

# Proof for paper titled Convergence Analysis of Client-Edge-Cloud Hierarchical Split Federated Learning

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## 1 Assumptions

**Assumption 1.** (*L-Smoothness*). The loss function  $f(x)$  is  $L$ -smooth with the Lipschitz constant  $0 < L < \infty$ , i.e., for any  $x$  and  $y$ , we have

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad (1)$$

**Assumption 2.** (*Variance of SGD*). The variances of stochastic gradients for client-side model and server-side model across each layer are bounded

$$\mathbb{E}_{\xi_n \sim \mathcal{D}_n} \left\| \tilde{\nabla} f_{r,n}(\mathbf{w}) - \nabla f_{r,n}(\mathbf{w}) \right\|^2 \leq \sum_{l=1}^{L_c} \sigma_l^2, \forall \mathbf{w}, \forall n, \quad (2)$$

$$\mathbb{E}_{\xi_n \sim \mathcal{D}_n} \left\| \tilde{\nabla} f_{s,n}(\mathbf{w}) - \nabla f_{s,n}(\mathbf{w}) \right\|^2 \leq \sum_{l=L_c+1}^{L^T} \sigma_l^2, \forall \mathbf{w}, \forall n, \quad (3)$$

where  $L_c$  is the number of client-side model layers, and  $L^T$  represents the total number of layers for the global model.  $\sigma_l^2$  denotes the bounded variance for the  $l$ -th layer of model.

**Assumption 3.** (*Second moments*). The expectation of the squared  $l_2$ -norm of the gradients for client-side models and server-side models across each layer are bounded

$$\mathbb{E}_{\xi_n \sim \mathcal{D}_n} \|\nabla f_{r,n}(\mathbf{w})\|^2 \leq \sum_{l=1}^{L_c} G_l^2, \forall \mathbf{w}, \forall n, \quad (4)$$

$$\mathbb{E}_{\xi_n \sim \mathcal{D}_n} \|\nabla f_{s,n}(\mathbf{w})\|^2 \leq \sum_{l=L_c+1}^{L^T} G_l^2, \forall \mathbf{w}, \forall n, \quad (5)$$

where  $G_l^2$  denotes the second order moments for the  $l$ -th layer of model.

## 2 Model Update

The evolution of the model parameters  $\mathbf{x}_r^{k+1}$ ,  $\mathbf{x}_s^{k+1}$  in the cloud are specified as follows:

$$\mathbf{x}_r^{k+1} = \mathbf{x}_r^k - \eta \sum_{z=1}^Z \frac{n^z}{N} \frac{1}{n^z} \sum_{\alpha=0}^{\tau_2-1} \sum_{j=1}^{n^z} \sum_{\beta=0}^{\tau_1-1} \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta}), \quad (6)$$

$$\mathbf{x}_s^{k+1} = \mathbf{x}_s^k - \eta \sum_{z=1}^Z \frac{n^z}{N} \frac{1}{n^z} \sum_{\alpha=0}^{\tau_1\tau_2-1} \sum_{j=1}^{n^z} \tilde{\nabla} f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha}). \quad (7)$$

Moreover, the evolution of the client-side model and server-side model parameters  $\mathbf{w}_{r,n}^{k,\alpha,\beta}$ ,  $\mathbf{w}_{s,n}^{k,\gamma}$  for client  $n$  are specified as follows:

$$\mathbf{w}_{r,n}^{k,\alpha,\beta} = \mathbf{x}_r^k - \eta \sum_{t_1=0}^{\beta-1} \tilde{\nabla} f_{r,n}(\mathbf{w}_{r,n}^{k,\alpha,t_1}) - \eta \sum_{t_2=0}^{\alpha-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \sum_{t_1=0}^{\tau_1-1} \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}), \quad (8)$$

$$\mathbf{w}_{s,n}^{k,\gamma} = \mathbf{x}_s^k - \eta \sum_{t=0}^{\gamma-1} \frac{1}{n^z} \sum_{j=1}^{n^z} \tilde{\nabla} f_{s,j}(\mathbf{w}_{s,j}^{k,t}), \quad (9)$$

where  $\gamma = \alpha \times \tau_1 + \beta$ .

### 3 Lemma 1

**Lemma 1.** (One Round of Cloud Aggregation). With Assumption 1, the following relationship between  $\mathbf{x}^{k+1}$  and  $\mathbf{x}^k$ :

$$\begin{aligned} \mathbb{E}[f(\mathbf{x}^{k+1})] &\leq \mathbb{E}[f(\mathbf{x}^k)] + \mathbb{E}[\langle \nabla f(\mathbf{x}^k), \mathbf{x}^{k+1} - \mathbf{x}^k \rangle] \\ &\quad + \frac{L}{2} \mathbb{E}[\|\mathbf{x}_r^{k+1} - \mathbf{x}_r^k\|^2] + \frac{L}{2} \mathbb{E}[\|\mathbf{x}_s^{k+1} - \mathbf{x}_s^k\|^2]. \end{aligned} \quad (10)$$

*Proof.* The proof directly follows from the property of the  $L$ -smoothness:

$$\mathbb{E}[f(\mathbf{x}^{k+1})] \leq \mathbb{E}[f(\mathbf{x}^k)] + \mathbb{E}[\langle \nabla f(\mathbf{x}^k), \mathbf{x}^{k+1} - \mathbf{x}^k \rangle] + \frac{L}{2} \mathbb{E}[\|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2]. \quad (11)$$

Note that,

$$\mathbb{E}[\|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2] = \mathbb{E}[\|\mathbf{x}_r^{k+1} - \mathbf{x}_r^k\|^2] + \mathbb{E}[\|\mathbf{x}_s^{k+1} - \mathbf{x}_s^k\|^2]. \quad (12)$$

Substituting Eq. (12) into Eq. (11), we complete the proof.

### 4 Lemma 2

**Lemma 2.** With Assumptions 1, 2, and 3,  $\mathbb{E}[\|\mathbf{x}_r^{k+1} - \mathbf{x}_r^k\|^2]$  is bounded as follows:

$$\mathbb{E}\|\mathbf{x}_r^{k+1} - \mathbf{x}_r^k\|^2 \leq \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E}\|\nabla f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta})\|^2 + \eta^2 \tau_1^2 \tau_2^2 \sum_{l=1}^{L_c} \sigma_l^2. \quad (13)$$

*Proof.*

$$\begin{aligned} &\mathbb{E}\|\mathbf{x}_r^{k+1} - \mathbf{x}_r^k\|^2 \\ &\stackrel{(a)}{=} \eta^2 \mathbb{E}\left\| \sum_{i=1}^N \frac{1}{N} \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \tilde{\nabla} f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta}) \right\|^2 \\ &\stackrel{(b)}{=} \eta^2 \mathbb{E}\left\| \sum_{i=1}^N \frac{1}{N} \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \nabla f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta}) \right\|^2 + \eta^2 \mathbb{E}\left\| \sum_{i=1}^N \frac{1}{N} \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \tilde{\nabla} f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta}) - \nabla f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta}) \right\|^2 \\ &\stackrel{(c)}{\leq} \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E}\|\nabla f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta})\|^2 + \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E}\|\tilde{\nabla} f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta}) - \nabla f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta})\|^2 \\ &\stackrel{(d)}{\leq} \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E}\|\nabla f_{r,i}(\mathbf{w}_{r,i}^{k,\alpha,\beta})\|^2 + \eta^2 \tau_1^2 \tau_2^2 \sum_{l=1}^{L_c} \sigma_l^2, \end{aligned} \quad (14)$$

where (a) follows from Eq. (6); (b) follows by applying the Eq.  $\mathbb{E}\|x\|^2 = \|\mathbb{E}x\|^2 + \text{Var}(x)^2$ ; (c) follows from the convexity of  $\|\cdot\|_2^2$ , and (d) follows from Eq. (2).

### 5 Lemma 3

**Lemma 3.** With Assumptions 1, 2, and 3,  $\mathbb{E}[\|\mathbf{x}_s^{k+1} - \mathbf{x}_s^k\|^2]$  is bounded as follows:

$$\mathbb{E}\|\mathbf{x}_s^{k+1} - \mathbf{x}_s^k\|^2 \leq \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \mathbb{E}\|\nabla f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha})\|^2 + \eta^2 \tau_1^2 \tau_2^2 \sum_{l=L_c+1}^{L^T} \sigma_l^2. \quad (15)$$

The proof process is similar to Lemma 2.

*Proof.*

$$\begin{aligned}
& \mathbb{E} \|\mathbf{x}_s^{k+1} - \mathbf{x}_s^k\|^2 \\
& \stackrel{(a)}{=} \eta^2 \mathbb{E} \left\| \sum_{i=1}^N \frac{1}{N} \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \tilde{\nabla} f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha}) \right\|^2 \\
& \stackrel{(b)}{=} \eta^2 \mathbb{E} \left\| \sum_{i=1}^N \frac{1}{N} \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \nabla f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha}) \right\|^2 + \eta^2 \mathbb{E} \left\| \sum_{i=1}^N \frac{1}{N} \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \tilde{\nabla} f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha}) - \nabla f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha}) \right\|^2 \\
& \stackrel{(c)}{\leq} \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \mathbb{E} \|\nabla f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha})\|^2 + \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \mathbb{E} \|\tilde{\nabla} f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha}) - \nabla f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha})\|^2 \\
& \stackrel{(d)}{\leq} \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_1 \tau_2 - 1} \mathbb{E} \|\nabla f_{s,i}(\mathbf{w}_{s,i}^{k,\alpha})\|^2 + \eta^2 \tau_1^2 \tau_2^2 \sum_{l=L_c+1}^{L^T} \sigma_l^2,
\end{aligned} \tag{16}$$

where (a) follows from Eq. (7); (b) follows by applying the Eq.  $\mathbb{E}\|x\|^2 = \|\mathbb{E}x\|^2 + \text{Var}(x)^2$ ; (c) follows from the convexity of  $\|\cdot\|_2^2$ , and (d) follows from Eq. (3).

Thus, substituting Eq. (13) and Eq. (15) into Eq. (12) yields

$$\mathbb{E} [\|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2] = \frac{\eta^2}{N} \sum_{i=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E} \|\nabla f_i(\mathbf{w}_i^{k,\alpha,\beta})\|^2 + \eta^2 \tau_1^2 \tau_2^2 \sum_{l=1}^{L^T} \sigma_l^2. \tag{17}$$

## 6 Lemma 4

**Lemma 4.** *With Assumption 1, 2, and 3,  $\mathbb{E} [\langle \nabla f(\mathbf{x}^k), \mathbf{x}^{k+1} - \mathbf{x}^k \rangle]$  is bounded as follows:*

$$\begin{aligned}
& \mathbb{E} [\langle \nabla f(\mathbf{x}^k), \mathbf{x}^{k+1} - \mathbf{x}^k \rangle] \\
& \leq -\frac{\eta \tau_1 \tau_2}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 - \frac{\eta \tau_1 \tau_2}{2} \mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \right\|^2 \\
& \quad + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2),
\end{aligned} \tag{18}$$

where  $C(\tau_1, \tau_2)$  and  $D(\tau_1, \tau_2)$  are functions of  $\tau_1$  and  $\tau_2$ ,

$$C(\tau_1, \tau_2) = \frac{\tau_1^2 (\tau_2 - 1) (2\tau_2 - 1)}{6} + \frac{(\tau_1 - 1) (2\tau_1 - 1)}{6} + \frac{\tau_1 (\tau_2 - 1) (\tau_1 - 1)}{2}, \tag{19}$$

$$D(\tau_1, \tau_2) = \frac{\tau_1^2 (\tau_2 - 1) (2\tau_2 - 1)}{6} + \frac{(\tau_1 - 1) (2\tau_1 - 1)}{6} - \frac{\tau_1 (\tau_2 - 1) (\tau_1 - 1)}{2}. \tag{20}$$

*Proof.*

By taking the expectation, we obtain:

$$\begin{aligned}
& \mathbb{E} [\langle \nabla f(\mathbf{x}^k), \mathbf{x}^{k+1} - \mathbf{x}^k \rangle] \\
& = -\eta \mathbb{E} \left[ \langle \nabla f(\mathbf{x}^k), \frac{1}{N} \sum_{j=1}^N \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \rangle \right] \\
& = -\eta \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E} \left[ \langle \nabla f(\mathbf{x}^k), \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \rangle \right] \\
& \stackrel{(a)}{=} \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \left[ -\frac{\eta}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 - \frac{\eta}{2} \mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \right\|^2 + \frac{\eta}{2} \mathbb{E} \left\| \nabla f(\mathbf{x}^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \right\|^2 \right],
\end{aligned} \tag{21}$$

where (a) follows by using the identity  $2\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2$ .

$$\begin{aligned}
& \mathbb{E} \|\nabla f(\mathbf{x}_r^k) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta})\|^2 \\
& \stackrel{(a)}{\leq} L^2 \eta^2 \mathbb{E} \left\| \sum_{t_1=0}^{\beta-1} \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) + \sum_{t_2=0}^{\alpha-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \sum_{t_1=0}^{\tau_1-1} \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) \right\|^2 \\
& \stackrel{(b)}{=} L^2 \eta^2 \mathbb{E} \left\| \sum_{t_1=0}^{\beta-1} \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) + \sum_{t_2=0}^{\alpha-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \sum_{t_1=0}^{\tau_1-1} \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) \right\|^2 \\
& \quad + L^2 \eta^2 \mathbb{E} \left\| \sum_{t_1=0}^{\beta-1} \left( \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) \right) + \sum_{t_2=0}^{\alpha-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \sum_{t_1=0}^{\tau_1-1} \left( \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) \right) \right\|^2 \\
& \stackrel{(c)}{\leq} L^2 \eta^2 \mathbb{E} \left\| \sum_{t_1=0}^{\beta-1} \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) + \sum_{t_2=0}^{\alpha-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \sum_{t_1=0}^{\tau_1-1} \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) \right\|^2 \\
& \quad + 2L^2 \eta^2 \left( \underbrace{\mathbb{E} \left\| \sum_{t_1=0}^{\beta-1} \left( \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) \right) \right\|^2}_{A_2} + \underbrace{\mathbb{E} \left\| \sum_{t_2=0}^{\alpha-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \sum_{t_1=0}^{\tau_1-1} \left( \tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) \right) \right\|^2}_{A_3} \right) \\
& \stackrel{(d)}{\leq} \underbrace{2L^2 \eta^2 \beta^2 \sum_{l=1}^{L_c} G_l^2 + 2L^2 \eta^2 \alpha^2 \tau_1^2 \sum_{l=1}^{L_c} G_l^2 + 2L^2 \eta^2 (A_2 + A_3)}_{A_1},
\end{aligned} \tag{26}$$

Now, we will bound the third term on the right hand side (RHS) of Eq. (21).

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta})\|^2 \\
& = \frac{\eta}{2} \mathbb{E} \|\nabla f(\mathbf{x}_r^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta})\|^2 + \frac{\eta}{2} \mathbb{E} \|\nabla f(\mathbf{x}_s^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha,\beta})\|^2,
\end{aligned} \tag{22}$$

where  $\frac{\eta}{2} \mathbb{E} \|\nabla f(\mathbf{x}_r^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta})\|^2$  can be bounded as

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E} \|\nabla f(\mathbf{x}_r^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta})\|^2 \\
& = \frac{\eta}{2} \mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \nabla f(\mathbf{x}_r^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta}) \right\|^2 \\
& \leq \frac{\eta}{2N} \sum_{j=1}^N \mathbb{E} \|\nabla f(\mathbf{x}_r^k) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta})\|^2,
\end{aligned} \tag{23}$$

where the bound on the RHS, Eq. (26), shown at the beginning of this page.

Now we continue to bound the two terms  $A_2$  and  $A_3$  in Eq. (26) as

$$A_2 \leq \beta \sum_{t_1=0}^{\beta-1} \mathbb{E} \|\tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1}) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,t_1})\|^2 \leq \beta^2 \sum_{l=1}^{L_c} \sigma_l^2. \tag{24}$$

$$A_3 \leq \alpha \tau_1 \sum_{t_2=0}^{\alpha-1} \sum_{t_1=0}^{\tau_1-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \mathbb{E} \|\tilde{\nabla} f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1}) - \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,t_2,t_1})\|^2 \leq \alpha^2 \tau_1^2 \sum_{l=1}^{L_c} \sigma_l^2. \tag{25}$$

Thus, substituting Eq. (26), Eq. (24), and Eq. (25) into Eq. (23) yields

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E} \left\| \nabla f(\mathbf{x}_r^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{r,j}(\mathbf{w}_{r,j}^{k,\alpha,\beta}) \right\|^2 \\
& \leq \frac{\eta}{2N} \sum_{j=1}^N \left[ 2L^2 \eta^2 \beta^2 \sum_{l=1}^{L_c} G_l^2 + 2L^2 \eta^2 \alpha^2 \tau_1^2 \sum_{l=1}^{L_c} G_l^2 + 2L^2 \eta^2 \sum_{l=1}^{L_c} \sigma_l^2 (\beta^2 + \alpha^2 \tau_1^2) \right] \\
& \leq L^2 \eta^3 \sum_{l=1}^{L_c} G_l^2 (\beta^2 + \alpha^2 \tau_1^2) + L^2 \eta^3 \sum_{l=1}^{L_c} \sigma_l^2 (\beta^2 + \alpha^2 \tau_1^2).
\end{aligned} \tag{27}$$

Next,  $\frac{\eta}{2} \mathbb{E} \left\| \nabla f(\mathbf{x}_s^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha,\beta}) \right\|^2$  can be bounded as

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E} \left\| \nabla f(\mathbf{x}_s^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha,\beta}) \right\|^2 \\
& = \frac{\eta}{2} \mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \nabla f(\mathbf{x}_s^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha,\beta}) \right\|^2 \\
& \leq \frac{\eta}{2N} \sum_{j=1}^N \mathbb{E} \left\| \nabla f(\mathbf{x}_s^k) - \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha,\beta}) \right\|^2,
\end{aligned} \tag{28}$$

the term on the RHS can be bounded as

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E} \left\| \nabla f(\mathbf{x}_s^k) - \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,\alpha,\beta}) \right\|^2 \\
& \stackrel{(a)}{\leq} \frac{\eta^3}{2} L^2 \mathbb{E} \left\| \sum_{t=0}^{\alpha\tau_1+\beta-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \tilde{\nabla} f_{s,j}(\mathbf{w}_{s,j}^{k,t}) \right\|^2 \\
& \stackrel{(b)}{=} \frac{\eta^3}{2} L^2 \mathbb{E} \left\| \sum_{t=0}^{\alpha\tau_1+\beta-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,t}) \right\|^2 + \frac{\eta^3}{2} L^2 \mathbb{E} \left\| \sum_{t=0}^{\alpha\tau_1+\beta-1} \sum_{j=1}^{n^z} \frac{1}{n^z} \left( \tilde{\nabla} f_{s,j}(\mathbf{w}_{s,j}^{k,t}) - \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,t}) \right) \right\|^2 \\
& \stackrel{(c)}{\leq} \frac{\eta^3}{2} L^2 (\alpha\tau_1 + \beta) \sum_{t=0}^{\alpha\tau_1+\beta-1} \frac{1}{n^z} \sum_{j=1}^{n^z} \mathbb{E} \left\| \nabla f_{s,j}(\mathbf{w}_{s,j}^{k,t}) \right\|^2 + \frac{\eta^3}{2} L^2 (\alpha\tau_1 + \beta)^2 \sum_{l=L_c+1}^{L^T} \sigma_l^2 \\
& \stackrel{(d)}{\leq} \frac{\eta^3}{2} L^2 (\alpha\tau_1 + \beta)^2 \sum_{l=L_c+1}^{L^T} G_l^2 + \frac{\eta^3}{2} L^2 (\alpha\tau_1 + \beta)^2 \sum_{l=L_c+1}^{L^T} \sigma_l^2,
\end{aligned} \tag{29}$$

where (a) follows from Eq. (9); (b) follows by applying the Eq.  $\mathbb{E} \|x\|^2 = \|\mathbb{E}x\|^2 + \text{Var}(x)^2$ ; (c) follows from the convexity of  $\|\cdot\|_2^2$  and Eq. (3), and (d) follows from Eq. (5). Thus, substituting Eq. (27), Eq. (28), and Eq. (29) into Eq. (22) yields

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E} \left\| \nabla f(\mathbf{x}^k) - \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \right\|^2 \\
& = L^2 \eta^3 \frac{(\alpha\tau_1 + \beta)^2}{2} \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + L^2 \eta^3 \frac{(\alpha\tau_1 - \beta)^2}{2} \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2)
\end{aligned} \tag{30}$$

Thus, substituting Eq. (30) into Eq. (21) yields

$$\begin{aligned}
& \mathbb{E} [\langle \nabla f(\mathbf{x}^k), \mathbf{x}^{k+1} - \mathbf{x}^k \rangle] \\
& \leq -\frac{\eta\tau_1\tau_2}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 - \frac{\eta\tau_1\tau_2}{2} \mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \right\|^2 + \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \left( L^2 \eta^3 \frac{(\alpha\tau_1 + \beta)^2}{2} \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) \right. \\
& \quad \left. + L^2 \eta^3 \frac{(\alpha\tau_1 - \beta)^2}{2} \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2) \right) \\
& = -\frac{\eta\tau_1\tau_2}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 - \frac{\eta\tau_1\tau_2}{2} \mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \nabla f_j(\mathbf{w}_j^{k,\alpha,\beta}) \right\|^2 + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) \\
& \quad + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2)
\end{aligned} \tag{31}$$

where

$$C(\tau_1, \tau_2) = \frac{\tau_1^2 (\tau_2 - 1) (2\tau_2 - 1)}{6} + \frac{(\tau_1 - 1) (2\tau_1 - 1)}{6} + \frac{\tau_1 (\tau_2 - 1) (\tau_1 - 1)}{2} \tag{32}$$

$$D(\tau_1, \tau_2) = \frac{\tau_1^2 (\tau_2 - 1) (2\tau_2 - 1)}{6} + \frac{(\tau_1 - 1) (2\tau_1 - 1)}{6} - \frac{\tau_1 (\tau_2 - 1) (\tau_1 - 1)}{2} \tag{33}$$

## 7 Theorem 1

**Theorem 1.** (Convergence of HierSFL for Non-Convex Loss Functions). The mean square gradient of cloud model after  $K$  aggregation rounds is bounded:

$$\begin{aligned}
& \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 \\
& \leq \frac{2}{\eta\tau_1\tau_2 K} (f(\mathbf{x}^0) - f^*) - \sum_{l=1}^{L^T} G_l^2 + L^2 \eta^2 C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + L\eta\tau_1\tau_2 \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) \\
& \quad + L^2 \eta^2 D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2),
\end{aligned} \tag{34}$$

where  $f^*$  represents the minimum value.

*Proof.*

By combining Lemmas 1 to 4, we now have the following:

$$\begin{aligned}
& \mathbb{E} [f(\mathbf{x}^{k+1})] \\
& \leq \mathbb{E} [f(\mathbf{x}^k)] - \frac{\eta\tau_1\tau_2}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 - \frac{\eta\tau_1\tau_2}{2} \frac{1}{N} \sum_{j=1}^N \mathbb{E} \|\nabla f_j(\mathbf{w}_j^{k,\alpha,\beta})\|^2 + \frac{L\eta^2}{2N} \sum_{j=1}^N \tau_1 \tau_2 \sum_{\alpha=0}^{\tau_2-1} \sum_{\beta=0}^{\tau_1-1} \mathbb{E} \|\nabla f_j(\mathbf{w}_j^{k,\alpha,\beta})\|^2 \\
& \quad + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2) + \frac{L}{2} \eta^2 \tau_1^2 \tau_2^2 \sum_{l=1}^{L^T} \sigma_l^2 \\
& \leq \mathbb{E} [f(\mathbf{x}^k)] - \frac{\eta\tau_1\tau_2}{2} \mathbb{E} \|\nabla f(\mathbf{x}^k)\|^2 - \frac{\eta\tau_1\tau_2}{2} \sum_{l=1}^{L^T} G_l^2 + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + \frac{L\eta^2}{2} \tau_1^2 \tau_2^2 \sum_{l=1}^{L^T} G_l^2 \\
& \quad + \frac{L^2 \eta^3}{2} \tau_1 \tau_2 D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2) + \frac{L}{2} \eta^2 \tau_1^2 \tau_2^2 \sum_{l=1}^{L^T} \sigma_l^2.
\end{aligned} \tag{35}$$

Dividing Eq. (35) both sides by  $\frac{\eta\tau_1\tau_2}{2}$  and rearranging terms yields

$$\begin{aligned}
\mathbb{E}\|\nabla f(\mathbf{x}^k)\|^2 &\leq \frac{2}{\eta\tau_1\tau_2} (\mathbb{E}[f(\mathbf{x}^k)] - \mathbb{E}[f(\mathbf{x}^{k+1})]) - \sum_{l=1}^{L^T} G_l^2 + L^2\eta^2C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + L\eta\tau_1\tau_2 \sum_{l=1}^{L^T} G_l^2 \\
&\quad + L^2\eta^2D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2) + L\eta\tau_1\tau_2 \sum_{l=1}^{L^T} \sigma_l^2
\end{aligned} \tag{36}$$

Summing over  $k = \{1, 2, \dots, K\}$  and dividing both sides by  $K$  yields

$$\begin{aligned}
&\frac{1}{K} \sum_{k=1}^K \mathbb{E}\|\nabla f(\mathbf{x}^k)\|^2 \\
&\leq \frac{2}{\eta\tau_1\tau_2K} (f(\mathbf{x}^1) - \mathbb{E}[f(\mathbf{x}^{K+1})]) - \sum_{l=1}^{L^T} G_l^2 + L^2\eta^2C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + L\eta\tau_1\tau_2 \sum_{l=1}^{L^T} G_l^2 \\
&\quad + L^2\eta^2D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2) + L\eta\tau_1\tau_2 \sum_{l=1}^{L^T} \sigma_l^2 \\
&\leq \frac{2}{\eta\tau_1\tau_2K} (f(\mathbf{x}^1) - f^*) - \sum_{l=1}^{L^T} G_l^2 + L^2\eta^2C(\tau_1, \tau_2) \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) + L\eta\tau_1\tau_2 \sum_{l=1}^{L^T} (G_l^2 + \sigma_l^2) \\
&\quad + L^2\eta^2D(\tau_1, \tau_2) \sum_{l=1}^{L_c} (G_l^2 + \sigma_l^2)
\end{aligned} \tag{37}$$