Typed Semantics for λ_i

1. Syntax

e	$::=$		Expression
	$\mid$	n	Integer Literal
	$\mid$	$e_1 + e_2$	Add
	$\mid$	x	Variable
	$\mid$	$\lambda x.\ e$	Abstraction
	$\mid$	$e_1\ e_2$	Application
	$\mid$	$e_1\ ,,\ e_2$	Merge
	$\mid$	$(e : A)$	Annotation

2. Semantics

$\boxed{v \Rightarrow A}$ Typed Values (Inf)

$$\frac{}{n \Rightarrow \mathsf{Int}}\ \text{VI\_INT} \qquad \frac{v_1 \Rightarrow A \qquad v_2 \Rightarrow B}{v_1\ ,,\ v_2 \Rightarrow A \,\&\, B}\ \text{VI\_MERGE} \qquad \frac{v \Leftarrow A}{(v : A) \Rightarrow A}\ \text{VI\_ANN}$$

$\boxed{v \Leftarrow A}$ Typed Values (Chk)

$$\frac{}{\lambda x.\ e \Leftarrow A \to B}\ \text{VC\_ABS}$$

$\boxed{e_1 \rightsquigarrow e_2}$ Reduction

$$\frac{e_1 \rightsquigarrow e_3}{e_1 + e_2 \rightsquigarrow e_3 + e_2}\ \text{R\_ADD1} \qquad \frac{e_1 \rightsquigarrow e_2}{n + e_1 \rightsquigarrow n + e_2}\ \text{R\_ADD2} \qquad \frac{r = n + m}{n + m \rightsquigarrow r}\ \text{R\_ADD3}$$

$$\frac{e_1 \rightsquigarrow e_3}{e_1\ e_2 \rightsquigarrow e_3\ e_2}\ \text{R\_APP1} \qquad \frac{e_1 \rightsquigarrow e_2}{v\ e_1 \rightsquigarrow v\ e_2}\ \text{R\_APP2} \qquad \frac{v \Rightarrow B \qquad A \neq B \qquad v \rightsquigarrow e \Leftarrow A}{(\lambda x.\ e : A \to C)\ v \rightsquigarrow (\lambda x.\ e : A \to C)\ e}\ \text{R\_APPSUB}$$

$$\frac{v \Leftarrow A}{(\lambda x.\ e : A \to B)\ v \rightsquigarrow (\lambda x.\ e : A \to B)\ (v : A)}\ \text{R\_APP4} \qquad \frac{v \Rightarrow A}{(\lambda x.\ e : A \to B)\ v \rightsquigarrow e[x \mapsto v]}\ \text{R\_APP5}$$

$$\frac{e_1 \rightsquigarrow e_3}{e_1\ ,,\ e_2 \rightsquigarrow e_3\ ,,\ e_2}\ \text{R\_MERGE1} \qquad \frac{e_1 \rightsquigarrow e_2}{v\ ,,\ e_1 \rightsquigarrow v\ ,,\ e_2}\ \text{R\_MERGE2} \qquad \frac{e_1 \rightsquigarrow e_2}{(e_1 : A) \rightsquigarrow (e_2 : A)}\ \text{R\_ANN1}$$

$$\frac{v \Rightarrow B \qquad A \neq B \qquad v \rightsquigarrow e \Leftarrow A}{(v : A) \rightsquigarrow (e : A)}\ \text{R\_ANNSUB} \qquad \frac{v \Rightarrow A}{(v : A) \rightsquigarrow v}\ \text{R\_ANN3}$$

$\boxed{v \rightsquigarrow e \Leftarrow A}$ Reduction (Sub)

$$\frac{v \Rightarrow A \to B}{v \rightsquigarrow (\lambda y.\ (v\ y : D) : C \to D) \Leftarrow C \to D}\ \text{RSUB\_FUN} \qquad \frac{}{v \rightsquigarrow (v : A)\ ,,\ (v : B) \Leftarrow A \,\&\, B}\ \text{RSUB\_ANDR}$$

$$\frac{\mathsf{ord}\ A \qquad v_1 \Rightarrow B \qquad B \leq A}{v_1\ ,,\ v_2 \rightsquigarrow v_1 \Leftarrow A}\ \text{RSUB\_ANDL1} \qquad \frac{\mathsf{ord}\ A \qquad v_2 \Rightarrow C \qquad C \leq A}{v_1\ ,,\ v_2 \rightsquigarrow v_2 \Leftarrow A}\ \text{RSUB\_ANDL2}$$

Remark 1: R_AppSub and R_AnnSub's condition $A \neq B$ prevents looping (subtyping is reflexive).
Remark 2: If there are no assumptions over v, then assume $\exists A.\ v \Rightarrow A$.

3. Metatheory

Lemma 3.1 (Subject reduction (sub)). `[safety_sub]` *If* $\Gamma \vdash v \Rightarrow A$ *and* $v \rightsquigarrow e \Leftarrow B$ *and* $A \leq B$, *then* $\Gamma \vdash e \Leftarrow B$.

Lemma 3.2 (Subject reduction (inf)). `[safety_inf]` *If* $\Gamma \vdash e_1 \Rightarrow A$ *and* $e_1 \rightsquigarrow e_2$, *then* $\Gamma \vdash e_2 \Rightarrow A$.

Lemma 3.3 (Subject reduction (chk)). `[safety_chk]` *If* $\Gamma \vdash e_1 \Leftarrow A$ *and* $e_1 \rightsquigarrow e_2$, *then* $\Gamma \vdash e_2 \Leftarrow A$.