r(X=x_i) * x_i * E[T] + \sum_{x_i=0}^{\infty} [\phi(s)]^{x_i} P_r(X=x_i)$$
$$E[T] = E[T] * E[X] + 1$$
$$E[T] = E[T]\mu + 1$$
$$E[T](1-\mu) = 1$$
$$E[T] = \frac{1}{1-\mu}$$

Question 3

4.26 In a lottery game, three winning numbers are chosen uniformly at random from $\{1, \ldots, 100\}$, sampling without replacement. Lottery tickets cost \$1 and allow a player to pick three numbers. If a player matches the three winning numbers they win the jackpot prize of \$1,000. For matching exactly two numbers, they win \$15. For matching exactly one number they win \$3.

(a) Find the distribution of net winnings for a random lottery ticket. Show that the expected value of the game is −70.8 cents.

(b) *Parlaying* bets in a lottery game occurs when the winnings on a lottery ticket are used to buy tickets for future games. Hoppe (2007) analyzes the effect of parlaying bets on several lottery games. Assume that if a player matches either one or two numbers they parlay their bets, buying respectively 3 or 15 tickets for the next game. The number of tickets obtained by parlaying can be considered a branching process. Find the mean of the offspring distribution and show that the process is subcritical.

$Pr(X_i = 1000) = \frac{r}{161700}$

$Pr(X_i \neq 1000)$ $\frac{\cancel{167}}{161700}$

(c) See Exercise 4.19. Let T denote the duration of the process, that is, the length of the parlay. Find $P(T = k)$, for $k = 1, \ldots, 4$.

$\frac{p}{1-m}$

(d) Hoppe shows that the probability that a single parlayed ticket will ultimately win the jackpot is approximately $p/(1 - m)$, where p is the probability that a single ticket wins the jackpot, and m is the mean of the offspring distribution of the associated branching process. Find this probability and show that the parlaying strategy increases the probability that a ticket will ultimately win the jackpot by slightly over 40%.

(a) let X be the net winnings then

$$P(X = -1) = \frac{\binom{97}{3}}{\binom{100}{3}} = \frac{147440}{161700}$$

$$P(X = -1+3) = P(X=2) = \frac{\binom{3}{1}\binom{97}{2}}{\binom{100}{3}} = \frac{13968}{161700}$$

$$P(X = -1+15) = P(X=14) = \frac{\binom{3}{2}\binom{97}{1}}{\binom{100}{3}} = \frac{291}{161700}$$

$$P(X = -1+1000) = P(X=999) = \frac{\binom{3}{3}}{\binom{100}{3}} = \frac{1}{161700}$$

$$E[X] = -1 * P(X=-1) + 2 * P(X=2) + 14 * P(X=14) + 999 * P(X=999)$$

$$= \frac{-147440}{161700} + \frac{2 \times 13968}{161700} + \frac{14 \times 291}{161700} + \frac{999}{161700}$$

$$= -0.708 = -70.8 \text{ cents}$$

(b) Let X be number of ticket by parlaying

Let Y be the money we gained after a single lottery game

then $Pr(Y=3) = Pr(\text{matching exactly one number}) = \frac{13968}{161700} = P(X=3)$

$Pr(Y=15) = \frac{291}{161700} = P(X=15)$

$Pr(Y=1000) = \frac{1}{161700}$ (when $Y=1000$ we will end the parlay process $\Rightarrow$

$Pr(Y=0) = \frac{147440}{161700}$

$\therefore P(X=0) = \frac{147441}{161700}$

$\Rightarrow E[X] = \frac{291}{161700} * 15 + \frac{13968}{161700} * 3 + \frac{147441}{161700} \times 0$

$= 0.286 < 1$,

then we have $\lim_{n\to\infty} E[Z_n] = (0.286)^n = 0$ subcritial.

(c) Firstly, $Z_0 = 1$

$Pr(T=1) = Pr(Z_1 = 0, Z_0 > 0) = \frac{147441}{161700}$

$G_X(S) = E[S^{X_i}] = S^0 * \frac{147441}{161700} + S^3 * \frac{13968}{161700} + S^{15} \times \frac{291}{161700}$

$= \frac{147441 + 13968 S^3 + 291 S^{15}}{161700}$

$G_X(G_X(S)) = \frac{147441}{161700} + \left[\frac{147441 + 13968 s^3 + 291 s^{15}}{161700}\right]^3 * \frac{13968}{161700} + \left[\frac{147441 + 13968 s^3 + 291 s^{15}}{161700}\right]^{15} * \frac{291}{161700}$

$\Rightarrow G_X(0) = \frac{147441}{161700}$ $\quad G_X(G_X(0)) = G_2(0) = \frac{147441}{161700} + \left[\frac{147441}{161700}\right]^3 * \frac{13968}{161700} + \left[\frac{147441}{161700}\right]^{15} * \frac{291}{161700}$

$= 0.0654847 + 0.911818 + 0.00045057$

$= 0.977753$

then $G_3(S) = G_X[G_X(G_X(S))]$

$= \frac{147441 + 13968 [G_X(G_X(S))]^3 + 291 * [G_X(G_X(S))]^{15}}{161700}$

$G_3(0) = \frac{147441 + 13968 \times [0.9777]^3 + 291 \times [0.9777]^{15}}{161700}$

$= \frac{147441 + 13054.22434 + 207.480466}{161700}$

$= 0.993832$

$$G_4(s) = G_x(G_x(G_x(G_x(s))))$$

$$= G_x(G_x[G_2(s)])$$

$$= \frac{147441}{161700} + \left[\frac{147441 + 13968\, G_x(G_x(s))^3 + 291\, G_x(G_x(s))^{15}}{161700}\right]^3 \times \frac{13968}{161700}$$

$$+ \frac{291}{161700}\left[\frac{147441 + 13968\, G_x(G_x(s))^3 + 291\, G_x(G_x(s))^{15}}{161700}\right]^{15}$$

$$\Rightarrow \left[\frac{147441 + 13968\, G_x(G_x(0))^3 + 291\, G_x(G_x(0))^{15}}{161700}\right]$$

$$= \left[\frac{147441 + 13968 \times (0.9938)^3 + 291 * (0.9938)^{15}}{161700}\right]$$

$$= 0.99824$$

$$\Rightarrow G_4(0) = \frac{147441}{161700} + (0.99824)^3 \times \frac{13968}{161700} + (0.99824)^{15} \times \frac{291}{161700}$$

$$= 161618.7902 / 161700$$

$$= 0.9994978$$

$\because Pr(T=n) = G_n(0) - G_{n-1}(0)$

$Pr(T=1) = \frac{147441}{161700} = 0.9118$

$Pr(T=2) = G_2(0) - G_1(0)$

$= 0.977753 - 0.9118 = 0.06595$

$Pr(T=3) = G_3(0) - G_2(0) = 0.993832 - 0.977753 = 0.016079$

$Pr(T=4) = G_4(0) - G_3(0) = 0.0056658$

(d) Let $A = \{$a single parlayed ticket will ultimately win the jackpot$\}$

$\because P = \frac{1}{161700}$ $\quad m = E[X_i] = 0.286$ $\quad$ then $P(A) = \frac{P}{1-m} = \frac{\frac{1}{161700}}{1-0.286} = \frac{1}{0.714 \times 161700}$

$\Rightarrow$ then $\frac{P(A)}{P} - 1 = \frac{1}{0.714 \times 161700} / \frac{1}{161700} - 1$

$= \frac{1}{0.714} - 1$

$\approx 40.05\,\%$

Question 4

5.7 Exhibit a Metropolis–Hastings algorithm to sample from a binomial distribution with parameters n and p. Use a proposal distribution that is uniform on $\{0, 1, \ldots, n\}$.

For the binomial distribution we have that $P = \binom{n}{i} p^i (1-p)^{n-i}$

then

$$\pi_i = \binom{n}{i} p^i (1-p)^{n-i}$$

& the proposal distribution is uniform on $\{0, 1, \ldots\ldots n\}$

$$\Rightarrow T_{ij} = \frac{1}{n+1}$$

Metropolis – Hastings algorithm.

① T & π has the same state space

$\Rightarrow$ state space of T is $S = \{0, 1, 2, \ldots\ldots n\}$

② Choose any starting state for X_0.

For $k = 1, 2, \ldots\ldots n$

③ Propose to move from $X_{k-1} = i$ to $X_k = j$ according to T. (with $T_{ij} = \frac{1}{n+1}$)

④ $$a(i,j) = \min\left(1, \frac{\pi_j}{\pi_i} \times \frac{T_{ji}}{T_{ij}}\right)$$

$$= \min\left(1, \frac{\binom{n}{j} p^j (1-p)^{n-j}}{\binom{n}{i} p^i (1-p)^{n-i}} \times \frac{\frac{1}{n+1}}{\frac{1}{n+1}}\right)$$

$$= \min\left(1, \frac{\binom{n}{j} p^j (1-p)^{n-j}}{\binom{n}{i} p^i (1-p)^{n-i}}\right)$$

then

$$P_{ij} = T_{ij} * a(i,j)$$

$$= \frac{1}{n+1} * \min\left\{1, \frac{\binom{n}{j} p^j (1-p)^{n-j}}{\binom{n}{i} p^i (1-p)^{n-i}}\right\}$$

$$= \frac{1}{n+1} * \min\left\{1, \frac{\binom{n}{j}}{\binom{n}{i}} \left(\frac{p}{1-p}\right)^{j-i}\right\}$$

$$= \min\left\{\frac{1}{n+1}, \frac{\binom{n}{j}}{(n+1)\binom{n}{i}} \left(\frac{p}{1-p}\right)^{j-i}\right\}$$

Repeat for $n+1$

Question 5

5.20 R : Consider a Poisson distribution with parameter $\lambda = 3$ conditioned to be nonzero. Implement an MCMC algorithm to simulate from this distribution, using a proposal distribution that is geometric with parameter $p = 1/3$. Use your simulation to estimate the mean and variance.

A poisson distribution conditioned to be non-zero :

$$\because P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$P(X=k \mid X\neq 0) = \frac{P(X=k,\ X\neq 0)}{P(X\neq 0)} = \frac{e^{-\lambda}\frac{\lambda^k}{k!}}{1-P(X=0)}$$

$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!}}{1-e^{-\lambda}} \quad \text{for } x>0$$

$$\Rightarrow \pi_i = \frac{e^{-3}3^i/i!}{1-e^{-3}} = \frac{3^i}{(e^3-1)\,i!}$$

the proposal distribution is Geometric distribution with $p=\frac{1}{3}$

then

$$P(X=k) = (1-p)^k p$$

$\because$ Conditioned on $X\neq 0$

$$P(X=k \mid X\neq 0) = \frac{P(X=k, X\neq 0)}{P(X\neq 0)} \quad \text{for } x>0$$

$$= \frac{(1-p)^k p}{1-P(X=0)} = \frac{(1-p)^k p}{1-(1-p)^0 * p} = \frac{p(1-p)^k}{1-p} = p(1-p)^{k-1}$$

Let $f(X=k) = p(1-p)^{k-1}$

then $T_{ij} = p(1-p)^{j-1}$

then
$$a(i,j) = \min\left\{1, \frac{\pi_j}{\pi_i} * \frac{T_{ji}}{T_{ij}}\right\}$$

$$= \min\left\{1, \frac{e^{-\lambda}\frac{\lambda^j}{j!}/(1-e^{-\lambda})\ p(1-p)^{i-1}}{e^{-\lambda}\frac{\lambda^i}{i!}/(1-e^{-\lambda})\ p(1-p)^{j-1}}\right\}$$

$$= \min\left\{1, \frac{e^{-\lambda}\frac{\lambda^j}{j!}}{e^{-\lambda}\frac{\lambda^i}{i!}} * \frac{p(1-p)^{i-1}}{p(1-p)^{j-1}}\right\}$$

$$= \min\left\{1, \frac{\text{dpois}(\lambda,j)}{\text{dpois}(\lambda,i)} * \frac{p(1-p)^i}{p(1-p)^j}\right\}$$

$$= \min\left\{1, \frac{\text{dpois}(\lambda,j)}{\text{dpois}(\lambda,i)} * \frac{\text{dgeom}(p,i)}{\text{dgeom}(p,j)}\right\}$$

5.20-code

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```
N<-50000
p<-1/3
lamda<- 3

X<-rep(0,N)
X[1]<-0
for(istep in 2:N){
#the current state
i<-X[istep-1]

#sample from proposal density
#since we want to generate from p(i-p)^(i-1)
#the proposed state
j<-rgeom(1,p) +1

#calculate the acceptance probability
#al <- (lamda^j)*factorial(i) *((1-p)^(j-i))/((lamda^i) * factorial(j))
accept<-min(1,(dpois(j,lamda)*dgeom(i,p))/(dpois(i,lamda)*dgeom(j,p)))
dgeom(i,p)

u<-runif(1)
if(u < accept){
        X[istep]<-j
    }else{
        X[istep]<-i
    }
}

mean(X)
```

```
## [1] 3.16622
```

```
var(X)
```

```
## [1] 2.682285
```