A.1 Differential Equations

A.1.1 Introduction to Differential Equations

Overview of Ordinary Differential Equations (ODEs)

What is a Differential Equation?

- A differential equation involves the derivatives of variables.
- If there is only one independent variable, it is called an ordinary differential equation (ODE).
- The order of an ODE is determined by the highest derivative of the equation.

- Example:
- $a_1 \cdot \dot{y}(t) + a_2 \cdot y(t) + x(t) = 0$
- $\dot{y}(t) = dy(t)/dt$, the derivative of $y(t)$ with respect to time
- a_1 and a_2 are constants, $x(t)$ is the forcing function.

First-Order Linear ODE with Constant Coefficients

- Equation:

$$a_1 \cdot \dot{y}(t) + a_2 \cdot y(t) + x(t) = 0$$

- If x(t) = a_3 (a constant), the equation is called **autonomous**.

- If x(t) = 0, the equation is called **homogeneous**.

Second-Order Linear ODE with Constant Coefficients

- Equation:

$$a_1 \cdot \ddot{y}(t) + a_2 \cdot \dot{y}(t) + a_3 \cdot y(t) + x(t) = 0$$

- This is a second-order linear ODE.

where $\quad \ddot{y}(t) \equiv d^2 y(t)/dt^2$

First-Order ODE with Variable Coefficients

Equation:

$$a_1 \cdot \dot{y}(t) + a_2(t) \cdot y(t) + x(t) = 0$$

- $a_2(t)$ is a time-dependent function, making this an ODE with variable coefficients.

Example of a Nonlinear First-Order ODE

Equation:

$$\log[\dot{y}(t)] + 1/y(t) = 0$$

- Nonlinear ODEs are more complex and don't follow the superposition principle.

Solving Differential Equations

- The objective is to determine the behavior of y(t) over time.

Methods:

1. Graphical Method (for autonomous equations)
2. Analytical Method (exact solutions for linear ODEs)
3. Numerical Methods (for complex equations)

A.1.2 First-Order Ordinary Differential Equations

Overview and Key Concepts

Introduction to First-Order ODEs

- First-order ordinary differential equations (ODEs) involve derivatives of the unknown function y(t) with respect to time t. The general form of a first-order ODE is:

$$\dot{y}(t) = f[y(t)]$$

- This is a first-order ODE because it involves the first derivative $\dot{y}(t)$

Graphical Solutions of First-Order ODEs

- Graphical solutions provide a qualitative understanding of first-order ODEs. We use the slope field to visualize the behavior of y(t) over time.

Consider an autonomous ODE:

$$\dot{y}(t) = f[y(t)] \tag{A.6}$$

- To solve this graphically, we plot f(y) against y and determine where $\dot{y}(t)$ is positive or negative by analyzing the slope of f(y).
- Positive slopes indicate increasing y, and negative slopes indicate decreasing y.

Graphical Solutions to First-Order ODE

- Autonomous Differential Equation:

$$\dot{y}(t) = f[y(t)]$$

Graphical Interpretation:

$f(\cdot) > 0 \rightarrow y(t)$ increases.

$f(\cdot) < 0 \rightarrow y(t)$ decreases.

- Steady State: Where $f[y(t)] = 0$.

Steady State

- Steady State occurs where f[y(t)] = 0.
- At this point, $\dot{y}(t) =$ dy(t)/dt = 0, so y(t) remains constant over time.
- Types of Steady States:

Stable: if y(t) returns to the steady state when disturbed.

Unstable: if y(t) moves away from the steady state when disturbed.

Linear ODE Example

- Linear ODE Example:

$$\dot{y}(t) = f[y(t)] = a \cdot y(t) - x$$

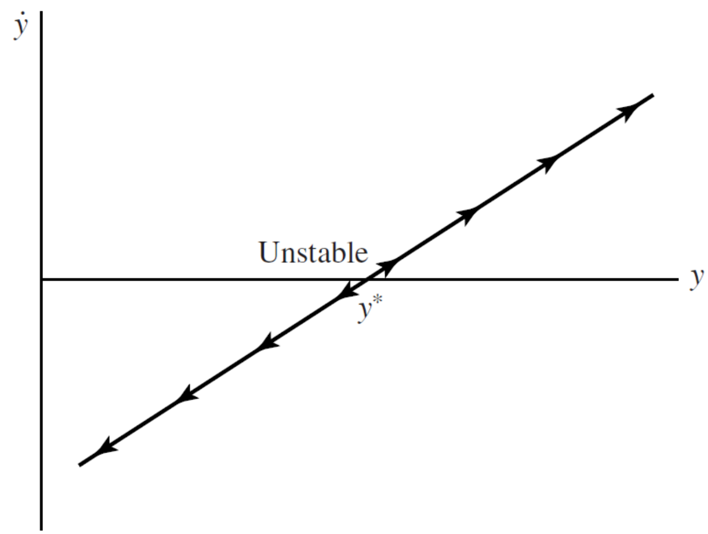
- a and x are constants.
- Graphical Representation: f(·) is a straight line that crosses the horizontal axis at y* = x/a.

Graphical Solutions of First-Order ODEs

Case 1: Unstable System

- If a > 0, the system is unstable.
- f(·) > 0 for y > x/a, and f(·) < 0 for y < x/a.

System moves away from steady state.



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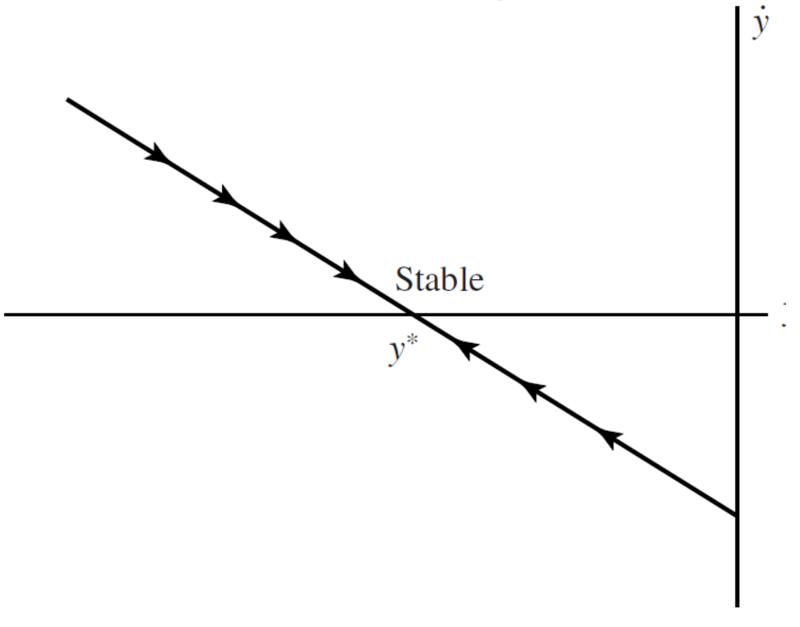
Graphical Solutions of First-Order ODEs

Case 2: Stable System

- If $a < 0$, the system is stable.
- $f(\cdot) < 0$ for $y > x/a$, and $f(\cdot) > 0$ for $y < x/a$.

System moves toward steady state.



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Nonlinear ODE Example

- Nonlinear ODE Example:

$$\dot{y}(t) = f[y(t)] = s \cdot [y(t)]^{\alpha} - \delta \cdot y(t)$$

- s, δ, and α are constants, and α < 1.
- Example from Solow-Swan growth model: y(t) represents capital stock.

Graphical Interpretation of Nonlinear ODE

f(·) is upward sloping for low values of y.

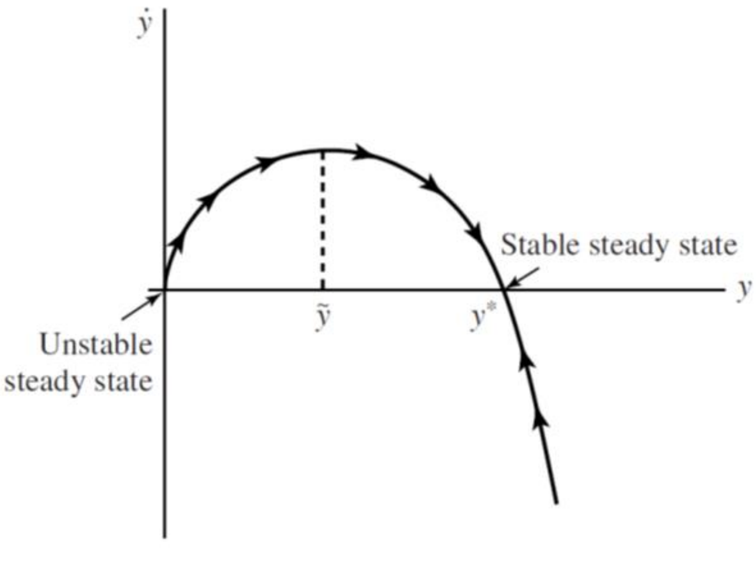
Reaches maximum and slopes downward for higher values of y.

- Two steady states:

  y = 0 (unstable).

$$y = y^* = (\delta/s)^{1/(\alpha-1)}$$

(stable).



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Stability Criterion

How to Determine Stability:

- Upward Slope: If $f(\cdot)$ slopes upward at steady state y^*, system is unstable.
- Downward Slope: If $f(\cdot)$ slopes downward at y^*, system is stable.

Mathematical Condition for Stability:

If $\partial \dot{y}/\partial y|_{y^*} > 0$, y is locally unstable

If $\partial \dot{y}/\partial y|_{y^*} < 0$, y is locally stable

Analytical Solutions: Linear ODEs

- Objective: Solve Ordinary Differential Equations (ODEs) analytically.
- Basic Form:

$$\dot{y}(t) + a \cdot y(t) + x(t) = 0$$

where a is a constant, and x(t) is a known function.

- Goal: Find y(t) that satisfies the equation.

Steps to Solve Linear ODEs (Step 1)

1. Rearrange the Equation:

$$\dot{y}(t) + a \cdot y(t) = -x(t)$$

- Move all terms involving y(t) and its derivatives to one side, and the rest to the other side.

Steps to Solve Linear ODEs (Step 2)

2. Multiply by the Integrating Factor:

$$e^{at}$$

- Multiply both sides of the equation by e^{at} and integrate:

$$\int e^{at} \cdot [\dot{y}(t) + a \cdot y(t)] \cdot dt = -\int e^{at} \cdot x(t) \cdot dt$$

Steps to Solve Linear ODEs (Step 3)

3. Simplify the Left-Hand Side:

The reason for multiplying by the integrating factor e^{at}

$$(d/dt)[e^{at} \cdot y(t) + b_0] \quad = \; e^{at} \cdot [\dot{y}(t) + a \cdot y(t)]$$

Steps to Solve Linear ODEs (Step 4)

4. Integrate Both Sides:

$$\int \frac{d}{dt}[e^{at} \cdot y(t) + b_0]dt = -\int e^{at} \cdot x(t)dt$$

- The left side integrates to $e^{at} \cdot y(t) + b_0$, and the right side becomes a function of t plus a constant, $\mathrm{INT(t)} + b_1$.

Steps to Solve Linear ODEs (Step 5)

5. Solve for y(t):

$$y(t) = -e^{-at} \int e^{at} \cdot x(t)dt - e^{-at}\, b_0$$

$$y(t) = -e^{-at} \cdot \mathrm{INT}(t) + be^{-at}$$

where $b = b_1 - b_0$

=> The general solution to the ODE

$$\dot{y}(t) + a \cdot y(t) + x(t) = 0$$

Example: Solving a Simple Linear ODE

- Consider the equation:

$$\dot{y}(t) - y(t) - 1 = 0$$

This is a simple first-order ODE where x(t) = -1 and a = -1.

Solution to Example ODE

1. Multiply by e^{-t}:

$$e^{-t}\,[\dot{y}(t) - y(t)] = e^{-t}$$

2.

$$\int e^{-t}[\dot{y}(t) - y(t)] \cdot dt = \int e^{-t}\,dt$$

3. Simplify:

   d/dt $[e^{-t} \cdot y(t)] = e^{-t}\,[\dot{y}(t) - y(t)]$

4. Integrate: $e^{-t} \cdot y(t) + b_0 = -e^{-t} + b_1$

5. Solve for $y(t)$: $y(t) = -1 + b \cdot e^{t}$

   where $b = b_1 - b_0$ is an arbitrary constant

Solution to Example ODE

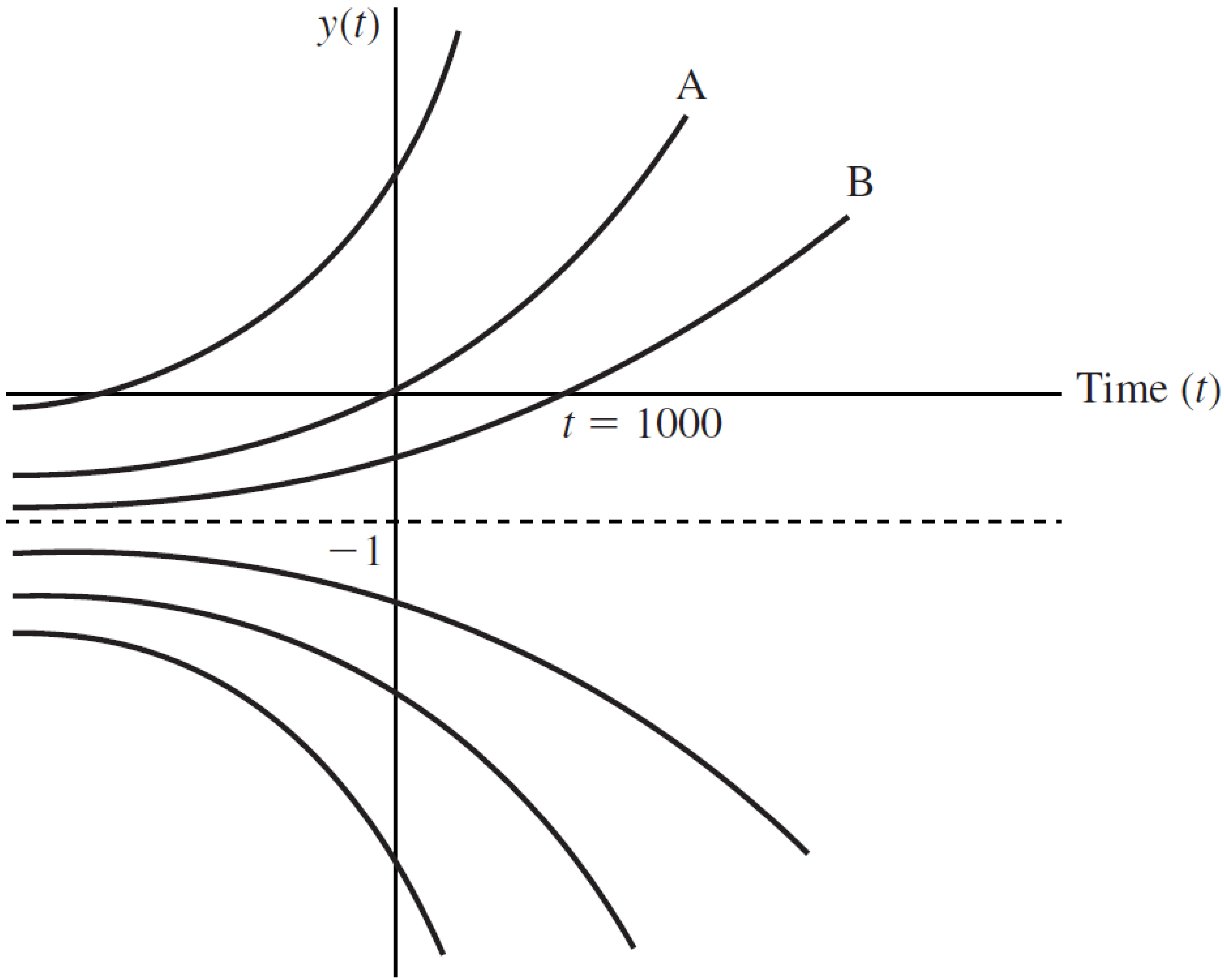
- **Solution:**

$$y(t) = -1 + b \cdot e^t$$

b is an arbitrary constant of integration

- **Verify the Solution:**
  - Take the derivative of y(t) :
  - $\dot{y}(t) = b \cdot e^t = y(t) + 1$

    $\dot{y}(t) - y(t) - 1 = 0$


y(t)
A
B
Time (t)
t = 1000
−1

Boundary Conditions

- To get a *particular* or *exact solution,* we have to specify the arbitrary constant of integration, b.
- To pin down which of the infinitely many possible paths applies, we need to know a value of $y(t)$ for at least one point in time.
- This ***boundary condition*** will determine the unique solution to the differential equation.

Boundary Conditions

Case 1: suppose we know the initial condition

- To find a particular solution, use the boundary condition $y(t)$ =0, as t=0. (initial condition)
- Substituting into the solution:

$$y(0) = -1 + b \cdot e^0 = 0$$

Solving for b : b =1

- The particular solution is:

$$y(t) = -1 + e^t$$

Boundary Conditions

Case 2: suppose we know the terminal condition

- suppose that the terminal date is t_1 =1000, and the value of y(t) at that time is 0, use the boundary condition $y(1000)$ =0. (terminal condition)
- Substituting into the solution:

$$y(1000) = -1 + b \cdot e^{1000} = 0$$

Solving for b : $b = e^{-1000}$

- The particular solution is:

$$y(t) = -1 + e^{-1000} \cdot e^{t}$$

ODE with variable coefficients

- Consider the differential equation

$$\dot{y}(t) + a(t) \cdot y(t) + x(t) = 0$$

where $a(t)$ is a known function of time but not a constant

We can follow the same steps as before.

The difference is that the integrating factor is

$$e^{\int_0^t a(\tau) d\tau}$$

Solution with Variable Coefficients

- The General Solution for the ODE:

$$y(t) = -e^{-\int_0^t a(\tau)d\tau} \cdot \int e^{\int_0^t a(\tau)d\tau} \cdot x(t) \cdot dt + b \cdot e^{-\int_0^t a(\tau)d\tau}$$

where b is an arbitrary constant of integration

- To find the particular or exact solution, we have to make use of a boundary condition.

A.1.3 Systems of Linear Ordinary Differential Equations

Introduction to Systems of Linear ODEs

- Form of the system:

$$\dot{y}_1(t) = a_{11} y_1(t) + \cdots + a_{1n} y_n(t) + x_1(t)$$

$$\cdots$$

$$\dot{y}_n(t) = a_{n1} y_1(t) + \cdots + a_{nn} y_n(t) + x_n(t)$$

- This is a system of first-order linear ODEs.

Matrix Notation for the System

- Matrix form:

$$\dot{y}(t) = A \cdot y(t) + x(t)$$

- $y(t)$ is a column vector of n functions of time.

$$\begin{bmatrix} y_1(t) \\ \cdots \\ y_n(t) \end{bmatrix}$$

- $\dot{y}(t)$ is the column vector of the n corresponding derivatives
- A is an n $\times$n square matrix of constant coefficients
- $x(t)$ is a vector of n functions

Three procedures for solving this system

- Phase Diagrams:

  Simple and provide a qualitative solution.

  Works for both linear and nonlinear systems.

  Drawbacks: Only applicable to 2x2 systems.

  Limited to autonomous equations with steady states.
- Analytical Solutions: Provides quantitative answers.

  Applicable to larger systems.

  Works only for linear equations.
- Numerical Solutions: For solving systems that cannot be handled analytically.

Introduction to Phase Diagrams

- Phase diagrams provide a graphical method for solving systems of differential equations.
- Advantages: Can be used for both linear and nonlinear systems.
- Drawbacks: Limited to 2x2 systems and autonomous equations with steady states.

Diagonal Systems

- System Form:

$$\dot{y}_1(t) = a_{11} \cdot y_1(t)$$

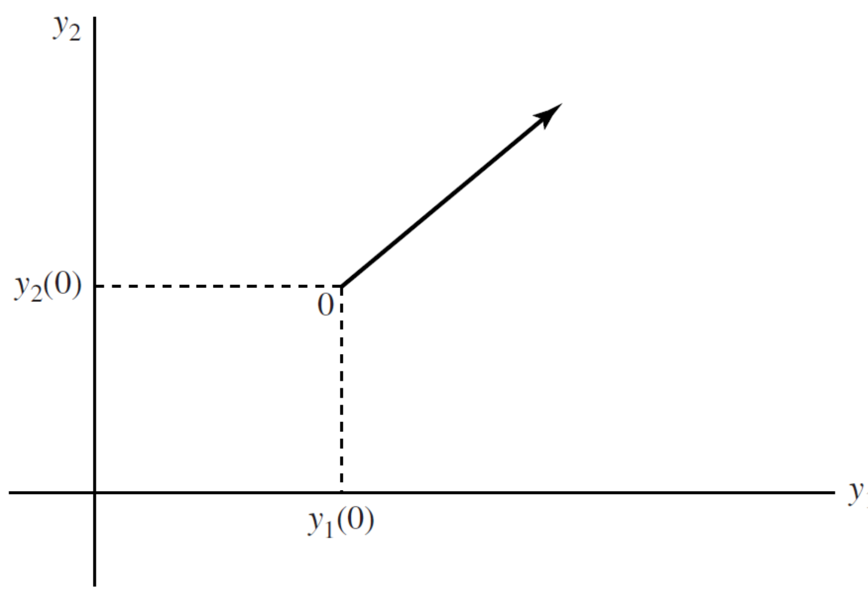
$$\dot{y}_2(t) = a_{22} \cdot y_2(t)$$

- a_{11} and a_{22} are real numbers
- This is a **2x2 diagonal matrix system** of first-order linear differential equations.

Diagonal Systems

Each point in the space represents the position of the system (y1, y2) at a given moment in time.

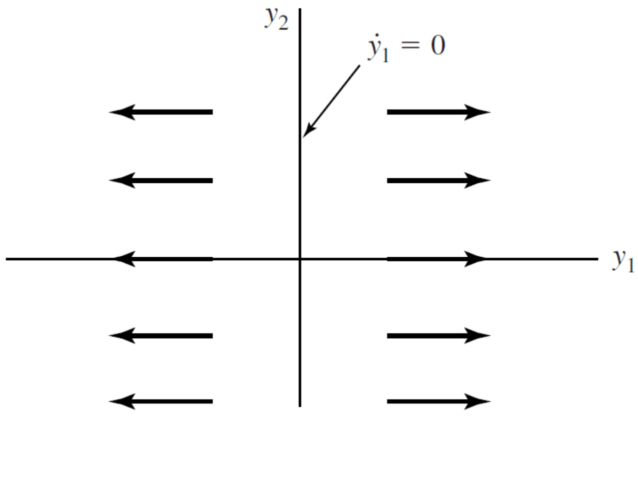
The object of a phase diagram is to translate the dynamics implied by the two differential equations into a system of arrows that describe the qualitative behavior of the economy over time.



Case 1: Unstable System ($a_{11} > 0$ and $a_{22} > 0$)

•**Steps to Construct the Phase Diagram**:

1. Plot the $\dot{y_1} = 0$ schedule (vertical axis).
2. Analyze the dynamics of y_1 to the right and left of this axis.
3. Plot the $\dot{y_2} = 0$ schedule (horizontal axis).
4. Analyze the dynamics of y_2 above and below this axis.
5. Combine the arrows from the two schedules.



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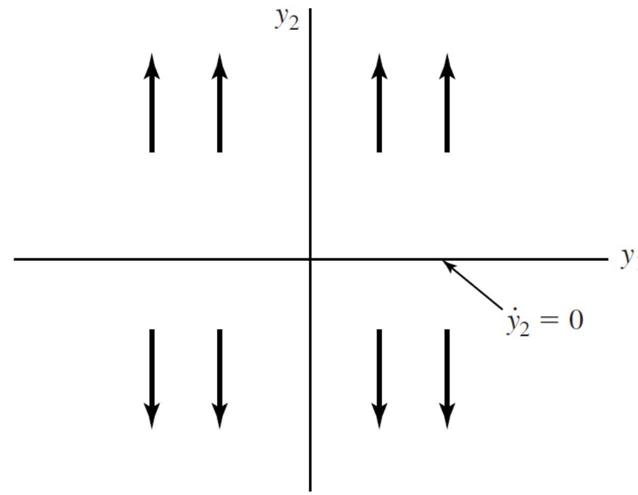

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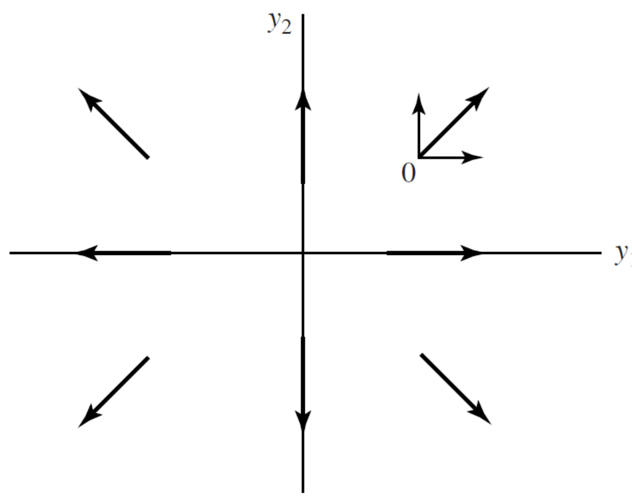

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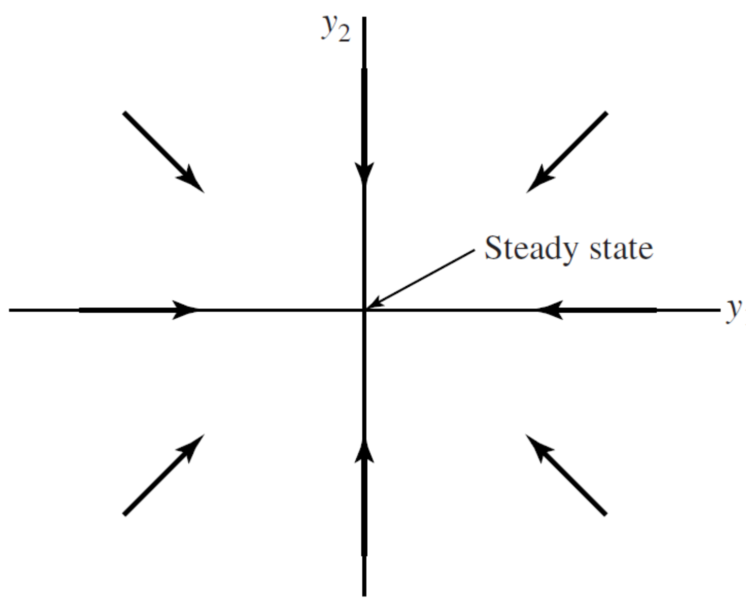
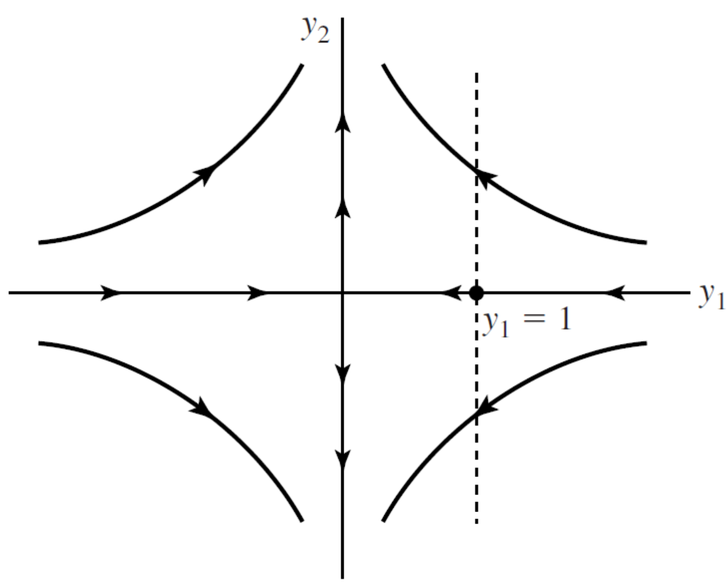
Case 2: Stable System ($a_{11} < 0$ and $a_{22} < 0$)

- Result: The system converges toward the steady state (0,0). The arrows in the phase diagram point inward.



Case 3: Saddle-Path Stability ($a_{11} < 0$ and $a_{22} > 0$)

- y_1 tends toward the steady state, while y_2 moves away.
- Result: The system is saddle-path stable.



The phase diagram shows a stable arm and an unstable arm.

Saddle-Path Stability

- Key Concept : The system is neither stable nor unstable.
- If the system starts at the steady state, it remains there.

Dynamics:

- If the system starts on the horizontal axis, it returns to the steady state.
- If the system starts away from the horizontal axis, it diverges away from the steady state.

Nondiagonal Example

- System Form:

$$\dot{y}_1(t) = 0.06 \cdot y_1(t) - y_2(t) + 1.4$$

$$\dot{y}_2(t) = -0.004 \cdot y_1(t) + 0.04$$

- Boundary conditions:

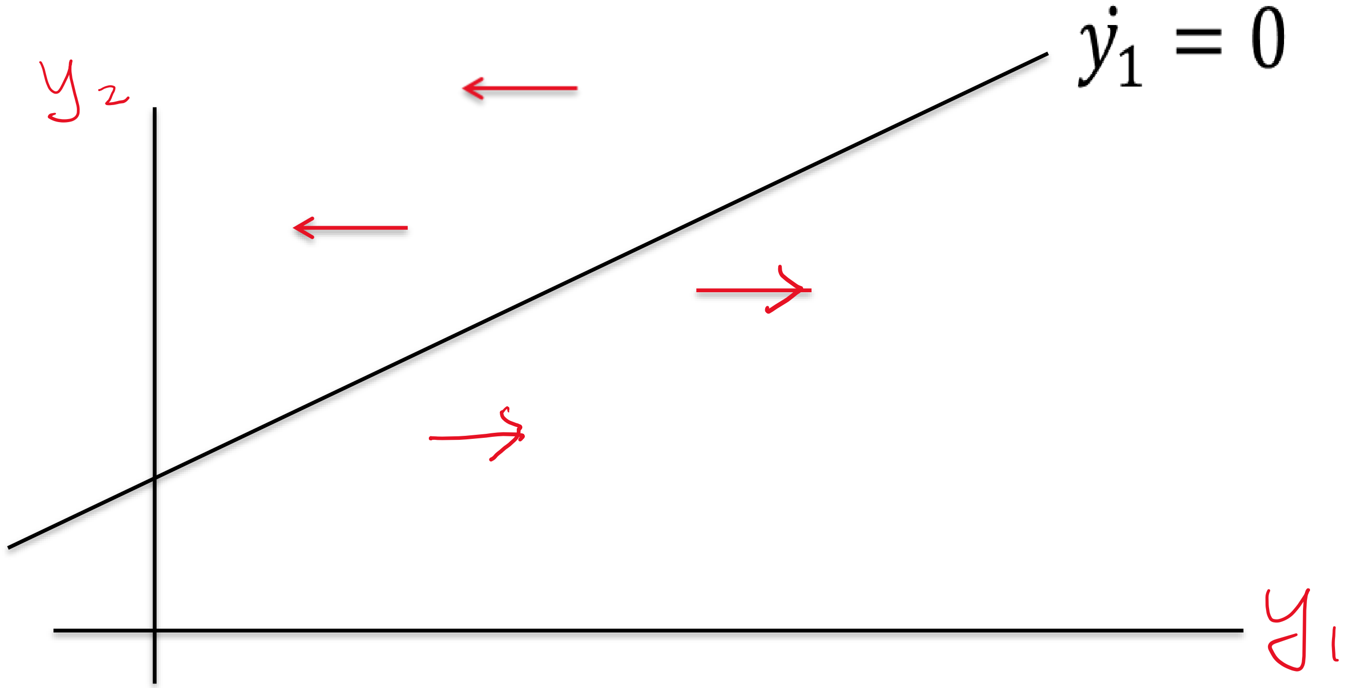
$$y_1(0) = 1 \text{ and } \lim_{t\to\infty}[e^{-0.06t} \cdot y_1(t)] = 0$$

- Phase Diagram: The system exhibits saddle-path stability.

Phase Diagram $\dot{y}_1 = 0$

- Upward-sloping line:

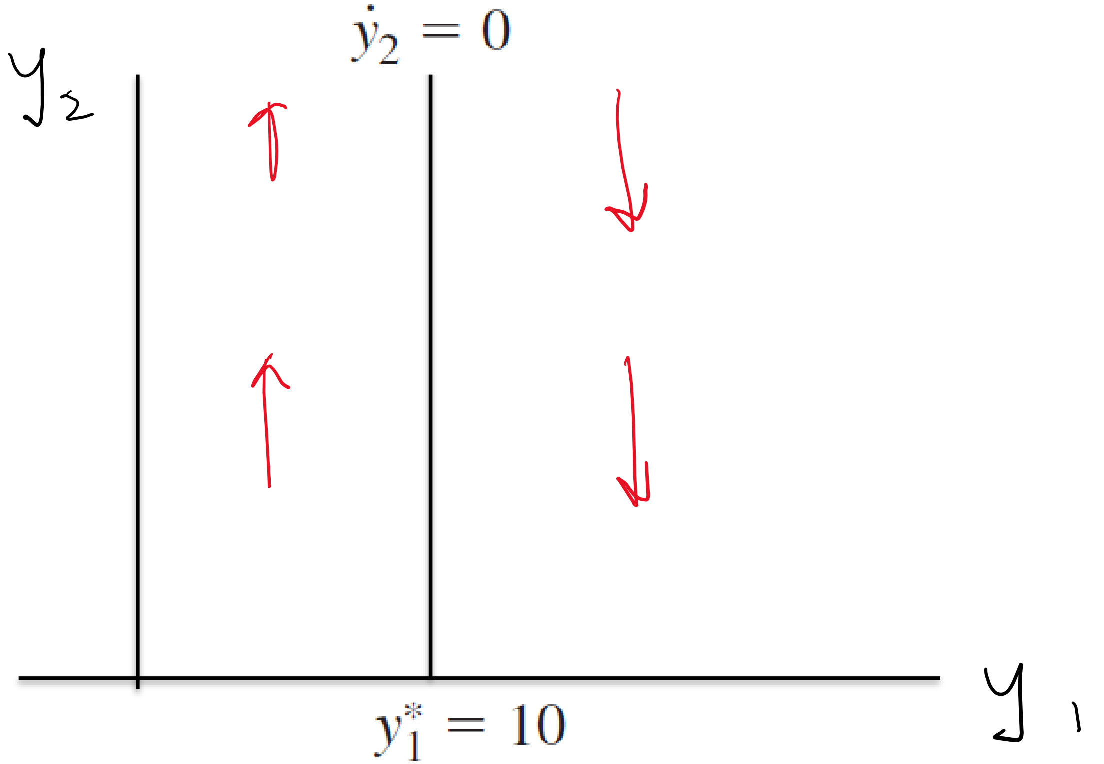
$$\dot{y}_1 = 0 \quad y_2 = 1.4 + 0.06 \cdot y_1$$



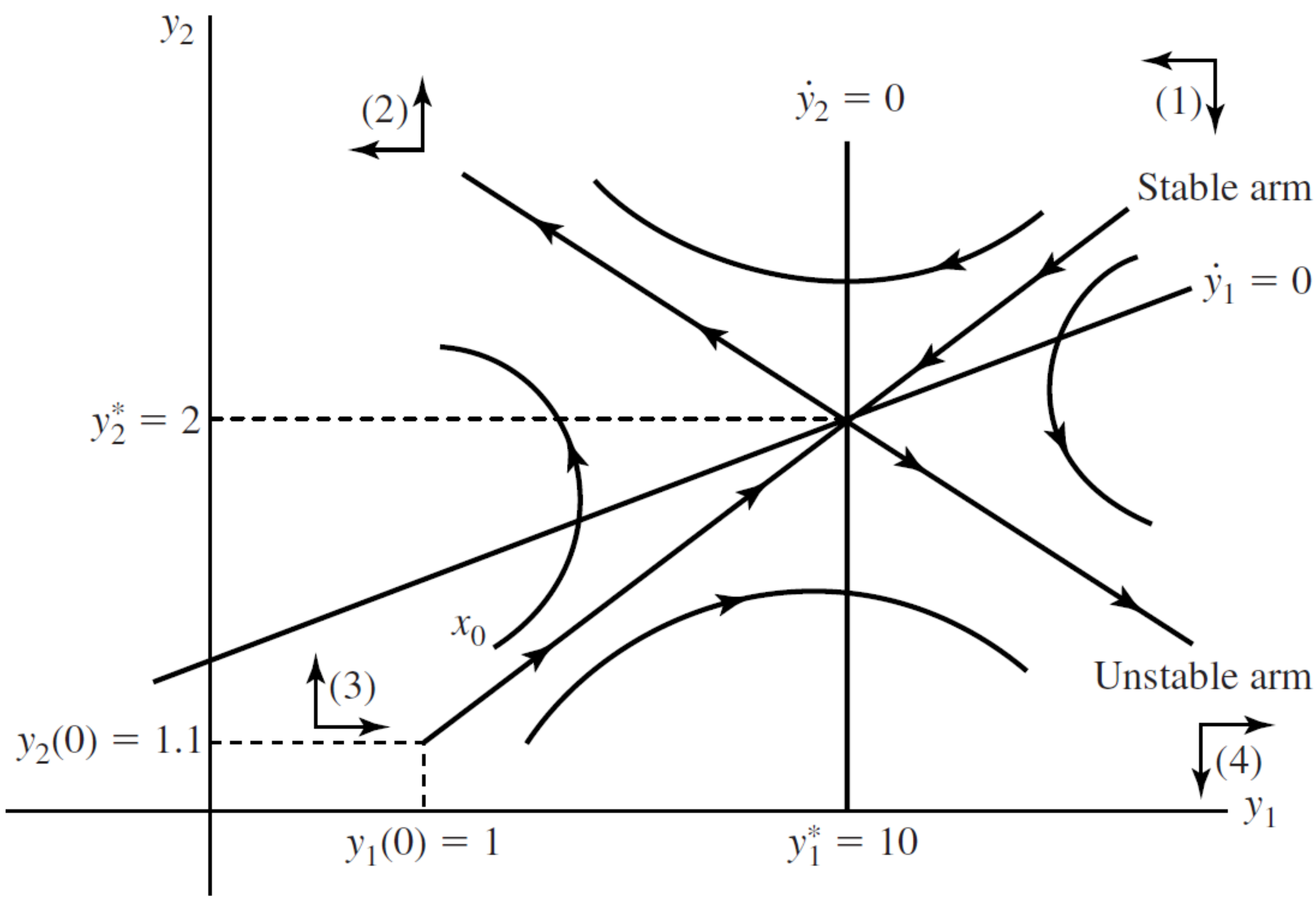
Phase Diagram $\dot{y_2} = 0$

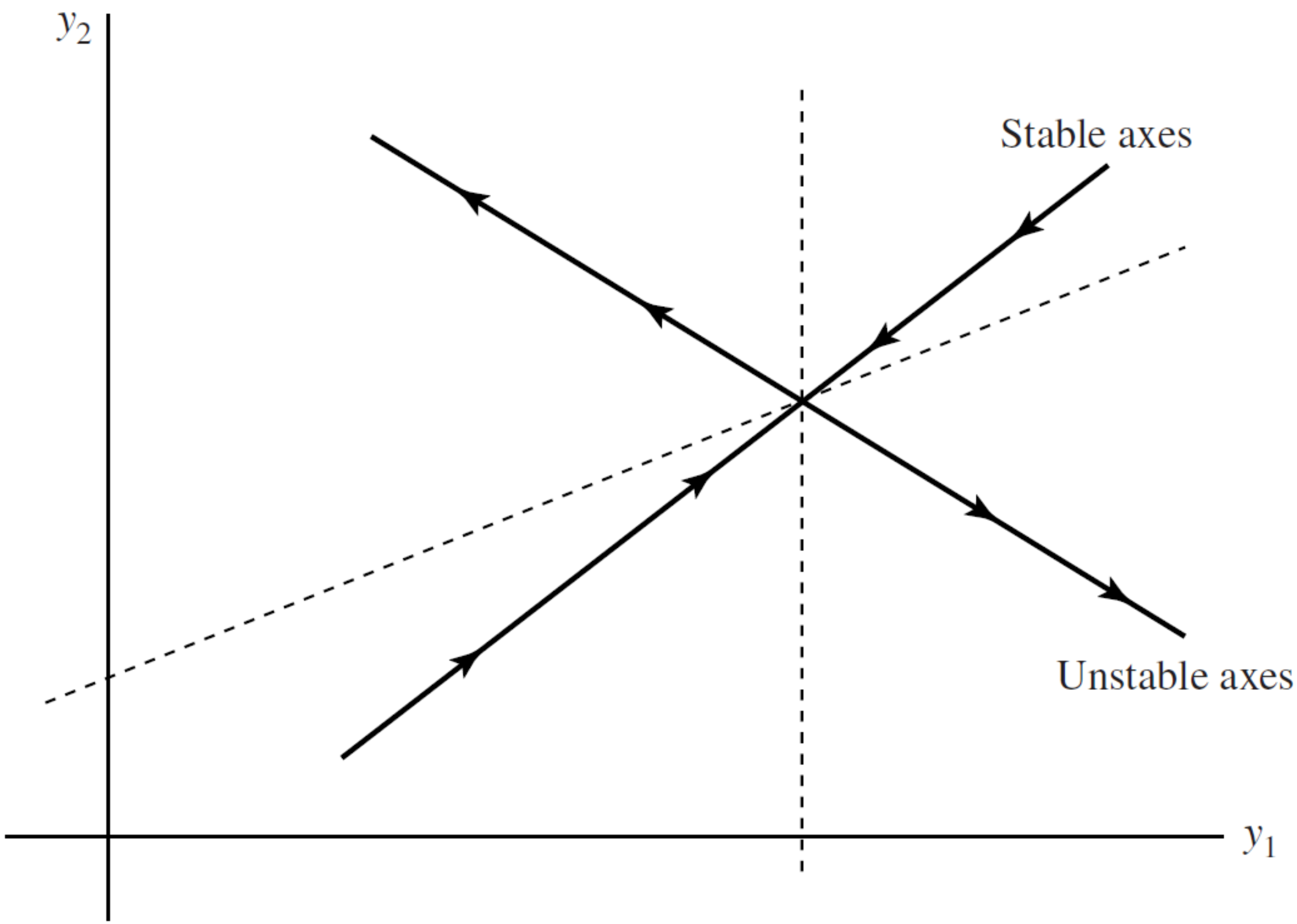
- Vertical line:

$\dot{y_2} = 0$ y_1=10



Combined Phase Diagram





y_2
Stable axes
Unstable axes
y_1

Saddle-Path Stability

- The system moves toward the steady state if it starts in regions 1 and 3.
- Saddle path: Located in these two regions.
- If the system starts on this path, it converges to the steady state.
- Starting slightly off the path leads to divergence.

A Nonlinear Example

- System Form:

$$\dot{k}(t) = k(t)^{0.3} - c(t)$$

$$\dot{c}(t) = c(t) \cdot [0.3 \cdot k(t)^{-0.7} - 0.06]$$

Boundary conditions:

$$k(0) = 1 \text{ and } \lim_{t\to\infty}[e^{-0.06t} \cdot k(t)] = 0$$

Phase Diagram: The system is saddle-path stable.

Phase Diagram

- Locus Equations:

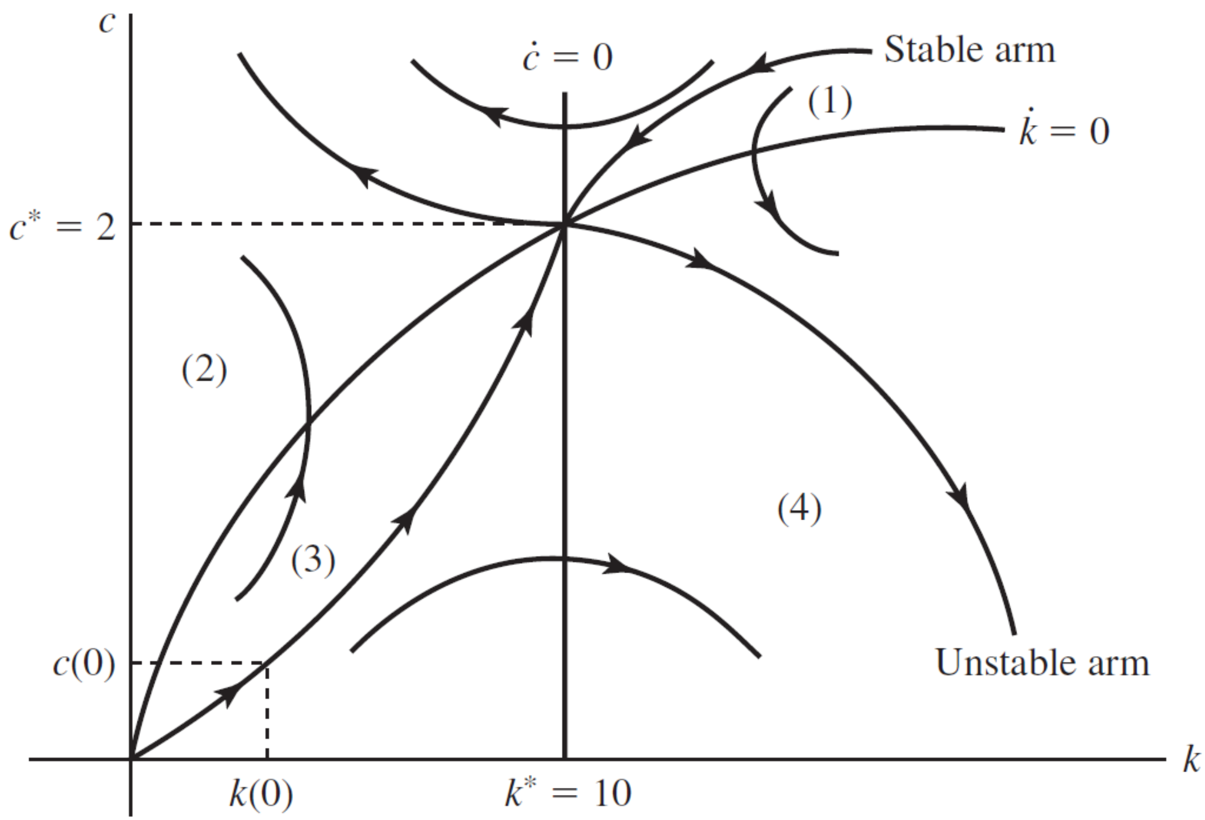
$$\dot{k} = 0 \qquad c = k^{0.3}$$

$$\dot{c} = 0 \qquad k = 10$$

Steady State:

$$c^* = 2, \qquad k^* = 10$$

The system is **saddle-path stable**.
The **stable arm** runs through regions 1 and 3,
while the **unstable arm** runs through regions 2 and 4.



Analytical Solutions of Linear, Homogeneous Systems

- System of Linear ODEs:

$$\dot{y}(t) = A \cdot y(t)$$

- y(t) is an $n \times 1$ column vector of functions of time.
- A is an $n \times n$ matrix of constant coefficients.

Matrix Diagonalization

- Key Step: Diagonalize the matrix A.
- Find a matrix V such that:

$$V^{-1}AV = D$$

- where D is a diagonal matrix with the eigenvalues of A on the diagonal, V is the matrix of eigenvectors.

Transformation to New Variables

- Define a new variable:

$$z(t) = V^{-1} \cdot y(t)$$

- The system becomes:

$$\dot{z}(t) = V^{-1} \cdot \dot{y}(t) = V^{-1} A \cdot y(t) = V^{-1} A V V^{-1} \cdot y(t) = D \cdot z(t)$$

- This results in n independent differential equations.

Solution for the Diagonal System

The transformed system consists of n independent equations:

$$\dot{z}_1(t) = \alpha_1 \cdot z_1(t)$$

$$\dot{z}_2(t) = \alpha_2 \cdot z_2(t)$$

$$\cdots$$

$$\dot{z}_n(t) = \alpha_n \cdot z_n(t)$$

Solution for the Diagonal System

Each independent equation is solved as:

$$\dot{z}_i(t) = \alpha_i \cdot z_i(t)$$

- The Solution for each equation is :

$$z_i(t) = b_i \cdot e^{\alpha_i t}$$

where b_i is an arbitrary constant of integration that is determined by the boundary conditions

<u>Matrix notation</u>

$$z(t) = Eb$$

Final Solution

- Transform back to the original variables:

$$y(t) = V \cdot z(t)$$

- The general solution is:

$$y(t) = V \cdot Eb$$

in nonmatrix notation

$$y_i(t) = v_{i1}e^{\alpha_1 t} \cdot b_1 + v_{i2}e^{\alpha_2 t} \cdot b_2 + \cdots + v_{in}e^{\alpha_n t} \cdot b_n \tag{A.29}$$

To solve a system of equations

1. Find the eigenvalues of the matrix A and call them $\alpha_1, \ldots, \alpha_n$.

2. Find the corresponding eigenvectors and arrange them as columns in a matrix V.

3. The solution takes the form of equation (A.29).

4. Use the boundary conditions to determine the arbitrary constants of integration (b_i).

Stability of the System

The stability of the system depends on the eigenvalues α_i :

- If all $\alpha_i < 0$: The system is stable.
- If all $\alpha_i > 0$: The system is unstable.
- Mixed signs: The system exhibits saddle-path stability.

Stability and Eigenvalues

Stability depends on the signs of the eigenvalues:

- Real positive eigenvalues: Unstable.
- Real negative eigenvalues: Stable.
- Mixed signs: Saddle-path stable.

The Relation Between the Graphical and Analytical Solutions

- The graphical solution is related to the analytical solution via the diagonalization of the matrix.
- Diagonalization finds a new set of axes, the eigenvectors.
- The eigenvalues are the elements of the diagonal matrix.

Eigenvectors and Stability

Stable and Unstable Arms:

- The graphical solution consists of stable and unstable arms, corresponding to the eigenvectors.
- Matrix A can be represented as a diagonal matrix of eigenvalues.

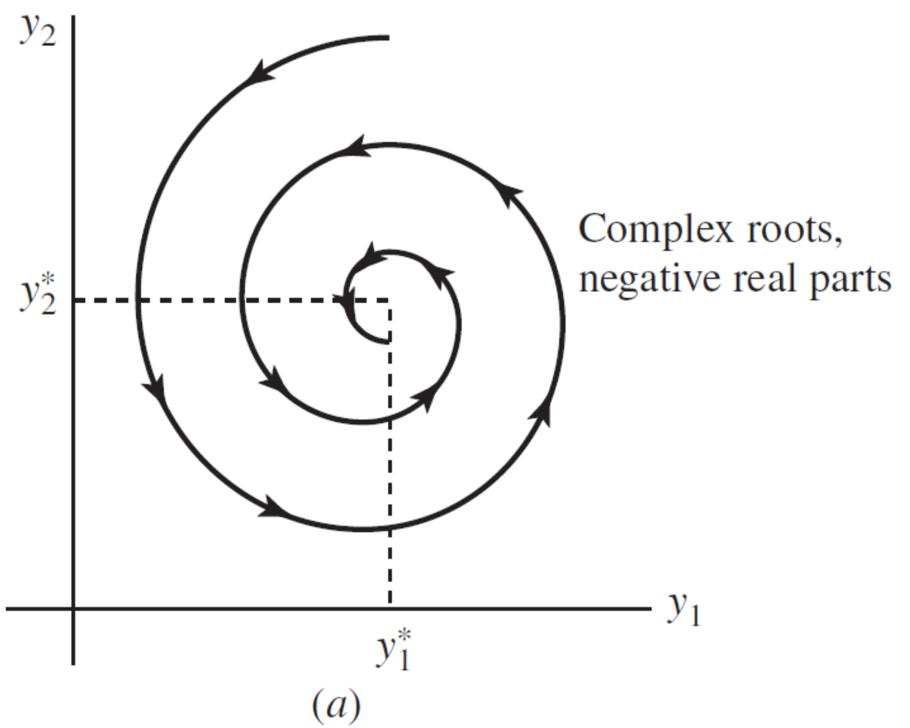
Stability Properties

- Case 1: Both eigenvalues positive → Unstable.
- Case 2: Both eigenvalues negative → Stable.
- Case 3: One positive, one negative eigenvalue → Saddle-path stable.

• Stable arm: Corresponds to the eigenvector with negative eigenvalue.

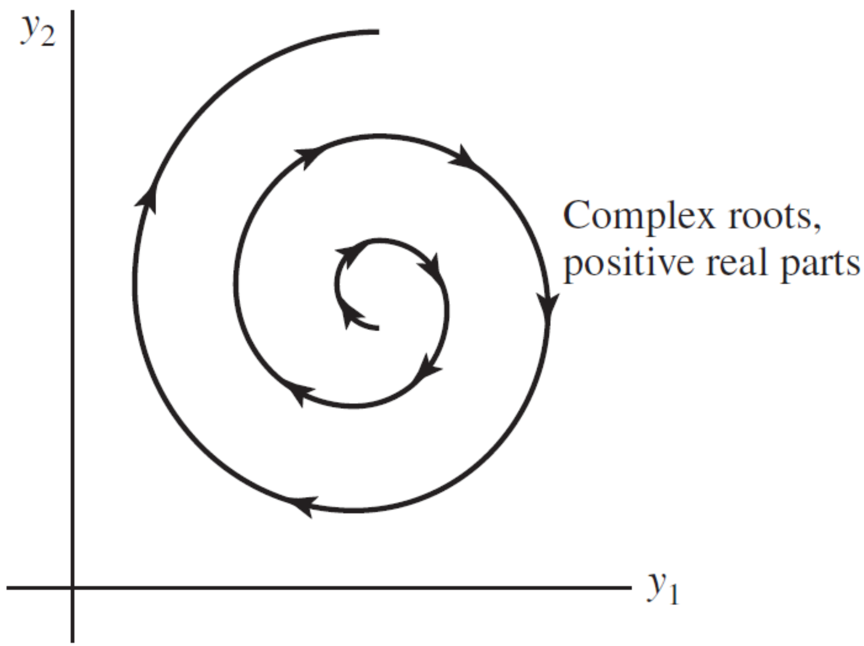
• Unstable arm: Corresponds to the eigenvector with positive eigenvalue.

Complex Eigenvalues and Oscillation

- Complex eigenvalues with negative real parts: Converging oscillations.
- 

Complex Eigenvalues and Oscillation

- Complex eigenvalues with positive real parts: Diverging oscillations.



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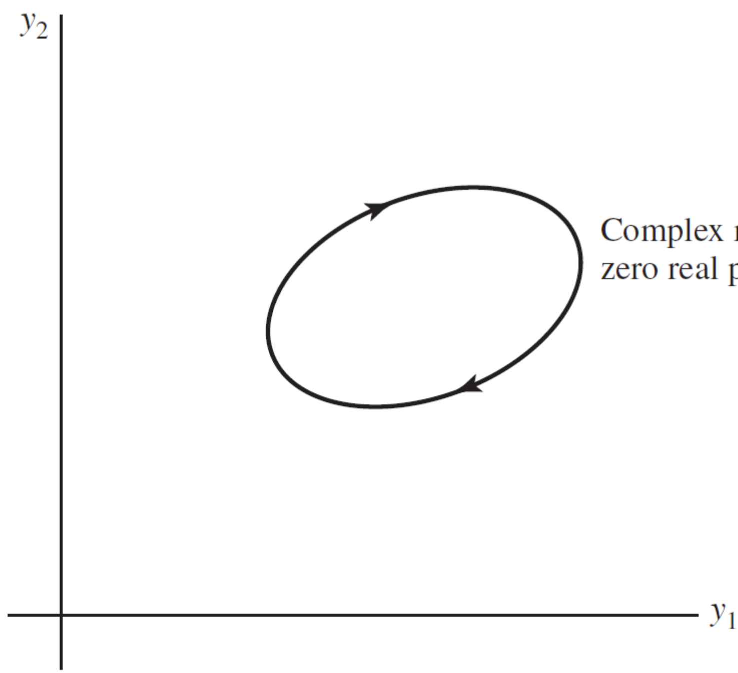
Elliptical Trajectories

- Complex eigenvalues with zero real parts: Elliptical trajectories around the steady state.



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Equal Eigenvalues

- When eigenvalues are equal, the solution takes the form:

$$y_i(t) = (b_{i1} + b_{i2} \cdot t) \cdot e^{\alpha t}$$

- The system can be stable or unstable depending on the sign of α. The solution is stable if $\alpha<0$ and unstable if $\alpha>0$

Analytical Solutions of Linear, Nonhomogeneous Systems

- System of Nonhomogeneous ODEs:

$$\dot{y}(t) = A \cdot y(t) + x(t)$$

- $y(t)$ is an $n \times 1$ vector of functions of time.
- A is an $n \times n$ matrix of constants.
- $x(t)$ is an $n \times 1$ vector of known functions of time.

Transforming the System

- Begin with matrix V composed of the eigenvectors of A such that:

$$V^{-1} A V = D$$

- Diagonalize the matrix A.
- Define a new variable:

$$z(t) = V^{-1} \cdot y(t)$$

System in New Variables

- The system in new variables becomes:

$$\dot{z} = V^{-1}\dot{y} = V^{-1} \cdot (Ay + x) = V^{-1}AVV^{-1}y + V^{-1}x = Dz + V^{-1}x$$

$$\dot{z}(t) = D \cdot z(t) + V^{-1} \cdot x(t)$$

- This results in independent differential equations in each z_i(t).

Solution for Each Differential Equation

$$\dot{z}_i(t) = \alpha_i \cdot z_i(t) + V_i^{-1} \cdot x(t)$$

where V_i^{-1} is the ith row of V^{-1}

- Solution for each z_i(t):

$$z_i(t) = e^{\alpha_i t} \cdot \int e^{-\alpha_i \tau} \cdot V_i^{-1} \cdot x(\tau) \cdot d\tau + e^{\alpha_i t} \cdot b_i$$

- b_i is an arbitrary constant of integration

Final Solution

- Solution in matrix form:

$$z = E\hat{X} + Eb$$

- E is a diagonal matrix with terms $e^{\alpha_i t}$
- $\hat{X}$ is a column vector containing integrals of the form

$$\int e^{-\alpha_i \tau} \cdot V_i^{-1} \cdot x(\tau) \cdot d\tau$$

Example

- Consider the system of ODEs

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} 0.06 & -1 \\ -0.004 & 0 \end{bmatrix} \bullet \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 1.4 \\ 0.04 \end{bmatrix}$$

with the boundary conditions $y_1(0) = 1$ and

$$\lim_{t \to \infty} [e^{-0.06 \cdot t} \cdot y_1(t)] = 0$$

- The solution is computed step by step using the eigenvalue approach.

Example

- In this example, x is a vector of constants.
- The diagonal matrix of eigenvalues, D, and the matrix of eigenvectors, V, are given by

$$D = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.4 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 1 \\ -0.04 & 0.1 \end{bmatrix}$$

where

$$V^{-1} = \begin{bmatrix} 0.1/0.14 & -1/0.14 \\ 0.04/0.14 & 1/0.14 \end{bmatrix}$$

Example -Transforming the System

- Define new variables

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V^{-1} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The transformed system becomes:

$$\dot{z}_1 = 0.1 \cdot z_1 + 10/14$$

$$\dot{z}_2 = -0.04 \cdot z_2 + 9.6/14$$

Example - Solution to the Transformed System

- The solutions for $z_1(t)$ and $z_2(t)$ are

$$z_1(t) = -100/14 + b_1 e^{0.1 \cdot t}$$

$$z_2(t) = 240/14 + b_2 e^{-0.04 \cdot t}$$

- By premultiplying z by V, we get the solutions for $y_1(t)$ and $y_2(t)$

$$y_1(t) = 10 + b_1 e^{0.1 \cdot t} + b_2 e^{-0.04 \cdot t}$$

$$y_2(t) = 2 - 0.04 \cdot b_1 e^{0.1 \cdot t} + 0.1 \cdot b_2 e^{-0.04 \cdot t}$$

Example -Determining Constants

- Using the initial condition $y_1(0) = 1$, we find that:

$$b_1 + b_2 = \text{-}9$$

multiply both sides of equation

$$y_1(t) = 10 + b_1 e^{0.1\cdot t} + b_2 e^{-0.04\cdot t} \text{ by } e^{-0.06\cdot t}$$

take limits as t goes to infinity, and use the terminal condition, $\lim_{t\to\infty}[e^{-0.06t} \cdot y_1(t)] = 0$

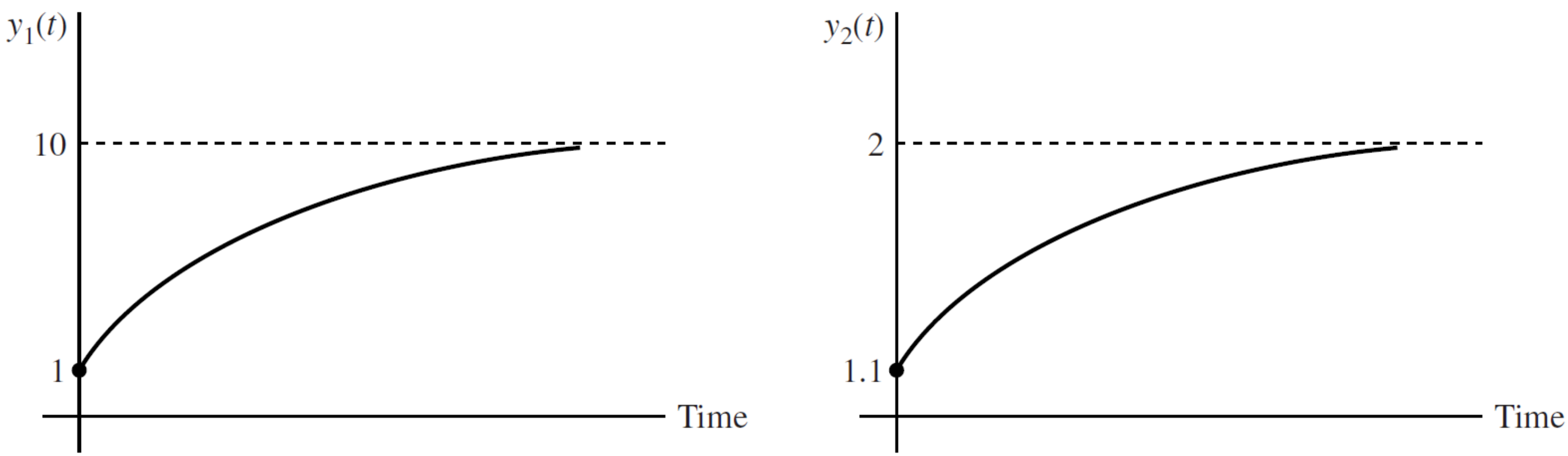
$$\lim_{t\to\infty}[e^{-0.06\cdot t} \cdot y_1(t)] = \lim_{t\to\infty}[10 \cdot e^{-0.06\cdot t} + b_1 e^{0.04\cdot t} + b_2 e^{-0.1\cdot t}] = 0$$

Example -Determining Constants

- Solving for $b_1 = 0$ and $b_2 = -9$, we get the final solution:

$$y_1(t) = 10 - 9 \cdot e^{-0.04 \cdot t}$$

$$y_2(t) = 2 - 0.9 \cdot e^{-0.04 \cdot t}$$



(a) (b)

Linearization of Nonlinear Systems

- Many nonlinear systems can be linearized near their steady states.
- Technique: Use a Taylor-series expansion to approximate the system near the steady state.
- This allows for the use of linear tools to analyze nonlinear systems.

System of Nonlinear ODEs

- Consider a system of nonlinear ODEs:

$$\dot{y}_1(t) = f^1[y_1(t), \ldots, y_n(t)]$$

$$\dot{y}_2(t) = f^2[y_1(t), \ldots, y_n(t)]$$

$$\ldots$$

$$\dot{y}_n(t) = f^n[y_1(t), \ldots, y_n(t)]$$

- Each function fif i is nonlinear.

First-Order Taylor Expansion

- Use the first-order Taylor expansion to linearize the system around its steady state:

$$\dot{y}_1(t) = f^1(\bullet) + (f^1)_{y_1}(\bullet) \cdot (y_1 - y_1^*) + \cdots + (f^1)_{y_n}(\bullet) \cdot (y_n - y_n^*) + R_1$$

$$\cdots$$

$$\dot{y}_n(t) = f^n(\bullet) + (f^n)_{y_1}(\bullet) \cdot (y_1 - y_1^*) + \cdots + (f^n)_{y_n}(\bullet) \cdot (y_n - y_n^*) + R_n$$

- Here, $f^1(\bullet), \ldots, f^n(\bullet)$ are evaluated at the steady state, and $(f^1)_{y_i}(\bullet), \ldots, (f^n)_{y_i}(\bullet)$ are partial derivatives at the steady state.

Linearized System in Matrix Form

- The linearized system can be written as:

$$\dot{y} = A \cdot (y - y^*)$$

where A is a $n \times n$ matrix of partial derivatives (evaluated at the steady state).

Example of Linearization

- Consider the following nonlinear system:

$$\dot{k} = k^{0.3} - c$$

$$\dot{c} = c \cdot (0.3 \cdot k^{-0.7} - 0.06)$$

with the boundary conditions:

$k(0) = 1$ and $\lim_{t\to\infty}[e^{-0.06t} \cdot k(t)] = 0$

The steady state is $k^* = 10$ and $c^* = 2$.

- Linearize around this point.

Example of Linearization

- After linearization, the system becomes:

$$\dot{k} = 0.3 \cdot (k^*)^{-0.7} \cdot (k - k^*) - (c - c^*) = 0.06 \cdot k - c + 1.4$$

$$\dot{c} = c^* \cdot [0.3 \cdot (-0.7) \cdot (k^*)^{-1.7}] \cdot (k - k^*) - 0 \cdot (c - c^*) = -0.008 \cdot k + 0.08$$

- This system is linear and can be analyzed using the tools developed for linear systems.