

# Generalized Symmetric Nonnegative Latent Factor Analysis for Large-scale Undirected Weighted Networks

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This is the supplementary file for the paper entitled Generalized Symmetric Nonnegative Latent Factor Analysis for Large-scale Undirected Weighted Networks. Additional convergence proofs of the proposed GSNL model are put into this file.

## 1 Proofs of a GSNL Model

### 1.1 Proof of Lemma 1

With Definition 3, the following inequality is given as,

$$\varphi(c_{(q)}^i) = \varepsilon(c_{(q)}^i, c_{(q)}^i) \geq \varepsilon(c_{(q)}^{i+1}, c_{(q)}^i) \geq \varphi(c_{(q)}^{i+1}) \quad (S1)$$

Then, it is easy to deduce the following sequence based on (S1),

$$\varphi(c_{\min(q)}) \leq \dots \leq \varphi(c_{(q)}^{i+1}) \leq \varphi(c_{(q)}^i) \leq \dots \leq \varphi(c_{(q)}^1) \leq \varphi(c_{(q)}^0) \quad (S2)$$

Considering  $m \in J$ ,  $d \in \{1 \sim D\}$ ,  $c_{m,d(q)} \in C$ , let  $\varphi_{c_{m,d(q)}}$  be the partial loss from  $L(C)$  related to  $c_{m,d(q)}$  only,

$$\varphi_{c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} = \sum_{y_{m,n} \in \Lambda} \left( -\frac{1}{\alpha_{(q)}^i \beta_{(q)}^i} \left( (y_{m,n})^{\alpha_{(q)}^i} (\tilde{y}_{m,n(q)}^i)^{\beta_{(q)}^i} - \frac{\alpha_{(q)}^i}{\alpha_{(q)}^i + \beta_{(q)}^i} (y_{m,n})^{\alpha_{(q)}^i + \beta_{(q)}^i} \right) - \frac{\beta_{(q)}^i}{\alpha_{(q)}^i + \beta_{(q)}^i} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i + \beta_{(q)}^i} \right) + \frac{\lambda_{(q)}^i}{2} \sum_{d=1}^D (c_{m,d(q)}^i)^2 \quad (S3)$$

where  $\tilde{y}_{m,n(q)}^i = c_{m,d(q)}^i c_{n,d(q)} + \sum_{l=1, l \neq d}^D c_{m,l(q)} c_{n,l(q)}$ .

Hence, the first-order and second-order derivatives of  $\varphi_{c_{m,d(q)}}$  with respect to  $c_{m,d(q)}$ ,

$$\begin{aligned} \varphi'_{c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} &= \frac{\partial L}{\partial c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} \\ &= \lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^i + \frac{1}{\alpha_{(q)}^i} \sum_{n \in \Lambda(m)} c_{n,d(q)} \left( (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} - (y_{m,n})^{\alpha_{(q)}^i} (\tilde{y}_{m,n(q)}^i)^{\beta_{(q)}^i - 1} \right) \end{aligned} \quad (S4)$$

$$\begin{aligned}
\varphi''_{c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} &= \frac{\partial^2 L}{\partial (c_{m,d(q)})^2}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} \\
&= \lambda_{(q)}^i |\Lambda(m)| + \frac{1}{\alpha_{(q)}^i} \sum_{n \in \Lambda(m)} (c_{n,d(q)})^2 \left( \frac{1}{\alpha_{(q)}^i + \beta_{(q)}^i - 1} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i + \beta_{(q)}^i - 2} - \frac{(y_{m,n})^{\alpha_{(q)}^i}}{\beta_{(q)}^i - 1} (\tilde{y}_{m,n(q)}^i)^{\beta_{(q)}^i - 2} \right)
\end{aligned} \tag{S5}$$

## 1.2 Proof of Proposition 1

Based on (16),  $\varepsilon(c, c) = \varphi_{c_{m,d(q)}}(c)$  holds. Then we focus on the proofs of  $\varepsilon(c, c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} \geq \varphi_{c_{m,d(q)}}(c) \Big|_{s_{(q)}=s_{(q)}^i}$ .

Firstly, we derive the quadratic approximation to  $\varphi_{c_{m,d(q)}}$  at  $c_{m,d(q)}^i$  under  $s_{(q)} = s_{(q)}^i$ ,

$$\begin{aligned}
\varphi_{c_{m,d(q)}}(c) \Big|_{s_{(q)}=s_{(q)}^i} &= \\
\varphi_{c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} &+ \varphi'_{c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} (c - c_{m,d(q)}^i) + \frac{1}{2} \varphi''_{c_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} (c - c_{m,d(q)}^i)^2
\end{aligned} \tag{S6}$$

By combining (16), (S5) and (S6), we see that  $\varepsilon(c, c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i}$  is an auxiliary function of  $\varphi_{c_{m,d(q)}}(c) \Big|_{s_{(q)}=s_{(q)}^i}$  if the following inequality holds,

$$\begin{aligned}
c_{m,d(q)}^i \sum_{n \in \Lambda(m)} (c_{n,d(q)})^2 (\tilde{y}_{m,n(q)}^i)^{\beta_{(q)}^i - 2} &\left( \frac{1}{\beta_{(q)}^i - 1} (y_{m,n})^{\alpha_{(q)}^i} + \frac{\beta_{(q)}^i - 1}{\alpha_{(q)}^i (\alpha_{(q)}^i + \beta_{(q)}^i - 1)} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i} \right) \geq 0 \\
\Rightarrow \frac{1}{\beta_{(q)}^i - 1} (y_{m,n})^{\alpha_{(q)}^i} &+ \frac{\beta_{(q)}^i - 1}{\alpha_{(q)}^i (\alpha_{(q)}^i + \beta_{(q)}^i - 1)} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i} \geq 0 \\
\Rightarrow \frac{\beta_{(q)}^i - 1}{\alpha_{(q)}^i (\alpha_{(q)}^i + \beta_{(q)}^i - 1)} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i} &\geq \frac{1}{1 - \beta_{(q)}^i} (y_{m,n})^{\alpha_{(q)}^i}
\end{aligned} \tag{S7}$$

## 1.3 Proof of Theorem 1

Based on (15), (16) and (S4), we have,

$$\begin{aligned}
c_{m,d(q)}^{i+1} &= \arg \min_c \varepsilon(c, c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} \\
&\Rightarrow \varphi'_{c_{m,d(q)}}(c_{m,d(q)}^i) + \frac{\lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^i + \frac{1}{\alpha_{(q)}^i} \sum_{n \in \Lambda(m)} c_{n,d(q)} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1}}{c_{m,d(q)}^i} (c - c_{m,d(q)}^i) = 0 \\
&\Rightarrow c_{m,d(q)}^{i+1} \leftarrow c_{m,d(q)}^i \frac{\sum_{n \in \Lambda(m)} (y_{m,n})^{\alpha_{(q)}^i} (\tilde{y}_{m,n(q)}^i)^{\beta_{(q)}^i - 1} c_{n,d(q)}}{\sum_{n \in \Lambda(m)} (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} c_{n,d(q)} + \alpha_{(q)}^i \lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^i}
\end{aligned} \tag{S8}$$

Based on (S8), it is clear that  $\varphi_{c_{m,d(q)}}$  is non-increasing with (13). Hence, *Theorem 1* holds.

Following Theorem 1, if the following condition is fulfilled,

$$(\alpha_{(q)}^i + \beta_{(q)}^i) (y_{m,n})^{\alpha_{(q)}^i} (\tilde{y}_{m,n(q)}^i)^{\beta_{(q)}^i} \leq \alpha_{(q)}^i (y_{m,n})^{\alpha_{(q)}^i + \beta_{(q)}^i} + \beta_{(q)}^i (\tilde{y}_{m,n(q)}^i)^{\alpha_{(q)}^i + \beta_{(q)}^i} \tag{S9}$$

then  $\varphi_{a_{m,d(q)}}(c_{m,d(q)}^i) \Big|_{s_{(q)}=s_{(q)}^i} \geq 0$  holds. Hence, it is easy to deduce that,

$$\begin{aligned}
& \lim_{i \rightarrow +\infty} \left( \varphi_{c_{m,d(q)}} \left( c_{m,d(q)}^{i+1} \right) \Big|_{s_{(q)} = s_{(q)}^{i+1}} - \varphi_{c_{m,d(q)}} \left( c_{m,d(q)}^i \right) \Big|_{s_{(q)} = s_{(q)}^i} \right) \rightarrow 0 \\
& \Rightarrow \\
& \lim_{i \rightarrow +\infty} \left( c_{m,d(q)}^{i+1} - c_{m,d(q)}^i \right) \rightarrow 0
\end{aligned} \tag{S10}$$

Hence, a sequence  $\{c_{m,d(q)}^i\}$  is bounded.

#### 1.4 Proof of Theorem 2

Let  $c_{(q)}^\#$  denote a stationary point of  $C_{(q)}$ . Then, the following KKT conditions of (3) regarding  $C_{(q)}$  should be satisfied, if  $c_{(q)}^\#$  is one of its stationary point.

$\forall m \in J, d \in \{1 \sim D\}$ :

$$\begin{aligned}
(a) \frac{\partial F}{\partial c_{m,d(q)}} \Big|_{c_{m,d(q)} = c_{m,d(q)}^\#, s_{(q)} = s_{(q)}^i} &= \sum_{n \in \Lambda(m)} \left( \frac{1}{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} c_{n,d(q)} + \lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^\# \right. \\
&\quad \left. - \frac{1}{\alpha_{(q)}^i} \left( y_{m,n} \right)^{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\beta_{(q)}^i - 1} c_{n,d(q)} \right) - \kappa_{m,d(q)}^\# = 0, \\
(b) \kappa_{m,d(q)}^\# c_{m,d(q)}^\# &= 0, \\
(c) c_{m,d(q)}^\# &\geq 0, \\
(d) \kappa_{m,d(q)}^\# &\geq 0.
\end{aligned} \tag{S11}$$

Note that following (4)-(7), Conditions (S11a) and (S11b) are naturally fulfilled with (13). Then we have,

$$\kappa_{m,d(q)}^\# = \sum_{n \in \Lambda(m)} \left( \frac{1}{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} c_{n,d(q)} + \lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^\# \right. \\
\left. - \frac{1}{\alpha_{(q)}^i} \left( y_{m,n} \right)^{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\beta_{(q)}^i - 1} c_{n,d(q)} \right) \tag{S12}$$

Thus, we focus on Conditions (S11c)-(S11d). We start with constructing  $h_{m,d(q)}^i$ ,

$$h_{m,d(q)}^i = \frac{\sum_{n \in \Lambda(m)} \left( y_{m,n} \right)^{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\beta_{(q)}^i - 1} c_{n,d(q)}}{\sum_{n \in \Lambda(m)} \left( \tilde{y}_{m,n(q)}^i \right)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} c_{n,d(q)} + \alpha_{(q)}^i \lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^i} \tag{S13}$$

From (S13), we can clearly see that  $\lim_{i \rightarrow +\infty} h_{m,d(q)}^i = h_{m,d(q)}^\# \geq 0$ .

Then we can rewrite (13) as follows,

$$c_{m,d(q)}^{i+1} = c_{m,d(q)}^i h_{m,d(q)}^i \Rightarrow c_{m,d(q)}^\# h_{m,d(q)}^\# - c_{m,d(q)}^\# = 0 \tag{S14}$$

Note that following (13),  $c_{m,d(q)}^\# \geq 0$  with a non-negatively initial hypothesis. Hence, we have the following inferences,

(1) **When**  $c_{m,d(q)}^\# > 0$ . Based on (S13) and (S14), we have,

$$\begin{aligned}
c_{m,d(q)}^\# h_{m,d(q)}^\# - c_{m,d(q)}^\# &= 0, c_{m,d(q)}^\# > 0 \\
\Rightarrow h_{m,d(q)}^\# &= 1 \\
\Rightarrow \sum_{n \in \Lambda(m)} \left( \tilde{y}_{m,n(q)}^i \right)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} c_{n,d(q)} + \alpha_{(q)}^i \lambda_{(q)}^i |\Lambda(m)| c_{m,d(q)}^\# \\
&\quad - \sum_{n \in \Lambda(m)} \left( y_{m,n} \right)^{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\beta_{(q)}^i - 1} c_{n,d(q)} = 0
\end{aligned} \tag{S15}$$

Combining (S12) and (S14), we can achieve  $\kappa_{m,d(q)}^\# = 0$ .

(2) **When**  $c_{m,d(q)}^\# = 0$ . We reformulate  $c_{m,d(q)}^\#$  into,

$$c_{m,d(q)}^\# = c_{m,d(q)}^0 \lim_{i \rightarrow +\infty} \prod_{z=0}^i h_{m,d(q)}^z \tag{S16}$$

Based on (S16), we further have the following inferences,

$$\begin{aligned}
c_{m,d(q)}^0 > 0, c_{m,d(q)}^0 \lim_{i \rightarrow +\infty} \prod_{z=0}^i h_{m,d(q)}^z &= c_{m,d(q)}^\# = 0 \\
\Rightarrow \lim_{i \rightarrow +\infty} \prod_{z=0}^i h_{m,d(q)}^z &= 0 \\
\Rightarrow h_{m,d(q)}^\# &= \frac{\sum_{n \in \Lambda(m)} \left( y_{m,n} \right)^{\alpha_{(q)}^i} \left( \tilde{y}_{m,n(q)}^i \right)^{\beta_{(q)}^i - 1} a_{n,d(q)}}{\sum_{n \in \Lambda(m)} \left( \tilde{y}_{m,n(q)}^i \right)^{\alpha_{(q)}^i + \beta_{(q)}^i - 1} a_{n,d(q)} + \alpha_{(q)}^i \lambda_{(q)}^i |\Lambda(m)| a_{m,d(q)}^\#} \leq 1 \\
\Rightarrow \kappa_{m,d(q)}^\# &\geq 0
\end{aligned} \tag{S17}$$

Hence, (S11) is fulfilled if  $c_{m,d(q)}^\# \geq 0$  with a non-negatively initial hypothesis.