A.3 Dynamic Optimization in Continuous Time

Introduction

- **Economists and Dynamic Optimization**
- Economists began focusing on dynamic problems in the 1920s with work from Hotelling and Ramsey.
- In the 1960s, dynamic mathematics became prevalent in economics, especially through the work of neo-classical growth theorists.

Techniques for Solving Dynamic Problems

- **Calculus of Variations**
  - The classical method used by mathematicians for solving dynamic problems.
  - This approach was later expanded in two main directions:
    - **Dynamic Programming** (Richard Bellman, 1950s)
    - **Maximum Principle of Optimal Control** (L. Pontryagin, 1950s)

Maximum Principle of Optimal Control

- A generalization of calculus of variations.
- Provides solutions to problems where constraints involve derivatives of state variables.
- Central to the theory of economic growth.

- Focus on Application, Not Proof

A.3.2 The Typical Problem

Overview

- The agent's objective is to maximize a function V(0) subject to constraints over time.
- This is framed as a dynamic optimization problem involving state variables k(t) and control variables c(t).

$$\max_{c(t)} V(0) = \int_0^T v[k(t), c(t), t] \cdot dt, \quad \text{subject to}$$

$$(a) \quad \dot{k}(t) = g[k(t), c(t), t]$$

$$(b) \quad k(0) = k_0 > 0$$

$$(c) \quad k(T) \cdot e^{-\bar{r}(T) \cdot T} \geq 0$$

Objective Function

$$\max_{c(t)} V(0) = \int_0^T v[k(t), c(t), t] \cdot dt$$

- V(0) is the value of the objective function from the initial moment.
- $v[k(t), c(t), t]$nts the instantaneous felicity function.

Constraints in the Problem

- Dynamic Equation for State Variable *k*(t):

$$\dot{k}(t) = g[k(t), c(t), t]$$

transition equation or equation of motion

Describes how k(t) evolves based on control variable c(t).

Boundary conditions

- Initial Condition:

$$k(0) = k_0 > 0$$

- Terminal Condition at Time T:

$$k(T) \cdot e^{-\bar{r}(T) \cdot T} \geq 0$$

- value of the state variable at the end of the planning horizon, $k(T)$, discounted at the rate $\bar{r}(T)$, must be nonnegative.
- For finite values of T, this constraint implies $k(T) \geq 0$, as long as the discount rate $\bar{r}(T)$ is positive and finite.

State and Control Variables

- State variable k(t) represents the state of the system, such as capital.
- Control variable c(t) represents decisions or actions, like consumption.
- The felicity function $v[k(t), c(t), t]$ integrates instantaneous utility over time.

Growth Model with Utility

- Instantaneous Utility Function:

$$v(k, c, t) = e^{-\rho t} \cdot u[c(t)]$$

- Capital Accumulation Equation:

$$\dot{k} = g[k(t), c(t), t] = f[k(t), t] - c(t) - \delta \cdot k(t)$$

- This equation describes how capital grows over time with production minus consumption and depreciation.

Key Takeaways

- Maximizing utility over a finite or infinite horizon.
- Choosing a path for control variables while adhering to constraints on state variables.
- Balancing current consumption against future capital growth in the example.

A.3.3 Heuristic Derivation of the First-Order Conditions

Lagrangian Construction

- Equation (A.58):

$$L = \int_0^T v[k(t), c(t), t] \cdot dt + \int_0^T \{\mu(t) \cdot (g[k(t), c(t), t] - \dot{k}(t))\} \cdot dt + \nu \cdot k(T) \cdot e^{\bar{r}(T) \cdot T}$$

$v[k(t), c(t), t]$: Instantaneous utility function.

$\mu(t)$: Costate variable (shadow price).

$g[k(t), c(t), t] - \dot{k}(t) = 0$:Accumulation constraint for capital.

$\nu \cdot k(T) \cdot e^{\bar{r}(T) \cdot T}$: Terminal condition for capital at time T.

Dynamic Lagrange Multipliers

- Costate Variables:
- μ(t): Shadow price or dynamic Lagrange multiplier.
- μ(t) represents the value of capital stock at time t in units of utility at time 0.
- Ensures that constraints are respected throughout the time horizon.

$$\int_0^T \{\mu(t) \cdot (g[k(t), c(t), t] - \dot{k}(t))\} \cdot dt = 0$$

Deriving First-Order Conditions

- rewrite the Lagrangian by integrating the term $\mu(t) \cdot k\dot{(t)}$ by parts to get Equation (A.59):

$$L = \int_0^T (v[k(t), c(t), t] + \mu(t) \cdot g[k(t), c(t), t])\, dt$$

$$+ \int_0^T \mu(t) k(t)\, dt + \mu(0)k_0 - \mu(T)k(T) + \nu k(T) e^{-\bar{r}(T)T}$$

- - μ(t) · $g[k(t), c(t), t]$: Contribution of shadow price and the transition of capital stock.
- - Boundary Conditions: $\mu(0)k_0$ and $\nu \cdot k(T) \cdot e^{\bar{r}(T) \cdot T}$

The Hamiltonian Function

- Equation (A.60):

$$H(k, c, t, \mu) \equiv v(k, c, t) + \mu \cdot g(k, c, t)$$

- Hamiltonian Function:
- Combines **direct** utility function (from consumption) and capital transition equation (The **indirect effect** through the capital stock's impact on future consumption).
- Represents the agent’s decision problem at any time t.

Introduction to Perturbation of Optimal Time Paths

- Optimal Time Paths for Control and State Variables
- Consider $\bar{\bar{c}}(t)$ *and* $\bar{\bar{k}}(t)$ as the optimal time paths for control and state variables.
- Introduce a perturbation to the optimal path using a perturbation function $p_1(t)$ for the control variable:

$$c(t) = \bar{\bar{c}}(t) + \epsilon \cdot p_1(t)$$
$$k(t) = \bar{\bar{k}}(t) + \epsilon \cdot p_2(t)$$

$$k(T) = \bar{\bar{k}}(T) + \epsilon \cdot dk(T)$$

Lagrangian with Perturbation ϵ

- The Lagrangian is rewritten to account for perturbation:

$$\overline{\overline{L}}(\cdot, \epsilon) = \int_0^T \{H[k(\bullet, \epsilon); c(\bullet, \epsilon)] + \dot{\mu}(\bullet) \cdot k(\bullet, \epsilon)\} \cdot dt$$

$$+ \mu(0) \cdot k_0 - \mu(T) \cdot k(T, \epsilon) + \nu \cdot k(T, \epsilon) \cdot e^{-\bar{r}(T) \cdot T}$$

- The derivative of the Lagrangian is taken with respect to ϵ and set to 0:

$$\partial \overline{\overline{L}} / \partial \epsilon = \int_0^T [\partial H / \partial \epsilon + \dot{\mu} \cdot \partial k / \partial \epsilon] \cdot dt + [\nu e^{-\bar{r}(T)T} - \mu(T)] \cdot \partial k(T, \epsilon) / \partial \epsilon = 0$$

Application of the Chain Rule to the Derivative

- Applying the chain rule:

$$\partial H/\partial\epsilon = [\partial H/\partial c]\cdot p_1(t) \ + \ [\partial H/\partial k]\cdot p_2(t)$$

$$\partial k(T,\epsilon)/\partial\epsilon = dk(T)$$

- Rearranged Derivative of Lagrangian

$$\partial L/\partial\epsilon = \int_0^T \{[\partial H/\partial c]\cdot p_1(t) + [\partial H/\partial k + \dot{\mu}]\cdot p_2(t)\}\cdot dt$$

$$+[\nu\cdot e^{-\bar{r}(T)T} - \mu(T)]\cdot dk(T) = 0$$

First-Order Conditions

Equation $A.62$A.62 holds for all perturbation paths if the components vanish.

Three main conditions:

$$\partial H / \partial c = 0$$

$$\partial H / \partial k + \dot{\mu} = 0$$

$$\nu \cdot e^{-\bar{r}(T) \cdot T} = \mu(T)$$

Maximum Principle

- First-Order Condition for Control Variables c(t) states:
- If $\bar{\bar{c}}(t)\ and\ \bar{\bar{k}}(t)$ are optimal, then:

$$\partial H/\partial c = 0$$

This ensures that the derivative of the Hamiltonian with respect to c equals zero for all t.

This result is called the Maximum Principle.

Euler Equation

- First-Order Condition for State Variables

$$\partial H / \partial k + \dot{\mu} = 0$$

- ensuring the correct adjustment of capital over time

Terminal Condition at Time T

At the terminal date T, the costate variable $\mu(T)$ equals ν, the Lagrange multiplier for the nonnegativity constraint on k.

- This is given by,

$$\nu \cdot e^{-\bar{r}(T)\cdot T} = \mu(T)$$

Summary of First-Order Conditions

- The Maximum Principle ensures the optimal control path $c(t)$.
- The Euler Equation governs the optimal trajectory of state variables like capital $k(t)$.
- The Terminal Condition ensures that the shadow price $\mu(T)$ aligns with the discounted terminal Lagrange multiplier.

A.3.4 Transversality Conditions

Understanding Transversality in Dynamic Optimization

Complementary-Slackness in Dynamic Optimization

- Transversality conditions are derived from Kuhn-Tucker first-order conditions.
- These conditions include complementary-slackness associated with inequality constraints in optimization.
- In dynamic problems, they govern the capital stock left at the end of the planning period.

Inequality Constraint on Capital

- In dynamic settings, the inequality constraint is:

$$k(T) \cdot e^{-\bar{r}(T) \cdot T} \geq 0$$

- This ensures that the discounted capital stock at the end of the period cannot be negative.

- Associated complementary-slackness condition:

$$\nu \cdot k(\mathrm{T}) \cdot e^{-\bar{r}(T) \cdot T} = 0 \text{ , with } \nu \geq 0$$

Transversality Condition

- Equation (A.66):

$$\mu(T) \cdot k(T) = 0$$

- At the terminal point, if the value of capital $k(T)$ is positive, the shadow price $\mu(T)$ must be zero, and vice versa.
- Ensures that no value is left unutilized at the end of the planning period.

Interpretation of the Transversality Condition

- The transversality condition governs the relationship between the remaining capital and its price at the terminal time T.

- If $k(T)>0$: $\mu(T)=0$
  – The capital stock has no value.
- If $\mu(T)>$: $k(T)=0$
  – No capital is left at the terminal time.

Transversality conditions ensure that capital constraints and terminal conditions are respected in dynamic optimization.

They provide essential boundary conditions that must hold for the solution to be optimal.

- First-order conditions ensure optimal control of capital and consumption over time.
- Lagrangian and Hamiltonian are central to dynamic optimization problems in economics.
- Transversality conditions determine behavior at the terminal period.

A.3.5 The Behavior of the Hamiltonian over Time

Understanding the Time Evolution of the Hamiltonian

Total Derivative of the Hamiltonian

- The behavior of the Hamiltonian over time can be analyzed by taking its total derivative of H with respect to t:

$$\frac{dH(k,c,\mu,t)}{dt} = \left[\frac{\partial H}{\partial k}\right] \cdot \dot{k} + \left[\frac{\partial H}{\partial c}\right] \cdot \dot{c} + \left[\frac{\partial H}{\partial \mu}\right] \cdot \dot{\mu} + \frac{\partial H}{\partial t}$$

This captures how the Hamiltonian changes with respect to time and its variables.

First-Order Conditions Implications

- Impact of First-Order Conditions on H

$$\frac{\partial H}{\partial c} = 0$$

- This implies that the second term in the total derivative $\left[\frac{\partial H}{\partial c}\right] \cdot \dot{c}$ equals zero.

Euler Condition for Capital Stock k

$$\frac{\partial H}{\partial k} = -\dot{\mu}$$

This cancels out with the third term in the total derivative:

$$\left[\frac{\partial H}{\partial \mu}\right] = g = \dot{k}$$

Autonomous Problems and Constant Hamiltonian

- When neither the objective function nor the constraints depend explicitly on time *t*, the total derivative simplifies to:

$$\frac{dH(k,c,\mu,t)}{dt} = \frac{\partial H}{\partial t}$$

If the problem is autonomous, this derivative is zero, implying that the Hamiltonian is constant over time.

A.3.6 Sufficient Conditions

Concavity and Dynamic Optimization

Sufficient Conditions in Static Problems

- In static, nonlinear maximization problems, the Kuhn-Tucker necessary conditions are also sufficient when:
  - The objective function is concave.
  - The constraints form a convex set.

- This makes the necessary conditions both necessary and sufficient for optimization.

Sufficiency in Dynamic Problems

- Mangasarian (1966) extended the Kuhn-Tucker result to dynamic problems.
- If the functions $v(\cdot)$ and $g(\cdot)$ are concave in both k and c, then the necessary conditions are also sufficient.
- This result is useful but considered somewhat restrictive.

More General Sufficiency Conditions

- Arrow and Kurz (1970) provided a more general sufficiency condition.
- Define $H^0(k, \mu, t)$ as the maximum of $H(k, c, \mu, t)$ with respect to c, for given k, μ, and t.
- If $H^0(k, \mu, t)$ is concave in k, for given μ and t, then the necessary conditions are also sufficient.

Concavity of $v(\cdot)$ and $g(\cdot)$

- Concavity of $v(\cdot)$ and $g(\cdot)$ is sufficient for the Arrow-Kurz condition to hold.
- However, the Arrow-Kurz condition requires checking the concavity of the derived function $H^0(k, \mu, t)$, which can be more difficult than checking $v(\cdot)$ and $g(\cdot)$ directly.

A.3.7 Infinite Horizons

Understanding the Optimization with Infinite Planning Horizons

Objective of Infinite Horizon Optimization

- The typical optimization problem with an infinite planning horizon is:

$$\max_{c(t)} V(0) = \int_0^\infty v[k(t), c(t), t] \cdot dt$$

- subject to

$$(a)\ \ \dot{k}(t) = g[k(t), c(t), t]$$

$$(b)\ \ k(t) = k_0$$

$$(c)\ \ \lim_{t\to\infty} [k(t) \cdot e^{-\bar{r}(t)\cdot t}] \geq 0$$

Difference from Finite Horizon Optimization

- The only difference is that the planning horizon is infinite.
- First-order conditions remain the same as in the finite-horizon case.
- The transversality condition differs and applies to $T \to \infty$ rather than a finite T.

Transversality Condition for Infinite Horizons

- The transversality condition in this case becomes:

$$\lim_{t \to \infty} [\mu(t) \cdot k(t)] = 0$$

- This condition ensures that capital does not leave something valuable unused asymptotically.
  - If $k(t)$ remains positive asymptotically, the shadow price $\mu(t)$ must approach zero.
  - If $k(t)$ grows forever, $\mu(t)$ must decrease at a faster rate for the product $\mu(t) \cdot k(t)$ to approach zero.

Michel’s Transversality Condition

- Another condition proposed by Michel (1982) is that the value of the Hamiltonian must approach zero as

$$\lim_{t\to\infty}[H(t)] = 0$$

- This condition ensures that the Hamiltonian vanishes at the optimum for infinite horizon problems.

A.3.8 Example:
The Neoclassical Growth Mode

Cobb-Douglas Production Function and Infinite Horizon Optimization

Optimization Problem Setup

- Objective: Maximize utility over time:

$$U(0) = \int_0^\infty e^{-\rho t} \cdot \log[c(t)] \cdot dt$$

- Subject to the following constraints

$$(a)\quad \dot{k}(t) = [k(t)^\alpha - c(t) - \delta \cdot k(t)]$$

$$(b)\quad k(0) = 1$$

$$(c)\quad \lim_{t \to \infty} [k(t) \cdot e^{-\bar{r}(t) \cdot t}] \geq 0$$

Model Assumptions

- α is a constant such that $0<\alpha<1$.
- $r(t)$ represents the interest rate.
- $k(0)=1$: Initial capital normalized.
- The agent maximizes utility through consumption while accumulating capital.

Hamiltonian for the Optimization Problem

- The Hamiltonian is:

$$H(c, k, t, \mu) = e^{-\rho t} \cdot \log(c) + \mu \cdot (k^{\alpha} - c - \delta k)$$

The first-order conditions are:

$$H_c = e^{-\rho t} \cdot (1/c) - \mu = 0$$

$$H_k = \mu \cdot (\alpha k^{\alpha-1} - \delta) = -\dot{\mu}$$

Eliminating the Costate Variable μ

- From the first-order condition H_c , we relate μ to c:

$$-\rho - \dot{c}/c = \dot{\mu}/\mu$$

- Substitute this into H_k to eliminate μ

$$\dot{c}/c = (\alpha k^{\alpha-1} - \rho - \delta)$$

- This condition says that consumption accumulates at a rate equal to the difference between the net marginal product of capital, $\alpha k^{\alpha-1} - \delta$, and the discount rate, ρ.

Steady-State Conditions

- In the steady state:

$$\alpha k^{\alpha-1} = \rho + \delta$$

The steady-state capital stock is:

$$k^* = [(\rho + \delta)/\alpha]^{-1/(1-\alpha)}$$

The steady-state consumption is:

$$c^* = (k^*)^{\alpha} - \delta k^*$$

Transversality Condition for the Neoclassical Growth Model

- The transversality condition is expressed as:

$$\lim_{t \to \infty} [e^{-\rho t} \cdot k(t)] = 0$$

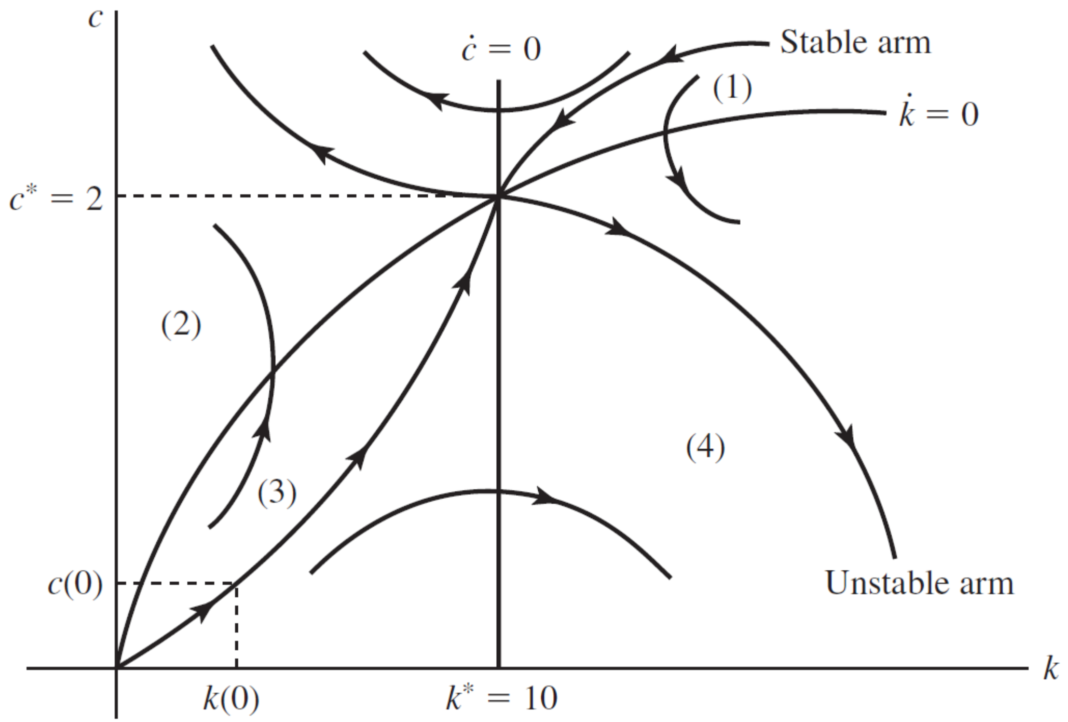
→provides a terminal condition, which, together with the initial condition $k(0) = 1$, yields the exact solution to the system of ODEs.

set ρ = 0.06, δ = 0, and α = 0.3

this system corresponds to the nonlinear system

$$\dot{k}(t) = k(t)^{0.3} - c(t)$$

$$\dot{c}(t) = c(t) \cdot [0.3 \cdot k(t)^{-0.7} - 0.06]$$



This system exhibits saddle-path stability, and the initial and terminal conditions ensure that the economy starts exactly on the stable arm

A.3.11 Present-Value and Current-Value Hamiltonians

Understanding the Different Forms of Hamiltonians

Present-Value Objective Function

- The typical objective function takes the form:

$$\int_0^T v[k(t), c(t), t] \cdot dt = \int_0^T e^{-\rho t} \cdot u[k(t), c(t)] \cdot dt$$

- Where ρ is the discount rate and $e^{-\rho t}$ is the discount factor.
- We can solve this by constructing the Hamiltonian.

Present-Value Hamiltonian Formulation

- The Hamiltonian is:

$$H = e^{-\rho t} \cdot u(k, c) + \mu \cdot g(k, c, t)$$

Here, $\mu(t)$ represents the value of the capital stock in units of time-zero utils.

Restructuring the Hamiltonian

- Sometimes it's useful to use current-value prices (in units of time-t utils):

$$H = e^{-\rho t} \cdot [u(k, c) + q(t) \cdot g(k, c, t)]$$

- where

$q(t) \equiv \mu(t) \cdot e^{\rho t}$ is the current-value shadow price

The **current-value Hamiltonian** is defined as:

$$\hat{H} \equiv u(k, c) + q(t) \cdot g(k, c, t)$$

First-Order Conditions for the Current-Value Hamiltonian

- The first-order conditions are:

$$\hat{H}_c = 0$$

$$\hat{H}_k = \rho q - \dot{q}$$

The transversality condition, $\mu(T) \cdot k(T) = 0$, can be expressed as

$$q(T) \cdot e^{-\rho T} \cdot k(T) = 0$$

Asset-Pricing Interpretation

- q represents the price of capital in terms of current utility.
- $\widehat{H}_k$ is the dividend (marginal contribution of capital to utility), and $\dot{q}$ is the capital gain.
- $\hat{H}_k = \rho q - \dot{q}$ at the optimum, the agent is indifferent between two types of investments because the total return to capital, $(\hat{H}_k + \dot{q})/q$ equals the return to consumption ρ.

A.3.12 Multiple Variables

Optimization with Multiple Control and State Variables

General Dynamic Problem with n Control and m State Variables

- Consider a dynamic problem with multiple variables:

$$\int_0^T u[k_1(t), \ldots, k_m(t); c_1(t), \ldots, c_n(t); t] \cdot dt$$

- subject to

$$\dot{k}_1(t) = g^1[k_1(t), \ldots, k_m(t), c_1(t), \ldots, c_n(t), t]$$

$$\dot{k}_2(t) = g^2[k_1(t), \ldots, k_m(t), c_1(t), \ldots, c_n(t), t]$$

$$\ldots$$

$$\dot{k}_m(t) = g^m[k_1(t), \ldots, k_m(t), c_1(t), \ldots, c_n(t), t]$$

$$k_1(0) > 0, \ldots, k_m(0) > 0, \quad \text{given}$$

$$k_1(T) \geq 0, \ldots, k_m(T) \geq 0, \quad \text{free}$$

Hamiltonian for Multiple Variables

- The Hamiltonian is given by:

$$H = u[k_1(t), \ldots, k_m(t); c_1(t), \ldots, c_n(t); t] + \sum_{i=1}^{m} \mu_i \cdot g^i(\bullet)$$

First-Order and transversality Conditions

$$\partial H / \partial c_i(t) = 0, \qquad i = 1, \ldots, n$$

$$\partial H / \partial k_i(t) = -\dot{\mu}_i, \qquad i = 1, \ldots, m$$

and the transversality conditions are

$$\mu_i(T) \cdot k_i(T) = 0, \qquad i = 1, \ldots, m$$

The structure remains similar to the case with one control and one state variable