SOLUTIONS - CHAPTER 5

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Exercise 5. 1. Show that if $Q \sim \chi^2_r$, then $\mathbb{E}[Q] = r$ and $Var[Q] = 2r$

First write $Q = \sum_{i=1}^{r} Z_i$ with $Z_i \sim \mathcal{N}(0,1)$ i.i.d.. Then $\mathbb{E}[Z_i^2] = 1$ and $\mathbb{E}[Z_i^4] = 3$ are easy to calculate.

Then $\mathbb{E}[Q] = r$ and

$$Var(Q) = rVar(Z_i^2) = r(\mathbb{E}[Z_i^4] - \mathbb{E}[Z_i^2]^2) = 2r$$

Exercise 5. 2. Show that if $e \sim \mathcal{N}(0, \mathbb{I}_n\sigma^2)$ and $H'H = \mathbb{I}_n$ then $u = H'e \sim \mathcal{N}(0, \mathbb{I}_n\sigma^2)$

By Theorem 5.3.3, we have

$$u = H'e \sim \mathcal{N}(H'0, H'\mathbb{I}_n\sigma^2 H) = \mathcal{N}(0, H'H\sigma^2) = \mathcal{N}(0, \mathbb{I}_n\sigma^2)$$

Exercise 5. 3. Show that if $e \sim \mathcal{N}(0, AA')$ then $u = A^{-1}e \sim \mathcal{N}(0, \mathbb{I}_n)$

By Theorem 5.3.3, we have

$$u = A^{-1}e \sim \mathcal{N}(A^{-1}0, A^{-1}AA'(A^{-1})' = \mathcal{N}(0, \mathbb{I}_n)$$

Exercise 5. 4.

Exercise 5. 5. Show that $\hat{Y}_i|X \sim \mathcal{N}(X_i'\beta, \sigma^2 h_i)$ where are the leverage values (3.40).

$$\hat{Y} = PY = X(X'X)^{-1}X'(X\beta + e) = X\beta + X(X'X)^{-1}X'e = X\beta + Pe$$

Then

$$\hat{Y} \sim \mathcal{N}(X\beta, PPVar(e)) \equiv \mathcal{N}(X\beta, \sigma^2 P)$$

Since h_i specifies the ith diagonal element of P, then this shows that $\hat{Y}_i|X \sim \mathcal{N}(X_i'\beta, \sigma^2 h_i)$

Date: May 2023.

Exercise 5. 6.

Exercise 5. 7. In the normal regression model show that the robust covariance matrices $V_{\hat{\beta}}^{HCi}$ are independent of the OLS estimator $\hat{\beta}$, conditional on X.

Recall

$$V_{\hat{\beta}}^{HC0} = (X'X)^{-1}(\sum_{i=1}^{n} X_i X_i' \widehat{e_i^2})(X'X^{-1})$$

$$V_{\hat{\beta}}^{HC1} = \frac{n}{n-k}(X'X)^{-1}(\sum_{i=1}^{n} X_i X_i' \widehat{e_i^2})(X'X^{-1})$$

$$V_{\hat{\beta}}^{HC2} = (X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-1} X_i X_i' \widehat{e_i^2})(X'X)^{-1}$$

$$V_{\hat{\beta}}^{HC3} = (X'X)^{-1}(\sum_{i=1}^{n}(1-h_i)^{-2} X_i X_i' \widehat{e_i^2})(X'X)^{-1}$$

Then it can be seen that the only stochastic term in each of the variance estimators is $\widehat{e_i^2}$ after conditioning on X. By Theorem 5.6, we have $\widehat{e_i^2}$ being independent of $\hat{\beta}$ conditional on X.

Exercise 5. 8. Let $F(u)$ be the distribution function of a random variable X whose density is symmetric about 0. Show that $F(-u) = 1 - F(u)$.

Let $f(u)$ be the p.d.f. of the distribution specified by $F(u)$. We know that $f(u)$ is symmetric about 0, then $F(0) = 0.5$ and for any x on the support of f, we have:

$$\int_{-u}^{0} f(x) = \int_{0}^{u} f(x) \Rightarrow F(0) - F(-u) = F(u) - F(0) \Rightarrow F(-u) = 1 - F(u)$$

Exercise 5. 9. Let $\hat{C}_\beta = [L, U]$ be a $1-\alpha$ confidence interval for β,and consider the transformation where $g()$ is monotonically increasing. Consider the confidence interval $\hat{C}_\theta = [g(L), g(U)]$ for θ. Show that $Pr(\theta \in \hat{C}_\theta) = Pr(\beta \in \hat{C}_\beta)$.

Note that since $g()$ is monotonically increasing, then $g(x) \leq g(L) \Leftrightarrow x \leq L$. Then

$$Pr(\beta \in \hat{C}_\beta) = Pr(\beta \leq U) - Pr(\beta \leq L) = Pr(g(\beta) \leq g(U)) - Pr(g(\beta) \leq g(L)) = Pr(\theta \in \hat{C}_\theta)$$

Exercise 5. 10.

Exercise 5. 11.

Exercise 5. 12. In the normal regression model let s^2 be the unbiased estimator of the error variance σ^2 from (4.31). Show that $Var(s^2) = 2\sigma^4/(n-k)$ and that it is strictly larger than the Cramér-Rao Lower Bound for σ^2

By Theorem 5.7 we have

$$\frac{(n-k)s^2}{\sigma^2} \sim \chi^2_{n-k}$$

Note that the variance of LHS is $2(n-k)$, and $(n-k)$ and σ are non-stochastic, so

$$\frac{(n-k)^2}{\sigma^4} Var(s^2) = 2(n-k)$$

This shows that $Var(s^2) = 2\sigma^4/(n-k)$. Note that $k > 0$ and the Cramér-Rao Lower Bound for σ^2 is $2\sigma^4/n$ from (5.20), so $Var(s^2) = 2\sigma^4/(n-k) > 2\sigma^4/n$