Question 1

Question 1 (20 points)

Let $\{Z_t\}$ be a sequence of independent random variables, each with mean 0 and variance σ^2, and let a, b, and c be constants. Which if, any, of the following processes are stationary? For each *stationary* process specify the mean and autocovariance function.

a) $X_t = a + bZ_t + cZ_{t-2}$

$$E[X_t] = E[a + bZ_t + cZ_{t-2}]$$
$$= a + bE[Z_t] + cE[Z_{t-2}]$$
$$= a \quad \text{is independent of } t$$

· For $h > 0$:

$$Cov(X_{t+h}, X_t) = \gamma_X(t+h, t)$$
$$= E[(a - X_{t+h})(a - X_t)]$$
$$= E[a^2 - aX_t - aX_{t+h} + X_t X_{t+h}]$$
$$= a^2 - a^2 - a^2 + E[(a + bZ_t + cZ_{t-2})(a + bZ_{t+h} + cZ_{t+h-2})]$$
$$= a^2 - a^2 - a^2 + a^2 = 0 = \gamma(h, 0)$$

· For $h = 0$:

$$Cov(X_t, X_t) = E[(a - Z_t)(a - Z_t)]$$
$$= E[a^2 - aZ_t - aZ_t + Z_t^2]$$
$$= a^2 + \sigma^2$$

then
$$\gamma(h, 0) = \begin{cases} a^2 & h = 0 \\ 0 & |h| > 0 \end{cases} \Rightarrow X_t \text{ is stationary.}$$

b) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$

$$E[X_t] = E[Z_1 \cos(ct) + Z_2 \sin(ct)] = 0$$

$$Cov(X_{t+h}, X_t) = E[(X_{t+h} - 0)(X_t - 0)]$$
$$= E[(Z_1 \cos(ct + ch) + Z_2 \sin(ct + ch))(Z_1 \cos(ct) + Z_2 \sin(ct))]$$

$$= E[\, Z_1^2 \cos(ct+ch)\cos(ct) + Z_2^2 \sin(ct+ch)\sin(ct)\,]$$

$$= \sigma^2 \cos(ct+ch)\cos(ct) + \sigma^2 \sin(ct+ch)\sin(ct)$$

$$= \sigma^2 \sin(ct+2ch)$$

$$Cov(X_t, X_t) = E[\, X_t^2 \,]$$

$$= E[\,(Z_1\cos(ct) + Z_2\sin(ct))(Z_1\cos(ct) + Z_2\sin(ct))\,]$$

$$= \cos^2(ct)\, E[Z_1^2] + \sin^2(ct)\, E[Z_2^2] = \sigma^2$$

then

$$\gamma(t+h, t) = \begin{cases} \sigma^2 \sin(ct+2ch) & \text{for } |h|>0 \\ \sigma^2 & \text{for } h=0 \end{cases}$$

$\Rightarrow$ Since $Cov(X_{t+h}, X_t)$ depends on t, X_t is not stationary.

c) $X_t = Z_t\cos(ct) + Z_{t-1}\sin(ct)$

$$E[X_t] = E[\, Z_t\cos(ct) + Z_{t-1}\sin(ct)\,]$$

$$= \cos(ct)\, E[Z_t] + \sin(ct)\, E[Z_{t-1}] = 0$$

- For $h \neq 0$:

$$Cov(X_t, X_{t+h}) = E[\, X_t \cdot X_{t+h}\,]$$

$$= E[\,(Z_t\cos(ct) + Z_{t-1}\sin(ct))(Z_{t+h}\cos(ct+ch) + Z_{t+h-1}\sin(ct+ch-1))\,]$$

Since $E[Z_t \cdot Z_{t+h}] = 0$, $E[Z_t\, Z_{t+h-1}] = 0$, $E[Z_{t-1}\, Z_{t+h}] = 0$, $E[Z_{t-1}\, Z_{t+h-1}] = 0$

$$\Rightarrow Cov(X_t, X_{t+h}) = 0 = \gamma(t+h, t) = \gamma(h, 0)$$

- For $h = 0$:

$$Cov(X_t, X_t) = E[\, X_t \cdot X_t\,]$$

$$= E[\,(Z_t\cos(ct) + Z_{t-1}\sin(ct))(Z_t\cos(ct) + Z_{t-1}\cdot\sin(ct))\,]$$

$$= \cos^2(ct)\, E[Z_t^2] + \sin^2(ct)\, E[Z_{t-1}^2]$$

$$= \sigma^2$$

so $\gamma(h,0) = \begin{cases} 0 & \text{for } h \neq 0 \\ \sigma^2 & \text{for } h = 0 \end{cases}$ $\Rightarrow$ X_t is stationary.

d) $X_t = a + bZ_0$

$E[X_t] = E[a + bZ_0] = a + bE[Z_0] = a$

- For $h \neq 0$:

$$\begin{aligned} Cov(X_{t+h}, X_t) &= E[(X_{t+h} - a)(X_t - a)] \\ &= E[(a + bZ_0 - a)(a + bZ_0 - a)] \\ &= E[b^2 Z_0^2] = b^2\sigma^2 = \gamma(t+h, t) = \gamma(h, 0) \end{aligned}$$

- For $h = 0$:

$$\begin{aligned} Cov(X_t, X_t) &= E[(X_t - a)(X_t - a)] \\ &= E[(a + bZ_0 - a)(a + bZ_0 - a)] \\ &= E[b^2 Z_0^2] = b^2\sigma^2 \end{aligned}$$

then $\gamma(h, 0) = b^2\sigma^2$ for $|h| \geq 0$

$\Rightarrow$ X_t is a stationary process.

e) $X_t = Z_0 \cos(ct)$

$E[X_t] = E[Z_0 \cos(ct)] = \cos(ct) E[Z_0] = 0$

- For $h \neq 0$:

$$\begin{aligned} Cov(X_t, X_{t+h}) &= E[X_t \cdot X_{t+h}] \\ &= E[Z_0 \cos(ct) \cdot Z_0 \cos(ct + ch)] \\ &= \cos(ct)\cos(ct+h) \cdot \sigma^2 = \gamma(t, t+h) \end{aligned}$$

- For $h = 0$:

$Cov(X_t, X_t) = E[X_t \cdot X_t]$

$$= E[Z_0 \cdot \cos(ct) \cdot Z_0 \cos(ct)]$$

$$= \cos^2(ct) \cdot E[Z_0^2] = \cos^2(ct) \cdot \sigma^2$$

then $\gamma(t, t+h) = \begin{cases} \cos(ct)\cos(ct+h)\,\sigma^2 & \text{for } h \neq 0 \\ \cos^2(ct) \cdot \sigma^2 & \text{for } h = 0 \end{cases}$

$\Rightarrow$ X_t is not stationary.

f) $X_t = Z_t Z_{t-1}$

$E[X_t] = E[Z_t \cdot Z_{t-1}] = E[Z_t] \cdot E[Z_{t-1}] = 0$

- For $h \neq 0$

$$Cov(X_t, X_{t+h}) = E[X_t \cdot X_{t+h}]$$

$$= E[Z_t \cdot Z_{t-1} \cdot Z_{t+h} \cdot Z_{t+h-1}]$$

$$= E[Z_t] \cdot E[Z_{t-1}] \cdot E[Z_{t+h}] \cdot E[Z_{t+h-1}] = 0 = \gamma(t+h, t) = \gamma(h, 0)$$

- For $h = 0$

$$Cov(X_t, X_t) = E[X_t \cdot X_t]$$

$$= E[Z_t \cdot Z_{t-1} \cdot Z_t \cdot Z_{t-1}]$$

$$= E[Z_t^2] \cdot E[Z_{t-1}^2] = \sigma^4$$

$\Rightarrow$ $\gamma(h, 0) = \begin{cases} 0 & h \neq 0 \\ \sigma^4 & h = 0 \end{cases}$ $\Rightarrow$ X_t is stationary.

Question 2

Question 2 (20 points)

Let $\{X_t\}$ be a seasonal series of monthly observations, for which the seasonal component at time t is denoted by $\{s_t\}$ where $s_t = s_{t-4}$ for all t. Denote a weakly stationary series of random deviations by $\{Y_t\}$.

(a) Consider the model $X_t = a + bt + s_t + Y_t$ having a global linear trend and additive seasonality. Prove that the seasonally differenced series $\nabla_4 X_t$ is weakly stationary.

$$\nabla_4 X_t = \nabla_4 (a + bt + S_t + Y_t)$$

$$= (a - a) + (bt - b(t-d)) + (S_t - S_{t-d}) + (Y_t - Y_{t-d})$$

$$= bd + Y_t - Y_{t-d} \quad = 4b + Y_t - Y_{t-d}$$

Since $\{Y_t\}$ is weakly stationary, then $E[Y_t]$ is independent of t and $Cov(X_{t+h}, X_t) = \gamma(t+h, t) = \gamma_X(h, 0)$ independent of t.

$\Rightarrow$

$$E[\nabla_4 X_t] = E[bd + Y_t - Y_{t-d}] = bd + E[Y_t] - E[Y_{t-d}]$$

$\because$ $E[Y_t]$ & $E[Y_{t-d}]$ are independent of t and $E[Y_t] = \mu$ for $\forall t$

$\therefore$ $E[\nabla_4 X_t] = bd = 4b$ is independent of t.

- For $h \geq 4$:

$Cov(\nabla_4 X_t, \nabla_4 X_{t+h})$

$$= E[(4b + Y_t - Y_{t-d} - 4b)(4b + Y_{t+h} - Y_{t+h-d} - 4b)]$$

$$= E[(Y_t - Y_{t-d})(Y_{t+h} - Y_{t+h-d})]$$

$$= E[Y_t Y_{t+h}] - E[Y_t Y_{t+h-d}] - E[Y_{t-d} \cdot Y_{t+h}] + E[Y_{t-d} \cdot Y_{t+h-d}]$$

Since $\gamma_Y(h) = Cov(Y_t, Y_{t+h}) = E[Y_t \cdot Y_{t+h}] - E[Y_t] \cdot E[Y_{t+h}]$

$$\Rightarrow E[Y_t \cdot Y_{t+h}] = \gamma_Y(h) + \mu^2$$

$\therefore$ $Cov(\nabla_4 X_t, \nabla_4 X_{t+h}) = (\gamma_Y(h) + \mu^2) - (\gamma_Y(h-d) + \mu^2) - (\gamma_Y(h+d) + \mu^2) + (\gamma_Y(h) + \mu^2)$

$$= 2\gamma_Y(h) - \gamma_Y(h-d) - \gamma_Y(h+d)$$

$$= 2\gamma_Y(h) - \gamma_Y(h-4) - \gamma_Y(h+4)$$

- For $0 < h < 4$:

$$Cov(\nabla_4 X_t, \nabla_4 X_{t+h}) = (\gamma_Y(h) + \mu^2) - (\gamma_Y(d-h) + \mu^2) - (\gamma_Y(h+d) + \mu^2) + (\gamma_Y(h) + \mu^2)$$

$$= 2\gamma_Y(h) - \gamma_Y(4-h) - \gamma_Y(4+h)$$

- For $h = 0$: $Cov(\nabla_4 X_t, \nabla_4 X_t) = E[(Y_t - Y_{t-d})(Y_t - Y_{t-d})]$

$$= E[Y_t^2 + Y_{t-d}^2 - 2Y_t Y_{t-d}]$$

$$= 2\gamma_Y(0) - 2\gamma_Y(4)$$

$$\therefore Cov(\nabla_4 X_t, \nabla_4 X_{t+h}) = \begin{cases} 2\gamma_Y(h) - \gamma_Y(h-4) - \gamma_Y(h+4) & h \geq 4 \\ 2\gamma_Y(h) - \gamma_Y(4-h) - \gamma_Y(h+4) & 0 < h < 4 \\ 2\gamma_Y(0) - 2\gamma_Y(4) & h = 0 \end{cases}$$

$\therefore$ Both $E[\nabla_4 X_t]$ & $Cov(\nabla_4 X_t, \nabla_4 X_{t+h})$ are independent of t

$\therefore$ $\nabla_4 X_t$ is weakly stationary.

(b) Consider the model $X_t = (a + bt)S_t + Y_t$ having a global linear trend and multiplicative seasonality. Find a differencing operator (or sequence of operators) that transforms X_t into a weakly stationary sequence.

Now consider $\nabla_4(\nabla_4 X_t)$

$$\nabla_4 X_t = X_t - X_{t-d}$$

$$= (a+bt)S_t + Y_t - (a+bt-bd)S_{t-d} - Y_{t-d}$$

$$= bd\,S_{t-d} + Y_t - Y_{t-d}$$

$$
\begin{aligned}
\nabla_4 X_{t-d} &= X_{t-d} - X_{t-2d} \\
&= (a+bt-bd)S_{t-d} + Y_{t-d} - (a+bt-2bd)S_{t-2d} - Y_{t-2d} \\
&= bd\,S_{t-2d} + Y_{t-d} - Y_{t-2d}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla_4(\nabla_4 X_t) &= \nabla_4(bd\,S_{t-d} + Y_t - Y_{t-d}) \\
&= bd\,S_{t-d} + Y_t - Y_{t-d} - bd\,S_{t-2d} - Y_{t-d} + Y_{t-2d} \\
&= Y_t + Y_{t-2d} - Y_{t-d} - Y_{t-d}
\end{aligned}
$$

Then $E[\nabla_4(\nabla_4 X_t)] = E[Y_t + Y_{t-2d} - Y_{t-d} - Y_{t-d}] = 2\cancel{\mu} - 2\cancel{\mu} = 0$

$$
\begin{aligned}
&\mathrm{Cov}(\nabla_4(\nabla_4 X_t),\ \nabla_4(\nabla_4 X_{t+h})] \\
&= E[(Y_t + Y_{t-2d} - 2Y_{t-d})(Y_{t+h} + Y_{t+h-2d} - 2Y_{t+h-d})] \\
&= E[Y_t \cancel{Y_{t+h}}] + E[Y_t Y_{t+h-2d}] - 2E[Y_t \cdot Y_{t+h-d}] + E[Y_{t-2d} \cdot Y_{t+h}] + E[\cancel{Y_{t-2d}} \cdot Y_{t+h-2d}] - 2E[Y_{t-2d} \cdot Y_{t+h-d}] \\
&\quad - 2E[Y_{t-d} \cdot Y_{t+h}] - 2E[Y_{t-d} \cdot Y_{t+h-2d}] + 4E[Y_{t-d} \cdot \cancel{Y_{t+h-d}}]
\end{aligned}
$$

$$
\mathrm{Cov}(\nabla_4(\nabla_4 X_t)) = \begin{cases}
6\gamma_Y(h) - 4\gamma_Y(h-d) - 4\gamma_Y(h+d) + \gamma_Y(h+2d) + \gamma_Y(h-2d), & h \geq 8 \\
6\gamma_Y(h) - 4\gamma_Y(h-d) - 4\gamma_Y(h+d) + \gamma_Y(h+2d) + \gamma_Y(2d-h), & 4 \leq h < 8 \\
6\gamma_Y(h) - 4\gamma_Y(d-h) - 4\gamma_Y(h+d) + \gamma_Y(h+2d) + \gamma_Y(2d-h), & h < 4 \\
6\gamma_Y(0) - 8\gamma_Y(4) + 2\gamma_Y(8), & h = 0
\end{cases}
$$

$\Rightarrow$ Both $E[\nabla_4(\nabla_4 X_t)]$ and $\mathrm{Cov}(\nabla_4(\nabla_4 X_t))$ are independent of t

$\Rightarrow$ $\nabla_4(\nabla_4 X_t)$ is weakly stationary.

Question 3

Question 3 (20 points)

Consider the MA(m) process, with equal weights $1/(m+1)$ at all lags, given by

$$X_t = \frac{1}{m+1}\sum_{k=0}^{m} Z_{t-k}$$

Show that the ACF of this process is:

$$\rho(k) = \begin{cases} (m+1-k)/(m+1) & k = 0, 1, ..., \\ 0 & k > m, \\ \rho(-k) & k <= 0. \end{cases}$$

$\{Z_t\} \sim WN(0, \sigma^2)$

$E[X_t] = E\left[\frac{1}{m+1}\sum_{k=0}^{m} Z_{t-k}\right] = \sum_{k=0}^{m}\frac{1}{m+1}E[Z_{t-k}] = 0$

- For $h \leq m$:

$Cov(X_{t+h}, X_t) = E[X_{t+h} \cdot X_t]$

$= E\left[\left(\frac{1}{m+1}\sum_{k=0}^{m} Z_{t-k}\right)\left(\frac{1}{m+1}\sum_{k=0}^{m} Z_{t+h-k}\right)\right]$

$= \frac{1}{(m+1)^2} E\left[\left(\sum_{k=0}^{m} Z_{t-k}\right)\left(\sum_{k=0}^{m} Z_{t+h-k}\right)\right]$

Since $\gamma_Z(h) = 0$ for $h \neq 0$, then $Cov(Z_{t+h}, Z_t) = E[Z_{t+h} \cdot Z_t] = 0$

for $h = 0$, $Cov(Z_t, Z_t) = E[Z_t \cdot Z_t] = \sigma^2$

$\Rightarrow Cov(X_{t+h}, X_t) = \frac{1}{(m+1)^2} E\left[\sum_{k=0}^{m-h} Z_{t-k}^2\right]$ & $\{Z_t^2\}$ are Independent

$\Rightarrow Cov(X_{t+h}, X_t) = \frac{1}{(m+1)^2}(m-h+1)\sigma^2 = \gamma_X(h)$

- For $h = 0$:

$Cov(X_{t+h}, X_t) = Cov(X_t \cdot X_t)$

$= E\left[\left(\frac{1}{m+1}\sum_{k=0}^{m} Z_{t-k}\right)\left(\frac{1}{m+1}\sum_{k=0}^{m} Z_{t-k}\right)\right]$

$= \frac{1}{(m+1)^2} E\left[\sum_{k=0}^{m} Z_{t-k}^2\right]$

$= \frac{1}{(m+1)^2}(m+1)\sigma^2 = \frac{\sigma^2}{m+1}$

- For $h > m$:

$$\begin{aligned}
Cov(x_{t+h}, x_t) &= E[X_{t+h} \cdot X_t] \\
&= E\left[\left(\frac{1}{m+1}\sum_{k=0}^{m} X_{t+h-k}\right)\left(\frac{1}{m+1}\sum_{k=0}^{m} X_{t-k}\right)\right] \\
&= \frac{1}{(m+1)^2} E\left[\sum_{k=0}^{m} \cdot \sum_{j=0}^{m} X_{t+h-k} \cdot X_{t-j}\right] \\
&= 0
\end{aligned}$$

- For $h < 0$

Let $a = -h$ then

$$\begin{aligned}
Cov(X_{t+h}, X_t) &= E[X_{t+h} \cdot X_t] \\
&= E[X_{t-a} \cdot X_t] \\
&= Cov(X_{t-a} \cdot X_t) = \gamma_x(a) = \gamma_x(-h)
\end{aligned}$$

$$\because \quad \rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)}$$

$$\therefore \quad \rho_x(h) = \begin{cases} 0 & \text{for } h > m \\ \frac{1}{(m+1)^2}(m+1-h)\cancel{\sigma^2} * \frac{\cancel{m+1}}{\cancel{\sigma^2}} = (m+1-h)/m+1 & \text{for } h \le m \\ 1 & \text{for } h = 0 \\ \rho_x(-h) & \text{for } h < 0 \end{cases}$$

$$\Rightarrow \quad \rho_x(k) = \begin{cases} (m+1-k)/m+1 & , \ k = 0, 1, \cdots, m \\ 0 & , \ k > m \\ \rho_x(-h) & , \ h \le 0 \end{cases}$$

Question 4

Question 4 (20 points)

(a) Let $\{Y_t\}$ be a white noise process and let

$$X_t = m_t + Y_t$$

where $m_t = a + bt$.

Find the mean and variance of $\hat{m}_t$

$$\hat{m}_t = \frac{1}{2q+1} \sum_{i=t-q}^{t+q} X_t \quad \nearrow \Delta \; ?$$

and show that $\hat{m}_t$ is an unbiased estimator of $a + bt$ with lower variance than X_t for estimating the m_t.

$\because$ $\{Y_t\}$ is white Noise Process

$\therefore$ Y_t are uncorrelated with each other, $E[Y_t] = 0$, $Var[Y_t] = \sigma^2 < \infty$

$\gamma_Y(h) = 0$ if $h \neq 0$

$$E[\hat{m}_t] = E\left[\frac{1}{2q+1} \sum_{i=t-q}^{t+q} X_i\right]$$

$$= \frac{1}{2q+1} \sum_{i=t-q}^{t+q} E[a + b \cdot i + Y_i]$$

$$= \frac{1}{2q+1} \sum_{i=t-q}^{t+q} a + b \cdot i + E[Y_i]$$

$$= \frac{1}{2q+1} \left[(2q+1)a + b \cdot \frac{(t-q+t+q)(2q+1)}{2}\right]$$

$$= a + \frac{b}{2} * 2t = a + bt$$

$$Var[\hat{m}_t] = E[\hat{m}_t^2] - E[\hat{m}_t]^2$$

$$= E\left[\left(\frac{1}{2q+1} \sum_{i=t-q}^{t+q} X_i\right)^2\right] - (a+bt)^2$$

$$= \left(\frac{1}{2q+1}\right)^2 E\left[\left(\sum_{i=t-q}^{t+q} a + bi + Y_i\right)^2\right] - (a+bt)^2$$

$$= \left(\frac{1}{2q+1}\right)^2 E\left[\left(\sum_{i=t-q}^{t+q} a + bi + Y_i\right)\left(\sum_{j=t-q}^{t+q} a + bj + Y_j\right)\right] - (a+bt)^2$$

$$= \left(\frac{1}{2q+1}\right)^2 E\left[\sum_{i=t-q}^{t+q} \sum_{j=t-q}^{t+q} (a+bi+Y_i)(a+bj+Y_i)\right] - (a+bt)^2$$

Now considering $E[X_i \cdot X_j]$ for $i \neq j$

$$E[X_i \cdot X_i] = E[(a+bi+Y_i)(a+bj+Y_i)]$$

$$= E[\,a^2 + abj + aY_j + abi + b^2ij + biY_j + aY_i + bjY_i + Y_iY_j\,]$$

$$= a^2 + abj + abi + b^2ij$$

If $i=j$ then

$$E[X_i \cdot X_i] = E[\,(a+bi+Y_i)(a+bi+Y_i)\,]$$

$$= E[\,a^2 + abi + aY_i + abi + b^2i^2 + biY_i + aY_i + biY_i + Y_i^2\,]$$

$$= a^2 + abi + abi + b^2i^2 + \sigma^2$$

$\Rightarrow$

$$Var[\hat{m}_t] = \left(\frac{1}{2q+1}\right)^2 E\Big[\,(2q+1)^2a^2 + ab\sum_{i=t-q}^{t+q}(4q+2)i + \sum_{i=t-q}^{t+q} b^2 i \sum_{j=t-q}^{t+q} j + (2q+1)\sigma^2\Big] - (a+bt)^2$$

$$= \left(\frac{1}{2q+1}\right)^2 E\Big[\,(2q+1)^2a^2 + ab(4q+2)\frac{(2q+1)2t}{2} + \sum_{i=t-q}^{t+q} b^2 i\,\frac{(2q+1)2t}{2} + (2q+1)\sigma^2\Big] - (a+bt)^2$$

$$= \left(\frac{1}{2q+1}\right)^2 \Big[\,(2q+1)^2a^2 + ab(4q+2)(2q+1)\cdot t + (2q+1)t\cdot b^2\cdot\frac{(2q+1)2t}{2} + (2q+1)\sigma^2\Big] - (a+bt)^2$$

$$= \left(\frac{1}{2q+1}\right)^2 \Big[\,(2q+1)^2a^2 + 2ab(2q+1)^2t + (2q+1)^2t^2b^2 + (2q+1)\sigma^2\Big] - (a+bt)^2$$

$$= \cancel{a^2} + \cancel{2abt} + t^2b^2 + \frac{\sigma^2}{2q+1} - \cancel{a^2} - b^2 - \cancel{2abt}$$

$$= (t^2-1)b^2 + \frac{\sigma^2}{2q+1}$$

Since $E[\hat{m}_t] = m_t \Rightarrow \hat{m}_t$ is an unbiased estimator

Now considering $Var[X_t]$:

$$E[X_t] = E[\,a + b\cdot t + Y_t\,] = a + bt$$

$$Var[X_t] = E[X_t^2] - E[X_t]^2$$

$$= E[\,(a+bt+Y_t)(a+bt+Y_t)\,] - (a+bt)^2$$

$$= E[\,a^2 + abt + aY_t + abt + b^2t^2 + btY_t + aY_t + btY_t + Y_t^2\,] - (a+bt)^2$$

$$= \cancel{a^2} + \cancel{2abt} + b^2t^2 + \sigma^2 - \cancel{a^2} - b^2 - \cancel{2abt}$$

$$= (t^2-1)\,b^2 + \sigma^2$$

Since $Var(X_t) = (t^2-1)b^2 + \sigma^2$ and $Var(\hat{m}_t) = (t^2-1)b^2 + \frac{\sigma^2}{2q+1}$

$\Rightarrow$ $\hat{m}_t$ has a much smaller variance than X_t.

(b) Let $\{Y_t\}$ be a white noise process. Show that a linear filter $\{a_j\}$ applied to $X_t = m_t + Y_t$ so that

$$\hat{m}_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j},$$

will yield an unbiased estimate of m_t for all $k-$th degree polynomials $m_t = c_0 + c_1 t + \cdots + c_k t^k$ if and only if the following two conditions hold:

i) $\sum_j a_j = 1$ and
ii) $\sum_j j^r a_j = 0$ for $r = 1, .., k$.

$$E[\hat{m}_t] = E[\sum_{j=-\infty}^{\infty} a_j X_{t-j}]$$
$$= E[\sum_{j=-\infty}^{\infty} a_j (m_{t-j} + Y_{t-j})]$$
$$= E[\sum_{j=-\infty}^{\infty} a_j m_{t-j}] + E[\sum_{j=-\infty}^{\infty} a_j Y_{t-j}]$$
$$= E[\sum_{j=-\infty}^{\infty} a_j (C_0 + C_1(t-j) + \cdots + C_k(t-j)^k)]$$

"$\Rightarrow$" If $E[\hat{m}_t] = m_t$

$$E[\sum_{j=-\infty}^{\infty} a_j (C_0 + C_1(t-j) + \cdots + C_k(t-j)^k)] = C_0 + C_1 t \cdots + C_k t^k$$
$$E[\sum_{j=-\infty}^{\infty} a_j \cdot C_0 + \cdots + a_j C_k (t-j)^k] = C_0 + C_1 t + \cdots + C_k t^k$$
$$(\sum_{j=-\infty}^{\infty} a_j) \cdot C_0 + \cdots + (\sum_{j=-\infty}^{\infty} a_j (t-j)^k) \cdot C_k = C_0 + C_1 t + \cdots + C_k t^k$$

$\therefore$ $\sum_{j=-\infty}^{\infty} a_j = 1$ and $\sum_{j=-\infty}^{\infty} a_j (t-j)^r = t^r$ for $r = 1, 2, \cdots k$

$\sum_{j=-\infty}^{\infty} a_j (t-j)^r = t^r$ should be hold for any $t \geq 0$

$$\therefore \sum_{j=-\infty}^{\infty} a_j (-j)^r = 0$$

$\sum_{j=-\infty}^{\infty} a_j \cdot j^r = 0$ for r ood or even

$\therefore$ Condition i) and ii) hold

"<=" Assume that $\sum_{j=-\infty}^{\infty} a_j \cdot j^r = 0$ holds for $r = 1, \cdots k$

and $\sum_{j=-\infty}^{\infty} a_j = 1$ hold

When $r=2$:

$$\sum_{j=-\infty}^{\infty} a_j \cdot j^2 = 0$$

Since $j^2 \geqslant 0$ always $\Rightarrow \begin{cases} a_j = 0 \text{ for } \forall j \neq 0 \\ \sum_{j=-\infty}^{\infty} a_j = 1 \end{cases} \Rightarrow a_0 = 1$

$$\begin{aligned}
\therefore E[\hat{m}_t] &= E\left[\sum_{j=-\infty}^{\infty} a_j \left(c_0 + c_1(t-j) + \cdots + c_k (t-j)^k\right)\right] \\
&= E\left[\sum_{j=-\infty}^{\infty} a_j c_0 + \cdots + a_j c_k (t-j)^k\right] \\
&= c_0 + \sum_{j=-\infty}^{\infty} a_j c_1 (t-j) + \cdots + a_j c_k (t-j)^k \\
&= c_0 + \left(\sum_{j=-\infty}^{\infty} a_j (t-j)\right) c_1 + \cdots + \left(\sum_{j=-\infty}^{\infty} a_j (t-j)^k\right) c_k \\
&= c_0 + a_0 (t-0) \cdot c_1 + \cdots + a_0 (t-0)^k \cdot c_k \\
&= c_0 + c_1 t + \cdots + c_k t^k
\end{aligned}$$

A1_Question5

For this question, you will analyze hypothetical sales data for company X measured in successive 4-week periods from 1995-1998. Download the Excel file called "Assign1Q5_sales.xlsx" from the Assignment 1 page at the myCourses website. You can create a time series object by first reading the data from the Excel file into a tibble data object and then converting the tibble into an appropriate time series object. To read in the Excel file, you can run the following code:

```
library(readxl)
sales_data<-read_excel("Assign1Q5_sales.xlsx")
```

To convert it into a time series object, you can run the following code:

```
library(forecast)
# ts is creating an object
sales_ts <- ts(sales_data, frequency=13)
```

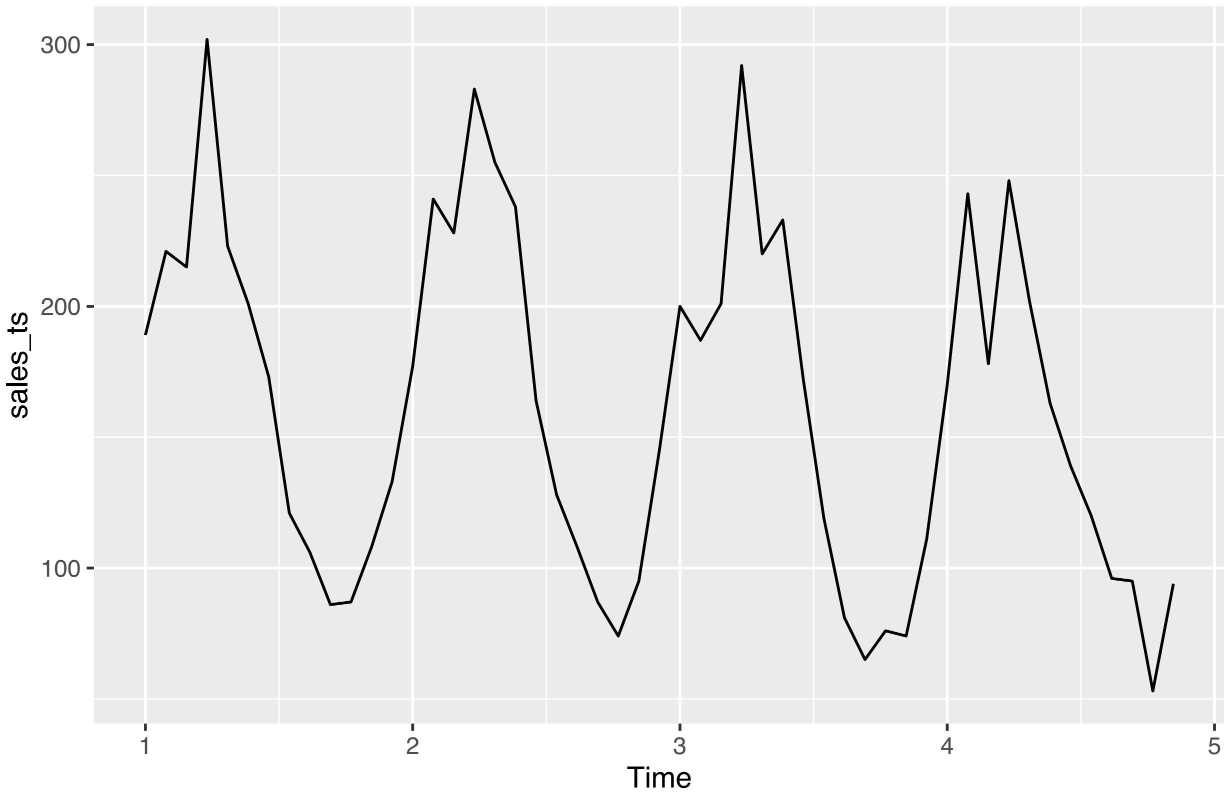
Now you can use the sales_ts time series object in the same way that I have used the LakeHuron and a10 times series objects in the class examples.

(a) First plot and describe the time series. Note any perceived trend and seasonal components. Do you believe that the sales data series is a stationary series? Explain your answer. Hint: You may want to use an ACF plot.

do the basic time plot

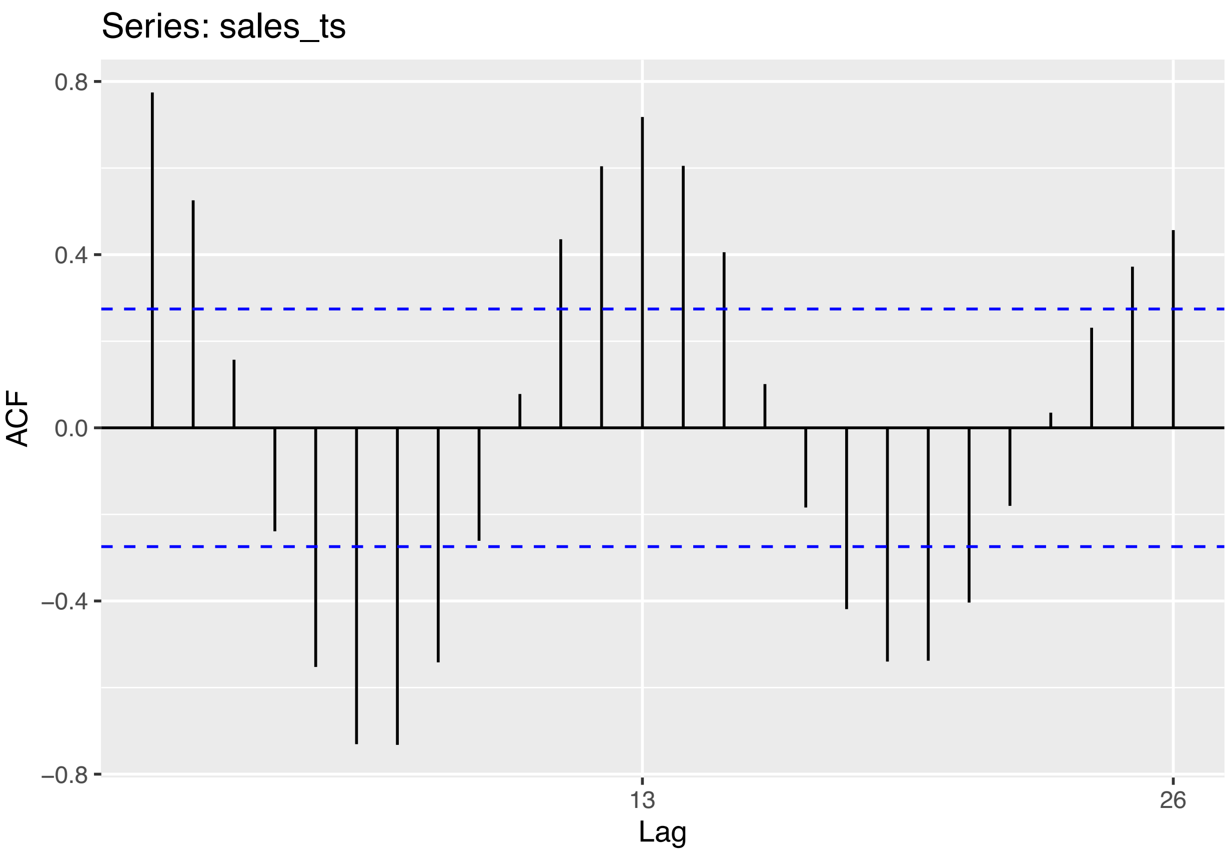
```
autoplot(sales_ts,facets=TRUE)
```

```
## Warning: Ignoring unknown parameters: facets
```



The above image shows the time series model might have an decreasing trend and seasonal component.

```
#ACF plot
ggAcf(sales_ts)
```

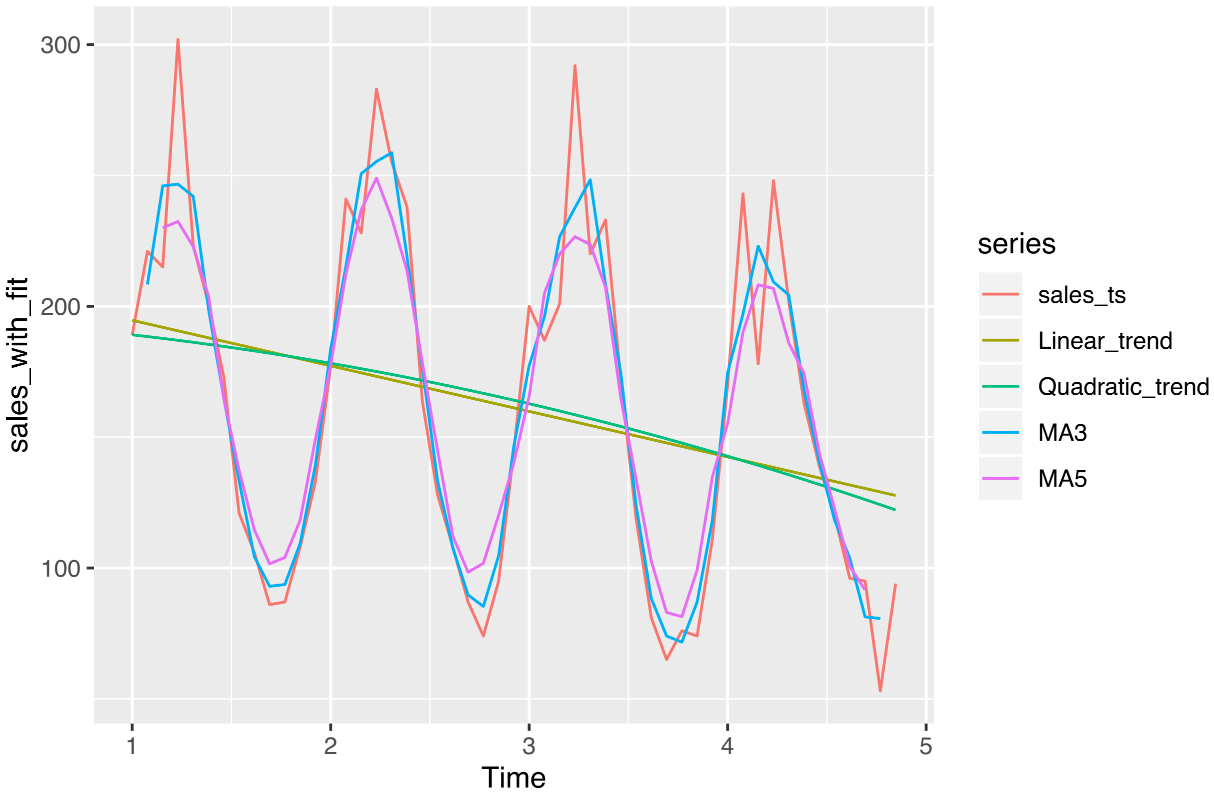


The above ACF plot shows the seasonality. Since the magnitude of autocorrelation are always larger under the seasonal lags (The selected time points y_{t+k} and y_t are sitting around the peak and the bottom respectively, then $(y_{t+k}-\overline{y})(y_t-\overline{y})$ will be large in magnitude). The smaller lags also give larger and positive autocorrelation which implies that there is a trend in these data. Because the nearby in time (smaller k) means nearby in value which can always give positive $(y_{t+k}-\overline{y})(y_t-\overline{y})$.
There is a slowly decreasing trend of magnitude in the ACF plot (Large autocorrelation value for large lags implies that the future data series is highly related to the past data series), so it might be a non-stationay process.

(b) Estimate trend and seasonal components for the time series. Do you find evidence of a trend and seasonal component in the data? Explain. Assess the residuals from your decomposition for evidence that they are resulting from a white noise or iid noise process.

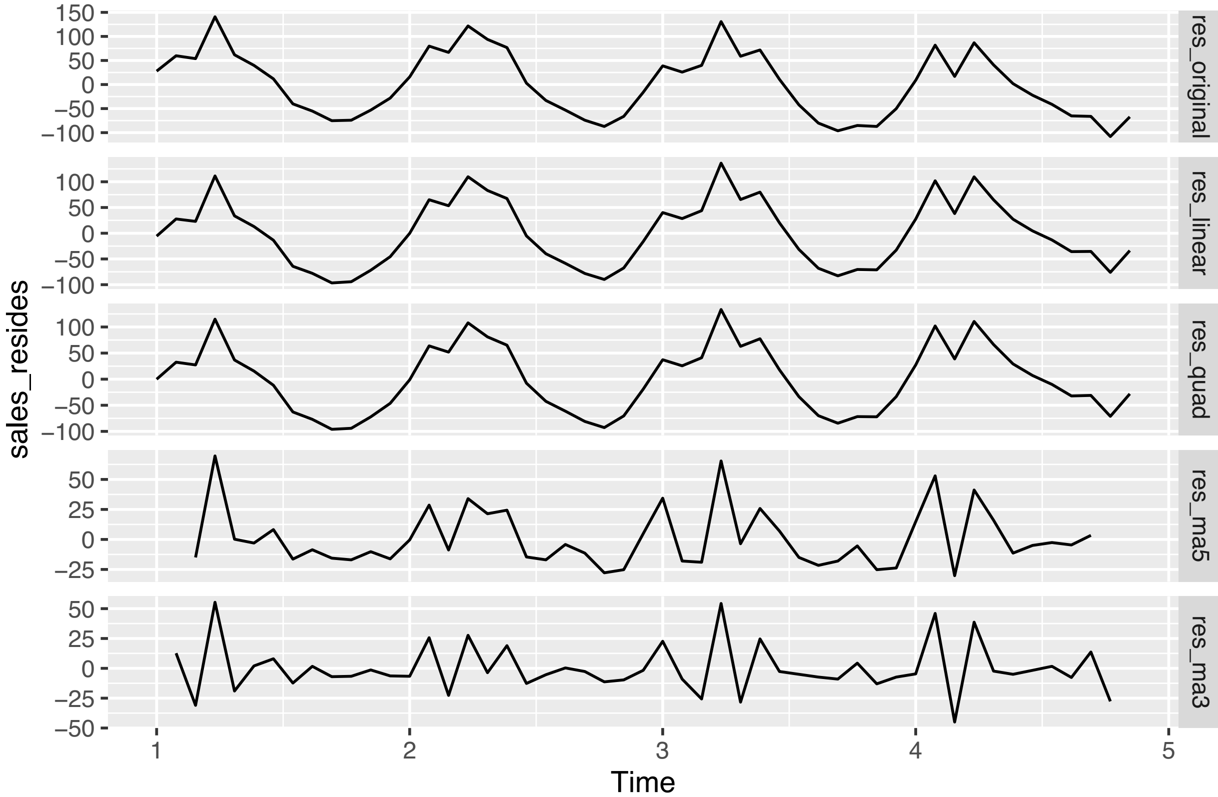
The following figure shows 5 different estimated trend on the sales dataset.

```
sales_linear<- tslm(sales_ts~trend)
sales_quadratic<- tslm(sales_ts~trend+I(trend^2))
sales_ma3<-ma(sales_ts,order=3)
sales_ma5<-ma(sales_ts,order=5)
#plot the fitted graph
sales_with_fit<-cbind(sales_ts,Linear_trend=fitted(sales_linear),Quadratic_trend=fitted(sales_quadratic)
autoplot(sales_with_fit)
```

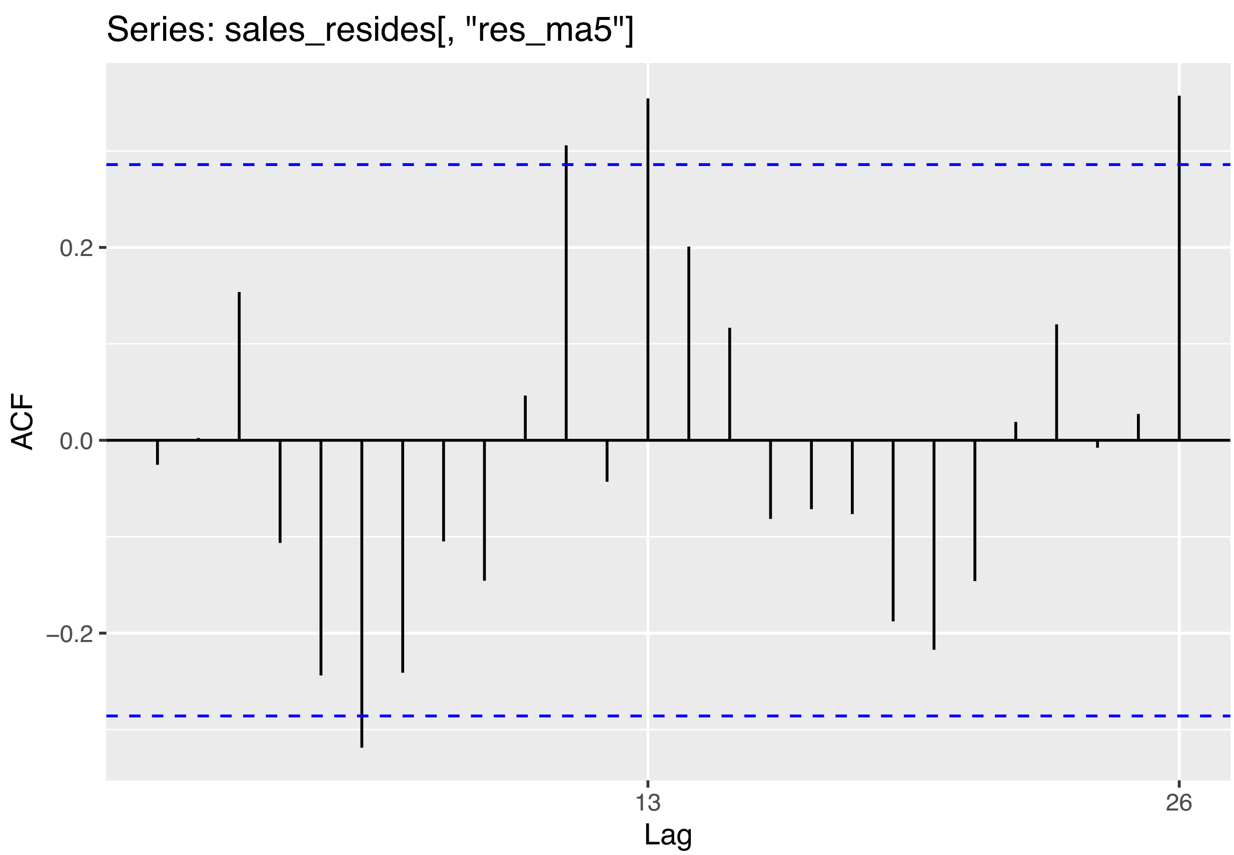


The residual plots by removing these 5 trends respectively from the dataset :

```
sales_resides<-cbind(res_original=sales_ts-mean(sales_ts),
                     res_linear=sales_ts-fitted(sales_linear),
                     res_quad=sales_ts-fitted(sales_quadratic),
                     res_ma5=sales_ts-sales_ma5,
                     res_ma3=sales_ts-sales_ma3)
autoplot(sales_resides,facet=TRUE)
```




```r
ggAcf(sales_resides[,"res_ma5"])
```

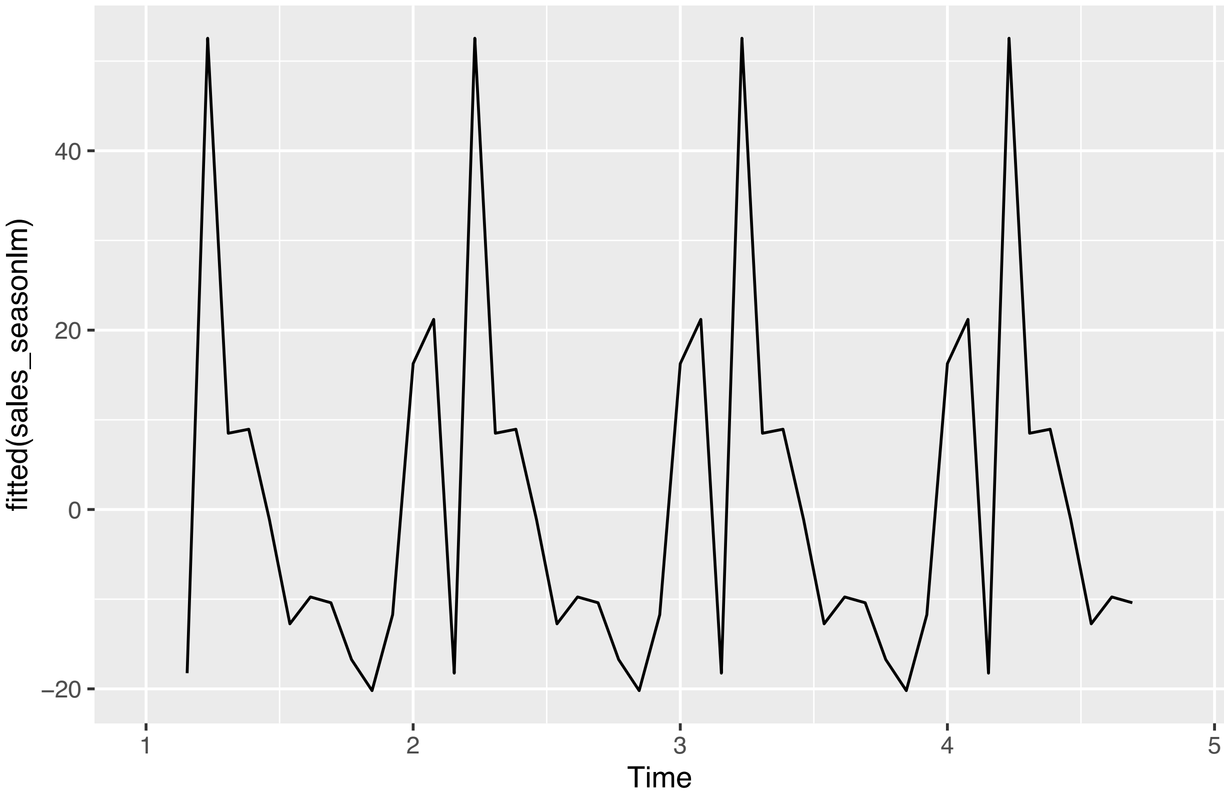


By removing the trend from dataset, the estimated autocorrelation becomes smaller.It gives the evidence that the dataset contains seasonal component. Now regard the MA5 as the estimated trend and evaluate the seasonality of the dataset:

```r
frequency(sales_resides[,"res_ma5"])
```

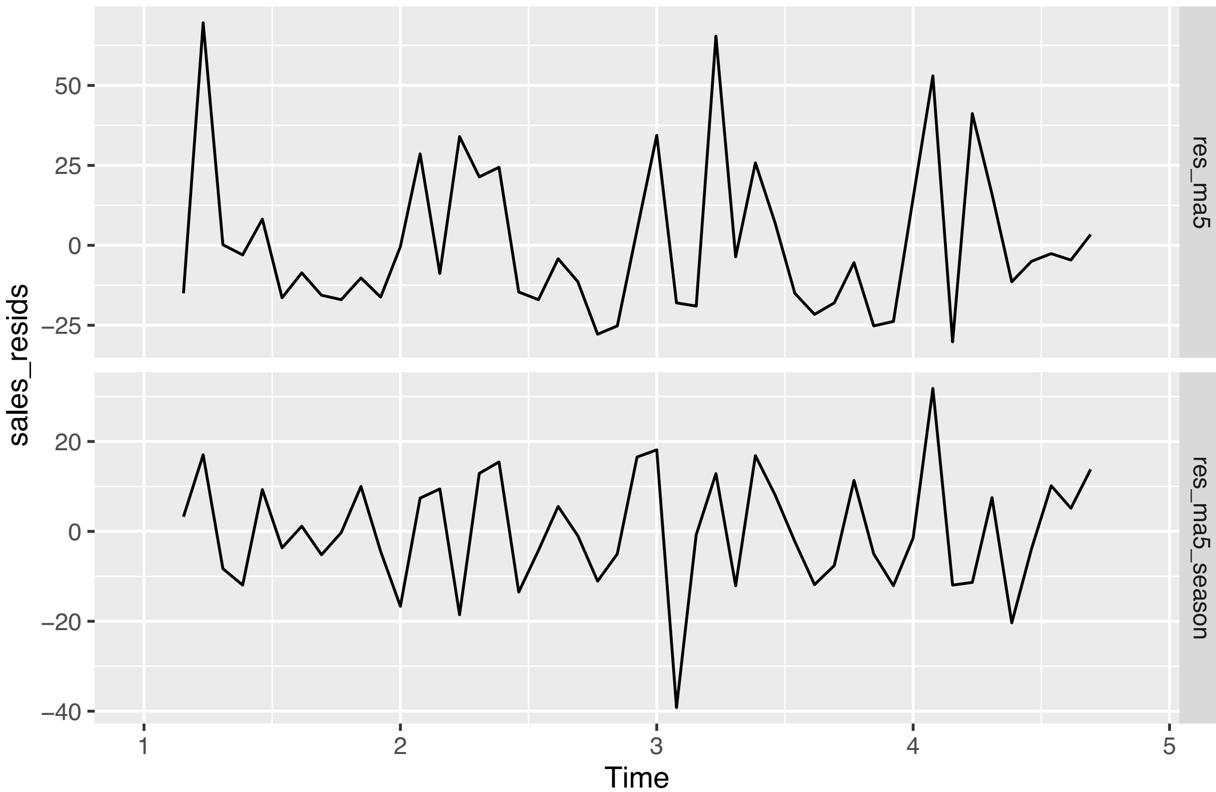
```
## [1] 13
```

```r
sales_seasonlm<-tslm(res_ma5~season,data=sales_resides)
autoplot(fitted(sales_seasonlm))
```



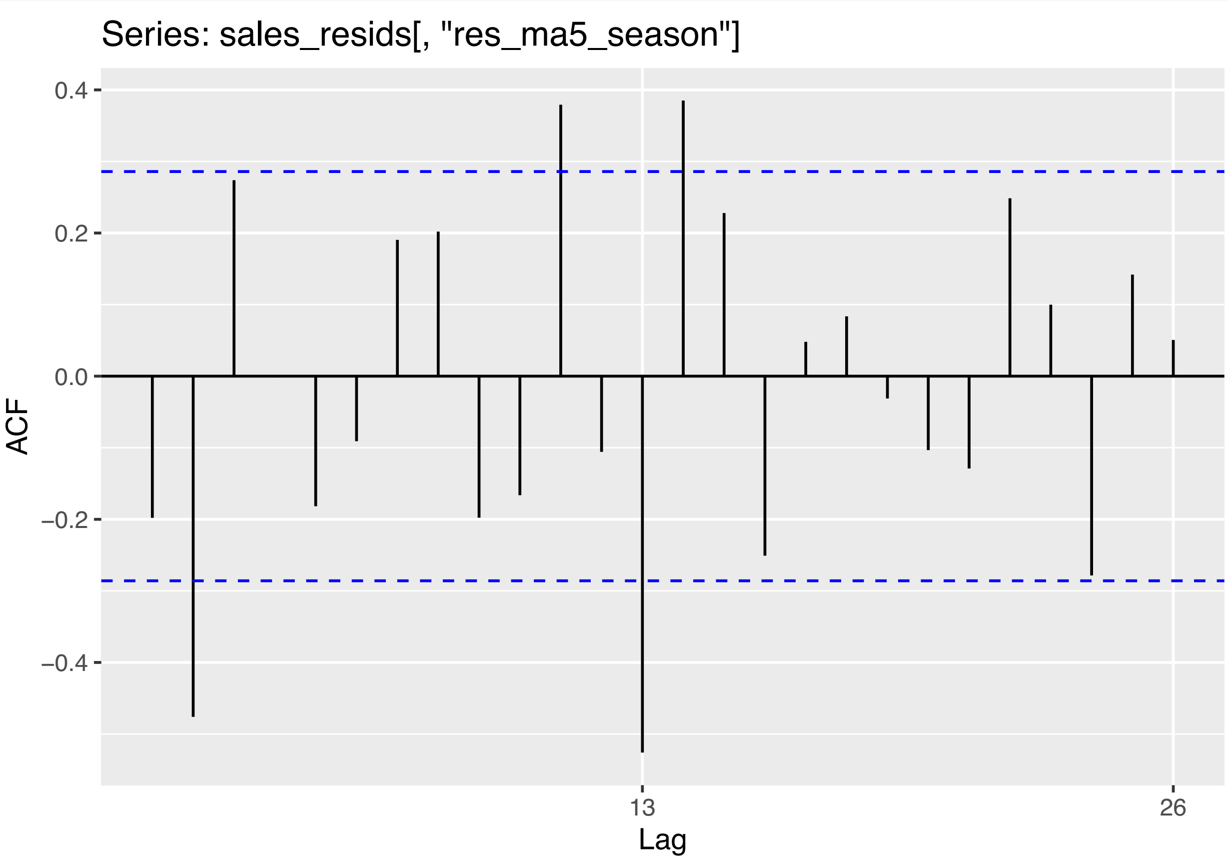
Visualize the data by removing both MA5 trend the seasonal component:

```r
sales_resids<-cbind(res_ma5=sales_ts-sales_ma5,
                    res_ma5_season=sales_ts-sales_ma5-fitted(sales_seasonlm))
autoplot(sales_resids,facet=TRUE)
```



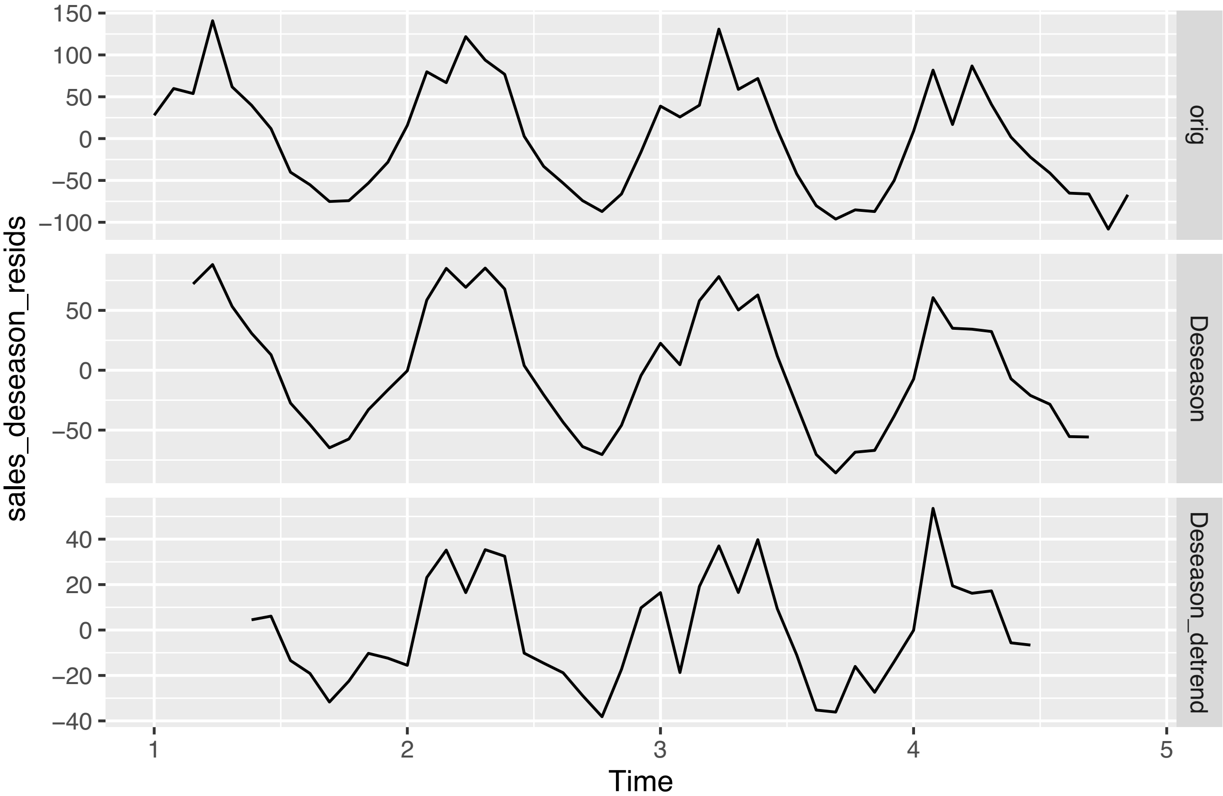
Do the ACF plot on the residual dataset by removing both trend and seasonal component:

```
ggAcf(sales_resids[,"res_ma5_season"])
```

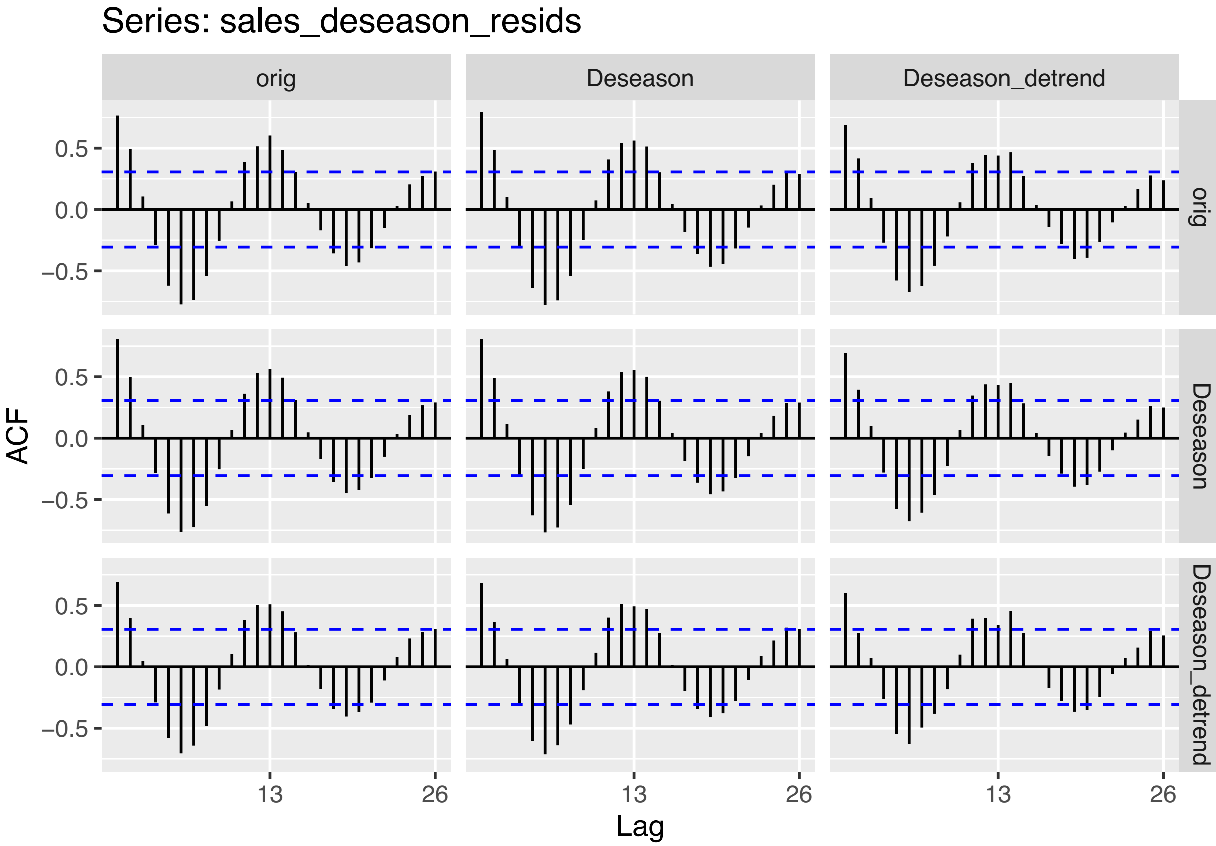


As the above figure shown, the correlation value getting smaller when the sesonal components are removed. This is the evidence of the seasonal component.Now try to remove the seasonal component first and then subtract the trend which gives the same conclusion.

```
sales_deseason<-sales_ts-mean(sales_ts)-fitted(sales_seasonlm)
sales_deseason_resids<-cbind(orig=sales_ts-mean(sales_ts),
                             Deseason=sales_ts-mean(sales_ts)-fitted(sales_seasonlm),
                             Deseason_detrend=sales_ts-mean(sales_ts)-fitted(sales_seasonlm)-ma(sales_de
autoplot(sales_deseason_resids,facet=TRUE)
```




```
ggAcf(sales_deseason_resids)
```



Now test the residuals to check if it is a white noise process or iid noise process:

```
library(forecast)
library(tidyverse)
```

```
## -- Attaching packages ------------------------------------------------------------------------

## v ggplot2 3.2.1     v purrr   0.3.3
## v tibble  2.1.3     v dplyr   0.8.4
## v tidyr   1.0.2     v stringr 1.4.0
## v readr   1.3.1     v forcats 0.4.0

## -- Conflicts ---------------------------------------------------------------------------------
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```
library(fpp2)
```

```
## Loading required package: fma

## Loading required package: expsmooth
```

```
library(tibbletime)
```

```
##
## Attaching package: 'tibbletime'

## The following object is masked from 'package:stats':
##
##     filter
```

```r
library(tsbox)
library(gridExtra)
```

```
##
## Attaching package: 'gridExtra'

## The following object is masked from 'package:dplyr':
##
##     combine
```

```r
library(knitr)
res_ma7<-sales_ts-ma(sales_ts,7)
sales_seasonlm<-tslm(res_ma7~season)
res_ma7_season<-res_ma7-fitted(sales_seasonlm)
test_acf_ma7<-tibble(Acf_vals=as.numeric(Acf(res_ma7_season,plot=FALSE)$acf),
      Lag = 0:(length(Acf_vals)-1),
      Z = 1.96/sqrt(length(res_ma7_season)),
      Reject=abs(Acf_vals) >Z)
test_acf_ma7 %>% filter(Lag>0) %>% count(Reject)
```

```
## # A tibble: 2 x 2
##   Reject     n
##   <lgl>  <int>
## 1 FALSE     21
## 2 TRUE       5
```

```r
kable(test_acf_ma7 %>% mutate_all(format,digits=3) %>% select(Lag,Acf_vals,Z,Reject))
```

Lag	Acf_vals	Z	Reject
0	1.0000	0.274	TRUE
1	-0.0298	0.274	FALSE
2	-0.4106	0.274	TRUE
3	-0.0199	0.274	FALSE
4	-0.0718	0.274	FALSE
5	-0.1642	0.274	FALSE
6	-0.0372	0.274	FALSE
7	0.2909	0.274	TRUE
8	0.1130	0.274	FALSE
9	-0.0964	0.274	FALSE
10	-0.0660	0.274	FALSE
11	0.2231	0.274	FALSE
12	-0.2209	0.274	FALSE
13	-0.4760	0.274	TRUE
14	0.3465	0.274	TRUE
15	0.2888	0.274	TRUE
16	-0.0679	0.274	FALSE
17	-0.0207	0.274	FALSE
18	0.0354	0.274	FALSE
19	-0.1470	0.274	FALSE
20	-0.1309	0.274	FALSE
21	0.0341	0.274	FALSE
22	0.2159	0.274	FALSE
23	0.0907	0.274	FALSE
24	-0.1958	0.274	FALSE
25	0.1089	0.274	FALSE

Lag	Acf_vals	Z	Reject
26	-0.0322	0.274	FALSE
The re	sult shows	21 False	and 5 True, so it might be a white noise process.
Now co	nduct the t	est to c	heck if it is a iid process:

```r
library(itsmr)
```

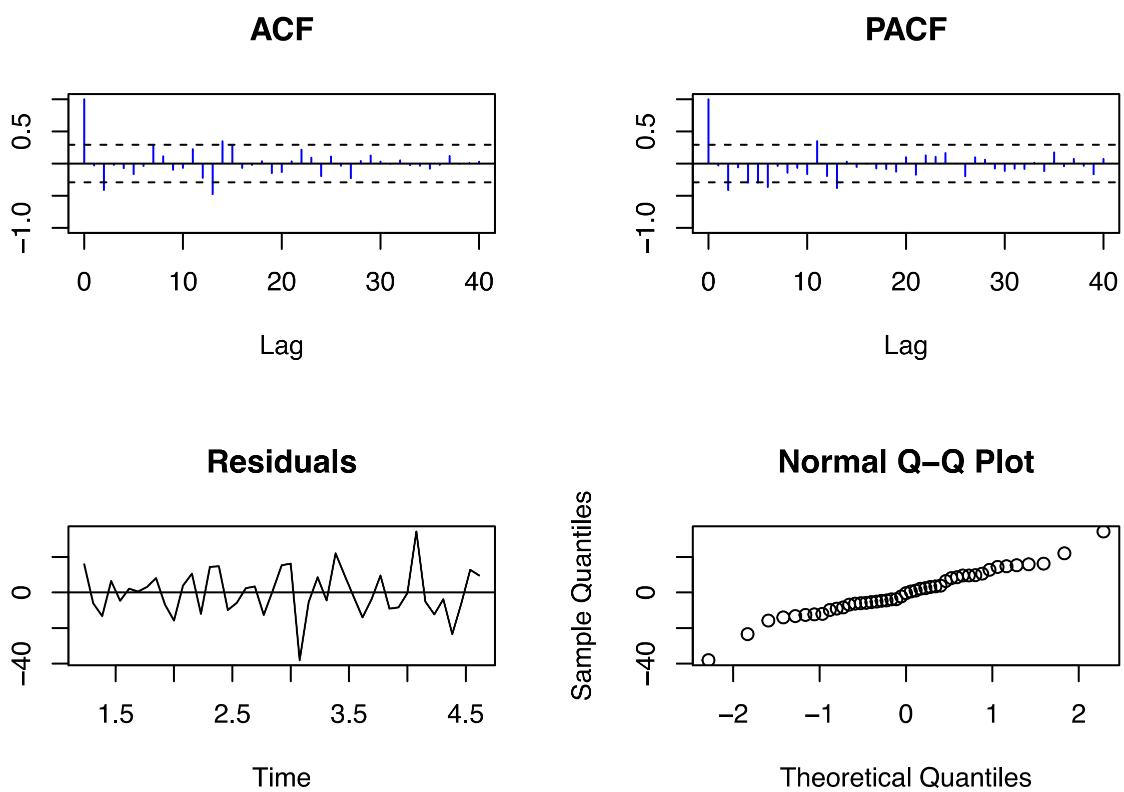
```
##
## Attaching package: 'itsmr'

## The following objects are masked from 'package:fma':
##
##     airpass, strikes

## The following object is masked from 'package:forecast':
##
##     forecast
```

```r
test(res_ma7_season %>% na.omit(.))
```

```
## Null hypothesis: Residuals are iid noise.
## Test                        Distribution Statistic   p-value
## Ljung-Box Q                Q ~ chisq(20)     55.31         0 *
## McLeod-Li Q                Q ~ chisq(20)     24.77    0.2104
## Turning points T   (T-28.7)/2.8 ~ N(0,1)        26    0.3359
## Diff signs S           (S-22)/2 ~ N(0,1)        25    0.1255
## Rank P            (P-495)/51.1 ~ N(0,1)       488    0.8911
```

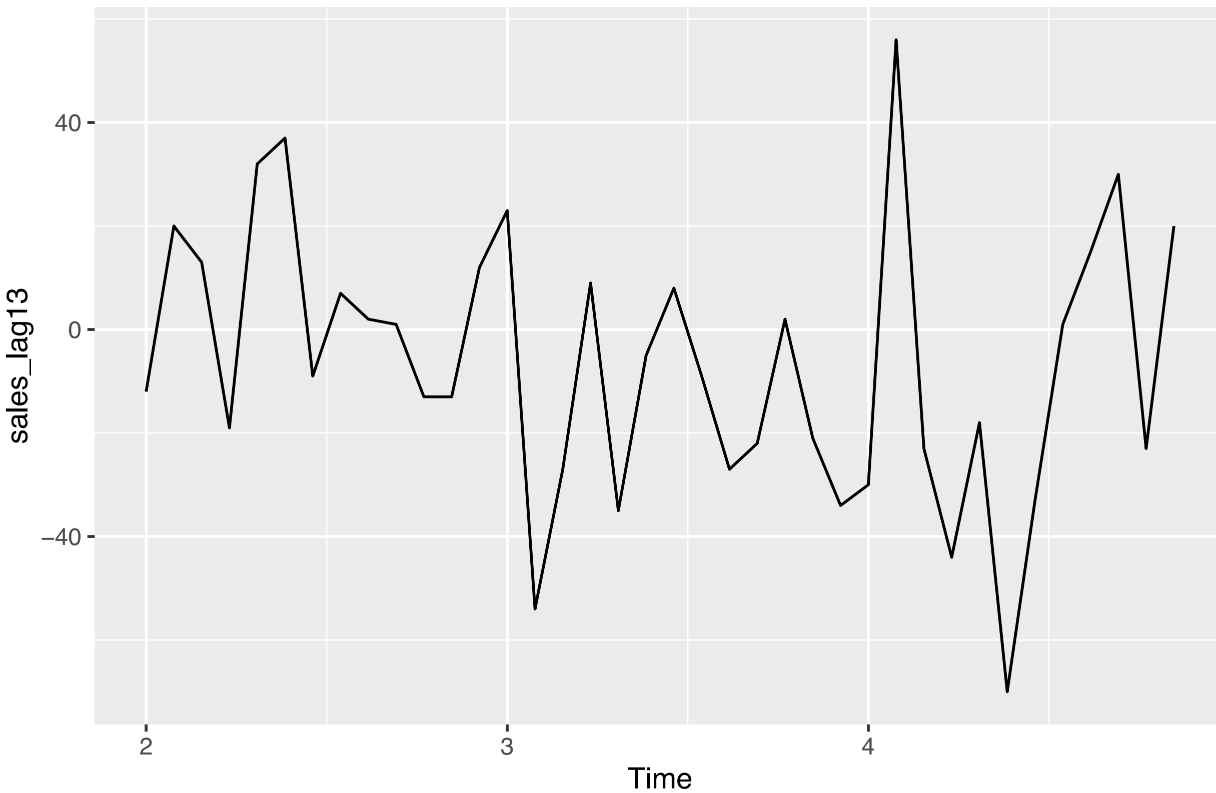


Most of the tests believe that it is a iid process except for Ljung-Box Q test.

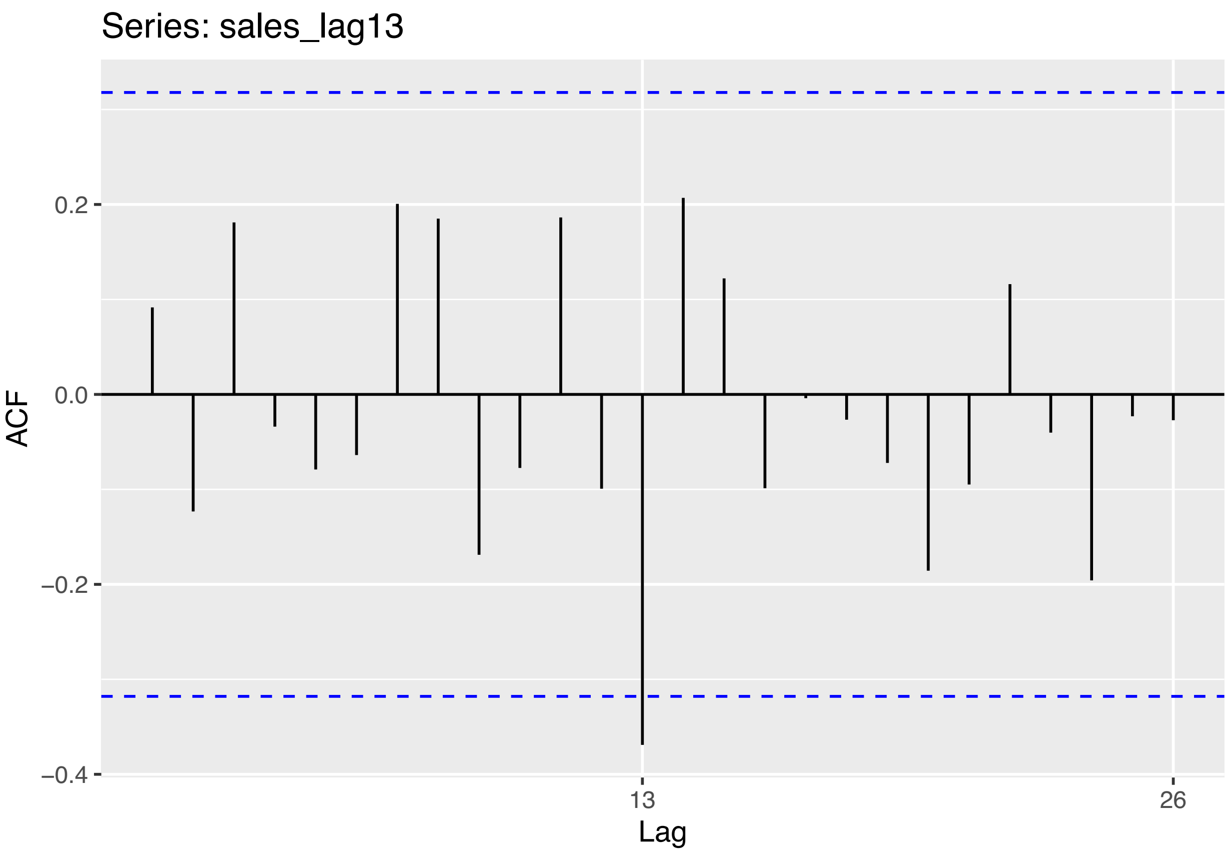
(c)Using an appropriate sequence of difference operators, try to eliminate any perceived trend and seasonal components from part (c). Assess the residuals from your decomposition for evidence that they are resulting from a white noise or iid noise process.

Gonna use $\nabla_d X_t$ to eliminate the trend and seasonal component:

```
sales_lag13<-diff(sales_ts,13)
autoplot(sales_lag13)
```




```
ggAcf(sales_lag13)
```




```
#sales_lag13_lag13<-diff(sales_lag13,13)
#autoplot(sales_lag13_lag13)
```

According to the ACF plot, we can see that the seasonal component and trend are almost removed.Now conduct the test to check if the difference is a white noise process or iid process.

```
test_acf_lag<-tibble(Acf_vals=as.numeric(Acf(sales_lag13,plot=FALSE)$acf),
      Lag = 0:(length(Acf_vals)-1),
      Z = 1.96/sqrt(length(sales_lag13)),
      Reject=abs(Acf_vals) >Z)
test_acf_lag %>% filter(Lag>0) %>% count(Reject)
```

```
## # A tibble: 2 x 2
##   Reject     n
##   <lgl>  <int>
## 1 FALSE     25
## 2 TRUE       1
```

```
kable(test_acf_lag %>% mutate_all(format,digits=3) %>% select(Lag,Acf_vals,Z,Reject))
```

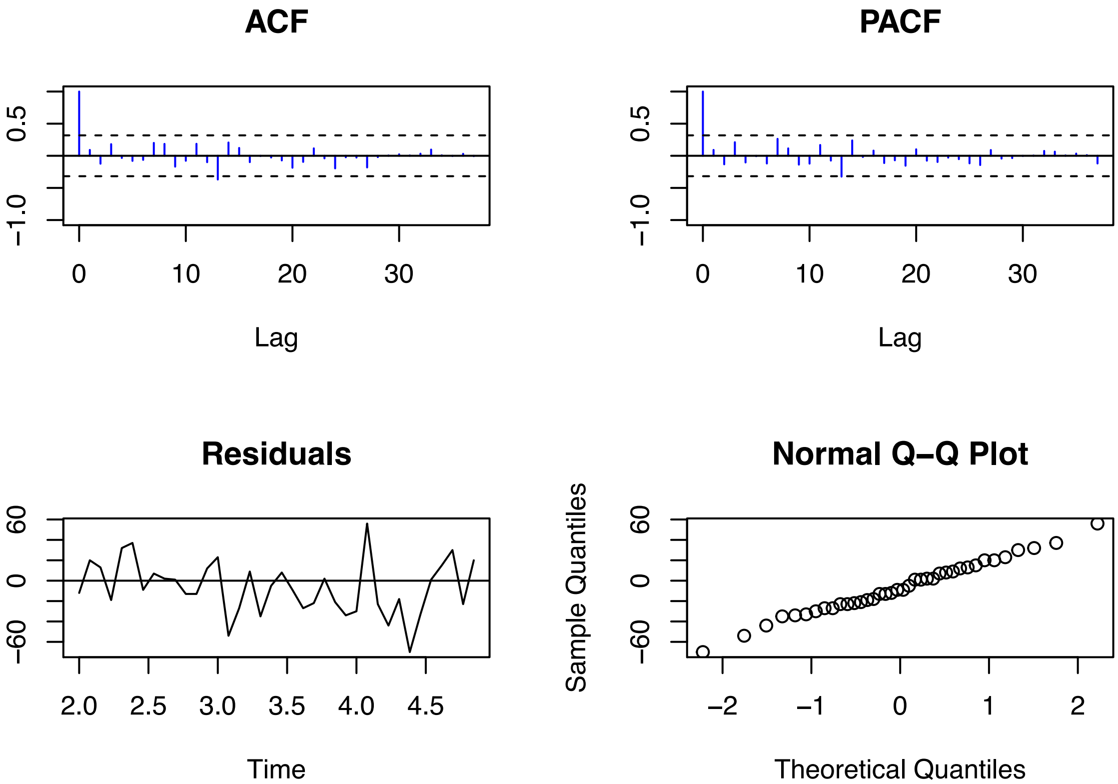
Lag	Acf_vals	Z	Reject
0	1.00000	0.318	TRUE
1	0.09159	0.318	FALSE
2	-0.12324	0.318	FALSE
3	0.18109	0.318	FALSE
4	-0.03391	0.318	FALSE
5	-0.07903	0.318	FALSE
6	-0.06384	0.318	FALSE

Lag	Acf_vals	Z	Reject
7	0.20064	0.318	FALSE
8	0.18513	0.318	FALSE
9	-0.16885	0.318	FALSE
10	-0.07740	0.318	FALSE
11	0.18633	0.318	FALSE
12	-0.09927	0.318	FALSE
13	-0.36903	0.318	TRUE
14	0.20694	0.318	FALSE
15	0.12214	0.318	FALSE
16	-0.09879	0.318	FALSE
17	-0.00398	0.318	FALSE
18	-0.02662	0.318	FALSE
19	-0.07214	0.318	FALSE
20	-0.18562	0.318	FALSE
21	-0.09486	0.318	FALSE
22	0.11610	0.318	FALSE
23	-0.04024	0.318	FALSE
24	-0.19581	0.318	FALSE
25	-0.02302	0.318	FALSE
26	-0.02721	0.318	FALSE

The results give 1 True and 25 False, so the difference is probably a white noise process.

```r
test(sales_lag13%>% na.omit(.))
```

```
## Null hypothesis: Residuals are iid noise.
## Test                        Distribution Statistic   p-value
## Ljung-Box Q                Q ~ chisq(20)     27.04    0.1342
## McLeod-Li Q                Q ~ chisq(20)     13.16    0.8706
## Turning points T      (T-24)/2.5 ~ N(0,1)       19    0.0487 *
## Diff signs S        (S-18.5)/1.8 ~ N(0,1)       20    0.4054
## Rank P            (P-351.5)/39.8 ~ N(0,1)      282    0.0806
```



4 out of 5 tests believe that the difference is an iid process except for the Turning points T test, so it is probably a iid process.