Useful Results in Calculus

Introduction

- Topics covered:
  - Implicit Function Theorem
  - Taylor's Theorem
  - L'Hôpital's Rule
  - Integration by Parts
  - Fundamental Theorem of Calculus
  - Rules of Differentiation of Integrals

A.5.1 Implicit Function Theorem

Understanding the Relationship between Variables in Equations

What is the Implicit Function Theorem?

- The Implicit Function Theorem relates to a bivariate function $f(x_1, x_2)$ in the real space, assumed to be twice continuously differentiable.
- It describes how the equation $\phi(x_1, x_2) = 0$ implicitly defines x_2 as a function of x_1.

Slope of the Implicit Function

- For the equation $\phi(x_1, x_2) = f(x_1, x_2) - a = 0$, where a is a constant:
- The slope of the implicit function $x_2 = x_2(x_1)$ is:
- $dx_2/dx_1 = -[\partial f(x_1, x_2)/\partial x_1] / [\partial f(x_1, x_2)/\partial x_2]$

$$\frac{d\tilde{x}_2}{dx_1} = -\frac{\partial f(x_1, x_2)/\partial x_1}{\partial f(x_1, x_2)/\partial x_2}$$

Applying the Theorem to a Simple Function

- Consider $f(x_1, x_2) = 3x_1^2 - x_2$ with the equation

  $\phi(x_1, x_2) = 3x_1^2 - x_2 - 1 = 0$.
- Explicitly, x_2 can be written as $\tilde{x}_2(x_1) = 3x_1^2 - 1$
- Using the implicit function theorem:
- $dx_2/dx_1 = -[6x_1] / [-1] = 6x_1$ $\quad d\tilde{x}_2/dx_1 = -(6x_1)/(-1) = 6x_1$

- Conclusion: The slope is $6x_1$.

Applying the Theorem to a More Complex Function

- Consider $f(x_1, x_2) = \log(x_1) + 3x_1^2x_2 + e^{x2}$ with the equation
- $\phi(x_1, x_2) = \log(x_1) + 3x_1^2x_2 + e^{x2} - 17 = 0$.
- Here, x_2 cannot be explicitly solved.
- Using the implicit function theorem:

$$d\tilde{x}_2/dx_1 = -[(1/x_1) + 6x_1x_2]/[3 \cdot (x_1)^2 + e^{x_2}]$$

- Conclusion: The theorem allows computation of derivatives even when explicit solutions are not possible.

Multivariate Implicit Function Theorem

- Extends to functions $f(x_1, ..., x_n)$ in real space, assumed to be twice continuously differentiable.
- For $\phi(x_1, ..., x_n) = 0$, where $x_n = \mathcal{x}_n(x_1, ..., x_{n-1})$, the derivatives are given by:
- $\partial x_n/\partial x_i = -[\partial f(x_1, ..., x_n)/\partial x_i] / [\partial f(x_1, ..., x_n)/\partial x_n]$, for $i = 1, ..., n-1$

$$\frac{\partial \tilde{x}_n}{\partial x_i} = -\frac{\partial f(\bullet)/\partial x_i}{\partial f(\bullet)/\partial x_n}, \quad i = 1, \ldots, n-1$$

Where is it Applied?

- Widely used in economics, engineering, and physics where equations often relate multiple variables implicitly.
- Example in economics: Determining equilibrium conditions where one variable is not easily solved in terms of another.

Key Takeaways

- The Implicit Function Theorem is a powerful tool for analyzing relationships between variables when explicit functions are difficult to derive.
- Understanding the partial derivatives is crucial for applying the theorem.
- Provides a method to assess how changes in one variable affect another in complex systems.

A.5.2 Taylor's Theorem

Approximating Functions with Polynomials

What is Taylor's Theorem?

- Taylor's Theorem allows the approximation of a function $f(x)$ around a point x^* using a polynomial of degree n.
- The approximation improves as the degree n increases.

Taylor Series Expansion Formula

- The Taylor series expansion of f(x) around x* is given by:

$$
\begin{aligned}
f(x) = {} & f(x^*) + (df/dx)|_{x^*} \cdot (x - x^*) + (d^2 f/dx^2)|_{x^*} \cdot (x - x^*)^2 \cdot (1/2!) \\
& + (d^3 f/dx^3)|_{x^*} \cdot (x - x^*)^3 \cdot (1/3!) + \cdots \\
& + (d^n f/dx^n)|_{x^*} \cdot (x - x^*)^n \cdot (1/n!) + R_n
\end{aligned}
$$

- Here, R_n is the residual that accounts for the error in approximation.

Taylor Series for x^3 Around $x^* = 1$

- Approximate x^3 around $x^* = 1$ using a third-degree polynomial:

$$\begin{aligned} x^3 &= 1^3 + (3 \cdot 1^2) \cdot (x-1) + (6 \cdot 1) \cdot (x-1)^2/2 + 6 \cdot (x-1)^3/6 + R_3 \\ &= 1 + (3x - 3) + 3 \cdot (x^2 - 2x + 1) + (x^3 - 3x^2 + 3x - 1) + R_3 \\ &= x^3 \end{aligned}$$

- Simplifying, we get: $x^3 = x^3$
- Conclusion: The residual R_3 is 0, meaning the approximation is exact for this case.

Taylor Series for e^x Around x* = 0

- Approximate e^x around x* = 0 using a fourth-degree polynomial:

$$\begin{aligned} e^x &= e^0 + e^0 \cdot x + e^0 \cdot (x^2/2) + e^0 \cdot (x^3/6) + e^0 \cdot (x^4/24) + R_4 \\ &= 1 + x + x^2/2 + x^3/6 + x^4/24 + R_4 \end{aligned}$$

- Conclusion: As the degree n increases, the approximation becomes more accurate.

Linearizing a Function

- Using a first-order Taylor expansion to approximate a function around x*.
- Example: Approximate $f(x) = \log(x)$ around $x^* = 1$:
- $\log(x) \approx \log(1) + 1/1*(x - 1) = x - 1$
- Conclusion: This linear approximation is useful in many empirical analyses.

Extending to Multiple Variables

- Taylor's Theorem can be extended to functions of two or more variables.
- Example: Approximate $f(x_1, x_2)$ around $(x_1{}^*, x_2{}^*)$ using a second-order expansion:

$$
\begin{aligned}
f(x_1, x_2) = {} & f(x_1^*, x_2^*) + f_{x_1}(\bullet) \cdot (x_1 - x_1^*) + f_{x_2}(\bullet) \cdot (x_2 - x_2^*) \\
& + (1/2) \cdot [f_{x_1 x_2}(\bullet) \cdot (x_1 - x_1^*)^2 + 2 \cdot f_{x_1 x_2}(\bullet) \cdot (x_1 - x_1^*) \cdot (x_2 - x_2^*) \\
& + f_{x_2 x_2}(\bullet) \cdot (x_2 - x_2^*)^2] + R_2
\end{aligned}
$$

- Include higher-order terms as necessary.

Applications of Taylor's Theorem

- Widely used in economics, physics, and engineering for approximating nonlinear functions.
- Example: Linearizing complex models to simplify analysis in economic theory.

Key Takeaways

- Taylor's Theorem provides a powerful tool for approximating functions using polynomials.
- The accuracy of the approximation increases with the degree of the polynomial.
- It is particularly useful in simplifying complex models for analysis.

A.5.3 L'Hôpital's Rule

L'Hôpital's Rule is a method in calculus for finding limits of indeterminate forms.

L'Hôpital's Rule Overview

- L'Hôpital's Rule is a method in calculus for finding limits of indeterminate forms.

- Purpose: Helps evaluate limits that result in forms like 0/0 or ∞/∞.

- Conditions:
- 1. Functions f(x) and g(x) are real functions, both twice continuously differentiable.
- 2. $\lim_{x \to x^*} [f(x)] = \lim_{x \to x^*} [g(x)] = 0$

L'Hôpital's Rule Statement

- Formulation:

$$\lim_{x \to x^*} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to x^*} \left(\frac{f'(x)}{g'(x)} \right)$$

- provided the limit on the right-hand side exists.

- Repeat Application: If the limit still results in 0/0 or ∞/∞, L'Hôpital's Rule can be applied repeatedly.

Applying L'Hôpital's Rule

Step-by-Step Application:

1. Identify Indeterminate Form: Check if the limit is 0/0 or ∞/∞.
2. Differentiate Numerator and Denominator: Differentiate f(x) and g(x) with respect to x.
3. Evaluate the Limit: Substitute and evaluate the limit using the derivatives.

As an example, consider $f(x) = 2x$ and $g(x) = x$. The limit of the ratio $f(x)/g(x)$ as x tends to 0 is

$$\lim_{x \to x^*} \left(\frac{f(x)}{g(x)} \right) = \frac{0}{0} = \lim_{x \to x^*} \left(\frac{f'(x)}{g'(x)} \right) = \frac{2}{1} = 2$$

Summary of L'Hôpital's Rule

- Key Takeaways:

- 1. L'Hôpital's Rule is a powerful tool for solving limits of indeterminate forms.
- 2. Ensure that functions are differentiable and limits of derivatives exist.
- 3. Practice with different examples to master its application.

A.5.4 Integration by Parts

a technique used to integrate products of functions

Introduction to Integration by Parts

- Integration by Parts is a technique used to integrate products of functions.

- Formula:

$$\int v_2 \cdot dv_1 = v_1 v_2 - \int v_1 \cdot dv_2$$

- Purpose: Simplifies the integration of products by reducing the integral into a simpler form.

Derivation of the Formula

- Start with the Product Rule:
- The derivative of a product of two functions $v_1(t)$ and $v_2(t)$:

$$d[v_1 v_2] = v_2 \cdot dv_1 + v_1 \cdot dv_2$$

where $dv_1 = v_1'(t) \cdot dt$ and $dv_2 = v_2'(t) \cdot dt$.

- Integrate Both Sides:

$$v_1 v_2 = \int v_2 \cdot dv_1 + \int v_1 \cdot dv_2$$

- Rearrange to Obtain:

$$\int v_2 \cdot dv_1 = v_1 v_2 - \int v_1 \cdot dv_2$$

Example Calculation

- Problem:

$$\int te^t \; dt$$

Solution Steps:

1. Choose $v_1 = t$ and $dv_2 = e^t \; dv$
2. Compute derivatives and integrals:

   - $dv_1 = dt$
   - $v_2 = e^t$

3. Apply the formula:

$$\int te^t dt = te^t - \int e^t dt = e^t \cdot (t-1)$$

Summary and Recap

- Key Points:
- 1. Integration by Parts is useful for integrating products of functions.
- 2. The formula rearranges the integral into a simpler form.
- 3. Practice and experience help in choosing the appropriate v_1 and dv_2.

A.5.5 Fundamental Theorem of Calculus

FTC links the concept of differentiation and integration.

Introduction to the Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus (FTC) links the concept of differentiation and integration.

Let $f(t)$ be continuous in $a \le t \le b$.

If $F(t) = \int f(t) \cdot dt$ is the indefinite integral of $f(t)$

so that $F'(t) = f(t)$

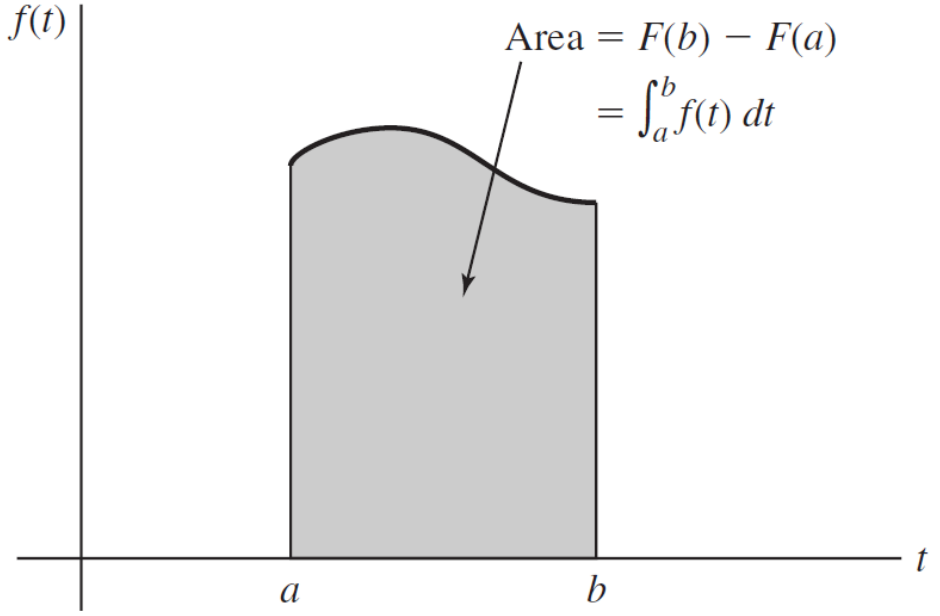
- The definite integral is

$$\int_a^b f(t)\,dt = \int_a^b F'(t)\,dt = F(b) - F(a)$$

Statement of the Fundamental Theorem of Calculus

Interpretation: The definite integral $\int_a^b f(t)\,dt$

represents the area under the curve of f(t) from t = a to t = b.



Applications in Real-World Problems

- Economics: Find the total accumulated profit or cost over time.

Summary and Key Points

- Key Points:

- 1. The Fundamental Theorem of Calculus connects differentiation and integration.
- 2. It simplifies the computation of definite integrals.
- 3. The theorem has wide-ranging applications in various fields.

A.5.6 Differentiation of Integrals

Differentiation w.r.t. Variable of Integration

- Rule: The condition $F'(t) = f(t)$ implies that the derivative of an indefinite integral with respect to the variable of integration, t, is the function $f(t)$ itself:

$$\frac{\partial}{\partial t}\left(\int f(t)\,dt\right) = \frac{\partial}{\partial t}[F(t)] = F'(t) = f(t)$$

- Implication: This expresses that the integral and differentiation are inverse operations.