Lecture 2
Cellular automata

28.10.2020

Definition

A dynamical system is a dynamic model that describes the evolution of a real-life system according to the specified rules.

example: evolutionary differential equations

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A discrete dynamical system - a dynamical system which operates in discrete time steps; it can be described as a set of numbers and rules of evolution in the following way:

$$\begin{array}{ll} X(t) = (x_1(t), \ldots, x_n(t), \ldots) & \textit{state variables} \\ X(t+1) = F(X(t)) & \textit{rules of evolution} \end{array} \quad (1)$$

State space - the set of all possible values of the state variables.

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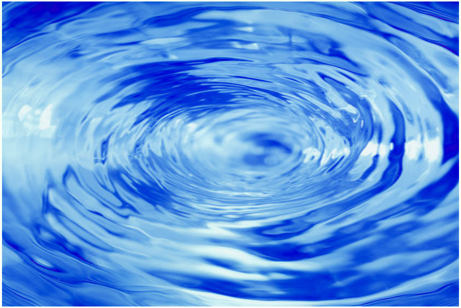
cellular automaton: discretized time and space

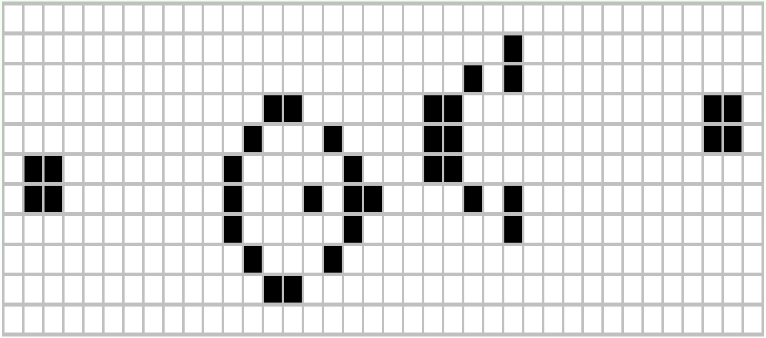
Cellular automata (CA) constitute a separate field of study.

1950s - Stanisław Ulam and John von Neumann applied CA to emulate microscopic behaviour of molecules in a fluid.



"The Game of Life"

The most popular CA - J.Conway's "Game of Life"
(a simple model of evolution of a colony of organisms).



Features of a cellular automaton

Cellular automata can differ by:

- number of states a cell can admit
- dimension and shape of a grid
- neighborhood of a cell
- boundary conditions
- rules of evolution (transition rules)
- initial configuration- configuration of the grid for $t = 0$ (initial generation).

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Transition rules are a function of

- cell state
- cell states in the neighborhood

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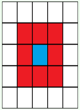
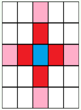
- Moore's neighborhood 
- von Neumann's neighborhood 

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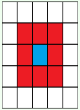
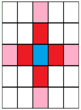
- Moore's neighborhood 
- von Neumann's neighborhood 

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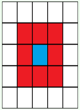
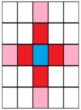
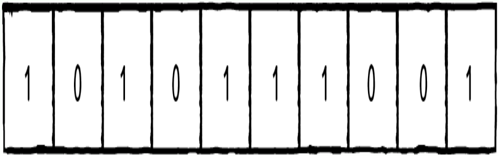
- Moore's neighborhood 
- von Neumann's neighborhood 

Elementary cellular automaton

Comprehensive studies regarding CA have been performed by S.Wolfram (creator of ”Mathematica” application) starting from 1980s and resulted in a book ”A New Kind of Science” (Wolfram, 2002).

Definition

Elementary cellular automaton is a one-dimensional binary (2-state) nearest-neighbor automaton.



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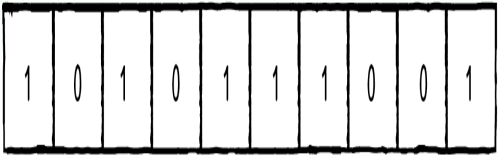

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Wolfram's codes of elementary CA

Two states (0-white 1-black), **2-cell** neighborhood.
There are $8 = 2^3$ possible configurations of a neighborhood of the cell

111 110 101 100 011 010 001 000

Each of them can result in 2 possible states of the middle cell in the next generation, so there are

$$2^{2^3} = 256$$

different rules.

A rule $d_7 d_6 d_5 d_4 d_3 d_2 d_1 d_0$ in a binary code ($d_0, ..., d_7 \in \{0, 1\}$) is transformed into its decimal representation (Rule#k):

$$k = d_7 2^7 + ... + d_1 2^1 + d_0 2^0.$$

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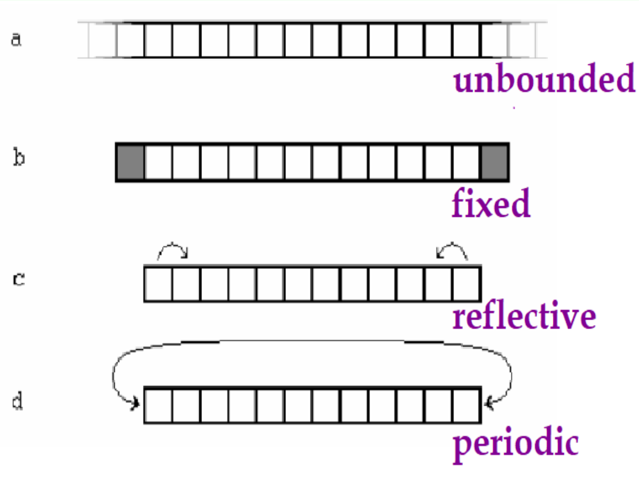
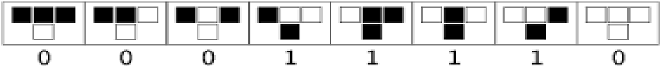
Different boundary conditions for elementary CA

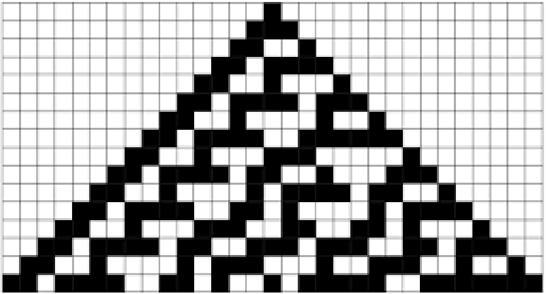


"Rule 30", spatio-temporal plot

Example of a cellular automaton: "rule 30"



Evolution starting from 1 black cell:



CA can generate random numbers

Although CA are deterministic, some of them have suitable properties for generating random numbers.

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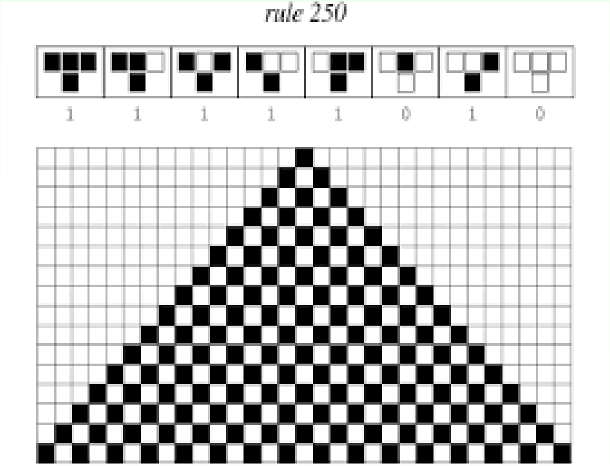
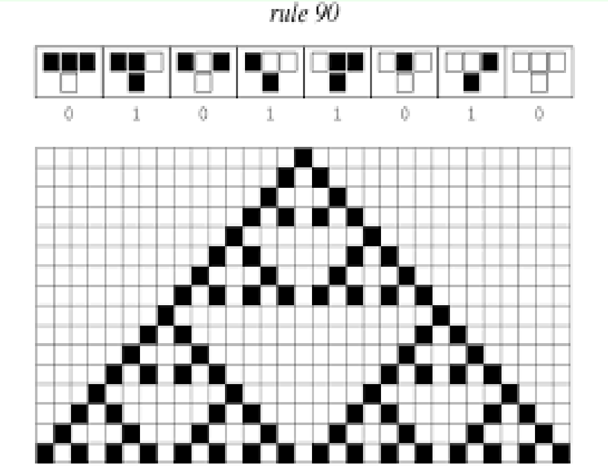
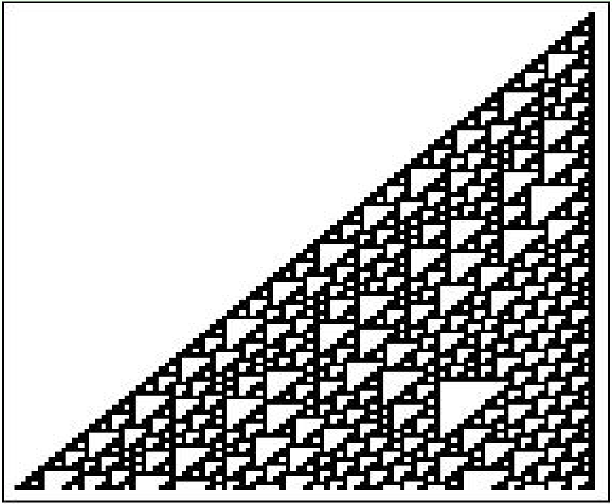
"Rule 250", spatio-temporal plot



"Rule 90", spatio-temporal plot



”Rule 110”, spatio-temporal plot



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Totalistic cellular automaton - its rules depend only on the total (or - equivalently - the average) of the values of the cell and its neigbors.

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One dimensional totalistic automaton

For a (one-dimensional, nearest neighbor) k-state totalistic automaton there are $3k-2$ possible values of the total of three cells.

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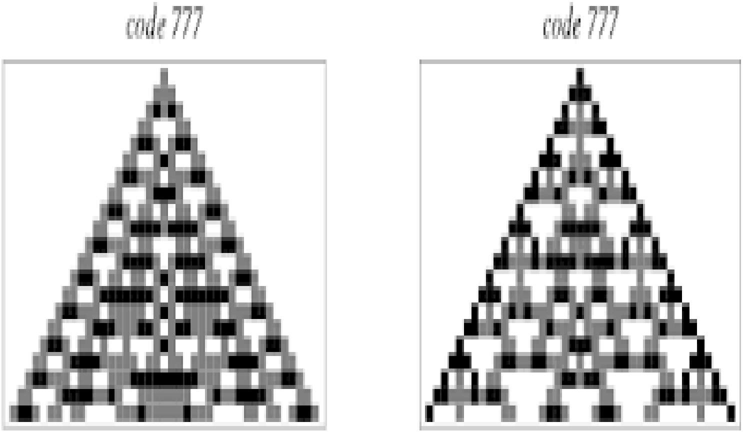
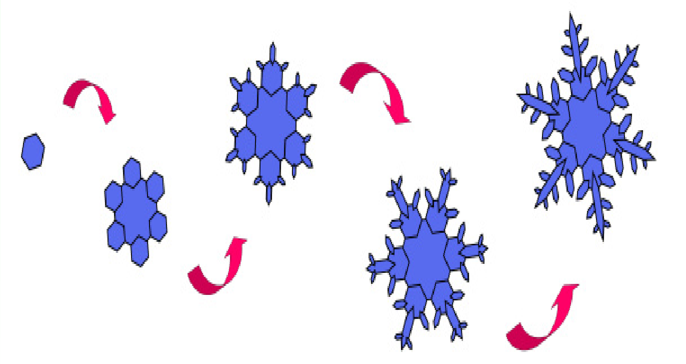
3-state 1-dimensional totalistic automaton: code777



Applications of CA

Cellular automata model a great variety of phenomena, like

- crystal growth



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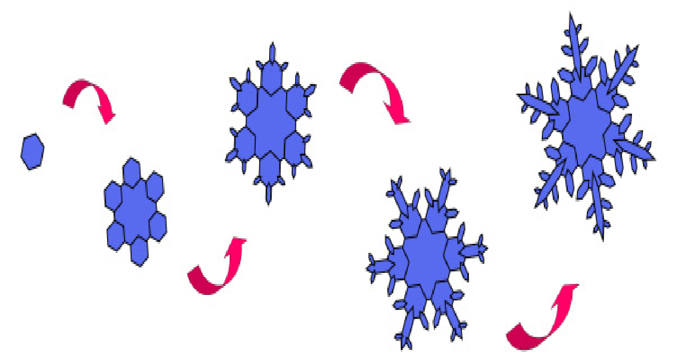
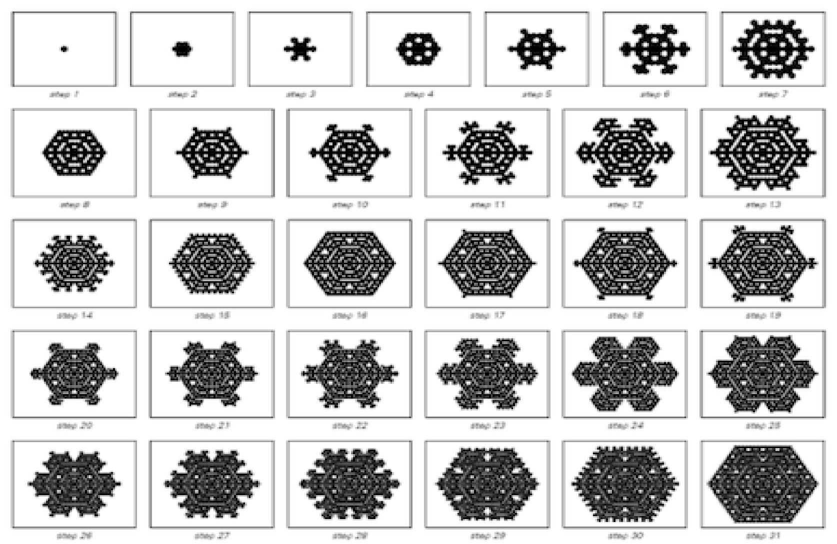
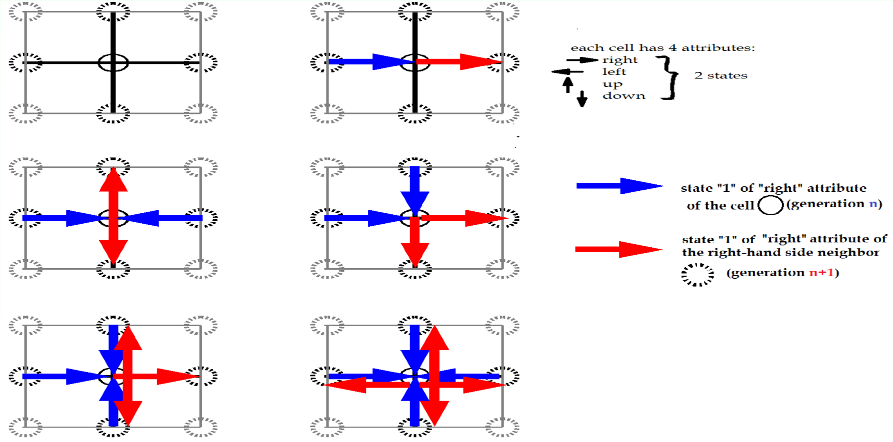




Some applications of CA

- fluid flow



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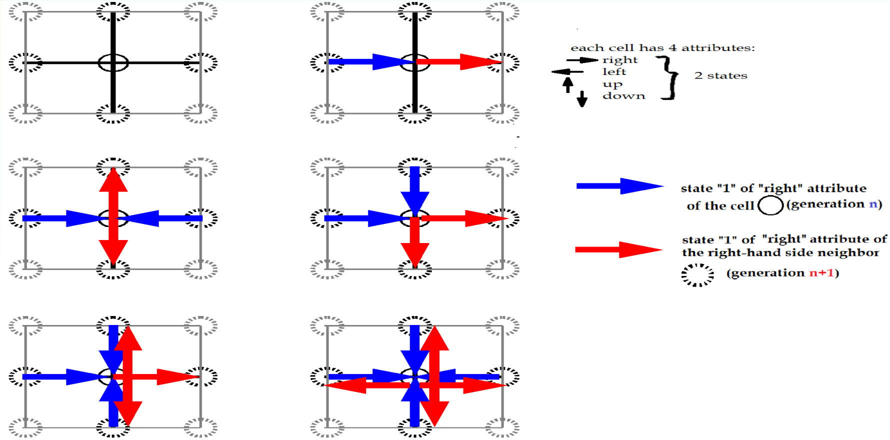

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- motion of loose substances
- breaking materials
- growth of plants
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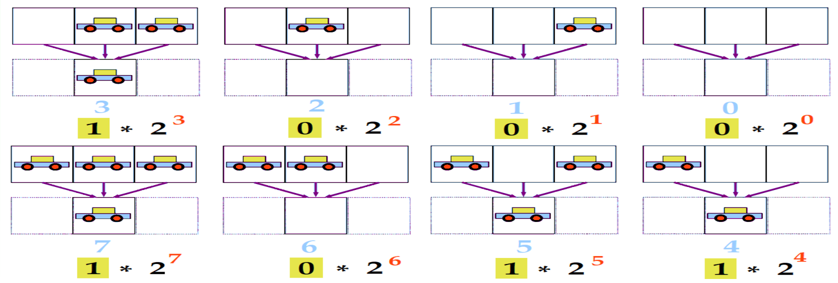
- traffic jams

Simplifying assumptions:

- automobiles are placed in a single lane of traffic
- they move to the right
- in each unit of time each car either moves to the right-hand side cell or stays in the current cell
- a car moves only if the right-hand side cell is empty



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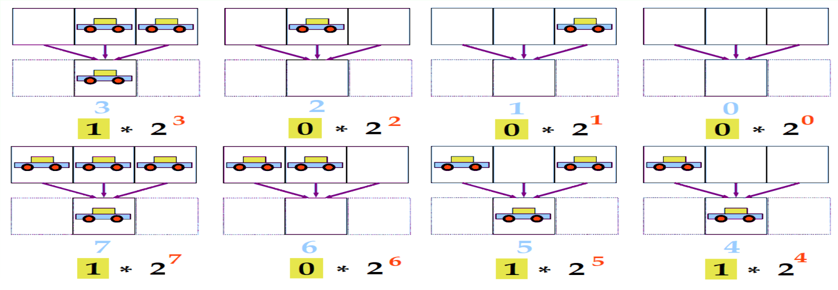

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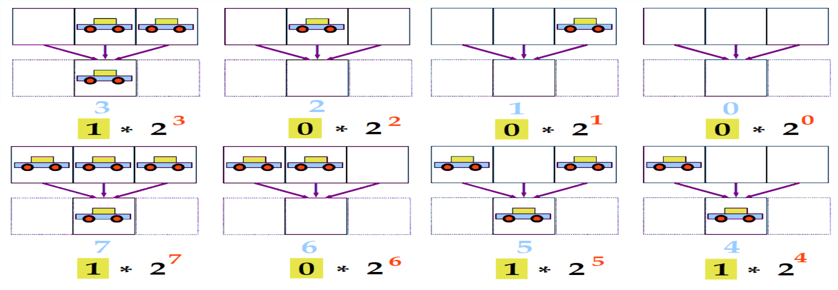

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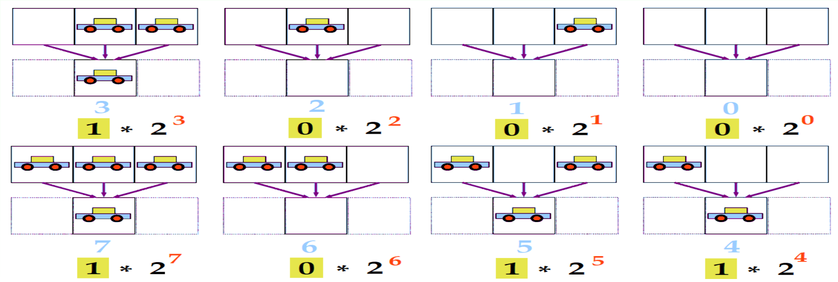

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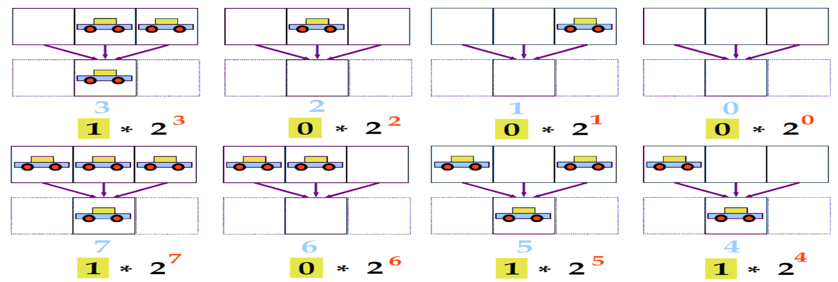

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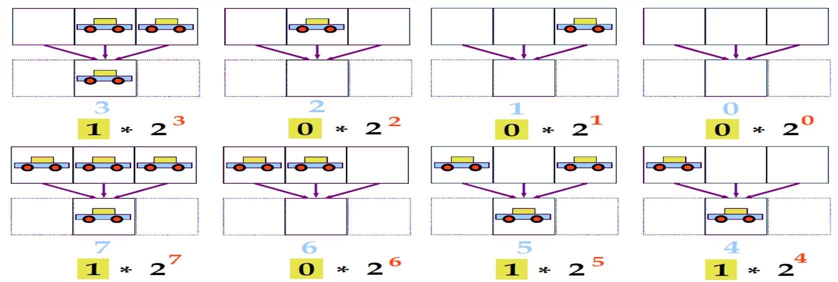

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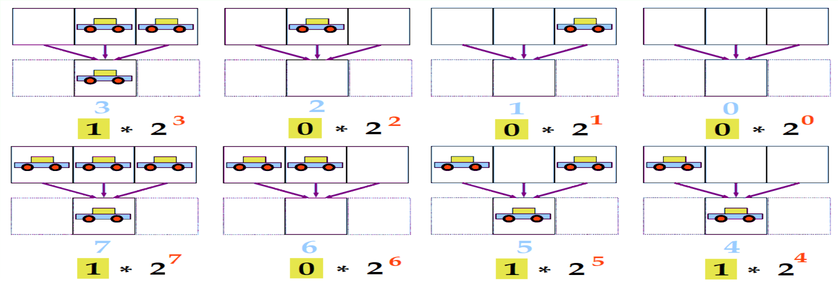

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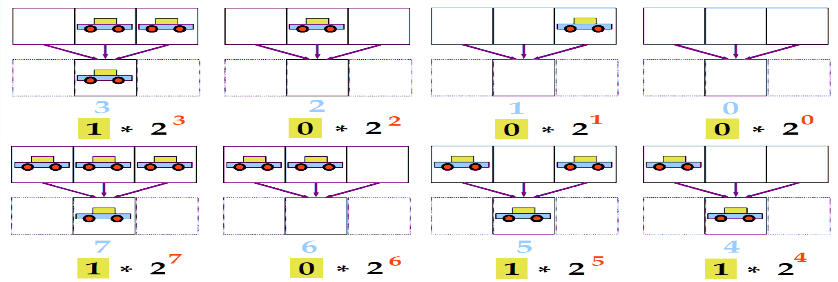

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