1. **[8 points]** Solve the following ordinary differential equation (ODE):

$$\dot{L}(t) = nL(t)$$

with the boundary condition $L(0) = L_0$.

2. **[49 points in total]** Suppose that the production function follows Cobb-Douglas function:

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{2}$$

and the capital accumulation follows

$$\dot{K}(t) = sY(t) - \delta K(t), \tag{3}$$

where s is the saving rate (fraction of output invested in capital) and δ is the depreciation rate of capital. The population growth and the technology growth follow

$$\dot{L}(t) = nL(t), \tag{4}$$

$$\dot{A}(t) = gA(t), \tag{5}$$

where n is the population growth rate and g is the technology growth rate.

(1) **[4 points]** Using the assumption of constant returns to scale to derive the intensive form of production function: $y = f(k) = k^{\alpha}$, where $y = \frac{Y}{AL}$ and $k = \frac{K}{AL}$.

(2) **[4 points]** Show that the marginal product of K and the marginal product of k are the same.

(3) **[5 points]** Use the chain rule to derive the dynamics of k.

(4) **[5 points]** Show the phase diagram of $\dot{k}$ as a function of k.

(5) **[8 points in total and 4 points for each]** Describe how, if at all, each of the following developments affects the break even and actual investment lines in our basic diagram for the Solow model:

(a) The rate of depreciation falls.

(b) Capital's share, α, rises.

(6) Consider a Solow economy that is on its balanced growth path.

(a) **[6 points in total and 2 points for each]** Find expressions for k^*, y^* and c^* as functions of the parameters of the model, s, n, δ, g and α.

(b) **[12 points in total and 3 points for each]** If the saving rate s decreases permanently at t_0, please sketch the paths of the following variables as the economy moves to its new balanced growth path: $\dot{k}$, k, $\dot{\left(\frac{Y}{L}\right)}$ and $\left(\frac{Y}{L}\right)$.

(c) **[5 points]** Following the previous question. How does it affect the consumption c?

3. **[8 points]** What is the Solow Residual?

4. **[35 points in total] [Maximum Principle of Optimal Control]**: Given the lifetime utility at t_0 as follows

$$U(0) = \int_0^\infty e^{-\rho t} \cdot \ln [c(t)] \, dt, \tag{6}$$

subject to the following constraints:

$$\dot{k}(t) = [k(t)]^\alpha - c(t) - \delta k(t), \tag{7}$$

$$k(0) = 1, \tag{8}$$

$$\lim_{t \to \infty} \left[k(t) \cdot e^{-\bar{r}(t) t} \right] \geq 0. \tag{9}$$

(1) **[5 points]** What are the control variables and the state variables?

(2) **[8 points in total and 4 points for each]** Show the Hamiltonian function. What is the economic intuition for the Hamiltonian function?

(3) **[6 points]** Show the first-order conditions and the transversality condition.

(4) **[6 points in total and 3 points for each]** Show the steady-state consumption c^* and the capital k^*.

(5) **[10 points]** Suppose that $\rho = 0.06$, $\delta = 0$ and $\alpha = 0.3$. Show the phase diagram of the dynamic system.