Understanding Limiting Distribution of Markov Chains

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Suppose there is an irreducible, aperiodic and positive recurrent Markov chain $\{X_n\}$ with k states. Assume the stationary distribution of the Markov chain is $\boldsymbol{\pi} = \begin{pmatrix}\pi_1 & \pi_2 & ... & \pi_k\end{pmatrix}$.

Then we have known that $\lim_{n\to\infty} P_{ij}^n = \pi_j$ for $i, j = 1, 2, ..., k$.

The n-step transition matrix is given by

$$\boldsymbol{P}^n = \begin{pmatrix} P_{11}^n & P_{12}^n & \dots & P_{1k}^n \\ P_{21}^n & P_{22}^n & \dots & P_{2k}^n \\ \vdots & \vdots & \ddots & \vdots \\ P_{k1}^n & P_{k2}^n & \dots & P_{kk}^n \end{pmatrix}$$

.

Then

$$\boldsymbol{P}^\infty = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_k \\ \pi_1 & \pi_2 & \dots & \pi_k \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1 & \pi_2 & \dots & \pi_k \end{pmatrix} = \begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi} \\ \vdots \\ \boldsymbol{\pi} \end{pmatrix}$$

.

Now assume the initial state X_0 has the distribution $\boldsymbol{a} = \begin{pmatrix}a_1 & a_2 & ... & a_k\end{pmatrix}$ where $a_i \geq 0$, $\sum_{i=1}^k a_i = 1$ and $a_i = P(X_0 = i)$ for $i = 1, 2, ..., k$.

Then $\boldsymbol{aP}$ is the distribution of X_1 with $P(X_1 = i) = (\boldsymbol{aP})_i$, $i = 1, 2, ..., k$, $\boldsymbol{aP}^2$ is the distribution of X_2 with $P(X_2 = i) = (\boldsymbol{aP}^2)_i$, $i = 1, 2, ..., k$, ...

$\forall n \in \{1, 2, ...\}$, $\boldsymbol{aP}^n$ is the distribution of X_n with $P(X_n = i) = (\boldsymbol{aP}^n)_i$, $i = 1, 2, ..., k$.

As a result, $\boldsymbol{aP}^\infty$ is the distribution of X_n when n is sufficiently large. (Actually we can use $\epsilon - N$ language to describe it, but we do not need.)

Now, we want to figure out what $\boldsymbol{aP}^\infty$ is exactly and whether $\boldsymbol{aP}^\infty$ depends on $\boldsymbol{a}$ or not.

Actually,

$$
\begin{aligned}
\boldsymbol{a}\boldsymbol{P}^{\infty} &= \boldsymbol{a}\begin{pmatrix}\boldsymbol{\pi}\\ \boldsymbol{\pi}\\ \vdots\\ \boldsymbol{\pi}\end{pmatrix} && (1)\\
&= \begin{pmatrix}a_1 & a_2 & \dots & a_k\end{pmatrix}\begin{pmatrix}\boldsymbol{\pi}\\ \boldsymbol{\pi}\\ \vdots\\ \boldsymbol{\pi}\end{pmatrix} && (2)\\
&= \sum_{i=1}^{k} a_i\boldsymbol{\pi} && (3)\\
&= (\sum_{i=1}^{k} a_i)\boldsymbol{\pi} && (4)\\
&= \boldsymbol{\pi} && (5)
\end{aligned}
$$

.

So we can conclude that no matter what initial distribution from which a Markov chain starts, the limiting distribution of the r.v. X_n will never change and will always be the stationary distribution $\boldsymbol{\pi}$ when the Markov chain is irreducible, aperiodic and positive recurrent.

In other words, when n is sufficiently large, $P(X_n = i) \approx (\boldsymbol{a}\boldsymbol{P}^{\infty})_i = \pi_i$ (or $\lim_{n\to\infty} P(X_n = i) = (\boldsymbol{a}\boldsymbol{P}^{\infty})_i = \pi_i$) for $i = 1, 2, ..., k$.

This conclusion depends on the fact that $\lim_{n\to\infty} P_{ij}^n = \pi_j$.