A Gridless Approach to the Histogram Filter

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Abstract

We present an analytical approach to the correct operation of a histogram filter, where measured points are matched against straight lines in a map. If both the initial state and the modeled measurement uncertainty are Gaussians with arbitrary covariance matrices, then the corrected state is given as a linear combination of Gaussians. This state can be approximated as one or more Gaussian states by calculating its moments. It is shown that, for each point-line pair, the six independent components of the first three moments can be expressed in terms of three one-dimensional integrals, which have analytical solutions in terms of the error function. An efficient and numerically stable algorithm is derived which computes the correct operation for each point-line pair in constant time, independent of the length of the line and the state and measurement covariances. For the case where the map contains relatively few distinct lines, the presented algorithm can outperform alternative approaches by an order of magnitude, using constant and negligibly small memory.

I. GRIDLESS CORRECT

We calculate the moments of the product of a Gaussian and a line blurred by a Gaussian kernel. We choose a coordinate system where the first Gaussian is centered. By virtue of the fact that the product of two Gaussians is a scaled Gaussian, the resulting probability distribution can be expressed as

$$p(\mathbf{x}) = G(\mathbf{0}, \mathbf{\Sigma}_1) \int d\ell\, G(\boldsymbol{\mu}_2(\ell), \mathbf{\Sigma}_2) = \int d\ell\, e^{-\frac{1}{2}Q(\ell)} G(\boldsymbol{\mu}_{12}(\ell), \mathbf{\Sigma}_{12}), \tag{1}$$

where

$$\boldsymbol{\mu}_{12}(\ell) \equiv \mathbf{\Sigma}_{12} \cdot \mathbf{\Sigma}_2^{-1} \cdot \boldsymbol{\mu}_2(\ell) \equiv \mathbf{B} \cdot \boldsymbol{\mu}_2(\ell), \tag{2}$$

$$\mathbf{\Sigma}_{12}^{-1} \equiv \mathbf{\Sigma}_1^{-1} + \mathbf{\Sigma}_2^{-1}, \tag{3}$$

$$Q(\ell) \equiv \boldsymbol{\mu}_2^T(\ell) \cdot \mathbf{\Sigma}_2^{-1} \cdot \boldsymbol{\mu}_2(\ell) - \boldsymbol{\mu}_{12}^T(\ell) \cdot \mathbf{\Sigma}_{12}^{-1} \cdot \boldsymbol{\mu}_{12}(\ell) \equiv \boldsymbol{\mu}_{12}^T(\ell) \cdot \mathbf{A} \cdot \boldsymbol{\mu}_{12}(\ell), \tag{4}$$

where

$$\mathbf{A} \equiv (\mathbf{\Sigma}_{12}^{-1} \cdot \mathbf{\Sigma}_2 - \mathbf{I}) \cdot \mathbf{\Sigma}_{12}^{-1} = (\mathbf{\Sigma}_1^{-1} \cdot \mathbf{\Sigma}_2 + \mathbf{I}) \cdot \mathbf{\Sigma}_1^{-1}. \tag{5}$$

is a constant positive definite symmetric matrix. The moments are

$$N \equiv \int d^2x\, p(\mathbf{x}) = N_{12} \int d\ell\, e^{-\frac{1}{2}Q(\ell)}, \tag{6}$$

$$N\boldsymbol{\mu} \equiv \int d^2x\, \mathbf{x}\, p(\mathbf{x}) = N_{12} \int d\ell\, e^{-\frac{1}{2}Q(\ell)} \boldsymbol{\mu}_{12}(\ell), \tag{7}$$

$$N(\mathbf{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T) \equiv \int d^2x\, \mathbf{x}\mathbf{x}^T p(\mathbf{x}) = N\mathbf{\Sigma}_{12} + N_{12} \int d\ell\, e^{-\frac{1}{2}Q(\ell)} \boldsymbol{\mu}_{12}(\ell)\boldsymbol{\mu}_{12}^T(\ell), \tag{8}$$

where

$$N_{12} \equiv \int d^2x\, G(\boldsymbol{\mu}_{12}(\ell), \mathbf{\Sigma}_{12}), \tag{9}$$

$$N_{12}\boldsymbol{\mu}_{12}(\ell) \equiv \int d^2x\, \mathbf{x}\, G(\boldsymbol{\mu}_{12}(\ell), \mathbf{\Sigma}_{12}), \tag{10}$$

$$N_{12}(\mathbf{\Sigma}_{12} + \boldsymbol{\mu}_{12}(\ell)\boldsymbol{\mu}_{12}^T(\ell)) \equiv \int d^2x\, \mathbf{x}\mathbf{x}^T G(\boldsymbol{\mu}_{12}(\ell), \mathbf{\Sigma}_{12}). \tag{11}$$

If the original line was parametrized as

$$\boldsymbol{\mu}_2(\ell) = \mathbf{u}_0 + \ell\mathbf{v}_0, \tag{12}$$

we have by linearity

$$\boldsymbol{\mu}_{12}(\ell) = \mathbf{u} + \ell\mathbf{v}, \tag{13}$$

where

$$\mathbf{u} \equiv \mathbf{B} \cdot \mathbf{u}_0, \tag{14}$$

$$\mathbf{v} \equiv \mathbf{B} \cdot \mathbf{v}_0. \tag{15}$$

After some transformations we have

$$N = K_0, \tag{16}$$

$$N(\boldsymbol{\mu} - \mathbf{u}) = N_{12} K_1 \mathbf{v}, \tag{17}$$

$$N(\boldsymbol{\Sigma} + (\boldsymbol{\mu} - \mathbf{u})(\boldsymbol{\mu} - \mathbf{u})^T) = N\boldsymbol{\Sigma}_{12} + N_{12} K_2 \mathbf{v}\mathbf{v}^T, \tag{18}$$

where

$$K_n \equiv \int dx\, x^n e^{-\frac{1}{2}Q(x)} \tag{19}$$

are integrals left to solve. We express

$$Q(x) = ax^2 + 2bx + c = t^2 + c - d^2, \tag{20}$$

where

$$a \equiv \mathbf{v}^T \cdot \mathbf{A} \cdot \mathbf{v}, \tag{21}$$

$$b \equiv \mathbf{u}^T \cdot \mathbf{A} \cdot \mathbf{v}, \tag{22}$$

$$c \equiv \mathbf{u}^T \cdot \mathbf{A} \cdot \mathbf{u}, \tag{23}$$

$$d \equiv \frac{b}{\sqrt{a}}, \tag{24}$$

$$t \equiv \sqrt{a}\, x + d. \tag{25}$$

Using the elementary integrals

$$I_0 \equiv \int dx\, e^{-\frac{1}{2}x^2} = \sqrt{\frac{\pi}{2}}\, \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right), \tag{26}$$

$$I_1 \equiv \int dx\, x\, e^{-\frac{1}{2}x^2} = -e^{-\frac{1}{2}x^2}, \tag{27}$$

$$I_2 \equiv \int dx (x^2 - 1) e^{-\frac{1}{2}x^2} = x I_1, \tag{28}$$

we finally have

$$K_0 = a^{-\frac{1}{2}} f I_0, \tag{29}$$

$$K_1 = a^{-1} f (I_1 - d I_0), \tag{30}$$

$$K_2 = a^{-\frac{3}{2}} f (I_2 - 2d I_1 + (d^2 + 1) I_0), \tag{31}$$

where

$$f \equiv e^{-\frac{1}{2}(c-d^2)}. \tag{32}$$