Option Valuation

Introduction

There are two main methods to price options:

- The partial differential equation method.
- The martingale method (also known as the risk-neutral method).

In this chapter both are discussed.

Also, there is a way to link the two methods: the so-called Feynman-Kac representation (the book explains a bit about it, I do not think we will get into that).

Introduction

Key concepts in option valuation:

- The self-financing replicating portfolio.
- The martingale probability (also known as the risk-neutral probability).

Introduction

In a financial market, a so-called arbitrage opportunity is a situation where an investor can make a guaranteed profit without incurring any risk.

Such situations arise regularly and there are people who specialize in spotting them, using sophisticated communication technology.

They then immediately initiate a trade which changes the supply-demand situation, and restores the market price to equilibrium.

Arbitrage opportunities are therefore very short lived. The fundamental condition for establishing the price of an option is that it should not permit an arbitrage

Absence of arbitrage is a highly realistic assumption, and more tangible than the equilibrium assumptions in economics.

Introduction

Notation:
$S(t)$ price of underlying stock at time t.
K strike price at which underlying can be bought or sold.
T maturity time of option contract.
r risk-free interest rate.
$V(t)$ option value at time t.
$S^*(t)$ stock price discounted by savings account, $e^{-rt}S(t)$.
$\hat{\mathbb{P}}$ probability distribution under which $S^*(t)$ is a martingale.
$\mathbb{E}_{\hat{\mathbb{P}}}$ expected value under $\hat{\mathbb{P}}$.

PDE Method

The value of an option on a stock is modelled as a function of two variables (t and stock price $S(t)$) and is denoted $V(t)$.

As usual:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$$

Then, by Itô:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2$$

Replacing and using the rule from stochastic calculus:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} (\mu S dt + \sigma S dB) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S^2 \sigma^2 dt$$

Or:

$$dV = (\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2)dt + \frac{\partial V}{\partial S}\sigma S dB$$

The partial differential equation method of option valuation is based on the insight that the option and the stock on which it is written have the same source of randomness.

Thus, by taking opposite positions in the option and the stock, the randomness of the one asset can offset the randomness of the other.

It is therefore possible to form a portfolio of stock and options in such proportion that the overall randomness of this portfolio is zero.

Moreover, if the proportion of stock and options in this portfolio is changed as the value of the stock changes, this portfolio can be maintained riskless at all times.

PDE Method

At time t, form a portfolio that is long λ shares and short 1 option. The value P of this portfolio at time t is

$$P(t) = \lambda S(t) - V(t)$$

and its differential:

$$dP = \lambda dS - dV$$

Plugging in we see that the random term is:

$$\lambda \sigma S - \sigma S \frac{\partial V}{\partial S}$$

which is 0 if $\lambda = \frac{\partial V}{\partial S}$.

PDE Method

The equation for P results:

$$dP = \left(\lambda \mu S - \frac{\partial V}{\partial t} - \mu S \frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

The portfolio is then riskless so its value must increase in accordance with the risk-free interest rate, otherwise there would be an arbitrage opportunity.

The interest accrued on 1 unit of money over a time interval of length dt is $1\, r\, dt$.

The value of the portfolio thus grows by $P\, r\, dt$ over dt.
Equating the two expressions for the change in the value of P gives:

PDE Method

Replacing the left hand side by $P\,r\,dt$, P by $V - \lambda S$ and λ by $\frac{\partial V}{\partial S}$ we obtain, after simplifying:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = r\,V$$

This is a second-order partial differential equation (PDE) in the unknown function V.

The fact that this PDE does not contain the growth rate μ of the stock price may be surprising at first sight.

PDE Method

The PDE must be accompanied by the specification of the option value at the time of exercise, the so-called option payoff.

The PDE method was developed by Black and Scholes using key insights by Merton.

Merton and Scholes were awarded the 1997 'Nobel prize' in Economics for this seminal work; Black had died in 1995.

The PDE derivation given here is attractive for its clarity. There are, however, other derivations which are considered to be more satisfactory in a technical mathematical sense.

We will see how to solve it (using PDE methods) on a different set of notes.

Martingale Method in One-Period Binomial Framework

The approach is to form a so-called replicating portfolio which comprises shares of the stock on which the option contract is based and an amount of risk-free borrowing.

If this portfolio can replicate the value of the option at all times then the initial value of the portfolio must be the value of the option.

If that was not the case then there would be an arbitrage opportunity in the market and the price would not be stable.

The value of the replicating portfolio is a random process whose value is denoted V.

The initial portfolio consists of α shares of initial price $S(0) = S$ and a loan of β, where α and β are to be determined.

Martingale Method in One-Period Binomial Framework

$$V(0) = \alpha S + \beta$$

During the period $[0, T]$, using discrete compounding at rate r for the period, the amount β grows to $\beta\,(1 + r)$, and S becomes $S\,u$ or $S\,d$.

The terminal value of the portfolio, $V(T)$, is then

- Up-state: $V(T)_{up} = \alpha Su + \beta(1 + r)$.
- Down-state: $V(T)_{down} = \alpha Sd + \beta(1 + r)$.

On the other hand, at time T (expiration), it must be true that:

- Up-state: $V(T)_{up} = \max(Su - K, 0)$.
- Down-state: $V(T)_{down} = \max(Sd - K, 0)$.

Martingale Method in One-Period Binomial Framework

From here we get that:

$$\alpha = \frac{V(T)_{up} - V(T)_{down}}{(u-d)\,S}, \;\; \beta = \frac{V(T)_{down}\,u - V(T)_{up}\,d}{(1+r)(u-d)}$$

Plugging in:

$$V(0) = \frac{(1+r)-d}{u-d}\,\frac{V(T)_{up}}{1+r} + \frac{u-(1+r)}{u-d}\,\frac{V(T)_{down}}{1+r}$$

If we call

$$q = \frac{(1+r)-d}{u-d}, \;\; 1-q = \frac{u-(1+r)}{u-d}$$

(If $d < 1+r < u$ then q is in $(0,1)$)
then:

$$V(0) = q\,\frac{V(T)_{up}}{1+r} + (1-q)\,\frac{V(T)_{down}}{1+r}$$

Martingale Method in One-Period Binomial Framework

The discounted option value process is a discrete martingale when probability q is used.

For this reason q is also known as the martingale probability.

Probability q is an artificial probability and its sole use is in the valuation of the option. It is not the real probability of an up-movement of the stock price.

Multiplying both sides of the expression for q by S and rearranging gives $(1 + r)S = q(uS) + (1 - q)(dS)$. The left-hand side is the terminal value when an amount S is invested in a risk-free savings account. The right-hand side is the expected value of the stock price if the amount S is used to purchase a stock.

The equation says that the investor is, in expected value terms, indifferent to whether the amount S is invested in a risk-free savings account or whether it is invested in a stock. Because of this interpretation, q is also called the risk-neutral probability.

It is as if the investor is indifferent to the risk of the stock price increment, when probability q is used in the valuation of the option.

That, of course, is not the true attitude of an investor towards risk. The option value computation turns out to be the correct one if the investor is treated as being risk-neutral, hence the name risk-neutral probability.

Probability q was determined without using investors' views on the probability of an up-movement. If the expectation is taken under a value that is different from q then the result is an option value that permits arbitrage. So the risk-neutral probability is linked to the absence of arbitrage.

Martingale Method in One-Period Binomial Framework

No-Arbitrage Condition

Assume, without loss of generality, that $S = 1$.

If α and β can be chosen such that initial investment $V(0)$ is zero, and terminal values $V(T)_{up}$ and $V(T)_{down}$ are both non-negative, but either $V(T)_{up}$ or $V(T)_{down}$ is strictly positive, then this is a scheme which produces a non-negative return for certain without any down-side risk.

That is an arbitrage opportunity.

Martingale Method in One-Period Binomial Framework

The presence or absence of arbitrage is analyzed by looking at the quantities:

$$V(0) = \alpha + \beta$$

$$V_{up}(T) = \alpha\, u + \beta\,(1 + r)$$

$$V_{down}(T) = \alpha\, d + \beta\,(1 + r)$$

Martingale Method in One-Period Binomial Framework

It turns out that absence of arbitrage is equivalent to:

$$d \leq (1 + r) \leq u$$

which is intuitive:

- If both branches of the tree grow more than $(1 + r)$ one has to borrow money and buy the stock.
- If both branches of the tree grow less than $(1 + r)$ one has to short the stock and invest the cash.

Martingale Method in One-Period Binomial Framework

Another way to look at $d \leq (1+r) \leq u$ is to say that $(1+r)$ is a convex combination of d and u.

$$1 + r = q\,u + (1-q)\,d$$

and q ends up being what we called the 'risk-neutral probability'.

Martingale Method in One-Period Binomial Framework

Knowing that the discounted value process is a martingale makes it possible to determine the initial value of the option from the terminal values of the option which are known.

So the martingale concept is used in reverse. That means that the valuation process can be carried out the other way around, namely by first finding a risk-neutral probability, and then taking the expected value under that probability of the discounted terminal option value.

Martingale Method in One-Period Binomial Framework

One other feature has to be mentioned: the replicating portfolio must be self-financing, which means that the change in portfolio value over time should only come from the change in the value of the stock and the change in borrowing. No money is withdrawn or added freely.

The above showed that a martingale probability arises in a completely natural way. It can also be derived from the condition that the discounted stock price process must be a martingale under discrete probability q.

Martingale Method in One-Period Binomial Framework

The expectation of the terminal discounted stock price under q is

$$\mathbb{E}_q\left(\frac{S(T)}{1+r} \mid S\right) = S$$

Now, using q, what is the expectation of the discounted terminal portfolio value?

$$\mathbb{E}_q\left(\frac{V(T)}{1+r}\right) = q\frac{V_{up}(T)}{1+r} + (1-q)\frac{V_{down}(T)}{1+r} = V(0)$$

Which is checked just by plugging in $V_{up}(T)$ and $V_{down}(T)$ and cleaning up all the terms.

Martingale Method in One-Period Binomial Framework

Since the terminal option values $V(T)$ are known in every terminal state of the market, the unknown initial value of the replicating portfolio, $V(0)$, and thus of the option, can be determined by turning the martingale expression around

$$\text{Initial option value} = \mathbb{E}_q \left(\frac{\text{Terminal option value}}{1 + r} \right)$$

Martingale Method in Continuous-Time Framework

Let us consider the discounted stock price:

$$S^*(t) = \frac{S(t)}{e^{r\,T}}$$

This expresses the stock price value in terms of the savings account as the numeraire

We can, using Itô find the SDE for S^*:

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\,dt + \sigma\,dB(t)$$

The expected growth rate of S^* is $(\mu - r)$, which is r less than the growth rate of S, because it is measured against the riskless growth rate r of the savings account.

As this SDE has a drift, the process S^* is not a martingale under the probability distribution of Brownian motion B.

Martingale Method in Continuous-Time Framework

In the replicating portfolio of the above discrete-time model, the discounted stock price was a martingale.

That led to the discounted replicating portfolio also being a martingale.

The same recipe will be followed here. A probability distribution can be found under which S^* is a martingale.

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\,dt + \sigma\,dB(t) = \sigma\left(\frac{\mu - r}{\sigma}\,dt + \,dB(t)\right)$$

$$= \sigma\,(\varphi\,dt + \,dB(t))$$

Martingale Method in Continuous-Time Framework

The idea now is to make $(\varphi\, dt + dB(t))$ into a Brownian motion by changing the measure.

At time t if $B(t) = x$ then $\varphi\, t + B(t) = y$.

The density of $B(t)$ can be written in terms of $\varphi\, t + x$:

$$\frac{1}{\sqrt{t}\sqrt{2\,\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2} = \frac{1}{\sqrt{t}\sqrt{2\,\pi}} e^{-\frac{1}{2}\left(\frac{\varphi\, t+x}{\sqrt{t}}\right)^2} e^{\frac{1}{2}\varphi^2\, t+\varphi\, x}$$

If we call $y = \varphi\, t + x$ then the first exponential on the rhs is the density of a new Brownian motion, $\widehat{B}(t) = y$.

The value of $\widehat{B}(t)$ is $\varphi\, t + B(t)$ and it is Brownian motion provided we look at it in the appropriate measure.

Martingale Method in Continuous-Time Framework

So, the SDE for $S^*(t)$ results:

$$\frac{dS^*(t)}{S^*(t)} = \sigma \, d\widehat{B}(t)$$

This says that under the probability distribution of Brownian motion $\widehat{B}(t)$, S^* is a martingale. Let this probability be denoted $\widehat{\mathbb{P}}$.

Martingale Method in Continuous-Time Framework

The value of the replicating portfolio at time t is denoted $V(t)$.

The portfolio consists of a quantity $\alpha(t)$ of stock $S(t)$ and an amount $\beta(t)$ of risk-free borrowing.

The evolution of $\beta(t)$ is specified by the ordinary differential equation $d\beta(t) = \beta(t)rdt$. Then, the portfolio is:

$$V(t) = \alpha(t)\,S(t) + \beta(t)$$

and it should be self-financing, so the change in the value of the portfolio must only come from the change in the value of the stock and the change in the value of the borrowing.
This condition is represented by

$$dV(t) = \alpha(t)\,dS(t) + d\beta(t)$$

Martingale Method in Continuous-Time Framework

Considering the discounted value of $V(t)$:

$$V^*(t) = \frac{V(t)}{e^{r\,t}}$$

by Itô:

$$dV^*(t) = -r\,V^*(t)\,dt + e^{-r\,t}\,dV$$

Replacing now dV, $V^*(t)$ and collecting terms:

$$dV^*(t) = \alpha(t)\,dS^*(t)$$

As there is no drift term, random process $V^*(t)$ is a martingale under $\widehat{\mathbb{P}}$.

Martingale Method in Continuous-Time Framework

Then, under $\widehat{\mathbb{P}}$, we can find the current value by taking expectations of the values at some time in the future.

$$\mathbb{E}_{\widehat{\mathbb{P}}}(V^*(T)) = V^*(0)$$

or:

$$e^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}(V(T)) = V(0)$$

This argument is usually used to find option prices: if, given an option we can find a replicating, self-financed, portfolio, then we can find its value at time 0.

Overview of Risk-Neutral Method

In both the discrete and the continuous-time framework the discounted stock price is forced to become a martingale.

That produces a new probability distribution, which is called the martingale probability or risk-neutral probability. Using this new probability distribution, the expected value of the discounted stock price at some future time equals the known discounted stock price at an earlier time.

Then a self-financing portfolio is formed which replicates the value of an option at all times in all possible states of the market.

The discounted value of this self-financing replicating portfolio is also found to be a martingale under this new probability.

Overview of Risk-Neutral Method

Thus the expectation of the discounted value of the replicating portfolio, at the time of exercise of the option, equals the present value of the portfolio.

As all possible values of the option at the time of exercise are known, and the martingale probability is also known, their expected value can be computed.

At the present time, the discounted value of the replicating portfolio equals the value of the option that is to be determined.

Hence the fact that the discounted portfolio value is a martingale makes it possible to compute this present value of the option. The steps in the methodology are summarized below.

Overview of Risk-Neutral Method

Table 6.1

	Stock price	Portfolio
Value at time t	$S(t)$	$V(t) = \alpha(t)S(t) + \beta(t)$
Dynamics under $B(t)$	$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dB(t)$	self-financing $dV(t) = \alpha(t)\, dS(t) + d\beta(t)$
Discounted value at time t	$S^*(t) = \exp(-rt)S(t)$	$V^*(t) = \exp(-rt)V(t)$
Dynamics of discounted value under $B(t)$	$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\, dt + \sigma\, dB(t)$	$dV^*(t) = \alpha(t)\, dS^*(t)$
Property	$S^*(t)$ *not* martingale	$V^*(t)$ *not* martingale
Dynamics of discounted value under $\widehat{B}(t)$	$\frac{dS^*(t)}{S^*(t)} = \sigma\, d\widehat{B}(t)$	$dV^*(t) = \alpha(t)\, dS^*(t)$ $= \alpha(t)\sigma S^*(t)\, d\widehat{B}(t)$
Property	$S^*(t)$ *martingale* $\mathbb{E}_{\widehat{\mathbb{P}}}[S^*(T)\mid S(0)] = S(0)$	$V^*(t)$ *martingale* $\mathbb{E}_{\widehat{\mathbb{P}}}[V^*(T)\mid V(0)] = V(0)$

Martingale Method: A computation that appears often

In the risk-neutral world:

$$S^*(T) = S^*(0)\, e^{-\frac{1}{2}\sigma^2 T + \sigma \widehat{B}(T)}$$

where $S^*(t) = \frac{S(t)}{e^{rt}}$. Therefore:

$$\log(S(T)) = \log(S(0)) + (r - \frac{1}{2}\sigma^2)T + \sigma \widehat{B}(T)$$

Now, if $S(T) \geq K$ then

$$\log(S(0)) + (r - \frac{1}{2}\sigma^2)T + \sigma \widehat{B}(T) \geq \log(K)$$

or

$$\widehat{B}(T) \geq -\left(\frac{\log(\frac{S(0)}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma}\right) := a$$

Martingale Method: Digital Call

It pays \$1 if $S(T) > K$ and 0 otherwise.

The discounted call value process $\frac{c(t)}{e^{rt}}$ is a martingale under probability $\widehat{\mathbb{P}}$, so at time 0:

$$\frac{c(0)}{e^{r\,0}} = \mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{c(T)}{e^{rT}}\right) = e^{-r\,T}\,\widehat{\mathbb{P}}(S(T) > K)$$

$$e^{-r\,T}\,\widehat{\mathbb{P}}(\widehat{B}(T) > a)$$

Since we know that $\widehat{B}(T) \sim N(0, T)$ then $\frac{\widehat{B}(T)}{\sqrt{T}} \sim N(0, 1)$:

$$= e^{-r\,T}\,\widehat{\mathbb{P}}(X > \frac{a}{\sqrt{T}})$$

with X standard normal (under $\widehat{\mathbb{P}}$).

Martingale Method: Digital Call

It results that:

$$c(0) = e^{-r\,T}\,(1 - N(\frac{a}{\sqrt{T}})) = e^{-r\,T}\,N(-\frac{a}{\sqrt{T}})$$

where:

$$-\frac{a}{\sqrt{T}} = \left(\frac{\log(\frac{S(0)}{K}) + (r - \frac{1}{2}\sigma^2)\,T}{\sigma\,\sqrt{T}}\right) := d_2$$

$$c(0) = e^{-r\,T}\,N(d_2)$$

Martingale Method: Asset-or-Nothing Call

This option pays on the same set as the digitals but the payment is $S(T)$:

$$c(T) = S(T)\,1_{\{S(T)\geq K\}}$$

As before:

$$\frac{c(0)}{e^{r\,0}} = \mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{c(T)}{e^{rT}}\right) = e^{-r\,T}\,\mathbb{E}_{\widehat{\mathbb{P}}}(S(T)\,1_{\{S(T)\geq K\}})$$

$$= \mathbb{E}_{\widehat{\mathbb{P}}}(S^*(T)\,1_{\{S^*(T)\geq e^{-rT}K\}})$$

Martingale Method: Asset-or-Nothing Call

Now:

$$S^*(T) = S^*(0)\, e^{-\frac{1}{2}\sigma^2 T + \sigma \widehat{B}(T)}$$

and

$$1_{\{S^*(T) \geq e^{-rT} K} = 1_{\widehat{B}(T) \geq a}$$

Then:

$$\mathbb{E}_{\widehat{\mathbb{P}}}(S^*(T)\, 1_{\{S^*(T) \geq e^{-rT} K\}}) = \mathbb{E}_{\widehat{\mathbb{P}}}(S^*(0)\, e^{-\frac{1}{2}\sigma^2 T + \sigma \widehat{B}(T)}\, 1_{\{\widehat{B}(T) \geq a\}})$$

Computing the expectation:

$$\int_a^{\infty} S(0)\, e^{-\frac{1}{2}\sigma^2 T+\sigma x}\, \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2} dx$$

Changing variables $y = \frac{x}{\sqrt{T}}$ this can be written as:

$$S(0) \int_{a/\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sigma\sqrt{T})^2} dy = S(0) \int_{a/\sqrt{T}-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dy$$

$$= S(0)\,(1 - N(a/\sqrt{T} - \sigma\sqrt{T})) = S(0)\, N(-a/\sqrt{T} + \sigma\sqrt{T})$$

$$= S(0)\, N(d_2 + \sigma\sqrt{T})$$

We call $d_1 := d_2 + \sigma\sqrt{T}$. Putting everything together:

$$c(0) = S(0)\, N(d_1)$$

Martingale Method: Standard European Call

Exactly the same idea:

$$c(T) = \max(S(T) - K, 0)$$

$$c(0) = e^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}(c(T)) = \mathbb{E}_{\widehat{\mathbb{P}}}(\max(S^*(T) - e^{-rT}K, 0))$$

$$\int_a^{\infty} (S(0)e^{-\frac{1}{2}\sigma^2 T+\sigma x}\,\frac{1}{\sqrt{2\pi T}} - e^{-rT}K)e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2}$$

Because of the same reasons as in the previous examples.

(a is the same also).

Martingale Method: Standard European Call

Actually, this is exactly the combination of the previous two examples:

$$\int_a^\infty (S(0)e^{-\frac{1}{2}\sigma^2 T+\sigma x}\frac{1}{\sqrt{2\pi T}} - e^{-rT}K)e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2}dx$$

$$= \int_a^\infty S(0)e^{-\frac{1}{2}\sigma^2 T+\sigma x}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2}dx$$

$$-e^{-rT}K\int_a^\infty e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2}dx = S(0)N(d_1) - e^{-rT}KN(d_2)$$

Martingale Method: Standard European Call

This is the famous Black-Scholes pricing expression for a standard European call.

Setting $K = 0$ recovers the pricing expression for the asset-or-nothing call.

Note that the second term in the Black-Scholes expression equals K times the digital call price, and that the first term equals the all-or-nothing call price.

Thus a standard European call can be seen as a portfolio that is long K European digital calls, and short one all-or-nothing call.

Multi-Period Binomial Link to Continuous

By dividing the time to maturity T into n time-steps $\Delta t = \frac{T}{n}$, and repeating the binomial stock price increment, a tree like that we saw before is produced with a stock price at each node.

$n + 1$ stock prices at maturity.

As for the random walk, it is assumed that the stock price increments over successive time-steps are independent.

There are several choices for the tree parameters u and d. The original, and most widely known, is

$$u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}$$

Link Between Methods

Multi-Period Binomial Link to Continuous

It can be shown that for $n \to \infty$:

- the binomial scheme converges to the Black-Scholes PDE
- the terminal stock price converges to the standard continuous time terminal stock price
- the binomial value of an option converges to the Black-Scholes value.

Link Between Methods

Multi-Period Binomial Link to Continuous

The binomial option value is highly non-linear in n.

The convergence pattern depends greatly on the ratio K/S, and for the above choice of u and d it is highly irregular when $K \neq S$.

This makes it difficult to fix a suitable n. A better choice is

$$u = e^{\sigma\sqrt{\Delta t} + \frac{1}{n}\log(K/S)}, d = e^{-\sigma\sqrt{\Delta t} + \frac{1}{n}\log(K/S)}$$

Now the convergence pattern is gradual for all K/S.