SOLUTIONS - CHAPTER 4

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Exercise 4. 1. For some integer k, set $\mu_k = \mathbb{E}[Y^k]$.

(a) Construct an estimator $\hat{\mu_k}$ for μ_k.

Let Y_i be the realized values of random variable Y, then the moment estimator $\hat{\mu_k}$ for μ_k is $\hat{\mu_k} = n^{-1}\sum_{i=1}^{n}(Y_i^k)$

(b) Show that $\hat{\mu_k}$ is unbiased for μ_k.

$$\mathbb{E}[\hat{\mu_k}] = n^{-1}n\mathbb{E}[Y_i^k] = \mathbb{E}[Y_i^k] = \mu_k$$

(c) Calculate $Var(\hat{\mu_k})$. What assumption is needed for $Var(\hat{\mu_k})$ to be finite?

Note that my $\hat{\mu_k}$ can be seen as the single-variable (intercept) linear regression coefficient. Then Theorem 4.3 applies and states that $Var(\hat{\mu_k}) < \infty$ iff $2k < n - 1 + 1$ (i.e. $k < \frac{n}{2}$)

(d) Propose an estimator of $Var(\hat{\mu_k})$.

I propose the bias-corrected sample variance:

$$\widehat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i^2k - \overline{Y^k})^2$$

Exercise 4. 2.

Exercise 4. 3. Question see book.

The former is sample mean, which is a random variable, while the latter is population mean / expectation, which is a fixed parameter.

Exercise 4. 4. Question see book.

False. Note: since $X \in \mathbb{R}$, then $\hat{\beta_{OLS}} = \frac{\sum_{i=1}^{n} X_iY_i}{\sum_{i=1}^{n} X_i^2}$. Then

$$\sum_{i=1}^{n} X_i^2\widehat{e_i} = \sum_{i=1}^{n} X_i^2(Y_i - X_i\hat{\beta}) = \sum_{i=1}^{n} X_i^2Y_i - \sum_{i=1}^{n}(X_i^3\frac{\sum_{i=1}^{n} X_iY_i}{\sum_{i=1}^{n} X_i^2})$$

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which is not 0 in general.

Exercise 4. 5. Prove (4.20) and (4.21).

(4.20) follows naturally from the unbiasedness property of OLS, as (4.17) specifies the standard OLS model $Y = X\beta + e$. To show (4.21):

$$\begin{aligned} Var(\hat{\beta}|X) &= Var((X'X)^{-1}X'Y|X) \\ &= (X'X)^{-1}X'Var(X\beta + e|X)X(X'X)^{-1} \\ &= (X'X)^{-1}X'Var(e|X)X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}(X'\Sigma X)(X'X)^{-1} \end{aligned}$$

with the last equation holding for we are given $Var(e|X) = \Sigma\sigma^2$.

Exercise 4. 6. Prove Theorem 4.5 under the restriction to linear estimators.

For any linear estimator $\tilde{\beta}$ there must exist $A_{n\times k}$ that is a function of X such that $\tilde{\beta} = A'Y$. Note that following the OLS G-M Theorem proof, we know that the unbiasedness of $\tilde{\beta}$ guarantees $A'X = I_k$. Also following the OLS-GM proof:

$$Var(\tilde{\beta}|X) = Var(A'Y|X) = A'Var(e|X)A = \sigma^2 A'\Sigma A$$

with the last equation holding for we are given $Var(e|X) = \Sigma\sigma^2$.

Now, let $C = \Sigma^{\frac{1}{2}}A - \Sigma^{-\frac{1}{2}}X(X'\Sigma^{-1}X)^{-1}$ so that $0 \leqslant C'C = A'\Sigma A - (X'\Sigma^{-1}X)^{-1}$. It follows that

$$Var(\tilde{\beta}|X) = \sigma^2 A'\Sigma A \geqslant \sigma^2(X'\Sigma^{-1}X)^{-1}$$

Exercise 4. 7.

Exercise 4. 8.

Exercise 4. 9. Show (4.32) in the homoskedastic regression model.

Note that with homoskedasticity, equation (4.27) holds, which states $\mathbb{E}[\widehat{e_i^2}|X] = (1-h_i)\sigma^2$. Then:

$$\begin{aligned}
\mathbb{E}[\overline{\sigma^2}|X] &= \mathbb{E}[n^{-1}\sum_{i=1}^{n}(1-h_i)^{-1}\widehat{e_i^2}|X] \\
&= n^{-1}(1-h_i)^{-1}\sum_{i=1}^{n}\mathbb{E}[\widehat{e_i^2}|X] \\
&= \sigma^2
\end{aligned}$$

Note that $(1-h_i)$ can be taken out of conditional expectations on X since h_i is a function of X

Exercise 4. 10. Prove (4.40)

Recall

$$V_{\hat{\beta}}^{HC0} = (X'X)^{-1}(\sum_{i=1}^{n} X_iX_i'\widehat{e_i^2})(X'X^{-1})$$

$$V_{\hat{\beta}}^{HC2} = (X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-1}X_iX_i'\widehat{e_i^2})(X'X)^{-1}$$

$$V_{\hat{\beta}}^{HC3} = (X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-2}X_iX_i'\widehat{e_i^2})(X'X)^{-1}$$

Then:

$$V_{\hat{\beta}}^{HC2} - V_{\hat{\beta}}^{HC0} = (X'X)^{-1}[\sum_{i=1}^{n}((1-h_i)^{-1}-1)\widehat{e_i^2}X_iX_i'](X'X^{-1})$$

$$V_{\hat{\beta}}^{HC3} - V_{\hat{\beta}}^{HC2} = (X'X)^{-1}[\sum_{i=1}^{n}((1-h_i)^{-2}-(1-h_i)^{-1})\widehat{e_i^2}X_iX_i'](X'X^{-1})$$

Note that $(1-h_i) \in [0,1]$ by Theorem 3.6, so $(1-h_i)^{-2} \geqslant (1-h_i)^{-1} \geqslant 1$, then the middle term in each of the last two equations is each a weighted sum of n PSD matrices X_iX_i' with no negative weights. As a result, each middle term is a PSD matrix so

$$V_{\hat{\beta}}^{HC3} \geqslant V_{\hat{\beta}}^{HC2} \geqslant V_{\hat{\beta}}^{HC0}$$

Exercise 4. 11. Show (4.41) in the homoskedastic regression model, i.e. $\mathbb{E}[V_{\hat{\beta}}^{HC2}|X] = \sigma^2$.

First note that in the book notations, $X'X = \sum X_i X_i'$. Then

$$\begin{aligned}
\mathbb{E}[V_{\hat{\beta}}^{HC2}|X] &= \mathbb{E}[(X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-1}X_iX_i'\widehat{e}_i^2|X] \\
&= (X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-1}\mathbb{E}[\widehat{e}_i^2|X]X_iX_i')(X'X)^{-1} \\
&= (X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-1}(1-h_i)\sigma^2X_iX_i')(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}(\sum_{i=1}^{n}X_iX_i')(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}(X'X)(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}
\end{aligned}$$

with third equality true due to equation (4.27) under homoskedasticity. Note that $(1-h_i)$ can be taken out of conditional expectations on X since h_i is a function of X.

Exercise 4. 12.

Exercise 4. 13.

Exercise 4. 14.

Exercise 4. 15. Question see book.

(a) Find $V_{\hat{\beta}} = Var(\widehat{\beta})$.

Since we know $X'X = n\mathbb{I}_k$, which means $(X'X)^{-1} = n^{-1}\mathbb{I}_k$. Also, since we assume an i.i.d. random sample, then $Var(e|X) = \sigma^2\mathbb{I}$. Now consider the conditional variance:

$$\begin{aligned}
Var(\widehat{\beta}|X) &= Var((X'X)^{-1}X'e|X) \\
&= (X'X)^{-1}X'Var(e|X)X(X'X)^{-1} \\
&= n^{-2}X'Var(e|X)X \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Then then unconditional variance is given by LOTV:

$$Var(\widehat{\beta}) = Var(\mathbb{E}[\widehat{\beta}|X] + \mathbb{E}[Var(\widehat{\beta}|X)] = \frac{\sigma^2}{n}$$