Question 1

3.3 Consider the infinite-order MA process $\{X_t\}$, defined by

$$X_t = Z_t + C(Z_{t-1} + Z_{t-2} + \cdots)$$

where C is a non-zero constant. Show that the process is non-stationary. Also show that the series of first differences $\{Y_t\}$ defined by

$$Y_t = X_t - X_{t-1}$$

is a first-order MA process and is stationary. Find the ac.f. of $\{Y_t\}$.

Since $Z_t \sim WN(0, \sigma^2)$, then $E[X_t] = E[Z_t + C(Z_{t-1} + Z_{t-2} + \cdots)] = 0$

$$\begin{aligned}
Cov(X_t, X_{t+h}) &= E[(X_t - E[X_t])(X_{t+h} - E[X_{t+h}])] \\
&= E[X_t \cdot X_{t+h}] \\
&= E[(Z_t + C(Z_{t-1} + Z_{t-2} + \cdots))(Z_{t+h} + C(Z_{t+h-1} + Z_{t+h-2} + \cdots))] \\
&= E[C Z_t^2 + C^2 Z_{t-1}^2 + C^2 Z_{t-2}^2 + \cdots] \\
&= C\sigma^2 + C^2\sigma^2 + C^2\sigma^2 + \cdots \\
&= C\sigma^2 + \sum_{-\infty}^{t-1} C^2\sigma^2
\end{aligned}$$

which is not independent of t

$\Rightarrow Z_t$ is not a stationary process.

$$\begin{aligned}
Y_t = X_t - X_{t-1} &= Z_t + C(Z_{t-1} + \cdots) - Z_{t-1} - C(Z_{t-2} + Z_{t-3} + \cdots) \\
&= Z_t + C Z_{t-1} - Z_{t-1} \\
&= Z_t + (C-1) Z_{t-1}
\end{aligned}$$

$\Rightarrow Y_t = Z_t + (C-1) Z_{t-1} = Z_t + \theta Z_{t-1}$, where $Z_t \sim WN(0, \sigma^2)$. According to the definition of First order moving average, $\{Y_t\}$ is MA 1.

$E[Y_t] = E[Z_t + (C-1)Z_{t-1}] = 0$ which is independent of t

$$\begin{aligned}
Cov(Y_{t+h}, Y_t) &= E[Y_t Y_{t+h}] \\
&= E[Z_t Z_{t+h} + (C-1) Z_{t+h} Z_{t-1} + (C-1) Z_{t+h-1} Z_t + (C-1)^2 Z_{t+h-1} Z_{t-1}] \\
&= \gamma_Z(h) + (C-1)\gamma_Z(h+1) + (C-1)\gamma_Z(h-1) + (C-1)^2 \gamma_Z(h)
\end{aligned}$$

then $\gamma_Y(h) = \begin{cases} 0 & \text{for } h>0 \ \& \ h \neq 1 \\ (c-1)\gamma_Z(0) & h=1 \\ (1+(c-1)^2)\gamma_Z(0) & \text{for } h=0 \\ \gamma_Y(-h) & \text{for } h<0 \end{cases}$

then $\rho_Y(h) = \begin{cases} 0 & \text{for } h>0 \ \& \ h\neq 1 \\ 1 & \text{for } h=0 \\ \frac{c-1}{1+(c-1)^2} & \text{for } h=1 \end{cases}$

Both $\gamma_Y(h)$ and $E[X_t]$ are independent of h $\Rightarrow$ $\{Y_t\}$ is stationary.

Question 2

2.11 Suppose that in a sample of size 100 from an AR(1) process with mean μ, $\phi = .6$, and $\sigma^2 = 2$ we obtain $\bar{x}_{100} = 0.271$. Construct an approximate 95 % confidence interval for μ. Are the data compatible with the hypothesis that $\mu = 0$?

$\because$ $\bar{X}_n \pm Z_{\alpha/2} \frac{\sqrt{V}}{\sqrt{n}}$ and $V = \sum_{|h|<\infty} \gamma(h) = \sum_{|h|<\infty} \frac{\sigma^2 \phi^{(h)}}{1-\phi^2} = \frac{\sigma^2}{(1-\phi)^2}$

So the 95% interval will be :

$L = \bar{X}_{100} - \frac{1.96}{\sqrt{100}} \frac{\sqrt{2}}{|1-0.6|} = 0.271 - 0.196 * \frac{\sqrt{2}}{0.4} = 0.271 - 0.693 = -0.422$

$R = \bar{X}_{100} + \frac{1.96}{\sqrt{100}} \frac{\sqrt{2}}{|1-0.6|} = 0.271 + 0.693 = 0.964$

$\Rightarrow$ 95% CI will be $[-0.422, 0.964]$

Since $H_0 : \mu = 0$ is included in the 95 % CI, the data is compatible with the hypothesis that $\mu = 0$.

Question 3

2.21 Let $X_1,\ X_2,\ X_4,\ X_5$ be observations from the MA(1) model

$$X_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \text{WN}\left(0, \sigma^2\right).$$

a. Find the best linear estimate of the missing value X_3 in terms of X_1 and X_2.

Let $Y = X_3$ and $W = (X_1, X_2)$

$\Gamma_{ij} = \gamma(i-j) \Rightarrow \Gamma = \begin{pmatrix} \gamma(2-2) & \gamma(2-1) \\ \gamma(1-2) & \gamma(1-1) \end{pmatrix} = \begin{pmatrix} \sigma^2(1+\theta^2) & \theta\sigma^2 \\ \theta\sigma^2 & \sigma^2(1+\theta^2) \end{pmatrix}$

$$\gamma = \begin{bmatrix} \theta\sigma^2 \\ 0 \end{bmatrix} \quad \text{then} \quad a^T = \begin{bmatrix} \sigma^2(1+\theta^2) & \theta\sigma^2 \\ \theta\sigma^2 & \sigma^2(1+\theta^2) \end{bmatrix}^{-1} \begin{bmatrix} \theta\sigma^2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sigma^4(1+\theta^2)^2 - \theta^2\sigma^4} \begin{bmatrix} \sigma^2(1+\theta^2) & -\theta\sigma^2 \\ -\theta\sigma^2 & \sigma^2(1+\theta^2) \end{bmatrix} \begin{bmatrix} \theta\sigma^2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sigma^4(1+\theta^2)^2 - \theta^2\sigma^4} \begin{bmatrix} \theta\sigma^4(1+\theta^2) \\ -\theta^2\sigma^4 \end{bmatrix}$$

$$\Rightarrow \quad X_3 = \frac{\theta\sigma^4(1+\theta^2)}{\sigma^4(1+\theta^2)^2 - \theta^2\sigma^4} X_2 - \frac{\theta^2\sigma^4}{\sigma^4(1+\theta^2)^2 - \theta^2\sigma^2} X_1$$

$$= \frac{\theta(1+\theta^2)}{(1+\theta^2)^2 - \theta^2} X_2 - \frac{\theta^2}{(1+\theta^2)^2 - \theta^2} X_1$$

b. Find the best linear estimate of the missing value X_3 in terms of X_4 and X_5.

Let $y = X_3 \quad W = (X_4, X_5)$

$$\Gamma = \begin{pmatrix} \gamma(5-5) & \gamma(5-4) \\ \gamma(4-5) & \gamma(4-4) \end{pmatrix} = \begin{pmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta \\ \sigma^2\theta & \sigma^2(1+\theta^2) \end{pmatrix}$$

$$\underset{\sim}{\gamma} = \begin{pmatrix} \theta\sigma^2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underset{\sim}{a}^T = \Gamma^{-1}\underset{\sim}{\gamma} = \frac{1}{\sigma^4(1+\theta^2)^2 - \sigma^4\theta^2} \begin{pmatrix} \sigma^2(1+\theta^2) & -\sigma^2\theta \\ -\sigma^2\theta & \sigma^2(1+\theta^2) \end{pmatrix} \begin{pmatrix} \theta\sigma^2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sigma^4(1+\theta^2)^2 - \sigma^4\theta^2} \begin{pmatrix} \theta\sigma^4(1+\theta^2) \\ -\sigma^4\theta^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\theta(1+\theta^2)}{(1+\theta^2)^2 - \theta^2} \\ \frac{\theta^2}{\theta^2 - (1+\theta^2)^2} \end{pmatrix}$$

$$X_3 = a^T W$$

$$= \frac{\theta(1+\theta^2)}{(1+\theta^2)^2 - \theta^2} X_5 + \frac{\theta^2}{\theta^2 - (1+\theta^2)^2} X_4$$

c. Find the best linear estimate of the missing value X_3 in terms of X_1, X_2, X_4, and X_5.

Let $Y = X_3$ and $W = (X_5 \quad X_4 \quad X_2 \quad X_1)$

$$\Gamma = \begin{pmatrix} \gamma(5-5) & \gamma(5-4) & \gamma(5-2) & \gamma(5-1) \\ \gamma(4-5) & \gamma(4-4) & \gamma(4-2) & \gamma(4-1) \\ \gamma(2-5) & \gamma(2-4) & \gamma(2-2) & \gamma(2-1) \\ \gamma(1-5) & \gamma(1-4) & \gamma(1-2) & \gamma(1-1) \end{pmatrix} = \begin{pmatrix} \sigma^2(1+\theta^2) & \theta\sigma^2 & 0 & 0 \\ \theta\sigma^2 & \sigma^2(1+\theta^2) & 0 & 0 \\ 0 & 0 & \sigma^2(1+\theta^2) & \theta\sigma^2 \\ 0 & 0 & \theta\sigma^2 & \sigma^2(1+\theta^2) \end{pmatrix}$$

then

$$\Gamma^{-1}\underset{\sim}{\gamma} = \begin{pmatrix} \sigma^2(1+\theta^2) & -\theta\sigma^2 & 0 & 0 \\ -\theta\sigma^2 & \sigma^2(1+\theta^2) & 0 & 0 \\ 0 & 0 & \sigma^2(1+\theta^2) & -\theta\sigma^2 \\ 0 & 0 & -\theta\sigma^2 & \sigma^2(1+\theta^2) \end{pmatrix} \begin{pmatrix} 0 \\ \theta\sigma^2 \\ \theta\sigma^2 \\ 0 \end{pmatrix} * \frac{1}{\sigma^4(1+\theta^2)^2 - \theta^2\sigma^4}$$

$$= \begin{pmatrix} -\theta^2\sigma^4 \\ \sigma^4\theta(1+\theta^2) \\ \sigma^4\theta(1+\theta^2) \\ -\theta^2\sigma^4 \end{pmatrix} \frac{1}{\sigma^4(1+\theta^2)^2 - \theta^2\sigma^4} = \begin{pmatrix} \frac{\theta^2}{\theta^2-(1+\theta^2)^2} \\ \frac{\theta(1+\theta^2)}{(1+\theta^2)^2-\theta^2} \\ \frac{\theta(1+\theta^2)}{(1+\theta^2)^2-\theta^2} \\ \frac{\theta^2}{\theta^2-(1+\theta^2)^2} \end{pmatrix}$$

$$\Rightarrow Y = a^T W = \frac{\theta^2}{\theta^2-(1+\theta^2)^2}(X_5+X_1) + \frac{\theta(1+\theta^2)}{(1+\theta^2)^2-\theta^2}(X_4+X_2)$$

d. Compute the mean squared errors for each of the estimates in (a)–(c).

(a) $E[(X_3 - PX_3)^2]$

$$= Var(X_3) - \underset{\sim}{a}^T \underset{\sim}{\gamma}$$

$$= \sigma^2(1+\theta^2) - \begin{pmatrix} \frac{\theta(1+\theta^2)}{(1+\theta^2)^2-\theta^2} \\ \\ \frac{\theta^2}{\theta^2-(1+\theta^2)^2} \end{pmatrix}^T \begin{pmatrix} \theta\sigma^2 \\ \\ 0 \end{pmatrix}$$

$$= \sigma^2(1+\theta^2) - \frac{\theta^2\sigma^2(1+\theta^2)}{(1+\theta^2)^2-\theta^2} = \frac{\sigma^2(1+\theta^2)^3 - 2\theta^2\sigma^2(1+\theta^2)}{(1+\theta^2)^2-\theta^2}$$

(b) $E[(X_3 - PX_3)^2] = \dfrac{\sigma^2(1+\theta^2)^3}{(1+\theta^2)^2-\theta^2}$

(c) $E[(X_3 - PX_3)^2]$

$$= Var(X_3) - a^T \underset{\sim}{\gamma}$$

$$= \sigma^2(1+\theta^2) - \begin{pmatrix} \frac{\theta^2}{\theta^2-(1+\theta^2)^2} \\ \frac{\theta(1+\theta^2)}{(1+\theta^2)^2-\theta^2} \\ \frac{\theta(1+\theta^2)}{(1+\theta^2)^2-\theta^2} \\ \frac{\theta^2}{\theta^2-(1+\theta^2)^2} \end{pmatrix}^T \begin{pmatrix} 0 \\ \theta\sigma^2 \\ \theta\sigma^2 \\ 0 \end{pmatrix}$$

$$= \sigma^2(1+\theta^2) - 2\frac{\theta^2\sigma^2(1+\theta^2)}{(1+\theta^2)^2-\theta^2}$$

$$= \frac{\sigma^2(1+\theta^2)^3 - 3\theta^2\sigma^2(1+\theta^2)}{(1+\theta^2)^2-\theta^2}$$