Exchanging Limit and Continuous Function

Cai Yufei (zhmgczh@foxmail.com)

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Definition: $f : [a, b] \to \mathbb{R}$ is continuous at $x_0 \in (a, b)$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

.

Consider the definition carefully, and we will see that $\lim_{x \to x_0} f(x) = f(x_0)$ implies $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in [a, b]$ and $0 < |x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. Since we know $|f(x_0) - f(x_0)| = 0 < \epsilon$, so in fact we can write alternatively $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in [a, b]$ and $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. (Just remove the condition $|x - x_0| > 0$.)

Claim: If g is an arbitrary function and

$$\lim_{x \to x_0} g(x) = u_0 \in (a, b)$$

, and if $f : [a, b] \to \mathbb{R}$ is continuous at u_0, then

$$\lim_{x \to x_0} f(g(x)) = f(u_0) = f(\lim_{x \to x_0} g(x))$$

.

Proof: Since $\lim_{x \to x_0} g(x) = u_0$, by the definition of the limit of a function, $\forall \sigma > 0$, $\exists \delta > 0$ s.t. $\forall x \in [a, b]$ and $0 < |x - x_0| < \delta$, $|g(x) - u_0| < \sigma$.

Since f is continuous at u_0, $\forall \epsilon > 0$, $\exists \sigma > 0$ s.t. $\forall u \in [a, b]$ and $|u - u_0| < \sigma$, $|f(u) - f(u_0)| < \epsilon$.

We can substitute the first expression into the second expression ($u \to g(x)$) and get $\forall \epsilon > 0$, $\exists \sigma > 0$, $\exists \delta > 0$ s.t. $\forall x \in [a, b]$ and $0 < |x - x_0| < \delta$, $|g(x) - u_0| < \sigma$, $|f(g(x)) - f(u_0)| < \epsilon$.

Omit the variable σ because it is a intermediate variable, and we can get finally $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in [a, b]$ and $0 < |x - x_0| < \delta$, $|f(g(x)) - f(u_0)| < \epsilon$, which is the definition of $\lim_{x \to x_0} f(g(x)) = f(u_0)$.

So $\lim_{x \to x_0} f(g(x)) = f(u_0) = f(\lim_{x \to x_0} g(x))$.

End of proof.

In conclusion, it has been proved that we can actually exchange the limit and a continuous function when the function is continuous at some specific point.