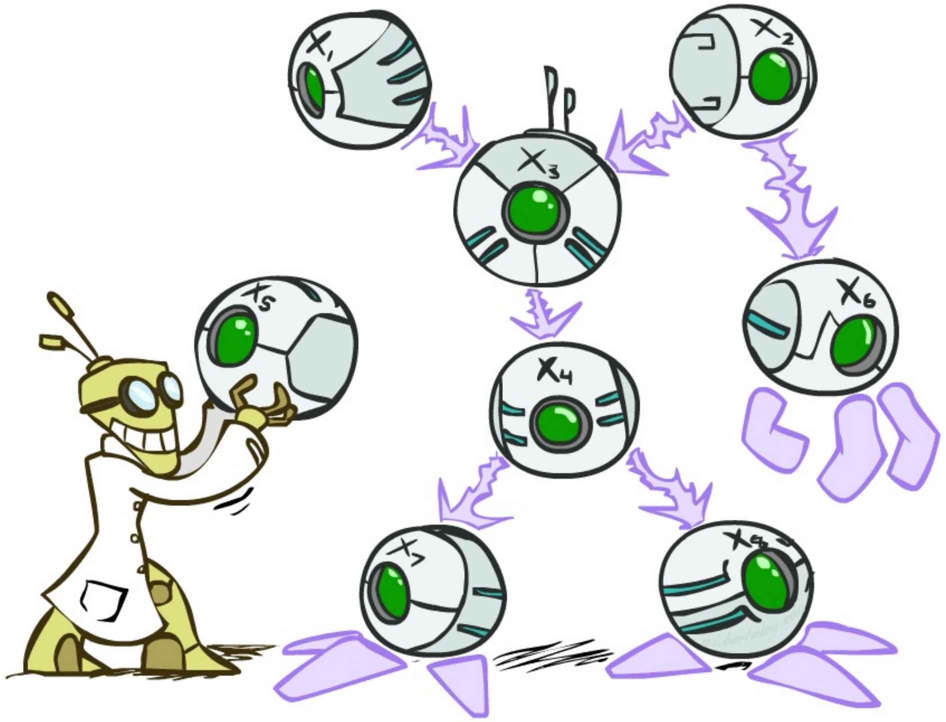
# Ve492: Introduction to Artificial Intelligence

## Bayesian Networks: Representation





Paul Weng

UM-SJTU Joint Institute


Slides adapted from http://ai.berkeley.edu, CMU, AIMA, UM

# Announcements

- Midterm
  - Graded midterms will be available this week
- Homework 5
  - Due June 30 at 11:59pm
- Project 3: RL
  - Due July 5 at 11:59pm

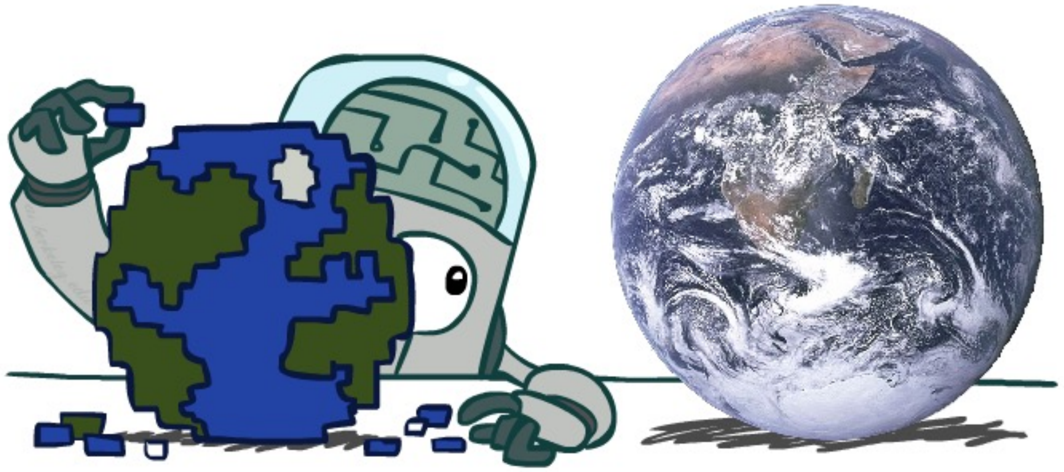
# Bayes' Nets

- Representation
- Conditional Independences
- Probabilistic Inference

# Probabilistic Models

- Models describe how (a portion of) the world works

- Models are always simplifications
  - May not account for every variable
  - May not account for all interactions between variables
  - "All models are wrong; but some are useful."
    – George E. P. Box



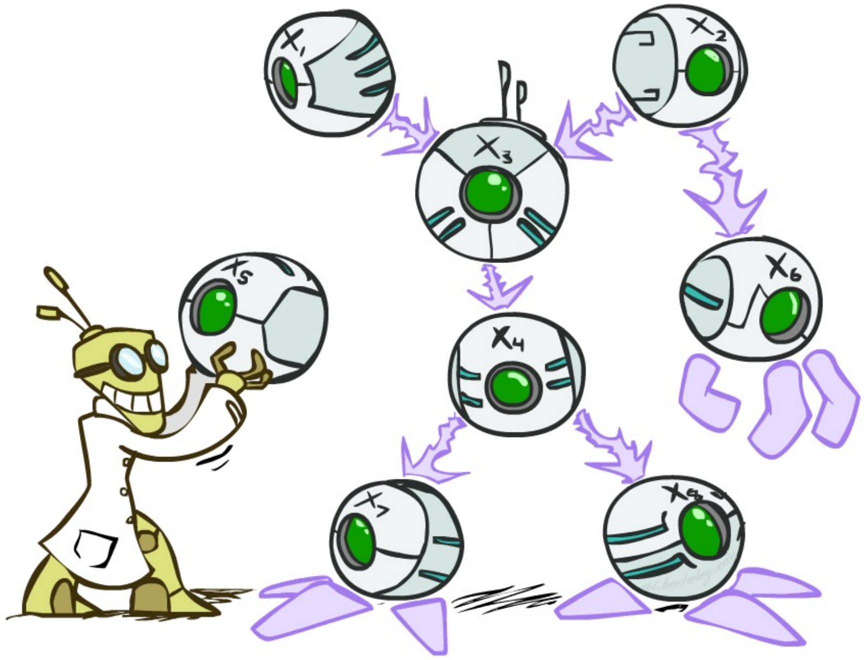
- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)

# Bayes' Nets: Big Picture

- Joint distribution can be used for inference

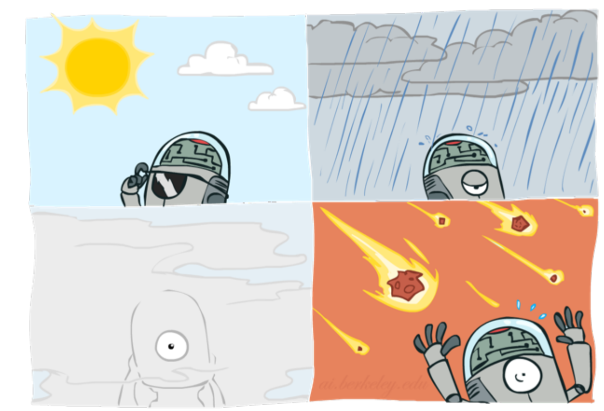
- Three problems with directly using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time
  - Computational complexity of inference

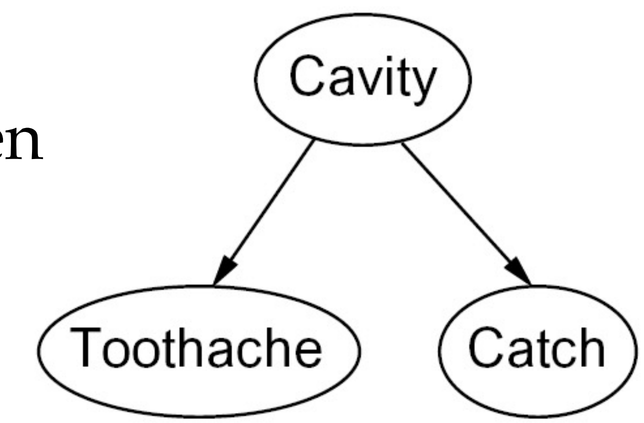
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - Instance of graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions

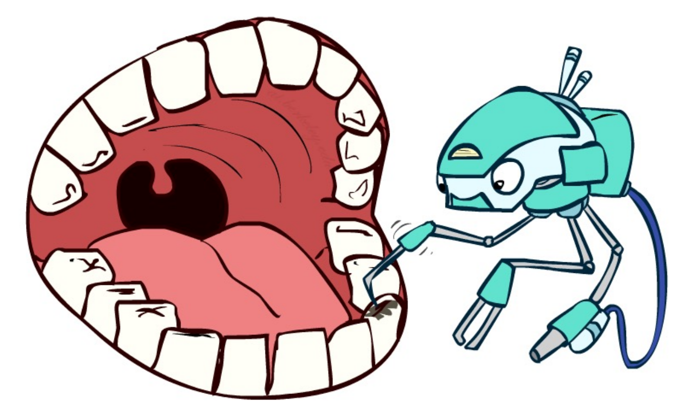


# Bayes' Net Notation

- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)
- Arcs: interactions
  - Similar to CSP constraints
  - Indicate "direct influence" between variables
  - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don't!)

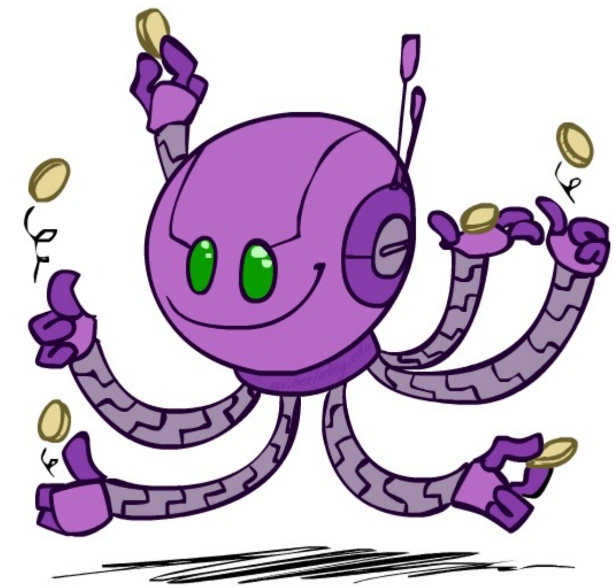





# Example: Coin Flips

- N independent coin flips

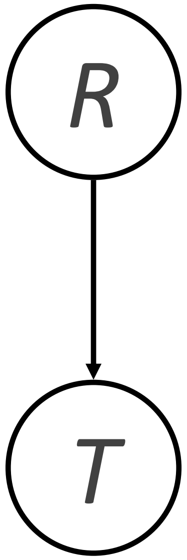
- No interactions between variables: absolute independence

# Example: Traffic

- Variables:
  - R: It rains
  - T: There is traffic

- Model 1: independence

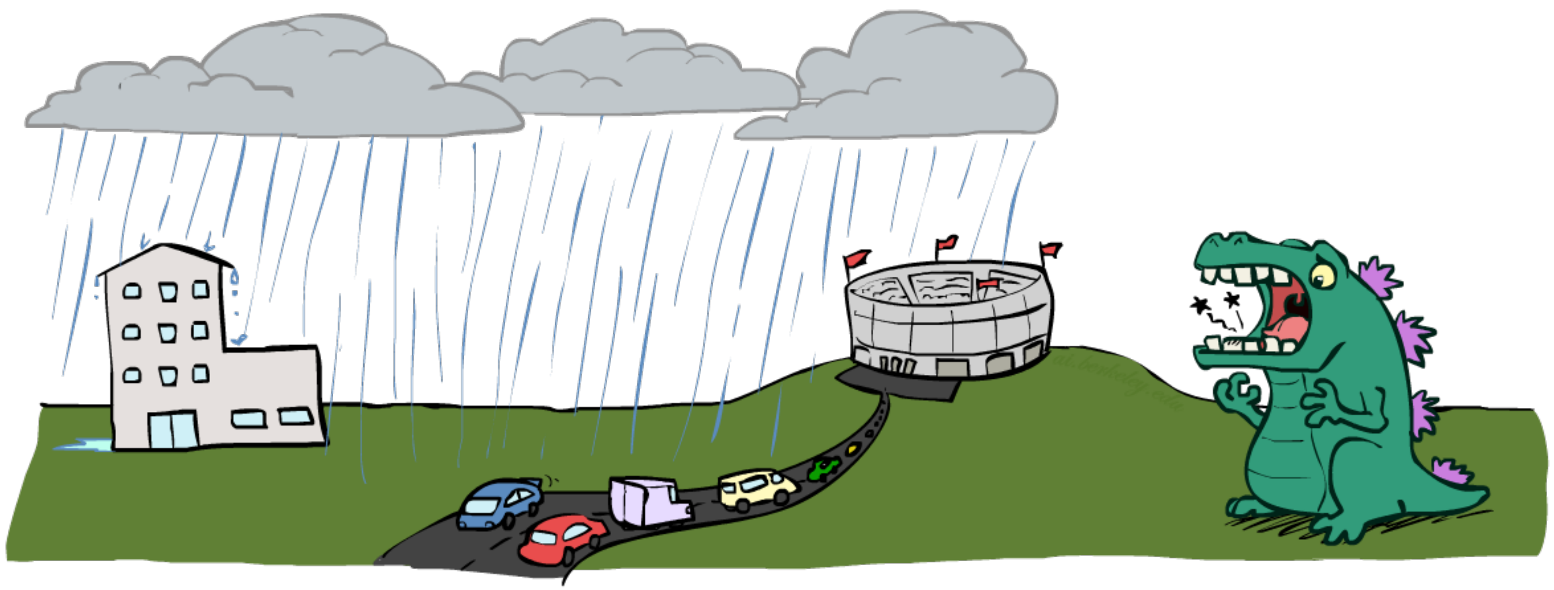

- Model 2: rain causes traffic




- Why is an agent using model 2 better?

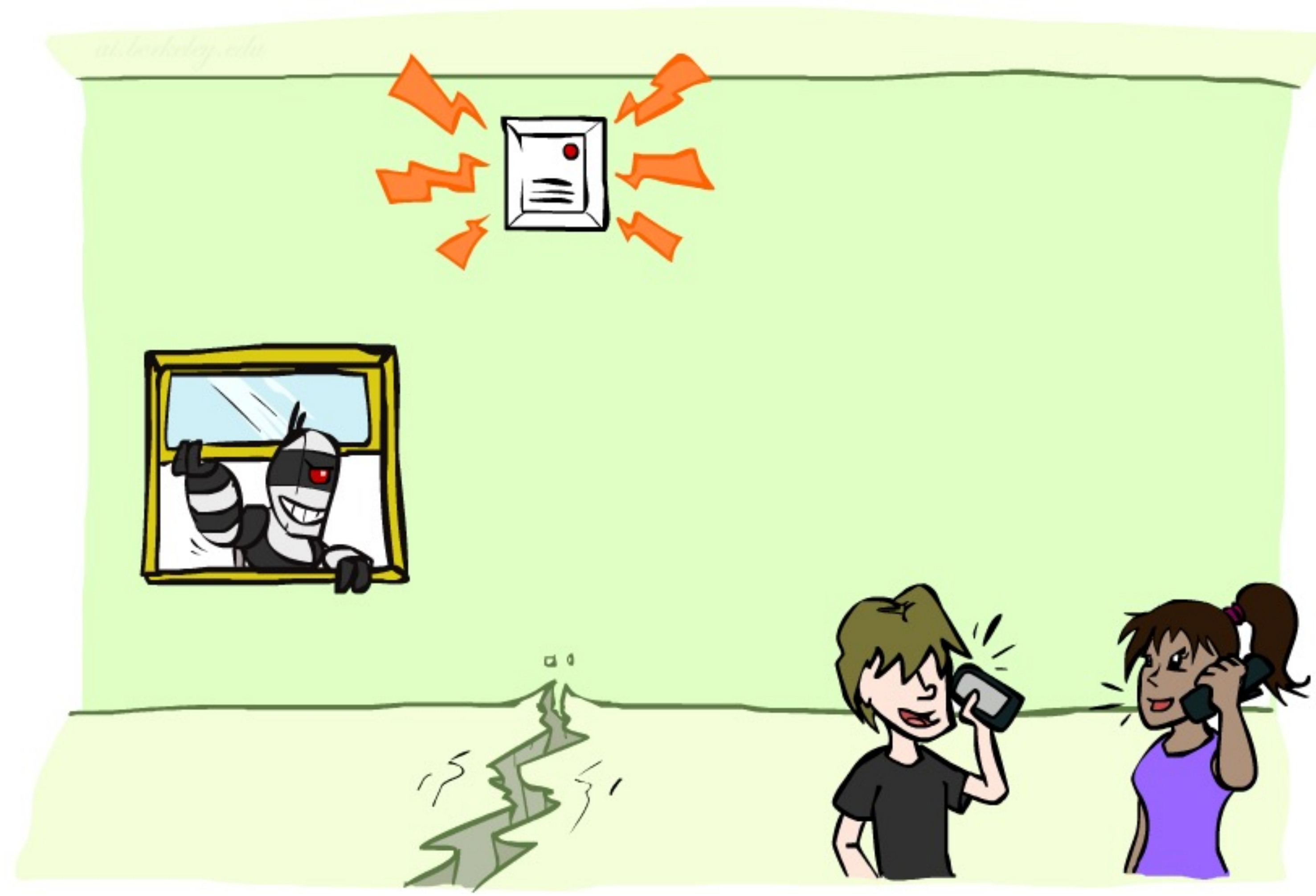
# Example: Traffic II

- Let's build a causal graphical model!
- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity

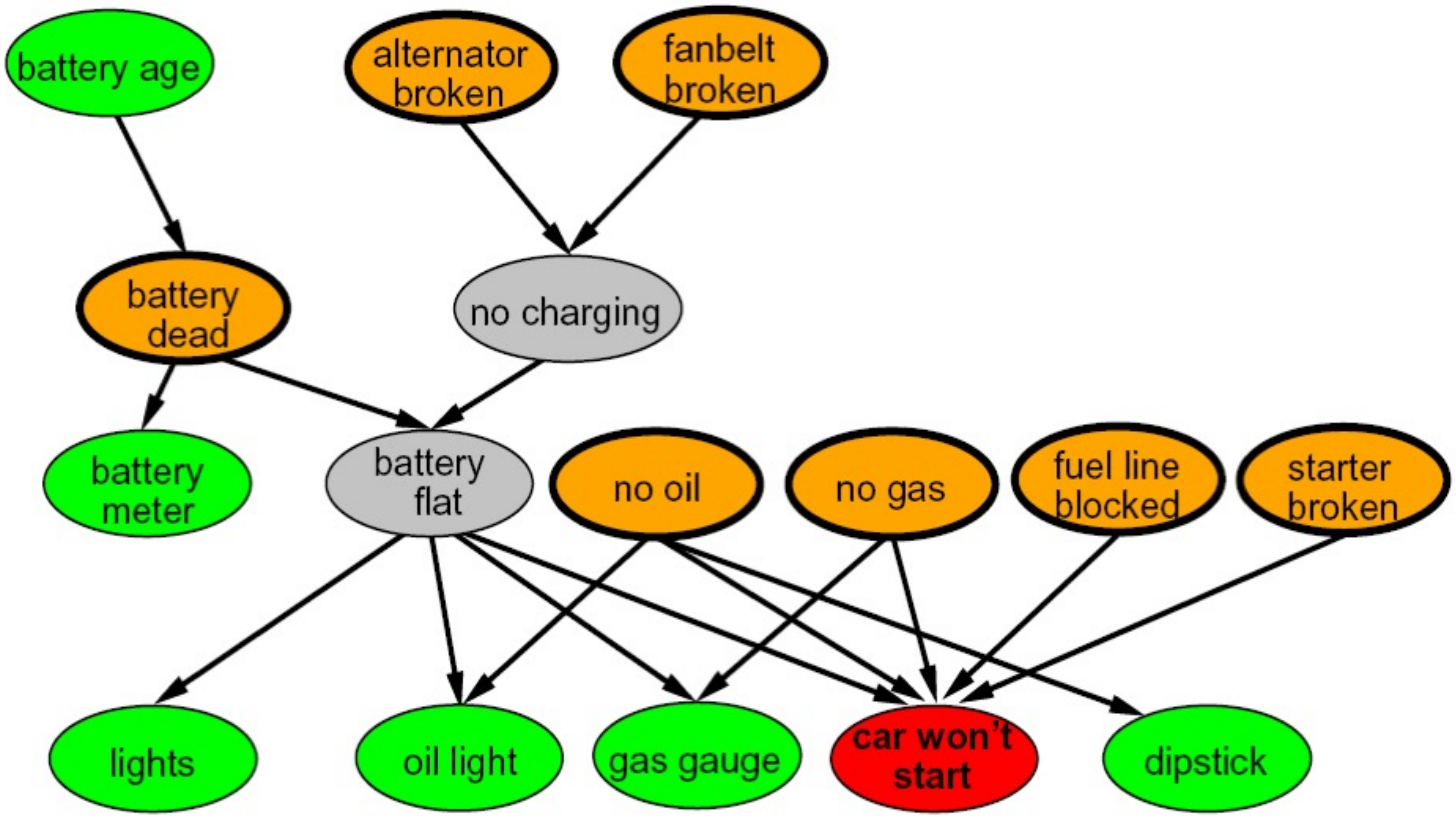


# Example: Alarm Network

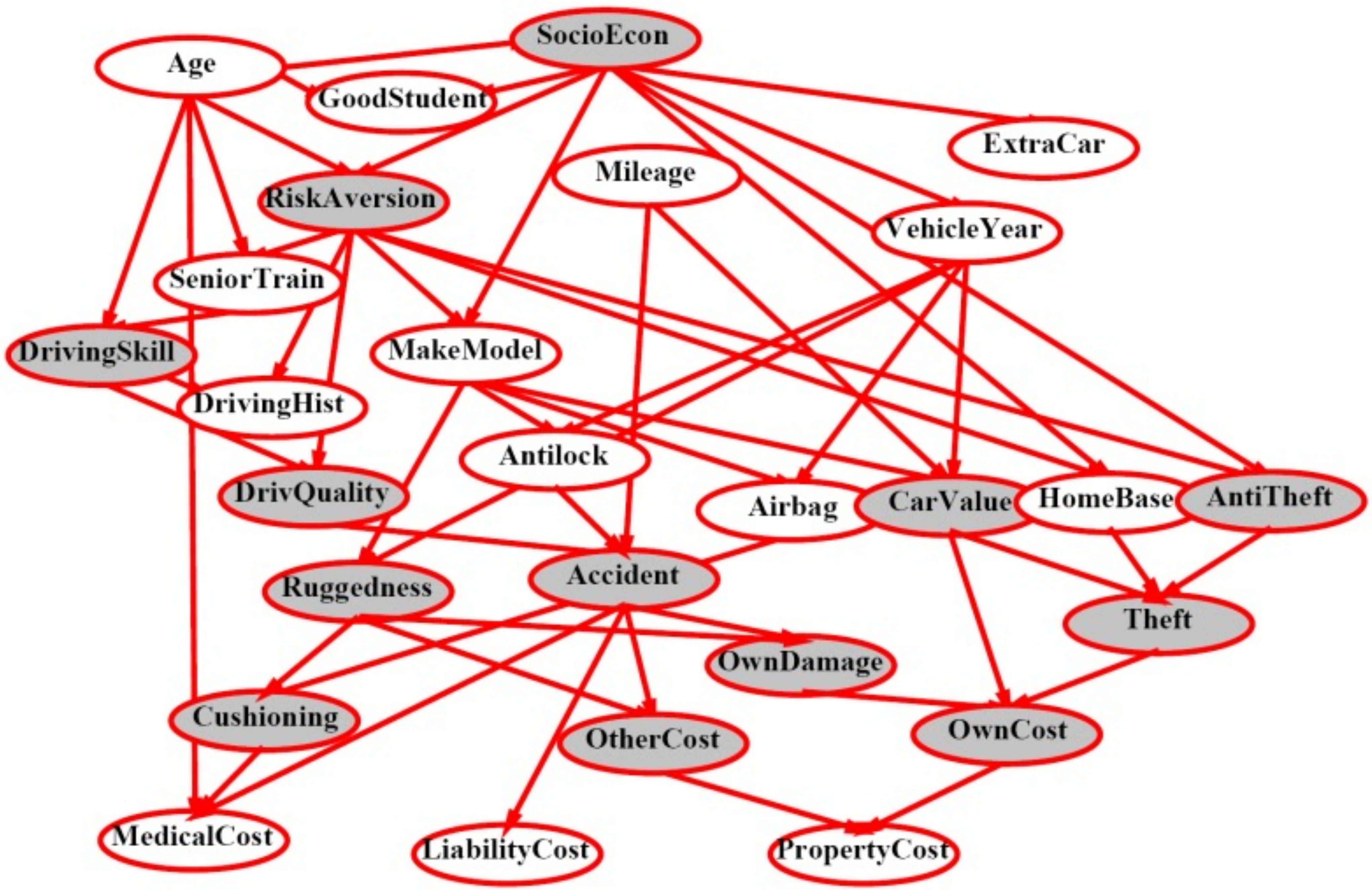
- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!



# Example Bayes' Net: Car

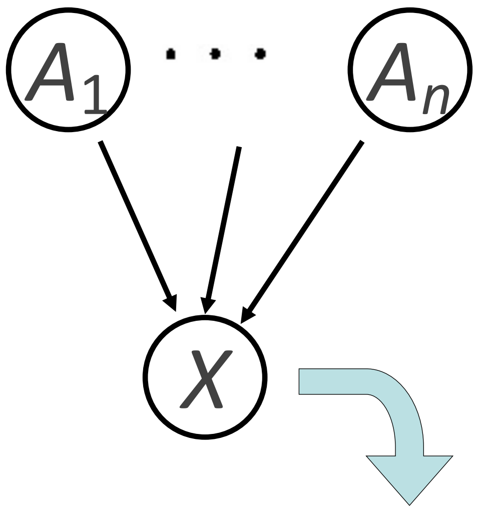


# Example Bayes' Net: Insurance



# Bayes' Net Definition and Semantics

- A set of nodes, one per variable
- A directed, acyclic graph
- A conditional distribution for each node




  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \dots a_n) \qquad\qquad P(X|A_1 \dots A_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process

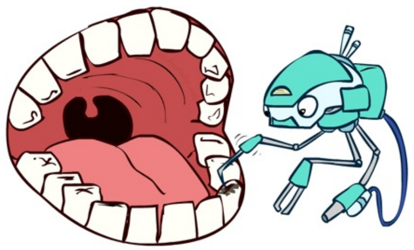
*A Bayes net = Topology (graph) + Local Conditional Probabilities*

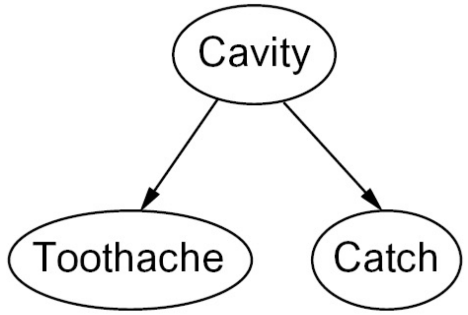
# Probabilities in BNs

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

  - Example:






$P(+\mathit{cavity}, +\mathit{catch}, -\mathit{toothache})$

# Probabilities in BNs

- Why are we guaranteed that setting

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1 \ldots x_{i-1})$

- Assume conditional independences: $P(x_i | x_1, \ldots x_{i-1}) = P(x_i | \mathit{parents}(X_i))$

  → Consequence: $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$

- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

# Example: Coin Flips

$X_1$ $\quad$ $X_2$ $\quad \cdots \quad$ $X_n$

$P(X_1)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

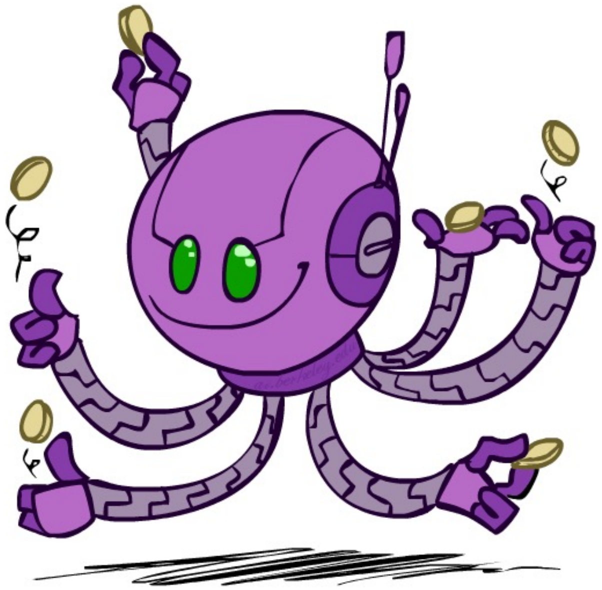
$P(X_2)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

$\cdots$

$P(X_n)$

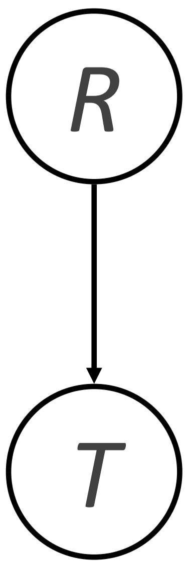
| | |
|---|---|
| h | 0.5 |
| t | 0.5 |



$$P(h, h, t, h) =$$

*Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.*

# Example: Traffic



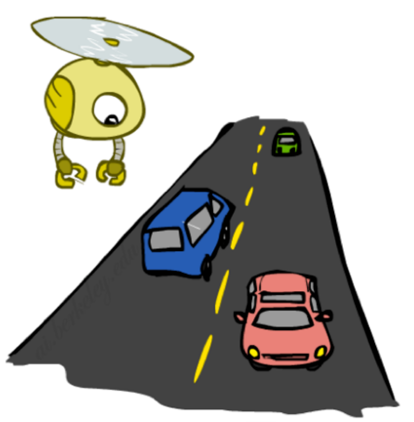

$P(R)$

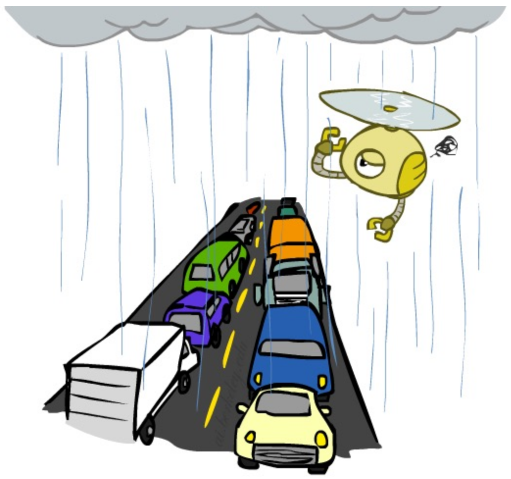
| | |
|---|---|
| +r | 1/4 |
| -r | 3/4 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 3/4 |
| | -t | 1/4 |
| -r | +t | 1/2 |
| | -t | 1/2 |

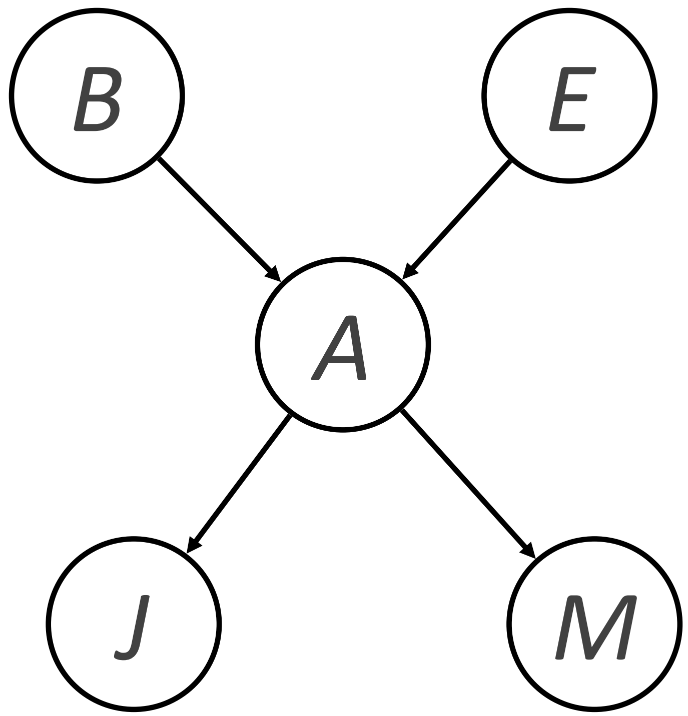
$$P(+r, -t) =$$



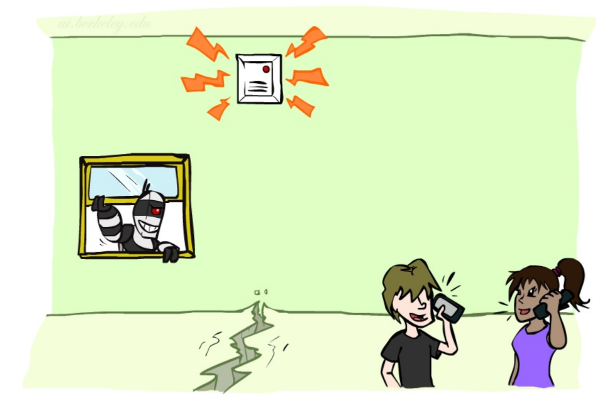


# Quiz: Alarm Network

| B | P(B) |
|---|---|
| +b | 0.001 |
| -b | 0.999 |




| E | P(E) |
|---|---|
| +e | 0.002 |
| -e | 0.998 |



| A | J | P(J\|A) |
|---|---|---|
| +a | +j | 0.9 |
| +a | -j | 0.1 |
| -a | +j | 0.05 |
| -a | -j | 0.95 |

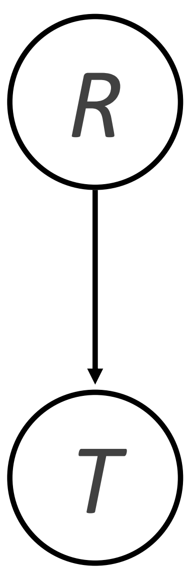
| A | M | P(M\|A) |
|---|---|---|
| +a | +m | 0.7 |
| +a | -m | 0.3 |
| -a | +m | 0.01 |
| -a | -m | 0.99 |

| B | E | A | P(A\|B,E) |
|---|---|---|---|
| +b | +e | +a | 0.95 |
| +b | +e | -a | 0.05 |
| +b | -e | +a | 0.94 |
| +b | -e | -a | 0.06 |
| -b | +e | +a | 0.29 |
| -b | +e | -a | 0.71 |
| -b | -e | +a | 0.001 |
| -b | -e | -a | 0.999 |

$$P(+b, -e, +a, -j, +m) =$$

# Example: Traffic

- Causal direction




$P(R)$

| | |
|---|---|
| +r | 1/4 |
| -r | 3/4 |

$P(T|R)$

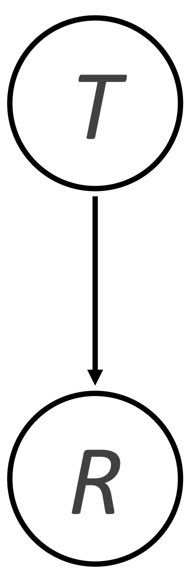
| | | |
|---|---|---|
| +r | +t | 3/4 |
| | -t | 1/4 |
| -r | +t | 1/2 |
| | -t | 1/2 |

$P(T, R)$

| | | |
|---|---|---|
| +r | +t | 3/16 |
| +r | -t | 1/16 |
| -r | +t | 6/16 |
| -r | -t | 6/16 |

# Example: Reverse Traffic

- Reverse causality?




$P(T)$

| | |
|---|---|
| +t | 9/16 |
| -t | 7/16 |

$P(R|T)$

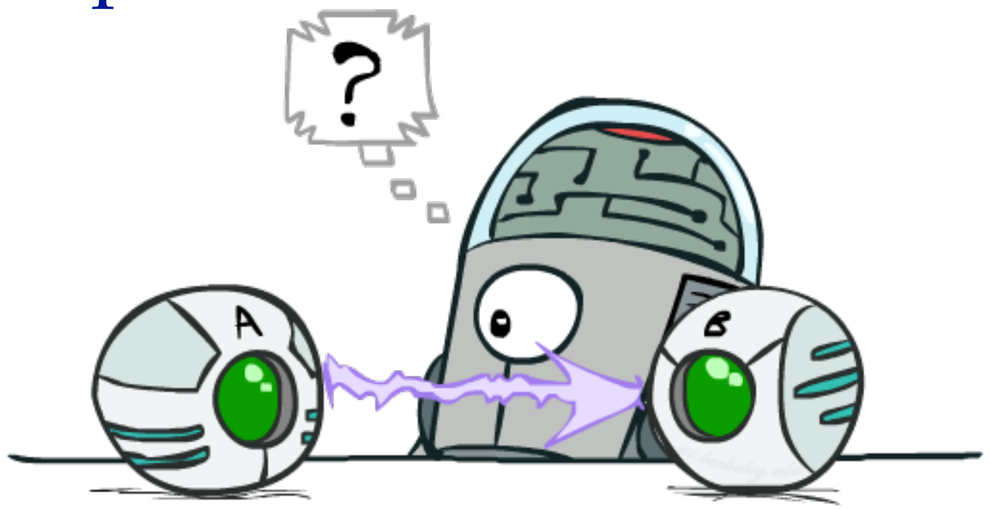
| | | |
|---|---|---|
| +t | +r | 1/3 |
| | -r | 2/3 |
| -t | +r | 1/7 |
| | -r | 6/7 |

$P(T,R)$

| | | |
|---|---|---|
| +r | +t | 3/16 |
| +r | -t | 1/16 |
| -r | +t | 6/16 |
| -r | -t | 6/16 |

# Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts




- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation

- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology really encodes conditional independence

$$P(x_i|x_1, \ldots x_{i-1}) = P(x_i|\mathit{parents}(X_i))$$

# Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

  $2^N$

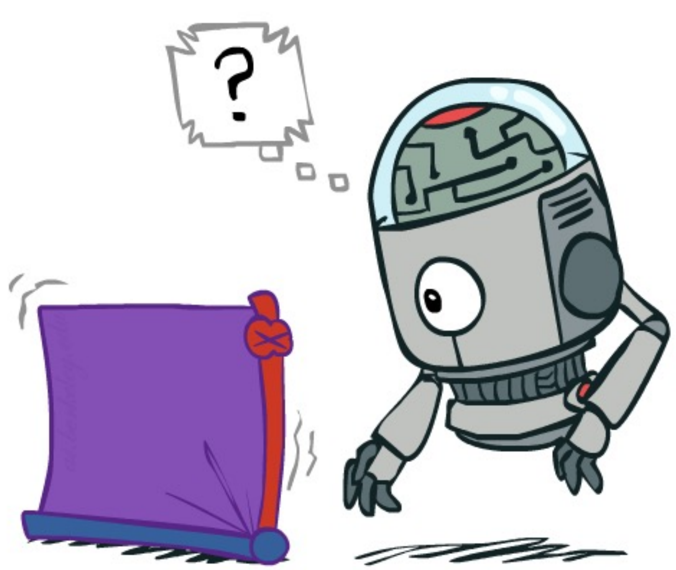
- How big is an N-node net if nodes have up to k parents?

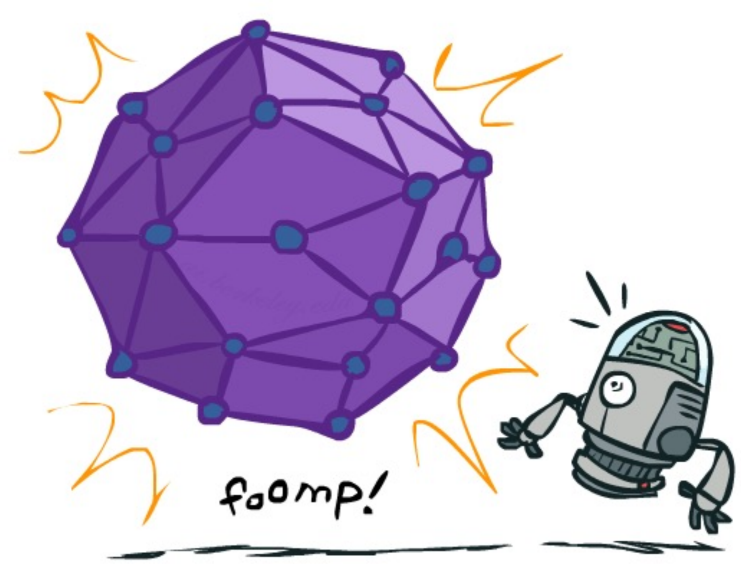
  $O(N * 2^{k+1})$

- Both give you the power to calculate

  $$P(X_1, X_2, \ldots X_n)$$

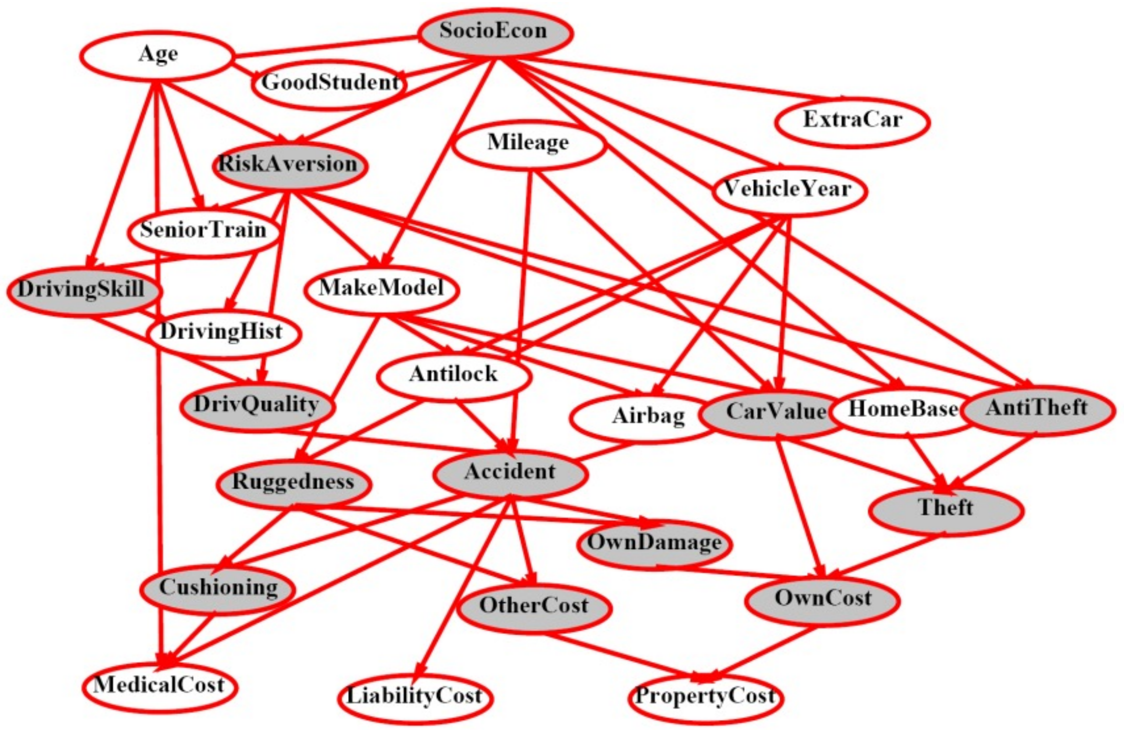
- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



# Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain




- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

# Bayes' Nets

- ✔ Representation
- Conditional Independences
- Probabilistic Inference

# Conditional Independence

- X and Y are independent if

$$\forall x, y \;\; P(x,y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

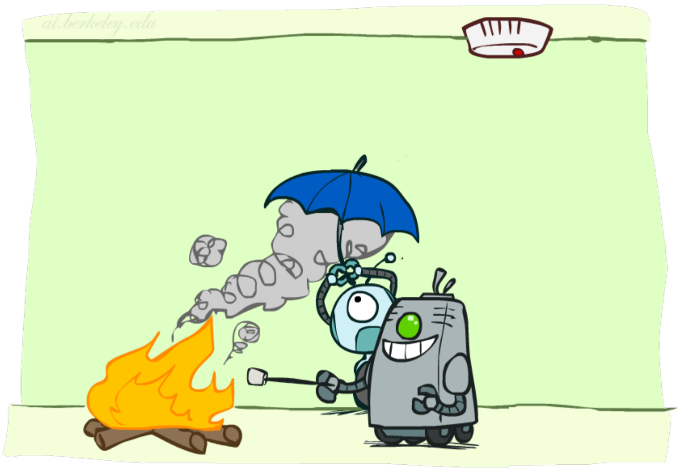
- X and Y are conditionally independent given Z

$$\forall x, y, z \;\; P(x,y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y | Z$$

- (Conditional) independence is a property of a joint distribution

- Example:

$$Alarm \perp\!\!\!\perp Fire | Smoke$$



# Bayes' Nets: Assumptions

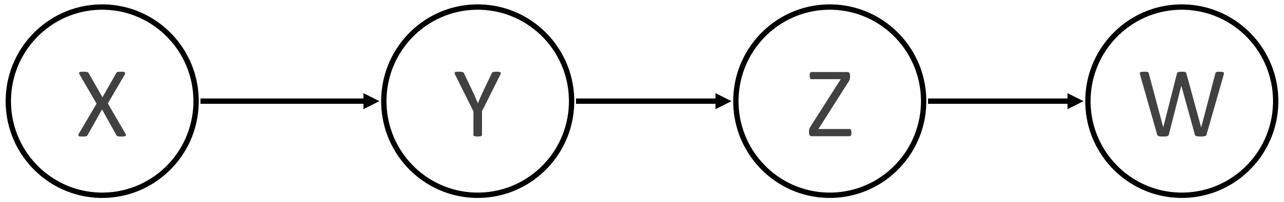
- Assumptions we are required to make to define the Bayes' net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$


- Beyond the above conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes' net graph

# Example

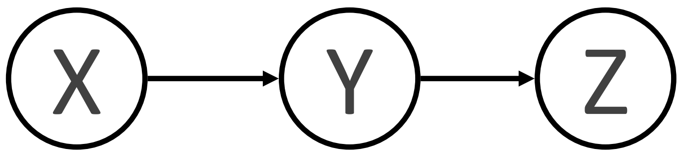
- Conditional independence assumptions directly from simplifications in chain rule:




- Additional implied conditional independence assumptions?

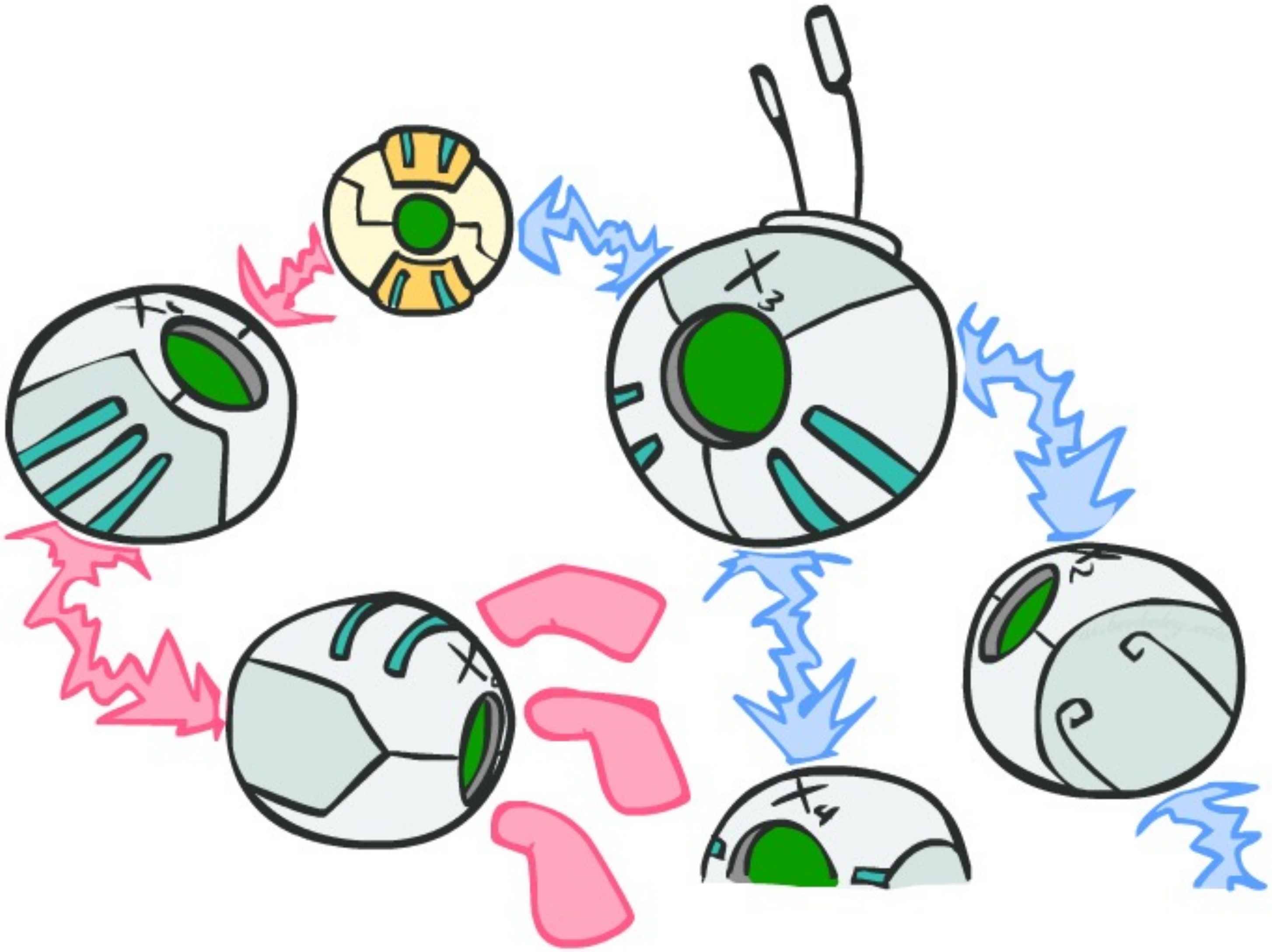
# Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:




  - Question: are X and Z necessarily independent?
    - Answer: No, e.g., low pressure causes rain, which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

# D-separation: Outline



# D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

# Serial Chains

- This configuration is a "serial chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Is X always independent of Z ?
  - *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Counter-example:
    - Low pressure => rain => traffic, high pressure => no rain => no traffic
    - In numbers:
      P( +y | +x ) = 1, P( -y | - x ) = 1,
      P( +z | +y ) = 1, P( -z | -y ) = 1

# Serial Chains

- This configuration is a "serial chain"


X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Is X always independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

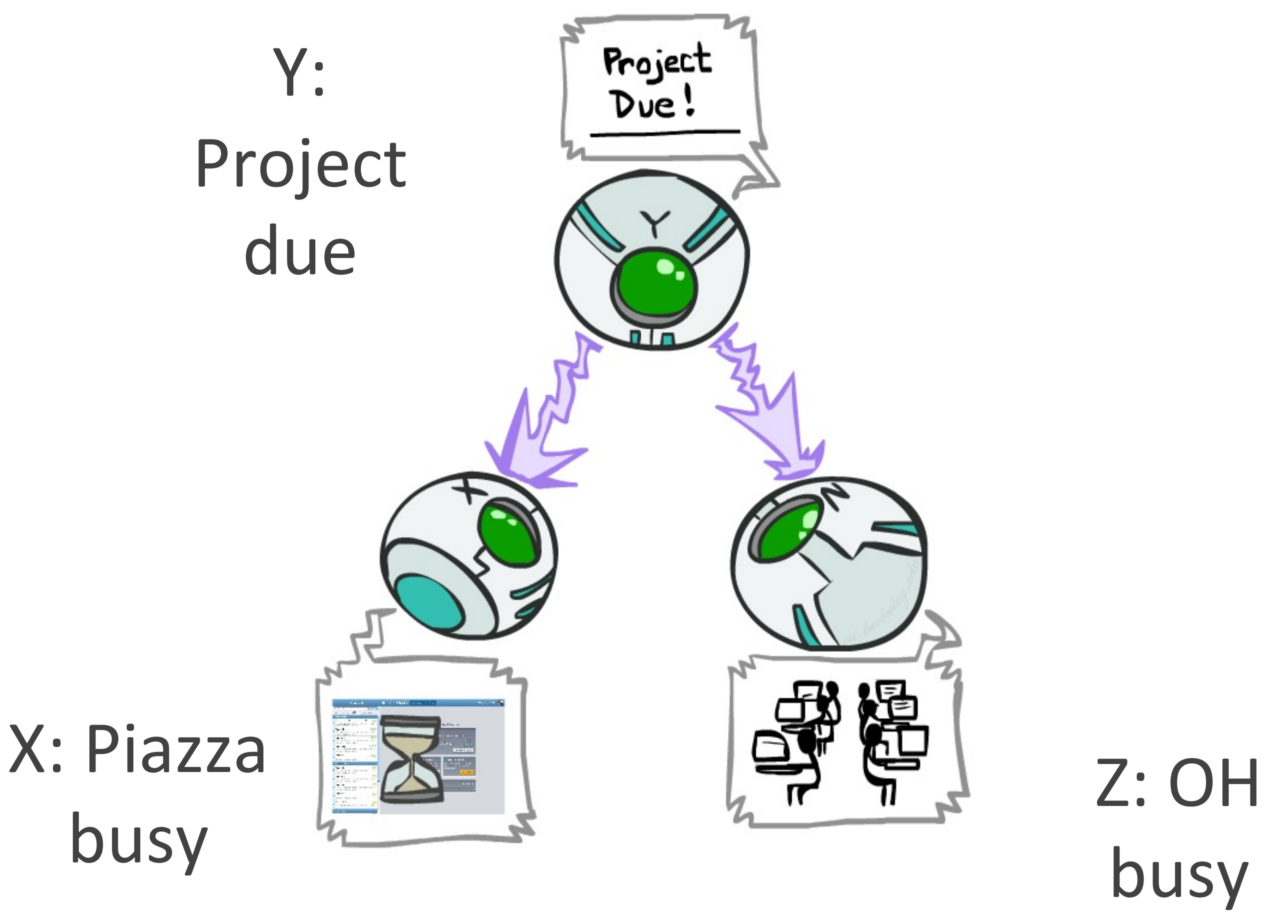
$$= P(z|y)$$

*Yes!*

- Evidence along the chain "blocks" the influence

# Divergent Chain

- This configuration is a "divergent chain"




$$P(x,y,z) = P(y)P(x|y)P(z|y)$$

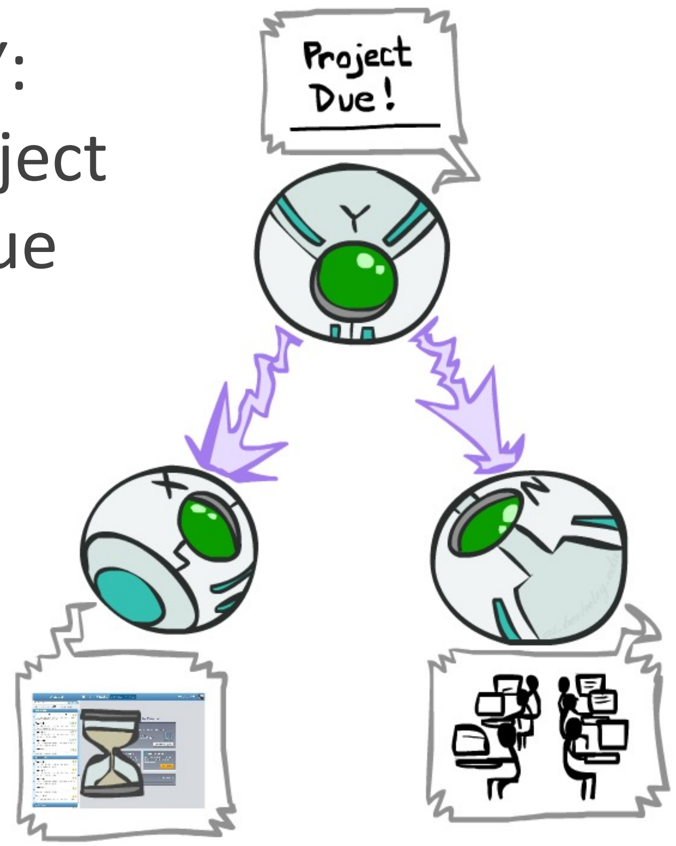
- Is X always independent of Z ?
  - *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Counter-example:
    - Project due => Piazza busy and OH busy
    - In numbers:

  P( +x | +y ) = 1, P( -x | -y ) = 1,
  P( +z | +y ) = 1, P( -z | -y ) = 1

# Divergent Chain

- This configuration is a "divergent chain"

Y: Project due




X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

$$= P(z|y)$$

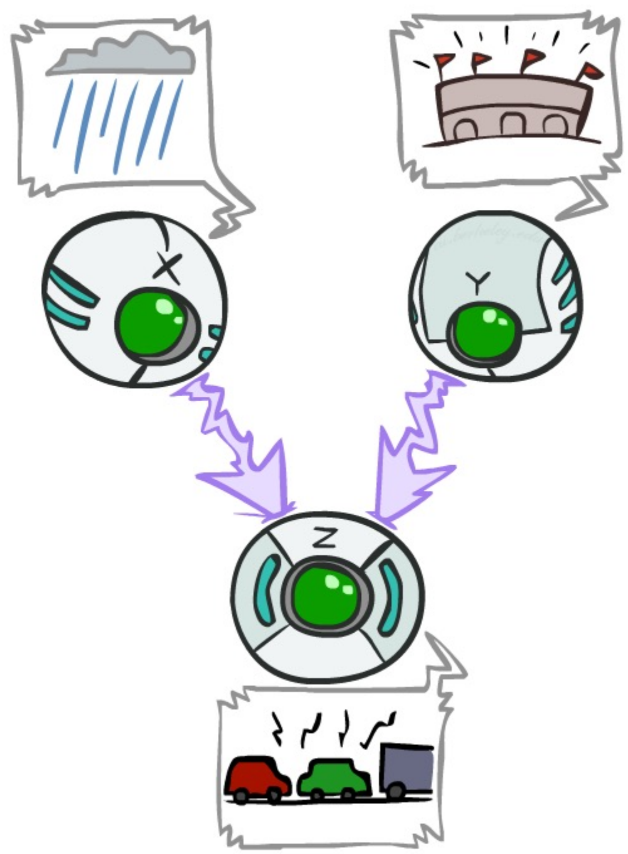
*Yes!*

- Observing the cause blocks influence between effects.

# Convergent Chain

- Last configuration: "convergent chain" (v-structures)

X: Raining Y: Ballgame




Z: Traffic

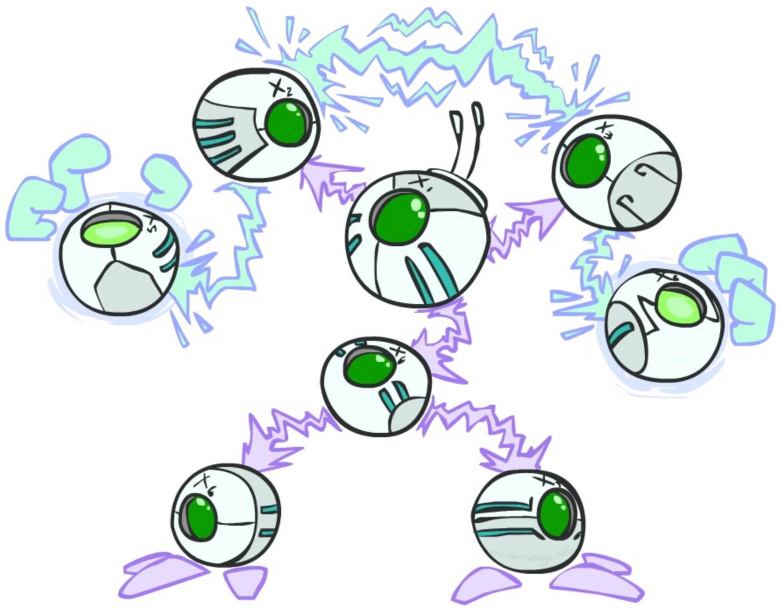
$$P(x, y, z) = P(x)P(y)P(z|x, y)$$

- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect activates influence between possible causes.

# The General Case

# The General Case

- **General question**: in a given BN, are two variables independent (given evidence)?



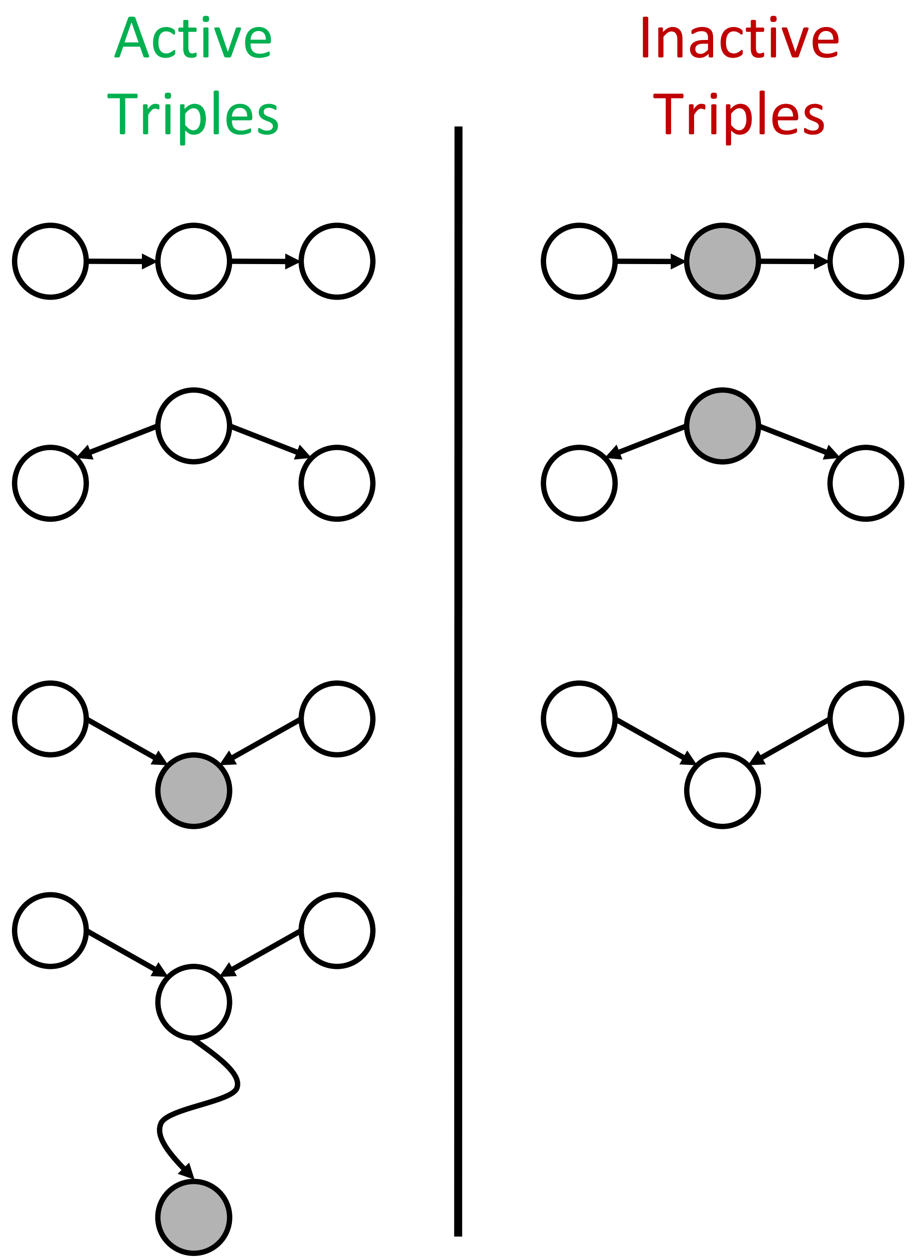
- **Solution**: analyze the graph

- Any complex example can be broken into repetitions of the three canonical cases

# Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!
- A path is active if each triple is active:
  - Serial chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
  - Divergent chain $A \leftarrow B \rightarrow C$ where B is unobserved
  - Convergent chain (aka v-structure)

  $A \rightarrow B \leftarrow C$ where B *or one of its descendants* is observed
- All it takes to block a path is a single inactive segment



# D-Separation

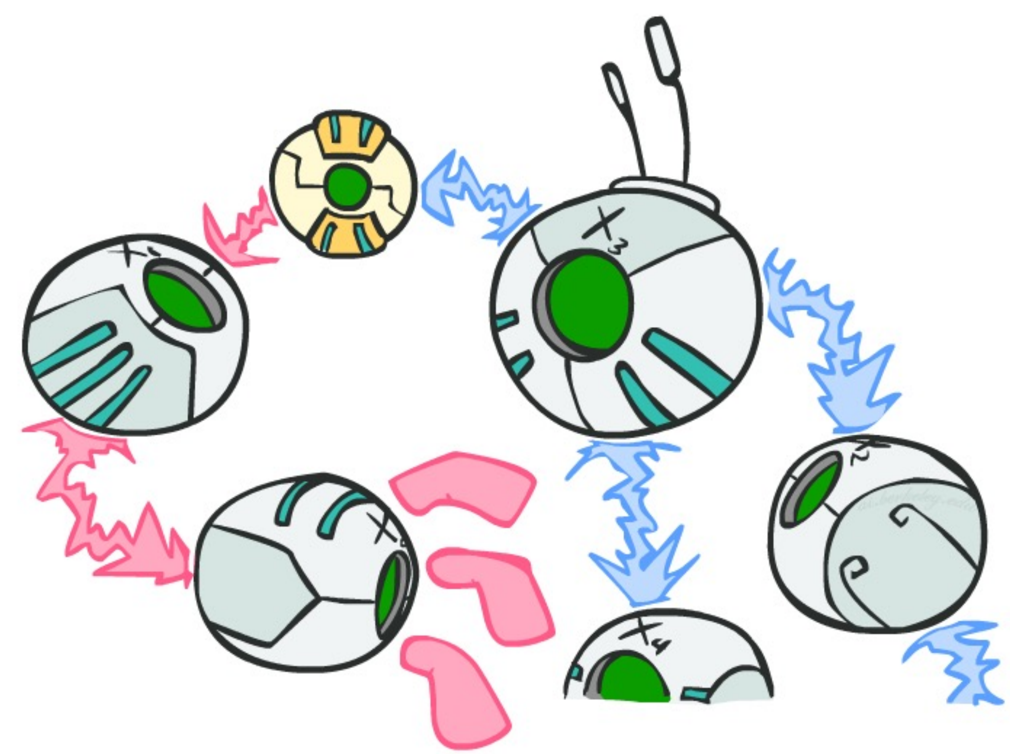
- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$ ?

- Check all (undirected!) paths between $X_i$ and $X_j$
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$



# Example

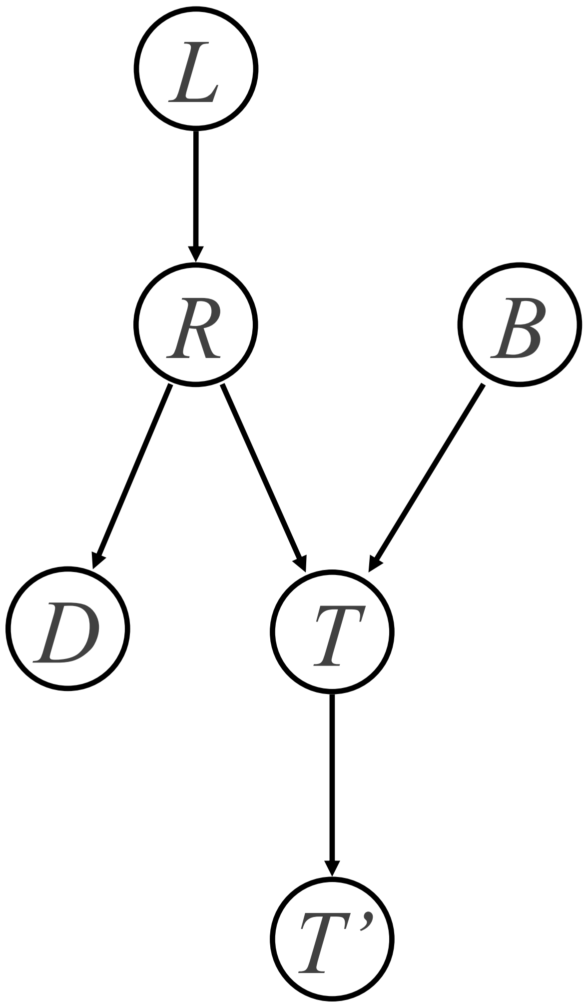
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$ *No*

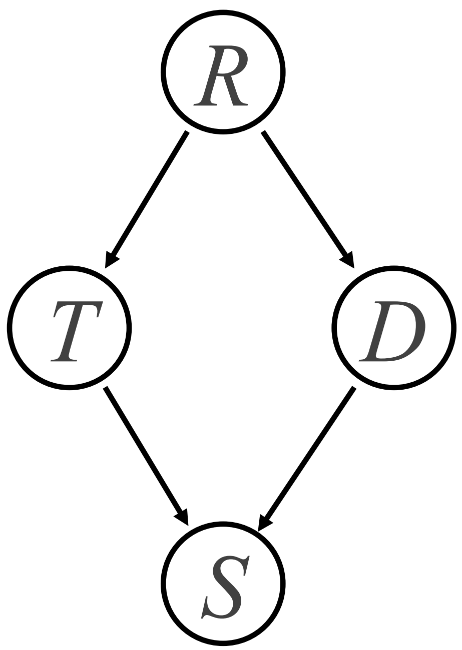
$L \perp\!\!\!\perp B | T'$ *No*

$L \perp\!\!\!\perp B | T, R$ *Yes*



# Quiz: Conditional Independence

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad




- Questions:

$$T \perp\!\!\!\perp D$$

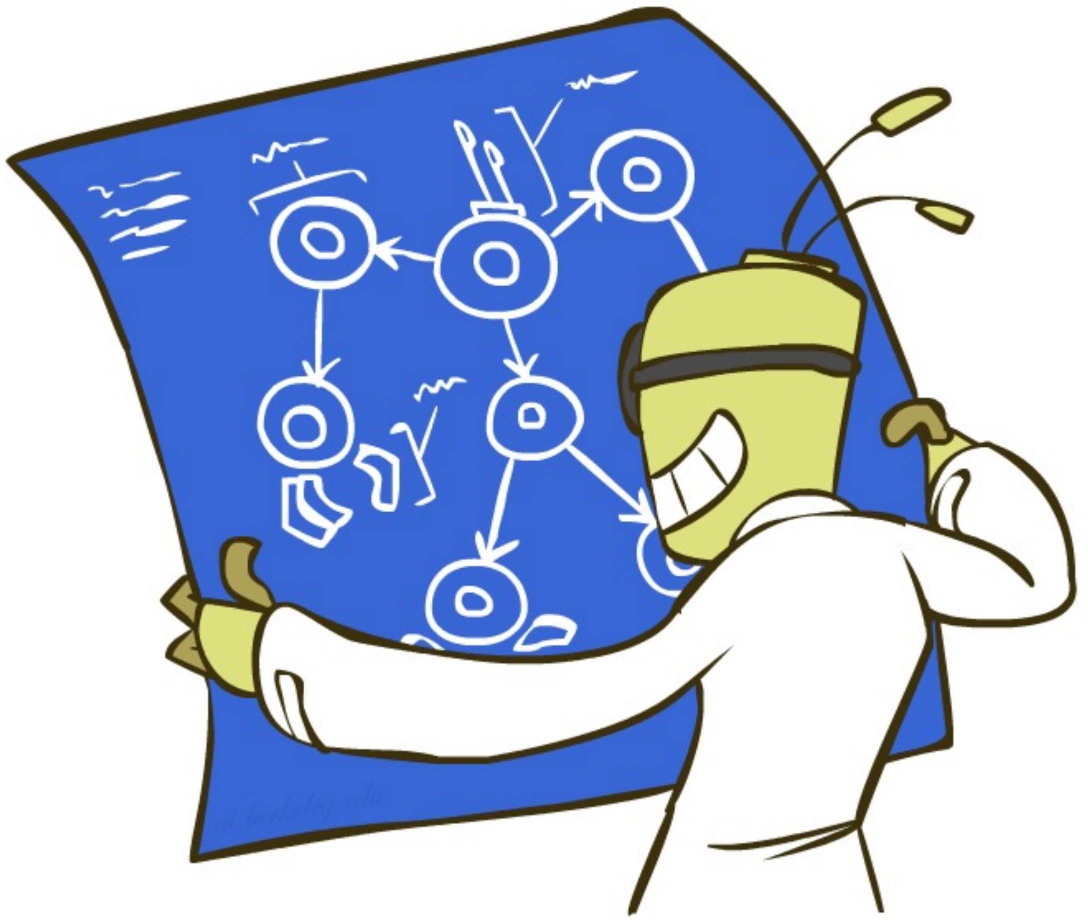
$$T \perp\!\!\!\perp D | R$$

$$T \perp\!\!\!\perp D | R, S$$

# Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

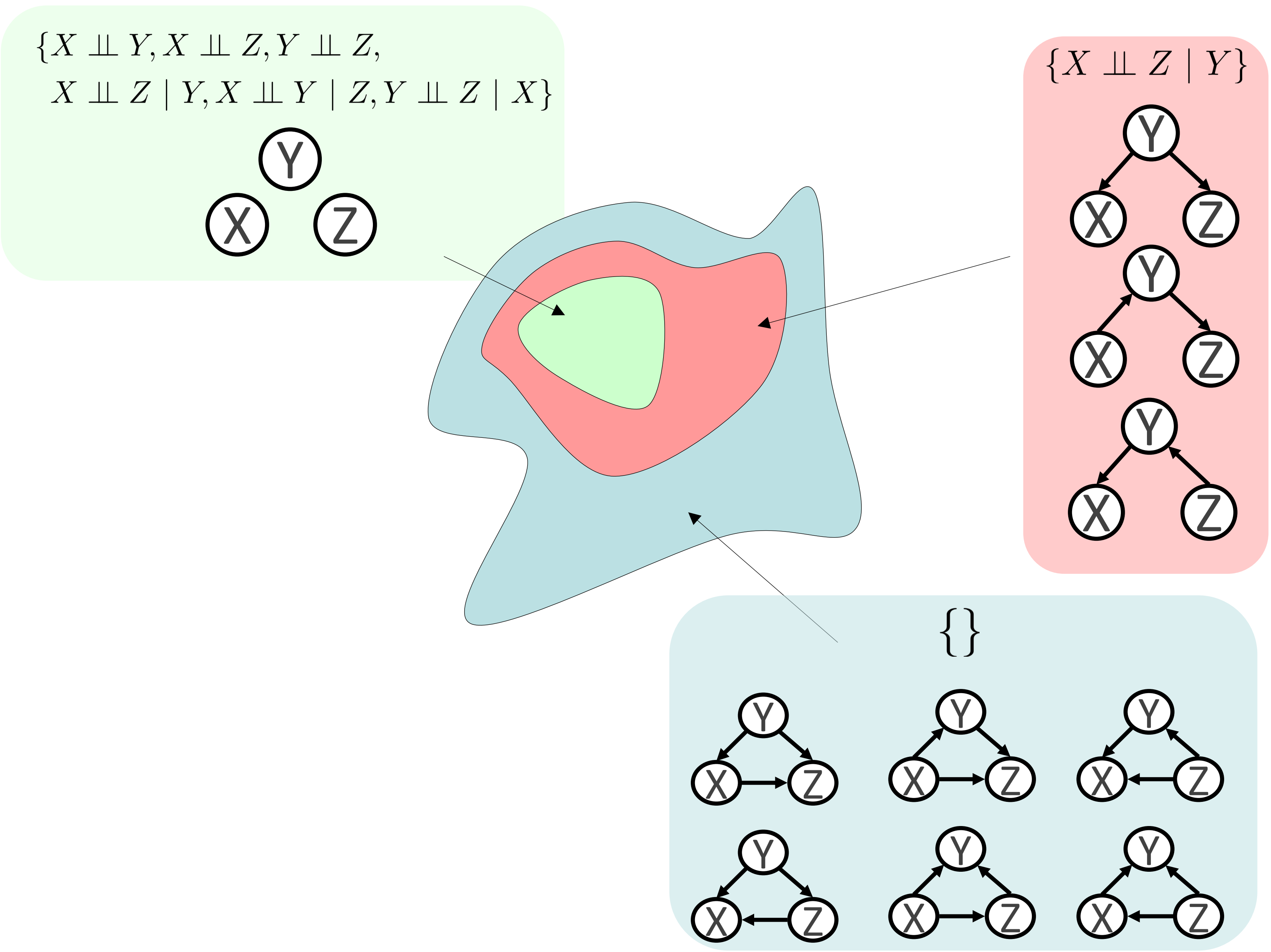
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$



- This list determines the set of probability distributions that can be represented

# Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



# Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

# Bayes' Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Probabilistic inference is NP-complete
  - Approximate inference (sampling)