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Assignment 2

MATH 447

$\boxed{\text{Question 1.} \quad (3.5)}$

$$P = \begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}$$

(a) let $\underset{\sim}{x} = (1, x_2, \ldots\ldots x_5)$

then $\underset{\sim}{\pi} = \frac{1}{1+x_2+x_3+x_4+x_5}(1, x_2, x_3, x_4, x_5)$

$\Rightarrow \quad (1, x_2, x_3, x_4, x_5) * P = (1, x_2, x_3, x_4, x_5)$

$$\Rightarrow \begin{cases} 0 + \frac{3}{4}\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 + \frac{1}{4}\cdot x_5 = 1 \\ \frac{1}{4}\cdot 1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 + \frac{3}{4}\cdot x_5 = x_2 \\ 0\cdot 1 + 0\cdot x_2 + 1\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 = x_3 \\ 0\cdot 1 + 0\cdot x_2 + 0\cdot x_3 + 1\cdot x_4 + 0\cdot x_5 = x_4 \\ \frac{3}{4}\cdot 1 + \frac{1}{4}\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 = x_5 \end{cases} \Rightarrow \left[\begin{array}{ccccc|c} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 1 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & x_2 \\ 0 & 0 & 1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & x_4 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & x_5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccccc|c} \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & x_2 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & -\frac{7}{3} & x_5 - \frac{1}{3} - 3x_2 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 4x_2 \\ 0 & 3 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & -7 & 3x_5 - 1 - 9x_2 \end{array}\right]$$

$\Rightarrow$ Let $x_3 = s$, $x_4 = t$

then $\begin{cases} x_1 + 3x_5 = 4x_2 \\ 3x_2 + x_5 = 4 \\ -7x_5 = 3x_5 - 1 - 9x_2 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_5 = 1 \end{cases}$ then $\underset{\sim}{\pi} = \frac{1}{1+x_2+x_3+x_4+x_5}(1, 1, s, t, 1)$

$= \frac{1}{3+s+t}(1, 1, s, t, 1)$

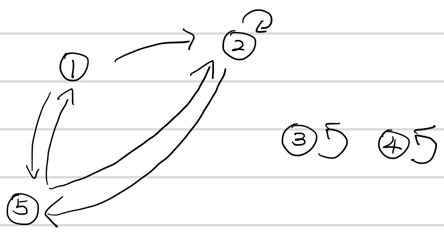
$\Rightarrow$ the set : $\left\{ \frac{1}{3+s+t}(1, 1, s, t, 1) \right\}$

(b) Use technology to find $\lim_{n\to\infty} P^n$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

So the result shows that this chain does not have a limiting distribution but as $n \to \infty$ the P^n converges to the above matrix gradually. and it has a limiting matrix.

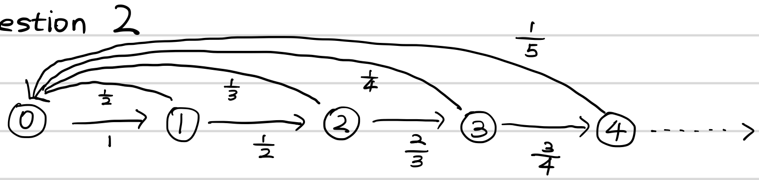
(c) Since a limiting distribution is the final states converges to π regardless of the initial state, but the above matrix doesn't show this property. Since state ③ and ④ are not accessible from each other and state ① ② ⑤ like $Pr(X_i = 3 \mid X_0 = 1) = 0$ always for $\forall i$



Then it is impossible for the probability of state ③ and ④ to converge to some value $[Pr(X_n = 3 \mid X_0 = 3) = 1$ and $Pr(X_n = 4 \mid X_0 = 4) = 1]$

But state ①, ② and ⑤ are limiting distribution

$\boxed{3.27}$ Question 2



(a) irreducible :

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 2 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & \cdots \\ 3 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & \cdots \\ 4 & \frac{1}{5} & 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & & & & \end{array}$$

proof by induction that $P_{0n} > 0$ & $P_{n0} > 0$

$P_{01} > 0$ & $P_{10} > 0$ $\Rightarrow$ 0 and 1 are communicate

$$\begin{cases} P^2_{02} = Pr(X_2 = 2 \mid X_0 = 0) = (0\ 1\ 0\ 0\ 0\ \cdots\cdots) * \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{2} > 0 \\ P_{20} > 0 \end{cases} \Rightarrow 0 \text{ and } 2 \text{ are communicative}$$

Assume $P_{0n} > 0$ and $P^n_{n0} > 0$ holds then $P_{n0} > 0$ is obvious

$$Pr^{n+1}_{0(n+1)} = Pr(X_{n+1} = n+1 \mid X_0 = 0)$$
$$> Pr(X_{n+1} = n+1 \mid X_n = n, X_0 = 0) * Pr(X_n = n \mid X_0 = 0)$$
$$= Pr(X_{n+1} = n+1 \mid X_n = n)\, P^n_{0n} > 0$$

$\Rightarrow$ $P_{0(n+1)} > 0$ and $P^{n+1}_{(n+1)0} > 0$ $\Rightarrow$ 0 and n+1 are communicate

$\Rightarrow$ 0, 1, 2, $\cdots\cdots$ are communicative

$\Rightarrow$ the chain is irreducible

• aperiodic : proof by induction that the set of steps needed to return back to state i be $S_i = \{i+1, i+2, \cdots\}$ for $i \geq 1$

for $k = 0$ we have :

$$\because \begin{array}{l} P(X_1 = 0 \mid X_0 = 0) = 0 \\ P(X_2 = 0 \mid X_0 = 0) > 0 \\ P(X_3 = 0 \mid X_0 = 0) > 0 \end{array} \Rightarrow \begin{cases} P_{00} = 0 \\ P^2_{00} > 0 \\ P^3_{00} > 0 \\ \vdots \end{cases} \Rightarrow \begin{array}{l} d(0) = \gcd\{n > 0 ;\ P^n_{00} > 0\} \\ = \gcd\{2, 3, 4, 5, 6, \cdots\} = 1 \end{array}$$

① for $k = 1$ we have $S_1 = \{2, 3, 4, 5, \cdots\}$ 5... } = 1

② Assume that $S_k = \{k+1, k+2, k+3, \cdots\}$ .... }

Since $Pr(X_1 = k+1 \mid X_0 = i) > 0$ if and only if $i = k$

so if we start from $k+1$, then one step before "returning back to $k+1$" must be k

and S_k also equals $\{n;$ # of steps that starts from $k+1$ and return back to $k\}$

then $S_{k+1} = S_k + 1 = \{k+2, k+3, \cdots\}$

$\Rightarrow$ then $\because S_k = \{k+1, k+2, \cdots\}$ for $k \geq 1$

then $d(k) = \gcd S_k = \gcd\{k+1, k+2, k+3, \cdots\} = 0$ for all k

$\Rightarrow$ aperiodic ▨

(b) Let $T_j = \min\{n > 0 : X_n = j\}$ given that start in j

$Pr(T_j = 0 \mid X_0 = 0) = 0$

$Pr(T_j = 1 \mid X_0 = 0) = P_{01} * P_{10}$

$Pr(T_j = 2 \mid X_0 = 0) = P_{01} P_{12} P_{20}$

$\cdots\cdots$

$\Rightarrow$

$f_0 = P(T_0 < \infty \mid X_0 = 0)$ be the probability that the chain started in 0 eventually returns to 0

$= P_{01} P_{10} + P_{01} P_{12} P_{20} + P_{01} P_{12} P_{23} P_{30} + \cdots$

$= \frac{1}{2} P_{01} + \frac{1}{3} P_{01} P_{12} + \frac{1}{4} P_{01} P_{12} P_{23} + \cdots$

$= \frac{1}{2} + \frac{1}{3} P_{12} + \frac{1}{4} P_{12} P_{23} + \cdots$

$= \frac{1}{2} \times 1 + \frac{1}{3} \times 1 \times \frac{1}{2} + \frac{1}{4} \times 1 \times \frac{1}{2} \times \frac{2}{3} + \cdots$

$= 1 \times \frac{1}{2} + 1 \times \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} + 1 \times \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5}$

$= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots$

$= (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots) = 1$

$\Rightarrow$ 0 is a recurrent state

Since this is a communication class

$\Rightarrow$ the chain is recurrent

(c) Now calculate $E(T_j \mid X_0 = j)$

$$E[T_0 \mid X_0 = 0] = \mu_0$$
$$= 1\cdot(1+\mu_1)$$
$$\mu_1 = \frac{1}{2} + \frac{1}{2}(1+\mu_2)$$
$$\mu_2 = \frac{1}{3} + \frac{2}{3}(1+\mu_3)$$
$$\vdots$$
$$\mu_n = \frac{1}{n+1} + \frac{n}{n+1}(1+\mu_{n+1})$$

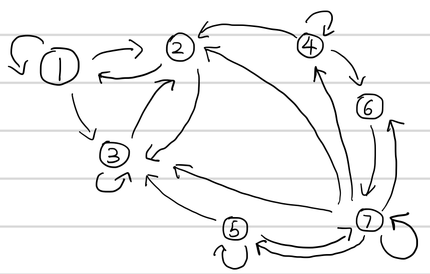
$$\Rightarrow \mu_0 = (1+\mu_1)$$
$$= (1+\frac{1}{2}+\frac{1}{2}(1+\mu_2))$$
$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\mu_2$$
$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\left[\frac{1}{3}+\frac{2}{3}(1+\mu_3)\right]$$
$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{6}+\frac{1}{3}(1+\mu_3)$$
$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{6}+\frac{1}{3}+\frac{1}{3}\mu_3.$$
$$= 2+\frac{1}{2}+\frac{1}{3}\left[\frac{1}{4}+\frac{3}{4}(1+\mu_4)\right]$$
$$= 2+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\mu_4$$
$$= 2+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\left(\frac{1}{5}+\frac{4}{5}(1+\mu_5)\right)$$
$$= 2+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\mu_5$$

$$\Rightarrow \begin{cases} \mu_0 = 1 + \underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{\text{infinite}} = \infty \\ \vdots \\ \mu_1 = \mu_0 - 1 = \infty \\ \vdots \\ \mu_n = \infty \end{cases}$$

Since $E[T_j \mid X_0 = j] = \infty$ for the chain

$\Rightarrow$ the chain is null recurrent

$\boxed{3.28}$



Communication class : $\{1, 2, 3\}$ $\{4, 5, 6, 7\}$

then recurrent states is $\{1, 2, 3\}$ transiant state is $\{4, 5, 6, 7\}$

determine $\lim_{n\to\infty} P_{ij}^{n}$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \end{array}$$

Since there is no absorbing state :& the transient state is $\{4, 5, 6, 7\}$

then

$$\Rightarrow \quad \tilde{P} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc|ccc} \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \hline 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right) \end{array} = \left[\begin{array}{c|c} Q & R \\ \hline 0 & B \end{array}\right] \text{ then}$$

$$P^2 = \begin{bmatrix} Q & R \\ 0 & B \end{bmatrix}\begin{bmatrix} Q & R \\ 0 & B \end{bmatrix} = \begin{bmatrix} Q^2 & QR+RB \\ 0 & B^2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} Q & R \\ 0 & B \end{bmatrix}\begin{bmatrix} Q^2 & QR+RB \\ 0 & B^2 \end{bmatrix} = \begin{bmatrix} Q^3 & Q(QR+RB)+B^2R \\ 0 & B^3 \end{bmatrix} \Rightarrow P^4 = \begin{bmatrix} Q^4 & Q^3R+Q^2RB+QRB^2+RB^3 \\ 0 & B^4 \end{bmatrix}$$

$$= \begin{bmatrix} Q^4 & R(Q^3+Q^2B+QB^2+B^3) \\ 0 & B^4 \end{bmatrix}$$

$$\Rightarrow \lim_{n\to\infty} P^n = \begin{bmatrix} 0 & (Q^{n-1}R + Q^{n-2}RB + \cdots RB^{n-1}) \\ 0 & B^n \end{bmatrix} = \begin{bmatrix} 0 & (I-Q)R \\ 0 & B^n \end{bmatrix}$$

Now consider the matrix $B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$$B^2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{9}+\frac{2}{9} & \frac{1}{9}+\frac{2}{9} & \frac{1}{9}\times 3 \\ \frac{2}{3}\times\frac{1}{3} & \frac{2}{3}\times\frac{1}{3}\times 2 & \frac{2}{3}\times\frac{1}{3}+\frac{1}{9} \\ \frac{2}{3}\times\frac{2}{3} & \frac{1}{3}\times\frac{2}{3} & \frac{2}{9}+\frac{1}{9} \end{pmatrix} > 0$$

$\Rightarrow \exists\ n = 2$ s.t $B^2 > 0$

$\Rightarrow$ B is a regular matrix & B is the transition matrix of Markov chain

$\Rightarrow \{1, 2, 3\}$ has a limiting distribution

$$(I - Q) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\ 0 & \frac{3}{4} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$ then

let $\pi = \frac{1}{1+x_1+x_2}(1, x_1, x_2)$ then

$$(1, x_1, x_2)\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (1, x_1, x_2)$$

$$\Rightarrow \left(\tfrac{1}{3}+\tfrac{2}{3}x_1,\ \tfrac{1}{3}+\tfrac{2}{3}x_2,\ \tfrac{1}{3}+\tfrac{1}{3}x_1+\tfrac{1}{3}x_2\right) = (1, x_1, x_2)$$

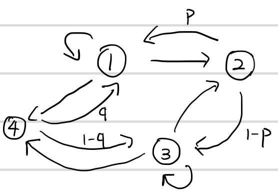
$$\Rightarrow \begin{cases} \frac{1}{3}+\frac{2}{3}x_1 = 1 & \Rightarrow x_1 = 1 \\ \frac{1}{3}+\frac{2}{3}x_2 = x_1 & \Rightarrow x_2 = 1 \\ \frac{1}{3}+\frac{1}{3}x_1+\frac{1}{3}x_2 = x_2 & \end{cases} \Rightarrow \pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$\Rightarrow \lim_{n\to\infty} P^n = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

3.43

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 2 & p & 0 & 1-p & 0 \\ 3 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 4 & q & 0 & 1-q & 0 \end{array}$$

(a) chain ergodic : ① irreducible ② aperiodic ③ all states have finite return time



Since irreducible ② needs to communicate with ①, ③ & ④

$\Rightarrow$ p, $1-p > 0$

q, $1-q > 0$ $\because$ ④ needed to be communicative

& p and q can not equal to 1 at the same time

$\because$ $d(i) = \gcd\{n>0 \,;\, P_{ii}^{n} > 0\} = \gcd\{n>0 ;\, Pr(X_n = i \mid X_0 = i) > 0\}$

$d(1) = \gcd\{1, 2, 4, \cdots\} = 1$

$d(2) = \gcd\{2, 3, \cdots\cdots\} = 1$

$d(3) = \gcd\{1, 2, \cdots\} = 1$

$d(4) = \gcd\{2, 3, \cdots\} = 1$ $\Rightarrow$ the chain is aperiodic

$\because$ the chain is a communication class

considering state $= 1$

If $Pr\{T_1 < \infty \mid X_0 = 1\} = 1$ $\Rightarrow$ Pr (state 1 will be revisited) $= 1$

$\Rightarrow$ p & q can not be 0 at the same time

$\Rightarrow$ $0 \le p, q \le 1$ but p and q can not be 0 and 1 at the same time

(b) let $\underset{\sim}{\pi} = \frac{1}{1+x_1+x_2+x_3}(1, x_1, x_2, x_3)$ then

$$\underset{\sim}{\pi} P = \underset{\sim}{\pi}$$

$$(1, x_1, x_2, x_3)\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ p & 0 & 1-p & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ q & 0 & 1-q & 0 \end{pmatrix} = (1, x_1, x_2, x_3)$$

$$\Rightarrow \left(\tfrac{1}{2} + x_1 p + x_3 q,\ \tfrac{1}{4} + \tfrac{1}{4}x_2,\ (1-p)x_1 + \tfrac{1}{2}x_2 + (1-q)x_3,\ \tfrac{1}{4} + \tfrac{1}{4}x_2\right) = (1, x_1, x_2, x_3)$$

$$\Rightarrow \begin{cases} \frac{1}{2} + x_1 p + x_3 q = 1 \\ \frac{1}{4} + \frac{1}{4}x_2 = x_1 \\ (1-p)x_1 + \frac{1}{2}x_2 + (1-q)x_3 = x_2 \\ \frac{1}{4} + \frac{1}{4}x_2 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 = \frac{1}{2(p+q)} \\ x_2 = 4x_1 - 1 = \frac{2}{(p+q)} - 1 \end{cases}$$

$$\underset{\sim}{\pi} = \frac{1}{1 + \frac{1}{p+q} + \frac{2}{(p+q)} - 1}\left(1, \frac{1}{2(p+q)}, \frac{2}{(p+q)} - 1, \frac{1}{2(p+q)}\right)$$

$$= \frac{p+q}{3}\left(1, \frac{1}{2(p+q)}, \frac{2}{(p+q)} - 1, \frac{1}{2(p+q)}\right)$$

$\because$ reversible $\Rightarrow$ $\pi_i P_{ij} = \pi_j P_{ji}$

$\pi_1 P_{12} = \pi_2 P_{21}$ $\Rightarrow$ $\frac{p+q}{3} \times \frac{1}{4} = \frac{1}{6} \times p$ $\Rightarrow$ $p + q = 2p$ $\Rightarrow$ $p = q$

$\pi_2 P_{23} = \pi_3 P_{32}$ $\Rightarrow$ $\frac{1}{6} \times (1-p) = \left(\frac{2}{3} - \frac{p+q}{3}\right) \times \frac{1}{4}$

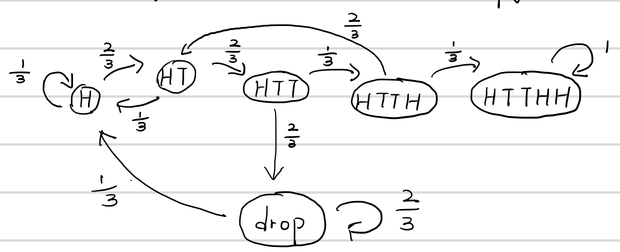
$\Rightarrow$ $2(1-p) = 2 - p - q$

$p = q$

$\rightarrow$ If $p = q$ then the chain is reversible

$\boxed{3.56}$ Since $T = \frac{2}{3}$ and $H = \frac{1}{3}$

Let state $= \{H, HT, HTT, HTTH, HTTHH, drop\}$



then

$$\begin{array}{c|cccccc} & drop & H & HT & HTT & HTTH & HTTHH \\ \hline drop & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ H & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ HT & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ HTT & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ HTTH & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ HTTHH & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

then the absorping state : $\{HTTHH\}$

transient state : $\{drop, H, HT, HTT, HTTH\}$

$$\Rightarrow \quad P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow (I - Q) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 1 & -\frac{2}{3} & 0 \\ -\frac{2}{3} & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{2}{3} & 0 & 1 \end{pmatrix}$$

then $F = (I - Q)^{-1}$

$$= \begin{pmatrix} 21 & 17.25 & 13.5 & 9 & 3 \\ 18 & 17.25 & 13.5 & 9 & 3 \\ 18 & 15.75 & 13.5 & 9 & 3 \\ 18 & 15 & 12 & 9 & 3 \\ 12 & 10.5 & 9 & 6 & 3 \end{pmatrix}$$

the expected steps before HTTHH first occur is the expected absorption time

$\Rightarrow$ from "drop" to "HTTHH"

$$\Rightarrow \sum_{k \in T} F_{drop, k} = 21 + 17.25 + 13.5 + 9 + 3 = 63.75$$