Generalized Symmetric Nonnegative Latent Factor Analysis for Large-scale Undirected Weighted Networks

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This is the supplementary file for the paper entitled Generalized Symmetric Nonnegative Latent Factor Analysis for Large-scale Undirected Weighted Networks. Additional convergence proofs of the proposed GSNL model are put into this file.

1 Proofs of a GSNL Model

1.1 Proof of *Lemma* 1

With Definition 3, the following inequality is given as,

$$\varphi\left(c_{(q)}^{i}\right)=\varepsilon\left(c_{(q)}^{i},c_{(q)}^{i}\right)\geq\varepsilon\left(c_{(q)}^{i+1},c_{(q)}^{i}\right)\geq\varphi\left(c_{(q)}^{i+1}\right) \tag{S1}$$

Then, it is easy to deduce the following sequence based on (S1),

$$\varphi\left(c_{\min(q)}\right)\leq\cdots\leq\varphi\left(c_{(q)}^{i+1}\right)\leq\varphi\left(c_{(q)}^{i}\right)\leq\cdots\leq\varphi\left(c_{(q)}^{1}\right)\leq\varphi\left(c_{(q)}^{0}\right) \tag{S2}$$

Considering $m\in J$, $d\in\{1\sim D\}$, $c_{m,d(q)}\in C$, let $\varphi_{c_{m,d(q)}}$ be the partial loss from $L(C)$ related to $c_{m,d(q)}$ only,

$$\varphi_{c_{m,d(q)}}\left(c_{m,d(q)}^{i}\right)\Big|_{s_{(q)}=s_{(q)}^{i}}=\sum_{y_{m,n}\in\Lambda}\left(\begin{aligned}&-\frac{1}{\alpha_{(q)}^{i}\beta_{(q)}^{i}}\left(\left(y_{m,n}\right)^{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\beta_{(q)}^{i}}-\frac{\alpha_{(q)}^{i}}{\alpha_{(q)}^{i}+\beta_{(q)}^{i}}\left(y_{m,n}\right)^{\alpha_{(q)}^{i}+\beta_{(q)}^{i}}\right.\\&\left.-\frac{\beta_{(q)}^{i}}{\alpha_{(q)}^{i}+\beta_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\alpha_{(q)}^{i}+\beta_{(q)}^{i}}\right)+\frac{\lambda_{(q)}^{i}}{2}\sum_{d=1}^{D}\left(c_{m,d(q)}^{i}\right)^{2}\end{aligned}\right) \tag{S3}$$

where $\tilde{y}_{m,n(q)}^{i}=c_{m,d(q)}^{i}c_{n,d(q)}+\sum_{l=1,\neq d}^{D}c_{m,l(q)}c_{n,l(q)}$.

Hence, the first-order and second-order derivatives of $\varphi_{c_{m,d(q)}}$ with respect to $c_{m,d(q)}$,

$$\begin{aligned}&\varphi'_{c_{m,d(q)}}\left(c_{m,d(q)}^{i}\right)\Big|_{s_{(q)}=s_{(q)}^{i}}=\frac{\partial L}{\partial c_{m,d(q)}}\left(c_{m,d(q)}^{i}\right)\Bigg|_{s_{(q)}=s_{(q)}^{i}}\\&=\lambda_{(q)}^{i}\left|\Lambda(m)\right|c_{m,d(q)}^{i}+\frac{1}{\alpha_{(q)}^{i}}\sum_{n\in\Lambda(m)}c_{n,d(q)}\left(\left(\tilde{y}_{m,n(q)}^{i}\right)^{\alpha_{(q)}^{i}+\beta_{(q)}^{i}-1}-\left(y_{m,n}\right)^{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\beta_{(q)}^{i}-1}\right)\end{aligned} \tag{S4}$$

$$\varphi''_{c_{m,d(q)}}\left(c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}} = \frac{\partial^2 L}{\partial\left(c_{m,d(q)}\right)^2}\left(c^i_{m,d(q)}\right)\Bigg|_{s_{(q)}=s^i_{(q)}}$$
$$= \lambda^i_{(q)}\left|\Lambda(m)\right| + \frac{1}{\alpha^i_{(q)}}\sum_{n\in\Lambda(m)}\left(c_{n,d(q)}\right)^2\left(\frac{1}{\alpha^i_{(q)}+\beta^i_{(q)}-1}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}+\beta^i_{(q)}-2} - \frac{\left(y_{m,n}\right)^{\alpha^i_{(q)}}}{\beta^i_{(q)}-1}\left(\tilde{y}^i_{m,n(q)}\right)^{\beta^i_{(q)}-2}\right) \tag{S5}$$

1.2 Proof of *Proposition* 1

Based on (16), $\varepsilon(c,c)=\varphi_{c_{m,d(q)}}(c)$ holds. Then we focus on the proofs of $\varepsilon\left(c,c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}} \geq \varphi_{c_{m,d(q)}}(c)\Big|_{s_{(q)}=s^i_{(q)}}$.

Firstly, we derive the quadratic approximation to $\varphi_{c_{m,d(q)}}$ at $c^i_{m,d(q)}$ under $s_{(q)}=s^i_{(q)}$,

$$\varphi_{c_{m,d(q)}}(c)\Big|_{s_{(q)}=s^i_{(q)}} =$$
$$\varphi_{c_{m,d(q)}}\left(c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}} + \varphi'_{c_{m,d(q)}}\left(c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}}\left(c-c^i_{m,d(q)}\right) + \frac{1}{2}\varphi''_{c_{m,d(q)}}\left(c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}}\left(c-c^i_{m,d(q)}\right)^2 \tag{S6}$$

By combining (16), (S5) and (S6), we see that $\varepsilon\left(c,c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}}$ is an auxiliary function of $\varphi_{c_{m,d(q)}}(c)\Big|_{s_{(q)}=s^i_{(q)}}$ if the following inequality holds,

$$c^i_{m,d(q)}\sum_{n\in\Lambda(m)}\left(c_{n,d(q)}\right)^2\left(\tilde{y}^i_{m,n(q)}\right)^{\beta^i_{(q)}-2}\left(\frac{1}{\beta^i_{(q)}-1}\left(y_{m,n}\right)^{\alpha^i_{(q)}} + \frac{\beta^i_{(q)}-1}{\alpha^i_{(q)}\left(\alpha^i_{(q)}+\beta^i_{(q)}-1\right)}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}}\right)\geq 0$$
$$\Rightarrow \frac{1}{\beta^i_{(q)}-1}\left(y_{m,n}\right)^{\alpha^i_{(q)}} + \frac{\beta^i_{(q)}-1}{\alpha^i_{(q)}\left(\alpha^i_{(q)}+\beta^i_{(q)}-1\right)}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}} \geq 0 \tag{S7}$$
$$\Rightarrow \frac{\beta^i_{(q)}-1}{\alpha^i_{(q)}\left(\alpha^i_{(q)}+\beta^i_{(q)}-1\right)}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}} \geq \frac{1}{1-\beta^i_{(q)}}\left(y_{m,n}\right)^{\alpha^i_{(q)}}$$

1.3 Proof of *Theorem* 1

Based on (15), (16) and (S4), we have,

$$c^{i+1}_{m,d(q)} = \arg\min_c \varepsilon\left(c,c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}}$$
$$\Rightarrow \varphi'_{c_{m,d(q)}}\left(c^i_{m,d(q)}\right) + \frac{\lambda^i_{(q)}\left|\Lambda(m)\right|c^i_{m,d(q)} + \frac{1}{\alpha^i_{(q)}}\sum_{n\in\Lambda(m)}c_{n,d(q)}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}+\beta^i_{(q)}-1}}{c^i_{m,d(q)}}\left(c-c^i_{m,d(q)}\right)=0 \tag{S8}$$
$$\Rightarrow c^{i+1}_{m,d(q)} \leftarrow c^i_{m,d(q)}\frac{\sum_{n\in\Lambda(m)}\left(y_{m,n}\right)^{\alpha^i_{(q)}}\left(\tilde{y}^i_{m,n(q)}\right)^{\beta^i_{(q)}-1}c_{n,d(q)}}{\sum_{n\in\Lambda(m)}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}+\beta^i_{(q)}-1}c_{n,d(q)} + \alpha^i_{(q)}\lambda^i_{(q)}\left|\Lambda(m)\right|c^i_{m,d(q)}}$$

Based on (S8), it is clear that $\varphi_{c_{m,d(q)}}$ is non-increasing with (13). Hence, *Theorem* 1 holds.

Following Theorem 1, if the following condition is fulfilled,

$$\left(\alpha^i_{(q)}+\beta^i_{(q)}\right)\left(y_{m,n}\right)^{\alpha^i_{(q)}}\left(\tilde{y}^i_{m,n(q)}\right)^{\beta^i_{(q)}} \leq \alpha\left(y_{m,n}\right)^{\alpha^i_{(q)}+\beta^i_{(q)}} + \beta^i_{(q)}\left(\tilde{y}^i_{m,n(q)}\right)^{\alpha^i_{(q)}+\beta^i_{(q)}} \tag{S9}$$

then $\varphi_{a_{m,d(q)}}\left(c^i_{m,d(q)}\right)\Big|_{s_{(q)}=s^i_{(q)}} \geq 0$ holds. Hence, it is easy to deduce that,

$$\begin{aligned}&\lim_{i\to+\infty}\left(\varphi_{c_{m,d(q)}}\left(c_{m,d(q)}^{i+1}\right)\Big|_{s_{(q)}=s_{(q)}^{i+1}}-\varphi_{c_{m,d(q)}}\left(c_{m,d(q)}^{i}\right)\Big|_{s_{(q)}=s_{(q)}^{i}}\right)\to 0\\&\Rightarrow\\&\lim_{i\to+\infty}\left(c_{m,d(q)}^{i+1}-c_{m,d(q)}^{i}\right)\to 0\end{aligned}\tag{S10}$$

Hence, a sequence $\left\{c_{m,d(q)}^{i}\right\}$ is bounded.

1.4 Proof of *Theorem* 2

Let $C_{(q)}^{\#}$ denote a stationary point of $C_{(q)}$. Then, the following KKT conditions of (3) regarding $C_{(q)}$ should be satisfied, if $C_{(q)}^{\#}$ is one of its stationary point.

$$\begin{aligned}&\forall m\in J, d\in\{1\sim D\}:\\&(a)\frac{\partial F}{\partial c_{m,d(q)}}\bigg|_{c_{m,d(q)}=c_{m,d(q)}^{\#},s_{(q)}=s_{(q)}^{i}}=\sum_{n\in\Lambda(m)}\left(\begin{aligned}&\frac{1}{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\alpha_{(q)}^{i}+\beta_{(q)}^{i}-1}c_{n,d(q)}+\lambda_{(q)}^{i}\left|\Lambda(m)\right|c_{m,d(q)}^{\#}\\&-\frac{1}{\alpha_{(q)}^{i}}\left(y_{m,n}\right)^{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\beta_{(q)}^{i}-1}c_{n,d(q)}\end{aligned}\right)-\kappa_{m,d(q)}^{\#}=0,\\&(b)\kappa_{m,d(q)}^{\#}c_{m,d(q)}^{\#}=0,\\&(c)c_{m,d(q)}^{\#}\geq 0,\\&(d)\kappa_{m,d(q)}^{\#}\geq 0.\end{aligned}\tag{S11}$$

Note that following (4)-(7), Conditions (S11a) and (S11b) are naturally fulfilled with (13). Then we have,

$$\kappa_{m,d(q)}^{\#}=\sum_{n\in\Lambda(m)}\left(\begin{aligned}&\frac{1}{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\alpha_{(q)}^{i}+\beta_{(q)}^{i}-1}c_{n,d(q)}+\lambda_{(q)}^{i}\left|\Lambda(m)\right|c_{m,d(q)}^{\#}\\&-\frac{1}{\alpha_{(q)}^{i}}\left(y_{m,n}\right)^{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\beta_{(q)}^{i}-1}c_{n,d(q)}\end{aligned}\right)\tag{S12}$$

Thus, we focus on Conditions (S11c)-(S11d). We start with constructing $h_{m,d(q)}^{i}$,

$$h_{m,d(q)}^{i}=\frac{\sum_{n\in\Lambda(m)}\left(y_{m,n}\right)^{\alpha_{(q)}^{i}}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\beta_{(q)}^{i}-1}c_{n,d(q)}}{\sum_{n\in\Lambda(m)}\left(\tilde{y}_{m,n(q)}^{i}\right)^{\alpha_{(q)}^{i}+\beta_{(q)}^{i}-1}c_{n,d(q)}+\alpha_{(q)}^{i}\lambda_{(q)}^{i}\left|\Lambda(m)\right|c_{m,d(q)}^{i}}\tag{S13}$$

From (S13), we can clearly see that $\lim_{i\to+\infty}h_{m,d(q)}^{i}=h_{m,d(q)}^{\#}\geq 0$.

Then we can rewrite (13) as follows,

$$c_{m,d(q)}^{i+1}=c_{m,d(q)}^{i}h_{m,d(q)}^{i}\Rightarrow c_{m,d(q)}^{\#}h_{m,d(q)}^{\#}-c_{m,d(q)}^{\#}=0\tag{S14}$$

Note that following (13), $c_{m,d(q)}^{\#}\geq 0$ with a non-negatively initial hypothesis. Hence, we have the following inferences,

(1) **When** $c_{m,d(q)}^{\#}>0$. Based on (S13) and (S14), we have,

$$
\begin{aligned}
& c^{\#}_{m,d(q)} h^{\#}_{m,d(q)} - c^{\#}_{m,d(q)} = 0, c^{\#}_{m,d(q)} > 0 \\
& \Rightarrow h^{\#}_{m,d(q)} = 1 \\
& \Rightarrow \sum_{n\in\Lambda(m)} \left(\tilde{y}^{i}_{m,n(q)}\right)^{\alpha^{i}_{(q)}+\beta^{i}_{(q)}-1} c_{n,d(q)} + \alpha^{i}_{(q)}\lambda^{i}_{(q)} \left|\Lambda(m)\right| c^{\#}_{m,d(q)} \\
& \quad - \sum_{n\in\Lambda(m)} \left(y_{m,n}\right)^{\alpha^{i}_{(q)}} \left(\tilde{y}^{i}_{m,n(q)}\right)^{\beta^{i}_{(q)}-1} c_{n,d(q)} = 0
\end{aligned} \tag{S15}
$$

Combining (S12) and (S14), we can achieve $\kappa^{\#}_{m,d(q)} = 0$.

(2) **When** $c^{\#}_{m,d(q)} = 0$. We reformulate $c^{\#}_{m,d(q)}$ into,

$$
c^{\#}_{m,d(q)} = c^{0}_{m,d(q)} \lim_{i\to+\infty} \prod_{z=0}^{i} h^{z}_{m,d(q)} \tag{S16}
$$

Based on (S16), we further have the following inferences,

$$
\begin{aligned}
& c^{0}_{m,d(q)} > 0, c^{0}_{m,d(q)} \lim_{i\to+\infty} \prod_{z=0}^{i} h^{z}_{m,d(q)} = c^{\#}_{m,d(q)} = 0 \\
& \Rightarrow \lim_{i\to+\infty} \prod_{z=0}^{i} h^{z}_{m,d(q)} = 0 \\
& \Rightarrow h^{\#}_{m,d(q)} = \frac{\sum_{n\in\Lambda(m)} \left(y_{m,n}\right)^{\alpha^{i}_{(q)}} \left(\tilde{y}^{i}_{m,n(q)}\right)^{\beta^{i}_{(q)}-1} a_{n,d(q)}}{\sum_{n\in\Lambda(m)} \left(\tilde{y}^{i}_{m,n(q)}\right)^{\alpha^{i}_{(q)}+\beta^{i}_{(q)}-1} a_{n,d(q)} + \alpha^{i}_{(q)}\lambda^{i}_{(q)} \left|\Lambda(m)\right| a^{\#}_{m,d(q)}} \le 1 \\
& \Rightarrow \kappa^{\#}_{m,d(q)} \ge 0
\end{aligned} \tag{S17}
$$

Hence, (S11) is fulfilled if $c^{\#}_{m,d(q)} \ge 0$ with a non-negatively initial hypothesis.