Machine Learning PS: 4

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You can see each problem at **??**, Problem 2, Problem 3, Problem 4, Problem 5.

Problem 2

(a)

$$\begin{aligned}
\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a) &= \sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)p(a|s) \\
&= \sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)\pi_0(s,a) \\
&= \sum_{(s,a)}\pi_1(s,a)R(s,a)p(s) \quad \text{since } \hat{\pi}_0=\pi_0 \\
&= \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}R(s,a)
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)}{\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} &= \frac{\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)p(a|s)}{\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}p(s)p(a|s)} \\
&= \frac{\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)\pi_0(s,a)}{\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}p(s)\pi_0(s,a)} \\
&= \frac{\sum_{(s,a)}\pi_1(s,a)R(s,a)p(s)}{\sum_{(s,a)}\pi_1(s,a)p(s)} \quad \text{since } \hat{\pi}_0=\pi_0 \\
&= \sum_{(s,a)}\pi_1(s,a)R(s,a)p(s) \quad \text{since } \sum_{(s,a)}\pi_1(s,a)p(s)=1 \\
&= \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}R(s,a)
\end{aligned}$$

(c) When in finite sample situations, we assume that we have only one seen value.

$$\begin{aligned}
\frac{\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)}{\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} &= \frac{\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)p(a|s)}{\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}p(s)p(a|s)} \\
&= \frac{\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)\pi_0(s,a)}{\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}p(s)\pi_0(s,a)} \\
&= R(s,a)
\end{aligned}$$

However,

$$\begin{aligned}
\mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}R(s,a) &= \sum_{(s,a)}\pi_1(s,a)R(s,a)p(s) \\
&= \pi_1(s,a)R(s,a)
\end{aligned}$$

So

$$\frac{\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)}{\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} \neq \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}R(s,a)$$

Note that **Importance Sampleing** is unbiased, when there is only one seen value, we have

$$\begin{aligned}
\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a) &= \sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)p(a|s) \\
&= \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}R(s,a)p(s)\pi_0(s,a) \\
&= \pi_1(s,a)R(s,a) \\
&= \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}R(s,a)
\end{aligned}$$

(d) i. Since $\mathbb{E}_{a\sim\pi_1(s,a)}\hat{R}(s,a)$ is a function w.r.t s but not a, so we have

$$\begin{aligned}
\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\left[\mathbb{E}_{a\sim\pi_1(s,a)}\hat{R}(s,a)\right] &= \mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\left[f(s)\right] \\
&= \mathbb{E}_{s\sim p(s)}\left[f(s)\right] \\
&= \mathbb{E}_{s\sim p(s)}\left[\mathbb{E}_{a\sim\pi_1(s,a)}\hat{R}(s,a)\right] \\
&= \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}\left[\hat{R}(s,a)\right]
\end{aligned}$$

Then

$$
\begin{aligned}
&\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\left\{\left[\mathbb{E}_{a\sim\pi_1(s,a)}\hat{R}(s,a)\right]+\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\left(R(s,a)-\hat{R}(s,a)\right)\right\}\\
=&\ \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}\left[\hat{R}(s,a)\right]+\sum_{(s,a)}\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\left(R(s,a)-\hat{R}(s,a)\right)p(s)\pi_0(s,a)\\
=&\ \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}\left[\hat{R}(s,a)\right]+\sum_{(s,a)}\pi_1(s,a)\left(R(s,a)-\hat{R}(s,a)\right)p(s)\quad\text{since }\hat{\pi}_0=\pi_0\\
=&\ \sum_{(s,a)}\pi_1(s,a)R(s,a)p(s)\\
=&\ \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}R(s,a)
\end{aligned}
$$

ii.

When $\hat{R}(s,a)=R(s,a)$, we have

$$
\begin{aligned}
&\mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\left\{\left[\mathbb{E}_{a\sim\pi_1(s,a)}\hat{R}(s,a)\right]+\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\left(R(s,a)-\hat{R}(s,a)\right)\right\}\\
=&\ \mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\left[\mathbb{E}_{a\sim\pi_1(s,a)}\hat{R}(s,a)\right]\\
=&\ \mathbb{E}_{s\sim p(s),a\sim\pi_0(s,a)}\left[\mathbb{E}_{a\sim\pi_1(s,a)}R(s,a)\right]\\
=&\ \mathbb{E}_{s\sim p(s),a\sim\pi_1(s,a)}\left[R(s,a)\right]
\end{aligned}
$$

(e)

i Importance sampling estimater is better. Since $\hat{R}(s,a)$ is hard to know, so we can find $\hat{\pi}_0(s,a)$, which is a simpler way.

ii Regression estimater is better. Since $\hat{\pi}_0(s,a)$ is hard to know, we instead estimate $\hat{R}(s,a)$.

Problem 3

First,

$$
\begin{aligned}
f_u(x)&=\arg\min_{v\in\mathcal{V}}\|x-v\|^2\\
&=\frac{u^Tx}{u^Tu}u\\
&=uu^Tx
\end{aligned}
$$

Second,

$$
\begin{aligned}
f(u,\lambda) &= \sum_{i=1}^{m} \|x^{(i)} - f_u(x^{(i)})\|^2 + \lambda(u^T u - 1) \\
&= \sum_{i=1}^{m} \|x^{(i)} - uu^T x^{(i)}\|^2 + \lambda(u^T u - 1) \\
&= \sum_{i=1}^{m} \left[x^{(i)} - uu^T x^{(i)}\right]^T \left[x^{(i)} - uu^T x^{(i)}\right] + \lambda(u^T u - 1) \\
&= \sum_{i=1}^{m} \left[{x^{(i)}}^T x^{(i)} - u^T x^{(i)} {x^{(i)}}^T u\right] + \lambda(u^T u - 1)
\end{aligned}
$$

Take derivative and set to 0,

$$
\begin{cases}
\frac{\partial f(u,\lambda)}{\partial u} & = -2\sum_{i=1}^{m} x^{(i)} {x^{(i)}}^T u + 2\lambda u = 0 \\
\frac{\partial f(u,\lambda)}{\partial \lambda} & = u^T u - 1 = 0
\end{cases}
$$

If we denote $\Sigma = \sum_{i=1}^{m} x^{(i)} {x^{(i)}}^T$, we have $\Sigma u = \lambda u$. And $f^*(u,\lambda) = \sum_{i=1}^{m} \left[{x^{(i)}}^T x^{(i)} - u^T x^{(i)} {x^{(i)}}^T u\right] = \sum_{i=1}^{m} {x^{(i)}}^T x^{(i)} - \lambda^*$. We can see that the larger λ is, the less f is. So this optimization gives the first principal component.

Problem 4

(a)

$$\frac{\partial \log(|W|)}{\partial W} = W^{-T}$$

For another part,

$$
\begin{aligned}
\frac{\partial \sum_{i=1}^{n} \sum_{j=1}^{d} \log g'(w_j^T x^{(i)})}{\partial w_l^T} &= \frac{\partial \sum_{i=1}^{n} \sum_{j=1}^{d} -\frac{1}{2}(w_j^T x^{(i)})^2}{\partial w_l^T} \\
&= \sum_{i=1}^{n} -w_l^T x^{(i)} {x^{(i)}}^T
\end{aligned}
$$

Then take derivative of $l(W)$ and set to 0:

$$
\begin{aligned}
&\frac{\partial l(W)}{\partial W} = -WX^T X + mW^{-T} = 0 \\
\implies &W^T W = m(X^T X)^{-1}
\end{aligned}
$$

If R is a rotational matrix and $W' = RW$, we have $W'^T W = (RW)^T (RW) = W^T R^T RW = W^T W = m(X^T X)^{-1}$. So W' is also a solution, which leads to ambiguity.

(b)

$$\begin{aligned}\frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{d}\log g'(w_j^T x^{(i)})}{\partial w_l^T} &= \frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{d} -|w_j^T x^{(i)}|}{\partial w_l^T} \\ &= \sum_{i=1}^{n} -sgn(w_l^T x^{(i)}) {x^{(i)}}^T\end{aligned}$$

Then take derivative of $l(W)$

$$\frac{\partial l(W)}{\partial W} = mW^{-T} - sgn(WX^T)X$$

So

$$W := W + \alpha \left(mW^{-T} - sgn(WX^T)X\right)$$

Problem 5

(a)

$$\begin{aligned}\|B(V_1) - B(V_2)\|_\infty &= \max_{s\in S} |B(V_1) - B(V_2)| \\ &= \gamma \max_{s\in S} |\max_{a\in A} \sum_{s'\in S} P_{sa}(s')V_1(s') - \max_{a\in A} \sum_{s'\in S} P_{sa}(s')V_2(s')| \\ &= \gamma \max_{s\in S} |\max_{a\in A} \sum_{s'\in S} P_{sa}(s')[V_1(s') - V_2(s')]| \\ &\le \gamma \max_{s\in S} \max_{a\in A} \sum_{s'\in S} P_{sa}(s')|V_1(s') - V_2(s')| \\ &\le \gamma \max_{s'\in S} |V_1(s') - V_2(s')| \quad \text{since } \sum_i p_i a_i \le \max_i a_i \\ &= \gamma \|V_1 - V_2\|_\infty\end{aligned}$$

(b) We are going to use contradiction. Assume that there are two fixed points V_1 and V_2, then

$$\begin{aligned}\|B(V_1) - B(V_2)\|_\infty &= \|V_1 - V_2\|_\infty \\ &\le \gamma \|V_1 - V_2\|_\infty\end{aligned}$$

Since $\gamma \le 1$. To make the inequality above true, we have $\|V_1 - V_2\|_\infty = \max_{s\in S} |V_1 - V_2| = 0$. Therefore, $V_1(s) = V_2(s)$ for any $s \in S$, which means that $V_1 = V_2$. So there is at most one fixed point.