Eigendecomposition and Linear Recurrence Relations

ACMS Teaching Seminars - Mock Lecture 2

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Definition 1.1 (Eigenvalues and Eigenvectors)

Suppose $A \in \mathbb{C}^{n\times n}$, and there exist $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$, $v \neq 0$ such that $Av = \lambda v$.
Then we call λ an eigenvalue of A, and v is a corresponding eigenvector of λ.

Definition 1.2 (Linear Independence)

A list of vectors $v_1, , v_2, \cdots, v_k \in \mathbb{C}^n$ *is linearly independent if*
$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ *(*$c_1, c_2, \cdots, c_k \in \mathbb{R}$*)* $\Leftrightarrow c_1 = c_2 = \cdots = c_k = 0$*.*

Review

Eigendecomposition

If $A \in \mathbb{C}^{n\times n}$ has n linearly independent eigenvectors $v_1, \cdots, v_n \in \mathbb{C}^n$ with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$, then

$$\begin{aligned} & A \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \\ &= \begin{pmatrix} Av_1 & Av_2 & \cdots & Av_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{pmatrix} \\ &= \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & v_n \end{pmatrix}, \end{aligned}$$

so $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^{-1} \triangleq S\Lambda S^{-1}$,

where $S = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \in \mathbb{C}^n$ and $\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

If $m \in \mathbb{Z}$ and $m > 0$, then

$$\begin{aligned}
A^m &= (S\Lambda S^{-1})^m \\
&= S\Lambda S^{-1} S\Lambda S^{-1} \cdots S\Lambda S^{-1} \\
&= S\Lambda\Lambda \cdots \Lambda S^{-1} \\
&= S\Lambda^m S^{-1} \\
&= \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^{-1}.
\end{aligned}$$

Definition 1.3 (Fibonacci Sequence)

The Fibonacci sequence $\{F_n\} \subset \mathbb{Z}$ *where* $n \in \mathbb{Z}$ *and* $n \geq 0$ *is defined as*

$$\begin{cases} F_0 = F_1 = 1, & \text{(Initial Values)}, \\ F_{n+1} = F_n + F_{n-1}, & \text{(Recurrence Relation)}. \end{cases}$$

Review

Recurrence Relations - Fibonacci Sequence

- $$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$
- $$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$
- $$\begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$
- $$\cdots$$

Review

Recurrence Relations - Fibonacci Sequence

If $n \geq 1$, then

$$\begin{aligned}
\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} \\
&= \cdots \\
&= \cdots \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}$$

which we can view as the explicit 2×2 matrix formula for the Fibonacci sequence.

Review

Recurrence Relations - Fibonacci Sequence

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = S\Lambda S^{-1}$$

$$= \begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \\ & \frac{1-\sqrt{5}}{2}\end{pmatrix}\left[\frac{1}{\sqrt{5}}\begin{pmatrix}1 & \frac{\sqrt{5}-1}{2}\\ 1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}\right],$$

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = S\Lambda^n S^{-1}$$

$$= \begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}(\frac{1+\sqrt{5}}{2})^n & \\ & (\frac{1-\sqrt{5}}{2})^n\end{pmatrix}\left[\frac{1}{\sqrt{5}}\begin{pmatrix}1 & \frac{\sqrt{5}-1}{2}\\ 1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}\right],$$

$$\begin{pmatrix}F_{n+1}\\ F_n\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}(\frac{1+\sqrt{5}}{2})^n & \\ & (\frac{1-\sqrt{5}}{2})^n\end{pmatrix}\left[\frac{1}{\sqrt{5}}\begin{pmatrix}1 & \frac{\sqrt{5}-1}{2}\\ 1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}\right]\begin{pmatrix}1\\ 1\end{pmatrix},$$

Review

Recurrence Relations - Fibonacci Sequence

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \\ \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right] \end{pmatrix}.$$

Then we get the explicit formula for the Fibonacci sequence as

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

when $n \geq 1$.

Definition 2.1 (General Linear Recurrence Relations)

A sequence $\{T_n\} \subseteq \mathbb{C}$ is generated from a general linear recurrence relation if there exist $k \in \mathbb{Z}, k \geq 0$ and $a_0, a_1, \cdots, a_k \in \mathbb{C}$ such that

$$\begin{cases} T_0, T_1, \cdots, T_k \text{ are known}, & \text{(Initial Values)}, \\ T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_k T_{n-k}, & \text{(Recurrence Relation)}. \end{cases}$$

- How can we derive an explicit formula for a general recurrence relation?

If $n \geq k$, then

$$
\begin{aligned}
\begin{pmatrix} T_{n+1} \\ T_n \\ T_{n-1} \\ T_{n-2} \\ \vdots \\ T_{n-k+2} \\ T_{n-k+1} \end{pmatrix}
&= \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} & a_k \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}_{(k+1)\times(k+1)}
\begin{pmatrix} T_n \\ T_{n-1} \\ T_{n-2} \\ T_{n-3} \\ \vdots \\ T_{n-k+1} \\ T_{n-k} \end{pmatrix} \\
&= \cdots \\
&= \cdots \\
&= \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} & a_k \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}^{n-k+1}_{(k+1)\times(k+1)}
\begin{pmatrix} T_k \\ T_{k-1} \\ T_{k-2} \\ T_{k-3} \\ \vdots \\ T_1 \\ T_0 \end{pmatrix}.
\end{aligned}
$$

But can this matrix be eigendecomposed? Does it have $k+1$ linearly independent eigenvectors?

Steps to calculate the eigenvalues and corresponding eigenvectors of a matrix $A \in \mathbb{C}^{n \times n}$:

1. Solve the equation $\det(A - \lambda I) = 0$ to get all distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$,
2. Solve the equations $(A - \lambda_i I)v = 0$ with respect to all distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, which is equivalent to find the bases for the null spaces of matrices $A - \lambda_1 I, A - \lambda_2 I, \cdots, A - \lambda_k I$, to get all linearly independent eigenvectors $v_1, v_2, \cdots, v_m$.

General Linear Recurrence Relations

Solution

$$\det(A-\lambda I) = \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}_{(k+1)\times(k+1)} \triangleq D_k \in \det \mathbb{C}^{(k+1)\times(k+1)}$$

Then

$$D_{k-1} = \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}_{k\times k} \in \det \mathbb{C}^{k\times k},$$

$$\cdots,$$

$$\cdots,$$

$$D_0 = \left|a_0-\lambda\right|_{1\times 1} = a_0-\lambda.$$

General Linear Recurrence Relations

Solution

Expand the determinant by the last row.

$$D_k = \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}_{(k+1)\times(k+1)}$$

$$= (-1)^{2k+1} \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}_{k\times k} + (-1)^{2k+2}(-\lambda) \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}_{k\times k}$$

Expand the determinant by the last row.

$$D_k = \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}_{(k+1)\times(k+1)}$$

$$= - \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}_{k\times k} - \lambda \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}_{k\times k}$$

Expand the first determinant by the last column.

$$D_k = -\begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}_{k\times k} - \lambda \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}_{k\times k}$$

$$= -(-1)^{k+1} a_k \begin{vmatrix} 1 & -\lambda & 0 & \cdots & 0 \\ 0 & 1 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}_{(k-1)\times(k-1)} - \lambda \begin{vmatrix} a_0-\lambda & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}_{k\times k}$$

$$= (-1)^{k+2} a_k - \lambda D_{k-1}$$

$$= (-1)^k a_k - \lambda D_{k-1}$$

General Linear Recurrence Relations

Solution

- $D_k = (-1)^k a_k - \lambda D_{k-1}$
- $D_{k-1} = (-1)^{k-1} a_{k-1} - \lambda D_{k-2}$
- $D_{k-2} = (-1)^{k-2} a_{k-2} - \lambda D_{k-3}$
- $D_{k-3} = (-1)^{k-3} a_{k-3} - \lambda D_{k-4}$
- $\cdots$
- $\cdots$
- $D_1 = (-1) a_1 - \lambda D_0$
- $D_0 = a_0 - \lambda$

General Linear Recurrence Relations

Solution

$$
\begin{aligned}
D_k &= (-1)^k a_k - \lambda D_{k-1} \\
&= (-1)^k a_k - \lambda \left[(-1)^{k-1} a_{k-1} - \lambda D_{k-2}\right] \\
&= (-1)^k \left[a_k + \lambda a_{k-1}\right] + \lambda^2 D_{k-2} \\
&= (-1)^k \left[a_k + \lambda a_{k-1}\right] + \lambda^2 \left[(-1)^{k-2} a_{k-2} - \lambda D_{k-3}\right] \\
&= (-1)^k \left[a_k + \lambda a_{k-1} + \lambda^2 a_{k-2}\right] - \lambda^3 D_{k-3} \\
&= \cdots \\
&= (-1)^k \left[a_k + \lambda a_{k-1} + \lambda^2 a_{k-2} + \cdots + \lambda^{m-1} a_{k-m+1}\right] + (-1)^m \lambda^m D_{k-m} \\
&= \cdots \\
&= (-1)^k \left[a_k + \lambda a_{k-1} + \lambda^2 a_{k-2} + \cdots + \lambda^{k-1} a_1\right] + (-1)^k \lambda^k D_0 \\
&= (-1)^k \left[a_k + \lambda a_{k-1} + \lambda^2 a_{k-2} + \cdots + \lambda^{k-1} a_1 + \lambda^k D_0\right] \\
&= (-1)^k \left[a_k + \lambda a_{k-1} + \lambda^2 a_{k-2} + \cdots + \lambda^{k-1} a_1 + \lambda^k (a_0 - \lambda)\right] \\
&= (-1)^k \left[-\lambda^{k+1} + a_0 \lambda^k + a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \cdots + a_{k-1} \lambda + a_k\right]
\end{aligned}
$$

If $\det(A - \lambda I) = D_k = 0$, then

$$\lambda^{k+1} - a_0\lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_{k-1}\lambda - a_k = 0.$$

So the eigenvalues of the matrix are the roots of the polynomial equation of order $k + 1$.
According to the fundamental theorem of algebra, the polynomial can be factorized as $(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{k+1})$, but $\lambda_1, \lambda_2, \cdots, \lambda_{k+1}$ can be duplicated.
We call this polynomial the “characteristic polynomial" of the linear recurrence relation

$$\begin{cases} T_0, T_1, \cdots, T_k \text{ are known,} & \text{(Initial Values),} \\ T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_k T_{n-k}, & \text{(Recurrence Relation).} \end{cases}$$

Theorem 1 (Sum of Eigenspaces is a Direct Sum)

Suppose $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then the bases of the null spaces of matrices $A - \lambda_1 I, A - \lambda_2 I, \cdots, A - \lambda_k I$ are linearly independent.

Theorem 2 (Conditions Equivalent to Diagonalizability)

Suppose $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. The following statements are equivalent:

- *A can be eigendecomposed.*
- *A has n linearly independent eigenvectors.*
- $\mathbb{C}^n = \mathrm{span}(N(A - \lambda_1 I), N(A - \lambda_2 I), \cdots, N(A - \lambda_k I))$, *where N denotes the null space of a matrix.*
- $n = \dim(N(A - \lambda_1 I)) + \dim(N(A - \lambda_2 I)) + \cdots + \dim(N(A - \lambda_k I))$.

The proof of the theorems can be found in Linear algebra done right (Axler, 2015) (Theorem 5.54 and Theorem 5.55), which is my favorite textbook on mathematics.

Thank you very much!

Questions or Comments?

REFERENCES I

Axler, S. (2015). ***Linear algebra done right*.** Springer.