Introduction to Mathematical Methods
Fianl Exam Presentation, Group V

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(a) Show that the feasible set is convex.

The production possibility frontier is

$$f_1(x, y) = x^2 + y^2 \leq 200$$

The environmental constraint is given by

$$f_2(x, y) = x + y \leq 20$$

The hessian matrix of $x^2 + y^2$ is

$$H = \nabla^2 f_1 = \begin{pmatrix} \frac{\partial^2 f1}{\partial x^2} & \frac{\partial^2 f1}{\partial x \partial y} \\ \frac{\partial^2 f1}{\partial y \partial x} & \frac{\partial^2 f1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

For Hessian matrix of f_1,

$$D_1 = 2 \geq 0, D_2 = 4 \geq 0.$$

Hence f_1 is PD(convex).

For f_2, it is a linear model, so it is affine.$\rightarrow$ Convex.

For the same reason, $x \geq 0$ and $y \geq 0$ are convex.

(b) Show that the the utility function is not concave.

The utility function is $u(x, y) = xy^3$. Its Hessian matrix is

$$H = \nabla^2 u(x, y) = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{pmatrix}$$

determinant $|H| = -9y^4 < 0$, so it is not concave.

(c) Since the utility function is not concave, for $x > 0$ and $y > 0$, take a natural logarithm of the utility function. Show that after this transformation, the utility function is concave.

Let $v(x, y) = \ln u(x, y) = \ln x + 3 \ln y$. The Hessian matrix of v(x,y) is

$$H = \nabla^2 v = \begin{pmatrix} \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial y \partial x} & \frac{\partial^2 v}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{x^2} & 0 \\ 0 & \frac{-3}{y^2} \end{pmatrix}$$

The first-order determinant of v is $\frac{-1}{x^2} \leq 0$ The second-order determinant of v is $\frac{3}{x^2 y^2} \geq 0$.

So v is ND(concave).

(d) Write down the Kuhn-Tucker first order conditions. Argue that it suffices for the global maximization.

Maximize $\ln u(x, y) = \ln x + 3 \ln y$ subject to

$$x^2 + y^2 \leq 200, \quad x + y \leq 20, \quad x, y \geq 0.$$

Lagrangian:

$$\mathcal{L} = \ln x + 3 \ln y + \lambda_1(200 - x^2 - y^2) + \lambda_2(20 - x - y) + \lambda_3(0 - x) + \lambda_4(0 - y)$$

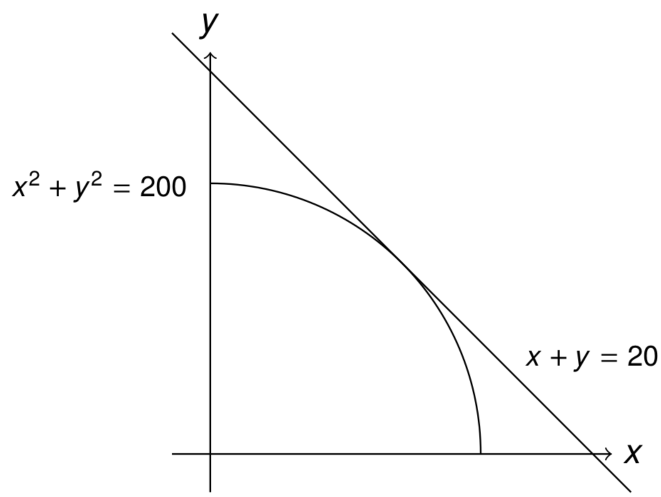
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = -\frac{1}{x} + 2\lambda_1 x + \lambda_2 - \lambda_3 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = -\frac{3}{y} + 2\lambda_1 y + \lambda_2 - \lambda_4 = 0 \\ x^2 + y^2 \leq 200, \quad x + y \leq 20, \quad x \geq 0, \quad y \geq 0 \\ \lambda_i \geq 0; \quad i = 1, 2, 3, 4 \\ \lambda_1(x^2 + y^2 - 200) = 0 \\ \lambda_2(x + y - 20) = 0 \\ \lambda_3(-x) = 0, \quad \lambda_4(-y) = 0 \end{cases}$$

Sufficiency

The objective function V is convex and the constraints define a convex set. A strictly feasible point exists, so Slater's condition holds. For example, consider the point $(x, y) = (5, 5)$:

$$x + y = 10 < 20, \quad x^2 + y^2 = 50 < 200, \quad x > 0, \quad y > 0$$

Hence, the KKT conditions are both necessary and sufficient for global optimality. At the maximizer we expect $x, y > 0$ and, as the utility is increasing in both x and y, the quadratic constraint binds while x + y → 20 is slack.



Solving the KKT conditions

We analyze the case where the first constraint is binding, but the linear constraint is not. Set $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and enforce $x^2 + y^2 = 200$. The stationarity conditions give:

$$\frac{1}{x} = 2\lambda_1 x, \quad \frac{3}{y} = 2\lambda_1 y \quad \Rightarrow \quad \frac{y^2}{x^2} = 3 \quad \Rightarrow \quad y = \sqrt{3}x.$$

Plugging this into $x^2 + y^2 = 200$ yields $4x^2 = 200 \Rightarrow x^* = 5\sqrt{2}$ and $y^* = 5\sqrt{6}$.

The linear constraint is slack, as we can verify:

$$x^* + y^* = 5(\sqrt{2} + \sqrt{6}) \approx 19.318 < 20.$$

Thus the unique global maximizer is

$$(x^*, y^*) = (5\sqrt{2}, 5\sqrt{6}).$$

The active constraint is $x^2 + y^2 = 200$, which is binding. The others are not.

(e) Solve the global maximization. How about the following points: (x, y) = (0, y) and (x, y) = (x, 0)? Identify which constraints are binding.

Global optimum:

$$(x^*, y^*) = (5\sqrt{2}, 5\sqrt{6}), \quad u^* = x^*(y^*)^3 = 7500\sqrt{3} \approx 1.299 \times 10^4.$$

Binding: $x^2 + y^2 = 200$. Slack: $x + y \leq 20,\ x \geq 0,\ y \geq 0$

Axis point:

$$(0, y) : best\, feasible\, is\, (0, 10\sqrt{2})\, with u = 0.(x, 0) : best\, feasible\, is\, (10\sqrt{2}, 0)\, with u = 0.$$

Any axis point has zero utility and thus cannot be optimal relative to $u^* > 0$.

(f) How to interpret the Lagrangean multiplier (at optimality), if you are considering a minimization problem, rather than a maximization problem?

Consider the parametric problem in standard inequality form

$$\min_x f(x_0) \quad s.t.\, g_i(x) \leq t_i,\ i = 1, ..., m,$$

with value function $p(t) = min f_0(x) : g(x) \leq t$ Under Slater and KKT, the optimal multiplier $\lambda^* \geq 0$ yields the first-order sensitivity

$$\frac{\partial p(t = 0)}{\partial t_i} = -\lambda_i^*$$

Interpretation: In minimization, λ_i^*, is the marginal increase in the optimal cost when the $i_t h$ constraint is tightened ($t_i \downarrow$), and the marginal decrease when it is relaxed ($ti \uparrow$). Non-binding constraints typically have $\lambda_i^* = 0$. Interpretation: In maximization, λ_i^* is the marginal increase in the optimal value when the $i_t h$ constraint is relaxed. If $\lambda_i^* = 0$, relaxing that constraint does not improve the optimum. Application to this problem. Only $x^2 + y^2 \leq 200$ is active at optimum and

$$\lambda_1^* = 0.01,\ \lambda_2^*,\ \lambda_3^*,\ \lambda_4^* = 0$$

Hence, for small $\varepsilon \geq 0$,

$$max \ln u(200 + \varepsilon) \approx max \ln u(200) + \lambda_1^* \varepsilon = max \ln u(200) + 0.01\varepsilon$$

Because $\frac{d}{dt} \ln u^* = \lambda_1^*$, we also have

$$\frac{d}{dt} \ln u^* = \lambda_1^* \approx 1.30 \times 10^2$$

so increasing the radius-squared budget by one unit raises the optimal utility by about 130.