A.2 Static Optimization

A.2.1 Unconstrained Maximan

- In optimization, we often need to find the maximum value of a function.
- We explore conditions under which a function achieves its local maxima.
- This section deals with finding maxima without any constraints (unconstrained).

Local Maximum of a Univariate Function

- Definition:
- Consider a univariate real function u(•). A function $u(x)$ has a local maximum at $\bar{\bar{x}}$ if: $u(\bar{\bar{x}})$≥$u(x)$ for all x in a neighborhood of $\bar{\bar{x}}$, for all x in the interval $[\bar{\bar{x}} - \epsilon, \bar{\bar{x}} + \epsilon]$ where $\epsilon$is some positive number.

- Absolute Maximum:$u(\bar{\bar{x}})$ is said to have an absolute maximum at $\bar{\bar{x}}$ if: $u(\bar{\bar{x}})$≥$u(x)$ for all x in the domain of the function.

Necessary Conditions for Local Maxima (Univariate Case)

- First Derivative Condition (Critical Points):
- For a function $u(x)$, the first derivative at a local maximum $\bar{\bar{x}}$ must satisfy: $u'(\bar{\bar{x}})=0$

This means the slope of the function at $\bar{\bar{x}}$ is zero, making the function flat at that point.

- Second Derivative Condition:
- To ensure that $\bar{\bar{x}}$ is a local maximum (and not a minimum or saddle point), the second derivative must satisfy: $u''(\bar{\bar{x}})\leq 0$
- If $u''(\bar{\bar{x}})<$, the function is concave at that point, confirming a local maximum.

Example: Univariate Maximization

- Suppose we have a function $u(x)$=$-x^2 + 4\text{x} + 5$
- First Derivative: $u'(x)$=$-2x$+4
- Setting $u'(x)$=: $-2x$+4=0 $\quad \Rightarrow \quad$ x=2
- Second Derivative: $u''(x)$=-2
- Since $u''(x)$<-2, the function is concave at x=2, confirming that x=2 is a local maximum.

Multivariate Case

- The multidimensional case is similar to the unidimensional case
- Consider a function $u : R^n \to R$, twice continuously differentiable.
- Local Maxima in Higher Dimensions:

For a function $u(x_1, x_{2,\ldots,} x_n)$, the necessary conditions for a local maximum are generalized to partial derivatives.

- First Derivative Condition:
- The partial derivatives must vanish at the local maximum $\bar{\bar{x}}$:$\frac{\partial u}{\partial x_i}(\bar{\bar{x}})$=0 for all i

A.2.2 Classical Nonlinear Programming: Equality Constraints

Problem Overview

- Objective: Maximize a function $u(x_1 ,x_{2,.,} x_n)$ subject to the constraint g$(x_1 ,x_{2,...,} x_n)$ =a.
- $$\max_{x_1,\, x_{2,...,}\, x_n} \left[u(x_1, x_{2,.,} x_n)\right]$$

  subject to g$(x_1 ,x_{2,...,} x_n)$ = a

where g: $R^n \rightarrow R$

We assume that *u(•)* and *g(•)* are twice continuously differentiable.

Implicit Function & Reformulation

- The constraint $g(x_1, ..., x_n) = a$ implicitly defines a function for x_1:
- $x_1 = \tilde{x}_1(x_{2,...,} x_n)$
- This reduces the problem to an unconstrained maximization:

$$u\left[\tilde{x}_1\,(x_{2,...,}\, x_n), x_{2,...,}\, x_n\right] \equiv \tilde{u}\,(x_{2,...,}\, x_n)$$

Necessary Condition for Maximum

- Necessary condition for the constrained maximum:

  All partial derivatives vanish for i = 2, ..., n:

$$\partial \tilde{u}(\bullet)/\partial x_i = [\partial u(\bullet)/\partial x_1] \cdot \partial \tilde{x}_1/\partial x_i + \partial u(\bullet)/\partial x_i = 0$$

Using the Implicit Function Theorem

- From the Implicit Function Theorem, the derivative $\partial \tilde{x}_1 / \partial x_i$ can be calculated:

$$\partial \tilde{x}_1/\partial x_i = -[\partial g(\bullet)/\partial x_i]/[\partial g(\bullet)/\partial x_1]$$

- Substituting into the necessary condition for maximum:

$$\frac{\partial g(\bullet)/\partial x_i}{\partial g(\bullet)/\partial x_1} = \frac{\partial u(\bullet)/\partial x_i}{\partial u(\bullet)/\partial x_1}$$

Proportionality in Partial Derivatives

- The partial derivatives of g with respect to x_i should be proportional to the partial derivatives of u with respect to x_i.
- The constant of proportionality, μ, is the same for all i.

Matrix Representation of the Condition

- The condition can be expressed in matrix form:

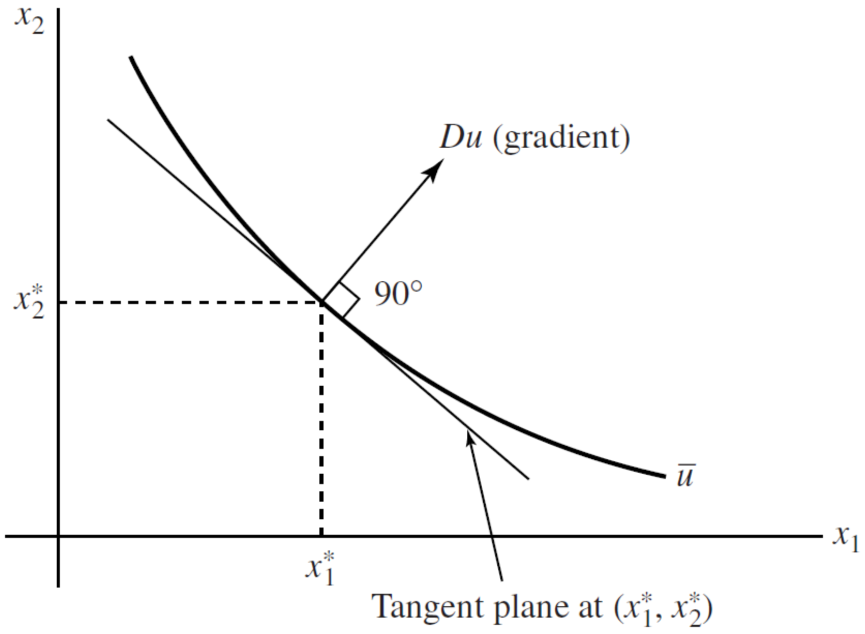
$$Du(\bar{\bar{x}}) = \mu \cdot Dg(\bar{\bar{x}})$$

where:

- $\bar{\bar{x}}$ is an n-dimensional vector.
- *Du* and *Dg* are gradients of the functions *u* and *g*, respectively.

Gradient Interpretation

- $Du = [\partial u/\partial x_1, ..., \partial u/\partial x_n]$, $Dg = [\partial g/\partial x_1, ..., \partial g/\partial x_n]$.
- The gradient of u at a point $\bar{\bar{x}}$ is perpendicular to the tangent line at that point.



Lagrange Multiplier Interpretation

- μ is the factor of proportionality.

- In economics, μ represents the familiar equality between marginal rates of substitution and marginal rates of transformation (or relative prices).

Lagrangian Function and First-Order Conditions

- The Lagrangian function is used to derive the first-order conditions:

$$L(\bullet) = u(x_1, \ldots, x_n) + \mu \cdot [a - g(x_1, \ldots, x_n)]$$

- First-order conditions are derived by differentiating with respect to each argument.

$$\frac{\partial L(\cdot)}{\partial x_1}=0,\ \frac{\partial L(\cdot)}{\partial x_2}=0,\ \ldots,\ \frac{\partial L(\cdot)}{\partial \mu}=0 \text{ (recovers the constraint)}$$

Economic Interpretation of Lagrange Multiplier

- Change in utility u(•) when income *a* changes:

$$du(\bullet)/da = \sum_{i=1}^{n} [\partial u(\bullet)/\partial x_i] \cdot \partial \overline{\overline{x}}_i/\partial a \tag{A.50}$$

where $\partial \overline{\overline{x}}_i/\partial a$ is the change in optimal quantity of x_i when the constraint is relaxed.

Expressing the Change in Utility

- Using first-order conditions from $Du(\overline{\overline{x}}) = \mu \cdot Dg(\overline{\overline{x}})$

we rewrite the utility change as:

$$du(\bullet)/da = \sum_{i=1}^{n} \mu \cdot [\partial g(\bullet)/\partial x_i] \cdot \partial \overline{\overline{x}}_i/\partial a$$

Differentiating the Budget Constraint

- Totally differentiate the budget constraint with respect to *a*:

$$dg(\bullet)/da = \sum_{i=1}^{n} [\partial g(\bullet)/\partial x_i] \cdot \partial \overline{\overline{x}}_i/\partial a = 1$$

- Substitute this result into (A.50) to get:

$$du(\bullet)/da = \mu$$

Lagrange Multiplier

- The Lagrange multiplier μ represents the additional utility gained when the constraint is relaxed.
- It is often referred to as the shadow price or shadow value of the constraint.

A.2.3 Inequality Constraints: The Kuhn–Tucker Conditions

Introduction to the Problem and Inequality Constraints

- The problem begins by introducing m inequality constraints of the form:

$$g_i(x_1, \ldots, x_n) \leq a_i \quad \text{for } i = 1, \ldots, m$$

- All functions $g_i(.)$ are twice continuously differentiable, and each a_i is constant.
- Optimization Problem: $$\max_{x_1,\ldots,x_n} [u(x_1, \ldots, x_n)]$$

- Subject to:

$$g_1(x_1, \ldots, x_n) \leq a_1$$

$$\ldots$$

$$g_m(x_1, \ldots, x_n) \leq a_m$$

Kuhn–Tucker Theorem and Lagrange Multipliers

- Solution to the Inequality Constraints:
- If $\bar{\bar{x}} = (\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_n)$ is a solution to the problem, there exists a set of m Lagrange multipliers μ_i.
- Conditions:

$$(a)\quad Du(\bullet) = \sum_{i=1}^{m} \mu_i \cdot [Dg_i(\bullet)]$$

$$(b)\quad g_i(\bullet) \leq a_i,\ \mu_i \geq 0$$

$$(c)\quad \mu_i \cdot [a_i - g_i(\bullet)] = 0$$

Interpretation of Kuhn–Tucker Conditions

- Condition (a): $$Du(\bullet) = \sum_{i=1}^{m} \mu_i \cdot [Dg_i(\bullet)]$$

The gradient of the objective function must be a linear combination of the gradients of the restrictions.

- Condition (b): $g_i(\bullet) \leq a_i,\ \mu_i \geq 0$

The constraints must be satisfied and the shadow prices must be nonnegative.

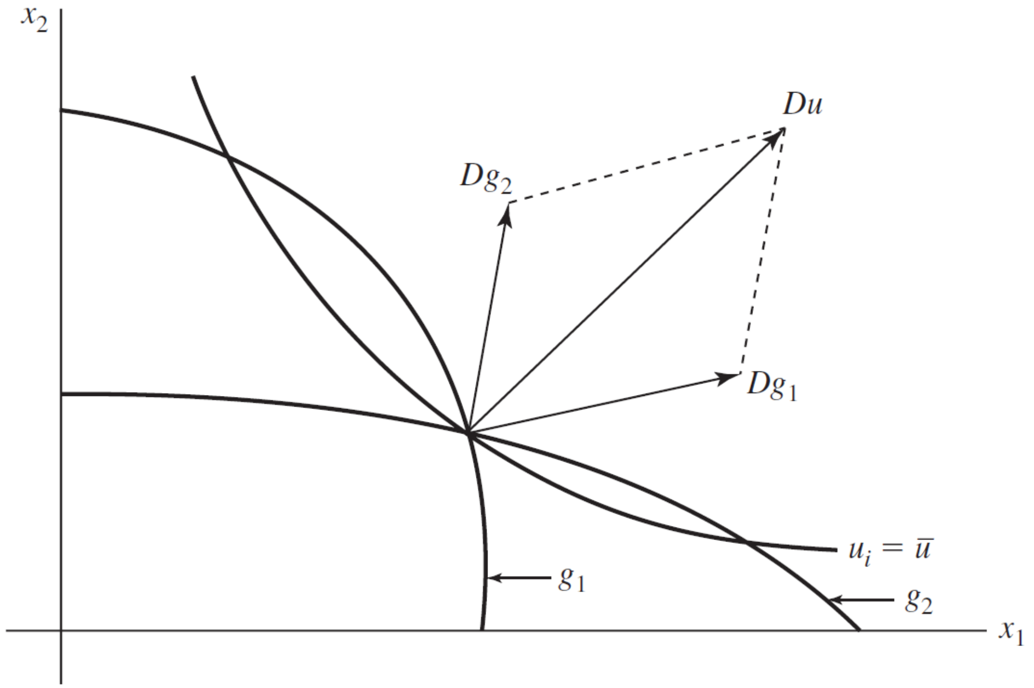
- Condition (c): $\mu_i \cdot [a_i - g_i(\bullet)] = 0$

Complementary-slackness condition: the product of the shadow price and the constraint is zero.

Graphical Representation of Constraints

- Figure A.12 illustrates the solution to the maximization problem with two inequality constraints, g_1 and g_2.

- Graph Elements:
- Gradients Dg_1 and Dg_2 show the direction of change in the constraints tied to the utility function Du.
- Maximum occurs where the gradient of the objective function is perpendicular to the constraints.

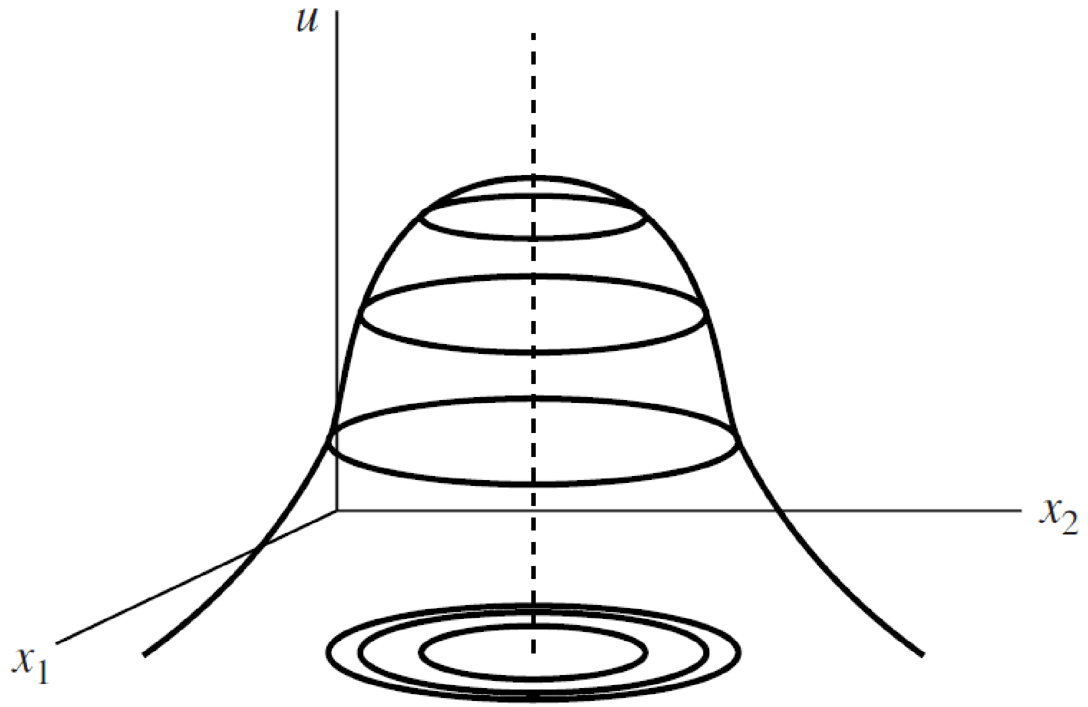


Complementary-Slackness Condition and Examples

- Condition:
- When the constraint is not binding, the shadow price must be zero, meaning Dg_i is not included in Du.
- If the constraint is binding, the shadow price is positive, and the constraint actively influences the solution.

Example (Figure A.13):

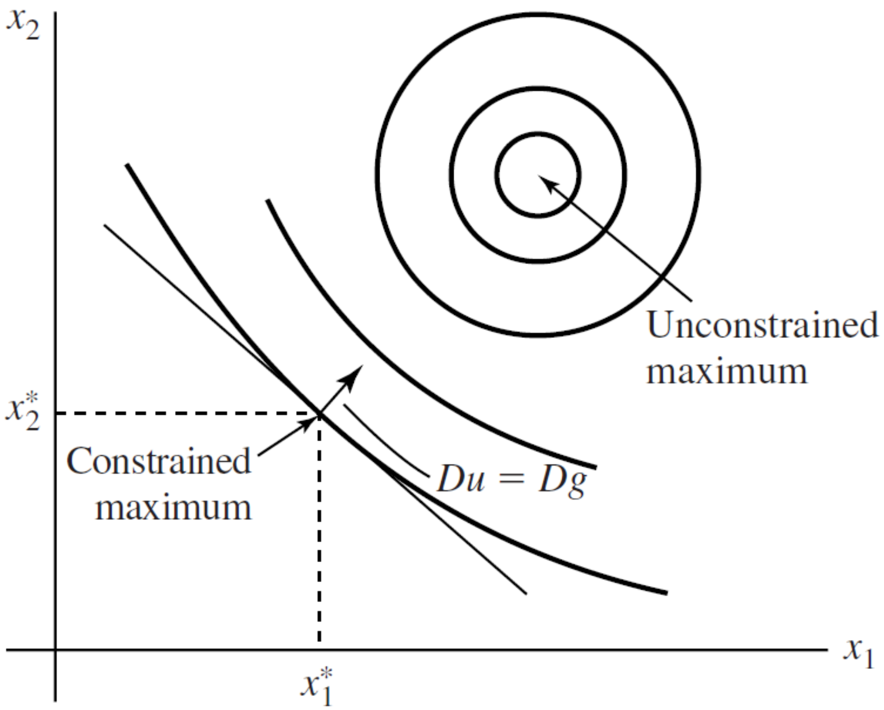
(a) **Preferences over two goods.** The indifference curves for x_1 and x_2 are assumed to take the form of a bell.



(a)

Example (Figure A.13):

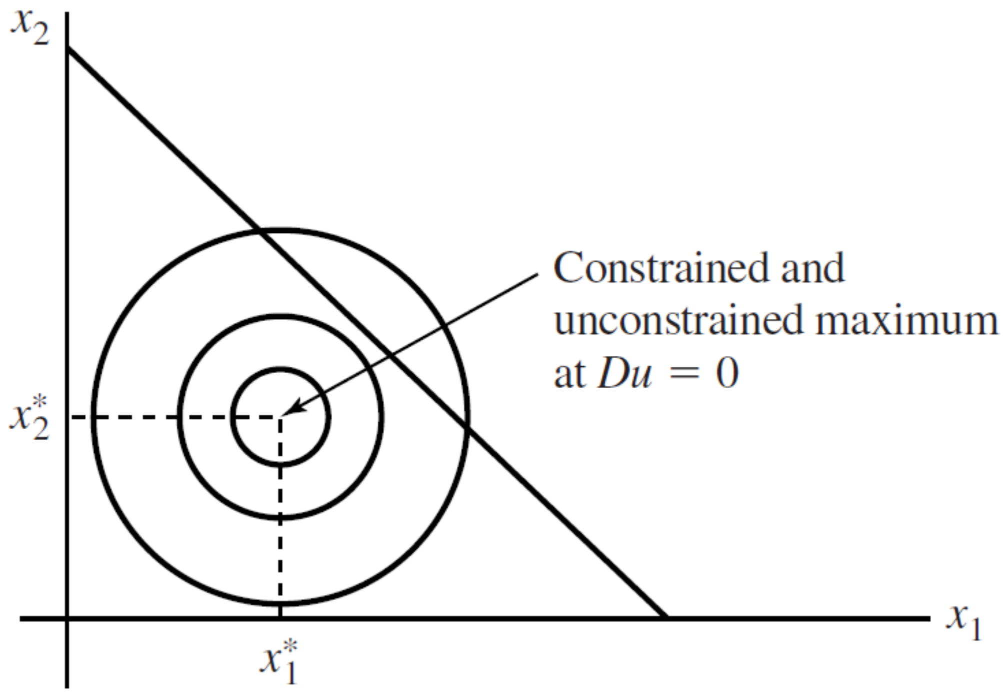
- (b) **Maximizing utility subject to a binding inequality constraint.** In this example, the budget constraint for x_1 and x_2 is binding



(b)

Example (Figure A.13):

- **Maximizing utility subject to a nonbinding inequality constraint.** In this example, the budget constraint for x_1 and x_2 is not binding.



(c)

The Lagrangian Formulation

- Lagrangian Function:

$$L(x_1, \ldots, x_n; \mu_1, \ldots, \mu_m) = u(x_1, \ldots, x_n) + \sum_{i=1}^{m} \mu_i \cdot [a_i - g_i(x_1, \ldots, x_n)]$$

- Necessary Conditions for Maximum:

For the solution $\bar{x}$ to be optimal, the Lagrangian must satisfy the Kuhn–Tucker conditions, i.e., be a saddle point $(\bar{x}, \mu)$, meaning it should be maximized with respect to $\bar{x}$ and minimized with respect to μ

Summary of Necessary and Sufficient Conditions

- Key Takeaway: Conditions (a), (b), and (c) of Kuhn–Tucker theorem are necessary for optimality.

Sufficiency:

- If the objective function u(.) is concave and the constraints form a convex set, the Kuhn–Tucker conditions are sufficient for an optimal solution.