A.4 Useful Results in Matrix Algebra

Eigenvalues, Eigenvectors, and Diagonalization of Matrices

Introduction to Matrix Algebra

- For an n × n square matrix $\boldsymbol{A}$, an eigenvalue $\boldsymbol{\alpha}$ and eigenvector $\boldsymbol{v}$ satisfy $\boldsymbol{A} \cdot \boldsymbol{v} = \boldsymbol{\alpha} \cdot \boldsymbol{v}$.
- Eigenvalues and eigenvectors simplify the matrix $\boldsymbol{A}$ for solving linear systems.
- Key question: Can we find a scalar $\boldsymbol{\alpha}$ and corresponding non-zero vector $\boldsymbol{v}$ such that

$$(A - \alpha I) \cdot v = 0 \quad ?$$

Introduction to Eigenvalues and Eigenvectors

- Given an n x n matrix A, eigenvalues α and eigenvectors v satisfy the equation

$$(A - \alpha I) \cdot v = 0$$

- For a non-trivial solution ($v \neq 0$), we must have:

$$\det(A - \alpha I) = 0$$

- This is the characteristic equation, leading to the eigenvalues.

- Importance: Eigenvalues and eigenvectors are crucial in understanding the properties of matrices, especially in transformations and stability analysis.

Characteristic Equation

- The determinant equation $\det(A - \alpha I) = 0$
  is a polynomial of degree n in α.

- The roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues.

- Example Calculation: Show an example calculation of eigenvalues using a simple 2 x 2 matrix.

Eigenvectors

- For each eigenvalue α_i, the eigenvector v_i satisfies $A\, v_i = v_i\, \alpha_i$.

- These eigenvectors can be arranged into a matrix V, and $A\, V = V\, D$, where D is a diagonal matrix of eigenvalues.

Example:

For a matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$,the characteristic equation is:

- $$\det(A - \alpha I) = det\left(\begin{bmatrix} 4-\alpha & 2 \\ 1 & 3-\alpha \end{bmatrix}\right)$$

$$=(4-\alpha)(3-\alpha) - 2 = \alpha^2 - 7\alpha + 10 = 0$$

The eigenvalues are the solutions to this polynomial: $\alpha_1 = 2$, $\alpha_2 = 5$.

Finding Eigenvectors

- Once eigenvalues are found, we can find the corresponding eigenvectors by solving:
- $(A - \alpha_i I) \cdot \nu_i$ = 0 for each eigenvalue α_i.

- Example (continued):
- - For α_1 = 2:
- $(A - 2I) \cdot \nu_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \cdot \nu_1$=0
- The solution gives ν_1 = [-1; 1]. eigenvectors

- Repeat for α_2 = 5 to find ν_2 = [2; 1]

Matrix Diagonalization

- If $\det(V) \neq 0$, the matrix A can be diagonalized as $V^{-1} A V = D$.

- This is useful in simplifying the solution of systems of linear differential equations.

- Example: A matrix A is diagonalized.

Matrix Diagonalization

- Construct matrix V from the eigenvectors as columns:

  $V = [v_1\ v_2]$.

- The diagonal matrix D has the eigenvalues on the diagonal:

$$D = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}.$$

- • The matrix A can be diagonalized as: $A = V\, D\, V^{-1}$.

Diagonalizing the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

$$V = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Verify $V^{-1} A V = D$.

Applications and Implications

- Diagonalization simplifies the process of solving linear systems, especially in differential equations.

- Eigenvalues indicate stability in systems dynamics:
  - Positive eigenvalues may indicate growth or instability.
  - Negative eigenvalues indicate decay or stability.

- Diagonalization is widely used in areas such as physics, engineering, economics, and more.

Example Calculation

- Given Matrix:
- $A = \begin{bmatrix} 0.06 & -1 \\ -0.004 & 0 \end{bmatrix}$

- Calculation Steps:
- 1. Find eigenvalues α_1 and α_2.
- 2. Find the corresponding eigenvectors.
- 3. Show how A is diagonalized using these vectors.

Concluding Remarks

- The importance of eigenvalues and eigenvectors in matrix algebra.
- Application in solving differential equations.