Introduction to Mathematical Methods

Final Exam Presentation, Group (V)

SU, Yi-Hsiang

WANG, Cheng-Ho

YANG, Shan-Tsun

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Problem Setup

Skippy produces two goods (x, y) with production set

$$x^2 + y^2 \leq 200,$$

and faces an environmental constraint

$$x + y \leq 20.$$

She consumes everything. Utility:

$$u(x, y) = x\,y^3.$$

Define the feasible set

$$\mathcal{F} = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \leq 200, \; x + y \leq 20, \; x \geq 0, \; y \geq 0\}.$$

(a) Show that the feasible set is convex

- $f_1(x, y) = x^2 + y^2$ is convex since $\nabla^2 f_1 = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_1}{\partial y \partial x} \\ \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_1}{\partial y^2} \end{bmatrix} = 2I \succeq 0$. Hence the set

  $\{(x, y) : f_1(x, y) \leq 200\}$ is convex.

  (Section 3.1.6, pp.75 of Boyd and Vandenberghe (2004).)

- $f_2(x, y) = x + y$ is affine $\Rightarrow \{(x, y) : f_2(x, y) \leq 20\}$ is convex.

- Nonnegativity constraints $\{x \geq 0\}, \{y \geq 0\}$ are convex.

- Intersection of convex sets is convex. Therefore $\mathcal{F}$ is convex.

  (Section 2.3.1 and Section 2.3.2, pp.36-38 of Boyd and Vandenberghe (2004).)

$\square$

(b) Show that the utility is not concave

Utility: $u(x, y) = xy^3$.

Compute the Hessian:

$$\nabla^2 u(x, y) = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y \partial x} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{bmatrix}.$$

Its determinant is

$$\det\left(\nabla^2 u\right) = -9y^4 < 0 \quad \text{for } y \neq 0.$$

Hence $\nabla^2 u$ is indefinite on any region with $y \neq 0$. Therefore u is neither concave nor convex on $\mathcal{F}$

(c) Log-transformation yields concavity

For $x > 0, y > 0$, define

$$v(x, y) = \ln u(x, y) = \ln x + 3 \ln y.$$

Then

$$\nabla^2 v(x, y) = \begin{bmatrix} \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y \partial x} \\ \frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 v}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -1/x^2 & 0 \\ 0 & -3/y^2 \end{bmatrix} \preceq 0,$$

so v is concave on $\mathbb{R}^2_{++}$.

(d) KKT conditions & sufficiency for global optimality

Because $\ln(\cdot)$ is strictly increasing, maximizing u over $\mathcal{F}$ is equivalent to

$$\max_{x,y} \; v(x,y) = \ln x + 3\ln y \quad \text{s.t.} \quad \begin{cases} x^2 + y^2 \leq 200, \\ x + y \leq 20, \\ x \geq 0, \;\; y \geq 0. \end{cases}$$

Let V(x,y) = -v(x,y), V is convex since we have shown that v is concave.

(d) KKT conditions & sufficiency for global optimality

Maximizing v over $\mathcal{F}$ is equivalent to

$$\min_{x,y} \mathbf{V}(x,y) = -\ln x - 3\ln y \quad \text{s.t.} \quad \begin{cases} x^2 + y^2 \leq 200, \\ x + y \leq 20, \\ x \geq 0, \; y \geq 0. \end{cases}$$

(Section 4.1.2, pp.129-130 of Boyd and Vandenberghe (2004).)

Lagrangian (multipliers $\lambda_i \geq 0$):

$$\mathcal{L} = -\ln x - 3\ln y + \lambda_1(x^2 + y^2 - 200) + \lambda_2(x + y - 20) + \lambda_3(0 - x) + \lambda_4(0 - y).$$

(d) KKT conditions & sufficiency for global optimality

KKT conditions. (Section 5.5.3, pp.244 of Boyd and Vandenberghe (2004).)

$$
\begin{cases}
\frac{\partial \mathcal{L}}{\partial x} = -\frac{1}{x} + 2\lambda_1 x + \lambda_2 - \lambda_3 = 0 \\
\frac{\partial \mathcal{L}}{\partial y} = -\frac{3}{y} + 2\lambda_1 y + \lambda_2 - \lambda_4 = 0 \\
x^2 + y^2 \leq 200, \; x + y \leq 20, \; x \geq 0, \; y \geq 0 \\
\lambda_i \geq 0; \quad i = 1, 2, 3, 4 \\
\lambda_1 (x^2 + y^2 - 200) = 0 \\
\lambda_2 (x + y - 20) = 0 \\
\lambda_3 (-x) = 0, \; \lambda_4 (-y) = 0
\end{cases}
$$

(d) KKT conditions & sufficiency for global optimality

Sufficiency:

The objective **V** is convex and the constraints are convex.

A strictly feasible point exists, Slater condition holds.

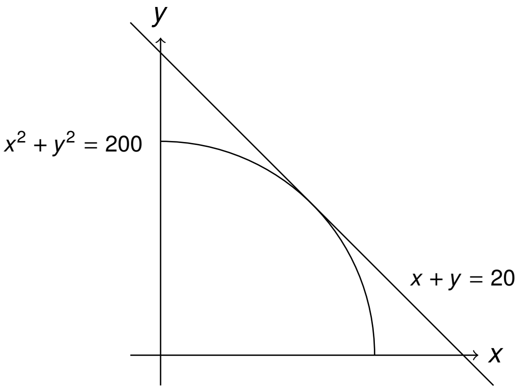
e.g. $(x, y) = (5, 5), \quad x + y = 10 < 20, \quad x^2 + y^2 = 50 < 200, \quad x > 0, \quad y > 0$

Hence KKT is necessary and sufficient for global optimality.

(Section 5.2.3, pp.226; and Section 5.5.3, pp.244 of Boyd and Vandenberghe (2004).)

(d) KKT conditions & sufficiency for global optimality

At the optimizer we expect $x, y > 0$ and, as the $\mathbf{V}$ is decreasing $((\ln x + 3 \ln y)$ increasing) in both x and y, the quadratic constraint binds while $x + y \leq 20$ doesn't.



(d) KKT conditions & sufficiency for global optimality

So $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $x^2 + y^2 = 200$. By KKT conditions we have

$$\frac{1}{x} = 2\lambda_1 x, \qquad \frac{3}{y} = 2\lambda_1 y \Rightarrow \frac{y^2}{x^2} = 3 \Rightarrow y = \sqrt{3}\,x.$$

Plugging into $x^2 + y^2 = 200$ yields $4x^2 = 200 \Rightarrow x^* = 5\sqrt{2}$ and $y^* = 5\sqrt{6}$.

The linear constraint is not binding:

$$x^* + y^* = 5(\sqrt{2} + \sqrt{6}) \approx 19.318 < 20.$$

(d) KKT conditions & sufficiency for global optimality

Thus the unique global optimizer is

$$(x^*, y^*) = (5\sqrt{2}, 5\sqrt{6}).$$

The active constraints: $x^2 + y^2 = 200$ is binding, the others are not.

(e) Global maximization value and axis comparisons

Global optimum.

$$(x^*, y^*) = (5\sqrt{2}, 5\sqrt{6}), \qquad u^* = x^*(y^*)^3 = 7500\sqrt{3} \approx 1.299 \times 10^4.$$

Binding: $x^2 + y^2 = 200$. Non-binding: $x + y \leq 20$, $x \geq 0$, $y \geq 0$.

Axis points.

- $(0, y)$: best feasible is $(0, 10\sqrt{2})$ with $u = 0$.
- $(x, 0)$: best feasible is $(10\sqrt{2}, 0)$ with $u = 0$.

Any axis point has zero utility and thus cannot be optimal relative to $u^* > 0$.

(f) Interpreting multipliers in a *minimization* problem

Consider the parametric problem in standard inequality form

$$\min_{x} \; f_0(x) \quad \text{s.t. } g_i(x) \leq t_i, \; i = 1, \ldots, m,$$

with value function $p(t) = \min\{f_0(x) : g(x) \leq t\}$.

Under KKT, the optimal multiplier $\lambda^* \geq 0$ yields the first-order sensitivity

$$\frac{\partial p(0)}{\partial t} = -\lambda^*$$

(f) Interpreting multipliers in a *minimization* problem

Interpretation. In minimization, λ_i^* is the marginal *increase* in the optimal cost when the ith constraint is *tightened* ($t_i \downarrow$), and the marginal *decrease* when it is relaxed ($t_i \uparrow$). Non-binding constraints typically have $\lambda_i^* = 0$.

(Section 5.6.2 and Section 5.6.3, pp.250-253 of Boyd and Vandenberghe (2004).)

(f) Multipliers in the *original* problem

Interpretation. In maximization, λ_i^* is the marginal *increase* in the optimal value when the ith constraint is *relaxed*. If $\lambda_i^* = 0$, relaxing that constraint does not improve the optimum.

Application to this problem. Only $x^2 + y^2 \leq 200$ is active at optimum and

$$\lambda_1^* = 0.01, \qquad \lambda_2^* = \lambda_3^* = \lambda_4^* = 0.$$

(f) Multipliers in the *original* problem

Because $\frac{d}{dt} \ln u^* = \lambda_1^* = \frac{1}{u^*} \frac{du^*}{dt}$, we also have

$$\frac{du^*}{dt} = \lambda_1^* u^* = 0.01 \times 7500\sqrt{3} \approx 1.30 \times 10^2,$$

so increasing the radius-squared budget by one unit raises the optimal utility by about 130.