Semantics

1. Syntax

$$
\begin{array}{lcll}
e & ::= & & \text{Expression} \\
 & | & n & \quad \text{Integer Literal} \\
 & | & e_1 + e_2 & \quad \text{Add} \\
 & | & x & \quad \text{Variable} \\
 & | & \lambda x.\ e & \quad \text{Abstraction} \\
 & | & e_1\ e_2 & \quad \text{Application} \\
 & | & e_1 ,, e_2 & \quad \text{Merge} \\
 & | & (e : A) & \quad \text{Annotation} \\
v & ::= & & \text{Value} \\
 & | & (n : A) & \\
 & | & (\lambda x.\ e : A \to B) & \\
 & | & v_1 ,, v_2 &
\end{array}
$$

2. Semantics

$\boxed{v \in A}$ Typed Values

$$\frac{}{(n : \text{Int}) \in \text{Int}}\ \text{T\_INT} \qquad \frac{}{(\lambda x.\ e : A \to B) \in A \to B}\ \text{T\_ABS} \qquad \frac{v_1 \in A \qquad v_2 \in B}{v_1 ,, v_2 \in A \,\&\, B}\ \text{T\_MERGE}$$

$\boxed{e_1 \rightsquigarrow e_2}$ Reduction

$$\frac{}{n \rightsquigarrow (n : \text{Int})}\ \text{R\_INT} \qquad \frac{e_1 \rightsquigarrow e_3}{e_1 + e_2 \rightsquigarrow e_3 + e_2}\ \text{R\_ADD1} \qquad \frac{v \in \text{Int} \qquad e_1 \rightsquigarrow e_2}{v + e_1 \rightsquigarrow v + e_2}\ \text{R\_ADD2}$$

$$\frac{r = m + n}{(m : \text{Int}) + (n : \text{Int}) \rightsquigarrow (r : \text{Int})}\ \text{R\_ADD3} \qquad \frac{e_1 \rightsquigarrow e_3}{e_1\ e_2 \rightsquigarrow e_3\ e_2}\ \text{R\_APP1} \qquad \frac{v \in A \to B \qquad e_1 \rightsquigarrow e_2}{v\ e_1 \rightsquigarrow v\ e_2}\ \text{R\_APP2}$$

$$\frac{v_1 \in A \to B \qquad v_2 \in C \qquad A \neq C}{v_1\ v_2 \rightsquigarrow v_1\ (v_2 : A)}\ \text{R\_APP3} \qquad \frac{v \in A \to B}{v\ (\lambda x.\ e) \rightsquigarrow v\ (\lambda x.\ e : A)}\ \text{R\_APP4}$$

$$\frac{v \in A}{(\lambda x.\ e : A \to B)\ v \rightsquigarrow (e[x \mapsto v] : B)}\ \text{R\_APP5} \qquad \frac{e_1 \rightsquigarrow e_3}{e_1 ,, e_2 \rightsquigarrow e_3 ,, e_2}\ \text{R\_MERGE1}$$

$$\frac{e_1 \rightsquigarrow e_2}{v ,, e_1 \rightsquigarrow v ,, e_2}\ \text{R\_MERGE2} \qquad \frac{(e_1 : A) \notin A \qquad e_1 \rightsquigarrow e_2}{(e_1 : A) \rightsquigarrow (e_2 : A)}\ \text{R\_ANN1} \qquad \frac{v \in \text{Int}}{(v : \text{Int}) \rightsquigarrow v}\ \text{R\_ANN2}$$

$$\frac{}{((\lambda x.\ e : A \to B) : C \to D) \rightsquigarrow (\lambda y.\ (((\lambda x.\ e : A \to B)\ (y : A)) : D) : C \to D)}\ \text{R\_ANN3}$$

$$\frac{}{((\lambda x.\ e : A \to B) : C \to D) \rightsquigarrow (\lambda x.\ e : A \to D)}\ \text{R\_ANN3ALT}$$

$$\frac{\text{ord}\ A \qquad v_1 ,, v_2 \in B \,\&\, C \qquad B \leq A}{(v_1 ,, v_2 : A) \rightsquigarrow (v_1 : A)}\ \text{R\_ANN4} \qquad \frac{\text{ord}\ A \qquad v_1 ,, v_2 \in B \,\&\, C \qquad C \leq A}{(v_1 ,, v_2 : A) \rightsquigarrow (v_2 : A)}\ \text{R\_ANN5}$$

$$\frac{v \in C}{(v : A \,\&\, B) \rightsquigarrow (v : A) ,, (v : B)}\ \text{R\_ANN6}$$

3. Metatheory

Lemma 3.1 (Subject reduction (inf)). `[safety_inf]` *If* $\Gamma \vdash e_1 \Rightarrow A$ *and* $e_1 \rightsquigarrow e_2$*, then* $\Gamma \vdash e_2 \Rightarrow A$*.*

Lemma 3.2 (Subject reduction (chk)). `[safety_chk]` *If* $\Gamma \vdash e_1 \Leftarrow A$ *and* $e_1 \rightsquigarrow e_2$*, then* $\Gamma \vdash e_2 \Leftarrow A$*.*

Lemma 3.3 (Progress (inf)). `[progress_inf]` *If* $\varnothing \vdash e \Rightarrow A$*, then* $e = v \land v \in A$ *or* $\exists\, e_1 .\, e \rightsquigarrow e_1$ *.*

Lemma 3.4 (Progress (chk)). `[progress_chk]` *If* $\varnothing \vdash e \Leftarrow A$*, then* $(e : A) = v \land v \in A$ *or* $\exists\, e_1 .\, (e : A) \rightsquigarrow e_1$ *.*

Lemma 3.5 (Reduction is determistic). `[red_unique]` *If* ($\Gamma \vdash e_1 \Leftarrow A$ *or* $\Gamma \vdash e_1 \Rightarrow A$)*, and* $e_1 \rightsquigarrow e_2$*, and* $e_1 \rightsquigarrow e_3$*, then* $e_2 = e_3$*.*