Assignment 1

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Question 1 (1.17)

Poisson random variable $\lambda = 3$ then $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-3}\frac{3^k}{k!}$

$$\Rightarrow E[X \mid X>2]$$

$$= \sum_{k=0}^{\infty} k * P(X=k \mid X>2)$$

$$= \sum_{k=0}^{2} k * P(X=k \mid X>2) + \sum_{k=3}^{\infty} k * P(X=k \mid X>2)$$

$$= \sum_{k=3}^{\infty} k * P(X=k \mid X>2)$$

$$= \sum_{k=3}^{\infty} k * \frac{P(X=k \cap X>2)}{P(X>2)}$$

$$= \sum_{k=3}^{\infty} k * \frac{P(X=k)}{P(X>2)} = \frac{1}{P(X>2)} \sum_{k=3}^{\infty} k * P(X=k)$$

$$= \frac{1}{P(X>2)} \left(E[X] - P(X=1) - 2P(X=2) \right)$$

$$= \frac{1}{P(X>2)} \left(3 - e^{-3} * 1 - 2 * e^{-3}\frac{3^2}{2!} \right)$$

$$= \frac{1}{1 - P(X=0) - P(X=1) - P(X=2)} \left(3 - e^{-3} - 9e^{-3}\right)$$

$$= \frac{1}{1 - e^{-3}\frac{3^0}{0!} - e^{-3}\frac{3^1}{1!} - e^{-3}\frac{3^2}{2!}} \left(3 - 10e^{-3}\right)$$

$$= \frac{3 - 10e^{-3}}{1 - e^{-3} - 3e^{-3} - \frac{9}{4}e^{-3}} = \frac{12e^3 - 40}{4e^3 - 22}$$

Question 2 (1.28)

$\because \Lambda$ is uniformly distributed

then $\Lambda \in [0, 3]$

Let X be the number of accident at a day

$f(\Lambda)$ be the PDF of Λ then $f(\Lambda) = \mathbb{1}_{(0,3)}(\Lambda) * \frac{1}{3-0}$

$$= \mathbb{1}_{(0,3)}(\Lambda) * \frac{1}{3}$$

then

$$E[X] = E_\Lambda \left[E_{X|\Lambda}(X \mid \Lambda) \right]$$

$$= E_\Lambda \left[\sum_{k=0}^{\infty} k * e^{-\Lambda}\frac{\Lambda^k}{k!} \right]$$

$$= E_\Lambda [\Lambda]$$

$$= \int_0^3 \Lambda * f(\Lambda)\, d\Lambda$$

$$= \int_0^3 \Lambda * \mathbb{1}_{(0,3)}(\Lambda) * \frac{1}{3-0}\, d\Lambda$$

$$= \frac{3+0}{2} = 1.5$$

$$\approx 1$$

$$Var(X) = Var_\Lambda \left[E_{X|\Lambda}[X \mid \Lambda] \right) + E_\Lambda \left(Var_{X|\Lambda}[X \mid \Lambda] \right)$$

$$= Var_\Lambda[\Lambda] + E_\Lambda \left(Var_{X|\Lambda}[X \mid \Lambda] \right)$$

$$= Var_\Lambda[\Lambda] + Var_\Lambda[\Lambda]$$

$$= 2 * \frac{(3-0)^2}{12} = \frac{9}{6} = \frac{3}{2}$$

Question 3 (2.2)

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

$\alpha = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ then

(a) $P(X_2 = 1 \mid X_1 = 3) = \frac{1}{3}$

(b) $P(X_1 = 3, X_2 = 1)$

$$= P(X_2 = 1 \mid X_1 = 3) * P(X_1 = 3)$$

$$= \frac{1}{3} P(X_1 = 3)$$

$$= \frac{1}{3} \sum_{k=1}^{3} P(X_1 = 3 \mid X_0 = k) P(X_0 = k)$$

$$= \left[\frac{1}{3} * \left(\frac{1}{2}, 0, \frac{1}{2}\right) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \right]_3$$

$$= \frac{1}{3} * \left(\frac{1}{2}, 0, \frac{1}{2}\right) * \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{3} * \left(\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \right)$$

$$= \frac{1}{12} + \frac{1}{18} = \frac{3}{36} + \frac{2}{36} = \frac{5}{36}$$

(c) $P(X_1 = 3 \mid X_2 = 1)$

$$= \frac{P(X_1 = 3, X_2 = 1)}{P(X_2 = 1)}$$

$$\because P(X_2 = 1) = \sum_{k=1}^{3} P(X_2 = 1 \mid X_0 = k) * P(X_0 = k)$$

$$= \left(\frac{1}{2}, 0, \frac{1}{2}\right) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \left(\frac{1}{2}, 0, \frac{1}{2}\right) \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} & \frac{1}{2} \times \frac{1}{3} & \frac{1}{2} \times \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} & \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} & \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} \end{pmatrix}$$

$$= \left[\left(\frac{1}{2}, 0, \frac{1}{2}\right) \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{4}{9} & \frac{5}{18} & \frac{5}{18} \end{pmatrix} \right]_1$$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{4}{9}$$

$$= \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$$

$$\Rightarrow P(X_1 = 3 \mid X_2 = 1) = \frac{P(X_1 = 3, X_2 = 1)}{P(X_2 = 1)} = \frac{\frac{5}{36}}{\frac{5}{9}} = \frac{1}{36} * 9 = \frac{1}{4}$$

(d) $P_r(X_9 = 1 \mid X_1 = 3, X_4 = 1, X_7 = 2)$

$$= \frac{P_r(X_1 = 3, X_4 = 1, X_7 = 2, X_9 = 1)}{P_r(X_1 = 3, X_4 = 1, X_7 = 2)}$$

$$= \frac{(\alpha P)_3 P_{31}^3 P_{21}^3 P_{12}^2}{(\alpha P)_3 P_{31}^3 P_{21}^3}$$

$$= (P^2)_{12} = \frac{1}{6}$$

Question 4 (2.7)

Since $Y_n = X_{3n}$ then for $\forall n$

$$P_r(Y_n = j \mid Y_{n-1} = i)$$

$$= P_r(X_{3n} = j \mid X_{3n-3} = i)$$

$$= \left(P^{3n-(3n-3)}\right)_{ij} = (P^3)_{ij}$$

We then have

$$\begin{pmatrix} P_r(Y_n = 1 \mid Y_{n-1} = 1) & \cdots & P_r(Y_n = k \mid Y_{n-1} = 1) \\ P_r(Y_n = 1 \mid Y_{n-1} = 2) & & \\ \vdots & & \\ P_r(Y_n = 1 \mid Y_{n-1} = k) & \cdots & P_r(Y_n = k \mid Y_{n-1} = k) \end{pmatrix} = \begin{pmatrix} (P^3)_{11} & \cdots & (P^3)_{1n} \\ (P^3)_{21} & \ddots & \vdots \\ \vdots & & \\ (P^3)_{n1} & \cdots & (P^3)_{nn} \end{pmatrix}$$

$\Rightarrow Y_0, Y_1, \cdots$ is a Markov chain

$\Rightarrow$ its transition matrix is

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & \cdots & k \end{array} \;] \; X_n \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ k \end{array} & \begin{pmatrix} (P^3)_{11} & \cdots & (P^3)_{1n} \\ \vdots & & \vdots \\ (P^3)_{n1} & \cdots & (P^3)_{nn} \end{pmatrix} \\ \underbrace{\qquad}_{X_{n-1}} & \end{array}$$

Question 5 (2.13)

Let (a, b, c) be that 3 books respectively

X_n be the order of a, b, c

$\Rightarrow$ Let P_a, P_b & P_c be the probability that a, b & c are chosen respectively

$$P = \begin{array}{c c} & \begin{array}{cccccc} abc & acb & bca & bac & cab & cba \end{array} \;] \; X_{n-1} \\ \begin{array}{c} abc \\ acb \\ bca \\ bac \\ cab \\ cba \end{array} & \begin{pmatrix} P_a & P_b & 0 & P_a & 0 & 0 \\ P_c & P_a & 0 & 0 & P_a & 0 \\ 0 & 0 & P_b & P_c & 0 & P_b \\ P_b & 0 & P_a & P_b & 0 & 0 \\ 0 & P_c & 0 & 0 & P_c & P_a \\ 0 & 0 & P_c & 0 & P_b & P_c \end{pmatrix} \\ \underbrace{\qquad}_{X_{n-1}} & \end{array}$$