MATH 447 A4

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Question 1.

6.12 Starting at noon, diners arrive at a restaurant according to a Poisson process at the rate of five customers per minute. The time each customer spends eating at the restaurant has an exponential distribution with mean 40 minutes, independent of other customers and independent of arrival times. Find the distribution, as well as the mean and variance, of the number of diners in the restaurant at 2 p.m. $= 120$ min

Let N_t be the number of diners arrived at restaurant in time t

B_t be the number of diners in restaurant at time t

then

$$P(B_t = k) = \sum_{n=k}^{\infty} P(B_t = k \mid N_t = n)\, P(N_t = n) = \sum_{n=k}^{\infty} P(B_t = k \mid N_t = n) * \frac{e^{-5t}(5t)^n}{n!}$$

Let $S_1, \dots, S_n$ be the arrival times of n dinners & $Z_1 \dots Z_n$ be the dining time of n customers

then the leaving time customers are : $S_1 + Z_1,\ S_2 + Z_2, \dots, S_n + Z_n$

then $\quad P(B_t = k \mid N_t = n)$

$= P(k$ of the diners departures time $> t \mid N_t = n)$

$= P(k$ of $S_1 + Z_1,\ S_2 + Z_2, \dots, S_n + Z_n$ exceed $t)$

$$= \binom{n}{k} [P(S_i + Z_i > t)]^k * [1 - P(S_i + Z_i > t)]^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$\Rightarrow \quad P(B_t = k)$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-5t}(5t)^n}{n!}$$

$$= \frac{p^k (5t)^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}(5t)^{n-k}}{(n-k)!}$$

$$= \frac{p^k (5t)^k}{k!} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (5t)^n}{n!}$$

$$= \frac{(5pt)^k}{k!} e^{-5pt}$$

where $p = P(S_i + Z_i > t) = \frac{1}{t}\int_0^t P(Z_i > t - x)\,dx = \frac{1}{120}\int_0^{120} P(Z_i > 120 - x)\,dx = \frac{1}{120}\int_0^{120} e^{-\frac{1}{40}(120 - x)}\,dx$

$$= \frac{1}{120}\left[e^{-3} \cdot 40 e^{\frac{x}{40}}\right]_0^{120} = \frac{1}{120}\left[40e^{-3} \cdot e^3 - e^{-3} 40\right]$$

$$= \frac{1}{3}\left[1 - e^{-3}\right]$$

$\Rightarrow \quad E[B_K] = Var[B_K] = 5pt$

$$= 5 * \tfrac{1}{3}(1 - e^{-3}) * 120 = 200\,(1 - e^{-3})$$

Question 2

(6.15) Failures occur for a mechanical process according to a Poisson process. Failures are classified as either major or minor. Major failures occur at the rate of 1.5 failures per hour. Minor failures occur at the rate of 3.0 failures per hour.

(a) Find the probability that two failures occur in 1 hour.

(b) Find the probability that in half an hour, no major failures occur.

(c) Find the probability that in 2 hours, at least two major failures occur or at least two minor failures occur.

(a) A Major failure : $\lambda_1 = 3$; B minor failure : $\lambda_2 = 1.5$; N_t = total number of failure in time t

$Pr(N_1 = 2)$

$= Pr(A=2, B=0) + Pr(A=1, B=1) + Pr(A=0, B=2)$

$= e^{-\lambda_1}\frac{\lambda_1^2}{2!} * e^{-\lambda_2}\frac{\lambda_2^0}{0!} + e^{-\lambda_1}\frac{\lambda_1}{1!} * e^{-\lambda_2}\frac{\lambda_2^1}{1!} + e^{-\lambda_1}\frac{\lambda_1^0}{0!} + e^{-\lambda_2}\frac{\lambda_2^2}{2!}$

$= e^{-3}\frac{3^2}{2} * e^{-1.5} + 3e^{-3} * e^{-1.5} * 1.5 + e^{-3} * e^{-1.5}\frac{1.5^2}{2}$

$= (\frac{9}{2} + 4.5 + \frac{2.25}{2}) * e^{-4.5}$

$= 10.125 * e^{-4.5} = 0.1125$

(b) $Pr(\text{Major failure} = 0, t = 0.5)$

$= e^{-\lambda_1 * 0.5}\frac{(0.5\lambda_1)^0}{0!}$

$= e^{-0.75}\frac{(1.5 \times 0.5)^0}{0!} = e^{-0.75}$

(c) $Pr(\text{Major failure} > 2 \cup \text{Minor failure} \geq 2, t = 2)$

$= Pr(\text{Major failure} \geq 2) + Pr(\text{Minor failure} \geq 2) - Pr(\text{Major failure} \geq 2, \text{Minor failure} \geq 2)$

$= [1 - Pr(\text{Major}=0) - Pr(\text{Major}=1)] + [1 - Pr(\text{Minor}=0) - Pr(\text{Minor}=1)] - Pr(\text{Major failure} \geq 2)\, Pr(\text{Minor failure} \geq 2)$

$= [1 - e^{-2\lambda_1} - e^{-2\lambda_1}\frac{(2\lambda_1)^1}{1!}] + [1 - e^{-2\lambda_2} - e^{-2\lambda_2}\frac{(2\lambda_2)^1}{1!}] - Pr(\text{Major} \geq 2)\, Pr(\text{Minor} \geq 2)$

$= [1 - e^{-6} - e^{-6} * 6] + [1 - e^{-3} - 3e^{-3}] - Pr(\text{Major} \geq 2)\, Pr(\text{Minor} \geq 2)$

$= (1 - 7e^{-6}) + (1 - 4e^{-3}) - (1 - 7e^{-6})(1 - 4e^{-3})$

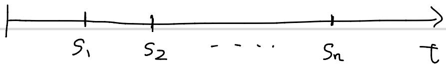
$= 2 - 7e^{-6} - 4e^{-3} - (1 - 4e^{-3} - 7e^{-6} + 28e^{-9})$

$= 1 - 28e^{-9}$

$= 0.9965$

Question 3

6.33 Let $S_1, S_2, \ldots$ be the arrival times of a Poisson process with parameter λ. Given the time of the nth arrival, find the expected time $E(S_1|S_n)$ of the first arrival.



$$Pr(S_1 = x_1 | S_n = x_n) = \frac{Pr(S_1 = x_1,\ S_n = x_n)}{Pr(S_n = x_n)} = \frac{Pr(S_n = x_n | S_1 = x_1)\ Pr(S_1 = x_1)}{Pr(S_n = x_n)} = \frac{Pr(S_{n-1} = x_n - x_1)\ Pr(S_1 = x_1)}{Pr(S_n = x_n)} \Rightarrow$$

$$Pr(S_1 = x_1 | S_n = x_n) = \frac{f_{S_{n-1}}(x_n - x_1)\ f_{S_1}(x_1)}{f_{S_n}(x_n)}$$

$$= \frac{\dfrac{\cancel{\lambda^{n-1}}(x_n - x_1)^{n-2}\cancel{e^{-\lambda(x_n - x_1)}}}{(n-1)!} * \dfrac{\cancel{\lambda^1} * (x_1)^0 * \cancel{e^{-\lambda x_1}}}{1!}}{\dfrac{\cancel{\lambda^n}\, x_n^{n-1}\cancel{e^{-\lambda x_n}}}{n!}}$$

$$= \frac{(x_n - x_1)^{n-2}}{(n-1)!} * \frac{n!}{x_n^{n-1}}$$

$$= \frac{n(x_n - x_1)^{n-2}}{x_n^{n-1}}$$

then $E[S_1 | S_n = x_n]$

$$= \sum_{k=0}^{x_n} k * Pr(S_1 = k | S_n = x_n)$$

$$= \int_0^{x_n} k * \frac{n(x_n - k)^{n-2}}{x_n^{n-1}}\, dk$$

$$= \frac{n}{x_n^{n-1}} \int_0^{x_n} k(x_n - k)^{n-2}\, dk$$

$$= \frac{n}{x_n^{n-1}} * \left[(x_n - k)^n \left(\frac{x_n}{(n-1)(k - x_n)} + \frac{1}{n} \right) \right]_0^{x_n}$$

$$= \frac{n}{x_n^{n-1}} * x_n^n \left(\frac{\cancel{x_n}}{-\cancel{x_n}(n-1)} + \frac{1}{n} \right)$$

$$= n x_n \left(\frac{1}{n-1} + \frac{1}{n} \right)$$

$$= \frac{(2n-1)\, x_n}{n-1}$$

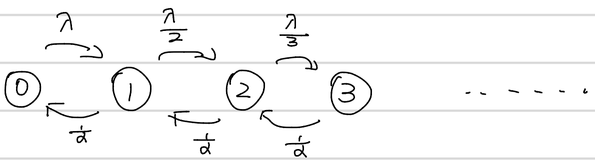
Question 4

7.24 Customers arrive at a busy food truck according to a Poisson process with parameter λ. If there are i people already in line, the customer will join the line with probability $1/(i+1)$. Assume that the chef at the truck takes, on average, α minutes to process an order.

(a) Find the long-term average number of people in line.

(b) Find the long-term probability that there are at least two people in line.

(a) N_t be the number of people



$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \left(\begin{array}{ccccc} -\lambda & \lambda & 0 & 0 & \cdots \\ \frac{1}{\alpha} & -(\frac{1}{\alpha}+\frac{\lambda}{2}) & \frac{\lambda}{2} & 0 & \cdots \\ 0 & \frac{1}{\alpha} & -(\frac{1}{\alpha}+\frac{\lambda}{3}) & \frac{\lambda}{3} & \cdots \\ 0 & 0 & \frac{1}{\alpha} & (-\frac{1}{\alpha}+\frac{\lambda}{4}) & \cdots \\ & & & & \end{array} \right) \end{array}$$

$$\frac{1}{\pi_0} = \sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \sum_{k=0}^{\infty} \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} = \sum_{k=0}^{\infty} \frac{\lambda \cdot (\frac{\lambda}{2}) \cdots (\frac{\lambda}{k})}{\frac{1}{\alpha} \cdot \frac{1}{\alpha} \cdots (\frac{1}{\alpha})} = \sum \frac{\lambda^k / k!}{1/\alpha^k} = \sum_{k=0}^{\infty} \frac{\lambda^k \alpha^k}{k!}$$

$$\Rightarrow \begin{cases} \pi_0 = e^{-\lambda\alpha} \\ \pi_k = e^{-\lambda\alpha} * \frac{(\lambda\alpha)^k}{k!} \end{cases}$$

$$\text{Average} = 0 \cdot \pi_0 + 1 \cdot \pi_1 + \cdots$$
$$= \sum_{k=0}^{\infty} k \cdot \pi_k = \lambda\alpha$$

(b) P_r (at leat 2 people in line)

$$= \pi_2 + \pi_3 + \cdots$$
$$= \sum_{k=2}^{\infty} \pi_k = 1 - \pi_0 - \pi_1$$
$$= 1 - e^{-\lambda\alpha} - e^{\lambda\alpha} \frac{(\lambda\alpha)}{1!}$$
$$= 1 - e^{-\lambda\alpha} - \lambda\alpha e^{-\lambda\alpha}$$

Question 5

7.27 Recall the discrete-time Ehrenfest dog–flea model of Example 3.7. In the continuous-time version, there are N fleas distributed between two dogs. Fleas jump from one dog to another independently at rate λ. Let X_t denote the number of fleas on the first dog.

(a) Show that the process is a birth-and-death process. Give the birth and death rates.

(b) Find the stationary distribution.

(c) Assume that fleas jump at the rate of 2 per minute. If there are 10 fleas on Cooper and no fleas on Lisa, how long, on average, will it take for Lisa to get 4 fleas?

(a)

The process is a birth and death process. Since in each state between 0 and n, the state variable i can only go to i-1 or i+1 state.

Since jump from one dog to another has rate $= \lambda$ then $q_i = 2\lambda$ $(0 < i < n)$

$\therefore$

$$\tilde{P} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots & N \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ N \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & \\ \frac{1}{N} & 0 & \frac{N-1}{N} & 0 & 0 & \cdots & \\ 0 & \frac{2}{N} & 0 & \frac{N-2}{N} & 0 & \cdots & \\ 0 & 0 & \frac{3}{N} & 0 & \frac{N-3}{N} & \cdots & \\ 0 & 0 & 0 & 0 & \frac{4}{N} & \cdots & \\ & & \cdots & & \cdots & & \\ & & & & & 1 & 0 \end{pmatrix} \end{array}$$

Since $p_{ij} = \frac{q_{ij}}{q_i} \Rightarrow q_{ij} = q_i \cdot p_{ij} \Rightarrow q_{01} = q_0 = \lambda$

$q_{10} = \frac{1}{N} * q_1 = \frac{1}{N} 2\lambda$ $\qquad q_{12} = \frac{N-1}{N} * q_1 = \frac{N-1}{N} 2\lambda$

$$\Rightarrow \begin{cases} q_{i(i-1)} = \frac{i}{N} q_i \\ q_{i(i+1)} = \frac{N-i}{N} q_i \end{cases}$$

X_n :

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "2\frac{1}{N}\lambda"] \arrow[r, bend left, "2\frac{N-1}{N}\lambda"] & 2 \arrow[l, bend left, "2\frac{2}{N}\lambda"] \arrow[r, bend left, "2\frac{N-2}{N}\lambda"] & 3 \arrow[l, bend left, "2\frac{3}{N}\lambda"] \arrow[r, bend left, "2\frac{N-3}{N}\lambda"] & 4 \arrow[l, bend left, "2\frac{4}{N}\lambda"] \arrow[r, bend left, "2\frac{N-4}{N}\lambda"] & 5 \arrow[l, bend left, "2\frac{5}{N}\lambda"] & \cdots \arrow[r, bend left] & N \arrow[l, bend left, "\lambda"] \end{tikzcd}$$

(b)

$$
Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ N \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & N \end{array} \\
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & \cdots & \\
\frac{1}{N}2\lambda & -2\lambda & \frac{N-1}{N}\cdot 2\lambda & 0 & 0 & \cdots & \\
0 & \frac{2}{N}2\lambda & -2\lambda & \frac{N-2}{N}\cdot 2\lambda & 0 & \cdots & \\
0 & 0 & \frac{3}{N}2\lambda & -2\lambda & \frac{N-3}{N}\cdot 2\lambda & \cdots & \\
0 & 0 & 0 & 0 \ \frac{4}{N}2\lambda & & \cdots & \\
 & & & \cdots & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & \lambda \ -\lambda
\end{pmatrix}
\end{array}
$$

then

$$\frac{1}{\pi_0} = \sum_{k=1}^{N} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \sum_{k=1}^{N} \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_0 \mu_1 \cdots \mu_k} = \sum_{k=1}^{N} \frac{2\lambda \cdot \frac{N-1}{N}2\lambda \cdots \frac{N-k+1}{N}2\lambda}{\frac{1}{N}2\lambda \cdots \frac{k}{N}2\lambda} = \sum_{k=1}^{N} \frac{1 * \frac{N-1}{N} * \cdots \frac{N-k+1}{N}}{\frac{1}{N} * \frac{2}{N} * \cdots \frac{k}{N}}$$

$$= \sum_{k=1}^{N} \frac{N(N-1)(N-2)\cdots(N-k+1)}{k!} = \sum_{k=1}^{N} \frac{N!}{k!\,(N-k)!} = \sum_{k=1}^{N} \binom{N}{k}$$

$$\pi_k = \pi_0 * \binom{N}{k} = \frac{1}{\left[\sum_{i=1}^{N} \binom{N}{i}\right]} * \binom{N}{k} = \binom{N}{k} / 2^N$$

$$\Rightarrow \tilde{\pi} = \left(\frac{\binom{N}{0}}{2^N}, \frac{\binom{N}{1}}{2^N}, \cdots \frac{\binom{N}{N}}{2^N} \right)$$

(c) $N = 10$; $\lambda = 2$

$$
Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
-2 & 2 & 0 & 0 & 0 \\
0.4 & -4 & 3.6 & 0 & 0 \\
0 & 0.8 & -4 & 3.2 & 0 \\
0 & 0 & 1.2 & -4 & 2.8 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$\Rightarrow F = -V^{-1}$$

$$= -\begin{pmatrix} -2 & 2 & 0 & 0 \\ 0.4 & -4 & 3.6 & 0 \\ 0 & 0.8 & -4 & 3.2 \\ 0 & 0 & 1.2 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5754 & 0.377 & 0.4464 & 0.357 \\ 0.0754 & 0.377 & 0.4464 & 0.357 \\ 0.0198 & 0.0992 & 0.4464 & 0.357 \\ 0.00595 & 0.02976 & 0.1339 & 0.357 \end{pmatrix},$$

then $a_0 = 0.5754 + 0.377 + 0.4464 + 0.357$

$= 1.7558$ min