SOLUTIONS - CHAPTER 7

ZHIZHONG PU

Exercise 7. 1. Question see book.

Given the DGS: $Y = X_1'\beta_1 + X_2'\beta_2 + e$ with $\mathbb{E}[Xe] = 0$, regressing Y on X_1 only yields

$$\begin{aligned}
\hat{\beta}_1 &= (X_1X_1')^{-1}X_1Y \\
&= (X_1X_1')^{-1}X_1(X_1'\beta_1 + X_2'\beta_2 + e) \\
&= \beta_1 + (X_1X_1')^{-1}(X_1X_2'\beta_2) + (X_1X_1')^{-1}(X_1e) \\
&= \beta_1 + (\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}')^{-1}(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{2i}')\beta_2 + (\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}')^{-1}(\frac{1}{n}\sum_{i=1}^{n} X_{1i}e_i)
\end{aligned}$$

Note that by WLLN, we have

(1) $(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}')^{-1} \to \mathbb{E}[X_1X_1']^{-1}$
(2) $\frac{1}{n}\sum_{i=1}^{n} X_{1i}e_i \to \mathbb{E}[X_1e] = 0$
(3) $\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{2i}' \to \mathbb{E}[X_1X_2']$ if $\mathbb{E}[X_1X_2'] < \infty$

Then $\hat{\beta}_1$ is generally not consistent, and it is consistent iff $\mathbb{E}[X_1X_2'] = 0$ or $\beta_2 = 0$

Exercise 7. 2.

Exercise 7. 3.

Exercise 7. 4.

Exercise 7. 5.

Exercise 7. 6. Full problem see book. Find the method of moments estimators $(\hat{\beta}, \hat{\Omega})$ for (β, Ω) where $\Omega = \mathbb{E}[XX'e^2]$.

Date: April 2023.

Since $\hat{\beta} = \mathbb{E}[XX']\mathbb{E}[XY]$, then we could substitute in the sample moment for population moment: $\hat{\beta}_{MoM} = (\frac{1}{n}\sum_{i=1}^{n} X_i X_i')^{-1}(\frac{1}{n}\sum_{i=1}^{n} X_i Y_i')$. And we can do the same for Ω:

$$\hat{\Omega}_{MoM} = (\frac{1}{n}\sum_{i=1}^{n} X_i X_i' e_i^2)$$

Exercise 7. 7.

Exercise 7. 8. Find the asymptotic distribution of $\sqrt{n}(\widehat{\sigma^2} - \sigma^2)$ as $n \to \infty$

First, we can re-write:

$$\begin{aligned}\sqrt{n}(\widehat{\sigma^2} - \sigma^2) &= \sqrt{n}(\widehat{\sigma^2} - \frac{n-k}{n}\sigma^2 - \frac{k}{n}\sigma^2) \\ &= \sqrt{n}(\widehat{\sigma^2} - \frac{n-k}{n}\sigma^2) - \frac{k}{\sqrt{n}}\sigma^2\end{aligned}$$

Note that $\widehat{\sigma^2} \equiv n^{-1}\sum_{i=1}^{n} \hat{e}'\hat{e}$ by definition and that in Chapter 4.11 we have established that

$$\mathbb{E}[\widehat{\sigma^2}] = \frac{n-k}{n}\sigma^2$$

Then we can view $\widehat{\sigma^2}$ and $\frac{n-k}{n}$ as a sample mean and its population mean, respectively. This warrants applying the CLT, which states that

$$\sqrt{n}(\widehat{\sigma^2} - \frac{n-k}{n}\sigma^2) \xrightarrow{d} \mathcal{N}(0, V)$$

where $V = \mathbb{E}[(\widehat{\sigma^2} - \frac{n-k}{n}\sigma^2)^2]$. Lastly, as $-\frac{k}{\sqrt{n}}\sigma^2$ is a constant that converges to 0, adding it is equivalent to a mean-shift of size 0. Then

$$\sqrt{n}(\widehat{\sigma^2} - \sigma^2) \xrightarrow{d} \mathcal{N}(0, V)$$