SOLUTIONS - CHAPTER 3

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Exercise 3. 1. Question see book.

Since

$$\bar{g_n}(y_i, \hat{\mu}, \hat{\sigma^2}) = \mathbf{0} = n^{-1}\sum_{i=1}^{n} g(y_i, \hat{\mu}, \hat{\sigma^2}) = n^{-1}\sum_{i=1}^{n}\begin{bmatrix} y_i - \hat{\mu} \\ (y_i - \hat{\mu})^2 - \hat{\sigma^2} \end{bmatrix}$$

This is equivalent to

$$\mathbf{0} = \begin{bmatrix} \sum_{i=1}^{n}(y_i - \hat{\mu}) \\ \sum_{i=1}^{n}((y_i - \hat{\mu})^2 - \hat{\sigma^2}) \end{bmatrix}$$

which then implies

$$\hat{\mu} = n^{-1}\sum_{i=1}^{n} y_i$$

$$\hat{\sigma^2} = n^{-1}\sum_{i=1}^{n}((y_i - \hat{\mu})^2$$

in which the RHS are the sample mean and variance.

Exercise 3. 2. Question see book.

First: regressing Y on X yields the OLS regression coefficient $\beta = (X'X)^{-1}X'Y$

Now let γ represent the regression coefficients of Y on $Z = XC$.Then

$$\gamma = (Z'Z)^{-1}Z'Y = (C'X'XC)^{-1}C'X'Y = C^{-1}(X'X)^{-1}C'^{-1}C'X'Y = C^{-1}(X'X)^{-1}X'Y = C^{-1}\beta$$

Exercise 3. 3. Using matrix algebra, show $X'\hat{e} = 0$

Since $Y = X\beta + e$, then $\hat{e} = Y - \hat{T} = Y - X\beta$

Then we can write:

$$X'\hat{e} = X'Y - X'X(X'X)^{-1}X'Y = X'Y - X'Y = 0$$

Date: March 2023.

Exercise 3. 4. Let $\hat{e}$ be the OLS residual from a regression of Y on $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$. Find $X_2'\hat{e}$.

Note that

$$X'\hat{e} = 0_{k\times 1} = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \hat{e} = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \hat{e} = \begin{bmatrix} X_1'\hat{e} \\ X_2'\hat{e} \end{bmatrix}$$

where RHS is $k \times 1$, so $X_2'\hat{e} = 0$

Exercise 3. 5. Question see book.

Note that

$$X'\hat{e} = X'(Y - \hat{Y}) = X'Y - X'X\hat{\beta} = X'Y - X'X(X'X)^{-1}X'Y = 0_{k\times 1}$$

Then we can write the OLS coefficient γ of $\hat{e}$ on X as:

$$\gamma = (X'X)^{-1}X'\hat{e} = 0_{k\times 1}$$

Exercise 3. 6. Let $\hat{Y} = X(X'X)^{-1}X'Y$. Find OLS regression coefficients of $\hat{Y}$ on X

Apply the regression coefficient formula, we can write the coefficient θ:

$$\theta = (X'X)^{-1}X'\hat{Y} = (X'X)^{-1}X'X(X'X)^{-1}X'Y = \mathbb{I}_n(X'X)^{-1}X'Y = (X'X)^{-1}X'Y$$

which is the coefficient vector from regressing Y on X

Exercise 3. 7. Show that if $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ then $PX_1 = X_1$ and $MX_1 = 0$

By (3.21) we know that $MX = 0$ where the RHS is $n \times k$, then

$$0_{n\times k} = MX = M \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} MX_1 & MX_2 \end{bmatrix}$$

This proves $MX_1 = 0$. Note that by definition $M = \mathbb{I}_n - P$, and thus naturally $PX_1 = X_1$

Exercise 3. 8. Show that $M = \mathbb{I}_n - P$ is idempotent: $MM = M$

Since P is idempotent:

$$MM = (\mathbb{I}_n - P)(\mathbb{I}_n - P) = \mathbb{I}_n - \mathbb{I}_n P - P\mathbb{I}_n + PP = \mathbb{I}_n - 2P + P = \mathbb{I}_n - P = M$$

Exercise 3. 9. Show that $tr(M) = n - k$.

By definition: $M = I_n - P$, so $M + P = I_n$, so $tr(M + P) = tr(M) + tr(P) = tr(I_n)$

Since $tr(I_n) = n$ and $tr(P) = k$ from Theorem 3.3.3, then $tr(M) = n - k$

Exercise 3. 10. Show that if $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ and $X_1'X_2 = 0$ then $P = P_1 + P_2$

By definition:

$$P = X(X'X)^{-1}X', P_1 = X_1(X_1'X_1)^{-1}X_1', P_2 = X_2(X_2'X_2)^{-1}X_2'$$

Then we can write

$$P = \begin{bmatrix} X_1 & X_2 \end{bmatrix} (\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix})^{-1} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix} (\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix})^{-1} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$$

Note that $X_1'X_2 = 0$ implies $X_2'X_1 = 0$, so

$$\begin{aligned} P &= \begin{bmatrix} X_1 & X_2 \end{bmatrix} (\begin{bmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{bmatrix})^{-1} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \\ &= \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} (X_1'X_1)^{-1} & 0 \\ 0 & (X_2'X_2)^{-1} \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \\ &= X_1(X_1'X_1)^{-1}X_1' + X_2(X_2'X_2)^{-1}X_2' \\ &= P_1 + P_2 \end{aligned}$$

Exercise 3. 11. Show that when X contains a constant, $n^{-1}\sum_{i=1}^{n} \hat{Y}_i = \bar{Y}$

We know from equation 3.17 (proof) that $n^{-1}\sum \hat{e}_i = 0$ when X contains a constant, then

$$\begin{aligned} &\Rightarrow n^{-1}\sum Y_i - \hat{Y}_i = 0 \\ &\Rightarrow n^{-1}\sum Y_i = \bar{Y} \end{aligned}$$

Exercise 3. 12.

Exercise 3. 13.

Exercise 3. 14. Question see book. Proof incomplete.

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\begin{aligned}
\beta_{n+1} &= \left(\begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix}' \begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix} \right)^{-1} \begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix}' \begin{pmatrix} Y_n \\ Y_{n+1} \end{pmatrix} \\
&= \left(\begin{pmatrix} X_n' & X_{n+1}' \end{pmatrix} \begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix} \right)^{-1} \begin{pmatrix} X_n' & X_{n+1}' \end{pmatrix} \begin{pmatrix} Y_n \\ Y_{n+1} \end{pmatrix} \\
&= (X_n' X_n + X_{n+1}' X_{n+1})^{-1} (X_n' X_n + X_{n+1}' Y_{n+1})
\end{aligned}
$$